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INTERPOLATION OF NON-ABELIAN LATTICE GAUGE FIELDS

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Abstract

We propose a method for interpolating non-abelian lattice gauge fields to the continuum, or to a finer lattice, which satisfies the properties of (i) transverse continuity, (ii) (lattice) rotation and translation covariance, (iii) gauge covariance, (iv) locality. These are the properties required for use in our earlier proposal for non-perturbative formulation and simulation of chiral gauge theories.

1 INTRODUCTION

Continuum interpolations of lattice gauge fields have been considered in the literature as a way of relating the topological aspects of continuum gauge theory to lattice gauge theory, and as an intermediate step in coupling lattice gauge fields to fermions in a manner that preserves chiral symmetry [1] [2] [3] [4] [5]. Recently, we have proposed a method for defining and simulating chiral gauge theories, with gauge fields on a coarse lattice coupled to fermions on a finer lattice via an interpolated gauge field [5]. We showed there, that in order for our method to succeed the interpolation procedure must satisfy the properties (i) – (iv) described below. The interpolation procedure we advocated was adapted from 't Hooft's recent suggestion [4] in the context of vector-like gauge theories. Unfortunately, this interpolation does not satisfy (i)*. Indeed we know of no procedure in the earlier literature which satisfies *all* the four properties we need. The purpose of this paper is to fill this gap. While it is possible to cure the problems of the interpolation of ref. [5], we will give an alternative procedure which is computationally more efficient. This procedure is very similar to the proposal of ref. [3], based on the earlier work of Lüscher [6]. However this earlier proposal does not satisfy properties (i) and (ii).

Given a four-dimensional non-abelian lattice gauge field configuration, $U_\mu(s)$, taking values in gauge group G, our procedure will give a continuum gauge field, $a_\mu(x)$. It is an interpolation in the sense that the parallel transport of $a_\mu(x)$ along the links of the lattice will equal the lattice link variables, $U_\mu(s)$. The interpolation, $a[U](x)$, has the following properties:

(i) Transverse Continuity: a_μ is differentiable *inside* each hypercube of the lattice and its components transverse to the normal of a boundary between adjacent hypercubes are continuous across the boundary.

(ii) Rotational and Translational Covariance: If T is a lattice (improper) rotation and/or translation of a lattice gauge field, U, then there exists a continuum gauge transformation, ω, such that

$$a[T[U]] = T[a[U]]^\omega. \tag{1}$$

(iii) Gauge Covariance: If Ω is a lattice gauge transformation of U, then there exists a continuum gauge transformation, ω, which interpolates Ω (the two agree on lattice vertices), such that

$$a[U^\Omega] = a^\omega[U]. \tag{2}$$

*We are grateful to R. Narayanan for bringing this to our attention.

(iv) Locality: In ref. [5] we defined a strong form of locality, namely that $a[U]$ in any hypercube be determined only by U on the bounding links. Here we shall only demand that the *gauge-invariant behavior* of $a[U]$ is determined locally. More precisely, we define locality to mean that (the trace of) any continuum Wilson loop obtained from $a[U]$ is determined only by U on links of hypercubes through which the Wilson loop passes. We have argued in ref. [5] that our proposal for formulating lattice chiral gauge theories is only sensitive to the gauge-invariant behavior of $a[U]$ in the continuum limit. We therefore expect that the present weaker form of locality is sufficient for $a[U]$ to be applied in that context.

A spacetime which is a four-dimensional torus is represented by periodic lattice gauge fields in a flat spacetime. The winding number of the interpolated continuum gauge field on the torus will be the winding number we assign to the lattice gauge field. This definition is essentially equivalent to Lüscher's [6]†. Our interpolation will not be periodic in general, since that is impossible for transversely continuous gauge fields with non-zero winding number. However for gauge fields with zero net winding number our interpolation will indeed be periodic. Only such fields are needed for our lattice formulation of chiral gauge theories, since we have shown that all physical effects can be obtained from the topologically trivial sector using cluster decomposition of the full theory [5]. For topologically non-trivial gauge fields the interpolation will be periodic in the $x_{1,2,3}$-directions, but will satisfy

$$a_{1,2,3}[x_1, x_2, x_3, L] = a^{\omega}_{1,2,3}[x_1, x_2, x_3, 0], \tag{3}$$

for some gauge transformation, ω, defined on the subspace $x_4 = L$, where L is the length of the torus in the x_4-direction. This represents a well-defined *connection* on the *torus*.

In section 2 we define our interpolation procedure to the continuum for the simplest example of non-abelian gauge group, $SU(2)$, proving properties (i) – (iv). In section 3, we work out the analog of our procedure for a compact $U(1)$ gauge field in two dimensions. This is the simplest example of most of the techniques in section 2. Section 4 contains some closing remarks.

