

Indirect determination of the Kugo-Ojima function from lattice data

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Abstract

We study the structure and non-perturbative properties of a special Green's function, $u(q^2)$, whose infrared behavior has traditionally served as the standard criterion for the realization of the Kugo-Ojima confinement mechanism. It turns out that, in the Landau gauge, $u(q^2)$ can be determined from a dynamical equation, whose main ingredients are the gluon propagator and the ghost dressing function, integrated over all physical momenta. Using as input for these two (infrared finite) quantities recent lattice data, we obtain an indirect determination of $u(q^2)$. The results of this mixed procedure are in excellent agreement with those found previously on the lattice, through a direct simulation of this function. Most importantly, in the deep infrared the function deviates considerably from the value associated with the realization of the aforementioned confinement scenario. In addition, the dependence of $u(q^2)$, and especially of its value at the origin, on the renormalization point is clearly established. Some of the possible implications of these results are briefly discussed.

PACS numbers: 11.15.Tk 12.38.Lg, 12.38.Aw,

I. INTRODUCTION

The problem of quark confinement and gluon screening is of central importance in QCD, and a large body of work has been dedicated to its understanding [1]. Some of the most widely explored mechanisms attempting to explain how quarks confine make concrete predictions about the non-perturbative behavior of the fundamental Green's functions of the theory. For example, a central ingredient in the center vortex picture of confinement put forth by Cornwall is the dynamical generation of a gluon mass [2] through the well-known Schwinger mechanism [3–5], implemented within the pinch technique (PT) framework [2, 6, 7]. In addition to taming the infrared divergences intrinsic to perturbation theory (Landau pole), this mass gives rise to a low energy effective theory [8] which supports quantum solitons (center vortices), not present in the massless theory, whose condensation furnishes an area law to the fundamental representation Wilson loop, thus confining quarks [9]. On the other hand, the adjoint potential shows a roughly linear regime followed by string breaking when the potential energy is about $2m$, where m is the induced mass of the gluon [10, 11], corresponding to gluon screening [12, 13]. At the level of the two fundamental Green's functions of the theory, namely the gluon and ghost propagators, the predictions of the above picture are very definite: the gluon propagator is infrared finite (due to the generation of the gluon mass [15], whose phenomenological value has been delimited in Ref. [14]), while, as has been shown recently [15, 16], in the Landau gauge the ghost remains massless, but with a finite dressing function (due to the saturation produced by the gluon mass) [17].

An entirely different set of predictions is obtained within the Kugo-Ojima (KO) scenario, which also establishes a highly non-trivial link between confinement and the infrared behavior of some of the most fundamental Green's functions of QCD. In the KO confinement picture (in covariant gauges), the absence of colored asymptotic states from the physical spectrum of the theory is due to the so-called “quartet mechanism” [18]. A sufficient condition for the realization of this mechanism (and the meaningful definition of a conserved BRST charge) is that a certain correlation function, to be denoted by $u(q^2)$, defined in Eq. (2.21), should satisfy the condition $u(0) = -1$ [19]. In addition, as first noted by Kugo [20], in the Landau gauge, $u(0)$ is related to the infrared behavior of the ghost dressing function $F(q^2)$ [see Eq. (2.7)] through the identity $F^{-1}(0) = 1 + u(0)$. Therefore, the KO confinement scenario predicts a divergent ghost dressing function, and vice-versa. Interestingly enough, the

same prediction about $F^{-1}(0)$ is obtained when implementing the Gribov-Zwanziger (GZ) horizon condition [21, 22]: in the IR region the ghost propagator diverges more rapidly than at tree-level [23]. Furthermore, it has been also argued that the Landau gauge gluon propagator should vanish in the same limit [24]. This alleged connection between confinement, the horizon condition, and an infrared “enhanced” ghost dressing function has served as the theoretical cornerstone of the so called “ghost-dominance” picture of QCD [25].

Turns out that recent large volume lattice simulations [26, 27] appear to be at odds with the original KO and GZ pictures described above, at least as far as their predictions about the infrared behavior of the Green’s functions are concerned [28]. Specifically, various lattice studies, both in $SU(2)$ and $SU(3)$, find (in the Landau gauge) an infrared finite gluon propagator [29] and an infrared finite (“non-enhanced”) ghost dressing function. Evidently, if taken at face value [30], these results furnish strong support for the PT picture of dynamical gluon mass generation and the ensuing confinement mechanism.

It is perfectly clear that further detailed scrutiny from all possible angles must be implemented before reaching a definite conclusion on any of these issues. In this vein, it is natural to ask what one really knows about the KO function $u(q^2)$. Turns out that $u(q^2)$ has been studied directly on the lattice using the field-theoretic definition of $u(q^2)$ appearing in the KO formulation. The first such study dates back to the work of Nakajima and Furui [31], who reported a value of $u(0)$ of about -0.8 . More recently, Sternbeck [32] presented large-volume lattice simulations of the KO function (renormalized within the MOM-scheme). As can be plainly seen from Sternbeck’s results (reproduced for convenience in Fig. 9 of this article), $u(0)$ deviates appreciably from its KO value of -1 ; specifically, the function $u(q^2)$ saturates in the deep infrared around approximately -0.6 . Interestingly enough, in a recent article [33] Kondo gave a simple derivation of this same value, after appropriately modifying the KO construction in order to self-consistently accommodate the GZ horizon condition.

Quite remarkably, in the (background) Landau gauge [34] the KO function coincides with a certain auxiliary function, usually denoted by $G(q^2)$, which constitutes a crucial ingredient in the modern formulation of the PT by means of the Batalin-Vilkovisky (BV) quantization formalism [35]. Specifically, $G(q^2)$ is the form-factor multiplying $g_{\mu\nu}$ in the Lorentz decomposition of a special Green’s function, denoted by $\Lambda_{\mu\nu}(q)$, which enters in all “background-quantum” identities [36, 37], *i.e.*, the infinite tower of non-trivial relations connecting the Green’s functions of the background field method (BFM) [38] to the con-

ventional ones (e.g. R_ξ gauges). Notice also that $G(q^2)$ plays a prominent role in the new Schwinger-Dyson (SD) equations derived within the PT framework [39], which, due to the special properties of the Green's functions involved, can be truncated in a manifestly gauge invariant way [40].

As has been shown in a recent article [41], one may derive a dynamical (SD-like) equation for $G(q^2)$, which, under mild assumptions, allows one to reconstruct $G(q^2)$ from the knowledge of the gluon and ghost propagators. Specifically, $G(q^2)$ is determined by integrating over all virtual momenta (k) a kernel involving the product $\Delta(k)F(k+q)$. We emphasize that the aforementioned dynamical equation is not a simple relation of several Green's functions at some special isolated point; instead, the value obtained for $G(q^2)$ at any point (such as $q^2 = 0$) must be compatible with the behavior of F and Δ in the entire range of their physical (euclidean) momenta. In particular, one must know their behavior not only in the IR, but also in the intermediate region of momenta (0.3-3 GeV), which appears rather difficult to obtain from SD studies [42]. This feature is very powerful, because it probes the details of the fundamental Green's functions over an extended range of momenta, rather than just a single point.

In the present work, we use the available lattice data on the gluon and ghost propagator as input into the aforementioned dynamical equation, thus obtaining an indirect determination of $G(q^2)$ in the entire range of available lattice momenta. Given the Landau gauge coincidence between $G(q^2)$ and $u(q^2)$, this procedure automatically determines the KO function as well. This, in turn, permits us to obtain the value of the KO parameter $u(0)$, as well as the GZ horizon function, and study their dependence on the renormalization point μ . Our analysis reveals an impressive self-consistency between the various ingredients entering into the calculation. In particular, the results obtained through our combined method (SD using lattice data as input) are in excellent agreement with those of [32], obtained through a direct simulation of the KO function.

The paper is organized as follows. In Section II we briefly introduce the BV framework for $SU(N)$ Yang-Mills theories, where the function $\Lambda_{\mu\nu}(q)$ appears naturally. Next, we review a number of relations where this function plays a key role: (i) the background-quantum identity relating the conventional and the BFM gluon propagators; (ii) the relation between the ghost dressing function $F(q^2)$ and the $\Lambda_{\mu\nu}(q)$ form factors $G(q^2)$ and $L(q^2)$; (iii) we establish the crucial equality $u(q^2) = G(q^2)$; (iv) the relation with the GZ horizon function.

In addition, in the last subsection we discuss the renormalization of the KO function and the resulting dependence on the renormalization point μ , focusing particularly on how this latter dependence manifests itself within the MOM scheme. The central results of this article are presented in Section III. Specifically, the Lorentz decomposition of $\Lambda_{\mu\nu}(q)$ gives rise to two form-factors, the $G(q^2)$, which in the previous section has been identified with the KO function $u(q^2)$, and the $L(q^2)$, which has the particular property of vanishing in the deep IR. After establishing the dynamical equations governing $G(q^2)$ and $L(q^2)$, we use the recent lattice data for the gluon and ghost propagators as input in these equations. In addition to the equations for $G(q^2)$ and $L(q^2)$, we consider the SD equation for the ghost, which is calibrated in order to be numerically compatible with the lattice data (at an impressive precision) simply by adjusting the gauge coupling to values that are slightly above the standard two-loop MOM prediction; the obtained value of the coupling is then used into the equations for $G(q^2)$ and $L(q^2)$. We use the multiplicative renormalizability of the gluon and ghost propagators in order to rescale the lattice data to different values of the renormalization point. Even though this procedure has an intrinsic limitation set by the relatively short reach of the available data into the UV, it amply demonstrates that $u(q^2)$ depends non-trivially on μ , in excellent agreement with the observation established in [32]. Finally, in Section IV we present our conclusions.

