

Locality properties of Neuberger's lattice Dirac operator

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Abstract

The gauge covariant lattice Dirac operator D which has recently been proposed by Neuberger satisfies the Ginsparg-Wilson relation and thus preserves chiral symmetry. The operator also avoids a doubling of fermion species, but its locality properties are not obvious. We now prove that D is local (with exponentially decaying tails) if the gauge field is sufficiently smooth at the scale of the cutoff. Further analytic and numerical studies moreover suggest that the locality of the operator is in fact guaranteed under far more general conditions.

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1. Introduction

Many technical complications in the standard formulation of lattice QCD have to do with the fact that chiral symmetry is violated at the scale of the cutoff. In particular, the quark masses are not protected from additive renormalizations and the leading lattice effects in physical amplitudes are proportional to the lattice spacing a rather than being of order a^2 .

Somewhat surprisingly it has recently turned out [2–6] that chiral symmetry can be preserved on the lattice, without fermion doubling, if the lattice Dirac operator D satisfies a certain algebraic relation,

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D, \quad (1.1)$$

originally due to Ginsparg and Wilson [1]. We shall not discuss the significance and consequences of this identity here, but refer the reader to the original papers quoted above and the rapidly growing literature on the subject [7–18]. A point which should be emphasized however is that the Ginsparg-Wilson relation only guarantees that the lattice theory has the same chiral symmetries as the continuum theory. Locality, the correct behaviour in the classical continuum limit and the absence of doubler modes are additional constraints which any decent lattice Dirac operator should satisfy.

Starting from the overlap formulation of chiral gauge theories, a relatively simple solution of the Ginsparg-Wilson relation has been found by Neuberger some time ago [4]. Explicitly it is given by

$$D = \frac{1}{a} \{1 - A(A^\dagger A)^{-1/2}\}, \quad A = 1 + s - a D_w, \quad (1.2)$$

where D_w denotes the standard Wilson-Dirac operator,

$$D_w = \frac{1}{2} \{ \gamma_\mu (\nabla_\mu^* + \nabla_\mu) - a \nabla_\mu^* \nabla_\mu \}, \quad (1.3)$$

and s is a real parameter in the range $|s| < 1$ which will be fixed later (cf. appendix A for unexplained notations). Neuberger's operator is manifestly gauge covariant and can be shown to have no doubler modes. One can also easily verify that it converges to the expected expression in the classical continuum limit, up to a finite normalization constant, but the requirement of locality is not obviously fulfilled. Evidently it is very important to check that the operator is local because the universality of the continuum limit depends on this fundamental property.

Before we begin with the detailed discussion of Neuberger's operator it may be helpful to state what precisely is meant if we say that D is local. Strict locality would imply that the non-zero contributions to the sum

$$D\psi(x) = a^4 \sum_y D(x,y)\psi(y) \tag{1.4}$$

come from the points y in a finite neighbourhood of x . Moreover the kernel $D(x,y)$ should only depend on the gauge field variables residing near x . From eqs. (1.2) and (1.3) it is obvious, however, that Neuberger's operator is not local in this restricted sense. A more general definition of locality is hence adopted here, where the kernel is allowed to have exponentially decaying tails at large distances. As long as the rate of decay can be shown to be proportional to the cutoff $1/a$, the sum in eq. (1.4) will be completely dominated by the contributions from a bounded region around x with a fixed diameter in lattice units. In particular, from the point of view of the continuum limit there is little doubt that this kind of locality is as good as strict locality.

It is clear that a non-locality of the Neuberger operator can only arise from the inverse square root of $A^\dagger A$ in eq. (1.2). Most of the time we shall thus be concerned with the properties of this operator. In section 2 we first consider the case where $A^\dagger A$ is bounded from below by a positive constant. Expanding the inverse square root of $A^\dagger A$ in a series of Legendre polynomials, the locality of D may then be proved straightforwardly. Moreover, the required lower bound on the spectrum of $A^\dagger A$ can be established rigorously if the gauge field is sufficiently smooth at the scale of the cutoff and Neuberger's operator is hence guaranteed to be local for all these fields.

One might expect that D becomes increasingly non-local when $A^\dagger A$ develops a zero mode. We briefly examine this question in section 2, using series expansions, and find that this is actually not so in general. The numerical studies reported in section 3 confirm this and they also provide a realistic estimate of the localization range of the operator at the gauge couplings of interest. All these results fit into a simple picture which suggests that Neuberger's operator (with an appropriate choice of the parameter s) is local for all statistically relevant gauge field configurations.

2. Spectrum of $A^\dagger A$ and locality of D

In this section some rigorous results are established which show that D is local (in the sense explained above) under certain conditions. Along the way we shall find that the locality of D is closely related to the spectral properties of $A^\dagger A$. This leads to interesting qualitative insights which allow us to be more confident about the interpretation of our numerical studies.

