

# A novel integral representation for the Adler function

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**Abstract.** New integral representations for the Adler  $D$ -function and the  $R$ -ratio of the electron-positron annihilation into hadrons are derived in the general framework of the analytic approach to QCD. These representations capture the nonperturbative information encoded in the dispersion relation for the  $D$ -function, the effects due to the interrelation between spacelike and timelike domains, and the effects due to the nonvanishing pion mass. The latter plays a crucial role in this analysis, forcing the Adler function to vanish in the infrared limit. Within the developed approach the  $D$ -function is calculated by employing its perturbative approximation as the only additional input. The obtained result is found to be in reasonable agreement with the experimental prediction for the Adler function in the entire range of momenta  $0 \leq Q^2 < \infty$ .

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## 1. Introduction

The hadronic vacuum polarization function  $\Pi(q^2)$  and the Adler function  $D(Q^2)$  are physical observables of central importance in various fields of elementary particle physics. In particular, these quantities are essential for the analysis of strong interaction processes such as electron–positron annihilation into hadrons [1, 2, 3, 4] and  $\tau$  lepton decay [5, 6]. Furthermore, the Adler function plays an important role when confronting precise experimental measurements of some electroweak observables with their theoretical predictions, giving rise to decisive tests of the Standard Model and furnishing stringent constraints on possible new physics beyond it. Specifically, due to the hadronic excitations in the photon vacuum polarization, the muon anomalous magnetic moment [7, 8] and the shift of the electromagnetic fine structure constant [9, 10] receive appreciable contributions from the strong interactions. The latter constitute in fact the main source of theoretical uncertainty when computing these observables, since the physical energy scales involved are such that the nonperturbative effects begin to be relevant.

To date, there is no systematic method for calculating the Adler function in the whole range of energies  $0 \leq Q^2 < \infty$ . Nonetheless, in the asymptotic ultraviolet region  $D(Q^2)$  may be approximated by the power series in the strong running coupling  $\alpha_s(Q^2)$  by making use of perturbation theory. However, unphysical singularities of  $\alpha_s(Q^2)$  (e.g., the one–loop Landau pole), being artifacts of the perturbative computations, invalidate this approach in the infrared domain. In turn, this significantly complicates the theoretical description of low–energy experimental data, and eventually forces one to resort to various models and phenomenologically inspired approximations.

The “semi–experimental” method for obtaining the Adler function in the infrared domain is to actually employ the data on the  $R$ –ratio of  $e^+e^-$  annihilation into hadrons, in conjunction with the dispersion relation [2]. Specifically, one first constructs  $R(s)$  by merging its low–energy experimental data with its high–energy perturbative prediction, and then integrates the dispersion relation for the Adler function [2]. This way of modeling of the  $R$ –ratio entails certain complications. In particular, one has to properly take into account the effects due to the analytical continuation of the perturbative results for  $D(Q^2)$  into the timelike domain (such as the resummation of the so–called  $\pi^2$ –terms, see Ref. [11]), and resolve the ambiguities related to the matching condition between the experimental and perturbative inputs for  $R(s)$ . In addition, the  $e^+e^-$  data are not always suitable for the extraction of the Adler function, basically due to large systematic uncertainties in the infrared domain. Nevertheless, recent progress in the study of the  $\tau$  lepton decays offers a way to overcome the latter difficulty. Specifically, for the energies below the mass of the  $\tau$  lepton, the  $e^+e^-$  data can be substituted (up to isospin breaking effects [12]) by a precise inclusive vector spectral function [13], extracted from the hadronic  $\tau$  decays. It is worthwhile to note also that some insights on the infrared behavior of  $D(Q^2)$  may be gained from chiral perturbation theory (see, e.g., paper [14] and references therein) and lattice simulations [15].