II. CONNECTING THE KUGO-OJIMA AND G FUNCTIONS

As already mentioned in the Introduction, in the Landau gauge the KO function $u(q^2)$ may be shown to be identical to the function $G(q^2)$, which appears in several formal contexts. In this section we first formulate Yang-Mills theories in the BV framework, which allows the derivation of a tower of identities, whose common ingredient is the function $G(q^2)$. Then we will show why $u(q^2) = G(q^2)$, and will review the connection between $u(q^2)$ and the GZ horizon. This main purpose of this section is to serve as a reminder and to bring together various seemingly disjoint pieces of information. For specific details on each topic the reader is referred to the corresponding extensive literature.

A. Batalin-Vilkoviski formalism

In the BV formulation of Yang-Mills theories [35], one starts by introducing certain sources (called anti-fields in what follows) that describe the renormalization of composite operators; the latter class of operator is in fact bound to appear in such theories due to the non-linearity of the BRST transformation of the elementary fields. In much the same way, the quantization of the theory in a background field type of gauge requires, in addition to the aforementioned anti-fields, the introduction of new sources which couple to the BRST variation of the background fields [36]. These sources are sufficient for implementing the full set of symmetries of a non-Abelian theory at the quantum level, and in the case of quarkless $SU(N)$ QCD, lead to the master equation

$$\int d^4x \left[\frac{\delta\Gamma}{\delta A_\mu^{*m}} \frac{\delta\Gamma}{\delta A_\mu^m} + \frac{\delta\Gamma}{\delta c^{*m}} \frac{\delta\Gamma}{\delta c^m} + B^m \frac{\delta\Gamma}{\delta \bar{c}^m} + \Omega_\mu^m \left(\frac{\delta\Gamma}{\delta \hat{A}_\mu^m} - \frac{\delta\Gamma}{\delta A_\mu^m} \right) \right] = 0. \quad (2.1)$$

In the formula above, Γ is the effective action, A^* and c^* the gluon and ghost anti-fields, \hat{A} is the gluon background field, and Ω the corresponding background source; finally B denotes the Nakanishi-Lautrup multiplier for the gauge fixing condition.

To determine the complete algebraic structure of the theory we need two additional equations. The first one is the Faddeev-Popov equation, that controls the result of the contraction of an anti-field leg with the corresponding momenta. In position space, it reads

$$\frac{\delta\Gamma}{\delta \bar{c}^m} + \left(\hat{\mathcal{D}}^\mu \frac{\delta\Gamma}{\delta A_\mu^*} \right)^m - (\mathcal{D}^\mu \Omega_\mu)^m - g f^{mrs} \hat{A}_\mu^r \Omega_s^\mu = 0, \quad (2.2)$$

where $(\mathcal{D}^\mu \Phi)^m = \partial^\mu \Phi^m + g f^{mnr} A_\mu^n \Phi^r$ [in the case of $(\hat{\mathcal{D}}^\mu \Phi)^m$ replace the gluon field A with a background gluon field \hat{A}]. The second one is the anti-ghost equation formulated in the background field Landau gauge, which reads [34]

$$\frac{\delta\Gamma}{\delta c^m} - \left(\hat{\mathcal{D}}^\mu \frac{\delta\Gamma}{\delta \Omega_\mu} \right)^m - (\mathcal{D}^\mu A_\mu^*)^m - g f^{mnr} c^{*n} c^r + g f^{mnr} \frac{\delta\Gamma}{\delta B^n} \bar{c}^r = 0, \quad (2.3)$$

This equation fully constrains the dynamics of the ghost field c , and implies that the latter will not get an independent renormalization constant. The local form of the anti-ghost equation (2.3) is only valid when choosing the background Landau gauge condition $(\hat{\mathcal{D}}^\mu A_\mu)^m = 0$; in the usual Landau gauge, $\partial^\mu A_\mu^m = 0$, an integrated version of this equation is available. In fact, even though the results that follow will be derived for convenience in the background Landau gauge, they are valid also in the conventional Landau gauge of the R_ξ .

$$\Lambda_{\mu\nu}(q) = \text{Diagram 1} + \text{Diagram 2}$$

$$H_{\sigma\nu}(k, q) = H_{\sigma\nu}^{(0)} + \text{Diagram 3}$$

FIG. 1: Diagrammatic representation of the functions Λ and H .

Now, differentiation of the functional (2.1) with respect to a combination of fields containing at least one ghost field or two ghost fields and one anti-field (and setting the fields and sources to zero afterwards) will provide the Slavnov-Taylor identities of the theory. Differentiation with respect to a background source and background or quantum fields will provide, instead, the so called background-quantum identities [36, 37], which relate 1PI Green's functions involving background fields with those involving quantum fields. Finally, differentiation of (2.2) and (2.3) with respect to fields and anti-fields or background sources give rise to relation among the different auxiliary ghost functions appearing in the theory.

The important point is that, when carrying out these differentiations, the following function appears (Fig. 1)

$$\begin{aligned} i\Lambda_{\mu\nu}(q) &= \Gamma_{\Omega_\mu A_\nu^*}(q) \\ &= g^2 C_A \int_k H_{\mu\rho}^{(0)} D(k+q) \Delta^{\rho\sigma}(k) H_{\sigma\nu}(k, q), \\ &= ig_{\mu\nu} G(q^2) + i \frac{q_\mu q_\nu}{q^2} L(q^2), \end{aligned} \quad (2.4)$$

and (in d -dimensions)

$$G(q^2) = \frac{1}{(d-1)q^2} (q^2 \Lambda_\mu^\mu - q^\mu q^\nu \Lambda_{\mu\nu}), \quad L(q^2) = \frac{1}{(d-1)q^2} (dq^\mu q^\nu \Lambda_{\mu\nu} - q^2 \Lambda_\mu^\mu). \quad (2.5)$$

In the equations above, the color factor δ^{mn} has been factored out (as always in what follows), C_A represents the Casimir eigenvalue of the adjoint representation [$C_A = N$ for $SU(N)$], and $\int_k \equiv \mu^{2\epsilon} (2\pi)^{-d} \int d^d k$, with $d = 4 - \epsilon$ the dimension of space-time. $\Delta_{\mu\nu}$ and D represents

the gluon and ghost propagator respectively, defined as

$$\Delta_{\mu\nu}(q) = -i \left[P_{\mu\nu}(q)\Delta(q^2) + \xi \frac{q_\mu q_\nu}{q^4} \right], \quad (2.6)$$

$$D(q^2) = \frac{iF(q^2)}{q^2}, \quad (2.7)$$

where ξ denotes the gauge-fixing parameter, and $P_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2$ is the usual transverse projector; notice that $\Delta^{-1}(q^2) = q^2 + i\Pi(q^2)$, with $\Pi_{\mu\nu}(q) = P_{\mu\nu}(q)\Pi(q^2)$ the gluon self-energy. $F(q^2)$ is the so called ghost dressing function. Finally, the function $H_{\mu\nu}(k, q)$ (see Fig. 1 again) is in fact a familiar object, since it appears in the all-order Slavnov-Taylor identity satisfied by the standard three-gluon vertex [43]

$$\begin{aligned} q^\alpha \Gamma_{\alpha\mu\nu}(q, k_1, k_2) &= iF(q^2)\Delta^{-1}(k_1^2)P_\mu^\rho(k_1)H_{\nu\rho}(k_2, k_1) \\ &\quad - iF(q^2)\Delta^{-1}(k_2^2)P_\nu^\rho(k_2)H_{\mu\rho}(k_1, k_2). \end{aligned} \quad (2.8)$$

It is also related to the full gluon-ghost vertex $\mathbf{\Gamma}_\mu(k, q)$ by the identity

$$q^\nu H_{\mu\nu}(k, q) = -i\mathbf{\Gamma}_\mu(k, q). \quad (2.9)$$

At tree-level, $H_{\mu\nu}^{(0)} = ig_{\mu\nu}$ and $\mathbf{\Gamma}_\mu^{(0)}(k, q) = \Gamma_\mu(k, q) = -q_\mu$.