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† In fact, the two definitions agree for "non-exceptional" gauge configurations as defined by Lüscher, for sufficiently small ϵ.

2 The Interpolation Procedure

For notational simplicity we will confine ourselves to the simplest non-abelian gauge group, $SU(2)$. Our procedure is straightforwardly generalized to any non-abelian group without $U(1)$ factors. We will work in units of the lattice spacing, so a lattice point s has integer valued components. We can uniquely write each lattice link variable in the form

$$U_\mu(s) = e^{i\vec{A}_\mu(s).\vec{\tau}}, \quad |\vec{A}_\mu(s)| < \pi, \tag{4}$$

where we are neglecting the measure-zero set of lattice fields where at least one of the link variables equals exactly -1. This assignment defines an obvious logarithm function for group elements different from -1,

$$A = -i \log U, \quad |\vec{A}| < \pi. \tag{5}$$

The basic strategy is to work separately in each hypercube of the lattice and choose a complete axial gauge fixing scheme [6] for the bounding link variables of the cell. The resulting link variables, $\overline{U}$, are interpolated in some smooth way, $\overline{a}[\overline{U}]$, to the whole cell, in the same gauge [3]. A gauge transformation is then applied to yield the final transversely continuous interpolation.

The lattice gauge field U will first be interpolated into lower dimensional sublattices, working up to 4-dimensional interpolation. This will help to ensure transverse continuity across hypercube boundaries. We will prove (i) – (iv) in each dimension.

2.1 1-dimensional interpolation

On the points along each lattice link, define

$$a_\mu(s + t\hat{\mu}) = A_\mu(s), \quad 0 \le t < 1. \tag{6}$$

The only nontrivial property to check is gauge covariance, (iii). Under a lattice gauge transformation, Ω, $U^\Omega_\mu(s) = \Omega(s)^{-1}U_\mu(s)\Omega(s+\hat{\mu}) \equiv e^{iA'}$ we can define a continuum gauge transformation,

$$\omega(s + t\hat{\mu}) = e^{-itA_\mu(s)}\Omega(s)e^{itA'_\mu(s)}, \tag{7}$$

which satisfies

$$a^\omega_\mu(s + t\hat{\mu}) = A'_\mu(s). \tag{8}$$

2.2 2-dimensional interpolation

Consider a $\mu - \nu$-oriented plane in the lattice, $\mu < \nu$. In each plaquette, $s_\mu \leq x_\mu \leq s_\mu + 1$, $s_\nu \leq x_\nu \leq s_\nu + 1$, we first define,

$$\overline{a}_\mu(x) = -i(x_\nu - s_\nu) \log U_{\mu\nu}(s), \quad \overline{a}_\nu(x) = 0, \tag{9}$$

where $U_{\mu\nu}(s) = U_\nu(s)\, U_\mu(s+\hat{\nu})\, U_\nu^{-1}(s+\hat{\mu})\, U_\mu^{-1}(s)$ is the standard product of link variables around the plaquette.

This interpolation is not in general transversely continuous across plaquette boundaries. We will fix this up by making a gauge transformation, ω, on $\overline{a}$ so that the result agrees on the bounding links with the earlier 1-dimensional interpolation made there, $a = A$. As is easily verified, this demand actually determines ω on the bounding links,

$$\begin{aligned}
\omega(s + t\hat{\mu}) &= e^{itA_\mu(s)}, \\
\omega(s + \hat{\mu} + t\hat{\nu}) &= e^{iA_\mu(s)} e^{itA_\nu(s+\hat{\mu})}, \\
\omega(s + t\hat{\nu}) &= e^{itA_\nu(s)}, \\
\omega(s + t\hat{\mu} + \hat{\nu}) &= e^{-t \log U_{\mu\nu}(s)} e^{iA_\nu(s)} e^{itA_\mu(s+\hat{\nu})},
\end{aligned} \tag{10}$$

where $0 \leq t \leq 1$. This continuous function from the boundary of the plaquette to $SU(2)$ is continuously extendable to the whole plaquette because $SU(2)$ is simply connected (in contrast to gauge groups with compact U(1) factors, which we discuss in section 3). We choose an extension of ω to the whole plaquette as follows. Define $\omega = e^\phi$, where ϕ is the solution of the 2-dimensional Laplace equation, with $\phi = \log \omega$ on the plaquette boundary. This extension to the whole plaquette is only continuous if $\omega \neq -1$ on the plaquette boundary, since we need to stay away from the discontinuity of our log function on $SU(2)$. This restriction will hold in all but a measure-zero set of lattice gauge fields, which we can safely neglect.

