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Atiyah-Manton Approach to Skyrmion Matter

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Abstract

We propose how to approach, and report on the first results in our effort for, describing nuclear matter starting from the solitonic picture of baryons which is supposed to represent QCD for large number of colors. For this purpose, the instanton-skyrmion connection of Atiyah and Manton is exploited to describe skyrmion matter. We first modify 't Hooft's multi-instanton solution so as to suitably incorporate proper dynamical variables into the skyrmion matter and then by taking these variables as variational parameters, we show that they cover a configuration space sufficient to adequately describe the ground state properties of nuclear matter starting from the skyrmion picture. Our results turn out to be comparable to those so far found in different numerical calculations, with our solution reaching stability at high density for a crystal structure and obtaining a comparable value for the energy per baryon at the minimum, thus setting the stage for the next step.

1 Introduction

Understanding from “first principles” the ground state of multi-baryon matter, namely, nuclear matter, is currently one of the most important issues in nuclear physics in connection with the mapping of the QCD phase structure. It is indispensable to the physics of hadronic matter under extreme conditions such as what is believed to be encountered in the interior of compact stars and in relativistic heavy-ion collisions. Of various approaches developed so far for the problem, one can cite, broadly, two classes of approaches to the problem: One deals directly with QCD variables and the other indirectly although both are anchored on the modern theory of strong interactions, QCD.

In the first class is the approach that deals directly with the quark and gluon fields but in a nonperturbative setting [1]. This is emerging in the most recent development and being in a seminal stage, does not yet lend itself to a direct confrontation with nature. We will have little to say on this in this paper. Let it suffice to briefly state that it exploits the possible “quark-hadron continuity” observed at high density [2] and extrapolated downward to low density in the “top-down” way.

In the second class are two approaches that resort to effective field theories of QCD that deal with *effective* degrees of freedom, namely, hadronic fields. Of the two, the one closest in spirit to the standard nuclear physics approach (SNPA) is based on chiral dynamics given in terms of local baryonic (nucleon, Δ , etc) and mesonic (pion, vector meson ρ , ω , etc.) fields. Here nuclear matter emerges from an effective chiral *action* as a sort of Q-matter or “chiral-liquid” nontopological soliton [3]. This description can then be related to Landau Fermi liquid structure [4] via Walecka’s model [5] when the notion of “sliding vacuum” in dense medium is implemented as discussed in [6, 7]. This approach allows a contact with nature and suggests how to extrapolate to densities higher than that of nuclear matter. Some of the recent developments along this line are reviewed by Song [8]. While this approach relies on an effective field theory strategy, thereby maintaining a link to “first principles” and enabling certain phase transitions like kaon condensation to be described within the same framework [9], numerous assumptions that are not directly accountable by the fundamental theory are invoked to arrive at the forms used for describing nature. As it stands, it is difficult to say to what extent this approach represents QCD and how reliable it is in confronting nature.

The other method in this second class which is in principle closest to QCD is the topological soliton approach based on an effective Lagrangian. In principle, given an effective chiral Lagrangian that is as “accurate” as can be, one should be able to quantitatively describe n -baryon systems for arbitrary n including nuclear matter as *solitons* of the Lagrangian. Presently we are far from being able to do this for the obvious reasons: First of all, such a Lagrangian is not available and secondly, even if it were available, we would not know how to obtain reliable soliton solutions. In the absence of an indication that this feat can be realized in the immediate future, our present aim is to use a “reasonable” but simple Lagrangian and arrive at a QCD-based description of nuclear matter that can be probed for any given density. The objective of this paper is to make a first such step toward that goal which is to pick a simple Lagrangian and develop a scheme which can be exploited to reach the ultimate goal, i.e., nuclear matter via skyrmion.

What we accomplish in this paper is admittedly quite modest but we feel that it presents a promising way to make the next step and hence deserves to be discussed. For this purpose, we shall take the simplest such effective Lagrangian known that is shown to be semi-realistic,

namely, the Skyrme model [11].

The Skyrme model describes the baryons with an arbitrary baryon number as static soliton solutions of an effective Lagrangian for pions. This model has been used to describe not only single baryon properties[12, 13], but also has served to derive the nucleon-nucleon interaction[11, 14], the pion-nucleon interaction[15], properties of light nuclei and of nuclear matter. In the case of nuclear matter, most of the developments [16, 17, 18, 19, 20, 21, 22, 23] involve the skyrmion crystal, except for a few exceptions[24] where the skyrmion fluid is considered. The first trial to understand dense skyrmion matter was made by Kutchera *et al.*[16]. In their work, a single hedgehog skyrmion is put into a spherical Wigner-Seitz cell without incorporating explicit information on the interaction.

The conventional approach so far studied to skyrmion matter is to assume a crystal symmetry and then to perform numerical simulations using the symmetry as a constraint. The first guess at this symmetry was made by Klebanov [17]. He considered a system where the skyrmions are located in the lattice site of a cubic crystal (CC) and have relative orientations in such a way that the pair of nearest neighbors attract maximally. Goldhaber and Manton [19] suggested that contrary to Klebanov's findings, the high density phase of skyrmion matter is to be described by a body-centered lattice (BCC) of half skyrmions. This suggestion was shown to be confirmed [20] and its energy per baryon of the ground state calculated, $(E/B)_{min} = 1.076 (6\pi^2 f_\pi/e)^1$ when the lattice size(L) is $L_{min} = 5.56 (\hbar c/ef_\pi)$.

Kugler and Shtrikman [22], using a variational method, investigated the ground state of the skyrmion crystal including not only the single skyrmion CC and half-skyrmion BCC but also the single skyrmion face-centered-cubic (FCC) and half-skyrmion CC. The ground state was found to be a half-skyrmion CC with the energy per baryon of $(E/B)_{min} = 1.038$ at $L_{min} = 4.72$. This simple cubic arrangement of half skyrmions undergoes a further phase transition at very high densities to a BCC crystal of half skyrmions.