B. Background-quantum identities

The first identities where the function $\Lambda_{\mu\nu}$ appears are the so-called background-quantum identities, *i.e.*, the infinite tower of non-trivial relations connecting the BFM Green's functions to the conventional ones [36, 37]. Consider, for example, the result of differentiating the functional (2.1) with respect to a background source and a background gluon, on the one hand, and a background source and a quantum gluon, on the other, one obtains two equations,

$$\begin{aligned} \Gamma_{\widehat{A}_\mu A_\nu}(q) &= [g_{\mu\rho} + \Lambda_{\mu\rho}(q)]\Gamma_{A^\rho A_\nu}(q), \\ \Gamma_{\widehat{A}_\mu \widehat{A}_\nu}(q) &= [g_{\mu\rho} + \Lambda_{\mu\rho}(q)]\Gamma_{A^\rho \widehat{A}_\nu}(q). \end{aligned} \quad (2.10)$$

Using the transversality of the gluon two-point function, these two equations can then be appropriately combined to yield the important identity

$$\Gamma_{\widehat{A}_\mu \widehat{A}_\nu}(q) = [1 + G(q^2)]^2 \Gamma_{A_\mu A_\nu}(q), \quad (2.11)$$

or, in terms of propagators

$$\widehat{\Delta}^{-1}(q^2) = [1 + G(q^2)]^2 \Delta^{-1}(q^2). \quad (2.12)$$

The quantity $\widehat{\Delta}(q^2)$ appearing on the left-hand side of the above equation captures the running of the QCD β function, exactly as happens with the QED vacuum polarization; this is a fundamental property of the BFM gluon self-energy, valid for every value of the (quantum) gauge-fixing parameter [38]. This can be easily checked to lowest order, where Eqs. (2.4) and (2.5) give (in the Landau gauge)

$$\begin{aligned} 1 + G(q^2) &= 1 + \frac{9 C_A g^2}{4 \cdot 48 \pi^2} \ln \left(\frac{q^2}{\mu^2} \right), \\ \Delta^{-1}(q^2) &= q^2 \left[1 + \frac{13 C_A g^2}{2 \cdot 48 \pi^2} \ln \left(\frac{q^2}{\mu^2} \right) \right], \end{aligned} \quad (2.13)$$

and thus

$$\widehat{\Delta}^{-1}(q^2) = q^2 \left[1 + b g^2 \ln \left(\frac{q^2}{\mu^2} \right) \right]. \quad (2.14)$$

where $b = 11 C_A / 48 \pi^2$ is the first coefficient in the QCD β function. Eq. (2.12) plays a central role in the derivation of a new set of SDEs [39] that can be truncated in a manifestly gauge invariant way [40].

Let us conclude this subsection by noticing that in more general identities the function $L(q^2)$ is also relevant. Consider *e.g.*, the identity relating the three-gluon proper vertices. One then has

$$\begin{aligned} \Gamma_{\widehat{A}_\mu A_\alpha A_\beta}(k_1, k_2) &= [g_\mu^\nu + \Lambda_\mu^\nu(q)] \Gamma_{A_\nu A_\alpha A_\beta}(k_1, k_2) + \dots \\ &= [1 + G(q^2)] \Gamma_{A_\mu A_\alpha A_\beta}(k_1, k_2) + \frac{q_\mu q^\nu}{q^2} L(q^2) \Gamma_{A_\nu A_\alpha A_\beta}(k_1, k_2) + \dots, \end{aligned} \quad (2.15)$$

where the omitted terms involve other auxiliary Green's functions (see [36, 37]), irrelevant to our discussion.

C. Two-point ghost sector

Let us now consider the two-point functions. Differentiating the ghost equation (2.2) with respect to a ghost field and a background source we get the relations

$$\begin{aligned} \Gamma_{c\bar{c}}(q) &= -i q^\nu \Gamma_{cA_\nu^*}(q), \\ \Gamma_{c\Omega_\mu}(q) &= q_\mu + q^\nu \Lambda_{\mu\nu}(q). \end{aligned} \quad (2.16)$$

$$-G_{\mu\nu}^{mn}(q) = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} + \begin{array}{c} \text{---} \bullet \text{---} \leftarrow \text{---} \circ \text{---} \leftarrow \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \leftarrow \text{---} \circ \text{---} \leftarrow \text{---} \bullet \text{---} \end{array}$$

Ω_{μ}^m A_{ν}^{*m} Ω_{μ}^m \bar{c}^s c^r A_{ν}^{*m}

FIG. 2: Connected components contributing to the function $G_{\mu\nu}^{mn}(q)$.

On the other hand, differentiating the anti-ghost equation (2.3) with respect to a gluon anti-field and an anti-ghost, one gets

$$\begin{aligned} \Gamma_{cA_{\nu}^*}(q) &= q_{\nu} + q^{\mu} \Lambda_{\mu\nu}(q), \\ \Gamma_{c\bar{c}}(q) &= -iq^{\mu} \Gamma_{c\Omega_{\mu}}(q). \end{aligned} \quad (2.17)$$

Next, contracting the first equation in (2.17) with q^{ν} , and making use of the first equation in (2.16), we see that the dynamics of the ghost sector is entirely captured by the $\Lambda_{\mu\nu}$ auxiliary function, since

$$i\Gamma_{c\bar{c}}(q) = q^2 + q^{\mu} q^{\nu} \Lambda_{\mu\nu}(q). \quad (2.18)$$

Introducing the Lorentz decompositions

$$\Gamma_{cA_{\mu}^*}(q) = q_{\mu} C(q^2), \quad \Gamma_{c\Omega_{\mu}}(q) = q_{\mu} E(q^2), \quad (2.19)$$

we find that Eq. (2.18) together with the last equation of (2.16) and (2.17) give the identities [20, 34]

$$\begin{aligned} C(q^2) &= E(q^2) = F^{-1}(q^2), \\ F^{-1}(q^2) &= 1 + G(q^2) + L(q^2). \end{aligned} \quad (2.20)$$

Finally, recalling that the dimension of the gluon anti-field A^* is three, while the dimension of the Ω source is one, power counting shows that (i) all functions appearing in Eqs. (2.16) and (2.17) are divergent, and (ii) the divergent part of $\Lambda_{\mu\nu}(q)$ can be proportional to $g_{\mu\nu}$ only [34, 41].

D. The (background) Landau gauge equality between $u(q^2)$ and $G(q^2)$

A crucial ingredient to our analysis is the equality between the KO function and the $G(q^2)$, in the (background) Landau gauge [34]. To see this, we start with the following

(Euclidean) two-point function of composite operators

$$\int d^4x e^{-iq \cdot (x-y)} \langle T [(\mathcal{D}_\mu c)_x^m (f^{nrs} A_\nu^n \bar{c}^s)_y] \rangle = P_{\mu\nu}(q) \delta^{mn} u(q^2), \quad (2.21)$$

which, due to the identity

$$\int d^4x e^{-iq \cdot (x-y)} \langle T [(\mathcal{D}_\mu c)_x^m \bar{c}^n] \rangle = -i \frac{q_\mu}{q^2} \delta^{mn} \quad (2.22)$$

can be related to the function $\langle [(\mathcal{D}_\mu c)_x^m (\mathcal{D}_\mu \bar{c})_y^n] \rangle$, through

$$\int d^4x e^{-iq \cdot (x-y)} \langle T [(\mathcal{D}_\mu c)_x^m (\mathcal{D}_\mu \bar{c})_y^n] \rangle = -\frac{q_\mu q_\nu}{q^2} \delta^{mn} + P_{\mu\nu}(q) \delta^{mn} u(q^2). \quad (2.23)$$

On the other hand, observe that in the background Landau gauge the function appearing on the lhs of the above equation is precisely given by

$$- \mathcal{G}_{\mu\nu}^{mn}(q) = \frac{\delta^2 W}{\delta \Omega_\mu^m \delta A_\nu^{*n}}, \quad (2.24)$$

where W is the generator of the connected Green's functions, and the two connected diagrams contributing to $\mathcal{G}_{\mu\nu}$ are shown in Fig. 2. Factoring out the color structure and making use of the identities (2.20) one has

$$\begin{aligned} -i \mathcal{G}_{\mu\nu}(q) &= \Lambda_{\mu\nu}(q) + \Gamma_{\Omega_\mu \bar{c}}(q) D(q^2) \Gamma_{A_\nu^* c}(q) \\ &= -\frac{q_\mu q_\nu}{q^2} + P_{\mu\nu}(q) G(q^2). \end{aligned} \quad (2.25)$$

Passing to the Euclidean formulation, and comparing with Eq. (2.23), we then arrive at the important equality

$$u(q^2) = G(q^2). \quad (2.26)$$

Then, the usual KO confinement criterion may be equivalently cast in the form: $1+G(0) = 0$. Evidently, if $L(0) = 0$ [see discussion after Eq. (3.15)], then from the identity (2.20) follows that if the KO criterion is satisfied then the ghost dressing function diverges in the IR.

E. Gribov-Zwanziger horizon

In order to avoid Gribov copies [21], in the GZ formulation of Yang-Mills theories the partition function assumes the form (in d -dimensional Euclidean space) [22]

$$Z_\gamma = \int [dA] \delta(\partial^\mu A_\mu) \text{Det} M \exp \left\{ -S_{\text{YM}} + \gamma \int d^d x h(x) \right\}, \quad (2.27)$$

where S_{YM} is the Yang-Mills action, $M = -\partial_\mu \mathcal{D}^\mu$ is the Faddeev-Popov operator, and the functional $h(x) = h[A](x)$ is the so-called GZ horizon function given by

$$h(x) = - \int d^d y g f^{amr} A_\mu^m(x) (M^{-1})^{rs}(x, y) g f^{asn} A_\nu^n(y); \quad (2.28)$$

thus, the action corresponding to the partition function above clearly contains a non-local term. The GZ parameter, γ , is determined through the so-called *horizon condition*, which for $SU(N)$ assumes the form

$$\langle h(x) \rangle_\gamma = d(N^2 - 1). \quad (2.29)$$

This condition can be rewritten in terms of the vev of the GZ horizon function if we integrate both sides over $d^d x$. In this way we get

$$\begin{aligned} \langle h(0) \rangle_\gamma &\equiv \frac{1}{V_d} \frac{\partial}{\partial \gamma} \ln Z_\gamma \\ &= \frac{1}{V_d} \int d^d x \langle h(x) \rangle_\gamma = d(N^2 - 1). \end{aligned} \quad (2.30)$$

On the other hand, assuming that γ is small, one can expand in powers of γ ; retaining the first order only, one gets

$$\langle h(0) \rangle_\gamma \simeq \langle h(0) \rangle_{\gamma=0} + \mathcal{O}(\gamma). \quad (2.31)$$

