Scrutinizing the Green's functions of QCD: Lattice meets Schwinger-Dyson

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The Green's functions of QCD encode important information about the infrared dynamics of the theory. The main non-perturbative tools used to study them are their own equations of motion, known as Schwinger-Dyson equations, and large-volume lattice simulations. We have now reached a point where the interplay between these two methods can be most fruitful. Indeed, the quality of the lattice data is steadily improving, while a recently introduced truncation scheme for the Schwinger-Dyson equations makes their predictions far more reliable. In this talk several of the above points will be reviewed, with particular emphasis on how to enforce the crucial requirement of gauge invariance at the level of the Schwinger-Dyson equations, the detailed mechanism of dynamical gluon mass generation and its implications for the ghost sector, the non-perturbative effective charge of QCD, and the indirect extraction of the Kugo-Ojima function from existing lattice data.

1. Introduction

The basic building blocks of QCD are the Green's (correlation) functions of the fundamental physical degrees of freedom, gluons and quarks, and of the unphysical ghosts. Even though it is well-known that these quantities are not physical, since they depend on the gauge-fixing scheme and the parameters used to renormalize them, it is widely believed that reliable information on their non-perturbative structure is essential for unraveling the infrared dynamics of QCD. Indeed, in addition to their relevance for phenomenology, the QCD Green's function encode information on confinement, albeit in a rather subtle way.

The two basic non-perturbative tools that permit the exploration of the infrared domain of QCD are (**i**) the lattice, where space-time is discretized and the quantities of interest are evaluated numerically, and (**ii**) the infinite set of integral equations governing the dynamics of the QCD Green's functions, known as Schwinger-Dyson equations (SDE). Given that both the lattice and the SDE aspire to describe essentially the same physics, it is important to advance their complementarity and strengthen their mutual interplay. In fact, it would seem that we have reached a point in time where the meaningful and systematic comparison between lattice and SDE results constitutes a tangible reality. Indeed, the quality of lattice data is steadily improving, while, due to several recent developments [1], we have at our disposal, for the first time, a manifestly gauge invariant truncation scheme for the SD series of QCD.

It it generally accepted by now that the lattice yields in the Landau gauge (LG) an infrared finite gluon *propagator* and an infrared finite (non-enhanced) ghost *dressing function*. This rather characteristic behavior has been firmly established recently using large-volume lattices, for pure Yang-Mills (no quarks included), for both $SU(2)$ [2] and $SU(3)$ [3]. In this talk we will review the SD part of this story, within the truncation scheme based on the pinch technique (PT) [4,5,6] and its connection with the background field method (BFM) [7]. We discuss the plethora of conceptual issues and the wide range of possibilities that emerge when the dynamical gluon mass generation picture of QCD is adopted. Moreover, we clarify some basic but subtle issues related with the non-perturbative definition of the QCD effective charge.

2. Problem(s) with the conventional SDEs

The SDEs provide a formal framework for tackling physics problems requiring a non-perturbative treatment. Given that the SDEs constitute an infinite system of coupled non-linear integral equations for all Green's functions of the theory, their practical usefulness hinges crucially on one's ability to devise a self-consistent truncation scheme that would select a tractable subset of these equations, without compromising the physics one hopes to describe. Devising such a scheme, however, is very challenging, especially in the context of non-Abelian gauge theories, such as QCD. For the purposes of this presentation we will only focus on the aspect of the problem related with gauge-invariance

In Abelian gauge theories the Green's functions satisfy naive Ward identities (WIs): the all-order generalization of a tree-level WI is obtained by simply replacing the Green's functions appearing in it by their all-order expressions. In general, this is not true in non-Abelian gauge theories, where the WIs are modified non-trivially beyond tree-level, and are replaced by more complicated expressions known as Slavnov-Taylor identities (STIs): in addition to the original Green's functions appearing at tree-level, they involve various composite *ghost operators*.

To appreciate why the WIs are instrumental for the consistent truncation of the SDEs, while the STIs complicate it, let us first consider how nicely things work in an Abelian case, namely *scalar* QED (photon), and then turn to the complications encountered in QCD (gluon).

The full photon (gluon) propagator, in a R_ξ-type of gauge, is given by

$$\Delta_{\mu\nu}(q) = -i\left[P_{\mu\nu}(q)\Delta(q^2) + \xi\frac{q_\mu q_\nu}{q^4}\right], \qquad (1)$$

where $P_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2$ is the transverse projector. The scalar function $\Delta(q^2)$ is related to the all-order (photon) gluon self-energy $\Pi_{\mu\nu}(q) = P_{\mu\nu}(q)\Pi(q^2)$ through $\Delta^{-1}(q^2) = q^2 + i\Pi(q^2)$. We also define the dimensionless vacuum-polarization $\mathbf{\Pi}(q^2)$, as $\Pi(q^2) = q^2\mathbf{\Pi}(q^2)$.

Note now a crucial point: local gauge-invariance (BRST in the case of the gluon) forces $\Pi_{\mu\nu}(q)$ (photon and gluon alike) to satisfy the fundamental transversality relation

$$q^\mu \Pi_{\mu\nu}(q) = 0\,, \qquad (2)$$

both perturbatively (to all orders) and non-perturbatively.