The final step of the interpolation is to define

$$a^{(2)}_{\mu,\nu}(x) = \overline{a}^{\omega}_{\mu,\nu}(x), \tag{11}$$

which is transversely continuous because it agrees with the one-dimensional interpolation previously performed. The superscript (2) reminds us that this is 2-dimensional interpolation, to save confusion later. It is obvious that the interpolation in each plaquette is determined completely by the values of U on the bounding links, so (iv) holds.

Under a general lattice gauge transformation Ω, $\overline{a}[U^{\Omega}]$ is given by

$$\begin{aligned}\overline{a}_{\mu} &= -i(x_{\nu}-s_{\nu})\log(\Omega(s)^{-1}U_{\mu\nu}(s)\Omega(s)) \\ &= -i\Omega(s)^{-1}\{(x_{\nu}-s_{\nu})\log U_{\mu\nu}(s)\}\Omega(s), \\ \overline{a}_{\nu} &= 0, \end{aligned} \tag{12}$$

which is clearly gauge equivalent to $\overline{a}[U]$. Therefore $a^{(2)}[U^{\Omega}]$ and $a^{(2)}[U]$ are also gauge equivalent, verifying (iii).

Rotational covariance was only broken by the dependence of $\overline{a}$ on the global coordinate frame which determines the particular complete axial gauge that $\overline{a}$ satisfies. A different global coordinate frame obtained by a rotation would produce a different $\overline{a}$, but it is easy to explicitly construct a gauge transformation that relates it to the old $\overline{a}$. Thus the old and new $a^{(2)}$ are also gauge equivalent, so property (ii) holds.

2.3 3-dimensional interpolation

Now let us consider a 3-dimensional sublattice of the 4-dimensional lattice. Unlike 2-dimensions, in 3-dimensions (and higher), referring to the global coordinate system in constructing $\overline{a}$ will break rotational covariance. To remedy this we will depart from ref. [3] and introduce a *local* coordinate system in each cube of the sublattice which depends on the gauge-invariant behaviour of U on the bounding links of the cube, as follows. The origin of the local coordinates is defined to be that corner of the cube which has the minimum sum of $\mathrm{tr}[U_{\mu\nu}(s)+U_{\mu\nu}(s)^{\dagger}]$ over all plaquettes of the cube which touch the corner. To each direction pointing from the local origin to one of the neighbouring corners of the cube we associate $\mathrm{tr}[U_{\mu\nu}(s)+U_{\mu\nu}(s)^{\dagger}]$ of the plaquette of the cube orthogonal to it and containing the local origin. We arrange these three directions in ascending order of their associated $\mathrm{tr}[U_{\mu\nu}+U_{\mu\nu}(s)^{\dagger}]$ and label them as the local $\hat{1},\hat{2},\hat{3}$ respectively. The reader may wish to ignore rotational invariance on a first reading, and use only the global coordinates on the whole lattice.

We will define a complete axial gauge inside each cube as follows. First we define a gauge transformation at each corner of the cube,

$$\Omega[U](z) = U_1(0)^{z_1}U_2(z_1,0,0)^{z_2}U_3(z_1,z_2,0)^{z_3}, \quad z_{1,2,3}=0,1, \tag{13}$$

where we are referring to the *local* coordinate system. Then define the gauge transformed lattice field on the bounding links,

$$U = \overline{U}^{\Omega[U]}, \tag{14}$$

which satisfies

$$\overline{U}_3 = \overline{U}_2(z_3 = 0) = \overline{U}_1(z_2 = z_3 = 0) = 1. \tag{15}$$

Now we interpolate $\overline{U}$ (not U!) to each of the bounding plaquettes of the cube using the 2-dimensional interpolation, thereby defining a gauge field $\overline{a} \equiv a^{(2)}_{loc-3}[\overline{U}]$ on each bounding plaquette. An important point here is that the $a^{(2)}$ functional was defined in the previous subsection by referring to the *global* coordinate system, but here we take it to be defined with reference to the *local* cube coordinate frame, hence the subscript $loc - 3$. We smoothly extend $\overline{a}$ to the interior of the cube by defining $\overline{a}_\mu(x)$ to be the solution to the 2-dimensional Laplace equation on the cross-section of the cube with fixed x_μ.