The right-hand side of the above equation can be related to the trace of the following Green's function (Euclidean space)

$$\mathcal{H}_{\mu\nu}^{mn}(q) = \delta^{mn} \left[P_\mu^\rho(q) + \frac{q_\mu q^\rho}{q^2} F(q^2) \right] \Lambda_{\rho\nu}(q), \quad (2.32)$$

in the limit $q^2 \rightarrow 0$ [33]. Specifically,

$$\begin{aligned} \langle h(0) \rangle_{\gamma=0} &= \frac{1}{V_d} \int d^d x \langle h(x) \rangle_{\gamma=0} = - \lim_{q^2 \rightarrow 0} \text{Tr} \{ \mathcal{H}_{\mu\nu}^{mn}(q) \} \\ &= -(N^2 - 1) \{ (d-1)G(0) + F(0) [G(0) + L(0)] \}. \end{aligned} \quad (2.33)$$

This result allows to rewrite the GZ horizon condition in terms of $G(0)$ (and therefore of the KO parameter $u(0)$); this will, in turn, restrict the allowed values of $u(0)$. In the limit of vanishing Gribov parameter, one can use the result of (2.33) to solve the horizon condition (2.29), in the approximation (2.31); if $L(0) = 0$, one finds (in $d = 4$) the following value of the KO parameter [33]

$$u(0) = G(0) = -\frac{2}{3}, \quad (2.34)$$

which is very close to that obtained directly from the lattice [32], and, as we will see in the next section, from our independent analysis.

F. Renormalization of $u(q^2)$: the MOM scheme and the associated μ -dependence

Before entering into the specifics of the KO function, let us briefly recall some basic facts about renormalization. In general, Green's functions in $d = 4$ must undergo renormalization. The renormalization procedure renders the renormalized quantities UV finite, introducing at the same time a dependence on the renormalization point, denoted in general by μ . This dependence, usually referred to as “ μ -dependence”, imposes non-trivial constraints on the asymptotic behavior of Green's functions, controlled by the renormalization group, and most concretely by the renormalization group equation corresponding to a given Green's function [50]. Specifically, a Green's function with n incoming fields ϕ , to be denoted (in momentum space) by $\Gamma^{(n)}(p_i, g, \mu)$, where g is the coupling constant, satisfies for asymptotically large momenta

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - n\gamma \right) \Gamma^{(n)}(p_i, g, \mu) = 0, \quad (2.35)$$

where $\beta = \mu(\partial g / \partial \mu)$, and γ is the so called “anomalous dimension” of the field ϕ , defined as $\gamma = \mu(\partial Z_\phi / \partial \mu)$. If the Green's function under consideration contains composite operators (*i.e.*, $\phi^2(x)$), then Eq. (2.35) must be appropriately modified (see, for example, [51]).

Note that the μ -dependence infests also Green's functions that are UV finite, *i.e.*, they do not need explicit subtraction to be rendered finite (*i.e.*, no new counter-terms need be introduced) For example, in the $(\phi^4)_4$ theory, all Green's functions with $n > 4$ are UV finite, but depend in general on μ . In this case, the μ -dependence enters through the dependence of the Green's functions on the propagators and vertices; since the latter depend explicitly on μ , $\Gamma^{(n)}$ (with $n > 4$) develops a μ -dependence through the higher loop corrections.

To be sure, a given Green's function may be renormalized at a fixed value of the incoming momenta, such as $p_i^2 = m_i^2$ (“on shell” scheme), in which case there is no *manifest* dependence on μ . Instead, within renormalization schemes such as the $\overline{\text{MS}}$ or the “MOM”, the intrinsic μ -dependence of Green's functions becomes manifest. Note also an additional important point: the exact functional dependence of a given Green's function on μ changes from one renormalization scheme to another; for example, the μ -dependence within the $\overline{\text{MS}}$ does not coincide with that of the MOM.

Turning now to $u(q^2)$, it is obvious from naive power-counting that it diverges logarithmically as the UV cutoff is taken to infinity. It is instructive to compute $u(q^2)$ (in the Landau gauge) at one loop, to be denoted by $u^{[1]}(q^2)$. To that end we compute the integral

in Eq. (2.4) at one loop, project out its $G(q^2)$ component using Eq. (2.5), and finally employ the crucial equality $u(q^2) = G(q^2)$ of Eq. (2.26). To avoid IR divergences, we introduce a hard gluon mass m (in the next section this will be done properly, using the IR-finite gluon propagator obtained from the lattice). Then, a straightforward calculation yields (setting $z \equiv q^2/m^2$)

$$u^{[1]}(z, \Lambda^2) = -\frac{3\alpha_s}{16\pi} f(z, \Lambda^2), \quad (2.36)$$

with

$$f(z, \Lambda^2) = 3 \ln \left(\frac{\Lambda^2}{m^2} \right) + \frac{19}{6} - \frac{1}{3z} - \left[3 + \frac{3}{z} - \frac{1}{3z^2} \right] \ln(1+z) + \frac{z}{3} \ln \left(\frac{1+z}{z} \right), \quad (2.37)$$

where Λ is the UV cutoff. Taking the limit of $z \rightarrow 0$ (expanding the logs), one finds that

$$u^{[1]}(0, \Lambda^2) = -\frac{9\alpha_s}{16\pi} \ln \left(\frac{\Lambda^2}{m^2} \right). \quad (2.38)$$

It is clear that a subtraction is sufficient to render $u^{[1]}(z, \Lambda^2)$ finite.

In the present work we will use the MOM scheme to renormalize the pertinent Green's functions, and will impose the necessary normalization conditions in the deep UV (or, at least, as far into the UV as permitted by the lattice data), where perturbation theory is reliable, and let the non-perturbative dynamics (captured by the lattice, the SDE, etc) determine what the IR behavior of the Green's functions will be. In fact, this latter renormalization procedure has been employed in practically all recent lattice studies; therefore, in order to be able to use self-consistently lattice data as input to our SDE equations, and compare meaningfully our results with those of [32], we have to use the MOM scheme, subject to an important constraint related with the preservation of the second identity in (2.20), as discussed in detail in subsection (III C).

Specifically, at the level of the one-loop example that we are considering in this subsection, the aforementioned constraint amounts to the statement that, due the validity of (2.20) and the fact that the function $L(z)$ does *not* vanish identically, one *cannot* impose the standard MOM condition *simultaneously* on both the one-loop dressing function, $F_R^{[1]}(z, \mu^2)$, and the KO function $u_R^{[1]}(z, \mu^2)$. In other words, one cannot have at the same time $F_R^{[1]}(z = \mu^2, \mu^2) = 1$ and $u_R^{[1]}(z = \mu^2, \mu^2) = 0$.

To see this explicitly, let us compute $L(z)$ at one-loop, to be denoted by $L^{[1]}(z)$, under the same assumptions employed before for $u^{[1]}(z)$; it is straightforward to obtain the finite

result

$$L^{[1]}(z) = \frac{\alpha_s}{4\pi} \left\{ \frac{1}{z^2} \left[\frac{z^2}{2} - m^2 z + m^4 \ln \left(1 + \frac{z}{m^2} \right) \right] + \frac{z}{m^2} \ln \left(1 + \frac{m^2}{z} \right) \right\}. \quad (2.39)$$

Note that $L^{[1]}(0) = 0$; however, for any other finite value of z , $L^{[1]}(z) \neq 0$. Given that (2.20) must remain valid, i.e.

$$\{F_R^{[1]}(z, \mu^2)\}^{-1} = 1 + u_R^{[1]}(z, \mu^2) + L^{[1]}(z), \quad (2.40)$$

it is clear that if one renormalizes the ghost according to the standard MOM prescription, $F^{[1]}(z = \mu^2, \mu^2) = 1$, then from (2.40) follows that

$$u_R^{[1]}(z = \mu^2, \mu^2) = -L^{[1]}(\mu^2), \quad (2.41)$$

which is the appropriate normalization condition for $u_R^{[1]}$; of course, $L^{[1]}(\mu^2) \neq 0$.

The exact form of $u_R^{[1]}(z, \mu^2)$ satisfying the normalization condition (2.41) is given by

$$u_R^{[1]}(z, \mu^2) = -\frac{3\alpha_s}{16\pi} f_R(z, \mu^2) - L^{[1]}(\mu^2), \quad (2.42)$$

where $f_R(q^2, \mu^2) = f(q^2, \Lambda^2) - f(\mu^2, \Lambda^2)$, namely (setting $t \equiv \mu^2/m^2$)

$$\begin{aligned} f_R(z, t) = & -\frac{1}{3z} - \left[3 + \frac{3}{z} - \frac{1}{3z^2} \right] \ln(1+z) + \frac{z}{3} \ln \left(\frac{1+z}{z} \right) \\ & + \frac{1}{3t} + \left[3 + \frac{3}{t} + \frac{1}{3t^2} \right] \ln(1+t) - \frac{t}{3} \ln \left(\frac{1+t}{t} \right). \end{aligned} \quad (2.43)$$

Evidently, $f_R(t, t) = 0$, or, equivalently $u(q^2 = \mu^2, \mu^2) = -L^{[1]}(\mu^2)$, a required by (2.41). The μ -dependence induced to $u(q^2)$ after imposing the renormalization condition given in (2.41) is shown in Fig.3.

