

A Unified Approach to High Density: Pion Fluctuations in Skyrmion Matter

Hee-Jung Lee^{a,b}, Byung-Yoon Park^{a,b}, Dong-Pil Min^c, Mannque Rho^{a,d}
and Vicente Vento^{a,e}

(a) *School of Physics, Korea Institute for Advanced Study, Seoul 130-012, Korea*

(b) *Department of Physics, Chungnam National University, Daejeon 305-764, Korea*

(E-mail: hjlee@phya.snu.ac.kr, bypark@chaosphys.cnu.ac.kr)

(c) *Department of Physics and Center for Theoretical Physics*

Seoul National University, Seoul 151-742, Korea

(E-mail: dpmin@phya.snu.ac.kr)

(d) *Service de Physique Théorique, CE Saclay*

91191 Gif-sur-Yvette, France

(E-mail: rho@spht.saclay cea.fr)

(e) *Departament de Física Teòrica and Institut de Física Corpuscular*

Universitat de València and Consejo Superior de Investigaciones Científicas

E-46100 Burjassot (València), Spain

(E-mail: Vicente.Vento@uv.es)

Abstract

As the first in a series of systematic work on dense hadronic matter, we study the properties of the pion in dense medium using Skyrme's effective Lagrangian as a unified theory of the hadronic interactions applicable in the large N_c limit. Dense baryonic matter is described as the ground state of a skyrmion matter which appears in two differentiated phases as a function of matter density: i) at high densities as a stable cubic-centered (CC) half-skyrmion crystal; ii) at low densities as an unstable face-centered cubic (FCC) skyrmion crystal. We substitute the latter by a stable inhomogeneous phase of lumps of dense matter, which represents a naive Maxwell construction of the phase transition. This baryonic dense medium serves as a background for the pions whose effective *in-medium* Lagrangian we construct by allowing time-dependent quantum fluctuations on the classical dense matter field. We find that the same parameter which describes the phase transition for baryonic matter, the expectation value of the σ field, also describes the phase transition for the dynamics of the *in-medium* pion. Thus, the structure of the baryonic ground state *crucially* determines the behavior of the pion in the medium. As matter density increases, $\langle\sigma\rangle$ decreases, a phenomenon which we interpret to signal, in terms of the parameters of the effective pion Lagrangian f_π^* and m_π^* , the restoration of chiral symmetry at high density. Our calculation shows also the important role played by the higher powers in the density as it increases and chiral symmetry is being restored. This feature is likely to be generic at high density although our ground state may not be the true ground state.

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1 Introduction

Understanding the properties of hadrons in a dense nuclear medium is currently an important issue in nuclear physics. The dynamically generated masses of the light hadrons reflect the symmetry breaking pattern of QCD. The quark condensate $\langle \bar{q}q \rangle$ which represents the order parameter drops as the density of the nuclear medium increases. Consequently all light-quark hadron masses (except that of the pions) are expected to follow the behavior of the condensate, a basis of the scaling proposed in [1] and predicted in an effective field theory with the vector manifestation fixed point [2]. The pion however is very special, because it is the Goldstone boson associated with the spontaneous symmetry breakdown of the chiral symmetry. The Hellmann-Feynman theorem [3, 4], a mean field approximation in chiral perturbation theory [5, 6, 7, 8], and the Nambu-Jona-Lasinio model [9, 10, 11] lead to the same linear dependence in the baryon number density ρ namely,

$$\frac{\langle \bar{q}q \rangle_\rho}{\langle \bar{q}q \rangle_{\rho=0}} = 1 - \frac{\Sigma_{\pi N}}{m_\pi^2 f_\pi^2} \rho + \dots, \quad (1)$$

where $\Sigma_{\pi N}$ is the pion-nucleon sigma term. If we assume that the Gell-Mann-Oakes-Renner (GMOR) relation holds for finite density, we have ¹

$$f_\pi^{*2} m_\pi^{*2} \approx -m_q \langle \bar{q}q \rangle_\rho, \quad (2)$$

where f_π^* is the in-medium pion decay constant and m_π^* the effective pion mass in the medium. There has been a lot of discussions on the physical consequence of eq. (2) with a decreasing quark condensate. The decrease in the *pion decay constant* is interpreted as a symptom of partial chiral restoration in the nuclear medium [10]. On the other hand, in refs. [5, 6, 7], the possibility of S-wave pion condensation resulting from the decrease of the *pion effective mass* as a consequence of the decrease of $\langle \bar{q}q \rangle$ in eq. (2) has been discussed. However, if one lets the pion decay constant change, then the effective pion mass can even increase with increasing nuclear density as discussed in ref. [8] and a completely different scenario can then arise.

The deeply bound pionic states observed in heavy nuclei (Pb, Sn etc.) at GSI [12, 13] have renewed interest in these issues [14, 15, 17, 18, 19, 20, 21]. The well defined 1s and 2p states with narrow width provide a new standard for the S-wave pion-nucleus optical potential $U_s(r)$, whose real part may be identified as the effective mass of the pions [14, 15]

$$m_\pi^{*2}(r) = m_\pi^2 + 2\omega \text{Re}U_s(r). \quad (3)$$

This formula would seem to imply that the pion mass increases a 7% at the center of the Pb nucleus and a 3% in symmetric nuclear matter. But other possibilities exist. For instance, if one assumes the pion mass to be unmodified for low density which is not unphysical ², then the measurements provide information on the scaling of the pion decay constant instead of that of the mass. See [13].

Theoretical studies on the effective mass of the pion in a nuclear medium follow two different schemes. One traditional approach relies on optical potentials. However, it is difficult to relate the phenomenological π -nucleus potential to the microscopic π - N interactions in a systematic

¹For the moment, we are not distinguishing the time and space components of the pion decay constant. Which one we are dealing with will be specified later.

²The pion mass of course remains strictly unmodified in the chiral limit but it is possible that even in the presence of small quark masses, the approximate chiral invariance protects the pion mass so that it is left more or less unaffected by density, at least at low density. An $SU(2)_c$ (2 color) QCD lattice calculation indicates that this is what happens [22].

way. The other more modern approach proceeds via chiral perturbation theory [16] with explicit baryon fields. By integrating out the baryon fields, using the mean field approximation, one obtains an effective theory whose parameters become dependent on the nuclear density ρ . These calculations consist of computing single-nucleon loops leading to a linear dependence on the density of the parameters. Higher orders in the nucleon loop expansion would provide us with higher powers of ρ , however, due to the strong couplings associated with the nuclear forces, it is not clear that the results will converge although chiral perturbation power counting is used. Perhaps more importantly an effective field theory Lagrangian properly matched to QCD at an appropriate scale as discussed by Harada and Yamawaki [2] *must* have a highly nontrivial “intrinsic” dependence on density (or temperature if in heat bath) in the parameters but this dependence is generally missing in the chiral Lagrangian so far used [8, 14].

We study the pion properties in dense baryonic matter in an approach based on Skyrme’s Lagrangian, which we use to describe, in a unified manner, the dynamics of baryons, pions and the interactions between them. We proceed by calculating initially the ground state of infinite skyrmion matter. At high density it is described by a skyrmion crystal. At lower density the crystal suffers a phase transition and becomes unstable. The system chooses then as its ground state an inhomogeneous phase where skyrmions condense to form lumps of matter of higher density. We adopt a naive Maxwell construction to describe such state. We incorporate the pion in this medium by performing quantum fluctuations in the pion field of the Lagrangian describing this state. We obtain in this way the effective dynamics of the pion in dense skyrmion matter. The parameters of this Lagrangian are dependent on matter density and contain *all higher orders in density* as dictated by Skyrme’s Lagrangian.

Before proceeding we must clearly lay down the scope of this work here. We do not claim that the result we obtain – which is quite striking and intriguing – is in any way reflecting Nature. In fact it is perhaps likely that the “ground state” we obtain is not close to what it should be. More work is needed to obtain from what we have a true ground state and we have some ideas as to how to proceed. What we do here is that we will simply assume that we have certain states given by exact classical solutions of the field equations of a theory that is considered to be valid at large N_c , namely, the crystal structure of different symmetries, and study what they imply for the excitations thereon. This work should be taken as representing the first exploratory step towards a more realistic treatment – that is in accordance with QCD – of highly dense matter, the objective of our effort.

This paper is organized as follows. In the next section, after a brief description of the model Lagrangian, we construct the classical skyrmion crystal solution using Fourier expansions. In sec. 3, we incorporate the pion fluctuations in the ground state and obtain the effective pion Lagrangian in the skyrmion medium. We discuss how the pion decay constant and the effective pion mass change in matter. Section 4 summarizes the most important results and contains our conclusions.