As in 2-dimensions, the resulting $\overline{a}$ is not generally transversely continuous across boundaries of neighbouring cubes. The cure is to apply a gauge transformation ω which ensures that, on the faces of the cube, the final 3-dimensional interpolation agrees with the 2-dimensional interpolation of lattice planes performed in the last subsection, $a^{(2)}[U]$ (where the absence of the *loc* subscript means that this interpolation was done with reference to the global coordinate frame). This demand determines ω on the boundary of the cube. To see this note that on any cube face, $\overline{a} = a^{(2)}_{loc-3}[\overline{U}]$ is gauge equivalent to $a^{(2)}_{loc-3}[U]$ by 2-dimensional gauge covariance. Now $a^{(2)}_{loc-3}[U]$ and $a^{(2)}[U]$ differ, if at all, only because of the choice of coordinate frame referred to, different choices being related by an (improper) two-dimensional rotation. By 2-dimensional rotational invariance, $a^{(2)}_{loc-3}[U]$ and $a^{(2)}[U]$ are gauge equivalent. Thus, $a^{(2)}_{loc-3}[\overline{U}]$ and $a^{(2)}[U]$ are also gauge equivalent. The associated gauge transformation, ω, can be readily determined as the composition of the various transformations associated with the above lattice gauge transformation and coordinate frame rotation, as described in the previous subsection.

Now that we have seen that ω is defined on the cube boundary we note that it can be extended to the whole cube because the second homotopy class of $SU(2)$ is trivial (and the boundary of the cube is topologically the same as the 2-sphere). We will use the explicit construction that $\omega = e^\phi$ where ϕ is the solution of the 3-dimensional Laplace equation with boundary condition $\phi = \log \omega$ on the cube boundary. Again this extension will *not* be continuous if $\omega = -1$ anywhere on the cube boundary because the log function is discontinuous there, but this situation will only arise for a measure-zero set of lattice gauge fields which we will neglect.

The final step of the 3-dimensional interpolation is to define

$$a^{(3)}[U] = \overline{a}^{\omega}[\overline{U}]. \tag{16}$$

It only depends on the link variables, U, on the links bounding the cube, so locality, (iv), is satisfied.

Clearly, $\overline{a}[\overline{U}]$ is a rotationally covariant functional of U, because a rotation of U will induce a rotation of the *local* coordinate frame, the frame exclusively referred to in defining $\overline{a}$ in 3-dimensions. Since $a^{(3)}[U]$ is gauge equivalent to $\overline{a}[\overline{U}]$, it satisfies (ii).

Under a *general* lattice gauge transformation, Ω, the local coordinate frame is invariant since its choice was based on the gauge invariant behavior of U. One can easily check that on the links of the cube,

$$\overline{U^{\Omega}} = \Omega^{-1}(0)\overline{U}\Omega(0), \tag{17}$$

where '0' is the origin of local coordinates. As a result,

$$\overline{a}[\overline{U^{\Omega}}] = \Omega^{-1}(0)\overline{a}[U]\Omega(0), \tag{18}$$

which is a (constant) gauge transformation. It follows that $a^{(3)}[U^{\Omega}]$ is gauge equivalent to $a^{(3)}[U]$, verifying (iii).

2.4 4-dimensional interpolation

As in three dimensions we define a local coordinate system in each hypercube of the lattice. The origin of the local coordinates is defined to be that corner of the hypercube which has the minimum sum of $\mathrm{tr}[U_{\mu\nu}(s) + U_{\mu\nu}(s)^{\dagger}]$ over all plaquettes of the hypercube which touch the corner. To each direction pointing from the local origin to one of the neighbouring corners of the hypercube we associate the sum of $\mathrm{tr}[U_{\mu\nu}(s) + U_{\mu\nu}(s)^{\dagger}]$ over all plaquettes of the hypercube orthogonal to it and containing the local origin. We arrange these four directions in ascending order of their associated plaquette sums and label them as the local $\hat{1}, \hat{2}, \hat{3}, \hat{4}$ respectively.

We again define a lattice gauge transformation on the hypercube to put U into a complete axial gauge [6],

$$\Omega[U](z) = U_1(0)^{z_1} U_2(z_1, 0, 0, 0)^{z_2} U_3(z_1, z_2, 0, 0)^{z_3} U_4(z_1, z_2, z_3, 0)^{z_4}, \tag{19}$$

where $z_{\mu} = 0, 1$, and we are employing the local coordinate system. Then define

$$U = \overline{U}^{\Omega}, \tag{20}$$

which satisfies

$$\overline{U}_4 = \overline{U}_3(z_4 = 0) = \overline{U}_2(z_3 = z_4 = 0) = \overline{U}_1(z_2 = z_3 = z_4 = 0) = 1. \qquad (21)$$

Next, interpolate $\overline{U}$ to each of the bounding cubes of the hypercube using the 3-dimensional interpolation, thereby defining a gauge field $\overline{a} \equiv a^{(3)}_{loc-4}[\overline{U}]$ on each bounding cube. It is important to note that in the previous subsection the $a^{(3)}[U]$ functional was defined with some reference to a *global* coordinate system, when a gauge transformation was performed to ensure agreement with the global 2-dimensional interpolation. By $a^{(3)}_{loc-4}[U]$ we will denote the 3-dimensional interpolation performed when the global coordinate frame is replaced by the 4-dimensional *local* coordinates. We smoothly extend $\overline{a}$ to the interior of the hypercube by defining $\overline{a}_\mu(x)$ to be the solution to the 3-dimensional Laplace equation on the cross-section of the hypercube with fixed x_μ.

