N–quantum approach to quantum field theory at finite T and μ: the NJL model

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Abstract

We extend the N–quantum approach to quantum field theory to finite temperature (T) and chemical potential (μ) and apply it to the NJL model. In this approach the Heisenberg fields are expressed using the Haag expansion while temperature and chemical potential are introduced simultaneously through a generalized thermal Bogoliubov transformation. Known mean field results are recovered using only the first term in the Haag expansion. In addition, we find that at finite T and in the broken symmetry phase of the model the mean field approximation can not diagonalize the Hamiltonian. Inclusion of scalar and axial vector diquark channels in the $SU(2)_f \otimes SU(3)_c$ version of the model can lead to a lowering of the vacuum energy density. We discuss how to go beyond the mean field approximation by including higher order terms in the Haag expansion.

I. INTRODUCTION

The N–quantum approach (NQA) to quantum field theory developed by O.W. Greenberg [1,2] is a method to solve the operator equations of motion by expanding the Heisenberg fields as a complete set of normal–ordered and on–shell asymptotic (in– or out–) fields, including the asymptotic fields for bound states. This expansion is known as the Haag expansion [3]. Although the NQA is yet to be applied to gauge theories successful applications thus far indicate that the method can be very powerful in some cases (see [2] for a summary of the applications and references). In particular NQA is well suited to describe bound states.

Greenberg and collaborators applied the NQA to study dynamical symmetry breaking involving the condensates of composite fields in the NJL model [4] both without [5] and with [6] isospin. Their method differed from the conventional approach to the NJL model [7] in several distinct ways:

- The NQA uses retarded amplitudes (for a Haag expansion using in–fields) with all but one leg on–shell instead of off–shell Feynman amplitudes.

- It allows one to work directly with the fields and states of broken symmetry without any reference to those of the unbroken–symmetry vacuum.

- The explicit structure of the vacuum need not be specified.

- Bound state amplitudes are calculated directly rather than searching for bound state poles in scattering amplitudes.

In both [5] and [6] it was shown that the well known gap equation for the NJL model can be obtained by using only the first term in the Haag expansion in the operator equation of motion. Analytical expressions for amplitudes and masses for various bound states were found by going one order higher in the expansion and the resulting masses agreed with the original results [4] except for the mass of the vector bound state. The form factor of the Nambu–Goldstone boson was also examined by extending the Haag expansion to third order in the one–loop approximation [6].

Since the NJL model has been used extensively to study the consequences of chiral symmetry breaking in hot and dense matter [7], it is interesting to ask whether the NQA to the NJL model can be extended to finite T and μ. In this letter we shall demonstrate that the answer to this question is affirmative. We stress that the method presented here to extend the NQA to finite T and μ is quite general and is not restricted to the study of the NJL model.

Our strategy is based on the extension of the thermal field theory formalism of Umezawa and collaborators [8] by including a finite chemical potential, and has been motivated by the application of the NQA to the BCS theory of superconductivity [9]. The basic idea is to apply a generalized thermal Bogoliubov transformation to the creation and annihilation operators of the asymptotic fields in the Haag expansion accompanied by a thermal doubling of the Hilbert space. In this way temperature and chemical potential are introduced simultaneously through the coefficients of the transformation and are treated as parameters on an equal footing. Quantities of interest are then obtained through the algebra of thermal creation and annihilation operators without specifiying the structure of the thermal vacuum state.

When the method is applied to the NJL model we recover the known mean field results for number and scalar densities, total chemical potential, gap equation and vacuum energy density using only the first term in the Haag expansion. Thus NQA in the lowest order Haag expansion is equivalent to the mean field approximation usually employed in the studies of the NJL model at finite T and μ. Furthermore, in the same approximation, we find that the Hamiltonian can not be diagonalized at finite T and in the broken symmetry phase of the model due to particle–hole and anti–particle and anti–hole excitations. This seems to be a general result valid for all versions of the NJL model at finite T and μ.

In the following section we start by summarizing the pertinent results for the U(1) NJL model for $(T, \mu) = (0, 0)$ in the NQA. In Section III the NQA is extended to finite T and μ which is used to recover the known mean field results in Section IV. In the same section we present the off–diagonal Hamiltonian and show that it can only vanish in the $T = 0$ or in the Breit–Wigner phase of the model. We repeat the same excercise in Section V using the $SU(2)_f \otimes SU(3)_c$ NJL model including the scalar and axial vector color $\bar{3}$ diquark channels with their respective coupling constants. By demanding that the total Hamiltonian is diagonalized at $T = 0$ a simple relation between these coupling constants is found. This relation is then used to show how the inclusion of the diquark channels can lower the vacuum energy density of the many–body system. In the final section we summarize the results and outline the procedure for going beyond the mean field approximation.

