FTUV-03-0114
IFIC-02-20

Pinch technique self-energies and vertices to all orders in perturbation theory

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Abstract

The all-order construction of the pinch technique gluon self-energy and quark-gluon vertex is presented in detail within the class of linear covariant gauges. The main ingredients in our analysis are the identification of a special Green's function, which serves as a common kernel to all self-energy and vertex diagrams, and the judicious use of the Slavnov-Taylor identity it satisfies. In particular, it is shown that the ghost-Green's functions appearing in this identity capture precisely the result of the pinching action at arbitrary order. By virtue of this observation the construction of the quark-gluon vertex becomes particularly compact. It turns out that the aforementioned ghost-Green's functions play a crucial role, their net effect being the non-trivial modification of the ghost diagrams of the quark-gluon vertex in such a way as to reproduce dynamically the characteristic ghost sector of the background field method. The gluon self-energy is also constructed following two different procedures. First, an indirect derivation is given, by resorting to the strong induction method and the assumption of the uniqueness of the S-matrix. Second, an explicit construction based on the intrinsic pinch technique is provided, using the Slavnov-Taylor identity satisfied by the all-order three-gluon vertex nested inside the self-energy diagrams. The process-independence of the gluon self-energy is also demonstrated, by using gluons instead of quark as external test particles, and identifying the corresponding kernel function, together with its Slavnov-Taylor identity. Finally, the general methodology for carrying out the renormalization of the resulting Green's functions is outlined, and various open questions are briefly discussed.

PACS numbers: 11.15.Bt,11.55.Fv,12.38.Bx,14.70.Dj

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I. INTRODUCTION

When quantizing gauge theories in the continuum one must invariably resort to an appropriate gauge-fixing procedure in order to remove redundant (non-dynamical) degrees of freedom originating from the gauge invariance of the theory [1]. Thus, one adds to the gauge invariant (classical) Lagrangian $\mathcal{L}_{\rm I}$ a gauge-fixing term $\mathcal{L}_{\rm GF}$, which allows for the consistent derivation of Feynman rules. At this point a new type of redundancy makes its appearance, this time at the level of the building blocks defining the perturbative expansion. In particular, individual off-shell Green's functions (n-point functions) carry a great deal of unphysical information, which disappears when physical observables are formed. S-matrix elements, for example, are independent of the gauge-fixing scheme and parameters chosen to quantize the theory, they are gauge-invariant (in the sense of current conservation), they are unitary (in the sense of conservation of probability), and well behaved at high energies. On the other hand Green's functions depend explicitly (and generally non-trivially) on the gauge-fixing parameter entering in the definition of $\mathcal{L}_{\rm GF}$, they grow much faster than physical amplitudes at high energies (e.g. they grossly violate the Froissart-Martin bound [2]), and display unphysical thresholds. Last but not least, in the context of the standard path-integral quantization by means of the Faddeev-Popov Ansatz, Green's functions satisfy complicated Slavnov-Taylor identities (STIs) [3] involving ghost fields, instead of the usual Ward identities (WIs) generally associated with the original gauge invariance.

The above observations imply that in going from unphysical Green's functions to physical amplitudes, subtle field theoretical mechanisms are at work, implementing vast cancellations among the various Green's functions. Interestingly enough, these cancellations may be exploited in a very particular way by the Pinch Technique (PT) [4, 5, 6, 7]: a given physical amplitude is reorganized into sub-amplitudes, which have the same kinematic properties as conventional n-point functions (self-energies,vertices,boxes) but, in addition, they are endowed with important physical properties. This has been accomplished diagrammatically, at the one- and two-loop level, by recognizing that *longitudinal momenta* circulating inside vertex and box diagrams generate (by "pinching" out internal fermion lines) propagator-like terms. The latter are reassigned to conventional self-energy graphs in order to give rise to effective Green's functions which manifestly reflect the properties generally associated with physical observables. In particular, the PT Green's function are independent of the

gauge-fixing scheme and parameters chosen to quantize the theory (ξ in covariant gauges, n_μ in axial gauges, etc.), they are gauge-invariant, *i.e.*, they satisfy simple tree-level-like WIs associated with the gauge symmetry of the classical Lagrangian $\mathcal{L}_{\rm I}$, they display only *physical thresholds*, and, finally, they are well behaved at high energies.

There are two basic questions that are of particular relevance in this context: (*i*) what are the conceptual and phenomenological advantages of being able to work with such special Green's functions, and (*ii*) how to achieve their systematic construction to all orders in perturbation theory. Before turning to the second question, which constitutes the main thrust of this paper, we will briefly discuss the first one, in an attempt to physically motivate the technical presentation that will follow [8].

- *QCD effective charge:* The unambiguous extension of the concept of the gauge-independent, renormalization group invariant, and process independent [9] effective charge from QED to QCD [4, 10], is of special interest for several reasons [11]. The PT construction of this quantity accomplishes the explicit identification of the conformally-(in)variant subsets of QCD graphs [12], usually assumed in the field of renormalon calculus [13]. In addition, the PT effective charge can serve as the natural scheme for defining the coupling in the proposed "event amplitude generators" based on the the light-cone formulation of QCD [14].

- *Breit-Wigner resummations, resonant transition amplitudes, unstable particles:* The Breit-Wigner procedure used for regulating the physical singularity appearing in the vicinity of resonances ($\sqrt{s} \sim M$) is equivalent to a *reorganization* of the perturbative series [15]. In particular, the Dyson summation of the self-energy $\Pi(s)$ amounts to removing a particular piece from each order of the perturbative expansion, since from all the Feynman graphs contributing to a given order n one only picks the part which contains n self-energy bubbles $\Pi(s)$, and then one takes $n \to \infty$. Given that non-trivial cancellations involving the various Green's function is generally taking place at any given order of the conventional perturbative expansion, the act of removing one of them from each order may distort those cancellations; this is indeed what happens when constructing non-Abelian *running widths*. The PT ensures that all unphysical contributions contained inside $\Pi(s)$ have been identified and properly discarded, *before* $\Pi(s)$ undergoes resummation [16].

- *Off-shell form-factors:* In non-Abelian theories their proper definition poses in general problems related to the gauge invariance [17]. Some representative cases have been the magnetic dipole and electric quadrupole moments of the W [18], the top-quark magnetic moment [19], and the neutrino charge radius [20]. The PT allows for an unambiguous definition of such quantities; most notably, the gauge-independent, renormalization-group- invariant, and target-independent neutrino charge radius constitutes a genuine *physical* observable, since it can be extracted (at least in principle) from experiments [21].

- *Schwinger-Dyson equations:* This infinite system of coupled non-linear integral equations for all Green's functions of the theory is inherently non-perturbative and can accommodate phenomena such as chiral symmetry breaking and dynamical mass generation. In practice one is severely limited in their use, and a self-consistent truncation scheme is needed. The main problem in this context is that the Schwinger-Dyson equations are built out of gauge-dependent Green's functions; since the cancellation mechanism is very subtle, involving a delicate conspiracy of terms from *all orders*, a casual truncation often gives rise to gauge-dependent approximations for ostensibly gauge-independent quantities [22, 23]. The role of the PT in this problem is to (eventually) trade the conventional Schwinger-Dyson series for another, written in terms of the new, gauge-independent building blocks [4, 24, 25]. The upshot of this program would then be to truncate this new series, by keeping only a few terms in a "dressed-loop" expansion, and maintain exact gauge-invariance, while at the same time accommodating non-perturbative effects.

- Other interesting applications include the gauge-invariant formulation of the ρ parameter at one-[26] and two-loops [27], various finite temperature calculations [28], a novel approach to the comparison of electroweak data with theory [29], resonant CP violation [30], the construction of the two-loop PT quark self-energy [31], and more recently the issue of particle mixings [32].

After this digression we return to the main issue to be addressed in this paper, namely the generalization of the PT to all orders. The original one-loop [4] and two-loop [33] PT calculations consist in carrying out algebraic manipulations inside individual box- and

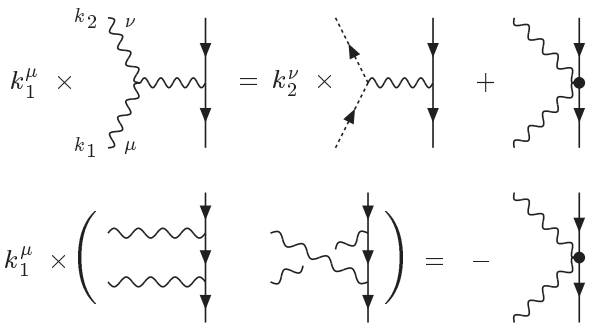


FIG. 1: The tree-level version of the fundamental s-t channel cancellation.

vertex-diagrams, following well-defined rules. In particular one tracks down the rearrangements induced when the action of (virtual) longitudinal momenta (k) on the bare vertices of diagrams trigger elementary WIs. Eventually a WI of the form $k_\mu \gamma^\mu = S^{-1}(\not{k}+\not{p}) - S^{-1}(\not{p})$ will give rise to propagator-like parts, by removing (pinching out) the internal bare fermion propagator $S(\not{k}+\not{p})$ [34]. Depending on the order and topology of the diagram under consideration, the final WI may be activated immediately, as happens at one-loop [4, 5], or as the final outcome of a sequential triggering of intermediate WIs, as happens at two-loops [33]. The propagator-like contributions so obtained are next reassigned to the usual gluon self-energies, giving rise to the PT gluon self-energy. The longitudinal momenta responsible for these rearrangements stem either from the bare gluon propagators or from the pinching part appearing in the characteristic decomposition of the tree-level (bare) three-gluon vertex.

As we will explain in detail in what follows, the aforementioned rearrangements are but lower-order manifestations of a fundamental cancellation taking place between graphs of distinct kinematic nature when computing the divergence of the four-point function $A^a_\mu\, A^b_\nu\, q^i\, \bar{q}^j$, with the gluons A^a_μ, A^b_ν off-shell, and the quarks q^i, $\bar{q}^j$ on-shell). The importance of this particular amplitude has first been recognized in the third paper of [16], where the tree-level version of this cancellation was considered: when the s-channel and t-channel diagrams of Fig.1 (*i.e.*, the tree-level contribution to the amplitude $A^a_\mu\, A^b_\nu\, q^i\, \bar{q}^j$) are contracted by a common longitudinal momentum, one obtains from either graph a common, propagator-like part, which eventually cancels against the other. These parts display the characteristic feature that, when depicted by means of Feynman diagrams, they contain unphysical vertices (Fig.1), *i.e.*, vertices which do not exist in the original Lagrangian [35] ; they correspond

precisely to the "pinch parts" mentioned above. It turns out that the aforementioned four-point function constitutes a common kernel to all self-energy and vertex diagrams appearing in the process $q^m \bar{q}^n \to q^i \bar{q}^j$. As has been shown in a recent brief communication [36] the judicious exploitation of the STI that this Green's function satisfies allows for the all-order generalization of the (S-matrix) PT procedure. We emphasize that the method outlined in [36], which will be explained in great detail in the present paper, does not constitute a new definition of the PT, but rather a new, far more expeditious way of carrying it out. Essentially one is trading off the tree-level WIs employed in the algebraic manipulations of individual Feynman graphs – following a well-defined, albeit cumbersome procedure which clearly does not lend itself for an all-order construction – for the formally more complicated, but operationally far more efficient, all-order STIs imposed on the (kernel) four-point function by the underlying Becchi-Rouet-Stora-Tyutin (BRST) symmetry [37].

In this paper we will focus on the following three main points: First, we will present in detail the various technical aspects of the all-order construction presented in [36], and further elaborate on the crucial role of the STI satisfied by the relevant four-point function. Second, we present the all orders generalization of the *intrinsic* PT procedure, which will allow for the explicit construction of the all-order PT gluon self-energy. Finally, we will show that the construction of the PT two-point function is *universal* (process-independence); this will be accomplished by studying the STI of the Green's function $A^a_\mu \, A^b_\nu \, A^{e_1}_{\sigma_1} \, A^{e_2}_{\sigma_2}$, which appears in the alternative on shell processes $q^m \, \bar{q}^n \to A^{e_1}_{\sigma_1} \, A^{e_2}_{\sigma_2}$ and $A^{d_1}_{\rho_1} \, A^{d_2}_{\rho_2} \to A^{e_1}_{\sigma_1} \, A^{e_2}_{\sigma_2}$.

The rest of the paper is organized as follows. In Section II we outline the general framework of the S-matrix PT, isolate the aforementioned particular Green's function which constitutes a kernel to all higher order diagrams, and derive in detail the STI it satisfies. In Section III we explain in detail why the usual fundamental PT cancellations are in fact encoded in this STI, an observation which eventually makes the all-order generalization possible. In Section IV we carry out explicitly the all-order construction of the PT gluon–quarks–anti-quark vertex. Section V is dedicated to the explicit all-order construction of the PT gluon self-energy, following the "intrinsic PT" algorithm. In Section VI we address the issue of the universality of the PT gluon self-energy, proving in a direct way that it is process-independent. In Section VII we discuss the general methodology that must be followed in order to carry out the renormalization of the effective PT Green's functions. Finally, in Section VIII we present our conclusions.

II. THE FOUR-POINT KERNEL AND ITS SLAVNOV-TAYLOR IDENTITY

In this section we will explain how the four-point function $A^a_\mu A^b_\nu\, q^i\, \bar{q}^j$ acquires its central role in the PT construction, and will derive in detail the STI that it satisfies. This STI will be instrumental in the study of the fundamental cancellations taking place between the (all-order) two- and three-point functions embedded into S-matrix elements, leading to the generalization of the S-matrix PT to all-orders.

Let us focus on the S-matrix element for the quark–anti-quark elastic scattering process $q^m(r_1)\bar{q}^n(r_2) \to q^i(p_1)\bar{q}^j(p_2)$ in QCD, typically considered in the PT construction. We set $q = r_2 - r_1 = p_2 - p_1$, with $s = q^2$ the square of the momentum transfer. The longitudinal momenta responsible for triggering the kinematical rearrangements characteristic of the PT stem either from the bare gluon propagator, $\Delta^{[0]}_{\mu\nu}(k)$, which in the covariant renormalizable gauges assumes the form

$$\Delta^{[0]}_{\mu\nu}(k) = -\frac{i}{k^2}\left[\, g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right], \tag{2.1}$$

or from the *external* tree-level three-gluon vertices, *i.e.*, the vertices where the physical momentum transfer q is entering [38]. The latter, to be denoted by $\Gamma^{eab\,[0]}_{\alpha\mu\nu}(q,k_1,k_2)$, is given by the following manifestly Bose-symmetric expression (all momenta are incoming, *i.e.*, $q + k_1 + k_2 = 0$)

$$\begin{aligned}
&\Gamma^{eab\,[0]}_{\alpha\nu\mu}(q,k_1,k_2) = g f^{eab}\Gamma^{[0]}_{\alpha\mu\nu}(q,k_1,k_2),\\
&\Gamma^{[0]}_{\alpha\mu\nu}(q,k_1,k_2) = (q-k_1)_\nu g_{\alpha\mu} + (k_1-k_2)_\alpha g_{\mu\nu} + (k_2-q)_\mu g_{\alpha\nu}.
\end{aligned} \tag{2.2}$$

$\Gamma^{[0]}_{\alpha\mu\nu}(q,k_1,k_2)$ may be then split into two parts [10]

$$\Gamma^{[0]}_{\alpha\mu\nu}(q,k_1,k_2) = \Gamma^{\rm F}_{\alpha\mu\nu}(q,k_1,k_2) + \Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2), \tag{2.3}$$

with

$$\begin{aligned}
\Gamma^{\rm F}_{\alpha\mu\nu}(q,k_1,k_2) &= (k_1-k_2)_\alpha g_{\mu\nu} + 2q_\nu g_{\alpha\mu} - 2q_\mu g_{\alpha\nu}\,,\\
\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2) &= k_{2\nu} g_{\alpha\mu} - k_{1\mu} g_{\alpha\nu}\,.
\end{aligned} \tag{2.4}$$

The vertex $\Gamma^{\rm F}_{\alpha\mu\nu}(q,k_1,k_2)$ is Bose-symmetric only with respect to the μ and ν legs. Evidently the above decomposition assigns a special role to the q-leg, and allows $\Gamma^{\rm F}_{\alpha\mu\nu}(q,k_1,k_2)$ to satisfy the WI

$$q^\alpha \Gamma^{\rm F}_{\alpha\mu\nu}(q,k_2,k_1) = (k_2^2 - k_1^2) g_{\mu\nu}\,. \tag{2.5}$$

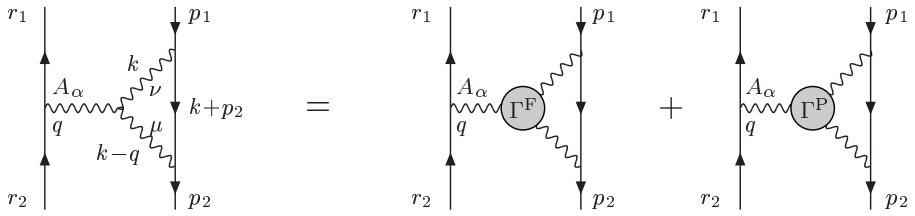


FIG. 2: Carrying out the fundamental PT vertex decomposition inside the tree-level three-gluon vertex.

where the right-hand side is the difference of two inverse propagators in the Feynman gauge [39]. The term $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2)$, which in configuration space corresponds to a pure divergence, contains the pinching momenta; as we will see in a moment, these momenta act on the amplitude $A^a_\mu A^b_\nu\, q\,\bar{q}$ and trigger its STI.