These classical crystalline structures are quite far from normal nuclear matter which is known to be in a Fermi liquid phase at low temperature[4, 25]. In order for the skyrmion matter to be identified with nuclear matter, we have to quantize and thermalize the classical system. Since it is a system of solitons in a pion field theory, it is not sufficient to quantize the pion fields only. We have to introduce and quantize proper collective variables *not only* to complement the broken symmetries of the whole skyrmion system *but also* to describe the dynamics of the single skyrmions in order to obtain a realistic picture of nuclear matter. The limitation of the works that include only the former has been discussed in Ref.[26]. In order to describe a system of extended objects, e.g. skyrmions, we need a large number of dynamical variables, such as, the position of their center of mass, their relative orientations, their size, their deformation, etc. Among them, those degrees of freedom that describe translations and rotations of the single skyrmion play the dominant role at low energy. Thus we need at least 6 variables for each skyrmion. For a single skyrmion, the collective variables that define the orientation in isospin space are quantized to give rise to nucleons and isobars[12]. For the multi-skyrmion system, the simplest way of introducing collective variables for the position and orientation of each single skyrmion is through the use of the product ansatz, the old idea of Skyrme to investigate the nucleon-nucleon force[11, 14]. In this case, a multi-skyrmion solution can be obtained by

¹We use throughout the paper the units in parenthesis to express the energy and the length. In these expressions f_π , the pion decay constant, and e are parameters appearing in Skyrme's model [11], as we shall show in the next section. We recall also that the energy of a single skyrmion is – in these units – 1.23 in the conventional parameterization [12].

products of single skyrmion solutions centered at the corresponding positions and rotated to have the corresponding orientation. However, the product ansatz works well only when the skyrmions are sufficiently separated. Furthermore, due to the non-commutativity of the matrix products, it is very difficult to apply the product ansatz to many-skyrmion systems.²

Another scheme which has been used to study multi-skyrmion systems is the procedure based on the Atiyah-Manton ansatz [27]. In this scheme, skyrmions of baryon number N are obtained by calculating the holonomy of Yang-Mills instantons of charge N . This ansatz has been used successfully in few-nucleon systems [30, 31, 32, 33, 34]. This procedure has been also applied to nuclear matter with the instanton solution on a four torus [23]. The energy per baryon was found to be $(E/B)_{min} = 1.058$ at $L_{min} = 4.95$, which is comparable to the variational result of Kugler and Shtrikman *et al*[22]. One advantage of the Atiyah-Manton ansatz over others is that it provides a natural framework to introduce the proper dynamical variables for the skyrmions through the parameters determining the multi-instanton configuration. Several useful ansätze for multi-instanton solutions are available in the literature[35, 36]. Contrary to the *product* rule of the multi-skyrmion solution as a product ansatz, the multi-instanton solutions are given in terms of an *additive* rule.

In our first effort to go from a skyrmion matter to nuclear matter, we analyze here whether the Atiyah-Manton scheme can be used for introducing the proper variables to describe the skyrmion system. As a multi-instanton solution, we adopt a modified 't Hooft ansatz[35] to ensure that each instanton has a finite size and relative orientations to others. This ansatz provides us with the smallest set of dynamical variables for the skyrmion system. We study the structure of the ground state as a function of density by minimizing with respect to the parameters describing the dynamical degrees of freedom. The method we used can be considered as a variational method. Once we are able to describe the ground state, we shall relax the parameters to go beyond skyrmion matter.

This summary has served not only to present a brief report on the status of skyrmion matter, but also to introduce our calculational techniques. For completeness, we should mention the rational map technique [43], which has been very successful for baryons up to $B = 27$. This technique has also been applied to skyrmion matter, but only in two dimensions for which a Skyrme lattice with hexagonal symmetry [44] has been obtained. However, applying this technique to skyrmion matter in three dimensions is difficult in its present form, since, although the parameters of this scheme may cover most of the configurations of the ground state of finite baryon-number systems, their physical meaning is obscure.

This paper is organized as follows; in the following section, we describe how to apply the Atiyah-Manton ansatz to the skyrmion crystals. The numerical results are presented in Section 3 and Section 4 contains our conclusions.

2 Ansatz for Skyrmion Matter

²The symmetrization principle haunts us: we have to construct a system of identical particles, at the classical level, such that the physical quantities are invariant under the exchange of any two of them.

2.1 The Skyrme model Lagrangian

Our starting point is the model of Skyrme [11] whose Lagrangian density reads ³

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{Tr}([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2), \quad (1)$$

where $U(\vec{r}, t)$ is the nonlinear realization of the pion fields $\pi_i(\vec{r}, t)$ ($i=1,2,3$); *viz.*

$$U(\vec{r}, t) = \exp(i\vec{\tau} \cdot \vec{\pi}/f_\pi), \quad (2)$$

with the Pauli matrices τ_i ($i=1,2,3$) generating the $SU(2)$ isospin space. Here, f_π is the pion decay constant and e is the so called *skyrme parameter*. The pions are assumed massless in this Lagrangian density.

It is more convenient and elegant to use dimensionless units for the length and energy ⁴ in terms of which the Lagrangian is expressed as

$$\mathcal{L} = \frac{1}{24\pi^2} \text{Tr}(L_\mu L^\mu) + \frac{1}{192\pi^2} \text{Tr}([L_\mu, L_\nu]^2), \quad (3)$$

where L_μ is the “left current” defined as

$$L_\mu \equiv \partial_\mu U^\dagger U \equiv i\vec{\tau} \cdot \vec{\ell}_\mu. \quad (4)$$

The Faddeev-Bogomoln’y bound of a soliton solution carrying the baryon number B is ⁵

$$E \geq |B|, \quad (5)$$

where E is the energy of the static soliton solution

$$E = \frac{1}{12\pi^2} \int d^3x \left[\vec{\ell}_i \cdot \vec{\ell}_j + \frac{1}{2} (\vec{\ell}_j \times \vec{\ell}_k)^2 \right]. \quad (6)$$