II. NQA TO THE U(1) NJL MODEL: $(T, \mu) = (0, 0)$

We first consider the U(1) invariant version of the NJL model defined by the Lagrangian

$$\mathcal{L} = i\bar{\Psi}\slashed{\partial}\Psi + g_0\Big[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2\Big]. \tag{2.1}$$

The charge conjugation symmetric form of the corresponding field equation of motion and Hamiltonian in momentum space are

$$\slashed{q}\Psi(q) = -\frac{g_0}{2}\int d^4p_1 d^4p_2 d^4p_3 \, \delta^4(q - p_1 - p_2 - p_3)$$
$$\otimes\left\{\Big[[\bar{\Psi}(p_1), \Psi(p_2)]_-, \Psi(p_3)\Big]_+ - \Big[[\bar{\Psi}(p_1), \gamma_5\Psi(p_2)]_-, \gamma_5\Psi(p_3)\Big]_+\right\} \tag{2.2}$$

and

$$\mathcal{H} = (2\pi)^3 \int d^4\bar{p}_1 d^4\bar{p}_2 \, \delta^3(\vec{p}_1 + \vec{p}_2)e^{-(p_1^0+p_2^0)t}\frac{1}{4}\Big[\bar{\Psi}(p_1), \vec{\gamma}\cdot(\vec{p}_1 - \vec{p}_2)\Psi(p_2)\Big]_-$$
$$-(2\pi)^3 \int d^4\bar{p}_1 d^4\bar{p}_2 d^4\bar{p}_3 d^4\bar{p}_4 \, \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)e^{-(p_1^0+p_2^0+p_3^0+p_4^0)t}$$
$$\otimes\frac{g_0}{4}\left\{\Big[\bar{\Psi}(p_1), \Psi(p_2)\Big]_-\Big[\bar{\Psi}(p_3), \Psi(p_4)\Big]_- - \Big[\bar{\Psi}(p_1), \gamma_5\Psi(p_2)\Big]_-\Big[\bar{\Psi}(p_3), \gamma_5\Psi(p_4)\Big]_-\right\}, \tag{2.3}$$

where $\bar{\Psi}(p) = \Psi^\dagger(-p)\gamma_0$, $[A, B]_\pm = AB \pm BA$ and we shall adapt the notation $d^n\bar{p} \equiv d^np/(2\pi)^3$. When $(T, \mu) = (0, 0)$ there are no contributions from the exchange (Fock) terms in the mean field approximation in this model.

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For the model defined by Eq. (2.1) the Haag expansion for the Heisenberg field $\Psi(p)$ is given by [5],

$$
\begin{aligned}
\Psi(q) = {} & \Psi_{\text{IN}}(q)\delta_m(q) \\
& + \sum_k \int d^4p_1 d^4p_2 \, \delta^4(q - p_1 - p_2) f^{(k)}(p_1, p_2) : \Psi_{\text{IN}}(p_1)\delta_m(p_1)F^{(k)}_{\text{IN}}(p_2)\delta_{m_k}(p_2) : \\
& + \int d^4p_1 d^4p_2 \, \delta^4(q - p_1 - p_2) e(p_1, p_2) C^{-1} : \bar{\Psi}_{\text{IN}}(p_1)\delta_m(p_1)E_{\text{IN}}(p_2)\delta_{m_k}(p_2) : \\
& + \cdots .
\end{aligned}
\tag{2.4}
$$

Here Ψ_{IN} is the asymptotic in–field of Ψ constrained to be on the mass–shell by the delta function $\delta_m(p) \equiv \delta^4(p^2 - m^2)$. $C = i\gamma_2\gamma_0$ is the charge conjugation operator and the symbol $:\ :$ denotes normal ordering. The k summation runs over the $\bar{\Psi}\Psi$, $\bar{\Psi}i\gamma_5\Psi$, and $\bar{\Psi}\gamma_\mu\Psi$ bound states with corresponding on–shell in–fields $F^{(k)}_{\text{IN}}$ and amplitudes $f^{(k)}(p_1, p_2)$. Similarly $e(p_1, p_2)$ is the amplitude for the $\Psi\Psi$ bound state with an on–shell asymptotic field E_{IN}. The ellipses represent higher order contributions involving terms with three or more products of Ψ_{IN}, $\bar{\Psi}_{\text{IN}}$ and asymptotic bound state fields together with corresponding amplitudes. An example of the third order contribution is given in [6].

As shown in [5], inserting the first term in the above expansion into the operator equation of motion Eq. (2.2) yields the well known gap equation

$$
m = 4g_0 \int d^3\bar{p} \, \frac{m}{\omega_p}
\tag{2.5}
$$

with $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$. Note that the asymptotic mass m plays the role of the chiral gap. There are three solutions to this gap equation; $m = 0$ corresponding to the phase with unbroken chiral symmetry and $m = \pm m_0$ for the broken symmetry phase. The vacuum energy densities for the two phases are

$$
\frac{1}{V}\langle\mathcal{H}\rangle^{m=0} = -2g_0 \int d^3\bar{p}d^3\bar{q} - 2\int d^3\bar{p} \, |\vec{p}|
\tag{2.6}
$$

$$
\frac{1}{V}\langle\mathcal{H}\rangle^{m=\pm m_0} = -2g_0 \int d^3\bar{p}d^3\bar{q} - \int d^3\bar{p} \, \frac{2|\vec{p}|^2 + m^2}{\omega_p}
\tag{2.7}
$$

To facilitate comparison with known results for finite T and μ we shall use the non–covariant regularization scheme and cut off the three momentum integral at $|\vec{p}| = \Lambda$. Then, as is well known, dynamical symmetry breaking occcurs only for those values of the coupling constant satisfying the relation $\pi^2/\Lambda^2 < g_0$. For these values of g_0 the vacuum energy density of the broken symmetry phase is lower than that of the unbroken phase, indicating that the phase with chiral condensates is the energetically favored phase.

III. EXTENSION TO FINITE T AND μ

In this work we shall explore the consequences of using only the first term in Eq. (2.4) to describe the NJL model at finite T and μ. At first sight this seems to be a trivial approximation, but as we shall see it turns out not to be the case. Extension of the present work to include the second order terms will be discussed below.