In what follows we will carry out the analysis starting directly from the renormalizable (linear) Feynman gauge (RFG), *i.e.* $\xi = 1$; this does not constitute a loss of generality, provided that one is studying the entire S-matrix, as we do [40]. This choice eliminates the longitudinal momenta from the tree-level propagators in Eq.(2.1), and allows us to focus our attention on the all-order study of the effects of the longitudinal momenta contained in $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2)$.

In order to appreciate the relevance of the amplitude $A^a_\mu A^b_\nu\, q\,\bar{q}$, let us remember the basic steps of the PT construction at one-loop. To begin with, in the RFG the box is completely inert, since there are no pinching momenta, and therefore the PT box coincides with the conventional one computed at $\xi = 1$. Then, in the non-Abelian vertex graph of Fig.2 we carry out the splitting of the elementary three-gluon vertex given in Eq.(2.3) (now $k_1 = k-q$ and $k_2 = -k$). Despite appearances, the part of the vertex graph containing $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k-q,-k)$ is

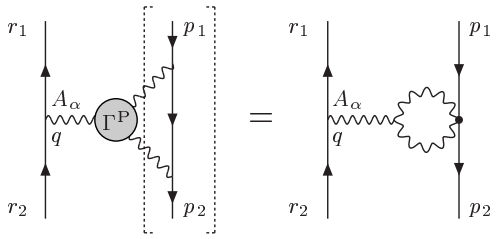


FIG. 3: The self-energy-like contribution coming from the pinching part of the tree-level three gluon vertex.

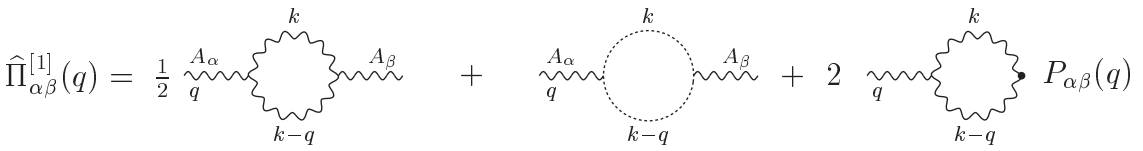


FIG. 4: Diagrammatic representation of the one-loop PT gluon self-energy $\widehat{\Pi}^{[1]}_{\alpha\beta}(q)$ as the sum of the conventional gluon self-energy and the pinch contributions coming from the vertex.

in fact a propagator-like contribution: the longitudinal momenta of $\Gamma^{\rm P}_{\alpha\mu\nu}(q, k-q, -k)$ trigger the elementary WI $k_\nu \gamma^\nu = S_0^{-1}(\not{k} + \not{p}) - S_0^{-1}(\not{p})$, whose first term removes (pinches out) the internal bare fermion propagator $S_0(\not{k}+\not{p})$ (see Fig.3), whereas the second term dies on shell. On the other hand, the part of the vertex graph containing $\Gamma^{\rm F}_{\alpha\mu\nu}(q, k-q, -k)$ remains unchanged, since it contains no longitudinal momenta; adding it to the usual Abelian-like graph (not shown) we obtain the one-loop PT vertex $\widehat{\Gamma}^{e\,[1]}_{\alpha}(q)$.

The propagator-like parts extracted from the vertex are subsequently reassigned to the conventional self-energy graphs, giving rise to the one-loop PT gluon self-energy $\widehat{\Pi}^{[1]}_{\alpha\beta}$ (Fig.4). Even though the answer is already contained in this sum, it is conceptually advantageous to trace down in more detail the exact fate of the pinch part. It turns out that this part cancels exactly against a corresponding term contained in the conventional self-energy graph. To expose this cancellation, one carries out the following standard rearrangement of the two

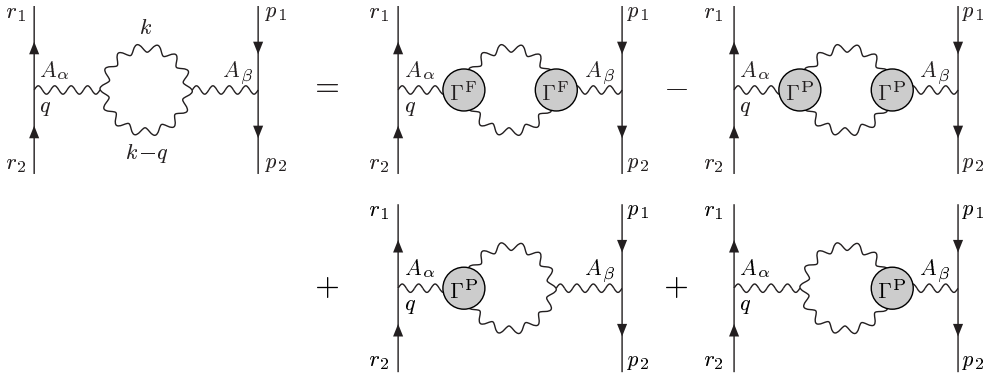


FIG. 5: The standard PT rearrangement of the two tree-level three-gluon vertices appearing in the self-energy diagram.

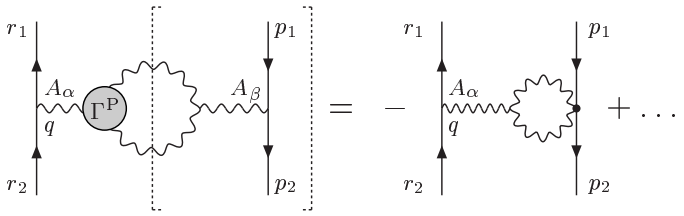


FIG. 6: The pinching term coming from the gluon self-energy diagram is minus the one that is extracted from the vertex diagrams.

elementary three-gluon vertices appearing in Fig.5:

$$\begin{aligned}\Gamma^{[0]}_{\alpha\mu\nu}\Gamma^{[0]\,\mu\nu}_{\beta} &= [\Gamma^{\rm F}_{\alpha\mu\nu}+\Gamma^{\rm P}_{\alpha\mu\nu}][\Gamma^{{\rm F}\,\mu\nu}_{\beta}+\Gamma^{{\rm P}\,\mu\nu}_{\beta}] \\ &= \Gamma^{\rm F}_{\alpha\mu\nu}\Gamma^{{\rm F}\,\mu\nu}_{\beta}-\Gamma^{\rm P}_{\beta\mu\nu}\Gamma^{{\rm P}\,\mu\nu}_{\beta}+\Big\{\Gamma^{\rm P}_{\alpha\mu\nu}\Gamma^{[0]\,\mu\nu}_{\beta}+\Gamma^{[0]}_{\alpha\mu\nu}\Gamma^{{\rm P}\,\mu\nu}_{\beta}\Big\}. \qquad (2.6)\end{aligned}$$

This particular splitting, usually associated with the "intrinsic" PT, allows for the identification of the term which will actually cancel against the pinch part coming from the vertex. All one needs to do is recognize that the terms of Eq.(2.6) appearing in curly brackets trigger the elementary WI

$$k^\nu\Gamma_{\alpha\mu\nu}(q,k-q,-k)=\Big[q^2g_{\alpha\nu}-q_\alpha q_\nu\Big]-\Big[\,(k-q)^2g_{\alpha\nu}-(k-q)_\alpha(k-q)_\nu\,\Big] \qquad (2.7)$$

together with its Bose-symmetric counter-part from $(k-q)^\mu\Gamma_{\alpha\mu\nu}(q,k-q,-k)$. Then it is elementary to verify that the term on the right-hand side proportional to $[q^2g_{\alpha\nu}-q_\alpha q_\nu]$ is the desired one (see Fig.6); incidentally, this is how the "intrinsic" PT works: one simply strips out all such terms from the conventional self-energy (first paper in [5]; see section V for more details). It must be clear from the above discussion that the PT rearrangement of terms between vertex- and self-energy graphs is actually encoded in the two graphs of Figs.2 and 5. Both graphs have the term $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k-q,-k)$ common, whereas their terms in dotted brackets are the tree-level t-channel and s-channel contributions, respectively, to the four-particle amplitude $A^a_\mu A^b_\nu\, q\,\bar{q}$. As we will see in what follows, dressing the above amplitude with higher order corrections, and exploiting its STI, will provide us with the way of generalizing the PT to all orders.

With this intention in mind, of all the diagrams contributing to the QCD amplitude under consideration we will focus on the subset of those graphs which will receive the action of the longitudinal momenta stemming from $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2)$, to be denoted by $\mathcal{A}^{mnij}(r_1,r_2,p_1,p_2)$.

It is given by

$$\mathcal{A}^{mnij}(r_1,r_2,p_1,p_2) = ig^2\bar{u}^m(r_1)\frac{(\lambda^e)_{mn}}{2}\gamma_\alpha v^n(r_2) f^{eab}\Gamma^{\alpha\mu\nu}(q,k_1,k_2)\mathcal{T}^{abij}_{\mu\nu}(k_1,k_2,p_1,p_2), \quad (2.8)$$

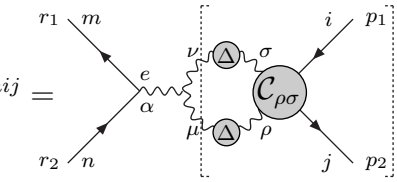
where $m,n,i,j = 1,\ldots,N$, are fundamental $SU(N)$ indices, λ^e are the Gell-Mann matrices, and $\mathcal{T}^{abij}_{\mu\nu}$ is the sub-amplitude $A^a_\mu(k_1)A^b_\nu(k_2) \to q^i(p_1)\bar{q}^j(p_2)$, with the gluons *off-shell* and the fermions on-shell; for the latter

$$\bar{v}(p_2)S^{-1}(p_2)\big|_{\not{p}_2=m} = S^{-1}(p_1)u(p_1)\big|_{\not{p}_1=m} = 0, \quad (2.9)$$

where $S(p)$ is the (full) quark propagator, related to the corresponding quark self-energy $\Sigma(p)$ through

$$S(p) = \frac{i}{\not{p} - m - i\Sigma(p)}. \quad (2.10)$$

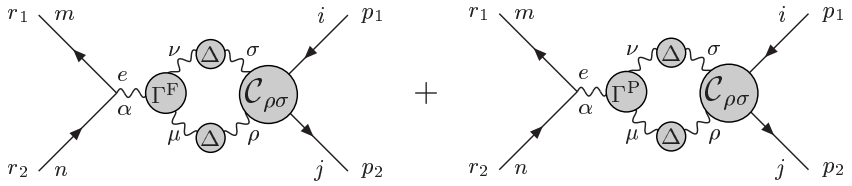
Diagrammatically we have

$\mathcal{A}^{mnij} =$ 

so that in terms of Green's functions the amplitude in brackets can be written as

$$\mathcal{T}^{abij}_{\mu\nu} = \bar{v}(p_2)\big[\mathcal{C}^{abij}_{\rho\sigma}(k_1,k_2,p_1,p_2)\Delta^\rho_\mu(k_1)\Delta^\sigma_\nu(k_2)\big]u(p_1). \quad (2.11)$$

We next carry out the vertex decomposition of Eq.(2.3), *i.e.* we write

$\mathcal{A}^{mnij} =$ 

Clearly, there is an equal contribution from the $\Gamma^{\rm P}$ situated on the right hand-side of the $\mathcal{T}^{abij}_{\mu\nu}$ amplitude.

Let us then focus on the STIs satisfied by the amplitude of Eq.(2.11). For deriving them, we start from the following identities [45]

$$\begin{aligned}\left\langle T\left[\bar{c}^a(x)A^b_\nu(y)q^i(z)\bar{q}^j(w)\right]\right\rangle &= 0\,,\\ \left\langle T\left[A^a_\mu(x)\bar{c}^b(y)q^i(z)\bar{q}^j(w)\right]\right\rangle &= 0\,,\end{aligned} \quad (2.12)$$

valid due to ghost-charge conservation. We then apply to the above equations the BRST operator s, which acts on the fields as follows (recall that we work in the RFG, *i.e.*, $\xi = 1$)

$$\begin{aligned}
sA^a_\mu(x) &= \partial_\mu c^a(x) + g f^{acd} A^c_\mu(x) c^d(x), \\
s\bar{c}^a(x) &= \partial^\mu A^a_\mu(x), \\
sq^i(x) &= ig \left[T^d\right]^{ik} c^d(x) q^k(x), \\
s\bar{q}^i(x) &= -ig\bar{q}^k(x) \left[T^d\right]^{ki} c^d(x),
\end{aligned} \tag{2.13}$$

where T^d are the $SU(N)$ generators. From Eq.(2.12) we then find the identities

$$\begin{aligned}
\partial^\mu_x C^{abij}_{\mu\nu} + \partial^y_\nu G^{abij}_1 + g f^{bcd} Q^{acdij}_{1\nu} + igX^{abij}_{1\nu} - ig\bar{X}^{abij}_{1\nu} &= 0, \\
\partial^\nu_y C^{abij}_{\mu\nu} + \partial^x_\mu G^{abij}_2 + g f^{acd} Q^{cdbij}_{2\mu} + igX^{abij}_{2\mu} - ig\bar{X}^{abij}_{2\mu} &= 0,
\end{aligned} \tag{2.14}$$

where we have introduced the following Green's functions (in configuration space)

$$\begin{aligned}
C^{abij}_{\mu\nu}(x,y,z,w) &= \left\langle T\left[A^a_\mu(x) A^b_\nu(y) q^i(z) \bar{q}^j(w)\right]\right\rangle, \\
Q^{acdij}_{1\nu}(x,y,z,w) &= \left\langle T\left[\bar{c}^a(x) A^c_\nu(y) c^d(y) q^i(z) \bar{q}^j(w)\right]\right\rangle, \\
Q^{cdbij}_{2\mu}(x,y,z,w) &= \left\langle T\left[A^c_\mu(x) c^d(x) \bar{c}^b(y) q^i(z) \bar{q}^j(w)\right]\right\rangle, \\
G^{abij}_1(x,y,z,w) &= \left\langle T\left[\bar{c}^a(x) c^b(y) q^i(z) \bar{q}^j(w)\right]\right\rangle, \\
G^{abij}_2(x,y,z,w) &= \left\langle T\left[c^a(x) \bar{c}^b(y) q^i(z) \bar{q}^j(w)\right]\right\rangle, \\
X^{abij}_{1\nu}(x,y,z,w) &= \left\langle T\left\{\bar{c}^a(x) A^b_\nu(y) \left[T^d\right]^{ik} c^d(z) q^k(z) \bar{q}^j(w)\right\}\right\rangle, \\
\bar{X}^{abij}_{1\nu}(x,y,z,w) &= \left\langle T\left\{\bar{c}^a(x) A^b_\nu(y) q^i(z) \bar{q}^k(w) \left[T^d\right]^{kj} c^d(w)\right\}\right\rangle, \\
X^{abij}_{2\mu}(x,y,z,w) &= \left\langle T\left\{A^a_\mu(x) \bar{c}^b(y) \left[T^d\right]^{ik} c^d(z) q^k(z) \bar{q}^j(w)\right\}\right\rangle, \\
\bar{X}^{abij}_{2\mu}(x,y,z,w) &= \left\langle T\left\{A^a_\mu(x) \bar{c}^b(y) q^i(z) \bar{q}^k(w) \left[T^d\right]^{kj} c^d(w)\right\}\right\rangle,
\end{aligned} \tag{2.15}$$

After Fourier transform, the above quantities define the following momentum-space Green's functions

$$\begin{aligned}
C^{abij}_{\mu\nu}(k_1,k_2,p_1,p_2) &= \mathcal{C}^{abij}_{\rho\sigma}(k_1,k_2,p_1,p_2) \Delta^\rho_\mu(k_1) \Delta^\sigma_\nu(k_2) S(p_1) S(p_2), \\
Q^{acdij}_{1\nu}(k_1,k_2,p_1,p_2) &= \mathcal{Q}^{acdij}_{1\nu}(k_1,k_2,p_1,p_2) D(k_1) S(p_1) S(p_2), \\
Q^{cdbij}_{2\mu}(k_1,k_2,p_1,p_2) &= \mathcal{Q}^{cdbij}_{2\mu}(k_1,k_2,p_1,p_2) D(k_2) S(p_1) S(p_2), \\
G^{abij}_1(k_1,k_2,p_1,p_2) &= \mathcal{G}^{abij}_1(k_1,k_2,p_1,p_2) D(k_1) D(k_2) S(p_1) S(p_2),
\end{aligned}$$