³As a side remark, we should clarify our point of view regarding the Skyrme Lagrangian (1). For mathematicians, the Lagrangian (1) *defines* the problem and the goal for them is to *solve* the problem. This endeavor led to some beautiful results for the structure of “mathematical baryons” with the baryon number B . For physicists who are interested in understanding the physics of B -body systems, (1) is only an approximation to nature and possibly a poor one for certain processes. While the first term of (1) is the current algebra term and hence is a legitimate ingredient of QCD, what comes after that term is not generally known, and depends in practice upon what one wants to study. It is understood that in the limit of large number of colors, $N_C \rightarrow \infty$, QCD can be represented in terms of the pion fields but in an infinite series. For studying low-energy interactions with pions, one can write the series systematically in terms of derivatives of the U field; this leads to chiral perturbation theory. However if one wants to study high energy processes or hadronic excitations under extreme conditions (high temperature and/or high density), such a large N_c Lagrangian in the simplest form does not work [10]. The same difficulty arises when one wants to obtain the baryons as solitons from the effective Lagrangian. There is no workable systematic way known to write down the appropriate Lagrangian that can quantitatively describe baryons. It is therefore more the reason for surprise that although there is no theoretical justification from “first principles”, the Skyrme Lagrangian (1) with the given quartic term turns out to give qualitatively correct description of the baryons [13]. What we are doing here is that we simply assume that (1) applies to nuclear matter as well and take it (with the above caveat in mind) to study how to go from the skyrmion matter structure to the nuclear matter structure that we wish to describe. We are not concerned here with the accuracy of the effective Lagrangian itself. Our point of view is that once we know how to do this with the Skyrme Lagrangian, we will be able to do a realistic calculation once a Lagrangian that represents QCD realistically is found.

⁴The following scaling is implied by our choice of units: the lengths change as $[ef_\pi]x \rightarrow x$ and the energies as $E[e/(f_\pi 6\pi^2)] \rightarrow E$

⁵Unlike in the case of the magnetic monopole, the equality in (5) is never satisfied for a nontrivial configuration for U .

and the baryon number is given by

$$B = \frac{1}{24\pi^2} \varepsilon^{ijk} \int d^3x \text{Tr}(L_i L_j L_k) = \frac{1}{12\pi^2} \varepsilon^{ijk} \int d^3x \vec{\ell}_i \cdot (\vec{\ell}_j \times \vec{\ell}_k). \quad (7)$$

For the pion decay constant and the Skyrme parameter, the following two sets of values have been widely used in the literature. One[28] is

$$\begin{cases} f_\pi = 93 \text{ MeV}, & \text{the empirical pion decay constant,} \\ e = 4.75, & \text{from a fitting to } g_A = 1.25. \end{cases}$$

This choice leads to a value for the skyrmion mass $M_{B=1} = 1.23 \times 6\pi^2 f_\pi / e \sim 1425 \text{ MeV}$ ⁶ and the length unit becomes $197 / (93 \times 4.75) \sim 0.45 \text{ fm}$. The other is obtained by a fit to the N - Δ mass difference [12] that arises after quantizing the classical soliton to order N_c , to include $1/N_c$ effects, which requires the following values

$$\begin{cases} f_\pi = 64.5, \\ e = 5.45, \end{cases}$$

In this case, the length unit is $197 / (64.5 \times 5.45) \sim 0.56 \text{ fm}$.

2.2 From instantons to skyrmions

Atiyah and Manton[27] proposed a remarkable ansatz, with which a multi-skyrmion solution of the Skyrme Lagrangian carrying the baryon number N is expressed in terms of a multi-instanton solution of charge N .⁷ Explicitly, the soliton solution is taken to be given by

$$U(\vec{x}) = C \mathcal{S} \left\{ \mathcal{P} \exp \left[\int_{-\infty}^{\infty} -A_4(\vec{x}, t) dt \right] \right\} C^\dagger, \quad (8)$$

where the time component of the gauge potential of an instanton field of charge N is integrated along the time direction. Here, \mathcal{P} denotes the time-ordering, \mathcal{S} is a constant matrix to make U approach 1 at infinity and C describes an overall $SU(2)$ rotation. The homotopy ensures that the static soliton configuration carries the same baryon number as the total charge of the instanton.

An exact solution for the multi-instanton of charge N is known and takes the form

$$A_4(\vec{x}, t) = \frac{i}{2} \frac{\vec{\nabla} \Phi}{\Phi} \cdot \vec{\tau}, \quad (9)$$

Here τ 's are the Pauli matrices generating the *same* $SU(2)$ space as those appearing in $U(\vec{r}, t)$. The ansatz reduces the equations of motion, $\partial_\mu F^{\mu\nu} = 0$, together with the self-duality relation, $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, to $\partial^2 \Phi = 0$. Two scalar functions Φ , yielding instanton charge N , are well-known in the literature.

⁶This looks much too big for a nucleon. However as explained in [29], there is a term which is not included in [28] of $\mathcal{O}(N_c^0)$ Casimir energy that comes out to be of order $\sim -500 \text{ MeV}$.

⁷One should not confuse this instanton solution with that of the color gauge field in QCD. We are just borrowing the mathematical structure. Thus, differently from the massive Yang-Mills theory, the presence of the pion mass term in the Skyrme Lagrangian does not cause any problem in using the 't Hooft instanton solution.

A first choice, known as 't Hooft instanton solution [35], is given by

$$\Phi = 1 + \sum_{n=1}^N \frac{\lambda_n^2}{(x - X_n)^2}, \quad (10)$$

which contains $5N$ parameters for the positions $X_n^\mu (\mu = 1, 2, 3, 4)$ of the n -th instanton in 4-dimensional Euclidean space and their sizes λ_n . In the $N = 1$ case, the substitution of this 't Hooft instanton solution into the Atiyah-Manton ansatz[27] leads to the hedgehog skyrmion solution

$$U(\vec{r}) = \exp(i\vec{\tau} \cdot \hat{r}F(r)), \quad (11)$$

with the profile function given by

$$F(r) = \pi \left(1 - 1/\sqrt{1 + \frac{\lambda^2}{r^2}} \right). \quad (12)$$

The minimum energy of the approximate solution comes out to be $E = 1.24$ when $\lambda = 2.11$, which is very close to that of the exact solution $E_{B=1}^{\text{exact}} = 1.23$. When substituted into the Atiyah-Manton ansatz, the 't Hooft instanton solution provides $5N - 1^8$ parameters to the skyrmion system, these parameters becoming dynamical variables in the procedure. This number is much smaller than the minimum number of variables required, i.e., $6N$.