2 The Model Lagrangian and the FCC Skyrmion Crystal

The description of baryonic matter we are going to investigate is based on the Skyrme model Lagrangian [23] for the massive pion fields, which reads

$$\mathcal{L} = -\frac{f_\pi^2}{4}\text{Tr}(U^\dagger\partial_\mu U U^\dagger\partial^\mu U) + \frac{1}{32e^2}\text{Tr}[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2 + \frac{f_\pi^2 m_\pi^2}{4}\text{Tr}(U + U^\dagger - 2), \quad (4)$$

where $U \in SU(2)$ is a nonlinear realization of the pion fields. f_π , e and m_π are the pion decay constant, Skyrme’s parameter and the pion mass, respectively. Note that this Lagrangian

(4) is described in terms of U , which contains only pionic degrees of freedom. We will not consider the isospin symmetry breaking due to the u- and d-quark mass difference, since it is of no relevance for the present study. Our aim is to study the properties of the pion in a dense baryonic medium. Since this is an effective Lagrangian, if matched to QCD [2], its parameters should depend on the properties of the medium, i.e. matter density in our case as discussed in [24]. However we follow a rather different philosophy, namely we shall assume that the above Lagrangian embodies a complete description of *all* physical processes of interest, i.e., free pions, free baryons, many baryon states, dense baryonic matter and moreover, it also describes the pions interacting with any of these baryonic systems. We fix therefore the parameters of the Lagrangians in one of these processes, for example by describing the free pion and nucleon systems, and calculate how these values change as the density of the baryonic matter increases. From the recent understanding [2, 24] of effective theories that purport to match to QCD, we know that this is not totally consistent since EFTs can make sense only for given scales and the parameters of the theory necessarily flow. This issue which is quite intricate will be dealt with in a forthcoming publication.

For completeness we should also mention that (4) is the most economical effective Lagrangian that one can take. To be truly realistic, massive fields – such as the light-quark vector mesons, scalar mesons or “dilaton”³ associated with trace anomaly etc – and/higher derivative terms need be incorporated but for our purpose it suffices to take the simplest yet sufficiently realistic form given by (4).

A similar procedure has been followed for example in chiral perturbation theory with explicit nucleon fields [7, 8]. They start with a Lagrangian containing all the degrees of freedom and free parameters, they integrate out the nucleons in and out of an à priori assumed Fermi sea and in the process they get a Lagrangian density describing the pion in the medium which corresponds to the above Skyrme Lagrangian except that the quadratic (current algebra) and the mass terms pick up a density dependence of the form

$$-\frac{f_\pi^2}{4} \left(g^{\mu\nu} + \frac{D^{\mu\nu} \rho}{f_\pi^2} \right) \text{Tr}(U^\dagger \partial_\mu U U^\dagger \partial^\nu U) + \frac{f_\pi^2 m_\pi^2}{4} \left(1 - \frac{\Sigma_{\pi N}}{f_\pi^2 m_\pi^2} \rho \right) \text{Tr}(U + U^\dagger - 2), \quad (5)$$

where ρ is the density of the nuclear matter and $D^{\mu\nu}$ and σ are physical quantities obtained from the pion-nucleon interactions. Note that in this scheme, nuclear matter is assumed ab initio to be a Fermi sea devoid of the intrinsic dependence mentioned above. We shall discover in the next section that our free pion Lagrangian will be also modified in a similar manner but in the crudest approximation. At a more advanced level, there will be an intricate nontrivial density dependence missing in the chiral perturbative calculations.

Having stated our scheme we proceed to describe dense baryonic matter in the Skyrme model. Consider a static field configuration $U_0(\vec{x}) = \sigma + i\vec{\tau} \cdot \vec{\pi}$ with $\sigma^2 + \vec{\pi}^2 = 1$. It is a nonlinear realization of the pion fields and a mapping, S^3 , from real space R^3 to the internal $SU(2)$ space. If we take for U_0 at infinity a constant $SU(2)$ matrix, the configurations can be classified according to their winding number

$$B = \int d^3x \rho_B(\vec{x}), \quad (6)$$

with

$$\rho_B(\vec{x}) = \frac{1}{24\pi^2} \epsilon_{ijk} \text{Tr}(U_0^\dagger \nabla_i U_0 U_0^\dagger \nabla_j U_0 U_0^\dagger \nabla_k U_0). \quad (7)$$

³Implementing dilatons as was done in [1] will play the role of incorporating the “intrinsic density dependence” inherent in Harada-Yamawaki theory of hidden local symmetry with the vector manifestation fixed point [2]. This point is being pursued and will be discussed in a forthcoming publication.

Skyrme [23] used this winding number to describe the baryon number. It is therefore through topology that this Lagrangian, written in terms of pion fields, describes baryonic systems.

Various solutions of static field configurations which describe infinite matter can be obtained from skyrmions by assigning specific symmetries [25, 26, 27, 28, 29, 30]. We proceed to describe in detail one of these solutions as a first step to study the behavior of the elementary pion in skyrmion matter. We adopt the method developed by Kugler and Shtrikman [30]. They construct as an Ansatz for skyrmion matter a crystal whose configurations are such that each neighboring pair is rotated in isospin space, relative to the other with respect to the line joining them, by π . Our starting crystal is FCC (face centered cubic) and the local baryon number takes its maximum at the FCC lattice sites. At high density, this configuration makes a phase transition to a CC (cubic centered) crystal, so called “*half-skyrmion*” [27] crystal, because only half of the baryon number carried by the single skyrmion is concentrated at the original FCC sites, while the other half is concentrated at the center of the links connecting those points, where the baryon number density was negligible initially. Thus the phase transition leads to a CC crystal made up of half-skyrmions.

Let us construct the initial skyrmion matter solution. Consider a point in space $\vec{x} = (x, y, z)$ at which the field is given by $(\sigma, \pi_1, \pi_2, \pi_3)$. Then, the FCC configuration is defined by the following symmetries:

- (S1) under reflection in space $\vec{x} \rightarrow (-x, y, z)$, the field is also reflected in isospin space according to $(\sigma, -\pi_1, \pi_2, \pi_3)$,
- (S2) under a rotation around the threefold axis in space $\vec{x} \rightarrow (y, z, x)$, the field is simultaneously rotated in isospin space about the corresponding axis in isospin space according to $(\sigma, \pi_2, \pi_3, \pi_1)$,
- (S3) under a rotation around the fourfold axis in space $\vec{x} \rightarrow (x, z, -y)$, the field is rotated around the corresponding fourfold axis in isospin space according to $(\sigma, \pi_1, \pi_3, -\pi_2)$,
- (S4) under a translation from a corner of a cube to the center of a face $\vec{x} \rightarrow (x + L, y + L, z)$, the field becomes rotated by π about the axis perpendicular to the face according to $(\sigma, -\pi_1, -\pi_2, \pi_3)$.

Here, $2L$ is the size of the single FCC unit cell which contains 4 skyrmions. Thus, the baryon number density is $\rho = 1/2L^3$. The normal nuclear matter density $\rho_0 = 0.17/\text{fm}^3$ corresponds to $L \sim 1.43$ fm.

We initially introduce “*unnormalized*” fields $(\bar{\sigma}, \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$, which are then “*normalized*” to define properly $U_0 = \sigma + i\vec{\tau} \cdot \vec{\pi}$ by

$$\sigma = \frac{\bar{\sigma}}{\sqrt{\bar{\sigma}^2 + \bar{\pi}_1^2 + \bar{\pi}_2^2 + \bar{\pi}_3^2}}, \quad (8)$$

with similar equations being satisfied by π_i ($i = 1, 2, 3$). The field configurations obeying the above symmetries can be easily found by expanding the unnormalized fields as Fourier series, i.e.

$$\bar{\sigma} = \sum_{a,b,c} \bar{\beta}_{abc} \cos(a\pi x/L) \cos(b\pi y/L) \cos(c\pi z/L), \quad (9)$$

and

$$\bar{\pi}_1 = \sum_{h,k,l} \bar{\alpha}_{hkl} \sin(h\pi x/L) \cos(k\pi y/L) \cos(l\pi z/L), \quad (10)$$

$$\bar{\pi}_2 = \sum_{h,k,l} \bar{\alpha}_{hkl} \cos(l\pi x/L) \sin(h\pi y/L) \cos(k\pi z/L), \quad (11)$$

$$\bar{\pi}_3 = \sum_{h,k,l} \bar{\alpha}_{hkl} \cos(k\pi x/L) \cos(l\pi y/L) \sin(h\pi z/L). \quad (12)$$

The symmetry relations (S1)-(S4) restrict the modes appearing in eqs. (9-12) as follows;

(M1) if h is even, then k, l are restricted to odd numbers and a, b, c are to even numbers,

(M2) if h is odd, then k, l are restricted to even numbers and a, b, c are to odd numbers.

Furthermore, the coefficients of the expansion satisfy $\bar{\alpha}_{hkl} = \bar{\alpha}_{hlk}$ and $\bar{\beta}_{abc} = \bar{\beta}_{bca} = \bar{\beta}_{cab} = \bar{\beta}_{acb} = \bar{\beta}_{cba} = \bar{\beta}_{bac}$. Note that the normalization process (8) does not spoil any symmetries that the unnormalized fields have, while the expansion coefficients α_{hkl} and β_{abc} lose their meaning as Fourier coefficients in the normalized fields.