An important source of the nonperturbative information about the hadron dynamics at low energies is provided by the relevant dispersion relations. The latter, being based on the general principles of the local Quantum Field Theory (QFT), supply one with the definite analytic properties in a kinematic variable of a physical quantity at hand. The idea of employing this information together with the perturbative treatment of the renormalization group (RG) method forms the underlying concept of the so-called “analytic approach” to QFT. It was first proposed in the framework of the Quantum Electrodynamics and applied to the study of the invariant charge of the theory [16]. Later on, it was argued (see, e.g., papers [4, 17, 18, 19, 20] and references therein) that a similar method can also be useful for studying the non-Abelian theories. Eventually, proceeding from these motivations, the analytic approach to Quantum Chromodynamics (QCD) has been developed [20]. Some of the main advantages of this method are the absence of the unphysical singularities and a fairly good higher-loop and scheme stability of outgoing results. The analytic approach has been successfully employed in studies of the strong running coupling [20, 21], perturbative series for QCD observables (the so-called “Analytic perturbation theory” (APT), see papers [22, 23, 24, 25] and references therein), hadron spectrum [26], pion form factor [27], and some other intrinsically nonperturbative aspects of the strong interaction [21, 28].

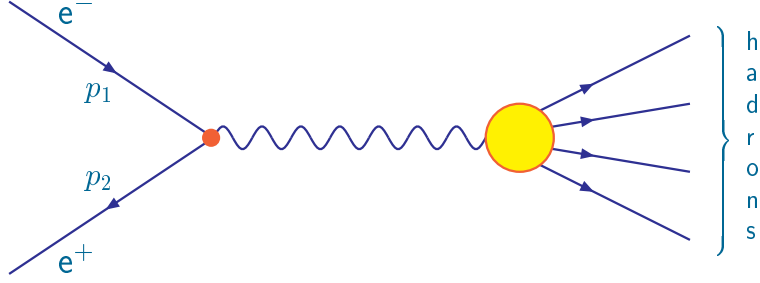
The primary objective of this paper is to derive novel integral representations for the Adler function and the  $R$ -ratio in a general framework of the analytic approach to QCD. These representations contain the nonperturbative information captured in the dispersion relation for  $D(Q^2)$ , the effects due to the interrelation between spacelike and timelike domains, and the effects due to the nonvanishing pion mass. In addition, we compute the Adler function within the approach at hand, by employing its perturbative approximation as the only additional input, and compare the obtained result with the relevant experimental data.

The layout of the paper is as follows. In Sec. 2 the dispersion relation for the Adler function  $D(Q^2)$  and its relation to the  $R$ -ratio of the  $e^+e^-$  annihilation into hadrons are briefly reviewed. In Sec. 3 new integral representations for  $D(Q^2)$  and  $R(s)$ , which involve a common spectral function, are derived. The calculation of the Adler function within the developed approach, its comparison with the experimental prediction for  $D(Q^2)$ , and a discussion of the obtained results are presented in Sec. 4.

## 2. Dispersion relation for the Adler function

Let us consider the process of the electron-positron annihilation into hadrons in the lowest order of the electromagnetic interaction (see Fig. 1). The corresponding Feynman amplitude is given by

$$\bar{v}(p_2)e\gamma^\mu u(p_1)\frac{1}{q^2}\langle\Gamma|J_\mu(q)|0\rangle, \quad (1)$$



**Figure 1.** The Feynman diagram for the electron–positron annihilation into hadrons.

where  $u(p_1)$  and  $\bar{v}(p_2)$  are the Dirac spinors of the electron and positron respectively,  $s = q^2 = (p_1 + p_2)^2$  is the timelike momentum transferred,  $\Gamma$  denotes a final hadron state, and  $J_\mu(q)$  stands for the quark current. The probability of the transition of an electron–positron pair into hadrons is proportional to the square of the amplitude (1) summed over the states  $\Gamma$ . The hadronic part of the latter can be represented as

$$\Delta_{\mu\nu}(q^2) = \sum_{\Gamma} \langle 0 | J_\mu(-q) | \Gamma \rangle \langle \Gamma | J_\nu(q) | 0 \rangle = \langle 0 | J_\mu(-q) J_\nu(q) | 0 \rangle, \quad (2)$$

where the completeness  $\sum_{\Gamma} |\Gamma\rangle\langle\Gamma| = 1$  has been employed. It is worth emphasizing here that  $\Delta_{\mu\nu}(q^2)$  exists only for  $q^2 \geq 4m_\pi^2$ , since otherwise no hadron state  $\Gamma$  could be excited. In particular,  $\Delta_{\mu\nu}(q^2)$  vanishes identically below the two–pion threshold (i.e., for  $q^2 < 4m_\pi^2$ ), see Ref. [1] for the details.