A second choice is the Jackiw-Nohl-Rebbi (JNR) instanton solution[36]

$$\Phi = \sum_{n=1}^{N+1} \frac{\lambda_n^2}{(x - X_n)^2}, \quad (13)$$

which nontrivially generalizes the 't Hooft solution to contain 4 more parameters. This increase in the number of parameters produces the toroidal minimum energy solution for the $B = 2$ skyrmion[31]. (Note that $5N + 3 > 6N$ only when $N < 3$.) On the other hand, here, X_n^μ loses its meaning as the position of the instanton so that it is quite difficult to guess the geometry of the resulting skyrmion system. The number of parameters is still less than $6N$.

2.3 Instanton crystal and skyrmion crystal

We proceed to modify the 't Hooft ansatz so as to be applicable to many-skyrmion systems. In its present form, the $1/x^2$ tail of the instanton, when summed over the infinite number of instantons in a crystal, makes the Φ diverge and hence the 't Hooft ansatz cannot be applied. To proceed, we shall slightly modify the 't Hooft multi-instanton structure, the time component of which, $A_4(x)$, can be written explicitly as

$$A_4(\vec{x}, t) = \frac{1}{2} \frac{\sum_{n=1}^N i\vec{\tau} \cdot \vec{\nabla} \phi(x, X_n, \lambda_n)}{1 + \sum_{n=1}^N \phi(x, X_n, \lambda_n)}, \quad (14)$$

⁸Note that in the time integration procedure to obtain a skyrmion out of the ansatz one can choose, for example, $X_{4,n=1} = 0$ without loss of generality, and therefore the skyrmion has one parameter less.

with

$$\phi(x, X_n, \lambda_n) = \frac{\lambda_n^2}{(x - X_n)^2}. \quad (15)$$

The modification consists of introducing the relative orientation of each instanton without changing the instanton charge:

$$A_4(\vec{x}, t) = \frac{1}{2} \frac{\sum_{n=1}^N C_n i\vec{\tau} \cdot \vec{\nabla} \phi(x, X_n, \lambda_n) C_n^\dagger}{1 + \sum_{n=1}^N \phi(x, X_n, \lambda_n)}, \quad (16)$$

with $C_n \in SU(2)$ ($n = 1, 2, \dots, N$). The C_n 's may be parameterized by the Euler angles (α, β, γ) ,

$$C = e^{i\alpha\tau_z/2} e^{i\beta\tau_x/2} e^{i\gamma\tau_z/2}. \quad (17)$$

We can further change the instanton profile function ϕ to be finite ranged by multiplying it with a suitable ‘‘truncation function’’ $h(x, X_n)$; *viz.*

$$\phi(x, X_n, \lambda_n) = \frac{\lambda_n^2}{(x - X_n)^2} h(x, X_n). \quad (18)$$

We have checked numerically that, as long as $h(x, X_n)$ satisfies the constraint, $h = 1$, when $x = X_n$, and is finite at infinity, the baryon number of the corresponding skyrmion field does not change. For simplicity we choose a spatially truncated function such as

$$h(x, X_n) = \begin{cases} \left(1 - \frac{|\vec{x} - \vec{X}_n|^2}{R^2}\right)^p, & |\vec{x} - \vec{X}_n| < R, \\ 0 & |\vec{x} - \vec{X}_n| > R, \end{cases} \quad (19)$$

which is continuous and differentiable up to p -th order. In our numerical analysis, we fix p to 3.

Due to this modification, Eq.(16) is no longer an exact solution of the instanton self-duality equation. However since the Atiyah-Manton strategy amounts to borrowing the topological structure of the instanton configuration via the holonomy, we believe that the use of this *approximate* solution would not lead to serious problems at the skyrmion level. We note that this modified formula resembles that figuring in the instanton liquid model[39] used to study QCD vacuum structure[40, 41, 42], for which the corresponding $A_4(\vec{x}, t)$ is of the form

$$A_4(\vec{x}, t) = \frac{1}{2} \sum_{n=1}^N \frac{C_n i\vec{\tau} \cdot \vec{\nabla} \phi(x, X_n, \lambda_n) C_n^\dagger}{1 + \phi(x, X_n, \lambda_n)}. \quad (20)$$

The difference is in the way the sums are performed.

Now, each instanton is described by 9 parameters: X_n^μ ($\mu=0,1,2,3$) defines its position in 4-dimensional Euclidean space, $C_n \in SU(2)$ the relative orientation, λ^2 the its strength and R the range. Except for the temporal position of the instantons, the other instanton parameters will preserve their physical meaning when transformed to the skyrmion configuration with the particles well-separated. Thus, it is rather easy to guess and construct an instanton crystal

#	X_0	X_1	X_2	X_3	α	β	γ	λ	R
1	$-T/2$	$+L_C/2$	$+L_C/2$	$+L_C/2$	0	0	0	λ	R
2	$+T/2$	$+L_C/2$	$+L_C/2$	$-L_C/2$	0	π	0	λ	R
3	$+T/2$	$+L_C/2$	$-L_C/2$	$+L_C/2$	π	0	0	λ	R
4	$-T/2$	$+L_C/2$	$-L_C/2$	$-L_C/2$	π	π	0	λ	R
5	$+T/2$	$-L_C/2$	$+L_C/2$	$+L_C/2$	π	π	0	λ	R
6	$-T/2$	$-L_C/2$	$+L_C/2$	$-L_C/2$	π	0	0	λ	R
7	$-T/2$	$-L_C/2$	$-L_C/2$	$+L_C/2$	0	π	0	λ	R
8	$+T/2$	$-L_C/2$	$-L_C/2$	$-L_C/2$	0	0	0	λ	R

Table 1: Values of the input parameters for the CC structure ansatz

corresponding to the skyrmion crystal of an expected symmetry. Explicitly, the ansatz for a given instanton crystal reads as

$$A_4(\vec{x}, t) = \frac{1}{2} \frac{\sum_{\text{box}} \sum_{n=1}^N C_n i\vec{\tau} \cdot \vec{\nabla} \phi(x, X_{n,\text{box}}, \lambda_n) C_n^\dagger}{1 + \sum_{\text{box}} \sum_{n=1}^N \phi(x, X_{n,\text{box}}, \lambda_n)}, \quad (21)$$

where the summation runs now over instantons in a periodic box and *for all the boxes* as far as the instanton in that box is within the range. The spatial periodicity of the crystal requires that