III. EXTRACTING THE KUGO-OJIMA FUNCTION FROM THE LATTICE

In this section we study the behavior of the function $G(q^2)$ by using the available lattice data on the gluon and ghost propagators. Specifically, we will first write down the dynamical equations that govern the functions $G(q^2)$ and $L(q^2)$, which involve both $\Delta(q^2)$ and $D(q^2)$, and using as an input the lattice results for these propagators we will obtain an indirect determination of $G(q^2)$. Of course, by virtue of the fundamental equality $u(q^2) = G(q^2)$, any information on $G(q^2)$ translates automatically to the Kugo-Ojima function.

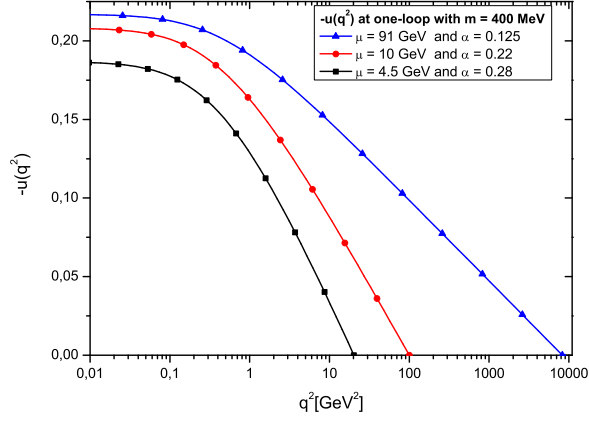


FIG. 3: The μ -dependence of $u(q^2)$ at one loop, under the normalization condition of (2.41).

A. Dynamical equations

The dynamical equations governing $G(q^2)$ and $L(q^2)$ may be obtained directly from the defining equation (2.4), by appropriately contracting it and taking its trace. Specifically, from Eq. (2.4) one has [41] (in d -dimensions, and setting $G(q^2) = u(q^2)$)

$$\begin{aligned} u(q^2) &= \frac{g^2 C_A}{d-1} \left[\int_k \Delta^{\rho\sigma}(k) H_{\sigma\rho}(k, q) D(k+q) + i \frac{1}{q^2} \int_k q^\rho \Delta_{\rho\sigma}(k) \Gamma^\sigma(k, q) D(k+q) \right], \\ L(q^2) &= -\frac{g^2 C_A}{d-1} \left[i \frac{d}{q^2} \int_k q^\rho \Delta_{\rho\sigma}(k) \Gamma^\sigma(k, q) D(k+q) + \int_k \Delta^{\rho\sigma}(k) H_{\sigma\rho}(k, q) D(k+q) \right]. \end{aligned} \quad (3.1)$$

Adding the above equations by parts, and employing (2.9), one may easily demonstrate [41], in the Landau gauge, the validity of the second identity in Eq. (2.20), given that the standard SD equation for the ghost propagator (Fig. 4) reads

$$iD^{-1}(q^2) = q^2 + ig^2 C_A \int_k \Gamma^\mu \Delta_{\mu\nu}(k) \Gamma^\nu(k, q) D(q+k). \quad (3.2)$$

The two equations in (3.1) involve five basic ingredients: the two-point functions $\Delta(q^2)$ and $D(q^2)$, the vertex functions $\Gamma_\mu(k, q)$ and $H_{\mu\nu}(q, k)$, and, eventually, the value of the (renormalized) coupling g^2 , at different renormalization points. Knowledge (direct or indirect) of these ingredients (for example from the lattice) would, in turn, determine fully the functions $u(q^2)$ and $L(q^2)$.

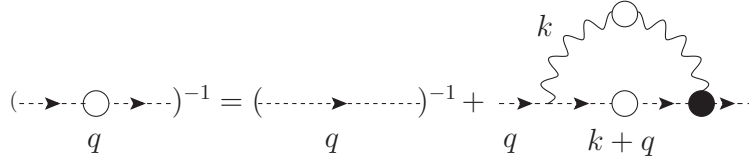


FIG. 4: The SDE satisfied by the ghost propagator.

B. The vertices

Let us begin with $\Gamma_\mu(k, q)$ and $H_{\mu\nu}(q, k)$; their most general Lorentz decomposition is given by

$$\begin{aligned}
 -\Gamma_\mu(k, q) &= B_1(k, q)q_\mu + B_2(k, q)k_\mu. \\
 -iH_{\mu\nu}(k, q) &= A_1(k, q)g_{\mu\nu} + A_2(k, q)q_\mu q_\nu + A_3(k, q)k_\mu k_\nu + A_4(k, q)q_\mu k_\nu + A_5(k, q)k_\mu q_\nu
 \end{aligned}
 \tag{3.3}$$

and from Eq. (2.9) we obtain two constraints for the various form-factors, namely

$$\begin{aligned}
 B_1(k, q) &= A_1(k, q) + q^2 A_2(k, q) + (k \cdot q)A_4(k, q), \\
 B_2(k, q) &= (k \cdot q)A_3(k, q) + q^2 A_5(k, q).
 \end{aligned}
 \tag{3.4}$$

Of course, since we work in the Landau gauge, due to the transversality of the gluon propagator, the only relevant form factors are

$$\begin{aligned}
 -\Gamma_\mu(k, q) &= B_1(k, q)q_\mu, \\
 -iH_{\mu\nu}(k, q) &= A_1(k, q)g_{\mu\nu} + A_2(k, q)q_\mu q_\nu.
 \end{aligned}
 \tag{3.5}$$

In the Landau gauge, the form factor B_1 of Eq. (3.3) is ultraviolet finite at one-loop, and therefore, no infinite renormalization constant needs to be introduced at that order; of course, B_2 must be ultraviolet finite in all gauges, and to all orders, otherwise the theory would be non-renormalizable. In order to obtain information about the ultraviolet behavior of B_1 beyond one-loop, one usually invokes the non-renormalization theorem of Taylor [53], which states that for vanishing ghost momentum one has that $B_1(q, -q) + B_2(q, -q) = 1$, to all orders in perturbation theory. Given that B_2 is finite to all orders (for any kinematic configuration), it follows that $B_1(-q, q)$ is also *finite* to all orders.

Turns out that the vertex $\Gamma_\mu(k, q)$ has been studied on the lattice in the Landau gauge, for the Taylor kinematics, both for $SU(2)$ [44] and $SU(3)$ [45]. According to these studies,

$B_1(-q, q)$ deviates very mildly from 1 (the tree-level value). Even though the integration over the gluon momentum in the integrals of Eq. (3.1) moves one away from the Taylor kinematics, the IR regime is well-represented, and we will approximate $B_1(k, q)$ by its tree-level value.

On the other hand, to the best of our knowledge, the vertex $H_{\mu\nu}(q, k)$ has not been studied on the lattice yet. Thus, the only constraints available are those coming from Eq. (3.4); we will simply satisfy it by setting $A_1(k, q) = B_1(k, q) = 1$ and $A_2(k, q) = 0$.

Then, under these approximations, the equations in (3.1) become

$$\begin{aligned} u(q^2) &= \frac{g^2 C_A}{d-1} \int_k \left[(d-2) + \frac{(k \cdot q)^2}{k^2 q^2} \right] \Delta(k) D(k+q), \\ L(q^2) &= \frac{g^2 C_A}{d-1} \int_k \left[1 - d \frac{(k \cdot q)^2}{k^2 q^2} \right] \Delta(k) D(k+q). \end{aligned} \quad (3.6)$$

C. Renormalization

Now, as discussed in detail in [41], the (unrenormalized) equations in (3.6) must be properly renormalized, *i.e.*, in such a way as to preserve the validity of the (BRST-induced) second identity in (2.20). Specifically, the quantities $G(q^2) = u(q^2)$, $L(q^2)$, and $F(q^2)$ appearing in Eq. (2.20) are unrenormalized (we have suppressed the corresponding subscript “0” for simplicity). Note in fact that Eq. (2.20) constrains the cutoff-dependence of the quantities involved; it is easy to recognize, for example, by substituting into (3.6) and (3.2) tree-level expressions, that $F^{-1}(q^2)$ and $u(q^2)$ have the same leading dependence on the UV cutoff Λ , namely [*viz.* (2.37)]

$$F_{\text{UV}}^{-1}(q^2) = u_{\text{UV}}(q^2) = \frac{9\alpha_s}{16\pi} \ln \left(\frac{\Lambda^2}{q^2} \right), \quad (3.7)$$

while $L(q^2)$ is finite (independent of Λ).

Let us now denote by Z_u the (yet unspecified) renormalization constant relating the bare and renormalized functions, $\Lambda_0^{\mu\nu}$ and $\Lambda^{\mu\nu}$, through

$$g^{\mu\nu} + \Lambda_0^{\mu\nu}(q) = Z_u^{-1} [g^{\mu\nu} + \Lambda^{\mu\nu}(q)]. \quad (3.8)$$

Note that the inclusion of the “zeroth-order” term $g^{\mu\nu}$ on both sides of (3.8) is absolutely essential for the self-consistency of the entire renormalization procedure. To be sure, the $g^{\mu\nu}$ term appears naturally, given, for example, the form of the BQI in (2.10); indeed, the

multiplicative renormalizability of (2.10) requires that the combination given in (3.8) should be renormalized as a whole.