Again, the resulting $\overline{a}$ is not generally transversely continuous across boundaries of neighbouring hypercubes. We will *try* to correct this by applying a gauge transformation ω which ensures agreement with the interpolation performed in the previous subsection on the 3-dimensional sublattices of the whole lattice, $a^{(3)}[U]$. And again, this demand determines ω on the boundary of the hypercube: (a) $\overline{a} = a^{(3)}_{loc-4}[\overline{U}]$ is gauge equivalent to $a^{(3)}_{loc-4}[U]$ by 3-dimensional gauge covariance, and (b) $a^{(3)}_{loc-4}[U]$ refers to the local hypercube coordinate system while $a^{(3)}[U]$ refers to the global coordinates. The two frames are related by a rotation, so by 3-dimensional rotational covariance, $a^{(3)}_{loc-4}[U]$ and $a^{(3)}[U]$ are gauge equivalent. Therefore $\overline{a}$ and $a^{(3)}[U]$ are gauge equivalent on the bounding cubes, and the associated ω can be determined as the composition of the various gauge transformations discussed in the previous subsection, in this case associated with the lattice gauge transformation $\Omega[U]$ and the rotation of the local hypercube frame to the global frame.

What makes four dimensions special is that ω, thusfar defined only on the boundary of the hypercube, *cannot in general* be continuously extended to the interior of the hypercube. This is because the third homotopy group of $SU(2)$ is not trivial, but is isomorphic to the additive group of integers (and the hypercube boundary is topologically the three-sphere). (Indeed it is this fact that is responsible for the existence of instantons in four-dimensions.) Let us denote the winding number of ω: hypercube boundary $\rightarrow SU(2)$, by $N(s)$, where s is the global coordinate of the corner of the hypercube with minimal $s_1 + s_2 + s_3 + s_4$. It can be computed in global coordinates using

[6]

$$N(s) = -\frac{1}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\int_{\partial\mathrm{HC}} dS_\mu \mathrm{tr}(\omega^{-1}\partial_\nu\omega)(\omega^{-1}\partial_\alpha\omega)(\omega^{-1}\partial_\beta\omega), \tag{22}$$

where the integral is over the boundary of the hypercube and dS_μ is the boundary volume element with an outward-pointing normal direction associated to it. For a non-zero measure of lattice gauge fields, there will be hypercubes for which $N(s) \neq 0$, so that ω cannot be continuously extended into the hypercube interior.

We will proceed by using the following trick. We first choose an integer-valued solution to the equation

$$\sum_\mu N_\mu(s+\hat{\mu}) - N_\mu(s) = N(s), \tag{23}$$

expressed in global coordinates, and associate $N_\mu(s)$ to the cube orthogonal to $\hat{\mu}$, with s being the global coordinates of the cube corner with minimal $s_1 + s_2 + s_3 + s_4$. Before discussing the global existence and choice of such a solution, let us explain what to do with it. In each cube of the lattice we define a continuum gauge transformation

$$\tilde{\omega} \equiv \omega\sigma^{N_\mu(s)}, \tag{24}$$

where in *global* coordinates,

$$\sigma(s + t_1\hat{\alpha} + t_2\hat{\beta} + t_3\hat{\gamma}) \equiv \frac{(\sum_a \tau_a \cot(\pi t_a) + i\epsilon_{\mu\alpha\beta\gamma})^2}{\sum_a \cot^2(\pi t_a) + 1}, \tag{25}$$

and $\alpha < \beta < \gamma$ are the global frame directions orthogonal to $\hat{\mu}$, and $0 \leq t_a \leq 1$. Note that $\sigma = 1$ on the boundaries of each cube and wraps around $SU(2)$ once [‡]. The $\sigma^{N_\mu(s)}$ define a continuous map on the boundary of each hypercube, which we will denote by $\tilde{\sigma}$. It may be readily verified that it has a winding number of $\sum_\mu N_\mu(s) - N_\mu(s+\hat{\mu}) = -N(s)$. Therefore $\tilde{\omega}$ is also a continuous map on a hypercube boundary, which clearly has trivial winding, so it can be extended to the whole hypercube. We will choose this extension to be the one which minimizes the action

$$S = \mathrm{tr}\int d^4x \;\; \partial_\mu\tilde{\omega}^{-1}\partial_\mu\tilde{\omega} \tag{26}$$

[‡]This map was obtained by composing a map from the unit cube onto infinite 3-dimensional space followed by a map from this space onto $SU(2)$. The first map is obtained by 1-dimensional sterographic projection of the unit interval (thought of as a circle) onto the real line. The second map is obtained by the inverse of 3-dimensional stereographic projection (recall that $SU(2)$ is a 3-sphere).

in each hypercube.