- i) all the image particles in each box share the same values of the parameters for λ_n^2 , C_n and $X_{0,n}$;
- ii) only the spatial position parameters depend on the position of the box.

$$\vec{X}_{n,\text{box}} = \vec{X}_n + \text{box position}. \quad (22)$$

We show in Fig.1 two of the crystals that can be constructed in the way described. Figure 1a illustrates the simple cubic crystal (CC) with the lattice constant L_C . This configuration carries one unit of baryon number in a volume of L_C^3 . The instanton parameters that we have used in searching the ground state(s) are listed in Table 1 and 2. The symmetry allows us to fix the size and the range of each instanton to be equal. Although the ground state is found for $T = 0$ later, we impose the most general condition for the temporal parameter consistent with the CC structure. Strictly speaking, it is not the conventional CC. As far as the field $U(\vec{r})$ is concerned, exactly the same field configuration is repeated with a period of $2L_C$. Thus, it is more convenient to work with a box of length $2L_C$ containing eight unit cells. However, with $T = 0$ and for large L_C , each unit cell shows the same baryon number distribution that forms a CC. If $T \neq 0$, it is somewhat like an NaCl crystal.

Fig.1b shows a Face-Centered Crystal (FCC) with lattice constant L_F . A box of volume L_F^3 contains 4 baryons. Here again, the nearest skyrmions are rotated by π with respect to the axis perpendicular to the line joining the pair. The choice of the other parameter is based on similar arguments to those of the CC crystal. In the remaining part of this paper, we will only work

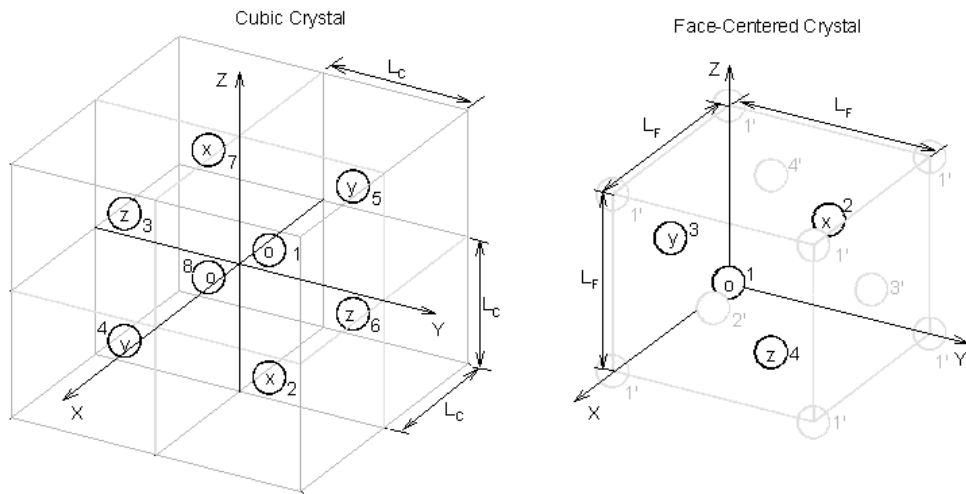


Figure 1: Face centered instanton crystal. The location of each instanton is denoted by a circle. The letters “x”, “y” and “z” inside the circle mean that the instanton is rotated by π around this axis. “o” means unrotated instanton. The image particles belonging to the neighboring boxes are presented by gray circles and labelled as “ i' ”, $i = 1, 2, \dots$.

#	X_0	X_1	X_2	X_3	α	β	γ	λ	R
1	$-T/2$	0	0	0	0	0	0	λ	R
2	$+T/2$	0	$+L_F/2$	$+L_F/2$	0	π	0	λ	R
3	$+T/2$	$+L_F/2$	0	$+L_F/2$	π	π	0	λ	R
4	$+T/2$	$+L_F/2$	$+L_F/2$	0	π	0	0	λ	R

Table 2: Values of the input parameters for the FCC structure ansatz

only with this FCC configuration. It fulfills the main purpose of this paper, which is to show that our technique, described in the previous section, is relevant to describe skyrmion matter.

It is important to realize that we are putting the starting crystal symmetry in our initial conditions in order to generate the required crystal structure for the ground-state configuration. In principle, all the parameters of our ansatz can be treated as free parameters or variables without imposing any restriction. This is the main advantage of our procedure over the more conventional ones [17, 20, 21, 22]. In the next section, we will illustrate the power of the present approach by evaluating the energy associated with the shift of a single skyrmion from its stable position (i.e., ground state) keeping all others fixed.

We are now ready to work out the Atiyah-Manton construction Eq.(8) with the modified instanton configuration (16). Time-ordering makes the construction somewhat complicated. In order to simplify the calculation, we introduce a skyrmion field generator $\tilde{U}(\vec{r})$ given by

$$\tilde{U}(\vec{r}, t) \equiv \mathcal{S}\mathcal{P} \exp \left(- \int_{-\infty}^t A_4(\vec{r}, t') dt' \right), \quad (23)$$

The generator satisfies the differential equation

$$\partial_t \tilde{U}(\vec{r}, t) = -A_4(\vec{r}, t) \tilde{U}(\vec{r}, t). \quad (24)$$

The skyrmion ansatz $U(\vec{r})$ can be obtained from the generator in the limit

$$\lim_{t \rightarrow \infty} \tilde{U} \quad (25)$$

We proceed therefore by solving the differential equation, which is a conventional time evolution problem, from $t_0 = -\infty$ where it takes the initial value

$$\tilde{U}(\vec{r}, -\infty) = \mathcal{S}, \quad (26)$$

to the final time t . The constant matrix \mathcal{S} is chosen so that

$$U(|\vec{r}| \rightarrow \infty) = 1. \quad (27)$$

In the numerical procedure of evaluating physical quantities, we only need the Sugawara variables L_μ . They can be directly obtained as a $t \rightarrow \infty$ limit of \tilde{L}_μ defined as

$$\tilde{L}_\mu = \partial_\mu \tilde{U}^\dagger \tilde{U} \quad (28)$$

which satisfies the differential equation

$$\partial_t \tilde{L}_\mu = [A_4(\vec{r}, t), \tilde{L}_\mu] - \partial_\mu A_4(\vec{r}, t). \quad (29)$$

The differential equation can be solved numerically by changing the integration variable as $t = \tan \theta$, which makes the integration range finite from $-\pi/2$ to $\pi/2$.