The lowest order term in the Haag expansion for Ψ is [1]

$$\Psi_{\text{IN}}(q)\delta_m(q) = \frac{1}{(2\pi)^3}\frac{m}{\omega_q}\sum_{s=\pm}\left[b(\vec{q},s)u(\vec{q},s)\delta(q^0-\omega_q) + d^\dagger(-\vec{q},s)v(-\vec{q},s)\delta(q^0+\omega_q)\right]. \quad (3.1)$$

Because this term alone reproduces the gap equation Eq. (2.5) and diagonalizes the Hamiltonian, the operators b and d correspond to quasi–particle and quasi–anti–particle annihilation operators, respectively, which annihilate the interacting vacuum state at zero temperature and chemical potential $|\mathcal{G}(0,0)\rangle$,

$$b(\vec{q},s)|\mathcal{G}(0,0)\rangle = d(\vec{q},s)|\mathcal{G}(0,0)\rangle = 0. \quad (3.2)$$

Thus using the first order term in the Haag expansion is equivalent to subjecting the creation and annihilation operators in the Heisenberg field Ψ to a Bogoliubov transformation into quasi–particle and quasi–anti–particle basis.

However, the b and d operators do not annihilate the interacting vacuum state at finite T and μ denoted as $|\mathcal{G}(T,\mu)\rangle$. In order to construct operators that annihilate $|\mathcal{G}(T,\mu)\rangle$ we apply a generalized thermal Bogoliubov transformation to the b and d operators as well as to the annihilation operators corresponding to quasi–holes $\tilde{b}$ and quasi–anti–holes $\tilde{d}$ as follows

$$b(\vec{q},s) = \alpha_q B(\vec{q},s) - s\beta_q \tilde{B}^\dagger(-\vec{q},s) \quad (3.3)$$
$$\tilde{b}(\vec{q},s) = \alpha_q \tilde{B}(\vec{q},s) + s\beta_q B^\dagger(-\vec{q},s) \quad (3.4)$$
$$d(\vec{q},s) = \gamma_q D(\vec{q},s) - s\delta_q \tilde{D}^\dagger(-\vec{q},s) \quad (3.5)$$
$$\tilde{d}(\vec{q},s) = \gamma_q \tilde{D}(\vec{q},s) + s\delta_q D^\dagger(-\vec{q},s) \quad (3.6)$$

Here B and $\tilde{B}$ annihilate a quasi–particle and a quasi–hole at finite T and μ, respectively, and D and $\tilde{D}$ are the thermal annihilation operators for quasi–anti–particles and quasi–anti–holes, respectively. These operators annihilate the interacting thermal vacuum state *for each T and μ.*

$$B(\vec{q},s)|\mathcal{G}(T,\mu)\rangle = \tilde{B}(\vec{q},s)|\mathcal{G}(T,\mu)\rangle = D(\vec{q},s)|\mathcal{G}(T,\mu)\rangle = \tilde{D}(\vec{q},s)|\mathcal{G}(T,\mu)\rangle = 0 \quad (3.7)$$

In the NQA it is not necessary to specify the structure of $|\mathcal{G}(T,\mu)\rangle$. We only have to assume that it is annihilated by the annihilation operators in the Haag expansion. The thermal doubling of the Hilbert space accompanying the thermal Bogoliubov transformation is implicit in Eq. (3.7) where a ground state which is annihlated by thermal operators B, $\tilde{B}$, D and $\tilde{D}$ is defined.

In addition, thermal operators satisfy the Fermion anti–commutation relations

$$(2\pi)^3\frac{\omega_p}{m}\delta^3(\vec{p}-\vec{q})\,\delta_{s_1s_2} = \left[B^\dagger(\vec{p},s_1),B(\vec{q},s_2)\right]_+ = \left[D^\dagger(\vec{p},s_1),D(\vec{q},s_2)\right]_+$$
$$= \left[\tilde{B}^\dagger(\vec{p},s_1),\tilde{B}(\vec{q},s_2)\right]_+ = \left[\tilde{D}^\dagger(\vec{p},s_1),\tilde{D}(\vec{q},s_2)\right]_+ \quad (3.8)$$

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[1]We use the conventions and normalizations of Itzykson and Zuber [10].

with vanishing anti–commutators for the remaining combinations. The coefficients of the transformation are $\alpha_q = \sqrt{1 - n_q^-}$, $\beta_q = \sqrt{n_q^-}$, $\gamma_q = \sqrt{1 - n_q^+}$ and $\delta_q = \sqrt{n_q^+}$, where $n_q^\pm = [e^{(\omega_q \pm \mu)/(k_B T)} + 1]^{-1}$ are the Fermi distribution functions for particles and anti–particles. They are chosen so that the total particle number densities are given by

$$n_q^- = \frac{1}{V} \frac{m}{\omega_q} \langle \mathcal{G}(T, \mu)|b^\dagger(\vec{q}, s)b(\vec{q}, s)|\mathcal{G}(T, \mu)\rangle \tag{3.9}$$

$$n_q^+ = \frac{1}{V} \frac{m}{\omega_q} \langle \mathcal{G}(T, \mu)|d^\dagger(\vec{q}, s)d(\vec{q}, s)|\mathcal{G}(T, \mu)\rangle \tag{3.10}$$

Hence in this approach temperature and chemical potential are introduced through the coefficients of the thermal Bogoliubov transformation and are treated as parameters.