2.1 Series expansion of $(A^\dagger A)^{-1/2}$

Following a suggestion of Bunk [19] the inverse square root of $A^\dagger A$ may be expanded in a series of Legendre polynomials. The expansion is suitable for numerical application, but here we use it as a theoretical tool to discuss the locality properties of the Neuberger operator. In the following lines the detailed form of $A^\dagger A$ does not matter. To ensure the convergence of the Legendre expansion we however assume that the bounds †

$$u \leq A^\dagger A \leq v \tag{2.1}$$

hold for some strictly positive constants $u < v$. Whether this is the case for a given gauge field is a separate issue which will be addressed later.

The Legendre polynomials $P_k(z)$ may be defined through the generating function

$$(1 - 2tz + t^2)^{-1/2} = \sum_{k=0}^{\infty} t^k P_k(z). \tag{2.2}$$

Usually z is taken to be a number, but eq. (2.2) remains meaningful if we substitute

$$z = (v + u - 2A^\dagger A)/(v - u). \tag{2.3}$$

It is easy to show that this operator has norm less than or equal to 1. In particular, using the operator calculus and the well-known properties of the Legendre polynomials (as quoted in ref. [20], for example), this implies

$$\|P_k(z)\| \leq 1. \tag{2.4}$$

The expansion (2.2) is hence norm convergent for all t satisfying $|t| < 1$.

† Here and below an inequality between operators stands for the corresponding inequality between the expectation values of the operators in arbitrary normalizable states

We now introduce a parameter θ through

$$\cosh \theta = (v + u)/(v - u), \quad \theta > 0, \quad (2.5)$$

and set $t = e^{-\theta}$. Eq. (2.2) then assumes the form

$$(A^\dagger A)^{-1/2} = \kappa \sum_{k=0}^{\infty} t^k P_k(z), \quad \kappa = \{4t/(v - u)\}^{1/2}, \quad (2.6)$$

since $1 - 2tz + t^2$ is proportional to $A^\dagger A$ for this choice of t .

2.2 Legendre expansion and locality of D

We now show that the convergence of the Legendre expansion (2.6) and the strict locality of A imply that D is local (with exponentially decaying tails). The lattice is here taken to be infinitely extended in all directions. In view of eq. (1.2) it suffices to establish the locality of the inverse square root of $A^\dagger A$.

The kernel $G(x, y)$ which is associated with this operator,

$$(A^\dagger A)^{-1/2} \psi(x) = a^4 \sum_y G(x, y) \psi(y), \quad (2.7)$$

is a matrix acting on the Dirac and colour indices of the fermion field at the point y . If we define the kernels $G_k(x, y)$ representing the operators $P_k(z)$ in the same way, we have

$$G(x, y) = \kappa \sum_{k=0}^{\infty} t^k G_k(x, y). \quad (2.8)$$

It is easy to show that the norm convergence of the Legendre expansion implies the absolute convergence of this series for all points x and y . Actually, from eq. (2.4) one infers that

$$a^4 \|G_k(x, y)\| \leq 1 \quad \text{for all } k, x, y, \quad (2.9)$$

where the norm here is the matrix norm in Dirac and colour space.

We now note that $G_k(x, y)$ vanishes unless x and y are sufficiently close to each other, because $P_k(z)$ is a polynomial in $A^\dagger A$ and A is a combination of nearest-neighbour difference operators. If we introduce the ‘‘taxi driver distance’’

$$\|x - y\|_1 = \sum_{\mu} |x_{\mu} - y_{\mu}|, \quad (2.10)$$

the precise statement is that

$$G_k(x, y) = 0 \quad \text{for all } k < \|x - y\|_1 / 2a. \quad (2.11)$$

Restricting the sum (2.8) to the non-zero terms and recalling eq. (2.9), the bound

$$a^4 \|G(x, y)\| \leq \frac{\kappa}{1-t} \exp\{-\theta \|x - y\|_1 / 2a\} \quad (2.12)$$

is thus obtained. In particular, the kernel is exponentially decaying at large distances with a rate proportional to the cutoff $1/a$. Moreover its dependence on the gauge field is local in a similar way, i.e. up to exponentially small tails.

A technical detail we wish to emphasize is that the localization range $2a/\theta$ and the proportionality constant in eq. (2.12) only depend on the bounds u and v . As long as these can be chosen uniformly in the gauge field, the Neuberger operator is guaranteed to behave essentially as a strictly local operator. The differentiability of the operator with respect to the gauge field can also be proved under these conditions (appendix B).