As already mentioned above, the origin of Eq. (2.20) is the BRST symmetry of the theory; in that sense, Eq. (2.20) has the same origin as the Slavnov-Taylor identities. Therefore, just as happens with the Slavnov-Taylor identities, Eq. (2.20) does not get deformed after renormalization. Of course, the prototype example of such a situation are the Ward identities of QED; the requirement that the fundamental Ward identity $q^\mu \Gamma_\mu = S^{-1}(p+q) - S^{-1}(p)$ should retain the same form before and after renormalization leads to the well-known textbook relation $Z_1 = Z_2$ between the corresponding renormalization constants. Similarly, for the case at hand, the renormalization must be carried out in such a way as to preserve the form Eq. (2.20). Specifically, denoting by Z_c the renormalization constant of the ghost dressing function, i.e.,

$$Z_c(\Lambda^2, \mu^2) F_0^{-1}(q^2, \Lambda^2) = F^{-1}(q^2, \mu^2), \quad (3.9)$$

and using the definition given in Eq. (3.8), it is clear that in order to preserve the relation (2.20) after renormalization, we must impose that

$$Z_u = Z_c. \quad (3.10)$$

As a result, one must renormalize Eq. (3.2) using Eq. (3.9), and Eq. (3.6) using the relations

$$Z_c(\Lambda^2, \mu^2)[1 + u_0(q^2, \Lambda^2) + L_0(q^2, \Lambda^2)] = 1 + u(q^2, \mu^2) + L(q^2, \mu^2). \quad (3.11)$$

To carry out the renormalization explicitly, let us introduce in addition

$$\begin{aligned} \Delta(q^2; \mu^2) &= Z_A^{-1}(\mu^2) \Delta_0(q^2), \\ g(\mu^2) &= Z_g^{-1}(\mu^2) g_0, \end{aligned} \quad (3.12)$$

and remember that, in the Landau gauge, due to Taylor's theorem, the vertex renormalization is 1. Thus, after imposing the MOM renormalization condition $F(\mu^2) = 1$, going to Euclidean space, setting $q^2 = x$, $k^2 = y$ and $\alpha_s = g^2/4\pi$, and implementing the standard angular approximation, one finds that the renormalized version of Eq. (3.2) reads

$$F^{-1}(x) = Z_c - \frac{\alpha_s C_A}{16\pi} \left[\frac{F(x)}{x} \int_0^x dy y \left(3 - \frac{y}{x} \right) \Delta(y) + \int_x^\infty dy \left(3 - \frac{x}{y} \right) \Delta(y) F(y) \right], \quad (3.13)$$

where the renormalization constant Z_c is given by

$$Z_c = 1 + \frac{\alpha_s C_A}{16\pi} \left[\frac{1}{\mu^2} \int_0^{\mu^2} dy y \left(3 - \frac{y}{\mu^2} \right) \Delta(y) + \int_{\mu^2}^\infty dy \left(3 - \frac{\mu^2}{y} \right) \Delta(y) F(y) \right]. \quad (3.14)$$

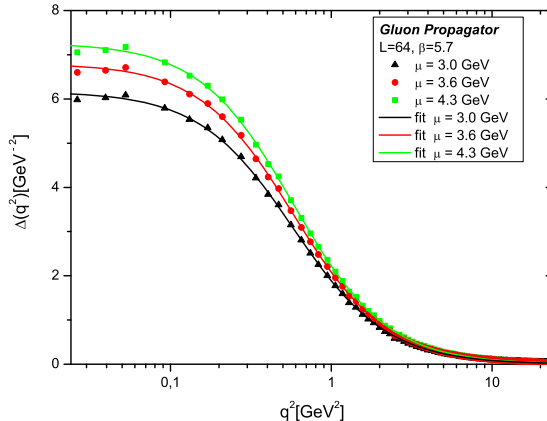


FIG. 5: Lattice results for the gluon propagator renormalized at three different renormalization points: $\mu = 3.0 \text{ GeV}$ (black curve), $\mu = 3.6 \text{ GeV}$ (red curve) and $\mu = 4.3 \text{ GeV}$ (green curve). We also show the corresponding fits using Eq.(3.17). The fitting parameters are: $a = 0.162 \text{ GeV}^2$, $b = 0.367 \text{ GeV}^{-1}$ and $c = 1.5$ ($\mu = 3.0 \text{ GeV}$); $a = 0.147 \text{ GeV}^2$, $b = 0.334 \text{ GeV}^{-1}$ and $c = 1.5$ ($\mu = 3.6 \text{ GeV}$); $a = 0.137 \text{ GeV}^2$, $b = 0.311 \text{ GeV}^{-1}$ and $c = 1.5$ ($\mu = 4.3 \text{ GeV}$).

Then, given Eq. (3.11), we have that the renormalized version of Eq. (3.6), under the same approximations, reads

$$\begin{aligned}
 1 + u(x) &= Z_c - \frac{\alpha_s C_A}{16\pi} \left[\frac{F(x)}{x} \int_0^x dy y \left(3 + \frac{y}{3x} \right) \Delta(y) + \int_x^\infty dy \left(3 + \frac{x}{3y} \right) \Delta(y) F(y) \right], \\
 L(x) &= \frac{\alpha_s C_A}{12\pi} \left[\frac{F(x)}{x^2} \int_0^x dy y^2 \Delta(y) + x \int_x^\infty dy \frac{\Delta(y) F(y)}{y} \right].
 \end{aligned} \tag{3.15}$$

From this last equation it is easy to see (*e.g.*, by means of the change of variables $y = zx$) that if Δ and F are IR finite, then $L(0) = 0$, as mentioned before [46]. Note that one cannot choose simultaneously the condition $u(\mu^2) = 0$ once $F(\mu^2) = 1$ has been imposed; indeed, given that $L(\mu^2) \neq 0$, such a choice would violate the identity of Eq. (2.20).

D. Numerical analysis

The starting point for our numerical analysis are the lattice results for the gluon propagator $\Delta(q^2)$ reported in [27]. In order to eventually study the dependence of the KO function on the renormalization point, we would like to obtain the lattice data at different renormalization points. Since the gluon propagator is multiplicatively renormalizable, the

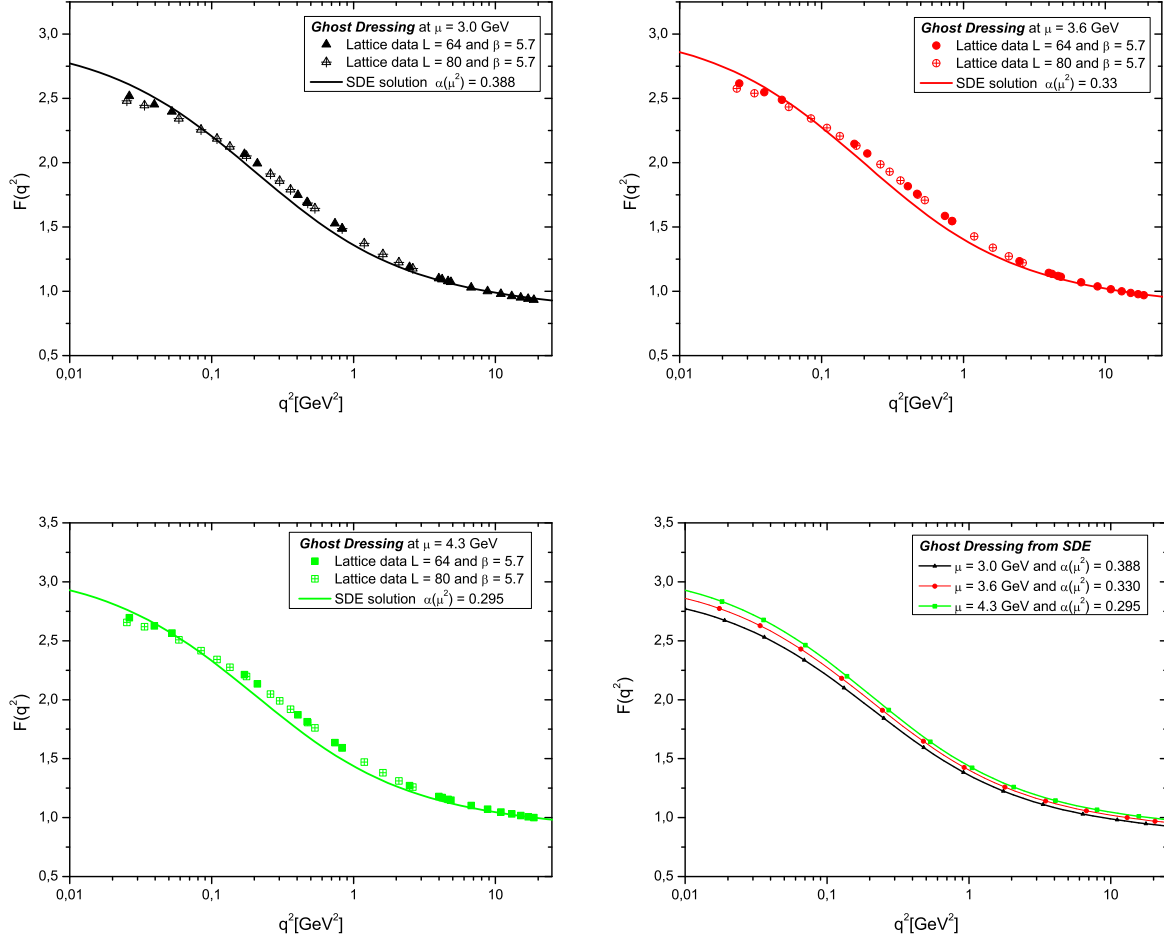


FIG. 6: *Top left panel:* The ghost dressing function $F(q^2)$ obtained from SDE (black continuous line) compared to the lattice data of [27] at $\mu = 3.0$ GeV. *Top right panel:* Same as in the previous panel but renormalized at $\mu = 3.6$ GeV. *Bottom left panel:* Same as before but renormalized at $\mu = 4.3$ GeV. *Bottom right panel:* The SDE solutions for the three different renormalization points all together.