The hadronic tensor  $\Delta_{\mu\nu}(q^2)$  is proportional to the discontinuity of the function

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T \{ J_\mu(x) J_\nu(0) \} | 0 \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2) \quad (3)$$

across the physical cut  $q^2 \geq 4m_\pi^2$  along the positive semiaxis of real  $q^2$ . In Eq. (3)  $\Pi(q^2)$  stands for the hadronic vacuum polarization function. The latter satisfies the once–subtracted dispersion relation [1, 2]

$$\Pi(q^2) = \Pi(s') - (q^2 - s') \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s - q^2)(s - s')} ds, \quad (4)$$

where  $m_\pi = (134.9766 \pm 0.0006)$  MeV [29] is the mass of the  $\pi^0$  meson. In Eq. (4)  $R(s)$  denotes the measurable ratio of two cross–sections

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} [\Pi(s - i\varepsilon) - \Pi(s + i\varepsilon)] = \frac{\sigma(e^+e^- \rightarrow \text{hadrons}; s)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-; s)}, \quad (5)$$

with  $s$  being the center–of–mass energy squared of the annihilation process. It is worth noting here that  $R(s) \equiv 0$  for  $s < 4m_\pi^2$ , because of the kinematic restrictions mentioned above.

For practical purposes it proves convenient to deal with the Adler function [2], which is defined as the logarithmic derivative of the hadronic vacuum polarization function (3)

$$D(Q^2) = \frac{d\Pi(-Q^2)}{d \ln Q^2}, \quad (6)$$

and, therefore, does not depend on the choice of subtraction point  $s'$  in dispersion relation (4). In Eq. (6)  $Q^2 = -q^2 \geq 0$  denotes a spacelike momentum. The Adler function (6) plays an indispensable role for the simultaneous processing of the timelike and spacelike experimental data, see Refs. [2, 3]. Indeed, since the perturbation theory and the RG method are not applicable directly to the study of the observables depending on the timelike kinematic variable, for the self-consistent description of the latter one first has to relate the timelike experimental data with the perturbative results. Here, the required link between the  $R$ -ratio (5) and the Adler function (6) can be obtained by differentiating Eq. (4), that results in the dispersion relation [2]

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds. \quad (7)$$

As it has been mentioned in the Introduction, this equation is commonly employed for restoring  $D(Q^2)$  from the  $e^+e^-$  experimental data. On the other hand, the inverse form of relation (7), which enables one to continue an explicit expression for the Adler function into timelike domain, can be derived by integrating Eq. (6) between finite limits. Ultimately, this leads to

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}, \quad (8)$$

where the integration contour lies in the region of analyticity of the integrand, see Ref. [11]. In particular, Eq. (8) provides the only known way for obtaining an explicit expression for  $R(s)$ , using theoretical prediction for  $D(Q^2)$  as input (see also Refs. [2, 3, 4] for details).