Without loss of generality, we can locate the centers of the skyrmions at the corners of the cube and at the centers of the faces by letting $\sigma = -1$ and $\pi_i (i = 1, 2, 3) = 0$ at those points. For the skyrmion field to have definite integer baryon number per cite, we should have $\sigma = +1$ and $\pi_i (i = 1, 2, 3) = 0$ at the points such as $(L, 0, 0)$. It gives the constraint

$$\sum_{a,b,c=\text{even}} \bar{\beta}_{abc} = 0. \quad (13)$$

If only the modes (M2) appear in the expansion, the configuration has an additional symmetry, namely that under the translation $\vec{x} \rightarrow (x+L, y, z)$, the field rotates under $O(4)$ by π in the σ, π_1 plane. This configuration corresponds to the *half-skyrmion* CC discussed above. Because of this additional symmetry, the physical quantities such as the local baryon number density and the local energy density become completely identical around the points with $\sigma = -1$ and the points with $\sigma = +1$. Thus, one half of the baryon number carried by a single skyrmion is concentrated at the sites where the centers of the single skyrmion is in the FCC crystal. The other half of the baryon number is now concentrated on the links connecting those points, where σ takes the value $+1$, while in this location in the FCC configuration the baryon number density is low. As a consequence the average value $\langle \sigma \rangle$ vanishes and we will see that this phenomenon signals that, in the dense medium, chiral symmetry restoration for the pion dynamics takes place. It is important to stress that it is the precise structure of the ground state which is responsible for the restoration. Both modes (M1) and (M2) are included in the calculation. The half-skyrmion crystal arises as the stable ground state at high density, because the expansion coefficients associated with the modes (M1) become suppressed.

The Fourier series expansion method works well for the unnormalized fields and only a few modes are necessary.⁴ The expansion coefficients are used as variational parameters and determined by minimizing the energy of the configuration. The coefficients depend on the box size L . In Table 1, we list a few modes below $E = 16(\pi/L)^2$ and we use only the modes with $E \leq 10(\pi/L)^2$ in our calculation.

In Fig. 1 we show the energy per baryon E/B as a function of the FCC box size parameter L . Each point denotes the minimum energy of the crystal configuration with a given value of L , which we find by using a proper minimization program such as “the down-hill simplex method” [31]. The solid circles correspond to the zero pion mass calculation and reproduce those of Kugler and Shtrikman [30]. In order for an easier comparison we present the results

⁴Of course, through the normalization process (8), infinitely many modes appear in the normalized fields. However, the expansions converge rapidly.

Table 1: A few modes used in the Fourier series expansion coefficients α_{hkl} and β_{abc} in eqs. (9-12). E is the energy of the modes in units $(\pi/L)^2$ and d is the degeneracy of the mode.

h	k	l	E	d	a	b	c	E	d
1	0	0	1	1	0	0	0	0	1
1	2	0	5	2	1	1	1	3	1
2	1	1	6	1	2	0	0	4	3
1	2	2	9	1	2	2	0	8	3
3	0	0	9	1					
2	3	1	12	2	3	1	1	11	3
3	2	0	13	2	2	2	2	12	1

with L given in units of $(ef_\pi)^{-1}$ ($\sim 0.45\text{fm}$) and E/B in units of $(6\pi^2 f_\pi)/e$ (~ 1160 MeV), respectively. The latter enable us to compare the numerical results on E/B easily with its Bogolmoln'y bound for the skyrmion in the chiral limit, which can be expressed as $E/B = 1$ in this convention.

By looking at the figure one sees that as we squeeze the system from $L = 6$ to around $L = 3.8$, the skyrmion system undergoes a phase transition from the FCC single skyrmion configuration to the CC half-skyrmion configuration. The system has a minimum energy at $L \sim 2.4$ with the energy per baryon $E/B \sim 1.038$, which is very close to the Bogolmoln'y bound. One can note in Fig. 1 that in the region $L > 3.8$, two different phases coexist. This coexistence leads to an interesting hysteresis phenomenon in the numerical process for finding the solutions, where one uses the found solution set for an L as the initial trial functions for the next L . If one starts from a small L , the numerical program might stay in a *quasi-stable* half-skyrmion CC configuration up to quite large values of $L > 6$, then suddenly it changes to the single skyrmion FCC configuration. This phase transition – and hysteresis – occurs in the unstable region where the system has negative pressure, so does not have any physical consequence.

The phase to the left of the minimum is referred to in this work as “*homogeneous*” and there the background fields are described by a crystal configuration. The phase to the right of the minimum is “*inhomogeneous*” since the pressure $P \equiv \partial E/\partial V$ is negative, the skyrmion matter is unstable, and to make it stable we will assume a realization in terms of an “*inhomogeneous phase*” where the skyrmions condense to form dense lumps with large empty spaces.

The open circles are the solutions found with a nonvanishing pion mass, $m_\pi = 140$ MeV.⁵ Comparing to the skyrmion system for massless pions, the energy per baryon is somewhat higher. Furthermore, there is no first order phase transition at low densities.

In Fig. 2, we present $\langle\sigma\rangle$, i.e. the average value of σ over space as a function of L . In the chiral limit, $\langle\sigma\rangle$ rapidly drops as the system shrinks. It reaches zero at $L \sim 3.8$. where the system goes to the half-skyrmion phase. This phase transition can be interpreted, once the pion fluctuations are incorporated, as a signal for chiral symmetry restoration. In the case of massive pions, the transition in $\langle\sigma\rangle$ is soft. Its value monotonically decreases and reaches zero asymptotically, as the density increases. Furthermore, as we can see in Fig. 3, where the local

⁵Incorporating the pion mass into the problem introduces a new scale in the analysis and therefore we are forced to give specific values to the parameters of the chiral effective Lagrangian, the pion decay constant and the Skyrme parameter, a feature which we have avoided in the chiral limit. In order to proceed, we simply take their empirical values, that is, $f_\pi = 93$ MeV and $e = 4.75$. Although the numerical results depend on these values, their qualitative behavior will not.

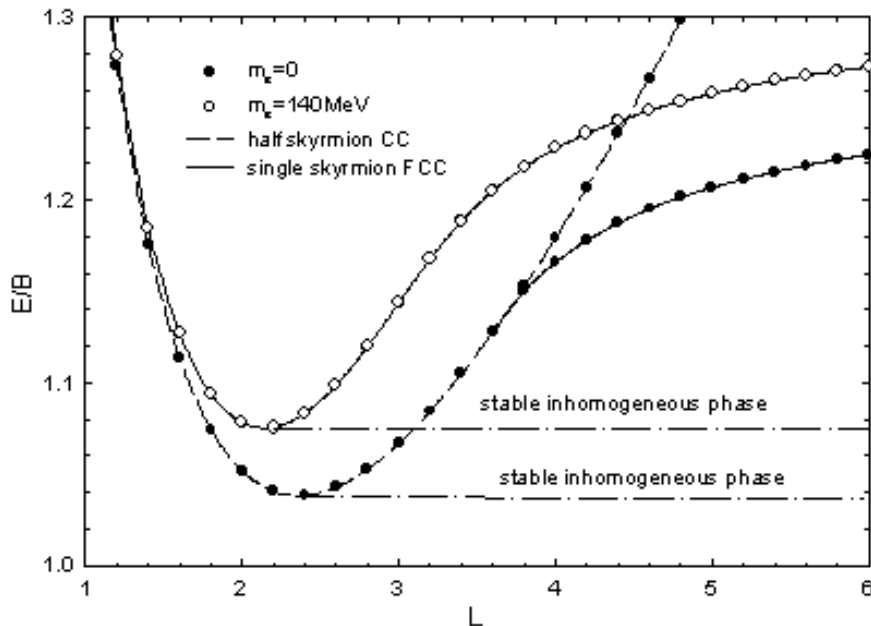


Figure 1: Energy per single skyrmion as a function of the size parameter L . The solid circles show the results for massless pions and the open circles are those for massive pions. Note the rapid phase transition around $L \sim 3.8$ for massless pions.

baryon number density is shown, for $L = 2$ (left) and $L = 3.5$ (right) in the $z = 0$ plane, the system is (almost) a half-skyrmion crystal at high density.