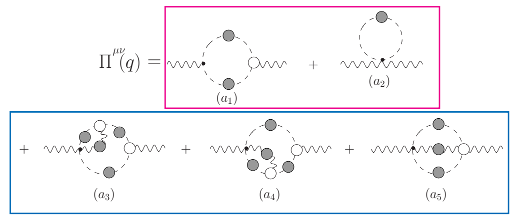

Figure 1. The SDE for the photon self-energy in scalar QED.

The SDE governing $\Pi_{\mu\nu}(q)$ in scalar QED is shown in Fig.1. The main question we want to address is the following: can one truncate the rhs of Fig.1, i.e. eliminate some of the graphs, without compromising the transversality of $\Pi_{\mu\nu}(q)$? The answer is shown already in Fig.1: the two blocks of graphs $[(a_1)+(a_2)]$ and $[(a_3)+(a_4)+(a_5)]$ are *individually transverse*, i.e.

$$q^\mu \sum_{i=1}^{2}(a_i)_{\mu\nu} = 0\,, \quad q^\mu \sum_{i=3}^{5}(a_i)_{\mu\nu} = 0\,. \qquad (3)$$

The reason for this special property are precisely the naive WIs satisfied by the full vertices appearing on the rhs of the SDE; for example, the first block is transverse simply because the *full* photon-scalar vertex Γ_μ [white blob in (a_1)] satisfies the WI

$$q^\mu \Gamma_\mu = e[\mathcal{D}^{-1}(k+q) - \mathcal{D}^{-1}(k)], \qquad (4)$$

where $\mathcal{D}(q)$ is the *full* propagator of the charged scalar. A similar WI relating the four-vertex with a linear combination of Γ_μ forces the transversality of the second block in Fig.1. Thus, due to the simple WIs satisfied by the vertices appearing on the SDE for $\Pi_{\mu\nu}(q)$, one may omit the second block of graphs and still maintain the transversality of the answer intact, i.e. the approximate $\Pi_{\mu\nu}(q)$ obtained after this truncation satisfies (2).

Let us now turn to the *conventional* SDE for

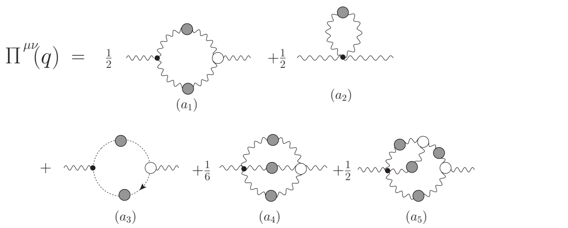

Figure 2. The conventional SDE for the gluon self-energy

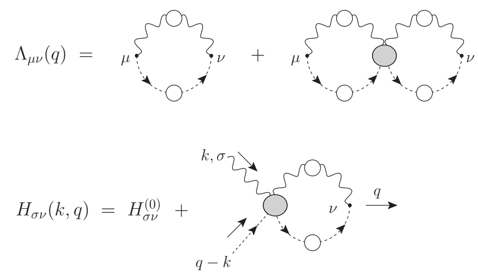

Figure 3. Diagrammatic representation of the functions Λ and H.

the gluon self-energy, in the R_ξ gauges, given in Fig.2. Clearly, by virtue of (2), we must have

$$q^{\mu}\sum_{i=1}^{6}(a_i)_{\mu\nu} = 0. \tag{5}$$

However, unlike the Abelian example, the diagrammatic verification of (5), i.e. through contraction of the individual graphs by q^μ, is very difficult, essentially due to the complicated STIs satisfied by the vertices involved. For example, the full three-gluon vertex $\Gamma_{\alpha\mu\nu}(q, k_1, k_2)$ satisfies the STI

$$\begin{aligned} q^{\alpha}\Gamma_{\alpha\mu\nu} &= F(q)\Delta^{-1}(k_2^2)P_{\nu}^{\gamma}(k_2)H_{\mu\gamma}(k_1,k_2) \\ &- F(q)\Delta^{-1}(k_1^2)P_{\mu}^{\gamma}(k_1)H_{\nu\gamma}(k_2,k_1), \end{aligned} \tag{6}$$

where $F(q) = q^2 D(q)$ is the ghost dressing function, $D(q)$ is the ghost propagator, and $H_{\mu\nu}$, defined in Fig.3, is related to the full gluon-ghost vertex by $q^\nu H_{\mu\nu}(k, q) = -i\Gamma_\nu(k, q)$; at tree-level, $H^{(0)}_{\mu\nu} = ig_{\mu\nu}$. In addition, some of the pertinent STIs are either too complicated, such as that of the conventional four-gluon vertex, or they cannot be cast in a particularly convenient form, such as the STI of the conventional gluon-ghost vertex. The main practical consequence of all this is that one *cannot* truncate the rhs of Fig.2 in any obvious way without violating the transversality of the resulting $\Pi_{\mu\nu}(q)$. For example, keeping only graphs (a_1) and (a_2) is not correct even perturbatively, since the ghost loop is crucial for the transversality of $\Pi_{\mu\nu}$ already at one-loop; adding (a_3) is still not sufficient for a SDE analysis, because (beyond one-loop) $q^\mu[(a_1) + (a_2) + (a_3)]_{\mu\nu} \neq 0$.