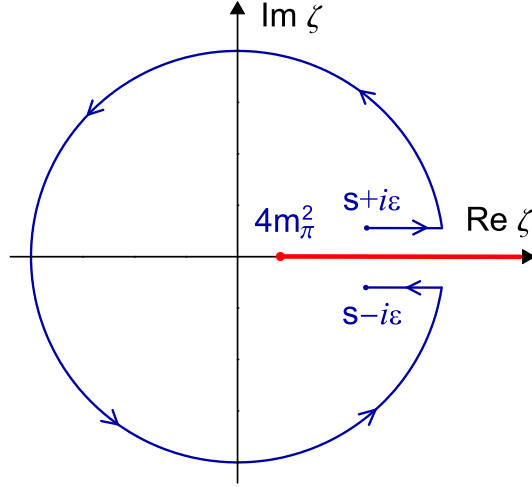
### 3. Novel integral representations for $D(Q^2)$ and $R(s)$

Before proceeding to the derivation of the integral representations for  $D(Q^2)$  and  $R(s)$ , it is worth emphasizing that the dispersion relation (7) imposes stringent constraints on the form of the Adler function. Specifically, Eq. (7) implies that  $D(Q^2)$  is an analytic function in the complex  $Q^2$ -plane with the only cut  $Q^2 \leq -4m_\pi^2$  along the negative semiaxis of the real  $Q^2$ . In addition, given that (i)  $R(s)$ , being a physical quantity, assumes finite values, and (ii) its asymptotic ultraviolet behavior  $R(s) \simeq 1 + \mathcal{O}[\ln^{-1}(s/\Lambda^2)]$  when  $s \rightarrow \infty$ , one concludes from Eq. (7) that  $D(Q^2)$  vanishes in the infrared limit  $Q^2 \rightarrow 0$ .

Let us for the moment turn off the strong interactions, and neglect all effects due to quark masses (these latter effects will be disregarded throughout the paper, even when the strong interaction are turned back on; a brief discussion of their possible relevance will be given in the next section). In this case, the  $R$ -ratio (5) is determined by the parton model prediction [1]:

$$R_0(s) = \theta(s - 4m_\pi^2), \quad s > 0, \quad (9)$$

where the overall factor  $N_c \sum_f Q_f^2$  is omitted throughout,  $N_c = 3$  is the number of colors,  $Q_f$  denotes the charge of the quark of the  $f$ th flavor, and  $\theta(x)$  stands for the



**Figure 2.** The integration contour in Eq. (8) for  $s > 4m_\pi^2$ . The physical cut of the Adler function  $D(-\zeta)$  is shown along the positive semiaxis of real  $\zeta$ .

Heaviside step-function. In turn, as it follows from the dispersion relation (7), Eq. (9) corresponds to the following zeroth order prediction for the Adler function:

$$D_0(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2}, \quad Q^2 > 0. \quad (10)$$

Obviously, this expression vanishes in the infrared limit  $Q^2 \rightarrow 0$ , and satisfies the physical condition  $D_0(Q^2) \rightarrow 1$  when  $Q^2 \rightarrow \infty$ . Thus, along with the “standard” intrinsically nonperturbative contributions [30, 31], the Adler function also receives power corrections due to the nonvanishing pion mass, which turn out to be important for  $Q \lesssim 2 \text{ GeV}$ :

$$D_0(Q^2) \simeq 1 + \sum_{n=1}^{\infty} \left( -\frac{4m_\pi^2}{Q^2} \right)^n, \quad Q^2 > 4m_\pi^2. \quad (11)$$

It is worth noting that the modification of the parton model prediction (10) due to the mass of the  $\pi$  meson is crucial for arriving at a form of  $D(Q^2)$  which agrees with the low-energy experimental data. Had one neglected the pion mass, one would have instead obtained as a zeroth order approximation  $D_0(Q^2) = 1$ , which would have been inconsistent with the infrared behavior of the Adler function extracted from the experiment (see Fig. 3). Remarkable as it may seem, to the best of our knowledge, Eq. (10) does not appear anywhere in the existing literature.