relation [47]

$$\Delta(q^2, \mu^2) = \frac{\Delta(q^2, \nu^2)}{\mu^2 \Delta(\mu^2, \nu^2)}, \quad (3.16)$$

can be used to connect a set of points renormalized at μ with the corresponding set renormalized at ν . Choosing the three different values $\mu = \{3.0, 3.6, 4.3\}$ GeV, we then obtain the three curves shown in Fig. 5. In the range of available momenta, a very accurate fit is

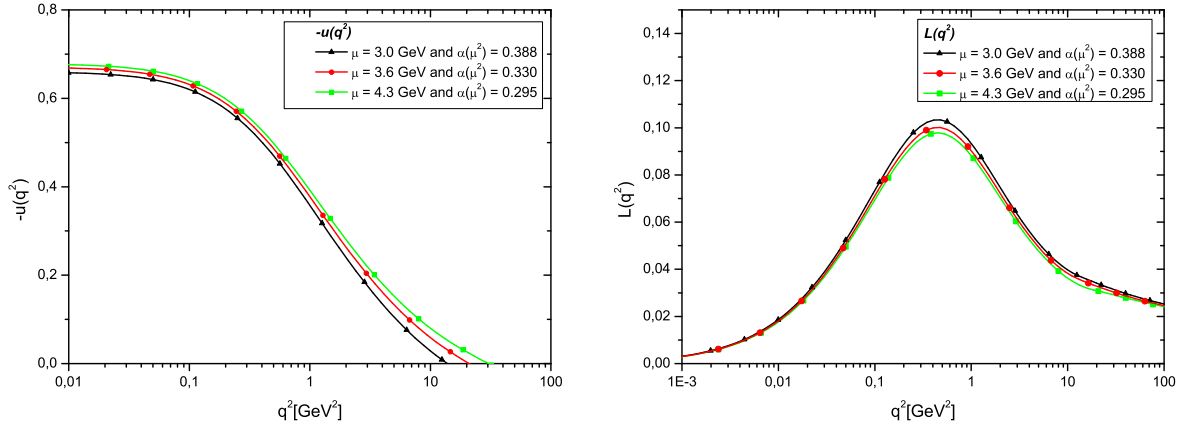


FIG. 7: *Left panel:* $-u(q^2)$ determined from Eq. (3.15), using the solutions for $\Delta(q^2)$ and $D(q^2)$ presented in Figs. 5 and 6 at the same renormalization points μ . *Right panel:* Same as in the previous panel but this time for $L(q^2)$.

provided by the expression

$$\Delta(q^2) = \frac{1}{a + bq^{2c}}, \quad (3.17)$$

as shown by the continuous line in Fig. 5 (the values of the fitting parameters a , b , and c are also reported there).

The next step is to employ the ghost SDE given in (3.13) in order to deduce the appropriate values that one must use for the gauge coupling. To that end we will follow the following steps: (i) employing once again the relation (3.16), with $\Delta \rightarrow D$, we generate from the lattice data on $F(q^2)$ reported in [27] the data sets for $F(q^2)$ corresponding to the renormalization points used previously for Δ ; (ii) using as input in (3.13) the different sets of results obtained in the previous step for Δ , we solve the integral equation (3.13) numerically, thus determining $F(q^2; \mu^2)$; (iii) the values of $\alpha_s(\mu^2)$ are fixed by demanding that the solutions obtained in (ii) match the different lattice sets generated at step (i).

The results of this procedure are displayed in Fig. 6, where we show both the comparison between the lattice data and the solutions of Eq. (3.13) (first three panels), as well as the dependence on μ^2 of these solutions (fourth panel). The couplings found are $\alpha(\mu^2) = \{0.388, 0.330, 0.295\}$ for $\mu = \{3.0, 3.6, 4.3\}$ GeV respectively. Note that the values of $\alpha_s(\mu^2)$ obtained by this procedure are about 20% higher than those found from the two-loop MOM calculation of [48].

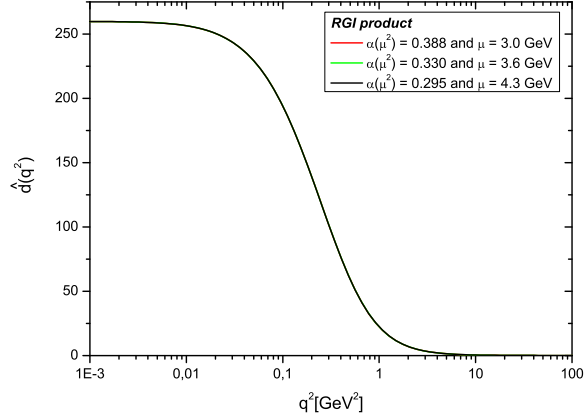


FIG. 8: The renormalization-group invariant product $\widehat{d}(q^2)$ obtained combining our results for $\Delta(q^2)$ and $G(q^2)$ according to Eq. (3.18).

At this point, all necessary ingredient for determining the functions $u(q^2)$ and $L(q^2)$ are available. Substituting them into the corresponding equations given in (3.15), we obtain the solutions shown in Fig. 7. Notice that $L(q^2)$ vanishes in the deep IR, as expected.

A very stringent test of the quality of the obtained solutions can be devised by observing that, on formal grounds, the combination

$$\widehat{d}(q^2) = \frac{g^2(\mu^2)\Delta(q^2; \mu^2)}{[1 + u(q^2; \mu^2)]^2}, \quad (3.18)$$

constitutes a renormalization group invariant (*i.e.*, μ -independent) quantity [49]. Indeed, it is well-known that, due to the Abelian Ward identities satisfied by the PT-BFM Green's functions, the propagator $\widehat{\Delta}^{-1}(q^2)$ absorbs all the RG logs, exactly as happens in QED with the photon self-energy. Specifically, if we define the renormalization constants of the gauge-coupling and the effective self-energy as

$$\begin{aligned} g(\mu^2) &= Z_g^{-1}(\mu^2)g_0, \\ \widehat{\Delta}(q^2; \mu^2) &= \widehat{Z}_A^{-1/2}(\mu^2)\widehat{\Delta}_0(q^2), \end{aligned} \quad (3.19)$$

then, since the renormalization constants above satisfy the QED-like relation

$$Z_g = \widehat{Z}_A^{-1/2}, \quad (3.20)$$

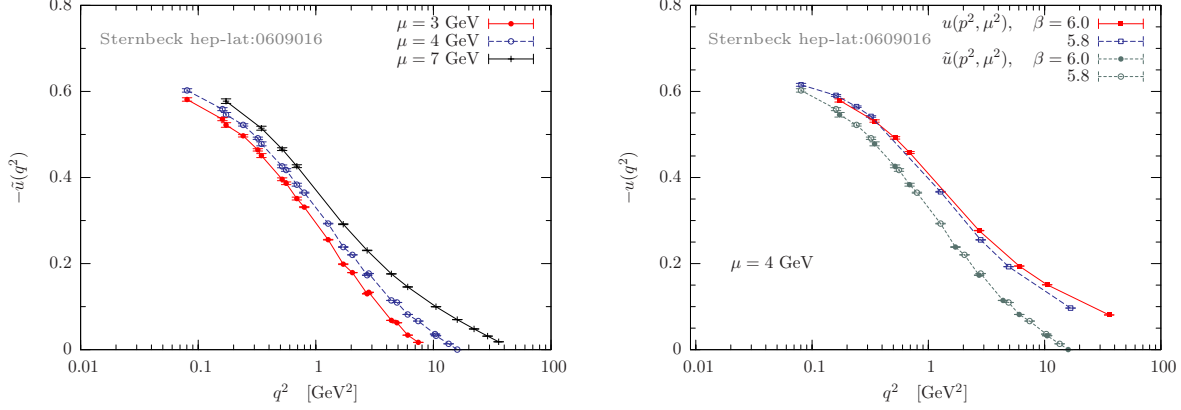


FIG. 9: The determination of the KO function obtained in [32]. *Left panel:* the function $-\tilde{u}(p^2) = 1 - F^{-1}(p^2)$, determined from Eq. (2.20) by neglecting the $L(p^2)$ component, plotted at three different renormalization points. *Right panel:* Comparison between the function $-u(p^2)$ measured directly on the lattice and its asymptotic behavior $-\tilde{u}(p^2)$.

the product

$$\hat{d}_0(q^2) = g_0^2 \hat{\Delta}_0(q^2) = g^2 \hat{\Delta}(q^2) = \hat{d}(q^2), \quad (3.21)$$

retains the same form before and after renormalization, *i.e.*, it forms a RG-invariant (μ -independent) quantity [2].

In Fig. 8 we plot the combination above for the three different values of μ chosen; evidently the product of $g^2(\mu^2)$, $\Delta(q^2; \mu^2)$ and $[1 + u(q^2; \mu^2)]^{-2}$ constructed from the solutions obtained is μ -independent to an extremely high degree of accuracy.

It is interesting to compare the curves plotted for $u(q^2)$ (Fig. 7 left panel) with those obtained by Sternbeck [32] (reproduced in Figs. 9 and 10), where the function (2.21) was studied in terms of Monte Carlo averages, and its asymptotic behavior was inferred from the identity (2.20). In that case however the extrapolation in the deep IR region was problematic, due to a lack of knowledge of the function $L(q^2)$ [there denoted by $q^2 v(q^2)$]; our analysis does not suffer from such a limitation, given that $L(q^2)$ is completely determined by its own equation.