3. Truncating gauge-invariantly: The PT-BFM framework

Recently a new truncation scheme for the SDEs of QCD has been proposed, that respects gauge invariance at every level of the "dressed-loop" expansion. This becomes possible due to the drastic modifications implemented to the building blocks of the SD series, *i.e.* the off-shell Green's functions themselves, following the general methodology of the PT [4,5,6]. The PT is a well-defined algorithm that exploits systematically the BRST symmetry in order to construct new Green's functions endowed with very special properties. In particular, the crucial property for our discussion is that they satisfy Abelian WIs instead of the usual STIs The PT may be used to rearrange systematically the entire SD series [1]. In the case of the gluon self-energy it gives rise *dynamically* to a new SDE, Fig.4, similar to that of Fig.2, but with two important differences:

(i) What appears on the lhs is *not* the conventional self-energy $\Pi_{\mu\nu}$, but rather the PT-BFM self-energy, denoted by $\widehat{\Pi}_{\mu\nu}$. Of course, $\widehat{\Pi}_{\mu\nu}$ is also transverse, i.e. $q^\mu\widehat{\Pi}_{\mu\nu}(q) = 0$. Quite interestingly, the two quantities are connected by a powerful formal identity [8] stating that

$$\Delta(q^2) = \left[1 + G(q^2)\right]^2 \widehat{\Delta}(q^2), \tag{7}$$

with $G(q^2)$ defined from $\Lambda_{\mu\nu}(q)$, Fig.3,

$$\begin{aligned} i\Lambda_{\mu\nu}(q) &= \lambda\int_k H^{(0)}_{\mu\rho}D(k+q)\Delta^{\rho\sigma}(k)\,H_{\sigma\nu}(k,q), \\ &= g_{\mu\nu}G(q^2) + \frac{q_\mu q_\nu}{q^2}L(q^2), \end{aligned} \tag{8}$$

where $\lambda = g^2 C_{\rm A}$, with $C_{\rm A}$ the Casimir eigenvalue of the adjoint representation [$C_{\rm A} = N$ for $SU(N)$], and $\int_k \equiv \mu^{2\varepsilon}(2\pi)^{-d}\int d^dk$, with $d = 4-\epsilon$ the dimension of space-time.

(ii) The graphs appearing on the rhs contain the conventional self-energy $\Pi_{\mu\nu}$ as before, but are composed out of new vertices (Fig.4). These new vertices correspond precisely to the Feynman rules of the BFM [7], *i.e.*, it is as if the external gluon had been converted dynamically into a background gluon. As a result, the full vertices $\widetilde{\Gamma}^{amn}_{\alpha\mu\nu}(q,k_1,k_2)$, $\widetilde{\Gamma}^{anm}_{\alpha}(q,k_1,k_2)$, $\widetilde{\Gamma}^{amnr}_{\alpha\mu\nu\rho}(q,k_1,k_2,k_3)$, and $\widetilde{\Gamma}^{amnr}_{\alpha\mu}(q,k_1,k_2,k_3)$ appearing on the rhs of the SDE shown in Fig.4 satisfy the simple WIs

$$\begin{aligned} q^\alpha \widetilde{\Gamma}^{amn}_{\alpha\mu\nu} &= g f^{amn}\left[\Delta^{-1}_{\mu\nu}(k_1) - \Delta^{-1}_{\mu\nu}(k_2)\right], \\ q^\alpha \widetilde{\Gamma}^{anm}_{\alpha} &= igf^{amn}\left[D^{-1}(k_1) - D^{-1}(k_2)\right], \\ q^\alpha \widetilde{\Gamma}^{amnr}_{\alpha\mu\nu\rho} &= g f^{adr}\Gamma^{drm}_{\nu\rho\mu}(q+k_2,k_3,k_1) + \text{cp}, \\ q^\alpha \widetilde{\Gamma}^{amnr}_{\alpha\mu} &= g f^{aem}\Gamma^{enr}_{\mu}(q+k_1,k_2,k_3) + \text{cp}. \end{aligned} \quad (9)$$

where "cp" stands for "cyclic permutations" The implication of this for the truncation of the SDE are far-reaching. Indeed, using these WIs, it is elementary to show that [1]

$$\begin{aligned} q^\mu \sum_{i=1}^{2}(a_i)_{\mu\nu} = 0\,, \quad q^\mu \sum_{i=3}^{4}(a_i)_{\mu\nu} = 0\,, \\ q^\mu \sum_{i=5}^{6}(a_i)_{\mu\nu} = 0\,, \quad q^\mu \sum_{i=7}^{10}(a_i)_{\mu\nu} = 0\,. \end{aligned} \quad (10)$$

Evidently, the SDE is composed of "one-loop" and "two-loop" dressed blocks that are *individually transverse*, exactly as happened in the Abelian case. In fact, the resulting pattern is even more spectacular: the gluon and ghost diagrams form separate transverse blocks!

4. Gluon mass generation: the big picture

Even though the gluon is massless at the level of the fundamental QCD Lagrangian, and remains massless to all order in perturbation theory, the non-perturbative QCD dynamics generate an effective, momentum-dependent mass, without affecting the local $SU(3)_c$ invariance, which remains intact [4].