The next step is to turn the strong interactions back on, and derive new integral representations for  $R(s)$  and  $D(Q^2)$ , which involve a common spectral function, to be denoted by  $\rho_D(\sigma)$ . The inclusion of the effects due to the strong interaction modify Eqs. (9) and (10):

$$R(s) = \theta(s - 4m_\pi^2) + r(s), \quad (12)$$

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} + d(Q^2). \quad (13)$$

As it has been mentioned above, the ultraviolet behavior of the strong correction  $d(Q^2)$  can be computed in the framework of perturbation theory, and the corresponding  $r(s)$  may be obtained from it through Eq. (8). Assuming that an explicit “exact” expression for the Adler function (13) is available, one can restore  $R(s)$  (12) by making use of relation (8). For the energies below the two-pion threshold (i.e., for  $0 \leq s < 4m_\pi^2$ ) the integration of Eq. (8) leads to  $R(s) = 0$ , in conformity with the kinematic restrictions mentioned above. For the energies above the two-pion threshold it is convenient to choose the integration contour in Eq. (8) in the form presented in Fig. 2, since the strong correction  $d(Q^2)$  vanishes at the ultraviolet asymptotic. In this case the only nontrivial contribution into the integral along the circle of the infinitely large radius comes from the parton model prediction (10), whereas the strong correction  $d(Q^2)$  contributes only to the integral along the cut of  $D(Q^2)$ . Eventually, this results in

$$R(s) = \theta(s - 4m_\pi^2) \left[ 1 + \int_s^\infty \rho_D(\sigma) \frac{d\sigma}{\sigma} \right], \quad (14)$$

where the spectral function is determined as the discontinuity of the Adler function across the cut

$$\rho_D(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} [D(-\sigma - i\varepsilon) - D(-\sigma + i\varepsilon)], \quad \sigma > 4m_\pi^2. \quad (15)$$

At the same time,  $\rho_D(\sigma)$  (15) can also be related to  $R(s)$ , by differentiating Eq. (5) with respect to  $\ln s$ :

$$\rho_D(\sigma) = -\frac{dR(\sigma)}{d \ln \sigma}, \quad \sigma > 4m_\pi^2, \quad (16)$$

that allows one to extract  $\rho_D(\sigma)$  from the relevant experimental data. In turn, the Adler function can be represented in terms of the spectral function  $\rho_D(\sigma)$  by integrating the dispersion relation (7) with  $R(s)$  given by Eq. (14). Carrying out the integration by parts, one arrives at

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} \left[ 1 + \int_{4m_\pi^2}^\infty \rho_D(\sigma) \frac{\sigma - 4m_\pi^2}{\sigma + Q^2} \frac{d\sigma}{\sigma} \right]. \quad (17)$$

It is worth noting that in deriving the integral representations (14) and (17) we have employed only Eqs. (7) and (8), the parton model prediction (9), and the fact that the strong correction  $d(Q^2)$  vanishes in the asymptotic ultraviolet limit  $Q^2 \rightarrow \infty$ ; no additional approximations nor model-dependent assumptions were involved. Similarly to the perturbative approach, the QCD scale parameter  $\Lambda$  remains the basic characterizing quantity of the theory. Besides, all the effects due to the interrelation between the spacelike and timelike observables are automatically accounted for by the representations (14) and (17). It is important to mention also that in the limit of the vanishing pion mass  $m_\pi = 0$  the derived relations (14) and (17) become identical to those obtained within the massless APT [22], namely

$$R_{\text{APT}}(s) = 1 + \int_s^\infty \rho_D(\sigma) \frac{d\sigma}{\sigma}, \quad (18)$$

$$D_{\text{APT}}(Q^2) = 1 + \int_0^\infty \frac{\rho_D(\sigma)}{\sigma + Q^2} d\sigma. \quad (19)$$