2.3 Bounds on $A^\dagger A$

The proof of the locality of D given above depends on the convergence of the Legendre expansion and thus on the validity of the bounds (2.1) for some positive constants $u < v$. The upper bound is easily seen to hold for any gauge field if we rewrite A in the form

$$A = 1 + s + \sum_{\mu} \left\{ \frac{1}{2}(1 - \gamma_{\mu})a\nabla_{\mu} - \frac{1}{2}(1 + \gamma_{\mu})a\nabla_{\mu}^* \right\}. \quad (2.13)$$

Using the triangle inequality it then follows that

$$\|A\| = \|A^\dagger\| \leq 8 \quad (2.14)$$

for $|s| < 1$ and $A^\dagger A$ is hence uniformly bounded from above.

As far as the lower bound is concerned it is clear that $A^\dagger A$ is non-negative, but one knows that the operator can have zero modes for some gauge field configurations. A uniform lower bound is hence excluded. The strict positivity of $A^\dagger A$ may however be established if the gauge field is sufficiently smooth at the scale of the cutoff. To make this more precise, let us suppose that

$$\|1 - U(p)\| \leq \epsilon \quad \text{for all plaquettes } p, \quad (2.15)$$

where $U(p)$ denotes the product of the gauge field variables around p and the norm is the matrix norm in colour space. For $s = 0$ the inequality

$$A^\dagger A \geq 1 - 30\epsilon \quad (2.16)$$

may then be proved (appendix C) and a similar bound may be deduced from this for general s by substituting $A = s + A|_{s=0}$ and using triangle inequalities. $A^\dagger A$ is hence uniformly bounded from below by a positive constant if ϵ is sufficiently small. In particular, the locality of D is guaranteed under these conditions.

2.4 Locality of D in the presence of near-zero modes of $A^\dagger A$

When the gauge field strength is not small it can happen that some of the eigenvalues of $A^\dagger A$ are very close to zero or even equal to zero. Since the exponent θ defined in subsect. 2.1 is proportional to $u^{1/2}$ at small u , one is tempted to conclude that the locality of D is lost in this situation. We now show that this is not so in general.

Let us consider the case where the spectrum of $A^\dagger A$ is contained an interval $[u, v]$ except for an isolated eigenvalue λ in the range

$$0 < \lambda < \frac{1}{2}u. \quad (2.17)$$

As before the constants $u < v$ are some fixed positive numbers while λ is allowed to become arbitrarily small. The projector on the associated eigenspace is given by

$$P = \oint \frac{dw}{2\pi i} (w - A^\dagger A)^{-1}, \quad (2.18)$$

where the integration contour is a circle in the complex plane centred at the origin with radius $\frac{3}{4}u$. We now first prove that P is local by noting that the operator in the square bracket on the right-hand side of the identity

$$(w - A^\dagger A)^{-1} = (w^* - A^\dagger A) [(w^* - A^\dagger A)(w - A^\dagger A)]^{-1} \quad (2.19)$$

has eigenvalues between $(\frac{1}{4}u)^2$ and $(v + \frac{3}{4}u)^2$. Its inverse may thus be expanded in a rapidly convergent series of Chebyshev polynomials and it follows from this that the kernel associated with P is local with exponentially decaying tails. It should be emphasized that this remains true even if λ approaches zero, which is a regular case in the above equations. In particular, the localization range of P is determined by u and v alone.

To establish the locality of D we write

$$(A^\dagger A)^{-1/2} = (A^\dagger A)^{-1/2}P + (A^\dagger A)^{-1/2}(1 - P) \quad (2.20)$$

and expand the terms on the right-hand side of this equation in Legendre polynomials. In particular,

$$(A^\dagger A)^{-1/2}(1 - P) = \kappa \sum_{k=0}^{\infty} t^k P_k(z)(1 - P), \quad (2.21)$$

where t , z and κ are as given in subsect. 2.1. The convergence of the series is guaranteed since $\|P_k(z)(1 - P)\| \leq 1$. Proceeding essentially as in subsect. 2.2, this implies the locality of the operator (2.21). For the Legendre expansion of the first term in eq. (2.20), the spectral bounds u, v should be replaced by \tilde{u}, \tilde{v} , where

$$\tilde{v} = 2\tilde{u}, \quad \tilde{u} < \lambda < \tilde{v}. \quad (2.22)$$

The associated expansion parameter t is independent of \tilde{u} and the localization range of the operator is hence the same for all values of λ . Note that the divergence of the normalization factor κ cancels when the inverse square root of $A^\dagger A$ is multiplied with A . In particular, Neuberger's operator remains finite and local in the limit $\lambda \rightarrow 0$.

2.5 Summary

Perhaps the most important result obtained in this section concerns the small field region where the bound (2.15) holds for some ϵ strictly less than $\frac{1}{30}$. Neuberger's operator is local in this case with exponentially decaying tails. Moreover the localization range is uniformly bounded from above by a constant depending on ϵ only.