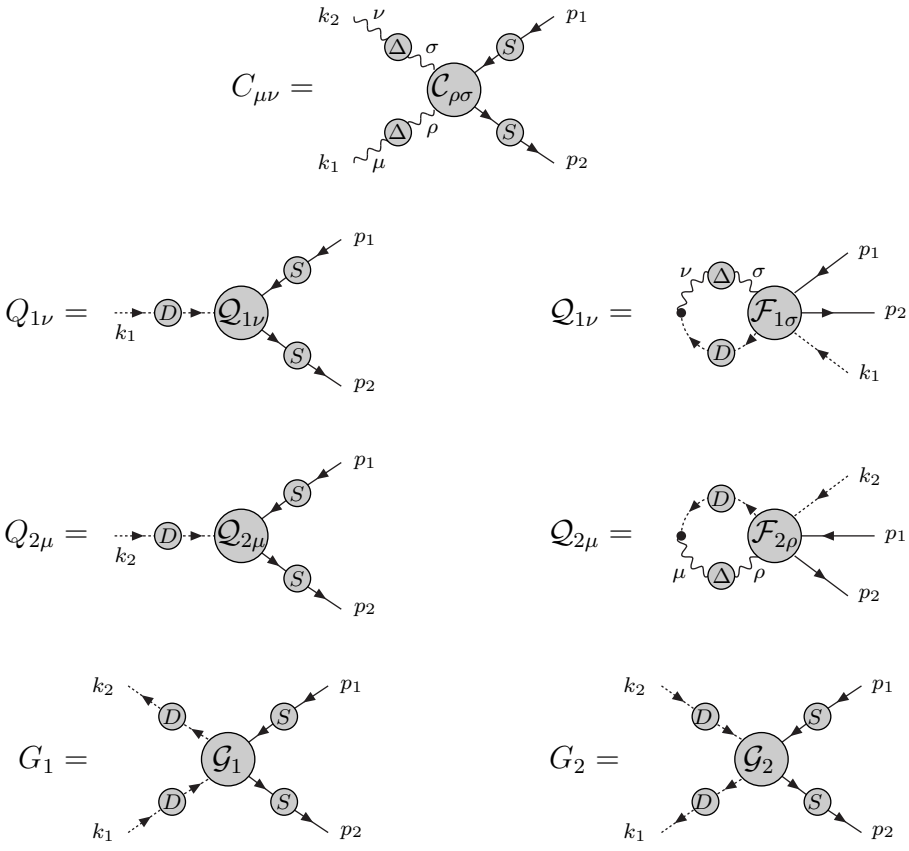
$$
\begin{aligned}
G_2^{abij}(k_1,k_2,p_1,p_2) &= \mathcal{G}_2^{abij}(k_1,k_2,p_1,p_2)D(k_1)D(k_2)S(p_1)S(p_2),\\
X_{1\nu}^{abij}(k_1,k_2,p_1,p_2) &= \mathcal{X}_{1\sigma}^{abij}(k_1,k_2,p_1,p_2)D(k_1)\Delta_\nu^\sigma(k_2)S(p_2),\\
\bar{X}_{1\nu}^{abij}(k_1,k_2,p_1,p_2) &= \bar{\mathcal{X}}_{1\sigma}^{abij}(k_1,k_2,p_1,p_2)D(k_1)\Delta_\nu^\sigma(k_2)S(p_1),\\
X_{2\mu}^{abij}(k_1,k_2,p_1,p_2) &= \mathcal{X}_{2\rho}^{abij}(k_1,k_2,p_1,p_2)\Delta_\mu^\rho(k_1)D(k_2)S(p_2),\\
\bar{X}_{2\mu}^{abij}(k_1,k_2,p_1,p_2) &= \bar{\mathcal{X}}_{2\rho}^{abij}(k_1,k_2,p_1,p_2)\Delta_\mu^\rho(k_1)D(k_2)S(p_1).
\end{aligned}
\tag{2.16}
$$

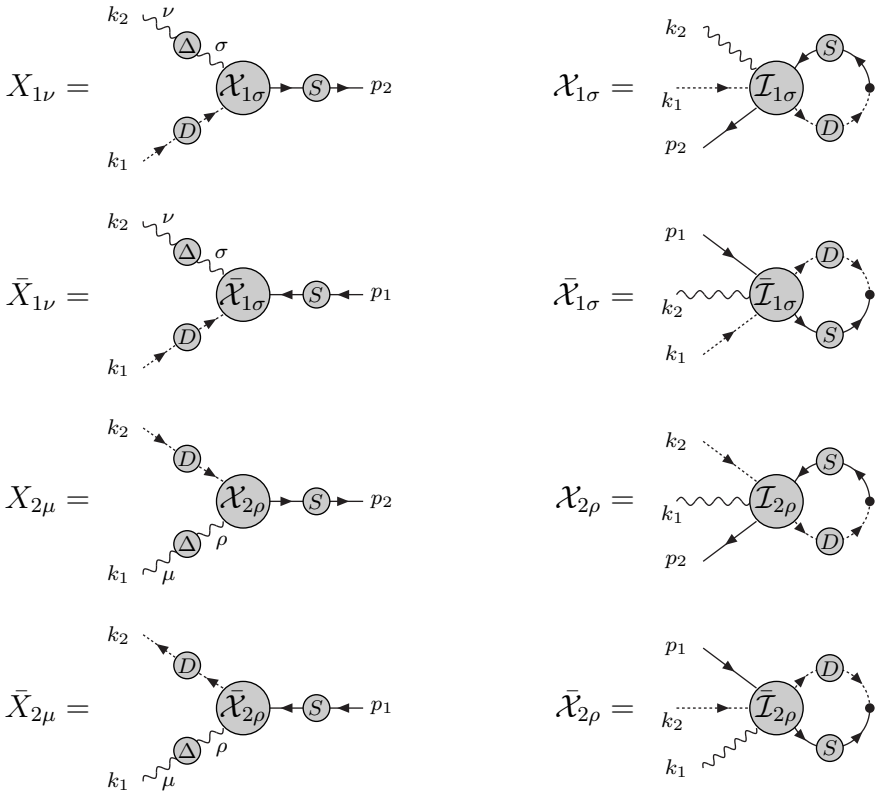
In the above formulas, all the momenta are supposed to be entering, *i.e.*, we have $k_1+k_2+p_1+p_2=0$; moreover we have denoted by $D(k)$ and $\Delta_{\mu\nu}(k)$ the (full) RFG ghost and gluon propagators which are related to the corresponding ghost and gluon self-energies $L(k)$ and $\Pi_{\mu\nu}(k)$ through

$$
\begin{aligned}
D(k) &= \frac{i}{k^2-iL(k)},\\
\Delta_{\mu\nu}(k) &= -i\left[\Delta(k^2)P_{\mu\nu}(k)+\frac{k_\mu k_\nu}{k^4}\right], \qquad \Delta(k^2)=\frac{1}{k^2+i\Pi(k^2)},
\end{aligned}
\tag{2.17}
$$

where $P_{\mu\nu}(k) = g_{\mu\nu} - k_\mu k_\nu/k^2$ represents the dimensionless projection operator, and $\Pi_{\mu\nu}(k)=\Pi(k^2)P_{\mu\nu}(k)$.

Then, the above Green's functions have the following diagrammatic representation





In terms of these quantities, the needed STIs read

$$\begin{aligned} k_1^\mu C^{abij}_{\mu\nu} + k_{2\nu} G_1^{abij} - igf^{bcd} Q^{acdij}_{1\nu} + g X^{abij}_{1\nu} - g\bar{X}^{abij}_{1\nu} &= 0, \\ k_2^\nu C^{abij}_{\mu\nu} + k_{1\mu} G_2^{abij} - igf^{acd} Q^{cdbij}_{2\mu} + g X^{abij}_{2\mu} - g\bar{X}^{abij}_{2\mu} &= 0. \end{aligned} \tag{2.18}$$

As can be clearly seen in the above diagrammatic representation, the terms $X_{1\nu}$, $\bar{X}_{1\nu}$, $X_{2\mu}$ and $\bar{X}_{2\mu}$, correspond to terms that die on-shell, since they are missing one fermion propagator; at lowest order they are simply the terms proportional to the inverse tree-level propagators $(\not{p}_1 + m)$ and $(\not{p}_2 - m)$ appearing in the PT calculations. Indeed, we multiply both sides of Eq.(2.18) by the product $S^{-1}(p_1)S^{-1}(p_2)$ of the two inverse propagators of the external fermions, and then sandwich the resulting amplitude between the on-shell spinors $\bar{v}(p_1)$ and $u(p_2)$. Since the fermion are assumed to be on-shell, by virtue of the Dirac equation the vanishing of the aforementioned terms follows. Thus we arrive at the on-shell STIs

$$\begin{aligned} k_1^\mu \mathcal{T}^{abij}_{\mu\nu}(k_1, k_2, p_1, p_2) &= \mathcal{S}^{abij}_{1\nu}(k_1, k_2, p_1, p_2), \\ k_2^\nu \mathcal{T}^{abij}_{\mu\nu}(k_1, k_2, p_1, p_2) &= \mathcal{S}^{abij}_{2\mu}(k_1, k_2, p_1, p_2), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
\mathcal{S}_{1\nu}^{abij}(k_1,k_2,p_1,p_2) &= \bar{v}(p_1)\Big[igf^{bcd}\mathcal{Q}_{1\nu}^{acdij}(k_1,k_2,p_1,p_2)D(k_1) \\
&- k_{2\nu}\mathcal{G}_1^{abij}(k_1,k_2,p_1,p_2)D(k_1)D(k_2)\Big]u(p_2), \\
\mathcal{S}_{2\mu}^{abij}(k_1,k_2,p_1,p_2) &= \bar{v}(p_1)\Big[igf^{acd}\mathcal{Q}_{2\mu}^{cdbij}(k_1,k_2,p_1,p_2)D(k_2) \\
&- k_{1\mu}\mathcal{G}_2^{abij}(k_1,k_2,p_1,p_2)D(k_1)D(k_2)\Big]u(p_2).
\end{aligned} \tag{2.20}$$

Perturbatively, the above equations are of the (schematic) form

$$\begin{aligned}
\mathcal{T}^{[n]} &= \mathcal{C}^{[n_1]}\Delta^{[n_2]}\Delta^{[n_3]}, \\
\mathcal{S}_1^{[n]} &= \mathcal{Q}_1^{[m_1]}D^{[m_2]} - \mathcal{G}_1^{[\ell_1]}D^{[\ell_2]}D^{[\ell_3]}, \\
\mathcal{S}_2^{[n]} &= \mathcal{Q}_2^{[m_1]}D^{[m_2]} - \mathcal{G}_2^{[\ell_1]}D^{[\ell_2]}D^{[\ell_3]},
\end{aligned} \tag{2.21}$$

with $n_1+n_2+n_3 = m_1+m_2 = \ell_1+\ell_2+\ell_3 = n$.

Since the external (on-shell) quarks are inert throughout our construction, and in order to avoid notational clutter, in what follows we will suppress both the color indices i and j of the fundamental $SU(N)$ representation, and denote through the label p_i ($i = 1, 2$) the dependence on the (on-shell) momenta p_1 and p_2.

III. THE FUNDAMENTAL CANCELLATIONS

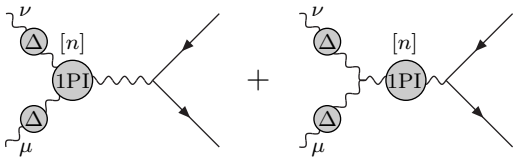
Having established the STIs of Eq.(2.19) we now turn to the main conceptual point related to the all orders PT construction. The basic observation is the following. In perturbation theory the quantities $\mathcal{T}_{\mu\nu}^{ab}$, $\mathcal{S}_{1\nu}^{ab}$, and $\mathcal{S}_{2\mu}^{ab}$ are given by Feynman diagrams, which can be separated into distinct classes, depending on their kinematic dependence and their geometrical properties. Graphs which do not contain information about the kinematical details of the incoming test-quarks are self-energy graphs, whereas those which display a dependence on the test quarks are vertex graphs. The former depend only on the variable s, whereas the latter on both s and the mass m of the test quarks; equivalently, we will refer to them as s-channel or t-channel graphs, respectively. In addition to this s/t decomposition, the Feynman diagrams allow for the distinction between one-particle irreducible (1PI) and one-particle reducible (1PR) graphs. Thus, 1PR graphs are those which, after cutting one line, get disconnected into two subgraphs none of which is a tree-level graph; if this does

not happen, then the graph is 1PI. The distinction between 1PI and 1PR is necessary for constructing eventually effective 1PI n-point functions. Thus, the Feynman graphs allow the following separation

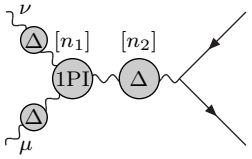
$$\begin{aligned}
\mathcal{T}^{ab}_{\mu\nu} &= [\mathcal{T}^{ab}_{\mu\nu}]_{s,\mathrm{I}} + [\mathcal{T}^{ab}_{\mu\nu}]_{s,\mathrm{R}} + [\mathcal{T}^{ab}_{\mu\nu}]_{t,\mathrm{I}} + [\mathcal{T}^{ab}_{\mu\nu}]_{t,\mathrm{R}},\\
\mathcal{S}^{ab}_{1\nu} &= [\mathcal{S}^{ab}_{1\nu}]_{s,\mathrm{I}} + [\mathcal{S}^{ab}_{1\nu}]_{s,\mathrm{R}} + [\mathcal{S}^{ab}_{1\nu}]_{t,\mathrm{I}} + [\mathcal{S}^{ab}_{1\nu}]_{t,\mathrm{R}},\\
\mathcal{S}^{ab}_{2\mu} &= [\mathcal{S}^{ab}_{2\mu}]_{s,\mathrm{I}} + [\mathcal{S}^{ab}_{2\mu}]_{s,\mathrm{R}} + [\mathcal{S}^{ab}_{2\mu}]_{t,\mathrm{I}} + [\mathcal{S}^{ab}_{2\mu}]_{t,\mathrm{R}}.
\end{aligned} \tag{3.1}$$

For example at order n one has the following decomposition

$$\big[\mathcal{T}^{[n]}_{\mu\nu}\big]_{s,\mathrm{I}} =$$

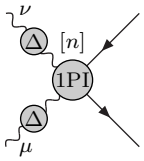


$$\big[\mathcal{T}^{[n]}_{\mu\nu}\big]_{s,\mathrm{R}} =$$

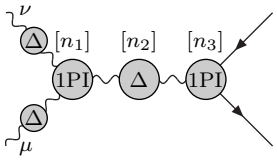


$$n_1 + n_2 = n,\ n_1 < n$$

$$\big[\mathcal{T}^{[n]}_{\mu\nu}\big]_{t,\mathrm{I}} =$$



$$\big[\mathcal{T}^{[n]}_{\mu\nu}\big]_{t,\mathrm{R}} =$$



$$n_1 + n_2 + n_3 = n,\ n_3 > 0 \tag{3.2}$$

(notice that when $n_2 = n$ in $[\mathcal{T}^{[n]}_{\mu\nu}]_{s,\mathrm{R}}$, one has to remove from the full propagator $\Delta^{[n]}$ the 1PI part).

The crucial point is that the action of the momenta k_1^μ or k_2^ν on $\mathcal{T}^{ab}_{\mu\nu}$ does not respect, in general, the above separation. In fact, whereas the characterization of graphs as propagator-like and vertex-like can be done unambiguously in the absence of longitudinal momenta (*e.g.*, in a purely scalar theory), their presence mixes propagator- with vertex-like terms, since the last ones generate (through pinching) effectively propagator-like terms. Similarly, the separation between 1PI and 1PR terms is no longer unambiguous, since 1PR diagrams are converted (through pinching) into effectively 1PI ones (the opposite cannot happen).

In particular this last effect is most easily seen in the so-called intrinsic PT construction (see Section V), where the order n 1PI self-energy diagrams receive 1PI contributions from the 1PR strings made of self-energy insertions of order less than n (incidentally notice that these last contributions are in fact instrumental for ensuring that in the Standard Model the PT resummed propagator does not shift the position of the pole, as has been shown in the second paper of [16]).