Our final interpolation will be defined as

$$a[U] = \overline{a}^{\tilde{\omega}}[\overline{U}]. \tag{27}$$

Note that on the cubes bounding the hypercube,

$$a[U] = (a^{(3)}[U])^{\tilde{\sigma}}, \tag{28}$$

so $a[U]$ is transversely continuous across hypercube boundaries.

Now let us return to the question of the global existence and uniqueness of $N_\mu(s)$ on a toroidal lattice. We are representing this lattice by global coordinates $s : 0 \leq s_\mu \leq L$ where we will eventually wish to identify $s_\mu = L$ with $s_\mu = 0$. We will demand solutions to eq.(23) which satisfy the boundary conditions,

$$\begin{aligned} N_j(s_j = L) &= N_j(s_j = 0), \quad j = 1, 2, 3, \\ N_4(s_4 = L) &= N_4(s_4 = 0) + \delta_{0s_1}\delta_{0s_2}\delta_{0s_3} N[U], \end{aligned} \tag{29}$$

where

$$N[U] \equiv \sum_{\text{hypercubes}} N(s). \tag{30}$$

Thus N_4 is *not* always periodic in the $\hat{4}$-direction.

N_μ solutions are not unique because of the invariance of eq.(23) under the 'gauge transformation'

$$N_\mu \rightarrow N_\mu + \epsilon_{\mu\nu\alpha\beta}\{N_{\alpha\beta}(s + \hat{\nu}) - N_{\alpha\beta}(s)\}, \tag{31}$$

where $N_{\alpha\beta}(s)$ is any integer-valued function associated with each plaquette. In order to have a definite interpolation algorithm we will pick a 'gauge-fixing' scheme for the N_μ on the lattice. For example, choose a path passing through each lattice hypercube once, starting from the hypercube at $(0, 0, 0, L-1)$, and not crossing the boundaries $x_\mu = 0, L$. Demand that $N_\mu = 0$ on all cubes *not* intersected by the path, with the exception that $N_4(0, 0, 0, L) = N[U]$ (see eq.(29)). This effectively provides a complete gauge-fixing scheme, compatible with the boundary conditions on N_μ.

With our boundary conditions it is easy to see that unless $N[U] = 0$, the gauge field interpolation $a[U]$ is not fully periodic, but rather is periodic modulo a gauge transformation,

$$a_j(x_4 = L) = a_j^{\tilde{\sigma}}(x_4 = 0). \tag{32}$$

Now $\tilde{\sigma}$ winds $N[U]$ times around $SU(2)$ on the boundary $x_4 = L$. This is nothing but a boundary condition for a continuum gauge field representing an $SU(2)$ connection with winding number,

$$N[U] = -\frac{1}{16\pi^2} \int d^4x Tr[f_{\mu\nu}\ \tilde{f}_{\mu\nu}], \tag{33}$$

where $f_{\mu\nu}$ is the field strength of the continuum field a_μ. This definition of winding number is very closely related to that given by Lüscher [6] §. In the event that $N[U] = 0$ it is easy to see that $a[U]$ is $fully$ periodic. Only such gauge fields are required in our earlier proposal for formulating chiral gauge theories [5], as mentioned in the introduction.

As in three dimensions it is clear that $\overline{a}[\overline{U}]$ is a rotationally and translationally covariant functional of U. Since $a[U]$ is gauge equivalent to $\overline{a}[\overline{U}]$ it must also satisfy (ii). And also as in three dimensions the fact that we took U to its completely gauge-fixed form, $\overline{U}$, prior to interpolating, ensures that gauge covariance, (iii), holds.

While it is clear that $\overline{a}[\overline{U}]$ is determined only by the U on the bounding links of the hypercube, $a[U]$ is *not* a local functional of U in general because $\tilde{\omega}$ is determined *locally* in terms of the fields U *and* $N_\mu(s)$, but $N_\mu(s)$ depends $non-locally$ on U through eq. (23). Nevertheless property (iv) is satisfied because the trace of any Wilson loop determined by $a[U]$ can be broken up into a product of segments that are each contained in a single hypercube, of the form

$$\begin{aligned} P\ e^{i\int_{x_0}^{x_1} dx^\mu a_\mu[U](x)} &= \tilde{\omega}^{-1}(x_0)\ P\ e^{i\int_{x_0}^{x_1} dx^\mu \overline{a}_\mu[\overline{U}](x)}\ \tilde{\omega}(x_1) \\ &= \tilde{\sigma}^{-1}(x_0)\omega^{-1}(x_0)\ P\ e^{i\int_{x_0}^{x_1} dx^\mu \overline{a}_\mu}\ \omega(x_1)\tilde{\sigma}(x_1), \end{aligned} \tag{34}$$

where the first equality follows from $a[U] = \overline{a}^{\tilde{\omega}}$ and the second from the fact that $\tilde{\omega} = \omega\tilde{\sigma}$ on hypercube boundaries (where $x_{0,1}$ lie). In the trace of the product of such segments making up a Wilson loop, the $\tilde{\sigma}$ dependence cancels out. Because $\overline{a}$ and ω in any hypercube are determined only by U on the bounding links of the hypercube, (iv) follows.