The first order Haag expansion for Ψ extended to finite T and μ is

$$\Psi_{\text{IN}}(q)\delta_m(q) = \frac{1}{(2\pi)^3} \frac{m}{\omega_q} \sum_{s=\pm} \left\{ \left[\alpha_q B(\vec{q}, s) - s\beta_p \tilde{B}^\dagger(-\vec{q}, s) \right] u(\vec{q}, s)\delta(q^0 - \omega_q) \right.$$
$$\left. + \left[\gamma_q D^\dagger(-\vec{q}, s) - s\delta_q \tilde{D}(\vec{q}, s) \right] v(-\vec{q}, s)\delta(q^0 + \omega_q) \right\}. \tag{3.11}$$

Note that this extension is independent of the structure of the Lagrangian and is therefore not restricted to the NJL model. If bosonic fields are present, as in the second order terms in Eq. (2.4), a thermal Bogoliubov transformation for bosonic creation and annihilation operators must be introduced together with a redefinition of the thermal ground state so that it is annihilated by both fermionic and bosonic annihilation operators. In the following sections we shall use the above ansatz for the Ψ field to study the NJL model at finite T and μ.

IV. NQA TO THE U(1) NJL MODEL: $(T, \mu) \neq (0, 0)$

Before extending the U(1) NJL model to finite T and μ we first calculate the number ($\mathcal{N}$) and scalar ($\mathcal{S}$) densities using the ansatz given in Eq. (3.11). It is a straightforward exercise to show that the number density is given by

$$\mathcal{N} = \frac{1}{V}(2\pi)^3 \int d^3\bar{p}_1 d^3\bar{p}_2 \; \delta^3(\vec{p}_1 + \vec{p}_2)\langle \mathcal{G}(T, \mu)|\frac{1}{2}\left[\Psi^\dagger(p_1), \Psi(p_2)\right]_-|\mathcal{G}(T, \mu)\rangle$$
$$= 2 \int d^3\bar{p} \left(\beta_p^2 - \delta_p^2\right) \tag{4.1}$$

while the scalar density is found to be

$$\mathcal{S} = \frac{1}{V}(2\pi)^3 \int d^3\bar{p}_1 d^3\bar{p}_2 \; \delta^3(\vec{p}_1 + \vec{p}_2)\langle \mathcal{G}(T, \mu)|\frac{1}{2}\left[\bar{\Psi}(p_1), \Psi(p_2)\right]_-|\mathcal{G}(T, \mu)\rangle$$
$$= -2 \int d^3\bar{p} \frac{m}{\omega_p}\left(1 - \beta_p^2 - \delta_p^2\right) \tag{4.2}$$

We now insert a bare chemical potential term $\mu_0\bar{\Psi}\gamma_0\Psi$ in the Lagrangian Eq. (2.1) and use Eq. (3.11) in the corresponding equation of motion and renormal order. Keeping only

the linear terms in the thermal field operators, one finds that the equation of motion is given by

$$\left(m + \mu_0 \gamma_0\right) \Psi(q) = \left\{ \left[4g_0 \int d^3\bar{p} \, \frac{1}{\omega_p} \left(1 - \beta_p^2 - \delta_p^2\right) \right] m + \left(g_0 \mathcal{N}\right) \gamma_0 \right\} \Psi(q) \tag{4.3}$$

Whereas the exchange terms did not contribute at $(T, \mu) = (0, 0)$, when introducing T and μ through the thermal Bogoliubov transformation one finds contributions to the equation of motion from both the direct and exchange terms. In fact, the exchange terms generate a contribution to the chemical potential proportional to the number density. From Eq. (4.3) we see that the total chemical potential is given by $\mu = \mu_0 - g_0 \mathcal{N}$.

The gap equation for $(T, \mu) \neq (0, 0)$ is found by equating the scalar terms in Eq. (4.3). The results is

$$m = 4g_0 \int d^3\bar{p} \, \frac{m}{\omega_p} \left(1 - \beta_p^2 - \delta_p^2\right) \tag{4.4}$$

and reduces to Eq. (2.5) in the $(T, \mu) \to (0, 0)$ limit. To obtain the vacuum energy density $\epsilon(T, \mu)$ we use Eq. (2.3) (plus the μ_0 term) and keep terms with two contractions after renormal ordering. We find

$$\epsilon(T, \mu) = \frac{1}{V} \langle \mathcal{G}(T, \mu) | H | \mathcal{G}(T, \mu) \rangle$$

$$= -\left\{ 2g_0 \int d^3\bar{p} d^3\bar{q} + \frac{1}{2} g_0 \mathcal{N}^2 + 2 \int d^3\bar{p} \, \omega_p \left(1 - \beta_p^2 - \delta_p^2\right) - \frac{m^2}{4g_0} + \mu_0 \mathcal{N} \right\} \tag{4.5}$$

Above expressions for the number and scalar densities, total chemical potential, the gap equation and the vacuum energy density all agree with the standard mean field results for the U(1) NJL model [7]. We note that unlike in the variational method the explicit structure of the thermal vacuum state need not be specified when obtaining these results. Instead they were determined by the algebra of the thermal fields defined by Eq. (3.7) and (3.8). We now turn to the discussion of the diagonalization of the model Hamiltonian.