3 Pions in Skyrmion Matter

Let $U_0(\vec{x}) = \sigma + i\vec{\tau} \cdot \vec{\pi}$ be the static skyrmion crystal configuration obtained so far. Then, we incorporate on it fluctuating *time-dependent* pion fields through the ansatz [32]

$$U(\vec{x}, t) = \sqrt{U_\pi} U_0(\vec{x}) \sqrt{U_\pi}, \quad (14)$$

where

$$U_\pi = \exp\left(i\vec{\tau} \cdot \vec{\phi}(x)/f_\pi\right), \quad (15)$$

with $\vec{\phi}$ describing the fluctuating pions. Substituting this ansatz into the Skyrme Lagrangian (4) and expanding the Lagrangian in terms of pions up to second order, we obtain the Lagrangian for the pion in the skyrmion matter which is given by

$$\mathcal{L} = \mathcal{L}_B(U_0) + \mathcal{L}_M(U_\pi) + \mathcal{L}_I, \quad (16)$$

where $\mathcal{L}_B(U_0)$ is the Lagrangian density for the skyrmion matter described by the static configuration U_0 , \mathcal{L}_M is for the Lagrangian density for the pions, and finally \mathcal{L}_I describes the interaction between the fluctuating pions and the skyrmion matter. Up to second order in the pion fields, \mathcal{L}_M is simply the free pion Lagrangian to

$$\mathcal{L}_M = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m_\pi^2 \phi_a^2. \quad (17)$$

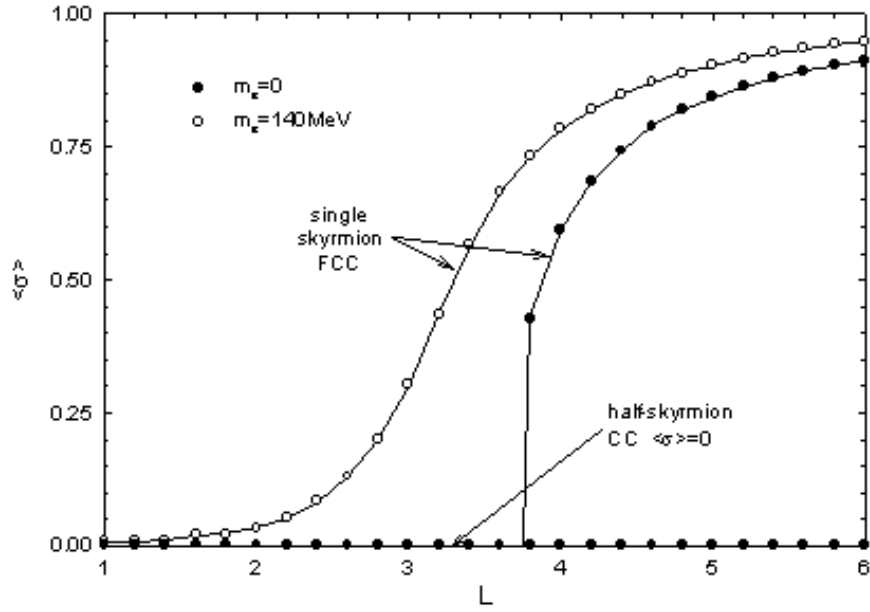


Figure 2: $\langle \sigma \rangle$ as a function of the size parameter L . The notation is the same as in Fig.1

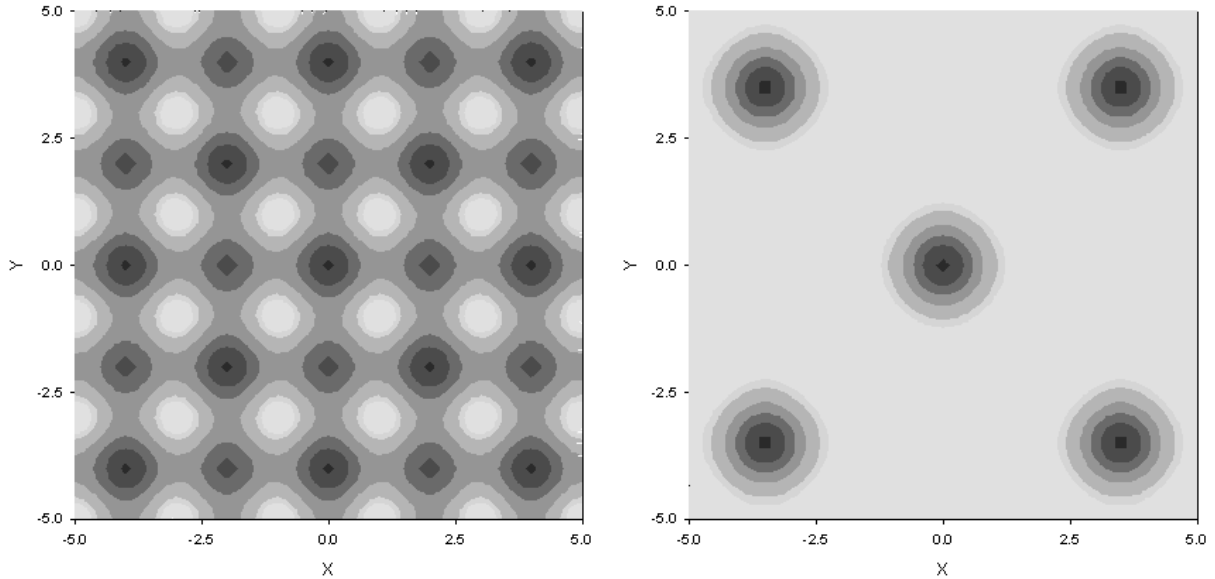


Figure 3: Local baryon number densities at $L = 3.5$ and $L = 2.0$ with massive pions. For $L = 2.0$ the system is (almost) a half-skyrmion in a CC crystal configuration.

\mathcal{L}_I is quite complicated and is given by

$$\begin{aligned}\mathcal{L}_I &= \frac{1}{2}\dot{\phi}_a G^{ab}(\vec{x})\dot{\phi}_b - \frac{1}{2}(\partial_i \phi_a) H_{ij}^{ab}(\vec{x})(\partial_j \phi_b) \\ &\quad - \frac{1}{f_\pi}(\partial_i \phi_a) A_i^a(\vec{x}) - \frac{1}{2f_\pi^2} \epsilon_{abc}(\partial_i \phi_a)\phi_b V_i^c(\vec{x}) \\ &\quad + \frac{1}{4}m_\pi^2 \phi_a^2 \text{Tr}(1 - U_0) + \frac{1}{2}m_\pi^2 f_\pi \phi_a \text{Tr}(i\tau_a U_0)\end{aligned}\quad (18)$$

where

$$G^{ab}(\vec{x}) = \frac{1}{4}\text{Tr}\left(\tau^a U_0 \tau^b U_0^\dagger - \tau^a \tau^b\right) + \frac{1}{32e^2 f_\pi^2} \text{Tr}\left([R_i, \tilde{\tau}^a][R_i, \tilde{\tau}^b]\right), \quad (19)$$

$$H_{ij}^{ab}(\vec{x}) = G^{ab}\delta_{ij} + \frac{1}{32e^2 f_\pi^2} \text{Tr}\left([R_i, R_j][\tilde{\tau}^a, \tilde{\tau}^b] - [R_i, \tilde{\tau}^b][R_j, \tilde{\tau}^a]\right), \quad (20)$$

$$V_i^a(\vec{x}) = \frac{i}{4}f_\pi^2 \text{Tr}[(L_i + R_i)\tau_a] + \frac{i}{16e^2} \text{Tr}\left([L_j, \tau_a][L_i, L_j] + [R_j, \tau_a][R_i, R_j]\right), \quad (21)$$

$$A_i^a(\vec{x}) = \frac{i}{4}f_\pi^2 \text{Tr}[(L_i - R_i)\tau_a] + \frac{i}{16e^2} \text{Tr}\left([L_j, \tau_a][L_i, L_j] - [R_j, \tau_a][R_i, R_j]\right) \quad (22)$$

with $L_i = (\partial_i U_0^\dagger)U_0$ and $R_i = (\partial_i U_0)U_0^\dagger$ and $\tilde{\tau}^a = \tau^a + U_0 \tau^a U_0^\dagger$. The currents V_i^a and A_i^a produced by the static background field satisfy the following conservation equations:

$$\vec{\nabla} \cdot \vec{V}^a = 0, \quad (23)$$

$$\vec{\nabla} \cdot \vec{A}^a = -\frac{1}{2}m_\pi^2 f_\pi^2 \text{Tr}(i\tau_a U_0). \quad (24)$$

The structure of our Lagrangian comes out to be very similar to that of chiral perturbation theory, eq. (5), and refs. [7, 8]. Let us use this fact to make some crude estimates.