Then, even though Eq.(2.19) holds for the entire amplitude $\mathcal{T}^{ab}_{\mu\nu}$, it is not true for the individual sub-amplitudes defined in Eq.(3.1), *i.e.*, we have

$$\begin{aligned} k_1^\mu [\mathcal{T}^{ab}_{\mu\nu}]_{x,\mathrm{Y}} &\neq [\mathcal{S}^{ab}_{1\nu}]_{x,\mathrm{Y}}, \\ k_2^\nu [\mathcal{T}^{ab}_{\mu\nu}]_{x,\mathrm{Y}} &\neq [\mathcal{S}^{ab}_{2\mu}]_{x,\mathrm{Y}}, \qquad x = s,t;\ \mathrm{Y} = \mathrm{I},\mathrm{R}, \end{aligned} \tag{3.3}$$

where I (respectively R) indicates the one-particle *irreducible* (respectively *reducible*) parts of the amplitude involved. The reason for this inequality, are precisely the propagator-like terms, such as those encountered in the one- and two-loop calculations (see Fig.7).

In particular, for individual sub-amplitudes we have that

$$\begin{aligned} k_1^\mu [\mathcal{T}^{ab}_{\mu\nu}]_{s,\mathrm{R}} &= [\mathcal{S}^{ab}_{1\nu}]_{s,\mathrm{R}} + [\mathcal{R}^{ab}_{1\nu}]^{\mathrm{int}}_{s,\mathrm{I}}, \\ k_1^\mu [\mathcal{T}^{ab}_{\mu\nu}]_{s,\mathrm{I}} &= [\mathcal{S}^{ab}_{1\nu}]_{s,\mathrm{I}} - [\mathcal{R}^{ab}_{1\nu}]^{\mathrm{int}}_{s,\mathrm{I}} + [\mathcal{R}^{ab}_{1\nu}]^{\mathrm{ext}}_{s,\mathrm{I}}, \\ k_1^\mu [\mathcal{T}^{ab}_{\mu\nu}]_{t,\mathrm{R}} &= [\mathcal{S}^{ab}_{1\nu}]_{t,\mathrm{R}} + [\mathcal{R}^{ab}_{1\nu}]^{\mathrm{int}}_{t,\mathrm{I}}, \end{aligned}$$

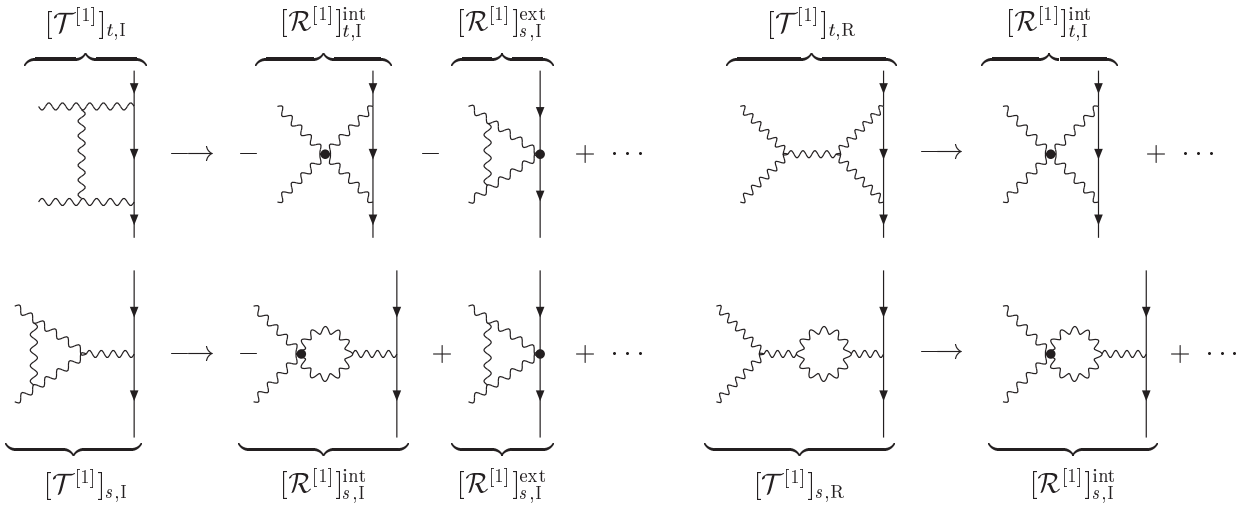
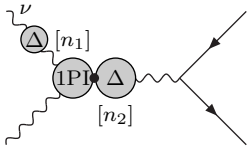


FIG. 7: Some two-loop examples of the $\mathcal{R}$ terms, together with the Feynman diagrams from which they originate.

$$k_1^\mu [\mathcal{T}^{ab}_{\mu\nu}]_{t,\mathrm{I}} \;=\; [\mathcal{S}^{ab}_{1\nu}]_{t,\mathrm{I}} - [\mathcal{R}^{ab}_{1\nu}]^{\mathrm{int}}_{t,\mathrm{I}} - [\mathcal{R}^{ab}_{1\nu}]^{\mathrm{ext}}_{s,\mathrm{I}}\,, \tag{3.4}$$

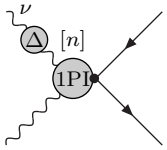
with similar equations holding when acting with the momentum k_2^ν. In the above equations the superscript "ext" and "int" stands for "external" and "internal" respectively, and refers to whether or not the pinching part of the diagram at hands has touched the external on-shell fermion legs. At order n, some example of the $\mathcal{R}^{[n]}$ terms introduced in the equations above are the following

$$[\mathcal{R}^{[n]}_{1\nu}]^{\mathrm{int}}_{s,\mathrm{I}} \subset$$



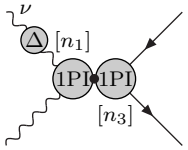
$$, \cdots$$

$$[\mathcal{R}^{[n]}_{1\nu}]^{\mathrm{ext}}_{s,\mathrm{I}} \subset$$



$$, \cdots$$

$$[\mathcal{R}^{[n]}_{1\nu}]^{\mathrm{int}}_{t,\mathrm{I}} \subset$$



$$, \cdots \tag{3.5}$$

where the black dot indicates that a tree-level propagator has been removed through pinching. The structure of the $[\mathcal{R}^{ab}_{1\nu}]_{s,\mathrm{Y}}$ terms is very characteristic: kinematically they only depend on s; whereas this is obviously true for the first two equations in (3.4) (since it comes from the action of k_1^μ on an s-dependent piece $[\mathcal{T}^{ab}_{\mu\nu}]_{s,\mathrm{Y}}$), is far less obvious for the third and fourth equations, since it stems from the action of k_1^μ on an t-dependent piece $[\mathcal{T}^{ab}_{\mu\nu}]_{t,\mathrm{Y}}$. In addition, all the $\mathcal{R}$ pieces share a common feature, *i.e.*, that they *cannot* be written in terms of conventional Feynman rules; instead they are built from unphysical vertices which do not correspond to any term in the original Lagrangian. All these terms are precisely the ones that cancel *diagrammatically* against each other when carrying out the PT procedure.

Thus, after the PT cancellations have been enforced, we find that for the 1PI t-channel part of the amplitude, we have the equality

$$\begin{aligned} [k_1^\mu \mathcal{T}^{ab}_{\mu\nu}]^{\mathrm{PT}}_{t,\mathrm{I}} \;&=\; [\mathcal{S}^{ab}_{1\nu}]_{t,\mathrm{I}}, \\ [k_2^\nu \mathcal{T}^{ab}_{\mu\nu}]^{\mathrm{PT}}_{t,\mathrm{I}} \;&=\; [\mathcal{S}^{ab}_{2\mu}]_{t,\mathrm{I}}. \end{aligned} \tag{3.6}$$

What is crucial in the above result is that it automatically takes care of both the s-t as well as the 1PR and 1PI cancellations of the $\mathcal{R}$ terms, which is characteristic of the PT, without having to actually trace them down. Thus, on hindsight, the PT algorithm as applied in the past has amounted to enforcing diagrammatically (essentially by hand) the vast, BRST-driven s-t channel cancellations, without making use of the all-order STIs. Evidently, tracing down the action of the longitudinal diagrams and the resulting exchanges between vertex and self-energy graphs, is equivalent to deriving (loop-by-loop) Eq.(3.6). Therefore, the non-trivial step for generalizing the PT to all orders is to recognize that the result obtained after the implementation of the PT algorithm on the left hand-side of Eq.(3.6) is already encoded on the right-hand side! Indeed, the right-hand side involves only conventional (ghost) Green's functions, expressed in terms of normal Feynman rules, with no reference to unphysical vertices.

This last point merits some additional comments. It should be clear that no pinching is possible when looking at the t-channel irreducible part of the right hand-side of Eq.(2.19). So, if we were to enforce the PT cancellations on both sides of the t irreducible part of Eq.(2.19), on the right hand-side there would be nothing to pinch (all the vertices are internal), and therefore, there would be no unphysical vertices generated. Therefore, on the left-hand side, where pinching is possible, all contributions containing unphysical vertices must cancel. The only way to distort this equality is to violate the PT rules, allowing for example the generation of additional longitudinal momenta by carrying out sub-integrations, or by splitting internal vertices. Violating these rules will result in undesirable consequences: in the first case the appearance of terms of the form $q \cdot k$ in the denominators will interfere with the manifest analyticity of the PT Green's functions constructed, whereas, in the second, the special unitarity properties of the PT Green's functions will be inevitably compromised.

In the next section we will study in detail how to use the considerations presented here in order to accomplish the all-order PT construction.

IV. THE PT GLUON–QUARK–ANTI-QUARK VERTEX TO ALL ORDERS

This section contains one of the central result of the present paper, namely the all-order PT construction of the gluon–quark–anti-quark vertex, with the gluon off-shell and the quarks "on-shell". By virtue of the observations made in the previous section, the

derivation presented here turns out to be particularly compact.

Before entering into the actual construction, some additional comments are in order. Once the effective Green's functions have been derived, they will be compared to the corresponding Green's functions obtained in the Feynman gauge of the background field method [41]. The latter is a special gauge-fixing procedure, implemented at the level of the generating functional. In particular, it preserves the symmetry of the action under ordinary gauge transformations with respect to the background (classical) gauge field $\widehat{A}^a_\mu$, while the quantum gauge fields A^a_μ appearing in the loops transform homogeneously under the gauge group, *i.e.*, as ordinary matter fields which happened to be assigned to the adjoint representation [42]. As a result of the background gauge symmetry, the n-point functions $\langle 0|T\left[\widehat{A}^{a_1}_{\mu_1}(x_1)\widehat{A}^{a_2}_{\mu_2}(x_2)\ldots\widehat{A}^{a_n}_{\mu_n}(x_n)\right]|0\rangle$ are gauge-invariant, in the sense that they satisfy naive, QED-like WIs. Notice however that they are *not* gauge-independent, because they depend *explicitly* on the quantum gauge-fixing parameter ξ_Q used to define the tree-level propagators of the quantum gluons. In theories with spontaneous symmetry breaking this dependence on ξ_Q gives rise to *unphysical* thresholds inside these Green's functions for $\xi_Q \neq 1$, a fact which limits their usefulness for resummation purposes [16]. Only the case of the background Feynman gauge (BFG) (*i.e.* background field method with $\xi_Q = 1$) is free from unphysical poles, and the results of these Green's functions collapse to those of the PT, at one- [43] and two-loops [33]. As we will see, this correspondence between the PT Green's functions and the ones obtained using the BFG persists to all orders. This fact provides a valuable book-keeping scheme, since, once the equality between the Green's functions obtained using either schemes has been established (and *only* then), the background field method Feynman rules may be directly employed. We note in passing that the PT construction goes through unaltered under circumstances where the background field method Feynman rules cannot even be applied. Specifically, if instead of an S-matrix element one were to consider a different observable, such as a current-current correlation function or a Wilson loop (as was in fact done by Cornwall in the original formulation [4], and more recently in [31]) one could not start out using the background Feynman rules, because *all* fields appearing inside the first non-trivial loop are quantum ones. Instead, by following the PT rearrangement inside these physical amplitudes one would "dynamically" arrive at the BFG answer. After these clarifying comments we proceed with the actual all-order construction.

To begin with, it is immediate to recognize that, in the RFG, box diagrams of arbitrary

order n, to be denoted by $B^{[n]}$, coincide with the PT boxes $\widehat{B}^{[n]}$, since all three-gluon vertices are "internal", *i.e.*, they do not provide longitudinal momenta. Thus, they coincide with the BFG boxes, $\tilde{B}^{[n]}$, *i.e.*,

$$\widehat{B}^{[n]} = B^{[n]} = \tilde{B}^{[n]} \tag{4.1}$$

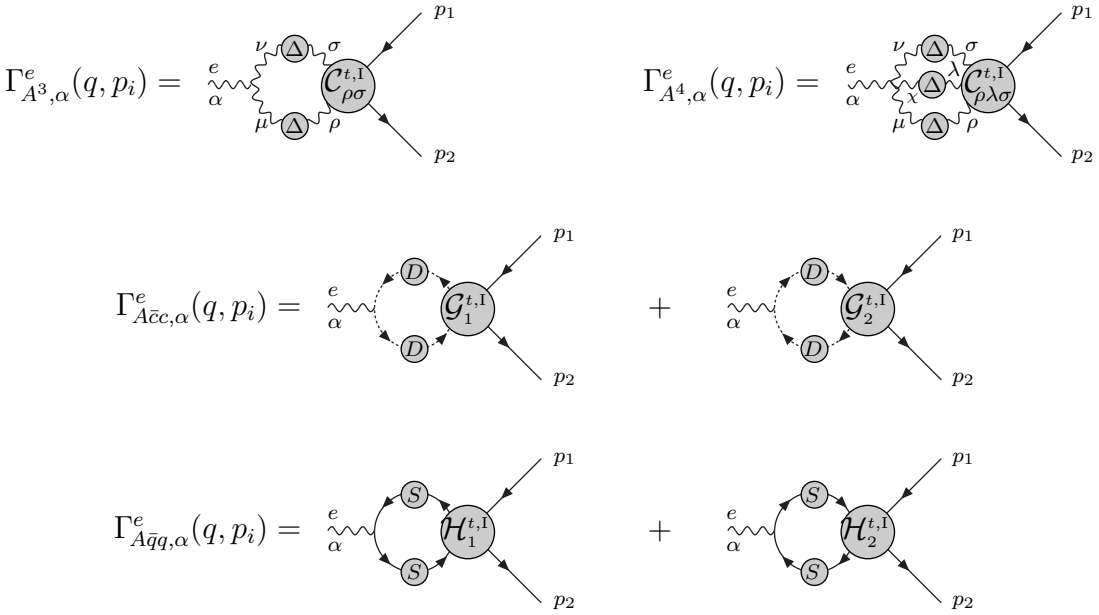
for every n. The same is true for the PT quark self-energies; for exactly the same reason, they coincide with their RFG (and BFG) counterparts, *i.e.*

$$\widehat{\Sigma}^{ab\,[n]} = \Sigma^{ab\,[n]} = \tilde{\Sigma}^{ab\,[n]}. \tag{4.2}$$

The next step will be the construction of the all orders PT gluon–quark–anti-quark 1PI vertex $\widehat{\Gamma}^{e}_{\alpha}(q, p_i)$. We start by classifying all the diagrams that contribute to this vertex in the RFG, into the following categories: (*i*) those containing an external three-gluon vertex, *i.e.*, those containing a three-gluon vertex where the momentum q is incoming, and (*ii*) those which do not have such an external three-gluon vertex. This latter set contains graphs where the incoming gluon couples to the rest of the diagram with any other type of interaction vertex other than a three-gluon vertex. Thus we write [33]

$$\Gamma^{e}_{\alpha}(q, p_i) = \Gamma^{e}_{A^3,\alpha}(q, p_i) + \Gamma^{e}_{A^4,\alpha}(q, p_i) + \Gamma^{e}_{A\bar{c}c,\alpha}(q, p_i) + \Gamma^{e}_{A\bar{q}q,\alpha}(q, p_i). \tag{4.3}$$