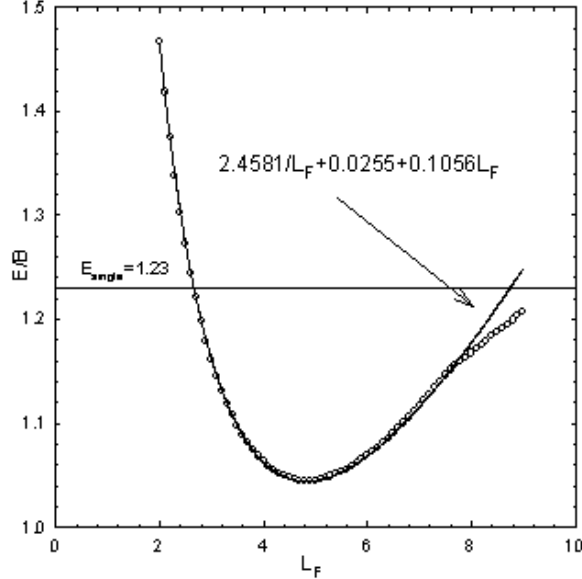


Figure 2: E/B as a function of L_F the lattice constant of the FCC crystal.

3 Results

Our calculation starts from an ansatz, at low density, constructed from an FCC lattice with 4 skyrmions per site. At first, for a given L_F , we search for the lowest energy configuration(s) by adjusting the other parameters such as λ , R and T . Shown in Fig.2 is E/B obtained in that way as a function of L_F . As the density increases from the dilute skyrmion system, the energy per baryon decreases to a minimum.

A best fit to this curve in the region around the minimum gives

$$E/B = 0.1056L_F + 0.0255 + 2.4581/L_F. \quad (30)$$

Therefore

$$(E/B)_{min} = 1.044, \quad (31)$$

and

$$L_{min} = 4.82. \quad (32)$$

This result is comparable to the lowest value $E/B = 1.038$ obtained by Kugler and Shtrikman[22].⁹ This indicates that our calculation has reached the true minimum of the skyrmion crystal at high density and that the instanton parameters so determined can be taken as the appropriate variables needed for describing the dynamics of the multi-skyrmion system.

To see the configuration of the final skyrmion crystal, we present in Fig. 3 the baryon density on the $x - y$ plane ($z = 0$) at two different values of L_F ; (a) $L_F = 8.0$ and (b) $L_F = 5.0$.

⁹For a rough estimation, if we take $f_\pi = 93$ MeV and $e = 4.65$, our values correspond to ~ 2.18 fm and to $B/V = 0.38$ nucleons/fm³. On the other hand, if we take $f_\pi = 64.5$ MeV and $e = 5.45$, our values correspond to $L_F \sim 2.72$ fm and $B/V = 0.20$ nucleons/fm³. Thus, it is compatible to the normal nuclear matter density.

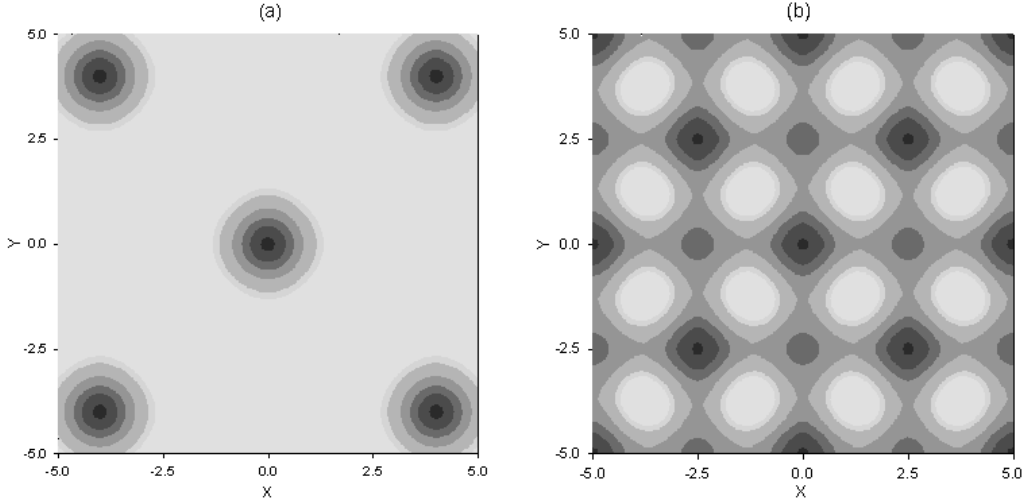


Figure 3: The baryon number density of the FCC Skyrme crystal on the $z = 0$ plane at (a) $L_F = 8.0$ and (b) $L_F = 5.0$. The baryon number density is normalized to the most dense region. Because of the overlap, part of the baryon number is accumulated at the places in between the FCC lattice sites and the configuration becomes that of the half-skyrmion CC crystal.

When it is dilute (Fig.3a), the system is composed of well separated single skyrmions lodged on each FCC lattice site. We may call this the phase of “single skyrmion FCC”. At high density (including the minimum energy configuration), the individual skyrmion loses its identity because of the overlap among the skyrmions. The dense centers of the single skyrmion split into two and the final configuration forms a CC crystal whose unit cells carry one-half unit of baryon number. This can be interpreted as the “half-skyrmion CC” obtained in Ref.[22]. However, the shape of the instanton tail described by Eq.(18) and the cut-off function Eq.(19) are not accurate enough to reproduce the exact half-skyrmion picture. Note that this half-skyrmion phase has arisen without any additional changes in our procedure: while our initial instanton configuration is still FCC, the final skyrmion configuration turns out to be a CC lattice with a fractional skyrmion per site.