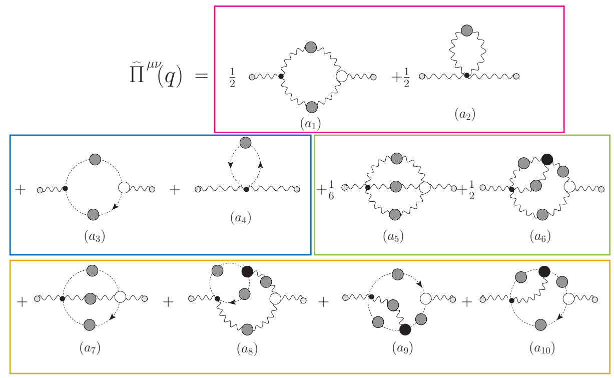

Figure 4. The SDE for the gluon self-energy in the PT-BFM framework.

The gluon mass generation is a purely non-perturbative effect. At the level of the SDEs the generation of such a mass is associated with the existence of infrared finite solutions for the gluon propagator [4,9], i.e. solutions with $\Delta^{-1}(0) > 0$. Such solutions may be fitted by "massive" propagators of the form $\Delta^{-1}(q^2) = q^2 + m^2(q^2)$; $m^2(q^2)$ is not "hard", but depends non-trivially on the momentum transfer q^2. In addition, one obtains the non-perturbative generalization of $\alpha(q^2)$, the QCD running coupling (effective charge), in the form [4]

$$\alpha^{-1}(q^2) = b\ln\left(\frac{q^2 + 4\,m^2(q^2)}{\Lambda^2}\right). \quad (11)$$

The $m^2(q^2)$ in the argument of the logarithm tames the Landau pole, and $\alpha(q^2)$ freezes at a finite value in the IR, namely $\alpha^{-1}(0) = b\ln(4m^2(0)/\Lambda^2)$. Moreover, the gluon mass forces $F(q^2)$ to stay IR-finite (non-enhanced) [9,10].

(i) The Schwinger mechanism

In order to obtain massive solutions *gauge-invariantly*, it is necessary to invoke the well-known Schwinger mechanism [11]: if, for some reason, $\mathbf{\Pi}(q^2)$ acquires a pole at zero momentum transfer, then the vector meson becomes massive, even if the gauge symmetry forbids a mass at the level of the fundamental Lagrangian. Indeed, it is clear that if the vacuum polarization $\mathbf{\Pi}(q^2)$ has a pole at $q^2 = 0$ with positive residue μ^2, i.e. $\mathbf{\Pi}(q^2) = \mu^2/q^2$, then (in Eu-

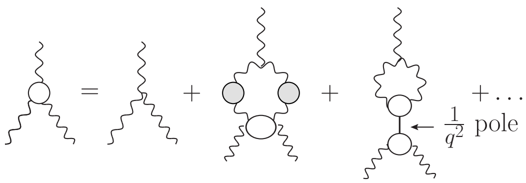

Figure 5. Vertex with non-perturbative massless excitations triggering the Schwinger mechanism.

clidean space) $\Delta^{-1}(q^2) = q^2 + \mu^2$. Thus, the vector meson becomes massive, $\Delta^{-1}(0) = \mu^2$, even though it is massless in the absence of interactions ($g = 0$). There is *no* physical principle which would preclude $\mathbf{\Pi}(q^2)$ from acquiring such a pole. In a *strongly-coupled* theory like QCD this may happen for purely dynamical reasons, since strong binding may generate zero-mass bound-state excitations [12]. The latter act *like* dynamical Nambu-Goldstone bosons, in the sense that they are massless, composite, and *longitudinally coupled*; but, at the same time, they differ from Nambu-Goldstone bosons as far as their origin is concerned: they do *not* originate from the spontaneous breaking of any global symmetry [4].

As we will see in a moment, the exact way how the Schwinger mechanism is integrated into the SDE is through the form of the three-gluon vertex. In particular, one assumes that the vertex contains *dynamical poles* $\sim 1/q^2$ [see Fig.5], which will trigger the Schwinger mechanism when inserted into the SDE for the gluon self-energy (other vertices, e.g, the four-gluon vertex, may have similar poles).

(ii) Gluon mass and confinement

It has been occasionally argued that the concept of a massive gluon may be at odds with confinement, because a massive gauge field does not give rise to the long-range potential necessary for this latter phenomenon to occur. This is, however, not true: *the gluon mass does not obstruct confinement, it enables it!*

Of course, the exact way how this happens is very intricate, and is inextricably connected with the notion of a *quantum soliton*. A quantum soliton is a localized finite-energy configuration of gauge potentials arising from an effective action

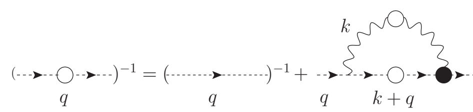

Figure 6. The SDE for the ghost propagator.