Then, the above Green's functions have the following diagrammatic representation



As a second step, we next carry out inside the class (i) diagrams the vertex decomposition given in Eq.(2.3); thus we write

$$\Gamma^{e}_{A^3,\alpha}(q,p_i) = \Gamma^{\mathrm{F},e}_{A^3,\alpha}(q,p_i) + \Gamma^{\mathrm{P},e}_{A^3,\alpha}(q,p_i), \tag{4.4}$$

where

$$\begin{aligned}
\Gamma^{\mathrm{F},e}_{A^3,\alpha}(q,p_i) &= gf^{eba}\int \Gamma^{\mathrm{F},\nu\mu}_{\alpha}(q,-k,k-q)\left[\mathcal{T}^{ab}_{\mu\nu}(-k+q,k,p_i)\right]_{t,\mathrm{I}}, \\
\Gamma^{\mathrm{P},e}_{A^3,\alpha}(q,p_i) &= gf^{eba}\int [(k-q)^{\mu}g^{\nu}_{\alpha} + k^{\nu}g^{\mu}_{\alpha}]\left[\mathcal{T}^{ab}_{\mu\nu}(-k+q,k,p_i)\right]_{t,\mathrm{I}},
\end{aligned} \tag{4.5}$$

and we have defined the integral measure

$$\int \equiv \mu^{2\varepsilon}\int \frac{d^dk}{(2\pi)^d}, \tag{4.6}$$

with $d = 4-2\varepsilon$ and μ the 't Hooft mass. Following the discussion presented in the previous section, the pinching action amounts to the replacement

$$k^{\nu}[\mathcal{T}^{ab}_{\mu\nu}]_{t,\mathrm{I}}(-k+q,k,p_i) \to [k^{\nu}\mathcal{T}^{ab}_{\mu\nu}(-k+q,k,p_i)]_{t,\mathrm{I}} = \left[\mathcal{S}^{ab}_{2\mu}(-k+q,k,p_i)\right]_{t,\mathrm{I}} \tag{4.7}$$

and similarly for the term coming from the momentum $(k-q)^{\mu}$, or, equivalently,

$$\Gamma^{\mathrm{P},e}_{A^3,\alpha}(q,p_i) \to gf^{eba}\int\left\{\left[\mathcal{S}^{ab}_{2\alpha}(-k+q,k,p_i)\right]_{t,\mathrm{I}} - \left[\mathcal{S}^{ab}_{1\alpha}(-k+q,k,p_i)\right]_{t,\mathrm{I}}\right\}. \tag{4.8}$$

At this point the construction of the effective PT vertex $\widehat{\Gamma}^{e}_{\alpha}$ has been completed, and we have

$$\begin{aligned}
\widehat{\Gamma}^{e}_{\alpha}(q,p_i) &= \Gamma^{\mathrm{F},e}_{A^3,\alpha}(q,p_i) + \Gamma^{e}_{A^4,\alpha}(q,p_i) + \Gamma^{e}_{A\bar{c}c,\alpha}(q,p_i) + \Gamma^{e}_{A\bar{q}q,\alpha}(q,p_i) \\
&+ gf^{eba}\int\left\{\left[\mathcal{S}^{ab}_{2\alpha}(-k+q,k,p_i)\right]_{t,\mathrm{I}} - \left[\mathcal{S}^{ab}_{1\alpha}(-k+q,k,p_i)\right]_{t,\mathrm{I}}\right\}.
\end{aligned} \tag{4.9}$$

We emphasize that in the construction presented thus far we have never resorted to the background formalism, but have only used the BRST identities of Eq.(2.19), together with Eq.(3.6). The next crucial question will be then to establish the connection between the effective PT vertex and the gluon–quark–anti-quark vertex $\widetilde{\Gamma}^{e}_{\alpha}(q,p_i)$ written in the BFG. For doing this we first of all observe that all the "inert" terms contained in the original RFG $\Gamma^{e}_{\alpha}(q,p_i)$ vertex carry over the same sub-groups of BFG graphs. In order to facilitate this identification we remind the reader that in the background field method the bare three-

and four-gluon vertices involving background and quantum gluons depend on the gauge-fixing parameter ξ_Q. In particular, the former involving one background gluon (carrying four-momentum q) and two quantum ones (carrying four-momenta k_1 and k_2), reads

$$\widetilde{\Gamma}_{\alpha\mu\nu}(q,k_1,k_2) = (q-k_1-\frac{1}{\xi_Q}k_2)_\nu g_{\alpha\mu} + (k_1-k_2)_\alpha g_{\mu\nu} + (k_2-q+\frac{1}{\xi_Q}k_1)_\mu g_{\alpha\nu}, \tag{4.10}$$

which can be rewritten as

$$\widetilde{\Gamma}_{\alpha\mu\nu}(q,k_1,k_2) = \Gamma^{\rm F}_{\alpha\mu\nu}(q,k_1,k_2) - \left(\frac{1-\xi_Q}{\xi_Q}\right)\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2), \tag{4.11}$$

or

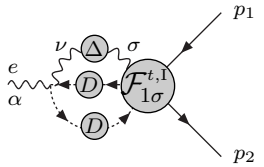
$$\widetilde{\Gamma}_{\alpha\mu\nu}(q,k_1,k_2) = \Gamma_{\alpha\mu\nu}(q,k_1,k_2) - \frac{1}{\xi_Q}\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2). \tag{4.12}$$

Eq.(4.11) implies then that in the BFG ($\xi_Q = 1$) the bare vertex of Eq.(4.10) coincides with the $\Gamma^{\rm F}_{\alpha\mu\nu}(q,k_1,k_2)$ of Eq.(2.4). Similarly, the four-particle vertex involving two background and two quantum gluons reduces at $\xi_Q = 1$ to the usual four-gluon vertex. Thus we have

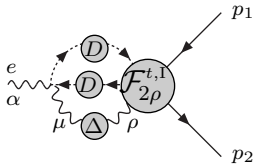
$$\begin{aligned}
\Gamma^{{\rm F},e}_{A^3,\alpha}(q,p_i) &\equiv \widetilde{\Gamma}^e_{\widetilde{A}A^2,\alpha}(q,p_i),\\
\Gamma^e_{A^4,\alpha}(q,p_i) &\equiv \widetilde{\Gamma}^e_{\widetilde{A}A^3,\alpha}(q,p_i),\\
\Gamma^e_{A\bar{q}q,\alpha}(q,p_i) &\equiv \widetilde{\Gamma}^e_{\widetilde{A}\bar{q}q,\alpha}(q,p_i),
\end{aligned} \tag{4.13}$$

where $\widetilde{A}$ is the background gluon. The only exception are the ghost diagrams $\Gamma^e_{A\bar{c}c,\alpha}(q,p_i)$. The important step is to recognize that the BFG ghost sector is provided precisely by combining the RFG ghosts with the right-hand side of Eq.(3.6). Specifically, one arrives at both the *symmetric* vertex $\widetilde{\Gamma}^e_{\widetilde{A}\bar{c}c}(q,p_i)$, characteristic of the BFG, as well as at the four-particle ghost vertex $\widetilde{\Gamma}^e_{\widetilde{A}A\bar{c}c}(q,p_i)$, with

$\widetilde{\Gamma}^e_{\widetilde{A}A\bar{c}c,\alpha}(q,p_i) =$



$+$



which is totally absent in the conventional formalism. Indeed, using Eq.(2.20), we find (omitting the spinors)

$$\begin{aligned}
\int \left[\mathcal{S}^{ab}_{1\alpha}(-k+q,k,p_i)\right]_{t,\rm I} &= -\int k_\alpha \left[\mathcal{G}^{ab}_1(-k+q,k,p_i)\right]_{t,\rm I} D(-k+q)D(k)\\
&\quad + igf^{bcd}\int \left[\mathcal{Q}^{acd}_{1\alpha}(-k+q,k,p_i)\right]_{t,\rm I} D(-k+q),\\
\int \left[\mathcal{S}^{ab}_{2\alpha}(-k+q,k,p_i)\right]_{t,\rm I} &= \int (k-q)_\alpha \left[\mathcal{G}^{ab}_2(-k+q,k,p_i)\right]_{t,\rm I} D(-k+q)D(k)\\
&\quad + igf^{acd}\int \left[\mathcal{Q}^{cdb}_{2\alpha}(-k+q,k,p_i)\right]_{t,\rm I} D(k).
\end{aligned} \tag{4.14}$$

Then it is easy to check that

$$\begin{aligned}
\widetilde{\Gamma}^{e}_{\widetilde{A}\bar{c}c,\alpha}(q,p_i) &= \Gamma^{e}_{A\bar{c}c,\alpha}(q,p_i) + gf^{eba}\int k_\alpha \left[\mathcal{G}_1^{ab}(-k+q,k,p_i)\right]_{t,\mathrm{I}} D(-k+q)D(k) \\
&+ gf^{eba}\int (k-q)_\alpha \left[\mathcal{G}_2^{ab}(-k+q,k,p_i)\right]_{t,\mathrm{I}} D(-k+q)D(k), \\
\widetilde{\Gamma}^{e}_{\widetilde{A}A\bar{c}c,\alpha}(q,p_i) &= -ig^2 f^{eba}f^{bcd}\int \left[\mathcal{Q}_{1\alpha}^{acd}(-k+q,k,p_i)\right]_{t,\mathrm{I}} D(-k+q) \\
&+ ig^2 f^{eba}f^{acd}\int \left[\mathcal{Q}_{2\alpha}^{cdb}(-k+q,k,p_i)\right]_{t,\mathrm{I}} D(k),
\end{aligned} \tag{4.15}$$

which gives us the final identity

$$\widehat{\Gamma}^{e}_{\alpha}(q,p_i) \equiv \widetilde{\Gamma}^{e}_{\alpha}(q,p_i). \tag{4.16}$$

Once again, we emphasize that the sole ingredient in the above construction has been the STIs of Eq.(2.19); in particular, at no point have we employed *a priori* the background field method formalism. Instead, its special ghost sector has arisen *dynamically*, once the PT rearrangement has taken place. An immediate consequence of the above correspondence between PT and BFG is that $\widehat{\Gamma}^{e}_{\alpha}(q,p_i)$ satisfies the QED-like WI

$$q^\alpha \widehat{\Gamma}^{e}_{\alpha}(q,p_1,p_2) = f^{ebc}\Big[\widehat{\Sigma}^{bc}(p_1) - \widehat{\Sigma}^{bc}(p_2)\Big]. \tag{4.17}$$

The final step, to be undertaken in detail in the next section, is to construct the all orders PT gluon self-energy $\widehat{\Pi}^{ab}_{\mu\nu}(q)$. Notice that at this point one would expect that it too coincides with the BFG gluon self-energy $\widetilde{\Pi}^{ab}_{\mu\nu}(q)$, since both the boxes $\widehat{B}$ and the vertex $\widehat{\Gamma}^{e}_{\alpha}(q,p_i)$ do coincide with the corresponding BFG boxes $\widetilde{B}$ and vertex $\widetilde{\Gamma}^{e}_{\alpha}(q,p_i)$, and the S-matrix is unique. We will end this section showing that this is indeed the case. To that end we will present a more detailed version of a proof based on the strong induction principle, which first appeared in [36]. This principle states that a given predicate $P(n)$ on $\mathbb{N}$ is true $\forall\, n \in \mathbb{N}$, if $P(k)$ is true whenever $P(j)$ is true $\forall\, j \in \mathbb{N}$ with $j < k$.

In order to avoid notational clutter, we will use the schematic notation introduced in Eq.(2.21), suppressing all the group, Lorentz, and momentum indices. At one- and two-loop (*i.e.*, $n = 1, 2$), we know that the result is true due to explicit calculations [4, 33]. Let us then assume that the PT construction has been successfully carried out up to the order $n-1$ (strong induction hypothesis): we will show then that the PT gluon self-energy is equal to the BFG gluon self-energy at order n, *i.e.*, $\widehat{\Pi}^{[n]} \equiv \widetilde{\Pi}^{[n]}$; hence, by the strong

induction principle, this last result will be valid at any given n, showing finally that the PT construction holds true to all orders.

From the strong inductive hypothesis, we know that

$$\begin{aligned}\widehat{\Pi}^{[\ell]} &\equiv \widetilde{\Pi}^{[\ell]},\\ \widehat{\Gamma}^{[\ell]} &\equiv \widetilde{\Gamma}^{[\ell]},\\ \widehat{B}^{[\ell]} &\equiv \widetilde{B}^{[\ell]} \equiv B^{[\ell]},\end{aligned} \tag{4.18}$$

where $\ell = 1, \ldots, n-1$.

Now, the S-matrix element of order n, to be denoted as $S^{[n]}$, assumes the form

$$S^{[n]} = \{\Gamma\Delta\Gamma\}^{[n]} + B^{[n]}. \tag{4.19}$$

Moreover, since it is unique, regardless if it is written in the Feynman gauge, in the BFG, as well as before and after the PT rearrangement, we have that $S^{[n]} \equiv \widehat{S}^{[n]} \equiv \widetilde{S}^{[n]}$. Using then Eq.(4.1) (which is all orders, implying that the last equation in (4.18) holds true also when $\ell = n$), we find that

$$\{\Gamma\Delta\Gamma\}^{[n]} \equiv \{\widehat{\Gamma}\widehat{\Delta}\widehat{\Gamma}\}^{[n]} \equiv \{\widetilde{\Gamma}\widetilde{\Delta}\widetilde{\Gamma}\}^{[n]}. \tag{4.20}$$

The above amplitudes can then be split into 1PR and 1PI parts; in particular, due to the strong inductive hypothesis of Eq.(4.18) the 1PR part after the PT rearrangement coincides with the 1PR part written in the BFG since

$$\{\Gamma\Delta\Gamma\}^{[n]}_{\rm R} = \Gamma^{[n_1]}\Delta^{[n_2]}\Gamma^{[n_3]}, \qquad \begin{cases} n_1, n_2, n_3 < n, \\ n_1 + n_2 + n_3 = n. \end{cases} \tag{4.21}$$

Then Eq.(4.20) state the equivalence of the 1PI parts, *i.e.*,

$$\{\widehat{\Gamma}\widehat{\Delta}\widehat{\Gamma}\}^{[n]}_{\rm I} \equiv \{\widetilde{\Gamma}\widetilde{\Delta}\widetilde{\Gamma}\}^{[n]}_{\rm I}, \tag{4.22}$$

which implies

$$\left(\widehat{\Gamma}^{[n]} - \widetilde{\Gamma}^{[n]}\right)\Delta^{[0]}\Gamma^{[0]} + \Gamma^{[0]}\Delta^{[0]}\left(\widehat{\Gamma}^{[n]} - \widetilde{\Gamma}^{[n]}\right) + \Gamma^{[0]}\Delta^{[0]}\left(\widehat{\Pi}^{[n]} - \widetilde{\Pi}^{[n]}\right)\Delta^{[0]}\Gamma^{[0]} \equiv 0. \tag{4.23}$$

At this point we do not have the equality we want yet, but only that

$$\begin{aligned}\widehat{\Gamma}^{[n]} &= \widetilde{\Gamma}^{[n]} + \mathcal{K}^{[n]}\Gamma^{[0]},\\ \widehat{\Pi}^{[n]} &= \widetilde{\Pi}^{[n]} - 2iq^2\mathcal{K}^{[n]},\end{aligned} \tag{4.24}$$

with $\mathcal{K}^{[n]}$ an arbitrary function of q^2. However, by means of the *explicit* construction of the PT vertex just presented, we have the all orders identity of Eq.(4.16), so that the second equation in (4.18) actually holds true even when $\ell = n$, *i.e.*, $\widehat{\Gamma}^{[n]} \equiv \widetilde{\Gamma}^{[n]}$; then one immediately gets

$$\widehat{\Pi}^{[n]} \equiv \widetilde{\Pi}^{[n]}. \tag{4.25}$$

Hence, by strong induction, the above relation is true for any given order n, *i.e.*, inserting back the Lorentz and gauge group structures, we arrive at the announced result

$$\widehat{\Pi}^{ab}_{\mu\nu}(q) \equiv \widetilde{\Pi}^{ab}_{\mu\nu}(q). \tag{4.26}$$

In the next section we will carry out the construction of the PT gluon self-energy in detail, and will see how the above conclusion is explicitly realized.

V. THE PT GLUON SELF-ENERGY TO ALL ORDERS

As we have seen in the previous sections, and as has been explained in detail in the literature, when constructing the PT two-point function various well-defined propagator-like contributions are moved from the three-point function to the two-point function. These pinch terms are always missing one or more propagators corresponding to the external legs of the two-point function under construction. Pictorially this characteristic structure gives rise to the appearance of the unphysical effective vertices, mentioned earlier. Of course, all such contributions, when re-alloted to the original two-point function will cancel exactly against analogous contributions concealed inside it. Reversing the order, the normal Feynman diagrams (*i.e.* with both external legs present) contributing to the two-point function must contain pieces that are effectively proportional to the inverse propagators of the external legs, a fact which allows them to communicate (and eventually cancel) against the pinch parts coming from the three-point function (or the boxes, when away from the Feynman gauge). Thus, when constructing the PT gluon self-energy one may follow two equivalent procedures. First, one may determine explicitly the pinch terms coming from the vertex and add them to the conventional graphs; this would correspond to the usual "S-matrix" PT construction. Second, one may isolate from the conventional self-energy all the aforementioned terms that are proportional to inverse propagators, and simply discard them; this would correspond to the "intrinsic PT". In this latter one avoids the embedding of the

PT objects into S-matrix elements, and manipulates only off-shell self-energy corrections. The two constructions are absolutely equivalent: discarding the aforementioned terms in the "intrinsic" construction is justified because we know that inside an S-matrix element they will eventually cancel (to all orders) against similar pieces stemming from the vertices.