In the large field region the situation appears to be more complicated and it could be that the locality of D cannot be guaranteed for all fields. Nevertheless we have been able to show that the presence of near-zero modes of $A^\dagger A$ does not by itself imply any non-locality. Our analytical investigations rather suggest that D remains local as long as the continuous spectrum of $A^\dagger A$ is separated from zero by a positive gap.

3. Numerical studies of Neuberger's operator

In numerical simulations of quenched lattice QCD, using the Wilson plaquette action, the representative gauge field configurations at the gauge couplings of interest have relatively large average plaquette values. As a consequence the locality of D cannot be guaranteed on the basis of the results of section 2 alone. The purpose of the numerical studies reported here is to obtain some direct evidence for (or against) the locality of D in this situation and to check whether the qualitative picture is as suggested by the theoretical analysis.

The lattices that we have considered are of size L in all directions with periodic boundary conditions. L/a has been set to 12 or 16. The gauge group is taken to be $SU(3)$, with three values of the bare coupling $g_0^2 = 6/\beta$ corresponding to $\beta = 6.0, 6.2$ and 6.4 . Following standard procedures, a representative ensemble of statistically independent gauge field configurations has been generated for each lattice. It should be emphasized that all results refer to quenched QCD. In the full theory the situation could be different, although there is currently no reason to expect this.

3.1 Localization range of D

Let us consider the source field

$$\eta_\alpha(x) = \begin{cases} 1 & \text{if } x = y \text{ and } \alpha = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

where y is some particular point on the lattice and α runs over the colour and Dirac indices of the field. We are then interested in the decay properties of

$$\psi(x) = A(A^\dagger A)^{-1/2}\eta(x) \quad (3.2)$$

at large distances $\|x - y\|_1$ [cf. eq. (2.10)]. It is implicitly understood here that the coordinate differences $x_\mu - y_\mu$ are taken modulo L so as to minimize the distance. In particular, the largest possible distance is $2L$.

In fig. 1 we plot the expectation value of the function

$$f(r) = \max \{ \|\psi(x)\| \mid \|x - y\|_1 = r \} \quad (3.3)$$

for various values of s . The norm $\|\psi(x)\|$ in this definition is the usual vector norm. To compute $\psi(x)$ we have used a Chebyshev approximation for the inverse square root of $A^\dagger A$, with coefficients adjusted so that a relative accuracy better than 10^{-9} is achieved [21,22]. One needs to know the extremal eigenvalues of $A^\dagger A$ for this, but

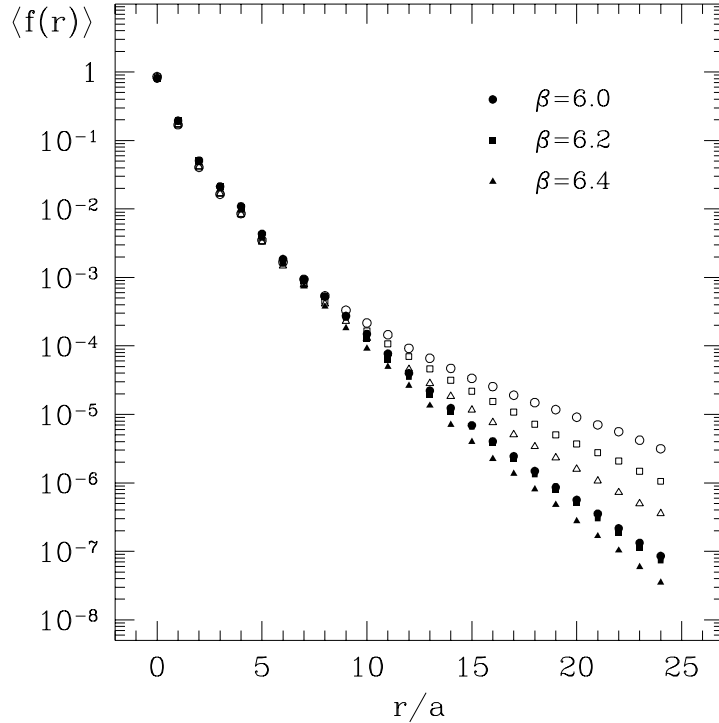


Fig. 1. Expectation value of $f(r)$ [eq. (3.3)] as a function of the distance r . Open symbols correspond to $s = 0$ while the filled symbols represent the data at $s = 0.4$ ($\beta \leq 6.2$) and $s = 0.2$ ($\beta = 6.4$) respectively. The statistical errors are not visible on the scale of this plot.

as discussed below they can be calculated reliably with a modest effort. A technical point we wish to emphasize is that the relatively high numerical precision quoted above is required to avoid systematic effects in the calculated values of $f(r)$ at large distances.