that summarizes quantum effects not present in the classical action; in our case, the quantum effect is a gauge-invariant dynamical gluon mass. Specifically, an effective low-energy field theory for describing the gluon mass is the gauged non-linear sigma model known as "massive gauge-invariant Yang-Mills" [13], with Lagrangian density $\mathcal{L}_m$

$$\mathcal{L}_m = \frac{1}{2}G^2_{\mu\nu} - m^2 \mathrm{Tr}\left[A_\mu - g^{-1}U(\theta)\partial_\mu U^{-1}(\theta)\right]^2 , \quad (12)$$

where $A_\mu = \frac{1}{2i}\sum_a \lambda_a A^a_\mu$, the λ_a are the SU(3) generators (with $\mathrm{Tr}\lambda_a\lambda_b = 2\delta_{ab}$), and the $N \times N$ unitary matrix $U(\theta) = \exp\left[i\frac{1}{2}\lambda_a\theta^a\right]$ describes the scalar fields θ_a. $\mathcal{L}_m$ is locally gauge-invariant under the combined gauge transformation

$$\begin{aligned} A'_\mu &= VA_\mu V^{-1} - g^{-1}\left[\partial_\mu V\right]V^{-1} , \\ U' &= U(\theta') = VU(\theta) , \end{aligned} \quad (13)$$

for any group matrix $V = \exp\left[i\frac{1}{2}\lambda_a\omega^a(x)\right]$, where $\omega^a(x)$ are the group parameters. $\mathcal{L}_m$ admits vortex solutions – these are the aforementioned quantum solitons – with a long-range pure gauge term in their potentials, which endows them with a topological quantum number corresponding to the center of the gauge group [Z_N for $SU(N)$], and is, in turn, responsible for quark confinement and gluon screening. Specifically, center vortices of thickness $\sim m^{-1}$, where m is the induced mass of the gluon, form a condensate because their entropy (per unit size) is larger than their action. This condensation furnishes an area law to the fundamental representation Wilson loop, thus confining quarks [13].

5. Solutions of the SDEs and comparison with the lattice results

We are now in a position to incorporate into the SDE truncation scheme based on the PT-BFM, described above, the concrete dynamics

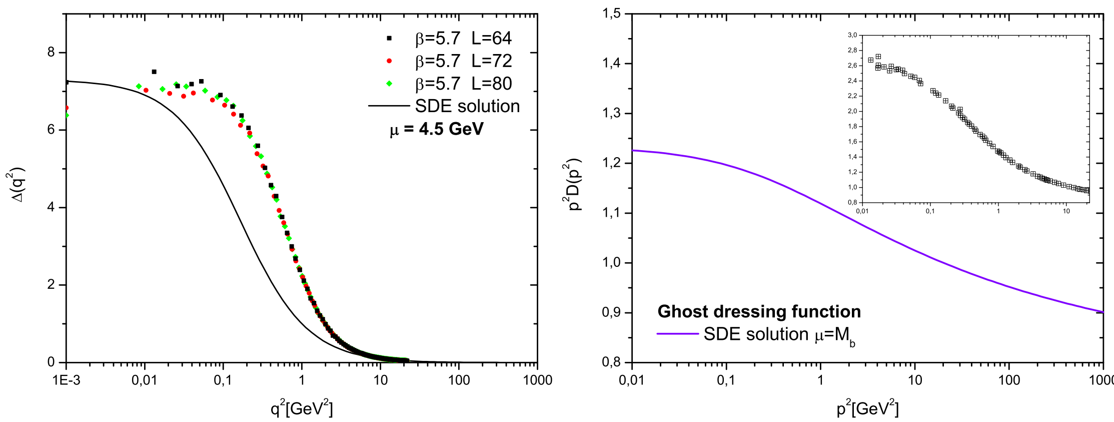

Figure 7. Comparison between SDE results of [9] and the lattice results of [3].

that will give rise to an infrared finite gluon propagator, signaling the dynamical generation of a gluon mass. To that end, we truncate (gauge-invariantly!) the SDE for the gluon propagator by keeping only the one-loop dressed contributions, i.e. the first two blocks of graphs in Fig.4. Then, we have [9]

$$\Delta^{-1}(q^2)\mathrm{P}_{\mu\nu}(q) = \frac{q^2\mathrm{P}_{\mu\nu}(q) + i\,\sum_{i=1}^{4}(a_i)_{\mu\nu}}{[1+G(q^2)]^2}. \quad (14)$$

We next express $\widetilde{\Gamma}_{\mu\alpha\beta}(q,k_1,k_2)$ and $\widetilde{\Gamma}_{\mu}(q,k_1,k_2)$ [appearing in (a_1) and (a_3) of Fig.4, respectively] as a function of the gluon and ghost self-energy, respectively, in such a way as to automatically satisfy the first two WIs of (9); failure to do so would invariably compromise the transversality of the answer. The Ansatz we will use for $\widetilde{\Gamma}_{\alpha\mu\nu}(q,k_1,k_2)$ is

$$\widetilde{\Gamma}_{\mu\alpha\beta} = \Gamma^{(0)}_{\mu\alpha\beta} + i\frac{q_\mu}{q^2}\left[\Pi_{\alpha\beta}(k_2) - \Pi_{\alpha\beta}(k_1)\right] , \quad (15)$$

and a similar expression for $\widetilde{\Gamma}_{\mu}$. The essential feature is the presence of massless pole terms, required for triggering the Schwinger mechanism. The resulting final expression for the SDE of (14) is too lengthy to report here [9]. The main point is that, as desired, $\Delta^{-1}(0) > 0$, i.e. the gluon self-energy is infrared finite.