In what follows we will present in detail the intrinsic construction, which, in addition to being more economical, it is intimately connected to the STI of $A^a_\mu\, A^b_\nu\, q^i\, \bar{q}^j$, employed in the previous sections. The important point is that the characteristic terms containing inverse propagators arise from the STI satisfied by the three-gluon vertex (of arbitrary order) appearing inside appropriate sets of diagrams, when it is contracted by longitudinal momenta. In fact, these terms are precisely the set of unphysical contributions $[\mathcal{R}^{ab}_{1\nu}]^{\rm ext}_{s,{\rm I}}$ produced by the action of a longitudinal momentum on the term $[\mathcal{T}^{ab}_{\mu\nu}]_{s,{\rm I}}$, as shown in Eq.(3.4). Evidently, the STI satisfied by the (full) three-gluon vertex gives independent knowledge, on the structure of the unphysical terms stemming from the 1PI self-energy contribution of a given order. Instead, we have no independent knowledge of the unphysical terms stemming from the 1PI vertex contribution; the latter may be deduced, if desirable through appropriate combination of the STI of $A^a_\mu\, A^b_\nu\, q^i\, \bar{q}^j$ and the STI of the (full) three-gluon vertex mentioned above. Roughly speaking, the unphysical contributions from the self-energy, which are known from the latter STI, must be canceled against the (unknown) unphysical contributions stemming from the vertex, since there are no unphysical contributions in the the STI of $A^a_\mu\, A^b_\nu\, q^i\, \bar{q}^j$, which is the sum of the two terms (as we will see in a moment, a minor refinement to this argument is necessary in order to account for 1PR contributions, but the general idea is essentially this).

In particular, denoting by $\mathbb{\Gamma}_{A_\alpha A_\mu A_\nu}(q,k_1,k_2)$ the all order gluon three-point function [with $\mathbb{\Gamma}^{[0]}_{A_\alpha A_\mu A_\nu} \equiv \Gamma^{[0]}_{\alpha\mu\nu}$ as defined in Eq.(2.2)] the STI triggered is [44]

$$
\begin{aligned}
k_1^\mu \mathbb{\Gamma}_{A_\alpha A_\mu A_\nu}(q,k_1,k_2) &= \left[i\Delta^{(-1)\,\rho}_\nu(k_2) + k_2^\rho k_{2\nu}\right]\left[k_1^2 D(k_1)\right] H_{\rho\alpha}(k_2,q) \\
&- \left[i\Delta^{(-1)\,\rho}_\alpha(q) + q^\rho q_\alpha\right]\left[k_1^2 D(k_1)\right] H_{\rho\nu}(q,k_2), \\
k_2^\nu \mathbb{\Gamma}_{A_\alpha A_\mu A_\nu}(q,k_1,k_2) &= \left[i\Delta^{(-1)\,\rho}_\alpha(q) + q^\rho q_\alpha\right]\left[k_2^2 D(k_2)\right] H_{\rho\mu}(q,k_1) \\
&- \left[i\Delta^{(-1)\,\rho}_\mu(k_1) + p_1^\rho k_{1\mu}\right]\left[k_2^2 D(k_2)\right] H_{\rho\alpha}(k_1,q),
\end{aligned}
\tag{5.1}
$$

where H represents the ghost Green's function appearing in the conventional formalism (see

for example [45]); at tree level

$$H^{[0]}_{\alpha\beta}(k_1,k_2) \;=\; \text{  } \;=\; -igg_{\alpha\beta}. \tag{5.2}$$

On the other hand, with the help of the Batalin-Vilkovisky formalism [46] formulated in the BFG, one can relate the BFG gluon two-point function $\widetilde{\mathbb{\Gamma}}_{\widetilde{A}_\alpha\widetilde{A}_\beta}$ with the conventional one $\mathbb{\Gamma}_{A_\alpha A_\beta}$ through a "background-quantum identity" (BQI) [47] of the form

$$\widetilde{\mathbb{\Gamma}}_{\widetilde{A}_\alpha\widetilde{A}_\beta}(q) = \mathbb{\Gamma}_{A_\alpha A_\beta}(q) + 2\mathbb{\Gamma}_{\Omega_\alpha A^*_\rho}(q)\mathbb{\Gamma}_{A^\rho A_\beta}(q) + \mathbb{\Gamma}_{\Omega_\alpha A^*_\rho}(q)\mathbb{\Gamma}_{A^\rho A^\sigma}(q)\mathbb{\Gamma}_{\Omega_\beta A^*_\sigma}(q), \tag{5.3}$$

where

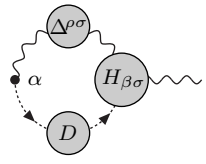
$$\mathbb{\Gamma}_{A^a_\alpha A^b_\beta}(q) = \delta^{ab}\left[iq_\alpha q_\beta - \Delta^{(-1)}_{\alpha\beta}(q)\right] \qquad \Longrightarrow \qquad \begin{cases} \mathbb{\Gamma}^{[0]}_{A_\alpha A_\beta}(q) = -iq^2 P_{\alpha\beta}(q), \\ \mathbb{\Gamma}^{[n]}_{A_\alpha A_\beta}(q) = \Pi^{[n]}_{\alpha\beta}(q^2), \end{cases} \tag{5.4}$$

and $\mathbb{\Gamma}_{\Omega A^*}$ represents an auxiliary (unphysical) two-point function connecting a background source Ω with a gluon anti-field A^* (see [48] for details).

The observation made in [48] was that, even though the auxiliary Green's function appearing in the STI of Eq.(5.1) is different from the one appearing in the BQI of Eq.(5.3), the two are related by a Schwinger-Dyson type of relation, which reads

$$i\mathbb{\Gamma}_{\Omega_\alpha A^*_\beta}(q) = C_A \int H^{[0]}_{\alpha\rho}(q,-k)D(k-q)\Delta^{\rho\sigma}(k)H_{\beta\sigma}(-q,k), \tag{5.5}$$

where C_A denotes the Casimir eigenvalue of the adjoint representation, *i.e.*, $C_A = N$ for $SU(N)$, and the integral measure is defined in Eq.(4.6). Diagrammatically, Eq.(5.5) reads

$$i\mathbb{\Gamma}_{\Omega_\alpha A^*_\beta}(q) \;=\; \text{  } \tag{5.6}$$

Evidently these last equations expresses the two-point Green's function $\mathbb{\Gamma}_{\Omega_\alpha A^*_\beta}(q)$, which is definable in the BV framework, entirely in terms of Green's functions definable in the conventional formalism; this will in turn connect the STI of Eq.(5.1) and the BQI of Eq.(5.3), which is what will finally allow to prove the correspondence between the PT an the BFG to all orders, using the intrinsic PT algorithm.

Following the two-loop case described in [48], we will now generalize the intrinsic PT procedure to all orders. The 1PI Feynman diagrams contributing to the conventional gluon

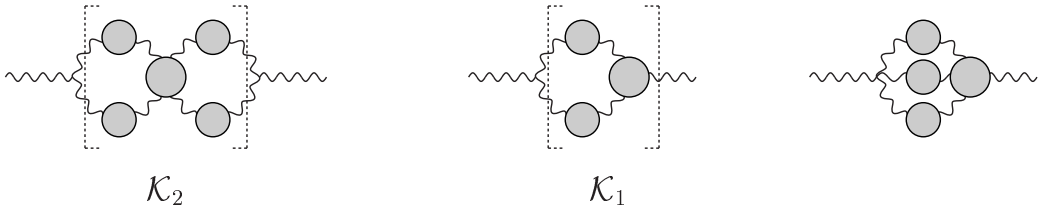


FIG. 8: Schematic representation of some 1PI diagrams, with their associated kernels, contributing to the all-order gluon self-energy.

self-energy in the R_ξ gauges can be always separated into three distinct sets (Fig.8): (*i*) the set of diagrams that have two external (tree-level) three-gluon vertices, and thus can be written schematically (suppressing Lorentz indices) as $\Gamma^{[0]}[\mathcal{K}_2]\Gamma^{[0]}$, where $\mathcal{K}_2$ is some kernel; (*ii*) the set of diagrams with only one external (tree-level) three-gluon vertex, and thus can be written as $\Gamma^{[0]}[\mathcal{K}_1]$ or $[\mathcal{K}_1]\Gamma^{[0]}$; (*iii*) all remaining diagrams, containing no external three-gluon vertices.

At this point we make the following observation: if we carry out the decomposition presented in Eq.(2.6) to the pair of external vertices appearing in the diagrams of the set (*i*), and the decomposition of Eq.(2.3) to the external vertex appearing in the diagrams of the set (*ii*), after a judicious rearrangement of the kernels $\mathcal{K}_2$ and $\mathcal{K}_1$ (together with their statistical factors), relabeling of internal momenta (the momenta k_1^α and k_2^β appearing in Eq.(5.1) will be in fact now related to virtual integration momenta appearing in the quantum loops), and taking into account the transversality of the gluon self-energy, we will end up with the result

$$\{\mathbb{\Gamma}_{AA}\}^{\rm P} = \quad + 2 \qquad (5.7)$$

Thus, the longitudinal terms k_1^α and k_2^β stemming from $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2)$ will be triggering the STIs of Eq.(5.1). For example, at the n-loop level, one would be triggering the (m)-loop version (with $m = 0, 1, \ldots, n-1$) of the aforementioned STIs. Therefore, the all order generalization of the intrinsic PT would amount to isolating from Eq.(5.7) the terms of the STI of Eq.(5.1) that are proportional to $[\Delta_\alpha^{(-1)\,\rho}(q)]$ ($[\Delta_\alpha^{(-1)\,\rho}(q)]^{[m]}$ in the n-loop case); we will denote such contributions by $\Pi^{\rm IP}_{\alpha\beta}(q)$. Thus the 1PI diagrams contributing to the gluon

self-energy can be cast in the form

$$\mathbb{\Gamma}_{A_\alpha A_\beta}(q) = G_{A_\alpha A_\beta}(q) + \Pi^{\rm IP}_{\alpha\beta}(q). \tag{5.8}$$

Notice however that the 1PR set $\mathbb{S}$ containing diagrams constructed from strings of lower order self-energy graphs (the set $\mathbb{S}^{[n]}$ containing the 2^{n-1} diagrams constructed from strings of self-energy insertions of order less than n, in our n-loop example), must also be rearranged following the intrinsic PT procedure, and converted into the equivalent set $\widehat{\mathbb{S}}$ containing strings involving PT self-energies. This treatment of the 1PR strings will give rise, in addition to the PT strings, to (*i*) a set of contributions which are proportional to the inverse tree-level propagator of the external legs $d^{-1}(q)$ (with $d(q) = -i/q^2$ the RFG tree-level gluon propagator), and (*ii*) a set of contributions which is *effectively* 1PI, and therefore also belongs to the definition of the 1PI PT gluon self-energy; we will denote these two sets of contributions collectively by $S^{\rm IP}_{\alpha\beta}(q)$. Thus the sum of the 1PI and 1PR contributions to the conventional gluon self-energy can be cast in the form

$$\mathbb{\Gamma}_{A_\alpha A_\beta}(q) + \mathbb{S}_{\alpha\beta}(q) = G_{A_\alpha A_\beta}(q) + \widehat{\mathbb{S}}_{\alpha\beta}(q) + \Pi^{\rm IP}_{\alpha\beta}(q) + S^{\rm IP}_{\alpha\beta}(q). \tag{5.9}$$

By definition of the intrinsic PT procedure, we will now discard from the above expression all the terms which are proportional to the inverse propagator of the external legs, thus defining the quantity

$$R^{\rm IP}_{\alpha\beta}(q) = \Pi'^{\rm IP}_{\alpha\beta}(q) + S'^{\rm IP}_{\alpha\beta}(q), \tag{5.10}$$

where the primed functions are defined starting from the unprimed ones appearing in Eq.(5.9) by discarding the aforementioned terms.

Thus, making use of Eqs.(5.8), (5.9) and (5.10), the intrinsic PT gluon self-energy, to be denoted as $\widehat{\mathbb{\Gamma}}_{A_\alpha A_\beta}(q)$, will be finally defined to all orders as

$$\begin{aligned}\widehat{\mathbb{\Gamma}}_{A_\alpha A_\beta}(q) &= G_{A_\alpha A_\beta}(q) + R^{\rm IP}_{\alpha\beta}(q) \\ &= \mathbb{\Gamma}_{A_\alpha A_\beta}(q) - \Pi^{\rm IP}_{\alpha\beta}(q) + R^{\rm IP}_{\alpha\beta}(q).\end{aligned} \tag{5.11}$$

We next proceed to the construction of the quantities $\Pi^{\rm IP}_{\alpha\beta}(q)$ and $R^{\rm IP}_{\alpha\beta}(q)$ discussed above.

A. 1PI diagrams

From Eq.(5.7) and the transversality of the gluon propagator, we find that the pinching contributions coming from the 1PI diagrams can be written as

$$\left\{\mathbb{\Gamma}_{A_\alpha A_\beta}\right\}^{\rm P} = -2C_A \int d(k)\Gamma^{\rm P}_{\alpha\mu\nu}(q,k-q,-k)\Delta^\nu_\sigma(k)\mathbb{\Gamma}_{A_\beta A^\mu A^\sigma}(q,k-q,-k). \tag{5.12}$$

Using then the definition of $\Gamma^{\rm P}$ given in Eq.(2.4) together with the tree-level value of the H Green's function [see Eq.(5.2)], we get

$$\left\{\mathbb{\Gamma}_{A_\alpha A_\beta}\right\}^{\rm P} = 2iC_A \int d(k)k^\mu H^{[0]}_{\alpha\nu}(q,-k)\Delta^{\nu\sigma}(k)\mathbb{\Gamma}_{A_\beta A_\mu A_\sigma}(q,k-q,-k). \tag{5.13}$$

To construct the PT quantity $\Pi^{\rm IP}_{\alpha\beta}(q)$, we now use the STI of Eq.(5.1) (with $k_1 = k-q$ and $k_2 = -k$) keeping only pinching terms; with the help of Eq.(5.4) we then find

$$\Pi^{\rm IP}_{\alpha\beta}(q) = 2iC_A \int H^{[0]}_{\alpha\nu}(q,-k)D(k-q)\Delta^{\nu\sigma}(k)H_{\rho\sigma}(-q,k)\mathbb{\Gamma}_{A^\rho A_\beta}(q), \tag{5.14}$$

which, using Eq.(5.5), can be finally cast in the form

$$\Pi^{\rm IP}_{\alpha\beta}(q) = -2\mathbb{\Gamma}_{\Omega_\alpha A^*_\rho}(q)\mathbb{\Gamma}_{A^\rho A_\beta}(q). \tag{5.15}$$

B. 1PR diagrams

From the 1PR set of diagrams $\mathbb{S}_{\alpha\beta}$, we need to identify the subset of contributions $S'^{\rm IP}_{\alpha\beta}$ which is effectively 1PI. In what follows, to avoid notational clutter we will suppress Lorentz indices.

The key observation for constructing the aforementioned quantity $S'^{\rm IP}$, is that at any order the only elements of the 1PR set $\mathbb{S}$ that can contribute to it are the strings that contains at most three self-energy insertions, *i.e.*, the subsets

$$\begin{aligned}\mathbb{S}_2 &= \mathbb{\Gamma}_{AA}\, d\, \mathbb{\Gamma}_{AA},\\ \mathbb{S}_3 &= \mathbb{\Gamma}_{AA}\, d\, \mathbb{\Gamma}_{AA}\, d\, \mathbb{\Gamma}_{AA}.\end{aligned} \tag{5.16}$$

To understand the reason for that, let us consider the order n set of 1PR diagrams $\mathbb{S}^{[n]}$, and suppose that the PT construction has been successfully carried out at order $n-1$. Then consider a generic string $\mathbb{S}^{[n]}_m \subset \mathbb{S}^{[n]}$ which contains m self-energy insertions

$$\mathbb{S}^{[n]}_m = \mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d \cdots d\, \mathbb{\Gamma}^{[i_\ell]}_{AA}\, d \cdots d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_m]}_{AA}, \tag{5.17}$$

where $\sum_{k=1}^{m} i_k \equiv n$.