In all cases considered $\langle f(r) \rangle$ is rapidly decaying when the distance r increases. Finite-volume effects appear to be negligible here and significant differences between the curves at different β and s are only seen when r/a is larger than 10 or so. For $r/a > 13$ the data can be represented by a single exponential,

$$\langle f(r) \rangle \propto e^{-\nu r/a}, \quad (3.4)$$

with exponents ν as listed in table 1. One may be worried at this point that the fluctuations of $f(r)$ are large, but our experience is that the mean deviations are at

Table 1. Values of exponent ν [eq. (3.4)]

β	L/a	s	ν
6.0	12	0.0	0.28
	12	0.4	0.49
	12	0.6	0.45
6.2	12	0.0	0.35
	12	0.4	0.49
	12	0.6	0.42
6.4	16	0.0	0.40
	12	0.0	0.40
	12	0.2	0.53
	12	0.4	0.49

most a fraction of the average value and would thus be barely visible in fig. 1.

From the figure and the table it is evident that $\langle f(r) \rangle$ becomes nearly independent of β if s is chosen appropriately. The lowest curve in fig. 1 is also practically matched by the curve that one obtains in the free quark theory. In other words, as far as $\langle f(r) \rangle$ is concerned, the localization properties of D at $\beta \geq 6.0$ and with a good choice of s are roughly the same as in the free case.

That some tuning of s is required to preserve the localization range of D does not come as a total surprise, because the spectrum of the Wilson-Dirac operator D_w moves to the right in the complex plane when β is decreased. In particular, the critical bare mass m_c is shifted to $-0.68/a$, $-0.74/a$ and $-0.82/a$ at $\beta = 6.4$, 6.2 and 6.0 respectively [23]. Since

$$A = -a(D_w + m_0), \quad m_0 = -(1 + s)/a, \quad (3.5)$$

a positive value of s partly compensates for this and ensures that m_0 keeps away from m_c by an appreciable margin.

3.2 Spectrum of $A^\dagger A$

To make contact with the theoretical discussion of section 2 we now proceed to examine the distribution of the low-lying eigenvalues of $A^\dagger A$. For any given gauge field configuration these eigenvalues can be computed by minimizing the Ritz functional using a conjugate gradient algorithm. The method has previously been applied and is described in detail in refs. [24,25].

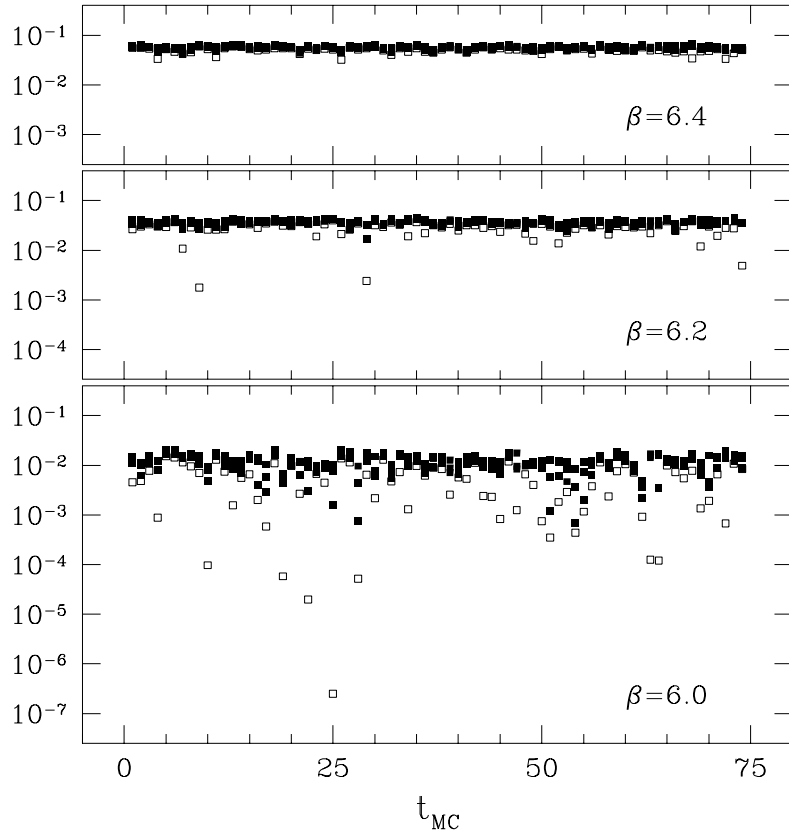


Fig. 2. Monte Carlo time evolution of the four lowest eigenvalues of $A^\dagger A$. The smallest eigenvalue is represented by an open square and the higher ones by filled squares. All data are from the 12^4 lattice with $s = 0$.