A rough analysis on the configuration shows that the phase transition from the single-skyrmion FCC to the half-skyrmion CC takes place at around $L_F \sim 7.5$. Such a phase transition can be explicitly seen in the parameter values that lead to the minimum energy configurations at the given L_F as shown in Fig.4. For the high density phase ($L_F \leq 7.5$), the parameters vary around

$$\lambda/L_F \sim 1.1, R/L_F \sim 1.4,$$

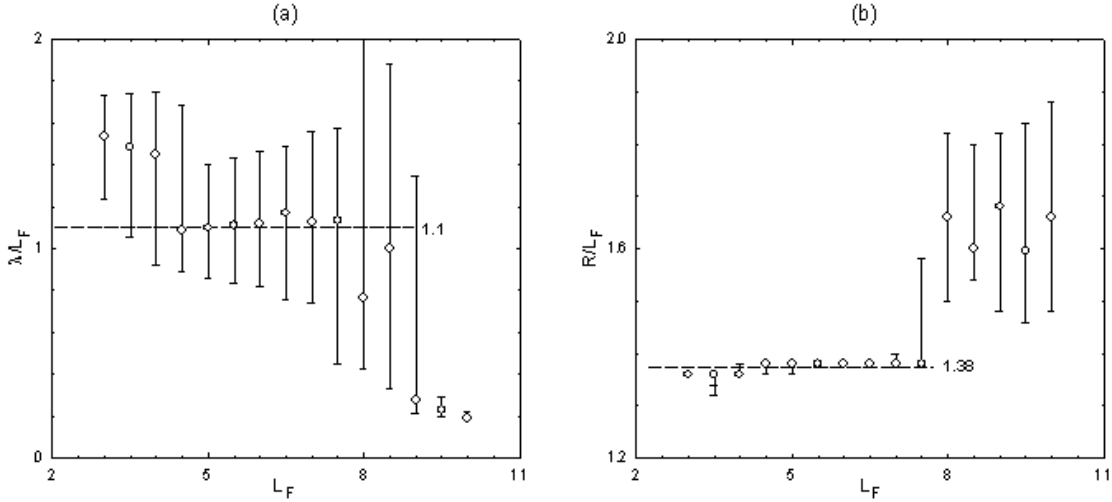


Figure 4: The parameters (a) λ/L_F and (b) R/L_F of the minimum energy configuration at the given L_F . The error bars denote the parameter range that yields the energy within 5% of the corresponding minimum energy.

to become a half-skyrmion CC, while for the low density phase ($L_F > 7.5$) the parameters change sharply to

$$\lambda \sim 2, R/L_F \sim 1.7$$

returning to the single skyrmion FCC crystal. The abrupt change which one might interpret as a phase transition is also observed in the E/B curve of Fig.2, where the slope of the curve suddenly changes to lower values around $L_F = 7.5$ making the approach to the asymptotic free value of 1.23 slower.

Though intriguing, it should however be pointed out that this “phase transition” may not be physical. If we look at the pressure defined as $P = \partial E / \partial V$, such a phase transition happens where the pressure is negative. In the density where the pressure is negative, the system is unstable and would prefer a disordered phase to the symmetric FCC or CC phases¹⁰.

It is interesting to see that all the configurations in the half-skyrmion phase show a certain scale invariance. In Fig.5 we plot the baryon number density normalized to its maximum value $\rho_B / \rho_{B,\max}$ as a function of the position normalized to the lattice constant x/L along the $y = z = 0$ line. The figure shows that $\rho_B / \rho_{B,\max}$ for $L_F = 7.5$ and $L = 4.8$ depends only on x/L . This type of scaling behavior repeats itself for all values of $L_F \leq 7.5$, i.e., for high densities all the configurations become almost half-skyrmion configurations. However if $L_F \geq 7.5$ they lose this scaling property and become like the initial FCC ansatz, i.e, a single skyrmion located at each FCC site and with the overlapping region more or less void. This scaling behavior may be related to the one found in the variational approach of Kugler and Shtrikman[22]: when the configuration of the skyrmion crystal is expanded in harmonics, the primary harmonics are found to play the dominant role. This implies that at high density, the symmetry of the system

¹⁰The phase transition that we have here should not be confused with that of skyrmions in the S_3 sphere [45, 46]. It happens in the range of density where the pressure appears negative and this is caused because the skyrmion system is constrained to be in FCC shape. By forming disordered clusters of a few skyrmions, the system could develop lower energy.

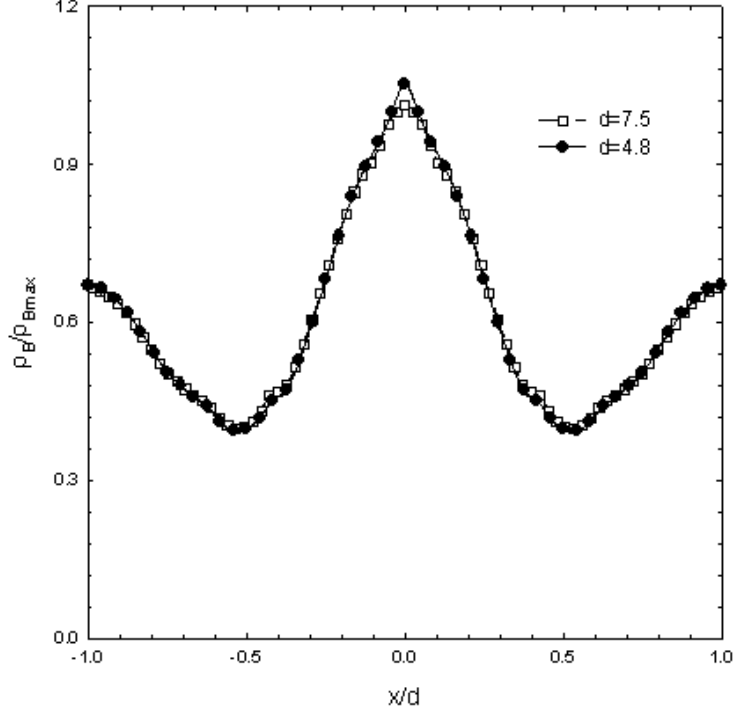


Figure 5: $\rho_B/\rho_{B,\max}$ as a function of x/L along the $y = z = 0$ line.

dominates over all dynamical details such as the interaction between two nearest neighbors.