The interactions of pions and skyrmions as given by eq. (18) modifies the mass term in eq. (17) as

$$\frac{1}{2}m_\pi^2 \sigma(\vec{x})\phi_a^2. \quad (25)$$

This implies that the *local* effective pion mass can be even imaginary in the central part of the skyrmions where σ is negative. By taking the average over space of σ , we obtain an effective mass of the pion in the baryonic medium as

$$m_\pi^* \sim m_\pi \langle \sigma \rangle^{1/2}. \quad (26)$$

The kinetic term is also modified by the background, with respect to the free case, with an additional factor $G^{ab}(\vec{x})$ given by

$$G^{ab}(\vec{x}) = -(\pi^2 \delta_{ab} - \pi_a \pi_b) + \frac{1}{e^2 f_\pi^2} \left\{ \left((\partial_i \sigma)^2 + (\partial_i \vec{\pi})^2 \right) \left(\sigma^2 \delta_{ab} + \pi_a \pi_b \right) \right. \quad (27)$$

$$\left. - \left(\partial_i \sigma \pi_a - \sigma \partial_i \pi_a \right) \left(\partial_i \sigma \pi_b - \sigma \partial_i \pi_b \right) \right\}. \quad (28)$$

The first term in $G^{ab}(\vec{x})$, $-\pi^2 \delta_{ab} + \pi_a \pi_b$, arises from the quadratic term of the Skyrme model Lagrangian, eq. (4), and corresponds to the term containing D^{00} in eq. (5). Let us denote it by $\Gamma^{ab}(\vec{x})$. It changes the kinetic term in eq. (17) to

$$\frac{1}{2}f_\pi^2 (1 + \Gamma^{ab}(\vec{x}))\dot{\phi}_a \dot{\phi}_b. \quad (29)$$

Ignoring that the inverse of this matrix Γ^{ab} becomes singular at the points where $\pi^2 = 1$ and taking the diagonal element, for example, Γ^{11} , eq. (29) implies that the pion decay constant behaves *locally* as

$$f_\pi^*(\vec{x}) \sim f_\pi(1 + \Gamma^{11}(\vec{x}))^{\frac{1}{2}}. \quad (30)$$

The average over the space leads to an effective pion decay constant in the medium f_π^* given by ⁶

$$f_\pi^* \sim f_\pi(1 - \frac{2}{3}\langle\pi^2\rangle)^{\frac{1}{2}}. \quad (31)$$

If we absorb this change of f_π in the fields to keep the functional form of the theory as in the free case, eq. (26) is modified to

$$m_\pi^* \sim m_\pi \left(\frac{\langle\sigma\rangle}{1 - \frac{2}{3}\langle\pi^2\rangle} \right)^{\frac{1}{2}}. \quad (32)$$

This equation shows the direct relation between the pion mass in the medium, m^* , and the parameter $\langle\sigma\rangle$ which signalled the phase transition in the case of skyrmion matter. We next show that chiral symmetry is restored in dense matter.

In Fig. 4 we show the estimates of f_π^*/f_π and m_π^*/m_π as a function of the density. As the density increases, f_π^* decreases to $\sim 0.65f_\pi$ and then it remains constant at that value ⁷. Note that $\langle\sigma\rangle^{\frac{1}{2}}$ has the same slope at low densities, which leads to $m_\pi^*/m_\pi \sim 1$ at low densities. Since at higher densities $\langle\pi^2\rangle$ becomes a constant, m_π^*/m_π decreases like $\langle\sigma\rangle^{1/2}$ with a factor which is greater than 1. As the density increases, higher order terms in ρ come to play important roles and m_π^*/m_π decreases.

The slope of $\langle\sigma\rangle$ at low density is approximately $1/3$. If we expand $\langle\sigma\rangle$ about $\rho = 0$ and compare it with eq. (5), we obtain

$$\langle\sigma\rangle \sim 1 - \frac{1}{3} \frac{\rho}{\rho_0} + \dots \sim 1 - \frac{\Sigma_{\pi N}}{f_\pi^2 m_\pi^2} \rho + \dots, \quad (33)$$

which yields $\Sigma_{\pi N} \sim m_\pi^2 f_\pi^2 / (3\rho_0) \sim 42$ MeV, which is comparable with the experimental value 45 MeV ⁸. This comparison is fully justified from the point of view of the $\frac{1}{N}$ expansion since both approaches should produce the same result to leading order in this expansion. The linear term is $\mathcal{O}(1)$.

The length scale is strongly dependent on our choice of the parameters f_π and e . Thus one should be aware that the ρ scale in Fig. 4 could change quantitatively considerably if one chooses another parameter set, however the qualitative behavior will remain unchanged.

We next proceed to a more rigorous derivation of these quantities using perturbation theory. The presence of the zero modes, associated with the collective motion of the skyrmion, and the complexity of G^{ab} make the quantization procedure non-trivial. Furthermore, G^{ab} becomes singular at the points where $\sigma(\vec{x}) = 0$.

Let us obviate the complications associated with the zero modes and proceed. The momenta conjugate to the pion field ϕ_a are given by

$$\Pi_a = (\delta_{ab} + G^{ab}(\vec{x})) \dot{\phi}_b \quad (34)$$

⁶Comparing eqs. (18), (19) and (20) with eq. (5) it is apparent that to second order \mathcal{L}_I leads to different time and space components for f_π^* . We keep the present discussion to leading order where they are equal.

⁷These estimates are based on the lowest order approximation and should be valid only for low densities.

⁸While this value is widely quoted, there is at present a considerable controversy on the precise value of this sigma term. In fact it can even be considerably higher than this. See [33] for a recent discussion.

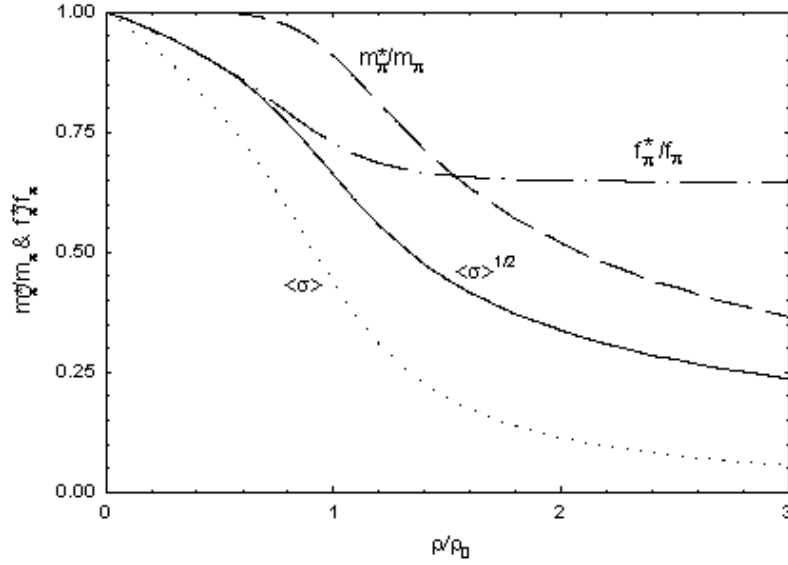


Figure 4: Naive estimates of m_π^*/m_π and f_π^*/f_π as functions of the baryon number density of skyrmion matter.

which leads to the following Hamiltonian

$$\mathcal{H} = \Pi_a \dot{\phi}_a - \mathcal{L}_M - \mathcal{L}_I. \quad (35)$$

It is convenient, as in chiral perturbation theory [34], to decompose it into the free pion Hamiltonian, \mathcal{H}_0 , plus the interacting Hamiltonian, \mathcal{H}_I , defined by

$$\mathcal{H}_0 = \frac{1}{2} \dot{\phi}_a \dot{\phi}_a + \frac{1}{2} (\vec{\nabla} \phi_a) \cdot (\vec{\nabla} \phi_a) + \frac{1}{2} m_\pi^2 \phi_a \phi_a, \quad (36)$$

$$\begin{aligned} \mathcal{H}_I &= \frac{1}{2} \dot{\phi}_a G^{ab}(\vec{x}) \dot{\phi}_b + \frac{1}{2} (\partial_i \phi_a) H_{ij}^{ab}(\vec{x}) (\partial_j \phi_b) \\ &\quad - \frac{1}{2f_\pi^2} \epsilon_{abc} (\partial_i \phi_a) \phi_b V_i^c(\vec{x}) + \frac{1}{2} m_\pi^2 \phi_a (\sigma(\vec{x}) - 1) \phi_a. \end{aligned} \quad (37)$$

Note that the momenta Π_a , defined by eq. (34), and used in the derivation of the Hamiltonian do not appear in \mathcal{H}_0 . This simplifies matters considerably since only $\dot{\phi}_a$ will define the canonical momenta of the quantization procedure. \mathcal{H}_0 defines the free pion propagator and \mathcal{H}_I the interaction potentials as summarized in Fig. 5.

We define the effective mass of the pion in skyrmion matter as the pole in the pion propagator in the limit of $\vec{p} \rightarrow 0$,

$$G(p) = \frac{1}{p^2 - m_\pi^2 - \Sigma(p)}, \quad (38)$$

where $\Sigma(p)$ is the pion self-energy in matter. Thus, the interactions associated with the spatial derivatives acting on the pion fields will not play any role in the process up to second order in the interactions. Only the interactions characterized by G^{ab} and σ will be active to this order.