One can see that the behavior is clearly the same encountered here (including the μ -dependence); in addition we notice that the remarkable agreement found between the KO function extracted using our method and the direct calculation on the lattice, shows a posteriori that our tree-level approximations for the vertices appearing in the SDEs (3.1) is

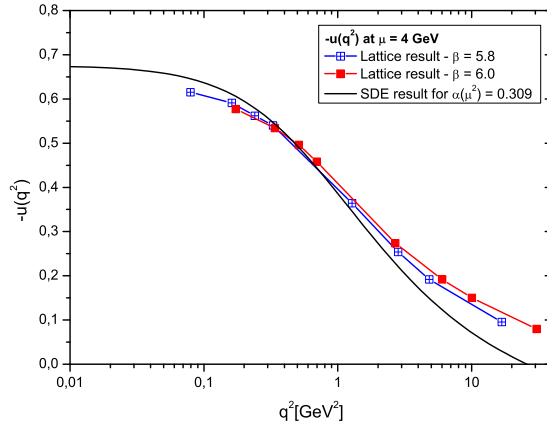


FIG. 10: The KO function, $-u(q^2)$, obtained from the solution of Eq. (3.15) (black continuous line) compared to the lattice data of [32] at $\mu = 4$ GeV.

indeed justified.

Now, the important point to emphasize is that the function $u(q^2)$ is *not* a μ -independent quantity; in fact, as we have established (within the MOM scheme) its μ -dependence is exactly what is needed in order to enforce the μ -independence of the RG-invariant expression given in (3.18). In fact, its value at $q^2 = 0$, *i.e.*, the KO parameter $u(0)$, *depends* on the renormalization point. Notice that in the case of an IR divergent ghost dressing function the possible μ -dependence would be inconsequential, since, due to the identity (2.20), $u(0) = -1$ irrespectively of the value of μ chosen. Evidently, the situation is different in the case of an IR finite ghost dressing function, since $u(0)$ acquires a non-trivial dependence on the renormalization scale. This dependence is plotted on the left panel of Fig. 11 for values of μ varying between 2.6 and 4.3 GeV (due to the limited number of UV lattice points of our data), where we see that $-u(0)$ varies in the interval $[0.65, 0.68]$, in good agreement with the prediction (2.34). The dependence of $u(0)$ on μ appears to be moderate, probably due to the rather narrow region of allowed μ values considered. A more detailed study with data sets extending deeper in the UV should allow one to explore the full extent of this dependence.

On the right panel Fig. 11 we plot finally the μ -dependence of the horizon function (2.33). Notice that both dependencies can be fitted with a function that is characteristic of a phase

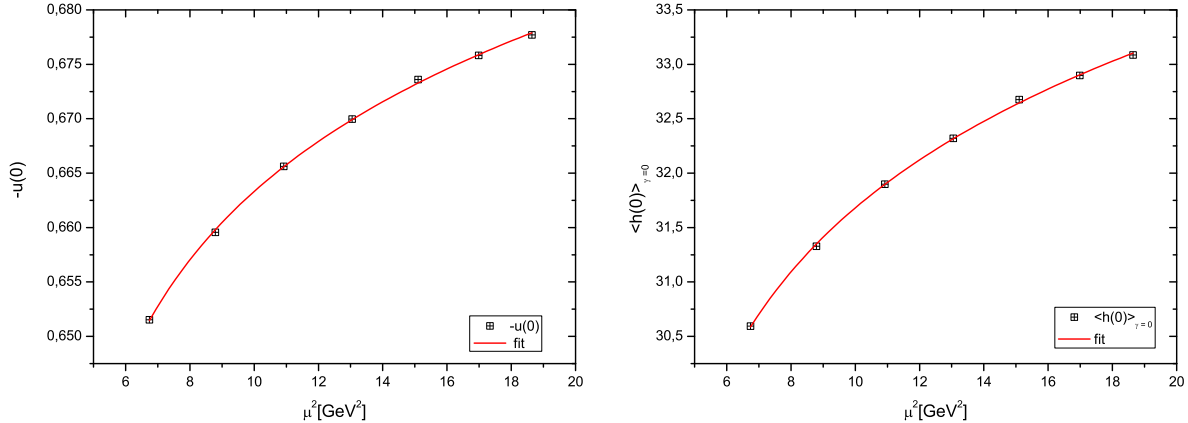


FIG. 11: *Left panel:* The dependence of the KO parameter $u(0)$ on the renormalization point μ . *Right panel:* Same as in the previous panel but for the horizon function. In both cases the continuous red line represents the fit given by Eq. (3.23) with $a_1 = 0.633$, $b_1 = 3.57$, $c_1 = 0.025$ and $a_2 = 28.61$, $b_2 = 3.25$, $c_2 = 0.05$ for $-u(0)$ and $\langle h(0) \rangle_{\gamma=0}$ respectively.

transition, namely

$$\begin{aligned}
 -u(0) &= a_1(\mu^2 - b_1)^{c_1}, \\
 \langle h(0) \rangle_{\gamma=0} &= a_2(\mu^2 - b_2)^{c_2},
 \end{aligned}
 \tag{3.22}$$

as shown by the continuous red curves appearing in Fig. 11 (the values of the fitting parameter are also reported there).

IV. CONCLUSIONS

In this work we have presented an indirect determination of the KO function from recent lattice data on the behavior of the QCD gluon and ghost propagators [26, 27] in the Landau gauge. The results obtained are in very good agreement with the original study of the same quantity presented in [32].

Of particular interest is the observed dependence of the KO function, and particular of its infrared value $u(0)$, on the renormalization point μ chosen within the MOM scheme. The μ -dependence of $u(0)$ within the latter scheme, mild as it may seem at first sight, is definitely there, as one would expect, given that the KO function $u(q^2)$ is *not* a renormalization-group invariant quantity, *i.e.*, it is not intrinsically μ -independent. In fact, the observed

μ -dependence is really sizeable when contrasted with the impressive absence of any μ -dependence displayed by a genuinely μ -independent quantity given in Eq.(3.18) which was computed using exactly the same sets of lattice data. We hope that the present work will contribute to the study of the possible effects that renormalization may have on the quantitative predictions of the KO formalism.

Let us take a closer look at the background-quantum identity given in Eq. (2.12), which relates the conventional gluon propagator Δ with the gluon propagator $\widehat{\Delta}$ of the BFM. Eq. (2.12) assumes that the corresponding gauge-fixing parameters, namely the ξ of the R_ξ and the ξ_Q used in the BFM to gauge-fix the quantum fields appearing inside the loops, are equal ($\xi = \xi_Q$). In the Landau gauge, $\xi = \xi_Q = 0$, due to the central equality of Eq. (2.26), we have that

$$u(q^2) = \sqrt{\frac{\Delta(q^2)}{\widehat{\Delta}(q^2)}} - 1. \quad (4.1)$$

Interestingly enough, this simple formula expresses the KO function in terms of two gluon propagators calculated in the Landau gauge of two very distinct gauge-fixing schemes, with no direct reference to the ghost sector of the theory. This observation opens up the possibility of deducing the structure of the KO function using an entirely different, and completely novel, approach. Specifically, one may envisage a lattice simulation of $\widehat{\Delta}$ [52]; then, $u(q^2)$ may be obtained from (4.1) by simply forming the ratio of the two gluon propagators. Given that $\Delta(0)$ is found to be finite on the lattice [26, 27], it is clear that, in order for the standard KO criterion to be satisfied (*i.e.*, $u(0) = -1$), $\widehat{\Delta}$ must diverge in the IR. Needless to say, we consider such a scenario highly unlikely. What is far more likely to happen, in our opinion, is to find a perfectly finite and well-behaved $\widehat{\Delta}$, which in the deep IR will be about an order of magnitude larger than $\Delta(0)$, furnishing a value $u(0) \sim -0.6$, namely what we have found in our analysis. In fact, one may turn the argument around: combining the results of this article with the lattice data for Δ [26, 27], one may use (4.1) to predict the outcome of the lattice simulation for $\widehat{\Delta}$; our prediction for the case of $SU(3)$ is shown in Fig. 12.

Acknowledgments

We thank Prof. K-I. Kondo for bringing to our attention reference [32]. The research of J. P. is supported by the European FEDER and Spanish MICINN under grant FPA2008-

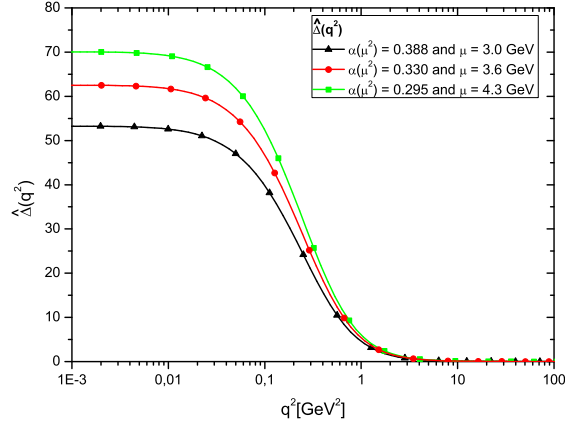


FIG. 12: The gluon propagator $\hat{\Delta}(q^2)$ of the BFM, renormalized at three different points: $\mu = 3.0 \text{ GeV}$ (black curve), $\mu = 3.6 \text{ GeV}$ (red curve) and $\mu = 4.3 \text{ GeV}$ (green curve).

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