We now concentrate on the self-energy insertion $\mathbb{\Gamma}^{[i_\ell]}_{AA}$ appearing in Eq.(5.17), and convert it into a PT self-energy insertion $\widehat{\mathbb{\Gamma}}^{[i_\ell]}_{AA}$. Since $i_\ell < n$, we know that $\widehat{\mathbb{\Gamma}}^{[i_\ell]}_{AA} \equiv \widetilde{\mathbb{\Gamma}}^{[i_\ell]}_{\widetilde{A}\widetilde{A}}$, and we can use the BQI of Eq.(5.3) to relate the BFG self-energy $\widetilde{\mathbb{\Gamma}}^{[i_\ell]}_{\widetilde{A}\widetilde{A}}$ to the conventional one $\mathbb{\Gamma}^{[i_\ell]}_{AA}$. Thus from the aforementioned PT conversion one will get the following extra terms

$$\begin{aligned}
&\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \left(\mathbb{\Gamma}^{[i_{\ell-1}]}_{AA}\mathbb{\Gamma}^{[i_\ell]}_{\Omega A^*}\right)\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_m]}_{AA},\\
&\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \left(\mathbb{\Gamma}^{[i_\ell]}_{\Omega A^*}\mathbb{\Gamma}^{[i_{\ell+1}]}_{AA}\right)\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_m]}_{AA},\\
&\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \left(-2\sum_{j=1}^{i_\ell-1}\mathbb{\Gamma}^{[i_\ell-j]}_{\Omega A^*}\mathbb{\Gamma}^{[j]}_{AA}\right)\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_m]}_{AA},\\
&\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \left(-\sum_{j_1=1}^{i_\ell-1}\sum_{j_2=0}^{j_1-1}\mathbb{\Gamma}^{[i_\ell-j_1]}_{\Omega A^*}\mathbb{\Gamma}^{[j_2]}_{AA}\mathbb{\Gamma}^{[j_1-j_2]}_{\Omega A^*}\right)\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_m]}_{AA}. \quad (5.18)
\end{aligned}$$

The first two comes from the part of the BQI of Eq.(5.3) proportional to $d^{-1}(q)$ and will contribute to cancel the terms one has to add in the conversion to the following two strings of the subset $\mathbb{S}^{[n]}_{m-1}$:

$$\begin{aligned}
&\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{\ell-2}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_{\ell-1}+i_\ell]}_{AA}\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{m-2}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA},\\
&\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{\ell-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_\ell+i_{\ell+1}]}_{AA}\, d\, \cdots\, d\, \mathbb{\Gamma}^{[i_{m-2}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_{m-1}]}_{AA}. \quad (5.19)
\end{aligned}$$

The last two terms in (5.18) will precisely cancel the terms leftover from the PT conversion of the string $\mathbb{S}^{[i_\ell]}$ appearing in the following subset of $\mathbb{S}^{[n]}_m$

$$\mathbb{\Gamma}^{[i_1]}_{AA}\, d\, \mathbb{\Gamma}^{[i_2]}_{AA}\, d\, \cdots\, d\, \mathbb{S}^{[i_\ell]}\, d\, \cdots\, d\mathbb{\Gamma}^{[i_{m-1}]}_{AA}\, d\, \mathbb{\Gamma}^{[i_m]}_{AA}. \quad (5.20)$$

We therefore see that the terms that one needs to add to a string of order n, which contains more than three self-energy insertions, will be canceled by other strings of the same order, but containing a different number of insertions. The only time that one will obtain terms that do not cancel and, as such, must be added to the 1PI gluon two-point function $\mathbb{\Gamma}^{[n]}_{A_\alpha A_\beta}$, is if the string contain two or three self-energy insertions ($\mathbb{S}^{[n]}_2$ and $\mathbb{S}^{[n]}_3$ respectively). In this case we will get

$$\begin{aligned}
\mathbb{S}^{[n]}_2 \;&\rightarrow\; \widehat{\mathbb{S}}^{[n]}_2 + 2\sum_{m=1}^{n-1}\mathbb{\Gamma}^{[n-m]}_{\Omega A^*}\mathbb{\Gamma}^{[m]}_{AA} + \sum_{m=1}^{n-1}\mathbb{\Gamma}^{[n-m]}_{\Omega A^*}\mathbb{\Gamma}^{[0]}_{AA}\mathbb{\Gamma}^{[m]}_{\Omega A^*} + 4\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1}\mathbb{\Gamma}^{[n-m]}_{\Omega A^*}\mathbb{\Gamma}^{[\ell]}_{AA}\mathbb{\Gamma}^{[m-\ell]}_{\Omega A^*},\\
\mathbb{S}^{[n]}_3 \;&\rightarrow\; \widehat{\mathbb{S}}^{[n]}_3 - 3\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1}\mathbb{\Gamma}^{[n-m]}_{\Omega A^*}\mathbb{\Gamma}^{[\ell]}_{AA}\mathbb{\Gamma}^{[m-\ell]}_{\Omega A^*}. \quad (5.21)
\end{aligned}$$

Thus we see that the total effective 1PI contribution coming from the conversion of the 2^{n-1} 1PR strings contributing to the gluon self-energy $I\!\Gamma^{[n]}_{A_\alpha A_\beta}$ at order n, into the corresponding 1PR PT strings, will be

$$S'^{\,\mathrm{IP}\,[n]} = 2\sum_{m=1}^{n-1} I\!\Gamma^{[n-m]}_{\Omega A^*} I\!\Gamma^{[m]}_{AA} + \sum_{m=1}^{n-1}\sum_{\ell=0}^{m-1} I\!\Gamma^{[n-m]}_{\Omega A^*} I\!\Gamma^{[\ell]}_{AA} I\!\Gamma^{[m-\ell]}_{\Omega A^*}. \tag{5.22}$$

On the other hand, Eq.(5.15) implies that

$$\Pi'^{\,\mathrm{IP}\,[n]} = -2\sum_{m=1}^{n-1} I\!\Gamma^{[n-m]}_{\Omega A^*} I\!\Gamma^{[m]}_{AA}, \tag{5.23}$$

so that adding by parts the last two equations and putting back Lorentz and momentum indices, we get the all order result

$$R^{\mathrm{IP}}_{\alpha\beta}(q) = I\!\Gamma_{\Omega_\alpha A^*_\mu}(q) I\!\Gamma_{A^\mu A^\nu}(q) I\!\Gamma_{\Omega_\beta A^*_\nu}(q). \tag{5.24}$$

Thus, making use of the BQI of Eq.(5.3), we have the identity

$$\begin{aligned}
\widehat{I\!\Gamma}_{A_\alpha A_\beta}(q) &= I\!\Gamma_{A_\alpha A_\beta}(q) - \Pi^{\mathrm{IP}}_{\alpha\beta}(q) + R^{\mathrm{IP}}_{\alpha\beta}(q) \\
&= I\!\Gamma_{A_\alpha A_\beta}(q) + 2I\!\Gamma_{\Omega_\alpha A^*_\rho}(q) I\!\Gamma_{A^\rho A_\beta}(q) + I\!\Gamma_{\Omega_\alpha A^*_\rho}(q) I\!\Gamma_{A^\rho A^\sigma}(q) I\!\Gamma_{\Omega_\beta A^*_\sigma}(q) \\
&= \widetilde{I\!\Gamma}_{\widetilde{A}_\alpha \widetilde{A}_\beta}(q),
\end{aligned} \tag{5.25}$$

a result that completes the all-order explicit construction of the PT gluon self-energy.

VI. PROCESS INDEPENDENCE OF THE PT ALGORITHM

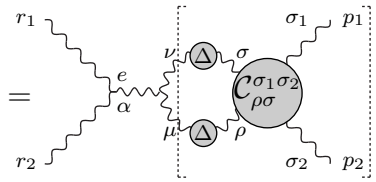
One important question to be addressed in the PT context, is whether this construction depends on the specific kind of external particles chosen. This issue was addressed in [49] by means of detailed calculations, and in [48] through the use of the BQIs. In both cases it was shown that, at the one-loop level, the gluon self-energy constructed by resorting to the S-matrix PT algorithm is universal, in the sense that its form does not depend on the specific process used for the embedding.

The purpose of this section is to demonstrate that this property holds true to all orders. Before proving this in the most general case, let us consider a specific example, *i.e.*, the construction of the PT gluon self-energy through the process $A^{d_1}_{\rho_1}(r_1) A^{d_2}_{\rho_2}(r_2) \to A^{e_1}_{\sigma_1}(p_1) A^{e_2}_{\sigma_2}(p_2)$, where $A^{d_i}_{\rho_i}(r_i)$ and $A^{e_i}_{\sigma_i}(p_i)$ represent on-shell gluons, *i.e.*, with $r_i^2 = p_i^2 = 0$ and $r_i^{\rho_i}\epsilon_{\rho_i}(r_i) = p_i^{\sigma_i}\epsilon_{\sigma_i}(p_i) = 0$.

As before we will denote by $\mathcal{A}^{d_1d_2e_1e_2}(r_1,r_2,p_1,p_2)$ the subset of graphs which will receive the action of the longitudinal momenta stemming from the pinch part $\Gamma^{\rm P}_{\alpha\mu\nu}(q,k_1,k_2)$ of the three-gluon vertex. We have then

$$\begin{aligned}\mathcal{A}^{d_1d_2e_1e_2}(r_1,r_2,p_1,p_2) \;&=\; g\epsilon^{\rho_1}(r_1)\epsilon^{\rho_2}(r_2)\Gamma^{ed_1d_2}_{\alpha\rho_1\rho_2}(q,r_1,r_2)f^{eab}\Gamma^{{\rm P},\alpha\mu\nu}(q,k_1,k_2)\,\times\\ &\times\;\mathcal{T}^{abe_1e_2}_{\mu\nu}(k_1,k_2,p_1,p_2),\end{aligned}\tag{6.1}$$

where now $\mathcal{T}^{abe_1e_2}_{\mu\nu}$ represents the sub-amplitude $g^a_\mu(k_1)g^b_\nu(k_2)\,\to\,g^{e_1}_{\sigma_1}(p_1)g^{e_2}_{\sigma_2}(p_2)$ with the initial gluons off-shell and final ones on-shell. Diagrammatically then,

$\mathcal{A}^{abe_1e_2}=$ 

so that in terms of Green's functions we have

$$\mathcal{T}^{abe_1e_2}_{\mu\nu}(k_1,k_2,p_1,p_2)=\left[\Delta^\rho_\mu(k_1)\Delta^\sigma_\nu(k_2)\mathcal{C}^{abe_1e_2}_{\rho\sigma\sigma_1\sigma_2}(k_1,k_2,p_1,p_2)\right]\epsilon^{\sigma_1}(p_1)\epsilon^{\sigma_2}(p_2).\tag{6.2}$$

Clearly there is an equal contribution coming from a mirror diagram where $\Gamma^{\rm P}$ is situated to the right-hand side of the $\mathcal{T}^{abe_1e_2}_{\mu\nu}$ amplitude. As in the quark–anti-quark case, we need to focus on the STI satisfied by the amplitude (6.2). To this end, we start from the trivial identities

$$\begin{aligned}\left\langle T\left[\bar{c}^a(x)A^b_\nu(y)A^{e_1}_{\lambda_1}(z)A^{e_2}_{\lambda_2}(w)\right]\right\rangle \;&=\;0,\\ \left\langle T\left[A^a_\mu(x)\bar{c}^b(y)A^{e_1}_{\lambda_1}(z)A^{e_2}_{\lambda_2}(w)\right]\right\rangle \;&=\;0,\end{aligned}\tag{6.3}$$

and then apply the BRST operator s of Eq.(2.13), to get the STIs

$$\begin{aligned}&\partial^\mu_x C^{abe_1e_2}_{\mu\nu\lambda_1\lambda_2}+\partial^y_\nu G^{abe_1e_2}_{1\lambda_1\lambda_2}+gf^{bcd}Q^{acde_1e_2}_{1\nu\lambda_1\lambda_2}\\ +\;&\partial^z_{\lambda_1}G^{abe_1e_2}_{1\nu\lambda_2}+\partial^w_{\lambda_2}G^{abe_1e_2}_{1\nu\lambda_1}+gf^{e_1cd}Q^{abcde_2}_{1\nu\lambda_1\lambda_2}+gf^{e_2cd}Q^{abe_1cd}_{1\nu\lambda_1\lambda_2}=0,\\ &\partial^\mu_y C^{abe_1e_2}_{\mu\nu\lambda_1\lambda_2}+\partial^x_\mu G^{abe_1e_2}_{2\lambda_1\lambda_2}+gf^{acd}Q^{cdbe_1e_2}_{2\mu\lambda_1\lambda_2}\\ +\;&\partial^z_{\lambda_1}G^{abe_1e_2}_{2\mu\lambda_2}+\partial^w_{\lambda_2}G^{abe_1e_2}_{2\mu\lambda_1}+gf^{e_1cd}Q^{abcde_2}_{2\mu\lambda_1\lambda_2}+gf^{e_2cd}Q^{abe_1cd}_{2\mu\lambda_1\lambda_2}=0,\end{aligned}\tag{6.4}$$

where the Green's functions appearing above (in configuration space) are obtained from the corresponding ones appearing in Eq.(2.15), through the replacements $\bar{q}^i(z)\,\to\,A^{e_1}_{\lambda_1}(z)$ and

$q^j(w) \to A^{e_2}_{\lambda_2}(w)$ plus an eventual suitable permutation of fields; for example,

$$\begin{aligned} G^{abe_1e_2}_{1\nu\lambda_2} &= \Big\langle T\left[\bar{c}^a(x)A^b_\nu(y)c^{e_1}(z)A^{e_2}_{\lambda_2}(w)\right]\Big\rangle, \\ Q^{abcde_2}_{2\mu\lambda_1\lambda_2} &= \Big\langle T\left[A^a_\mu(x)\bar{c}^b(y)A^c_{\lambda_1}(z)c^d(z)A^{e_2}_{\lambda_2}(w)\right]\Big\rangle. \end{aligned} \tag{6.5}$$

We can then Fourier transform the identity of Eq.(6.4) to obtain the momentum-space STIs

$$\begin{aligned} & k_1^\mu C^{abe_1e_2}_{\mu\nu\lambda_1\lambda_2} + k_{2\nu} G^{abe_1e_2}_{1\lambda_1\lambda_2} - igf^{bcd} Q^{acde_1e_2}_{1\nu\lambda_1\lambda_2} \\ + \; & p_{1\lambda_1} G^{abe_1e_2}_{1\nu\lambda_2} + p_{2\lambda_2} G^{abe_1e_2}_{1\nu\lambda_1} - igf^{e_1cd} Q^{abcde_2}_{1\nu\lambda_1\lambda_2} - igf^{e_2cd} Q^{abe_1cd}_{1\nu\lambda_1\lambda_2} = 0, \\ & k_2^\nu C^{abe_1e_2}_{\mu\nu\lambda_1\lambda_2} + k_{1\mu} G^{abe_1e_2}_{2\lambda_1\lambda_2} - igf^{acd} Q^{cdbe_1e_2}_{2\mu\lambda_1\lambda_2} \\ + \; & p_{1\lambda_1} G^{abe_1e_2}_{2\mu\lambda_2} + p_{2\lambda_2} G^{abe_1e_2}_{2\mu\lambda_1} - igf^{e_1cd} Q^{abcde_2}_{2\mu\lambda_1\lambda_2} - igf^{e_2cd} Q^{abe_1cd}_{2\mu\lambda_1\lambda_2} = 0, \end{aligned} \tag{6.6}$$

where the momentum-space Green's functions appearing above are obtained from the corresponding ones appearing in Eq.(2.16), by replacing the fermion propagators $S(p_i)$ with the gluon propagators $\Delta^{\tau_i}_{\lambda_i}(p_i)$, and adding the corresponding Lorentz index τ_i to the kernel involved in the definition.