Due to the crystal structure of the background fields in the homogeneous phase, $G^{ab}(\vec{x})$ and $\sigma(\vec{x})$ can be expressed as

$$\sigma(\vec{x}) = \sum_{\ell_x, \ell_y, \ell_z} \hat{S}(\vec{\ell}) \exp(i\vec{\ell} \cdot \vec{x}\pi/L), \quad (39)$$

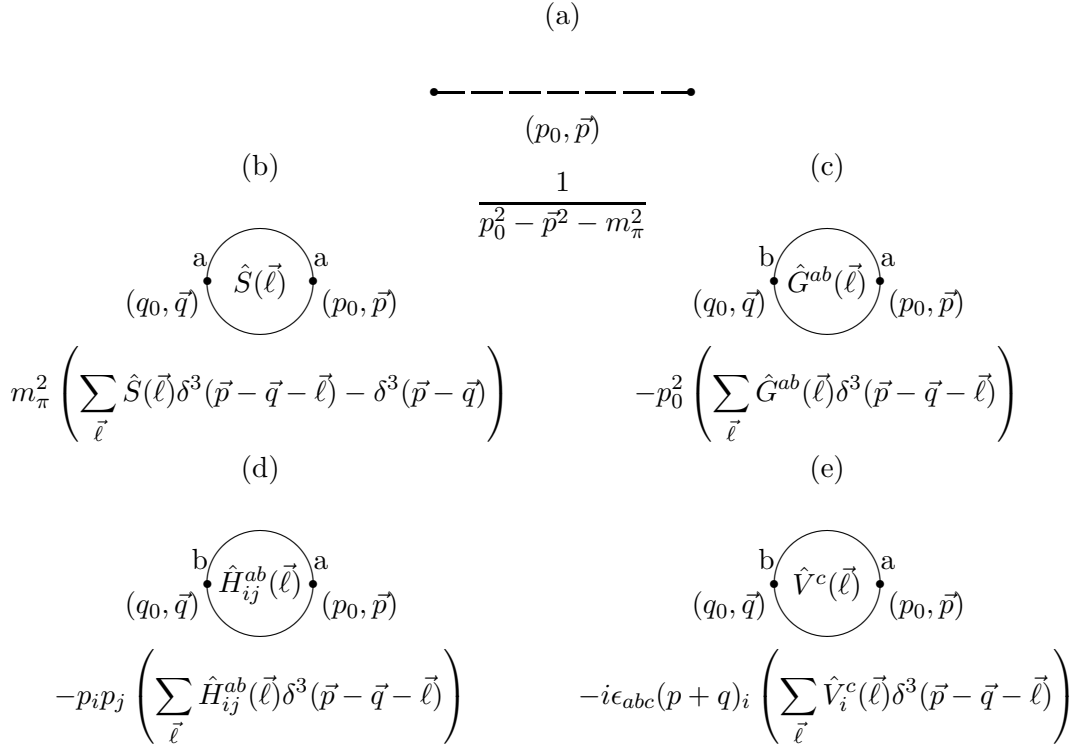


Figure 5: Free propagator and interactions between the pion fields and background skyrmion matter.

Table 2: Possible integer values for the expansion coefficients $\beta_{\vec{\ell}}$, $\gamma_{\vec{\ell}}^{11}$ and $\gamma_{\vec{\ell}}^{12}$.

$\beta_{\vec{\ell}}$	ℓ_x	ℓ_y	ℓ_z	$\gamma_{\vec{\ell}}^{11}$	ℓ_x	ℓ_y	ℓ_z	$\gamma_{\vec{\ell}}^{12}$	ℓ_x	ℓ_y	ℓ_z
(i)	even	even	even	(i)	odd	odd	odd	(i)	odd	odd	even
(ii)	odd	odd	odd	(ii)	even	even	even	(ii)	even	even	odd

$$G^{ab}(\vec{x}) = \sum_{\ell_x, \ell_y, \ell_z} \hat{G}^{ab}(\vec{\ell}) \exp(i\vec{\ell} \cdot \vec{x}\pi/L), \quad (40)$$

with the help of a set of integers $\vec{\ell} = (\ell_x, \ell_y, \ell_z)$. Thus, the corresponding $\hat{\sigma}(\vec{p})$ and $\hat{G}^{ab}(\vec{p})$ in momentum space have the following structure,

$$\hat{\sigma}(\vec{p}) = \sum_{\vec{\ell}} \hat{S}(\vec{\ell}) \delta^3(\vec{p} - \vec{\ell}), \quad \hat{\gamma}^{ab}(\vec{p}) = \sum_{\vec{\ell}} \hat{G}^{ab}(\vec{\ell}) \delta^3(\vec{p} - \vec{\ell}). \quad (41)$$

The FCC crystal yields non-vanishing expansion coefficients only for the integer set listed in Tab. 2. Furthermore, the coefficients have the following symmetries

$$\begin{aligned} \hat{S}(\ell_x, \ell_y, \ell_z) &= +\hat{S}(-\ell_x, \ell_y, \ell_z) = +\hat{S}(\ell_x, -\ell_y, \ell_z) = +\hat{S}(\ell_x, \ell_y, -\ell_z) \\ \hat{G}^{11}(\ell_x, \ell_y, \ell_z) &= \hat{G}^{11}(-\ell_x, \ell_y, \ell_z) = +\hat{G}^{11}(\ell_x, -\ell_y, \ell_z) = +\hat{G}^{11}(\ell_x, \ell_y, -\ell_z) \\ \hat{G}^{12}(\ell_x, \ell_y, \ell_z) &= -\hat{G}^{12}(-\ell_x, \ell_y, \ell_z) = -\hat{G}^{12}(\ell_x, -\ell_y, \ell_z) = +\hat{G}^{12}(\ell_x, \ell_y, -\ell_z) \end{aligned} \quad (42)$$

To the first order in the interactions, the self-energy gets contributions from $\sigma(\vec{0})$ and $G^{11}(\vec{0})$ given by

$$\Sigma^I(p_0, \vec{p} \rightarrow 0) = m_\pi^2 (\hat{S}(\vec{0}) - 1) - p_0^2 \hat{G}^{11}(\vec{0}). \quad (43)$$

To this order, the effective pion mass m_π^* becomes

$$m_\pi^{*I} = \left(\frac{\hat{S}(\vec{0})}{1 + \hat{G}^{11}(\vec{0})} \right)^{1/2} m_\pi. \quad (44)$$

Since $\hat{S}(\vec{0})$ is nothing but $\langle \sigma \rangle$, if we ignore the denominator, we can interpret our result as the naive estimate (26). Furthermore, keeping only the first order terms in $G^{11}(\vec{x})$, the result coincides with our estimate (32). Thus, eq. (44) differs from eq. (32) by the presence of the higher order term coming from the Skyrme term. Since the latter is positive definite, $\hat{G}^{11}(\vec{0})$ is always larger than $-\frac{2}{3}\langle \pi^2 \rangle$ appearing in eq. (32). According to our numerical results, however, $\hat{G}^{11}(\vec{0})$ takes negative values for a wide range of ρ down to the density where E/B has the minimum. Thus, eq. (44) tells us that m_π^*/m_π decreases slower than $\langle \sigma \rangle^{1/2}$ in this range of ρ , but faster than the naive estimate (32).

If we include the second order diagrams as shown in Fig. 6, we obtain the self-energy as

$$\begin{aligned} \Sigma^{II}(p_0, \vec{0}) &= \Sigma^I(p_0, \vec{0}) \\ &+ \sum_{\vec{\ell} \neq \vec{0}} \left(\frac{\hat{S}(\vec{\ell})\hat{S}(-\vec{\ell})m_\pi^4}{p_0^2 - m_\pi^2 - \ell^2\pi^2/L^2} - \frac{2\hat{S}(\vec{\ell})\hat{G}^{11}(-\vec{\ell})m_\pi^2 p_0^2}{p_0^2 - m_\pi^2 - \ell^2\pi^2/L^2} \right. \\ &\left. + \frac{(\hat{G}^{11}(\vec{\ell})\hat{G}^{11}(-\vec{\ell}) + 2\hat{G}^{12}(\vec{\ell})\hat{G}^{21}(-\vec{\ell}))p_0^4}{p_0^2 - m_\pi^2 - \ell^2\pi^2/L^2} \right). \end{aligned} \quad (45)$$

Figure 6: Diagrams used to evaluate the self-energy up to second order in the interactions. Here, b runs over 1,2,3 and $\vec{\ell} \neq 0$.

The resulting effective pion mass can be found numerically by solving the equation

$$p_0^2 - m_\pi^2 - \Sigma^{II}(p_0, \vec{0}) = 0, \quad (46)$$

after substituting $p_0 \equiv m_\pi^{*II}$.

In the chiral limit where $m_\pi = 0$, eq. (46) always provides us with the trivial solution $p_0 = 0$, which supports the consistency of our process. We show in Fig. 7 the numerical results on m_π^*/m_π as a function of the density ρ . We have taken in eq. (45) up to $|\vec{\ell}|^2 < 6$ Fourier components. Although, there is no apparent reason for the interactions to be weak, m_π^{*II} is very close to m_π^{*I} . The Fourier expansion for $\hat{S}(\vec{p})$ and $\hat{G}^{ab}(\vec{p})$ can only be done for the homogeneous skyrmion crystal.

Now that we have developed the perturbative expansion, we may check our estimate eq. (31) of the pion decay constant in the medium. We proceed by calculating it directly from the appropriate expectation value of the axial current.