#### 4. Discussion and conclusions

We hasten to emphasize that the new integral representation for the Adler function (17) incorporates the same nonperturbative constraints on  $D(Q^2)$  as those contained in the dispersion relation (7); namely,  $D(Q^2)$  of (17) possesses correct analytic properties in the  $Q^2$  variable, and vanishes in the infrared limit  $Q^2 \rightarrow 0$ . However,  $D(Q^2)$  of (17) has the advantage of being expressed as the discontinuity of itself across the cut (15). This fact brings about certain simplifications in the theoretical analysis of the Adler function. Specifically, as it has been noted in the Introduction, the standard extraction of  $D(Q^2)$  through Eq. (7) constitutes a two-step procedure: first, one has to construct  $R(s)$  for the entire energy range  $4m_\pi^2 \leq s < \infty$  (encountering the complications mentioned above), and only then integrate Eq. (7). Instead, Eq. (17) eliminates the intermediate step of building up  $R(s)$ , allowing the reconstruction of  $D(Q^2)$  through the spectral function (15). At the same time, one is still able to incorporate the relevant experimental information about  $R(s)$  into the spectral function  $\rho_D(\sigma)$  by virtue of Eq. (16), that, however, turns out to be a bit more complicated technically, since the numerical differentiation of the experimental data on  $R(s)$  is required.

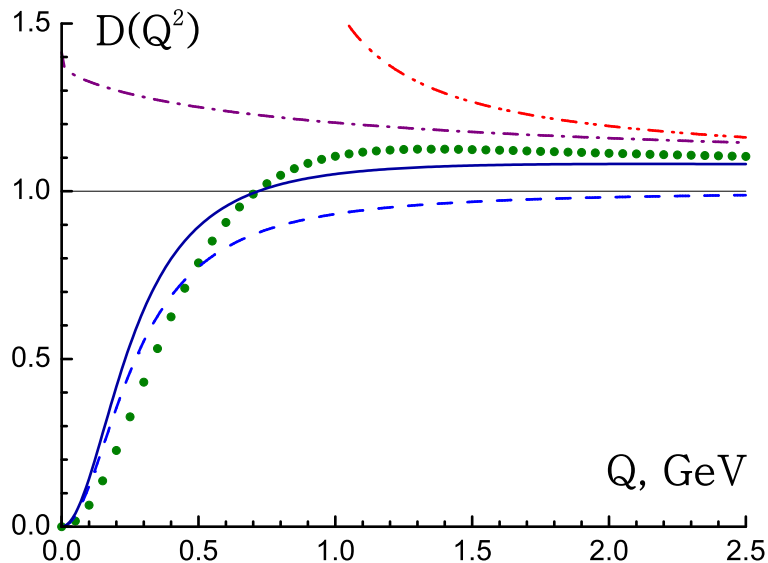
In order to exploit Eq. (17) one should furnish an input for the central quantity of the approach at hand, namely, the spectral function  $\rho_D(\sigma)$  (15). In general,  $\rho_D(\sigma)$  is supposed to embody all available information about functions  $D(Q^2)$  and  $R(s)$ . In this paper we restrict ourselves to the study of the perturbative contributions to  $\rho_D(\sigma)$  only. Following the same motivations as those of APT [22], the latter can be obtained by identifying  $D(Q^2)$  in Eq. (15) with its perturbative approximation, which at the  $\ell$ -loop level reads (see paper [32] and references therein)

$$D_{\text{pert}}^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j [\alpha_s^{(\ell)}(Q^2)]^j, \quad Q^2 \rightarrow \infty. \quad (20)$$

It is worth noting that the resulting perturbative spectral function  $\rho_{\text{pert}}(\sigma)$  receives no contributions from spurious singularities of  $\alpha_s(Q^2)$ , since the former is determined as the discontinuity of  $D_{\text{pert}}(Q^2)$  across the physical cut. Specifically, at the one-loop level ( $d_1 = 1/\pi$ ) one has  $\rho_{\text{pert}}^{(1)}(\sigma) = (4/\beta_0)[\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}$ , where  $\beta_0 = 11 - 2n_f/3$ , and  $n_f$  is the number of active quarks.