In order to obtain the off–diagonal terms in the Hamiltonian, denoted as $\mathcal{H}_{\text{OD}}$, to second order in the thermal creation operators we keep terms involving one contraction when renormal ordering Eq. (2.3) (again, with the μ_0 term), and exploit the fact that the thermal annihilation operators annihilate the thermal vacuum state. Using the gap equation Eq. (4.4) the result can be written as

$$\mathcal{H}_{\text{OD}} = m \sum_s \int d^3\bar{p} \left\{ \left[\alpha_p \beta_p \tilde{B}^\dagger(\vec{p}, s) B^\dagger(-\vec{p}, s) + \gamma_p \delta_p \tilde{D}^\dagger(-\vec{p}, s) D^\dagger(\vec{p}, s) \right] \right.$$

$$\left. - \frac{\mu}{\omega_p} \left[\alpha_p \beta_p \tilde{B}^\dagger(\vec{p}, s) B^\dagger(-\vec{p}, s) - \gamma_p \delta_p \tilde{D}^\dagger(-\vec{p}, s) D^\dagger(\vec{p}, s) \right] \right\} \tag{4.6}$$

We see that the off–diagonal terms consist of particle–hole and anti–particle–anti–hole excitations coupled to the spin triplet state. These terms vanish in the chirally symmetric phase when $m = 0$, or in the zero temperature limit when $\alpha_p \beta_p = \gamma_p \delta_p = 0$. However Eq. (4.6) *can not vanish for finite T in the broken symmetry phase.*

In the standard treatment of BCS theory of superconductivity [11] the gap equation is obtained by demanding that the second order off–diagonal Hamiltonian vanishes. We could also have followed this path and derived the gap equation from $\mathcal{H}_{\mathrm{OD}}$. Instead we used the result obtained from the equation of motion to simplify the second order off–diagonal Hamiltonian as a consistency check. Also, in the BCS theory the condensation takes place between particles and holes in the spin *singlet* state and thus contributions from spin *triplet* terms in the Hamiltonian are usually ignored [11]. In the present case the chiral condensate is also a spin singlet bound state and if we adapt an analogous approximation Eq. (4.6) can also be ignored.

V. SU(2)$_\mathrm{F}$ ⊗ SU(3)$_\mathrm{C}$ NJL MODEL AT FINITE T AND μ

We now apply the method presented in the preceeding sections to the SU(2)$_\mathrm{f}$ ⊗ SU(3)$_\mathrm{c}$ NJL model consisting of scalar $(0^+, T = 0)$ and pseudoscalar $(0^+, T = 1)$ color singlet $\bar{q}q$ channels as well as scalar and axial vector $(1^+, T = 1)$ color $\bar{3}$ qq channels. This model was used in [12] to study three quark bound states in the NJL model and is defined as

$$\mathcal{L} = i\bar{\Psi}\slashed{\partial}\Psi + g_1\Big[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\vec{\tau}\Psi)^2\Big] + g_2\frac{3}{2}\Big[(\bar{\Psi}\gamma_5 C\tau_2\lambda^A\bar{\Psi}^T)(\Psi^T C^{-1}\gamma_5\tau_2\lambda^A\Psi)\Big]$$
$$+ g_3\frac{3}{2}\Big[(\bar{\Psi}\gamma_\mu C\tau_a\tau_2\lambda^A\bar{\Psi}^T)(\Psi^T C^{-1}\gamma^\mu\tau_2\tau_a\lambda^A\Psi)\Big] \tag{5.1}$$

where λ^A with $A = 2, 5, 7$ project out the color $\bar{3}$ channel.[2] Summation convention is implied for repeated indices. Positive values for the coupling constants g_1, g_2 and g_3 implies attraction in color 1 and $\bar{3}$ channels.

Just as in the case of the U(1) NJL model we Haag expand the Heisenberg field operators and apply the thermal Bogoliubov transformation to the first order terms. Use of the equation of motion corresponding to Eq. (5.1) immediately yields the total chemical potential and the gap equation for finite T and μ

$$\mu = \mu_0 - 4g_1 \int d^3\bar{p} \left(\beta_p^2 - \delta_p^2\right) \tag{5.2}$$

$$m = 26g_1 \int d^3\bar{p}\, \frac{m}{\omega_p}\Big(1 - \beta_p^2 - \delta_p^2\Big) \tag{5.3}$$

These results for μ and the gap equation are identical to the Hartree–Fock results for the SU(2)$_\mathrm{f}$ ⊗ SU(3)$_\mathrm{c}$ NJL model *without* the diquark channels [7]. Indeed, we find that to the lowest order in the Haag expansion the diquark channels do not contribute to the equation of motion.

The diquark channels do contribute to the vacuum energy density which, with the use of Eq. (5.2) and (5.3), can be written as

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[2]The Gell–Mann matrices λ^a are normalized as $\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta_{ab}$.

$$\epsilon(T,\mu) = -6\left\{ 12g_1 \int d^3\bar{p}\,d^3\bar{q} + 2\int d^3\bar{p}\,\omega_p\left(1 - \beta_p^2 - \delta_p^2\right) - (26g_1 + 6g_2 - 72g_3)\left(\frac{m}{26g_1}\right)^2 \right.$$

$$\left. + \frac{(\mu - \mu_0)^2}{4g_1} - (6g_2 - 36g_3)\left[\int d^3\bar{p}\left(\beta_p^2 - \delta_p^2\right)\right]^2 \right\} \tag{5.4}$$