The total axial current of the system can be written as

$$\begin{aligned} \mathcal{A}_{\mu,a} &= f_\pi \partial_\mu \phi_a(x) + f_\pi G^{ab}(\vec{x}) \partial_\mu \phi_b(x) + f_\pi H_{\mu\nu}^{ab}(\vec{x}) \partial^\nu \phi_b(x) \\ &+ \frac{1}{2f_\pi} \epsilon_{abc} \phi_b(x) V_{\mu,c}(\vec{x}) + A_{\mu,a}(\vec{x}), \end{aligned} \quad (47)$$

where $A_{\mu,a}$ and $V_{\mu,a}$ are the axial and vector currents from the background skyrmion matter, respectively. The total axial current satisfies the following conservation relation,

$$\partial^\mu \mathcal{A}_{\mu,a} = -f_\pi m_\pi^2 \sigma(\vec{x}) \phi_a(x) - \frac{1}{2f_\pi} \epsilon_{abc} \partial^\mu \phi_b(x) V_{\mu,c}(\vec{x}) + \frac{f_\pi^2}{2} m_\pi^2 \text{Tr}(i\tau_a U_0). \quad (48)$$

The axial current of the fluctuating pion is obtained from eq. (47) by omitting the background (last) term of the equation. Let us use the symbol $A_{\mu,a}^\phi$ for the in medium pion axial current. Its conservation leads to the equation of motion,

$$\partial^\mu A_{\mu,a}^\phi = -f_\pi m_\pi^2 \sigma(\vec{x}) \phi_a(x) - \frac{1}{2f_\pi} \epsilon_{abc} \partial^\mu \phi_b(x) V_{\mu,c}(\vec{x}). \quad (49)$$

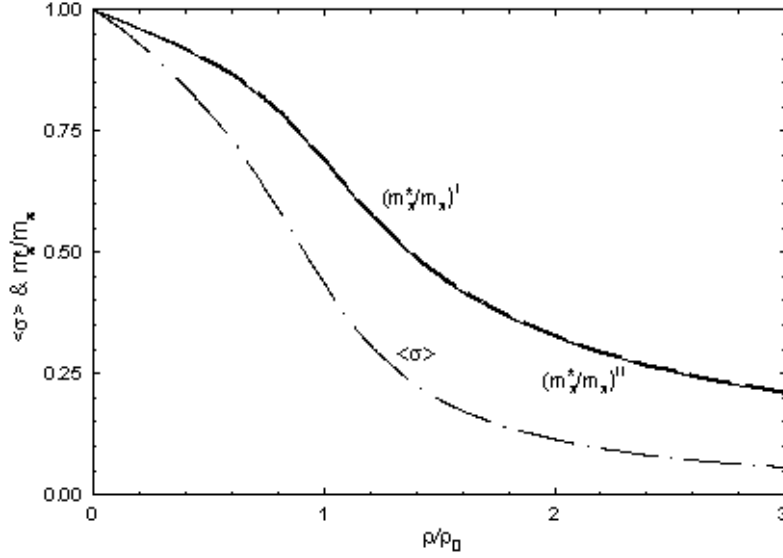


Figure 7: m_{π}^{*I} and m_{π}^{*II} as a function of the density. We have used $f_{\pi}=93\text{MeV}$ and $e = 4.75$ in the calculation.

Applying the LSZ reduction formalism, the appropriate matrix element of the pion axial current in the ground state of skyrmion matter is,

$$\langle \tilde{0} | A_{\mu,1}^{\phi}(x) | \pi_1(q) \tilde{0} \rangle = -i\sqrt{Z}f_{\pi}q_{\mu}(1 + G^{11}(\vec{x}))e^{-iq \cdot x}, \quad (50)$$

where we have used π^1 to perform the calculation, exploiting isospin symmetry, and Z is the wave function renormalization constant obtained as $Z^{-1} = (1 + G^{11}(\vec{0}))$ in eq. (43). The ground state of skyrmion matter is denoted by $|\tilde{0}\rangle$. In the spirit of our approximation we only consider the $\mu = 0$ component, to which $H_{\mu\nu}^{ab}$ does not contribute. If we regard the quantity appearing in the right hand side of the above equation as the in-medium pion decay constant and take the spatial average of $G^{11}(\vec{x})$, we obtain,

$$f_{\pi}^{*} = f_{\pi}(\sqrt{Z})^{-1} = f_{\pi}(1 + G^{11}(\vec{0}))^{1/2}, \quad (51)$$

which supports eq. (31). Similarly, the matrix element for the conservation relation becomes

$$\langle \tilde{0} | \partial^{\mu} A_{\mu,1}^{\phi}(x) | \pi_1(q) \tilde{0} \rangle = -\sqrt{Z}f_{\pi}m_{\pi}^{*2}\sigma(\vec{x}). \quad (52)$$

Again, if we take the quantity in the right hand as $-f_{\pi}^{*2}m_{\pi}^{*2}$, this equation gives the following relation between free and in-medium quantities,

$$\frac{m_{\pi}^{*2}}{m_{\pi}^2} \cdot \frac{f_{\pi}^{*}}{f_{\pi}} = \sqrt{Z}\sigma(\vec{0}). \quad (53)$$

Incorporating the relation eq. (51), we obtain

$$\frac{m_{\pi}^{*2}}{m_{\pi}^2} \cdot \frac{f_{\pi}^{*2}}{f_{\pi}^2} = \sigma(\vec{0}), \quad (54)$$

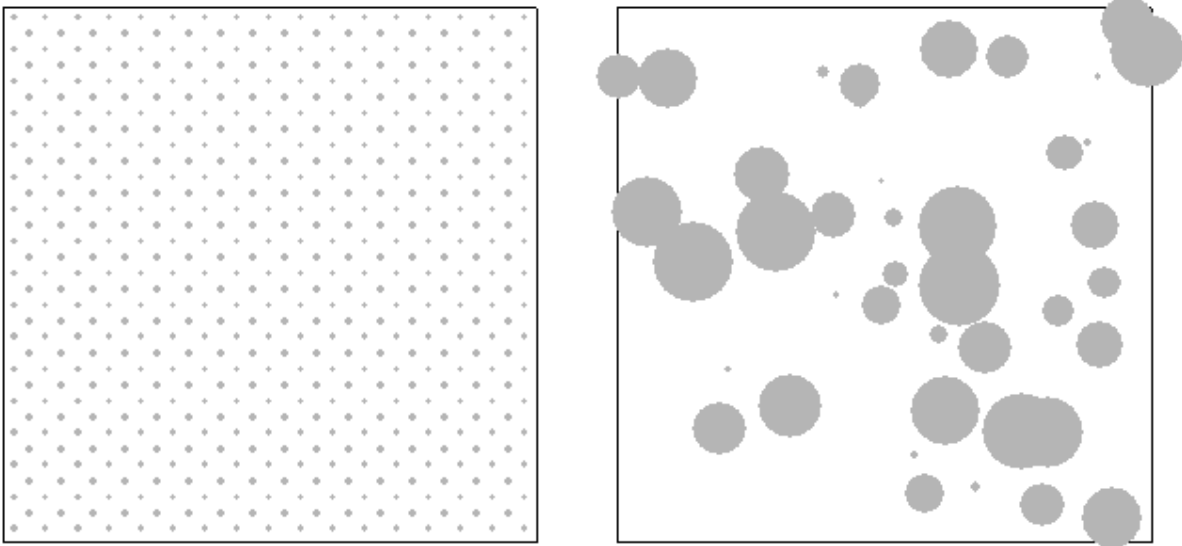


Figure 8: (a) The homogeneous but unstable crystal phase and (b) the inhomogeneous but stable skyrmion matter.

which is nothing but eq. (32).

Up to now, we have considered only the “*homogeneous*” phase where the background fields are in a crystal configuration. We next proceed to describe the “*inhomogeneous phase*”. In Fig. 8, we present the situation schematically; in (a) the skyrmions are in the homogeneous crystal phase, while in (b) they condense to form rather dense lumps of matter and empty spaces. Our crude approximation consists of describing the dense lumps of matter by the crystal structure at the minimum energy per baryon ignoring surface effects. In this approximation E/B of the inhomogeneous phase (b) will be the same as $(E/B)_{\min}$. That is, the inhomogeneous matter has a lower energy than the homogeneous one and becomes stable. In Fig. 1, such energy is represented by a horizontal line starting at the minimum. Our approximation can be interpreted as a Maxwell construction between the two phases.

In the inhomogeneous matter space is divided into two different phases. Let us distinguish these by using the subscripts “*vac*” denoting the (empty) *vacuum* phase and “*min*” denoting the lumps of skyrmion matter at the *minimum* energy. The expectation values $\langle\sigma\rangle$ can be approximated by

$$\langle\sigma\rangle^{(i)} \approx \frac{\langle\sigma\rangle_{\min}^{(h)} V_{\min} + \langle\sigma\rangle_{\text{vac}}^{(h)} V_{\text{vac}}}{V_{\min} + V_{\text{vac}}} \quad (55)$$

where the superscript “h” and “i” denote that the quantities are for the homogeneous and inhomogeneous phase, respectively.