3 The example of compact $U(1)$ in two dimensions

One may also wonder how to treat $U(1)$ factors of the gauge group when they are taken as compact¶. For this case, our construction breaks down in two

§In fact, the only difference is in the particular interpolation of ω used.

¶The non-compact case is much simpler, see [5] for an explicit 4-D interpolation.

dimensions because we cannot in general extend ω defined on the plaquette boundaries to the whole plaquette since $U(1)$ is not simply connected. In two-dimensions the problem is easily solved by constructing the analog of the $\tilde{\omega}$ map, needed in four dimensions for the non-abelian case. In three or four-dimensions we note that the compactness of $U(1)$ is only relevant in the continuum limit if the $U(1)$ is the result of spontaneous breaking of a non-abelian symmetry. Therefore one can handle this case by keeping the original non-abelian gauge group, and whatever matter fields lead to its spontaneous breakdown. The non-abelian lattice gauge fields are then to be treated by the methods of the previous section.

It is instructive to construct the explicit expression for the interpolation in compact QED in two dimensions, to illustrate some of the methods in section 2 in a simple setting, and to compare the method with previous proposals in the literature. According to the procedure of subsection 2.2, we get

$$\begin{aligned} \overline{a}_1(s+t_1\hat{1}+t_2\hat{2}) &= -i\, t_2 \log(U_{12}(s)) \\ \overline{a}_2(s+t_1\hat{1}+t_2\hat{2}) &= 0, \end{aligned} \tag{35}$$

while ω on the links is,

$$\begin{aligned} \omega(s+t_1\hat{1}) &= e^{it_1 A_1(s)}, \\ \omega(s+\hat{1}+t_2\hat{2}) &= e^{iA_1(s)+it_2 A_2(s+\hat{1})}, \\ \omega(s+t_2\hat{2}) &= e^{it_2 A_2(s)}, \\ \omega(s+t_1\hat{1}+\hat{2}) &= e^{-t_1 \log U_{12}(s)+iA_2(s)+it_1 A_1(s+\hat{2})}, \end{aligned} \tag{36}$$

Using the two dimensional analog of eq. (22),

$$N(s) = \frac{i}{2\pi}\epsilon_{\mu\nu}\int_{\partial \mathrm{P}} dS_\mu\, \omega^{-1}\partial_\nu \omega \tag{37}$$

we find,

$$N(s) = Int[\,\chi(s), 2\pi] \tag{38}$$

where $Int[, 2\pi]$ denotes the nearest integer part modulo 2π and

$$\chi(s) \equiv A_2(s) + A_1(s+\hat{2}) - A_2(s+\hat{1}) - A_1(s) \tag{39}$$

In general a non-measure zero set of lattice configurations will have $N(s) \neq 0$ and in this case ω cannot be extended smoothly to the interior of the plaquette. According to the rules of section 2.4, in order to proceed we must solve the integer equation (23) in the 'gauge' depicted in Fig. 1, where we take our original spacetime to be a torus. All the links that are not crossed by the path have the associated $N_\mu(s)$ set to zero (remember that $\hat{\mu}$ is *orthogonal* to the link in our notation). The $N_\mu(s)$ solution is then clearly unique (the explicit closed form expression is not very illuminating, and so is omitted). It is easy to check that the gauge transformation defined on each link by

$$\tilde{\sigma}(s + t_\nu \hat{\nu}) \equiv e^{i\, 2\pi\epsilon_{\mu\nu} N_\mu(s)\, t_\nu} \tag{40}$$

has winding number $-N(s)$ and that $\tilde{\omega}$ defined in (24) has zero winding. Thus it can be extended to the interior of the plaquette. It is easy to explicitly find the minimum of equation (26) in this case,

$$\tilde{\omega}(s + t_1 \hat{1} + t_2 \hat{2}) = e^{i\, \phi(t_1, t_2)}, \tag{41}$$

with

$$\begin{aligned} \phi(t_1, t_2) = & (A_1(s) - 2\pi N_2(s))\, t_1 + (A_2(s) + 2\pi N_1(s))\, t_2 \\ & + (A_2(s + \hat{1}) + 2\pi N_1(s + \hat{1}) - A_2(s) - 2\pi N_1(s))\, t_1 t_2 \end{aligned} \tag{42}$$

and the final expression for $\overline{a}^{\tilde{\omega}}(s + t_1 \hat{1} + t_2 \hat{2})$ is,

$$\begin{aligned} a_1 &= (1 - t_2)\, (A_1(s) - 2\pi N_2(s)) + t_2\, (A_1(s + \hat{2}) - 2\pi N_2(s + \hat{2})) \\ a_2 &= (1 - t_1)\, (A_2(s) + 2\pi N_1(s)) + t_1\, (A_2(s + \hat{1}) + 2\pi N_1(s + \hat{1})) \end{aligned} \tag{43}$$