For the conventional ghost-gluon vertex Γ_μ, appearing in the ghost SDE of Fig.6, we use its tree-level expression, *i.e.*, $\Gamma_\mu = -p_\mu$; this is perfectly legitimate, since in the PT-BFM formalism the two ghost vertices, $\widetilde{\Gamma}_\mu$ and Γ_μ, are different. Finally, for $H_{\mu\nu}$ we use its tree-level value, $ig_{\mu\nu}$.

Then, we obtain (Euclidean space) [9,15]

$$\begin{aligned} F^{-1}(q^2) \; &= \; 1 - \lambda \int_k \left[1 - \frac{(q\cdot k)^2}{q^2k^2}\right] \Delta(k)\, D(q+k), \\ G(q^2) \; &= -\frac{\lambda}{3}\int_k \left[2 + \frac{(q\cdot k)^2}{k^2q^2}\right]\Delta(k) D(k+q), \\ L(q^2) \; &= -\frac{\lambda}{3}\int_k \left[1 - 4\frac{(q\cdot k)^2}{k^2q^2}\right]\Delta(k) D(k+q). \end{aligned} \quad (16)$$

In fact, there exists a powerful formal identity relating $F(q^2)$, $G(q^2)$, and $L(q^2)$, namely [14]

$$F^{-1}(q^2) = 1 + G(q^2) + L(q^2)\,. \quad (17)$$

In addition to its formal derivation [14], the above relation has been recently obtained at the level of the SDEs defining these three quantities [15]. Adding by parts Eqs.(16) we can verify that Eq.(17) is indeed satisfied under the approximations employed.

Eq.(17) merits further analysis. The origin of this equation is the BRST symmetry of the theory [14]; in that sense, Eq.(17) has the same nature as the STIs. Therefore, just as happens with the latter, (17) must not get deformed after renormalization [15]. Thus, denoting by Z_u the renormalization constant relating the bare and renormalized functions, $\Lambda_0^{\mu\nu}$ and $\Lambda^{\mu\nu}$, [see (8)] through

$$Z_u[g^{\mu\nu} + \Lambda_0^{\mu\nu}(q)] = g^{\mu\nu} + \Lambda^{\mu\nu}(q), \tag{18}$$

the requirement of non-deformation imposes the crucial condition $Z_u = Z_c$ [15]. Thus, $Z_c[1 + G_0(q^2,\Lambda^2)] = 1 + G(q^2,\mu^2)$ and $Z_c L_0(q^2,\Lambda^2) = L(q^2,\mu^2)$.

The solutions obtained form the above system of SDEs is shown in Fig.7, where it is compared with the corresponding lattice data [$L(q^2)$ is numerically suppressed and is not shown here]. Note that while there is good qualitative agreement with the lattice, there is a significant discrepancy (a factor of 2) in the intermediate region of momenta. This of course may not come as a surprise, given that the "two-loop" dressed part of the SDE for Δ has been omitted [the last two blocks in Fig.4]. Even though this omission has not introduced artifacts (since it was done gauge-invariantly), the terms left out are expected to modify precisely the intermediate region, given that both the IR and UV limits of the solutions are already captured by the terms considered here.

6. The effective charge of QCD

Due to the Abelian WIs satisfied by the PT-BFM Green's functions, the new $\widehat{\Delta}^{-1}(q^2)$ absorbs all the renormalization-group (RG) logarithms, exactly as happens in QED with the photon self-energy [4,5,6]. As a result, the renormalization constants of the gauge-coupling and of the PT gluon self-energy, defined as

$$g(\mu^2) = Z_g^{-1}(\mu^2) g_0,$$

$$\widehat{\Delta}(q^2,\mu^2) = \widehat{Z}_A^{-1}(\mu^2)\widehat{\Delta}_0(q^2), \tag{19}$$

where the "0" subscript indicates bare quantities, satisfy the QED-like relation $Z_g = \widehat{Z}_A^{-1/2}$. Thus, regardless of the renormalization prescription chosen, the product

$$\widehat{d}_0(q^2) = g_0^2\widehat{\Delta}_0(q^2) = g^2(\mu^2)\widehat{\Delta}(q^2,\mu^2) = \widehat{d}(q^2) \tag{20}$$

retains the same form before and after renormalization, *i.e.*, it forms a RG-invariant (μ-independent) quantity [4]. Note that, by virtue of (7), we have that

$$\widehat{d}(q^2) = g^2(\mu^2)\frac{\Delta(q^2,\mu^2)}{[1+G(q^2,\mu^2)]^2}. \tag{21}$$

To see how beautifully the μ-independence of $\widehat{d}(q^2)$ is captured by the solutions of the SDEs we study, note in Fig.8 the explicit μ-dependence of the individual ingredients entering into its definition. However, when they are put together according to the rhs of (21), the resulting $\widehat{d}(q^2)$ is practically μ-independent!

The next step is to extract out of $\widehat{d}(q^2)$ a *dimensionless* quantity, that would correspond to the QCD *effective charge*. Perturbatively, i.e. for asymptotically large momenta, it is clear that the mass scale is saturated simply by q^2, the bare gluon propagator, and the effective charge is defined by pulling a $1/q^2$ out of the corresponding RG-invariant quantity, according to

$$\widehat{d}(q^2) = \frac{\overline{g}^2(q^2)}{q^2}, \tag{22}$$

where $\overline{g}^2(q^2)$ is the RG-invariant effective charge of QCD; at one-loop

$$\overline{g}^2(q^2) = \frac{g^2}{1 + bg^2\ln(q^2/\mu^2)} = \frac{1}{b\ln\left(q^2/\Lambda^2_{\text{QCD}}\right)}. \tag{23}$$

where Λ_{QCD} denotes an RG-invariant mass scale of a few hundred MeV.