As shown by fig. 2 the spectrum of $A^\dagger A$ depends quite strongly on the gauge coupling. At $\beta = 6.4$ the lower end of the spectrum appears to be clearly separated from zero and $A^\dagger A$ thus satisfies the bounds (2.1). This is not so at $\beta = 6.2$ and 6.0 , where one has a non-zero probability to find eigenvalues that are orders of magnitude below the rest of the spectrum. Moreover the band of the ordinary modes is wider at these couplings.

When s is set to higher values (such as those quoted in fig. 1) the qualitative features of the spectrum do not change, but the level of the ordinary low-lying eigenvalues is raised by a factor 2 or so. The maximal eigenvalue, on the other hand, hardly changes and stays around 41 for all β and s that we have considered. So far we have only been able to analyse a limited number of gauge field configurations

on the larger lattice and thus cannot make a detailed statement about the volume dependence of the spectrum at this point. There is, however, a clear tendency that the probability for near-zero modes increases with the lattice size.

3.3 Localization properties of the near-zero modes

The minimization of the Ritz functional not only yields the low-lying eigenvalues of $A^\dagger A$ but also the corresponding eigenfunctions. In particular, the localization properties of the near-zero modes can be studied straightforwardly. A possible definition of the localization range of a given wave function is discussed in ref. [27]. Proceeding along these lines we have found that all near-zero modes are well localized with exponentially decaying tails. Moreover we have observed that the localization ranges shrink significantly when s is increased. All this completely agrees with the results previously reported by Edwards et al. [26] (see also ref. [27]).

For illustration let us consider an eigenvector $\phi(x)$, suitably normalized, with maximal magnitude $\|\phi(x)\|$ at some point $x = y$. The function

$$h(r) = \max \{ \|\phi(x)\| \mid \|x - y\|_1 = r \} \quad (3.6)$$

then provides an upper bound on the wave function at distance r from the centre of its localization region. A typical result for $h(r)$ is plotted in fig. 3. The associated eigenvalue is nearly two orders of magnitude below the band of the ordinary low-lying modes in this example.

In fig. 3 the function $f(r)$, calculated for the same gauge field configuration and with the same choice of y , is also shown. Comparing with $\langle f(r) \rangle$ one clearly sees that the presence of the near-zero mode does not affect the localization properties of D in any significant way.

3.4 Synthesis

Taken together the theoretical and numerical results reported in this paper suggest that the spectrum of $A^\dagger A$ is clearly separated from zero at large β , for all statistically relevant gauge field configurations. The locality of Neuberger's operator is rigorously guaranteed in this situation. It is our experience, however, that the theoretical bounds tend to over-estimate the localization range by a large factor. At $\beta = 6.4$ and $s = 0$, for example, the exponent $\theta/2 = 0.03$ which one obtains by inserting the numerically determined spectral bounds in eq. (2.5) is much smaller than the exponent $\nu = 0.4$ quoted in table 1.

At the lower values of β , near-zero modes develop and become increasingly frequent, but they are well localized (if s is chosen appropriately) and thus do not

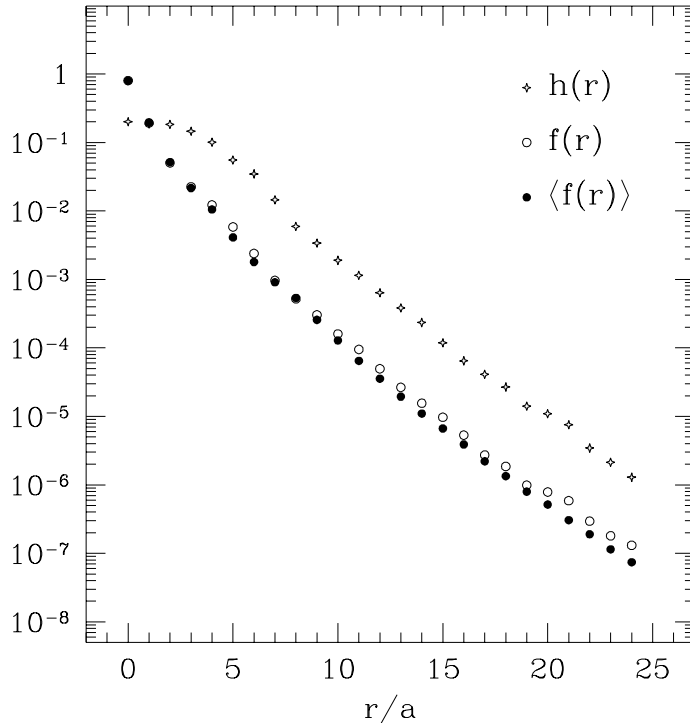


Fig. 3. Plot of the magnitude $h(r)$ of a typical near-zero mode at $\beta = 6.2$, $s = 0.4$ and $L/a = 12$, together with the function $f(r)$ and the corresponding ensemble average $\langle f(r) \rangle$.

destroy the locality of Neuberger’s operator. On the lattices that we have studied the near-zero modes are isolated from the rest of the spectrum. The observed locality properties are, therefore, completely in line with the theoretical discussion of section 2, although here again the analytical estimates of the localization radius are far too pessimistic.