Above expression reduces to the known vacuum energy density for the $SU(2)_f \otimes SU(3)_c$ NJL model when $g_2 = g_3 = 0$ [7]. Similarly, the second order off–diagonal Hamiltonian can be simplified to the following form

$$\mathcal{H}_{OD} = m \sum_{l,\epsilon} \int d^3\bar{p} \left\{ \sum_s s\left[1 + \frac{1}{26g_1}(6g_2 + 72g_3)\frac{m^2}{\omega_p^2}\right] \right.$$

$$\otimes \left[\alpha_p\beta_p \tilde{B}_{l\epsilon}^\dagger(\vec{p},s)B_{l\epsilon}^\dagger(-\vec{p},s) + \gamma_p\delta_p \tilde{D}_{l\epsilon}^\dagger(-\vec{p},s)D_{l\epsilon}^\dagger(\vec{p},s)\right]$$

$$- \sum_s s\frac{1}{\omega_p}\left[\mu + (6g_2 + 36g_3)\int d^3\bar{p}\left(\beta_p^2 - \delta_p^2\right)\right]$$

$$\otimes \left[\alpha_p\beta_p \tilde{B}_{l\epsilon}^\dagger(\vec{p},s)B_{l\epsilon}^\dagger(-\vec{p},s) - \gamma_p\delta_p \tilde{D}_{l\epsilon}^\dagger(-\vec{p},s)D_{l\epsilon}^\dagger(\vec{p},s)\right]$$

$$- \sum_{s_1,s_2}\left[\frac{1}{26g_1}(6g_2 + 72g_3)\frac{m^2}{\omega_p^2}\right]$$

$$\otimes \left[\alpha_p\gamma_p D_{l\epsilon}^\dagger(\vec{p},s_2)B_{l\epsilon}^\dagger(-\vec{p},s_1) - s_1 s_2\beta_p\delta_p \tilde{D}_{l\epsilon}^\dagger(-\vec{p},s_1)\tilde{B}_{l\epsilon}^\dagger(\vec{p},s_2)\right]$$

$$\otimes \left(\chi_1\vec{\sigma}\cdot\vec{p}\chi_2\right) \right\} \tag{5.5}$$

where $l = u,d$ and $\epsilon = 1,2,3$ are the flavor and color indices, respecitvely and χ_i is the two component spinor for spin s_i.

One sees immediately that the Hamiltonian for the $SU(2)_f \otimes SU(3)_c$ NJL model is exactly diagonalized in the Breit–Wigner phase when $m = 0$. However in the broken symmetry phase the first two terms in Eq. (5.5), which is similar to Eq. (4.6), vanish only in the zero temperature limit but the third term in general does not. This term consists of particle–anti–particle and hole–anti–hole excitations with zero total three momentum with the spins coupled to both 0 and 1. Since it contains spin singlet excitations it is desirable that this term vanishes exactly in order to obtain a true ground state of the many–body system at least in the zero temperature limit. This can be accomplished if we impose the condition $g_3 = -\frac{1}{12}g_2$ on the coupling constants for the scalar and axial vector diquark interactions. An immediate consequence of this restriction is that the scalar and axial vector qq channels can not both be attractive or repulsive.

In Figure 1a we plot the phase transition curve for the $SU(2)_f \otimes SU(3)_c$ NJL model using the Hartree approximation for the $\bar{q}q$ channel.[3] The input parameters for all our

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[3]In this approximation Eq. (5.2) becomes $\mu = \mu_0$ and the coefficient infront of the integral in Eq. (5.3) changes from 26 to 24.

numerical work are $\Lambda = 0.05$ GeV, $g_1 = 5.01$ GeV^{-2} and $g_2 = -12g_3 = 3.11$ GeV^{-2} which are taken from [13]. On the phase transition line $m = 0$ and the model Hamiltonian is exactly diagonalized. The corresponding vacuum energy densities, $\epsilon(T, \mu) - \epsilon(0, 0)$, are shown in Figure 1b. The upper curve corresponds to the energy density obtained with only the $\bar{q}q$ channel while the lower one includes both the $\bar{q}q$ and qq channels. For all values of temperature and chemical potential on the phase transition line the inclusion of the scalar and axial vector diquark channel lowers the vacuum energy density of the system.

The effect of the diquark channel on the energy density is similar in the broken symmetry phase at finite temperature. Figure 2a shows the behaviour of the asymptotic mass m as a function of μ for $T = 0.1$ GeV. The phase transition is second order, which is a consequence of the Hartree approximation for the $\bar{q}q$ channel, and the critical chemical potential is found to be $\mu_c = 0.253$ GeV. Below μ_c the Hamiltonian is only approximately diagonalized and in Figure 2b we plot the vacuum energy density as a function of μ. As in Figure 1b the upper curve is obtained with only the $\bar{q}q$ channel while the lower curve has contributions from both the $\bar{q}q$ and qq channels. Note that in both cases the values of the vacuum energy densities are largest at $\mu = \mu_c$ indicating that the broken symmetry phase is the energetically favored phase. We also find that the addition of the qq channel lowers the vacuum energy density for finite density at zero temperature. Hence the inclusion of the diquark channel can lower the vacuum energy density of the system when the Hartree approximation is invoked for the $\bar{q}q$ channel.

VI. SUMMARY AND OUTLOOK

In this work we have extended the NQA to quantum field theory to finite temperature and chemical potential. The basic idea is to subject the asymptotic fields in the Haag expansion to a thermal Bogoliubov transformation accompanied by a thermal doubling of the Hilbert space. Temperature and chemical potential are introduced non–linearly through the coefficients of the transformation. The NQA allows one to solve the operator equations of motion without explicitly specifying the structure of the ground state although it can be obtained a posteriori by diagonalizing the Hamiltonian. All quantities of interest are determined through the algebra of asymptotic fields. One only needs to assume the existence of a ground state which is annihilated by the annihilation operators used in the Haag expansion. The thermal doubling of the Hilbert space is implicit in Eq. (3.7) where a thermal ground state was introduced which is annihilated by the thermal annihilation operators.

We tested our formalism on the U(1) and SU(2)$_f$ $\otimes$ SU(3)$_c$ versions of the NJL model and found that the first term in the Hagg expansion can reproduce the mean field results for the number and scalar densities, the chemical potential, the gap equation and the vacuum energy density for finite T and μ. Furthermore we found that the second order Hamiltonian can not be diagonalized at finite T when the models are in the broken symmetry phase.