It is important to point out that even though the interpolated gauge fields are non-local, due to the N_μ "fields", the field strength of (43) is local, and equal to $-i \log(U_{12}(s))$. Consequently, all gauge invariant quantities (which in QED_2 can be constructed from the field strength) are local. Also, as expected the interpolation is transversely continuous and covariant under lattice gauge transformations, rotations and translations.

This interpolation has important differences with previous interpolations in the literature. The interpolation in ref. [3] for compact QED_2 differs in that the authors directly interpolate ω instead of $\tilde{\omega}$, thus getting singular gauge fields whenever any $N(s)$ is non-zero. Such gauge fields are unsuitable for our formulation of chiral gauge theories [5]. On the other hand, Flume and Wyler in ref. [1] get a smooth interpolation, but at the expense of breaking *compact* gauge covariance. This is also the case in the last reference of [2].

4 Concluding Remarks

In ref. [5] we actually needed an interpolation of U to a *finer* lattice, rather than the continuum. This is easily accomplished by taking $\overline{a}$ to live on the links of the finer lattice, and $\tilde{\omega}$ to live on the vertices of the fine lattice and replacing the various continuum Laplace equations by lattice equations on the fine lattice. The interpolated link variables u_μ are then given by

$$u_\mu(x) = \tilde{\omega}^{-1}(x) e^{if\overline{a}_\mu(x)} \tilde{\omega}(x + f\hat{\mu}), \tag{44}$$

where f is the lattice spacing for the fine lattice. The integral formula for $N(s)$ in terms of ω can be replaced by the lattice equivalent, rounded to the nearest integer. The resulting interpolation will agree with the continuum interpolation in the limit $f \to 0$.

One of the standard tests of success for any scheme which claims to preserve chiral symmetries in the continuum limit is to look at what happens at exactly $g = 0$, where g is the gauge coupling. The exponential of the Wilson action for gauge fields becomes a δ-function which only permits lattice gauge fields which are pure gauge. The test is then to see if the fermions coupled to the gauge field become free chiral fermions at $g = 0$ in the continuum limit (see for example [7]). It is interesting to see how this works in our formulation of lattice chiral gauge theory [5]. In our interpolation procedure, it is straightforward to verify that in the special case $U = 1$ everywhere, the interpolation is just $a_\mu = 0$ everywhere. So by property (iii) of gauge covariance, when U is pure gauge, the interpolation is a transversely continuous gauge field which is pure gauge. In ref. [5] we showed that in our continuum limit the fermions are gauge-invariantly coupled to transversely continuous gauge field interpolations. In particular therefore, when the interpolation is pure gauge the fermions are free.

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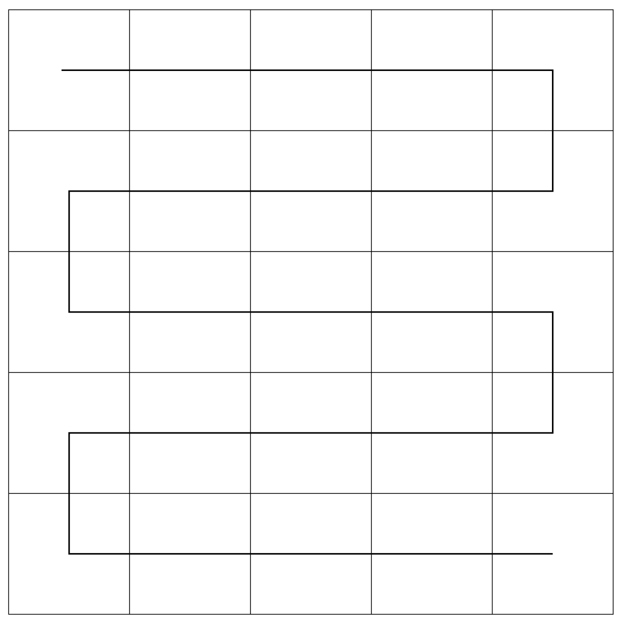

Figure 1: ‘Gauge choice’ used in solving (23) for compact QED_2. The N_μ corresponding to links that are not intersected by the path are zero.

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