It turns out that, even with the one-loop perturbative approximation of the spectral function,  $\rho_{\text{pert}}^{(1)}(\sigma)$ , representation (17) is capable of providing an output for  $D(Q^2)$  which is compatible with its experimental prediction. Indeed, as can be seen in Fig. 3, for the entire energy range,  $0 \leq Q^2 < \infty$ , the result obtained is in good agreement with the experimental behavior of  $D(Q^2)$  extracted from the ALEPH data on the inclusive vector spectral function [13], in the way described above. The value of the scale parameter  $\Lambda = 335 \text{ MeV}$  has been estimated for the case of  $n_f = 2$  active flavors by fitting  $D(Q^2)$  (17) to its experimental prediction by means of the least-squares method. It is worth mentioning that the Adler function (17) is stable with respect to the higher-loop perturbative corrections. Indeed, calculations up to four loops have revealed that the relative difference between the  $\ell$ -loop and the  $(\ell + 1)$ -loop expressions for  $D(Q^2)$  (17)





**Figure 3.** Comparison of the Adler function (17) corresponding to  $\rho_{\text{pert}}^{(1)}(\sigma)$  (solid curve) with its experimental behavior ( $\bullet$ ) extracted from [13]. The zeroth order prediction for  $D(Q^2)$  (10) is shown by the dashed curve, the massless APT case (19) is denoted by the dot-dashed curve, and the dot-dot-dashed curve corresponds to the one-loop perturbative approximation (20) of the Adler function.

is less than 4.9%, 1.5%, and 0.3%, for  $\ell = 1$ ,  $\ell = 2$ , and  $\ell = 3$ , respectively, for  $0 \leq Q^2 < \infty$  (the estimation [33] of the four-loop expansion coefficient  $d_4$  has been adopted here).

In order to illustrate the significance of the mass of the  $\pi$  meson within this approach, it is worth presenting the massless APT prediction of  $D(Q^2)$  (19) computed by making use of  $\rho_{\text{pert}}^{(1)}(\sigma)$  (dot-dashed curve in Fig. 3). In this case, one arrives at a result, which is free of infrared unphysical singularities (to be contrasted with the perturbative dot-dot-dashed curve in Fig. 3), but fails to describe the experimental behavior of the Adler function in the low-energy domain  $Q \lesssim 1$  GeV, where the effects due to the pion mass become appreciable. Of course, in the framework of the massless APT, the infrared behavior of the Adler function can be greatly improved following the procedure introduced in [34]; this consists essentially in carrying out an appropriate resummation of threshold singularities, and introducing into (18) and (19) effects from nonperturbative light quark masses. The necessary nonperturbative information on the quark masses is furnished from the study of Schwinger-Dyson equations and quark condensates. Specifically, one extracts a particular momentum-dependence for the dynamical (effective) quark mass, which interpolates between the constituent and current mass values. We believe however that the method developed here presents certain advantages compared to that of [34]. This is so because, in addition to the possible ambiguities stemming from the truncation and general treatment of the Schwinger-Dyson equations, the final numerical implementation of the procedure of [34] seems to require rather elevated constituent masses for the light quarks ( $m_u = m_d =$

250 MeV); instead, the only phenomenological parameter appearing in Eqs. (14) and (17) is the measurable mass of the pion. Finally, again in the framework of massless APT, the infrared behavior of the Adler function has been brought into qualitative agreement with experiment, but at the expense of resorting to additional assumptions, such as the vector meson dominance [35].

As mentioned in the previous section, in this paper we have omitted effects due to the quark masses. Of course, this is not to say that such effects are not relevant. As a matter of fact, and in addition to their role in bringing the predictions of APT in agreement with experiment, as exemplified in [34], they could be of relevance even within the approach at hand. In particular, in the present analysis such effects can be retained by employing the perturbative prediction for the Adler function, which accounts for the quark masses (see paper [36] and references therein), instead of  $D_{\text{pert}}(Q^2)$  given by Eq. (20). We hope to present a detailed study of this issue in the near future.

The results obtained in this paper can be further scrutinized through the incorporation of higher order perturbative contributions into the spectral function  $\rho_D(\sigma)$ , along with the effects due to the quark masses, power corrections due to gluon and quark condensates [30, 31], and restrictions imposed by the low-energy experimental data. It would also be of apparent interest to apply the present approach to the estimation of the hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant.

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