The last four terms of both the STIs of Eq.(6.6) will actually die due to the on-shell condition of the external gluons. In fact, we multiply both sides of Eq.(6.6) by the product $\Delta^{(-1)\lambda_1}_{\sigma_1}(p_1)\Delta^{(-1)\lambda_2}_{\sigma_2}(p_2)$ of the two inverse propagators of the external gluons, and then contract the resulting amplitudes with the polarization tensors $\epsilon^{\sigma_i}(p_i)$. Since the external gluon are assumed to be on-shell, we have that

$$\begin{aligned} & \epsilon^{\sigma_i}(p_i)\Delta^{(-1)\lambda_i}_{\sigma_i}(p_i) = 0, \\ & \epsilon^{\sigma_i}(p_i)p_{i\sigma_i} = 0, \end{aligned} \tag{6.7}$$

from which the vanishing of the aforementioned terms follows. Thus we arrive at the on-shell STIs

$$\begin{aligned} k_1^\mu \mathcal{T}^{abe_1e_2}_{\mu\nu}(k_1,k_2,p_1,p_2) &= \mathcal{S}^{abe_1e_2}_{1\nu}(k_1,k_2,p_1,p_2), \\ k_2^\nu \mathcal{T}^{abe_1e_2}_{\mu\nu}(k_1,k_2,p_1,p_2) &= \mathcal{S}^{abe_1e_2}_{2\mu}(k_1,k_2,p_1,p_2), \end{aligned} \tag{6.8}$$

with

$$\mathcal{S}^{abe_1e_2}_{1\nu}(k_1,k_2,p_1,p_2) = \big[igf^{bcd}\mathcal{Q}^{acde_1e_2}_{1\nu\sigma_1\sigma_2}(k_1,k_2,p_1,p_2)D(k_1)$$

$$
\begin{aligned}
&- \; k_{2\nu} G^{abe_1e_2}_{1\sigma_1\sigma_2}(k_1,k_2,p_1,p_2) D(k_1) D(k_2) \big] \, \epsilon^{\sigma_1}(p_1) \epsilon^{\sigma_2}(p_2), \\
\mathcal{S}^{abe_1e_2}_{2\mu}(k_1,k_2,p_1,p_2) \; &= \; \big[ig f^{acd} \mathcal{Q}^{cdbe_1e_2}_{2\mu\sigma_1\sigma_2}(k_1,k_2,p_1,p_2) D(k_2) \\
&- \; k_{1\mu} G^{abe_1e_2}_{2\sigma_1\sigma_2}(k_1,k_2,p_1,p_2) D(k_1) D(k_2) \big] \, \epsilon^{\sigma_1}(p_1) \epsilon^{\sigma_2}(p_2). \qquad (6.9)
\end{aligned}
$$

These STIs have exactly the same form of the ones shown in Eq.(2.19) derived in the quark–anti-quark case. The only difference is in the kernels which enters in the definitions of the amplitude $\mathcal{T}$ and the Green's functions $\mathcal{S}$. However the all orders PT algorithm constructed in the previous sections does not depend in any way on the kernels involved, so that it goes through unmodified also in the present case.

Notice that the reason for which the STIs of Eqs.(2.19) and (6.8) have the same form, is due to the fact that the BRST variation of an on-shell field (independently of it being a quark or a gluon) vanishes due to the on-shell condition. Thus, the particular STIs needed for the application of the PT algorithm are completely determined by the off-shell particles, which are fixed (*i.e.*, two gluons), regardless of the process in which we embed the two-point function we want to construct.

We thus conclude the construction of the PT gluon self-energy through the embedding into the process $A^{d_1}_{\rho_1}(r_1) A^{d_2}_{\rho_2}(r_2) A^{d_3}_{\rho_3}(r_3) \to A^{e_1}_{\sigma_1}(p_1) A^{e_2}_{\sigma_2}(p_2) A^{e_3}_{\sigma_3}(p_3)$ with on-shell initial and final gluons, proceeds in exactly the same way as in the case of final on-shell quarks. The only adjustments required are those pertaining to the kernels appearing in the corresponding STIs, while the construction algorithm itself remains unaltered.

VII. RENORMALIZATION

In this section we will discuss the renormalization of the PT Green's functions constructed in the previous sections. There is of course no doubt that if one supplies the correct counterterms within the conventional formulation, the entire S-matrix will continue being renormalized, even after the PT rearrangement of the (unrenormalized) Feynman graphs. The question addressed in this section is whether the new Green's function constructed through the PT rearrangement are *individually* renormalizable [50].

The general methodology for dealing with this issue has been established in the second paper of [33], where the two-loop case was studied in detail: One should start out with the counterterms which are necessary to renormalize individually the conventional Green's

functions contributing to the n-loop S-matrix in the RFG. Then, one should show that, by simply rearranging these counterterms, following the PT rules, one arrive at renormalized n-loop PT Green's functions. This section is meant to serve as a general framework for the all-order construction, putting particular emphasis on the various conceptual and methodological issues involved, rather than an explicit proof of renormalizability. We consider this discussion sufficient for convincing the reader that renormalization poses no problem whatsoever to the all-order PT construction. The basic points are the following:

(i) We will assume that the massless Yang-Mills theory, quantized in the RFG, is renormalizable to all-orders. We will use the following notation: Z_1 is the vertex renormalization constant for the quark-gluon vertex Γ_α, Z_2 is the wave-function renormalization for the (external) quarks, Z_A the gluon wave-function renormalization corresponding to the gluon self-energy Π, Z_3 is the vertex renormalization constant for the three-gluon vertex $\Gamma_{\alpha\mu\nu}$, $\bar{Z}_2$ is the usual ghost wave-function renormalization, and $\bar{Z}_1$ the ghost-gluon vertex renormalization constant; of course, all above quantities and renormalization constants are to be computed in the RFG. Notice also that, the BRST symmetry demands that $Z_3/Z_A = \bar{Z}_1/\bar{Z}_2$. Equivalently, one can carry out the renormalization program using appropriately defined counter-terms. The corresponding counterterms, which, when added to the above n-loop quantities render them UV finite, are, respectively $K_1^{[n]}$, $K_2^{[n]}$, $K_A^{[n]}$, $\widehat{K}_A^{[n]}$, $K_3^{[n]}$, $K_{3F}^{[n]}$, $\bar{K}_2^{[n]}$, and $\bar{K}_1^{[n]}$. The Z's and the K's are in general related by $Z = 1 + \sum_{j=1} K^{[j]}$. Of course, mass counterterms δm must also be supplied if the quarks are considered to be massive.

(ii) It is important to recognize that, even though the PT self-energies does not coincide with the ones appearing inside the loops (exactly as happens in the background field method) there is no conflict with renormalization. This point is rather subtle, and deserves some further clarification. At the level of the original Lagrangian (in the RFG) the counterterms will be furnished as usual, *i.e.* in such a way as to render the self-energy and vertices finite. At one-loop, for example, a counterterm of the form $(q^2 g_{\mu\nu} - q_\mu q_\nu) K_A^{[1]}$ must be provided to the self-energy $\Pi^{[1]}$, and a term $K_1^{[1]}(\lambda/2)\gamma_\alpha$ to the vertex $\Gamma_\alpha^{[1]}$. The PT self-energy $\widehat{\Pi}^{[1]}$ and vertex $\widehat{\Gamma}_\alpha^{[1]}$ have different renormalization properties than $\Pi^{[1]}$ and $\Gamma_\alpha^{[1]}$; therefore, the existing counterterms must be appropriately reshuffled. In particular, due to the fact that, unlike $\Gamma_\alpha^{[1]}$, the PT vertex satisfies the QED-like WI of Eq.(4.17) it becomes ultraviolet finite when the counterterms $K_2^{[1]}$, equal to that of the (external) quarks, is added to it. Consequently, the remaining vertex counterterm, *i.e.* $K_1^{[1]} - K_2^{[1]}$, together with an equal contribution from the

mirror-vertex, must be given to $\widehat{\Pi}^{[1]}$; this is accomplished by inserting, as usual, the unity as $q^2(1/q^2)$ and adding the missing longitudinal pieces for free. Thus, the resulting (effective) counterterm for $\hat{\Pi}^{[1]}$ will be $\widehat{K}_A^{[1]} = K_A^{[1]} - 2(K_1^{[1]} - K_2^{[1]})$, and is, of course, equal to the counterterm necessary to renormalize $\widetilde{\mathbb{\Gamma}}^{[1]}_{\widetilde{A}_\alpha \widetilde{A}_\beta}$. At this point $\widehat{K}_A^{[1]}$ can effectively be thought off as a new propagator-like counterterm. Of course, exactly as happens in the BFG, when going to the next order the counterterm alloted to $\Pi^{[1]}$ appearing inside loops will still be $K_A^{[1]}$ and not $\widehat{K}_A^{[1]}$, *i.e.* one must start out, at any given order, with the counterterms generated by the original Lagrangian defined in the RFG, and rearrange them appropriately. Notice also that, again due to the validity of Eq.(4.17), the renormalization constants before and after the PT rearrangements are related to the gauge coupling renormalization as follows:

$$Z_g^2 \;=\; Z_1^2 Z_2^{-2} Z_A^{-1} = \widehat{Z}_1^2 \widehat{Z}_2^{-2} \widehat{Z}_A^{-1} = \widehat{Z}_A^{-1}. \tag{7.1}$$

(*iii*) Of course, primitively divergent graphs which are inert under the PT rearrangement, such as the third graph of Fig.8, are rendered finite when their usual counterterms are furnished, without any need for further modifications. The same is true for the entire PT box, since it coincides with the conventional box in the RFG (and the BFG); therefore it has no primitive divergence, and all its sub-divergences are canceled by the normal counterterms.

(*iv*) The bare three-gluon vertices $\Gamma^{eab\,[0]}_{\alpha\mu\nu}$ associated to counter-terms do *not* undergo the PT splitting of Eq.(2.3). This is consistent with the general PT rules, simply because such terms are essentially furnished in order to cancel divergences stemming from sub-integrations; as we have explained earlier, longitudinal pieces induced by sub-integrations should *not* pinch, in order not to violate the manifest analyticity of the individual Green's functions. The simplest way to see that, once pinching induced by sub-integration has been forbidden, the counterterms proportional to $\Gamma^{eab\,[0]}_{\alpha\mu\nu}$ should not pinch either, is to consider the first one-loop vertex diagram appearing on the second row of Fig.7 (denoted by $[\mathcal{T}^{[1]}]_{s,\mathrm{I}}$), and imagine that the gluonic triangle has been replaced by a fermionic one. Evidently the resulting graph cannot furnish pinching momenta; on the other hand, its divergent part is proportional to $\Gamma^{eab\,[0]}_{\alpha\mu\nu}$, and so is the counterterm which must be supplied to render it finite. Clearly, splitting the counterterm, while the main digram is inert, will result in an obvious mismatch between its divergent parts and the corresponding counterterm.

(*v*) The fundamental STI employed in section II survives renormalization, simply because all counterterms necessary to render it finite are already furnished by the usual counterterms

of the RFG Lagrangian. This is, of course, a direct result of the basic assumption the the theory in the RFG is renormalizable: once all counterterms have been supplied in the RFG, the STI which is studied in the same gauge, will continue being valid.

(*vi*) As has been explained in [33], and as is obvious from the coincidence of the PT and BFG results, the basic structure which appears nested inside the PT Green's functions is the high-order generalization of the vertex quantity $\Gamma^{\mathrm{F}\,[0]}_{\alpha\mu\nu}$. This quantity, to be denoted by $\Gamma^{\mathrm{F}\,[n]}_{\alpha\mu\nu}(q,p_1,p_2)$ coincides with the all-order BFG Green's functions with one background ($\tilde{A}$) and two quantum (A) gluons incoming, *i.e.* $\Gamma^{[n]}_{\tilde{A}_\alpha A_\mu A_\nu}(q,p_1,p_2)$. $\Gamma^{\mathrm{F}\,[n]}_{\alpha\mu\nu}(q,p_1,p_2)$ satisfies the following WI

$$q^\alpha \Gamma^{\mathrm{F}\,[n]}_{\alpha\mu\nu}(q,p_1,p_2) = \Pi^{[n]}_{\mu\nu}(p_1) - \Pi^{[n]}_{\mu\nu}(p_2), \tag{7.2}$$

which is the exact one-loop analog of the tree-level Ward identity of Eq (2.5); indeed the RHS is the difference of two conventional n-loop self-energies computed in the RFG. Notice also that Eq. (7.2) dictates that the ultraviolet-divergent part of $\Gamma^{\mathrm{F}\,[1]}_{\alpha\mu\nu}$ is proportional to $\Gamma^{[0]}_{\alpha\mu\nu}$ rather than $\Gamma^{\mathrm{F}\,[0]}_{\alpha\mu\nu}$; had it been the other way around there would be no longitudinal ultraviolet-divergent pieces on the RHS of Eq. (7.2). As has been explained in [33], this "mismatch" will generate the pieces which, in the background field method language, give rise to the gauge-fixing renormalization of the vertices [see point (*vii*), below]. Clearly, due to the WI of Eq. (7.2), we must have $Z_{3F} = Z_A$, where Z_{3F} is the vertex renormalization constant for the $\Gamma^{\mathrm{F}\,[n]}_{\alpha\mu\nu}$.

(*vii*) After the rearrangements of the original counterterms (in the RFG), in such a way as to render the PT Green's functions finite, one should be able to verify that the resulting counterterms are in fact identical to those obtained when carrying out the background field method renormalization program as explained by Abbott in the eighth paper of [41], *i.e.* by renormalizing only the background gluons, the external quarks, the coupling constant g, and the quantum gauge-fixing parameter ξ_Q. Thus, the relevant renormalization constants are given by

$$g_0 = Z_g g\,, \qquad \tilde{A}_0 = Z_{\tilde{A}}^{1/2}\tilde{A}\,, \qquad \xi^0_Q = Z_{\xi_Q}\xi_Q\,, \qquad Z_{\xi_Q} = Z_A\,. \tag{7.3}$$

The renormalization of ξ_Q is necessary due to the fact that the longitudinal part of the quantum gluon propagator is not renormalized. As pointed out by Abbott, in the context of the background field method this step may be avoided if the calculation is carried out with an arbitrary ξ_Q rather than the BFG $\xi_Q = 1$. Of course, as we have seen, the PT brings us

effectively at $\xi_Q = 1$; thus, when attempting to interpret the resulting counterterm from the background field method point of view, one should keep in mind that gauge-fixing parameter renormalization is necessary. The renormalization of ξ_Q not only affects the propagator-lines, but also the longitudinal parts of the external vertices; it renormalizes precisely the Γ_P part, as can be seen from Eq.(4.12).

All the above ingredients must be combined appropriately in order to demonstrate the renormalizability of the PT effective Green's functions; for the purposes of this paper we shall not pursue this point any further.

VIII. DISCUSSION AND CONCLUSIONS

In this article we have presented in detail the construction to all orders in perturbation theory of three basic PT Green's functions, namely the off-shell gluon self-energy, the quark–anti-quark-gluon vertex, and the four-quark box. The PT procedure, through its very definition, is based on the systematic exploitation of a fundamental cancellation between the self-energy and vertex diagrams appearing in the amplitude of a physical process. This cancellation allows for the construction of gauge-independent and gauge-invariant effective Green's functions, with the variety of phenomenological uses outlined in the Introduction. The central result of the present paper is that this crucial cancellation can be carried out systematically and expeditiously to all orders by appealing to the STI satisfied by a special four-point function, which constitutes a common kernel to the self-energy and vertex diagrams involved in the pinching procedure. Therefore, all the important properties of the PT Green's functions, known from the one- and two-loop analysis, are valid to all orders.

As was first shown in [36], and in the present one in much more detail, the known correspondence between the PT Green's functions and those calculated in the BFG persists to all orders. This fact which provides a very convenient book-keeping scheme for the actual calculation of the former, in principle to any desired order. We emphasize that this correspondence has been established through an a-posteriori comparison of the PT results, derived in the RFG, to those of the BFG; all diagrammatic rearrangements leading to the latter scheme, and in particular to its very characteristic ghost sector, have proceeded dynamically, due to the appropriate exploitation of the corresponding STIs. It would be clearly very interesting to reach a deeper understanding of what singles out the value $\xi_Q = 1$.

One possibility would be to look for special properties of the BFM action at $\xi_Q = 1$ [51]; an interesting 3-d example of a field-theory, which, when formulated in the background Landau gauge ($\xi_Q = 0$), displays an additional (non-BRST related) rigid super-symmetry, is given in [52].

Despite the progress reported in the present article, various technical questions merit further study. To begin with, the general construction of higher PT n-point functions with all legs off-shell (for example, the all-order three-gluon vertex ($n = 3$), whose one-loop derivation was presented in the first paper of [5]) is lacking for the moment. In addition, our analysis has been restricted to the case of the linear covariant gauges, but it would be interesting to study what happens in the context of entirely different gauges, as for example is the case of the non-covariant axial gauges [53]. These gauges present the additional complication that the convenient Feynman gauge cannot be reached a priori by simply fixing appropriately the value of the gauge fixing parameter. Our experience from explicit one- and two-loops calculations (see for example [31], and the third paper of [28]) is that the application of the usual PT algorithm leads to a vast number of cancellations, which dynamically projects one to the $g_{\mu\nu}$ part of the gluon propagator. Thus, even if one uses a bare gluon propagator of the general axial gauge form, after the aforementioned cancellations have taken place one arrives effectively to the answer written in the RFG; it is an open question whether this fact persists to all orders. Needless to say, the generalization of the formalism developed here to the Electroweak sector of the Standard Model presents, as in the two-loop case [54], a significant technical challenge. Furthermore, at the conceptual level it is unknown whether a formal definition of the PT Green's functions in terms of fundamental fields, encoding "ab initio" their special properties, is possible. Finally, it would be interesting to explore possible connections with various related formalisms [55, 56, 57, 58, 59].

Acknowledgments

J.P. thanks R. Pisarski for his continuing encouragement, and F. del Aguila and N. Mavromatos for valuable comments. The work of D.B. is supported by the Ministerio of Educación, Cultura y Deporte, Spain, under Grant DGICYT-PB97-1227, and the research

of J.P. is supported by CICYT, Spain, under Grant AEN-99/0692.

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