Of course, given that in the IR the gluon propagator becomes effectively massive, *particular care is needed in deciding exactly what combination of mass-scales ought to be pulled out.* It would certainly be unwise, for example, to continue defining the effective charge by forcing out just a factor of $1/q^2$; such a procedure would furnish (trivially) a completely unphysical coupling, vanishing

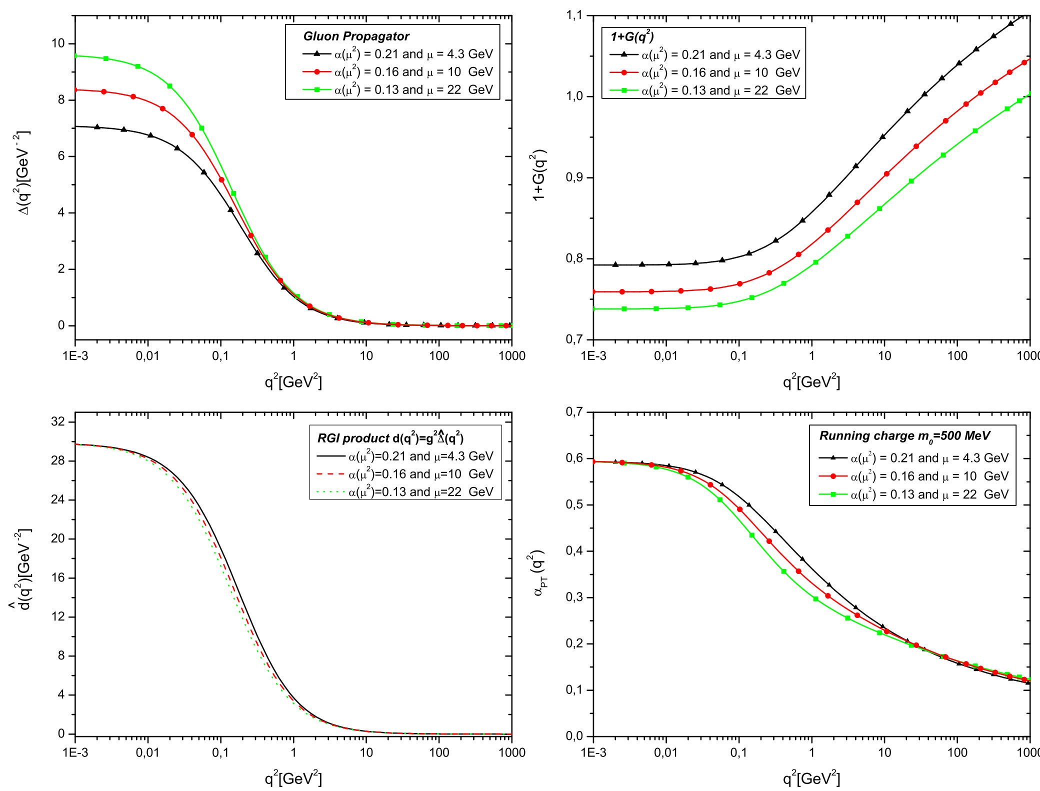

Figure 8. *Top left*: $\Delta(q^2)$ in the LG; *Top right*: the $1+G(q^2)$; *Bottom left*: the RG-invariant dimensionful $\widehat{d}(q^2)$; *Bottom right*: The $\alpha_{\rm PT}(q^2)$ obtained from $\widehat{d}(q^2)$, using (21) and (25), with $m_0 = 500\,\text{MeV}$.

in the deep infrared, where QCD is supposed to be strongly coupled ! The correct procedure, instead, is to factor out a "massive" propagator, of the form $[q^2 + m^2(q^2)]^{-1}$, i.e. one must set [4,15]

$$\widehat{d}(q^2) = \frac{\overline{g}^2(q^2)}{q^2 + m^2(q^2)}\,, \tag{24}$$

where $m^2(q^2)$ is a momentum-dependent gluon mass. Clearly, for $q^2 \gg m^2(q^2)$ the expression on the rhs of (24) goes over to that of (22). The effective charge, $\alpha_{\rm PT}(q^2) = \overline{g}^2(q^2)/4\pi$, obtained from the $\widehat{d}(q^2)$ after using (24), is also shown in Fig.8. We have assumed "power-law" running of the mass,

$$m^2(q^2) = \frac{m_0^4}{q^2 + m_0^2}\,, \tag{25}$$

consistent with various independent studies [16].