As stressed, the main advantage of our approach is not in the calculation of the ground state configuration but in the ability to describe *with the same ansatz* the skyrmion system *far* from the crystal structure without relying on symmetry arguments. For example, we can calculate how the energy of the whole skyrmion system changes when a single skyrmion is shifted from its stable lattice position. This can be done just by shifting the position of one instanton in Eq.(16) while leaving unchanged that of all other instantons in the same box and in the neighboring boxes (including its image particles). Since the instantons are chosen to have a finite range, the calculation is easy. Only the physical quantities inside this finite range are affected. Shown in Fig.6 is the energy change of the system when a skyrmion is shifted from its FCC lattice site by an amount d in the direction of the z -axis. The error bars are estimated by the numerical fluctuations in the baryon number, assuming that the energy is calculated with an error of the same magnitude. Two extreme cases are shown. In the case of a dense system ($L_F = 5.0$), the energy changes abruptly. For small d , it is almost quadratic in d . It implies that the dense system is in the crystal phase. On the other hand, in the case of a dilute system ($L_F = 10.0$), the system energy remains almost constant up to some large d , which implies that the system is in a gas (or liquid) phase. If we let all the variables vary freely, the system will prefer to change to a disordered phase – in which a few skyrmions will form clusters – to be more attractive.

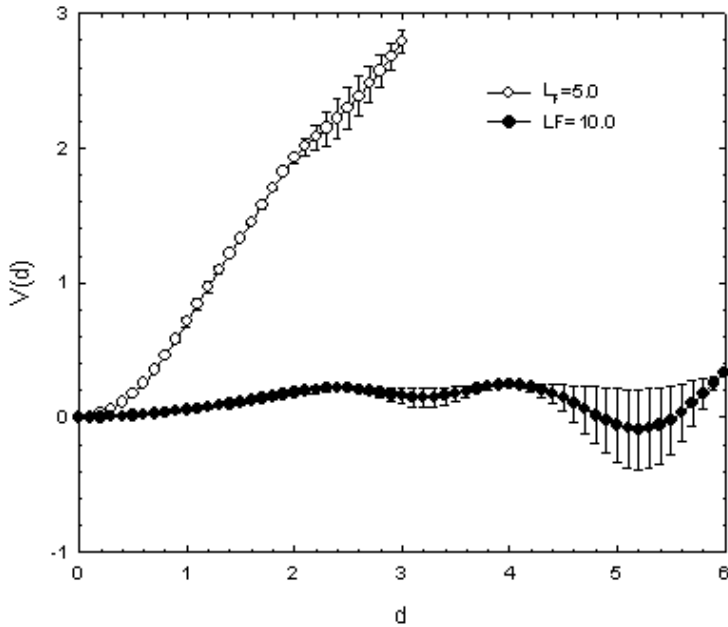


Figure 6: The energy cost to shift a single skyrmion from its stable position by an amount d in the direction of the z -axis.

4 Conclusion

In this paper, we propose a systematic way of introducing dynamical variables in skyrmion matter that would allow us to study nuclear matter with a link to large- N_c QCD. The method exploits the Atiyah-Manton ansatz to construct the pion field configuration for the multi-skyrmion system from the multi-instanton configuration. By modifying the 't Hooft instanton solution, we can introduce a set of basic dynamical variables, such as, the positions of the single skyrmions, the relative orientations in the isospin space, sizes, etc. to cover the necessary configuration space. To check whether these variables can indeed cover sufficiently the important configuration space of skyrmion matter, we look for the ground state of the system. Although not perfect in its details – e.g., the shape of the tails of the instanton, the numerical results show that our procedure is working satisfactorily.

The power of our approach is that we can go without much effort beyond the ground state of the skyrmion crystal which is not readily accessible to other approaches. As an example, we demonstrate how easily one can evaluate the energy change of the system when a single skyrmion is shifted from its stable position while the others are kept fixed. A careful analysis of the data enables us to investigate the vibrational energy spectrum of the Skyrme crystal and hence its thermodynamic properties, e.g., the specific heat, compressibility, etc. of skyrmion matter.

What is obtained in this paper sets the stage for the next step.

- First of all, the theory can be quantized. Substituting in the Atiyah-Manton skyrmion-instanton connection our modified 't Hooft instanton ansatz, making the dynamical variables “*time-dependent*,” and plugging the result into the Skyrme Lagrangian density, after integrating over the space variables, we obtain the Lagrangian describing the skyrmion

dynamics,

$$\int d^3r \mathcal{L}(U(\vec{x}, q_\xi(t))) = L(q_\xi, \dot{q}_\xi) = \frac{1}{2} \mathcal{M}_{\xi\eta} \dot{q}_\xi \dot{q}_\eta - V(q_\xi), \quad (33)$$

where $q_\xi(t)$ ($\xi = 1, 2, \dots, 9N$) denote the variables introduced as X_n , λ_n and $\mathcal{M}_{\xi\eta} = \mathcal{M}_{\xi\eta}(q_\zeta)$ are the inertia tensors conjugate to these dynamical variables. This Lagrangian yields the equations of motion for these variables. If these equations of motion are solved by applying molecular dynamics simulation techniques[47], we expect to see that the skyrmion crystal melts to a liquid at a certain density.

- By incorporating fluctuations in the pionic field[48], we should be able to study how pions (and also kaons) behave in dense medium. This will provide a self-consistent scheme to study the role of pions as Goldstone bosons in dense medium, in particular in connection with Goldstone boson condensation and chiral restoration. The presently available information on this matter is incomplete in that although chiral symmetry is maintained, the role of pions in the structure of baryons as described by e.g. chiral Lagrangian is obscure particularly when the quark condensate representing the QCD vacuum structure is modified by density.

These two issues will be addressed in the forthcoming publication.

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