An important point to note in this context is that the probability to find localized near-zero modes grows proportionally to the volume, because widely separated regions on the lattice basically behave as separate systems. Eventually there will be configurations with several near-zero modes and in the infinite volume limit these modes form a dense so-called pure point spectrum (cf. ref. [29], chapter VII). It seems unlikely that this statistical phenomenon has any relevance for the locality properties of Neuberger’s operator, but it is clearly desirable to check this by extending our studies to larger lattices.

4. Concluding remarks

At this point there is little doubt that Neuberger's operator is local at small gauge couplings g_0 , although a rigorous proof of the locality is only available if the gauge field is assumed to satisfy the smoothness condition (2.15) for some $\epsilon < \frac{1}{30}$. This constraint can, incidentally, be imposed on the system by choosing an appropriate gauge field action and one then obtains a lattice regularization of QCD which preserves the chiral and flavour symmetries without violating basic principles. As far as we know all other regularizations of QCD that have been considered to date break chiral symmetry and the folklore has been that this is in fact unavoidable.

For numerical simulations of lattice QCD with the standard gauge action, the locality of Neuberger's operator must be guaranteed for larger values of the gauge coupling as well. We have addressed this question in quenched QCD and did not find any indication that the locality is lost at the couplings of interest, provided the parameter s is chosen appropriately.

In principle one may now use Neuberger's operator to calculate the hadron spectrum etc., but this may still be somewhat premature, because there are other choices of A which lead to significantly smaller localization ranges, at least in the free case [8,28]. This may be important in practice, since one cannot afford to simulate very large lattices. Moreover, as has been pointed out by Niedermayer [28], the probability for near-zero modes may be very much suppressed for some of the proposed choices of A and this too could make the calculations easier.

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Appendix A

The notational conventions used in this paper are standard. We consider a four-dimensional hyper-cubic lattice with spacing a and variable size. If the lattice is finite we impose periodic boundary conditions although most results hold for other boundary conditions as well. The gauge field is represented by unitary matrices $U(x, \mu)$ where x runs through all lattice points and $\mu = 0, \dots, 3$ labels the space-time

directions. Dirac fields $\psi(x)$ carry a Dirac and a colour index as in the continuum theory. The gauge covariant forward and backward difference operators act on such fields according to

$$\nabla_\mu \psi(x) = \frac{1}{a} [U(x, \mu) \psi(x + a\hat{\mu}) - \psi(x)], \quad (\text{A.1})$$

$$\nabla_\mu^* \psi(x) = \frac{1}{a} [\psi(x) - U(x - a\hat{\mu}, \mu)^{-1} \psi(x - a\hat{\mu})], \quad (\text{A.2})$$

where $\hat{\mu}$ denotes the unit vector in direction μ . Since we are in euclidean space, the Dirac matrices can be taken to be hermitean,

$$\gamma_\mu^\dagger = \gamma_\mu, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad (\text{A.3})$$

and our conventions for γ_5 and $\sigma_{\mu\nu}$ are

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]. \quad (\text{A.4})$$

Repeated indices are always summed over unless stated otherwise.

Appendix B

The expansion in Legendre polynomials derived in sect. 2 is exponentially convergent. It is not obvious, however, that it may be differentiated with respect to the gauge field (or any other parameter), because the differentiated polynomials need not be uniformly bounded. In the following lines we establish a bound which excludes such an irregular behaviour.

We first derive an integral representation for the Legendre polynomials. Starting from eqs. (2.2) and (2.3) we have

$$P_k(z) = \oint \frac{dw}{2\pi i} w^{-k-1} (w^2 - 2wz + 1)^{-1/2}, \quad (\text{B.1})$$

where the integration runs along a circle in the complex plane centred at the origin. The radius r of the circle should be strictly less than 1 to avoid the singularities of the integrand. Because of the square root and since z is an operator, eq. (B.1) is

not easily differentiated. To overcome this difficulty we make use of a well-known identity to rewrite the integral in the form

$$P_k(z) = \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \oint \frac{dw}{2\pi i} w^{-k-1} (w^2 - 2wz + 1 + \sigma^2)^{-1}. \quad (\text{B.2})$$

Note that the denominator of the integrand can be factorized according to

$$w^2 - 2wz + 1 + \sigma^2 = (w - u^\dagger)(w - u), \quad u = z + i(1 + \sigma^2 - z^2)^{1/2}. \quad (\text{B.3})$$

Since $u^\dagger u = 1 + \sigma^2$ it is then immediately clear that

$$\|(w^2 - 2wz + 1 + \sigma^2)^{-1}\| \leq \{(1 + \sigma^2)^{1/2} - r\}^{-2}. \quad (\text{B.4})$$

In particular, the integral (B.2) is norm convergent.