Let us finally comment on some important conceptual issues. As already mentioned, the "freezing" of the coupling is a direct consequence of the appearance of a mass in the RG logarithm. This fundamental property of the strong coupling may be reformulated in terms of what in the language of the *effective field theories* is referred to as "decoupling": at energies sufficiently inferior to their masses, the particles appearing in the loops (in this case the gauge bosons) cease to contribute to the "running" of the coupling. Note, however, a crucial point, which has been the source of considerable confusion in the recent literature: the "decoupling", as described above, is *not* a synonym for non-interactive! In the electroweak sector, for example, such a "decoupling" takes place indeed, since below the mass of the W the gauge boson loops do not contribute to the running. However, this is by no means the same as saying that the theory is non-interactive (for one thing, the β decay still takes place.) Another central point is that when the QCD charge is constant (non-vanishing!) in the infrared (and the quark masses are ignored), QCD becomes conformally invariant, and the AdS/CFT correspondence becomes applicable [17]. In conclusion: *The gluon mass keeps QCD strongly coupled, and with the "conformal window" open!*

6.1. A case of SDE-lattice synergy: The Kugo-Ojima function.

The Kugo-Ojima (KO) scenario [18] claims to establish a highly non-trivial link between confinement and the infrared behavior of the ghost dressing function $F(q^2)$. In a nutshell, a sufficient condition for the realization of the KO confinement ("quartet") mechanism is that a certain correlation function, $u(q^2)$, defined as

$$\int d^4x \; \mathrm{e}^{-iq\cdot(x-y)}\langle T\left[\,(\mathcal{D}_\mu c)^m_x \,(f^{nrs}A^n_\nu \bar{c}^s)_y\,\right]\rangle = P_{\mu\nu}(q)\delta^{mn}u(q^2), \qquad (26)$$

should satisfy the condition $u(0) = -1$. Given that $u(0)$ is related to the infrared behavior of $F(q^2)$ through the identity $F^{-1}(0) = 1 + u(0)$, the KO confinement scenario predicts a divergent ghost dressing function, and vice-versa. As was already mentioned, however, the ghost dressing function is not enhanced on the lattice [2,3]. In addition, and perhaps not surprisingly, the lattice [19] finds no evidence of $u(0) = -1$ either: $u(q^2)$ saturates in the deep infrared around approximately -0.6.

Quite remarkably, even though their field-theoretic origin appears to be completely different, the KO function coincides (in the LG!) with the $G(q^2)$, introduced in (8), namely [14]

$$u(q^2) = G(q^2). \qquad (27)$$

One may then substitute into the SDE for $G(q^2)$, given in (16), the available lattice data on the gluon and ghost propagators (e.g. [3]), and obtain an *indirect* determination for $u(q^2)$ [20]. Note that from (16) one can verify explicitly [15] that $L(0) = 0$. The $u(q^2)$ so obtained may be then compared with the *direct* lattice determination of the same quantity [19], from its defining Eq.(26).

The outcome of this procedure is displayed in the left panel of Fig.9. Evidently, the coincidence between the result obtained from the combined (indirect) SDE-lattice appraoch and that of the pure (direct) lattice is rather good. Note in addition an important (and only recently appreciated) point: $u(q^2)$, and in particular $u(0)$, depend *explicitly* on the renormalization point μ, as one would expect, given that the KO function is *not* an intrinsically μ-independent quantity [unlike, e.g., $\widehat{d}(q^2)$]. The implications of this, and other facts, for the KO confinement mechanism (and approaches relying on it) are currently under intense scrutiny [21].

7. Conclusions

In this presentation we have outlined the salient features of the SDEs formulated within the PT-BFM framework, and have given several examples of the considerable potential offered by their interplay with the lattice simulations. Clearly several aspects need to be further investigated, e.g.
(i) the dependence of the infrared behavior on the gauge chosen (e.g., LG vs Feynman gauge);
(ii) study on the lattice the auxiliary (ghost) Green's functions appearing in the new SDE;
(iii) refinement of the SDE treatment to improve the agreement with lattice;
(iv) the possible connection between the SDEs and the Gribov horizon.
We hope that the ongoing effort of the lattice and SDE communities will soon shed light on these and many more questions.

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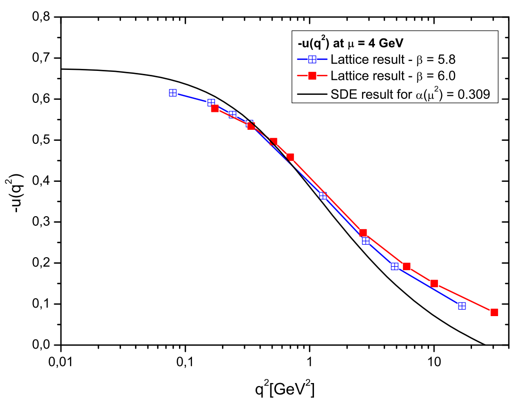

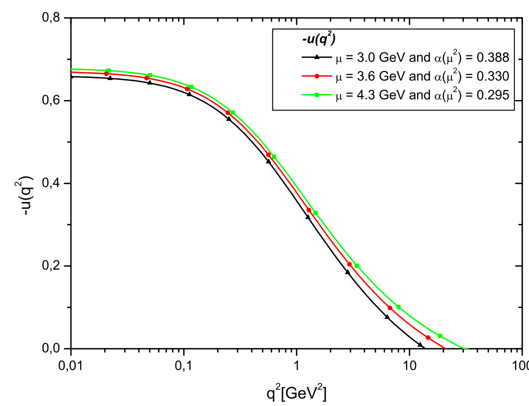

Figure 9. Left: $-u(q^2)$ from SDE-lattice [20] compared to [19]. Right: The μ-dependence of $-u(q^2)$.

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