When scalar and axial vector qq channels are added to the SU(2)$_f$ $\otimes$ SU(3)$_c$ NJL model we found it necessary to impose a relation on the scalar and axial vector coupling constants to diagonlize the Hamiltonian for zero temperature. As a consequence scalar and vector diquark channels can not simulateously be attractive or repulsive. For an attractive scalar diquark channel and in the Hartree approximation to the $\bar{q}q$ channel, this restriction can lead to the lowering of the vacuum energy density. These findings should be taken into account

when the work developed in [12] to study three quark bound states in the NJL model is extended to finite T and μ.

To go beyond the mean field approximation it is necessary to add higher order terms in the Haag expansion as shown in Eq. (2.4). Here the second order terms involve both fermionic and bosonic fields, the latter corresponding to various bound states in the model. It is necessary to subject both types of fields to their respective thermal Bogoliubov transformations accompanied by a redefinition of the thermal ground state so that it is annihilated by both the fermionic and bosonic annihilation operators. In the NJL model it is possible to analytically calculate the bound state amplitudes by exploiting their behaviour under parity, time–reversal and charge–conjugation transformations [5]. Once these amplitudes have been determined for finite T and μ they can be fed back into the Haag expansion of the Heisenberg fields and quantities of interest recalculated.

We emphasize that the formalism presented here is quite general and has wide applications. Recently the importance of the scalar diquark channel in the study of color superconductivity has been emphasized by several authors using an effective four–Fermion instanton induced interaction [14]. Our method is very well suited for this application and investigation into the phenomenology of color superconductivity at finite temperature and chemical potential beyond the mean field approximation is currently in progress. Other applications include studies of pion or kaon condensations or properties of pion gas in chiral perturbation theory. In fact, our method can be applied to any quantum field theory in Minkowski space with well defined asymptotic fields.