Let us now assume that $A^\dagger A$ depends on some parameter τ in a differentiable manner such that $\dot{z} = \partial z / \partial \tau$ has finite norm. An upper bound on the derivative of the Legendre polynomials with respect to τ is then obtained by differentiating eq. (B.2) and applying eq. (B.4). The right-hand side of the resulting inequality

$$\|\dot{P}_k(z)\| \leq 4 \|\dot{z}\| r^{-k} \int_0^\infty \frac{d\sigma}{\pi} \{(1 + \sigma^2)^{1/2} - r\}^{-4} \quad (\text{B.5})$$

can be evaluated by substituting

$$(1 + \sigma^2)^{1/2} - 1 = (1 - r)\rho^2, \quad 0 \leq \rho < \infty. \quad (\text{B.6})$$

After some algebra one then ends up with the bound

$$\|\dot{P}_k(z)\| \leq \text{constant} \times \|\dot{z}\| r^{-k} (1 - r)^{-4}, \quad (\text{B.7})$$

where the constant is independent of k and r .

So far the radius r has not been specified apart from the requirement that it should be in the range $0 < r < 1$. We may now adjust the radius so that the factor $r^{-k}(1 - r)^{-4}$ is minimized. Using simple estimates this leads to the bound

$$\|\dot{P}_k(z)\| \leq \text{constant} \times \|\dot{z}\| (1 + k)^4. \quad (\text{B.8})$$

The differentiated series (2.6) is hence exponentially convergent with the same exponent as the original series. Similar estimations show that this is also true when higher-order differential operators are applied (each differentiation gives rise to an additional factor of $(1 + k)^2$ in the bound on the Legendre polynomials).

Appendix C

To establish eq. (2.16) we first expand the product $A^\dagger A$ using simple identities for the covariant difference operators and the Dirac matrices. As a result one gets a sum of terms,

$$A^\dagger A = 1 + \frac{1}{4} \sum_{\mu \neq \nu} \{B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu}\}, \quad (\text{C.1})$$

which can be treated separately (setting $s = 0$ has been essential here to ensure the cancellation of some non-trivial diagonal terms). Explicitly they are given by

$$B_{\mu\nu} = a^4 \nabla_\mu^* \nabla_\mu \nabla_\nu^* \nabla_\nu, \quad (\text{C.2})$$

$$C_{\mu\nu} = \frac{1}{2} i \sigma_{\mu\nu} a^2 [\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu], \quad (\text{C.3})$$

$$D_{\mu\nu} = -\gamma_\mu a^2 [\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu]. \quad (\text{C.4})$$

The commutator terms (C.3) and (C.4) are proportional to the field strength and should thus be of order ϵ . To prove this we note that

$$a^2 [\nabla_\mu, \nabla_\nu] \psi(x) = \{U(x, \mu)U(x + a\hat{\mu}, \nu) - U(x, \nu)U(x + a\hat{\nu}, \mu)\} \psi(x + a\hat{\mu} + a\hat{\nu}). \quad (\text{C.5})$$

The curly bracket in this equation is equal to a unitary matrix times $1 - U(p)$ for some plaquette p . Eq. (2.15) thus implies the bound

$$\|a^2 [\nabla_\mu, \nabla_\nu]\| \leq \epsilon \quad (\text{C.6})$$

and the same inequality also holds if one or both forward difference operators are replaced by backward difference operators. In particular,

$$\|C_{\mu\nu}\| \leq 2\epsilon \quad \text{and} \quad \|D_{\mu\nu}\| \leq 4\epsilon. \quad (\text{C.7})$$

To bound the first term, eq. (C.2), we rewrite it in the form

$$B_{\mu\nu} = a^4 \nabla_\mu^* \nabla_\nu^* \nabla_\nu \nabla_\mu - a^3 \nabla_\mu^* [\nabla_\mu, \nabla_\nu^* - \nabla_\nu], \quad (\text{C.8})$$

which shows that it is equal to a non-negative operator plus another operator with norm less than 4ϵ . Taken together these estimates imply that the right-hand side of eq. (C.1) is bounded from below by $1 - 30\epsilon$ which proves eq. (2.16).

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