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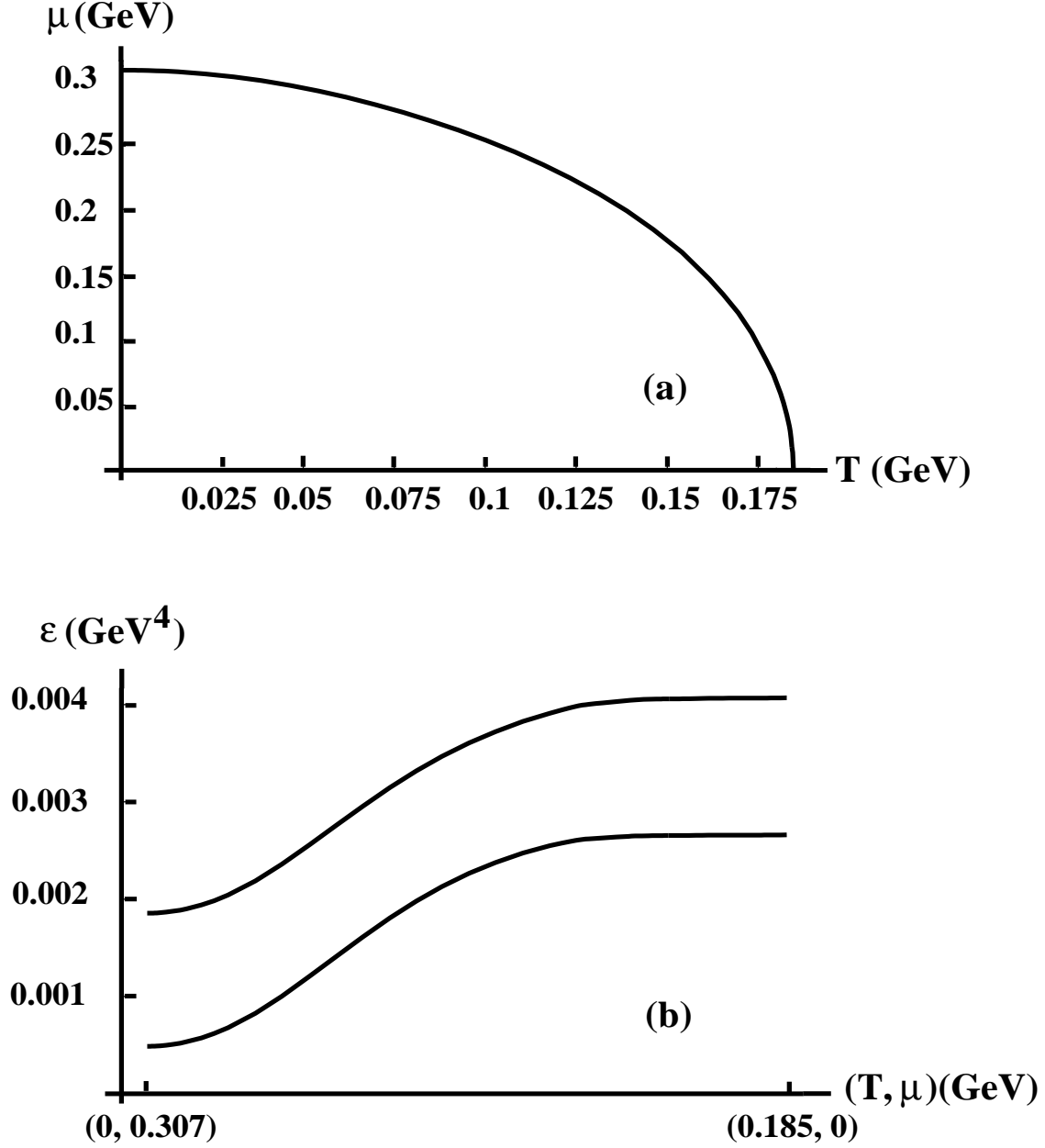
FIG. 1. a) Phase trasnsition line for the $SU(2)_f \otimes SU(3)_c$ NJL model with $\Lambda = 0.05$ GeV and $g_1 = 5.01$ GeV^{-2}. b) Vacuum energy density $\epsilon(T, \mu) - \epsilon(0, 0)$ on the phase transition line begining at $(T, \mu) = (0, 0.307)$ GeV and ending at $(T, \mu) = (0.185, 0)$ GeV. The upper and lower curves have been obtained with $g_1 = 5.01$ GeV^{-2}, $g_2 = g_3 = 0$ and $g_1 = 5.01$ GeV^{-2}, $g_2 = -12g_3 = 3.11$ GeV^{-2}, respectively.
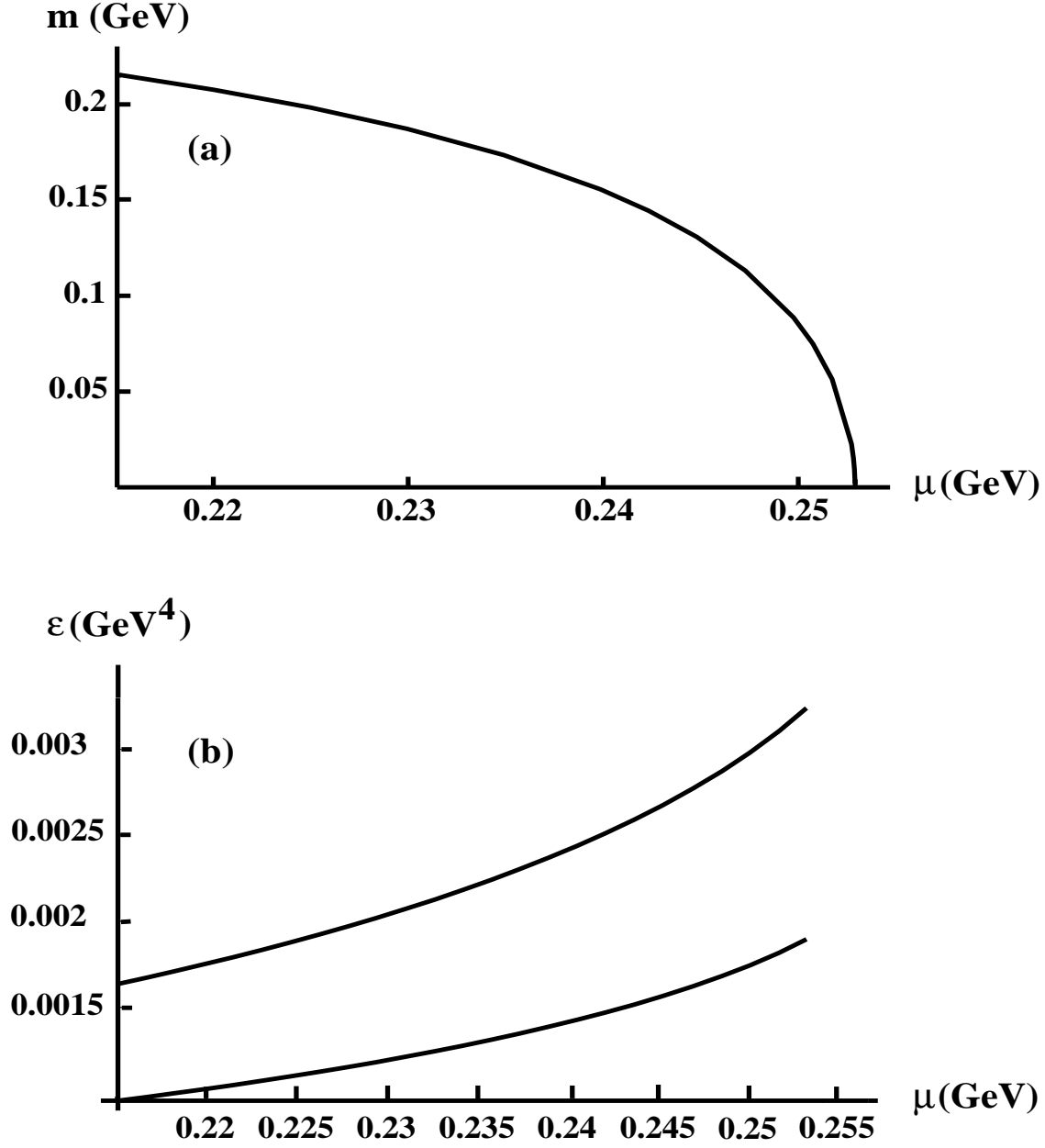
FIG. 2. a) Asymptotic mass m in the $SU(2)_f \otimes SU(3)_c$ NJL model as a function of chemical potential μ for $T = 0.1$ GeV with the same model parameters as in Figure 1a. b) Vacuum energy density $\epsilon(T, \mu) - \epsilon(0, 0)$ as a function of μ for $T = 0.1$ GeV. The upper and lower curves have been obtained with $g_1 = 5.01$ GeV^{-2}, $g_2 = g_3 = 0$ and $g_1 = 5.01$ GeV^{-2}, $g_2 = -12g_3 = 3.11$ GeV^{-2}, respectively.