In the inhomogeneous matter, the total volume of the dense droplets V_{\min} is nothing but L_{\min}^3 and the volume of the whole space is $V_{\min} + V_{\text{vac}}$. Thus, eq. (55) can be simply written as

$$\langle\sigma\rangle^{(i)} \approx \langle\sigma\rangle_{\text{vac}}^{(h)} - (\langle\sigma\rangle_{\text{vac}}^{(h)} - \langle\sigma\rangle_{\min}^{(h)}) (L_{\min}/L)^3. \quad (56)$$

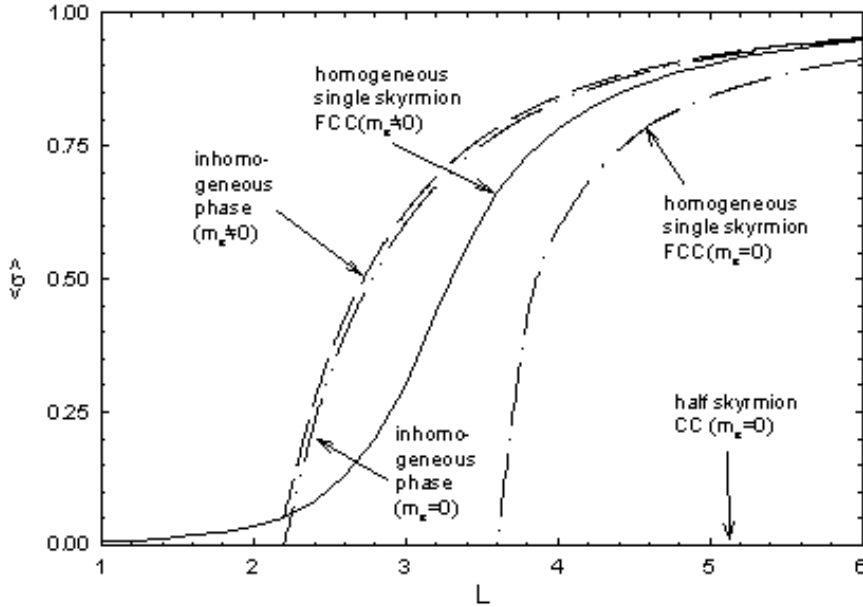


Figure 9: Maxwell construction for the homogeneous unstable crystal phase. We represent the change in $\langle \sigma \rangle$ as a function of lattice size.

The dash-dot lines in Fig. 9 represent the value of $\langle \sigma \rangle$ for the inhomogeneous phase obtained by using the approximate Maxwell construction defined by eq. (55). The change of $\langle \sigma \rangle$ in the chiral limit is remarkable. Chiral restoration occurs at the point $L = L_{min}$ where E/B has its lowest value through a phase transition from the inhomogeneous phase to the homogeneous half-skyrmion CC phase at $L = L_{min}$. The sharp phase transition may be an artifact of our naive approximation (55). In the case of $m_\pi \neq 0$ also a rapid change in $\langle \sigma \rangle$ occurs in the transition to the stable inhomogeneous phase, moreover there is almost no difference between the massive and massless cases.

We show in Fig. 10 the value of m_π^{*i} obtained by eq. (32) for the inhomogeneous phase.

We learn from our calculation that the dynamics of the pion in the medium depends not only on the density of the baryonic matter but also from its detailed structure. It may be possible to interpret this as simulating the effect of the “intrinsic” density dependence required in Harada-Yamawaki theory [2, 24].

4 Concluding Remarks

We have studied the dynamics of the pion in a baryonic environment by incorporating quantum fluctuations to classical skyrmion matter. The procedure has led to an effective Lagrangian for the pion in the medium defined in terms of the parameters, f_π^* and m_π^* , which are density-dependent. The fact that we have a unified hadronic scheme to calculate both baryonic matter and the interactions of the pion in the background of such matter, with only a few parameters, allows us to draw important conclusions. This contrasts with the conventional approach where

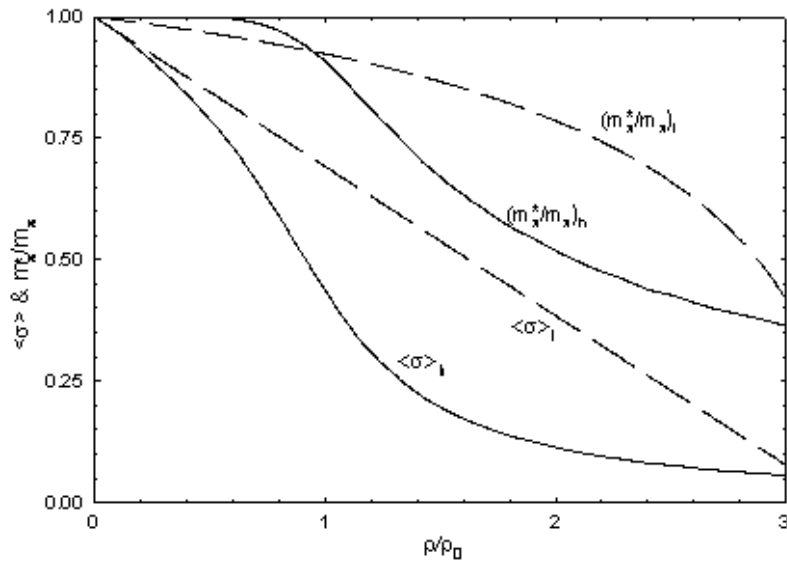


Figure 10: Maxwell construction for the homogeneous unstable crystal phase. We show m_{π}^{*h} , m_{π}^{*i} , $\langle \sigma \rangle_h$ and $\langle \sigma \rangle_i$ as a function of density.

the starting point is the matter-free vacuum, with the density effect taken into account perturbatively. While our scheme is still far from being realistic, it contains the right symmetries and the support of $\frac{1}{N}$ physics, which renders it qualitatively predictive if not yet quantitatively. We are able to describe interesting phenomena though we are not able to predict yet the scales at which they will occur.

The main result of our presentation is that, as the density of matter increases, chiral symmetry for the dynamics of the pion is restored, i.e. the effective parameters become those of a chiral symmetric phase. Two important dynamical mechanisms lead to this phenomenon. On the one hand the structure of the matter state is crucial in producing the phase transition. At high densities, the symmetries of the classical field describing the CC half skyrmion crystal, conspire to provide the σ components with adequately distributed values on the lattice so that its average tends to vanish as the crystal becomes more and more precisely defined. This phenomenon which occurs at the level of the baryonic state is carried over by the quantum fluctuations to the dynamics of the pion in this baryonic medium. Thus the important physical quantities of the pion become density-dependent in terms of the expectation values of the classical fields and therefore are essential to this phase transition, which tends to restore chiral symmetry in the dynamics of the pion. On the other hand our procedure produces a density dependence which is non-trivial, i.e., the classical field leads to higher orders in the density dependence of the parameters. This intricate density dependence is analogous, if not entirely equivalent, to the intrinsic density dependence found in hidden local symmetry theory with the vector manifestation fixed point [24]. The results show that these additional terms, which do not appear in more conventional calculations, play an important role in the phase transition, suggesting that one should not trust results of this phenomenon obtained with the linear approximation or in few-order low-density expansions.

It is important to stress that the results just exposed have required the development of a non-trivial perturbative procedure which has been solved beyond the leading order in the case of the effective pion mass.

Ours is not yet a realistic calculation. The model we have used describes skyrmion matter, not nuclear matter. For instance we do not have the liquid structure of normal nuclear matter. In fact it appears nontrivial to arrive at a Fermi liquid from the crystal structure. Specifically while the in-medium pion decay constant going proportional to $\langle\sigma\rangle^{1/2}$ up to near nuclear matter density is consistent with what one expects in the effective field theory with vector manifestation [24], its saturation at higher density and non-vanishing as one approaches the chiral transition indicate that we may be describing a phase which is not the standard Wigner-Weyl symmetry. It may be in a pseudo-gap phase where the gap is non-zero though chiral symmetry is restored, resembling what might be happening in the “normal phase” of high T_c superconductivity [35]⁹. This may account for the fact that the in-medium pion mass *decreases* as one goes beyond normal matter density, which is at variance with the results of other approaches.

It should be reiterated that the beauty of the whole procedure is that a unique Lagrangian is able to describe in a unified way *all* hadronic interactions, the ground state and fluctuations based on it. This hadronic theory provides us with results which are promising and exciting while exposing weaknesses of the more traditional approaches. Needless to say, there is still a long way to go before confronting Nature, such as, for instance, the properties of the pionic atoms that are offering a tantalizing view of what is going on in dense medium and of course the interior of compact stellar systems where chiral symmetry is presumably restored. Properly and realistically formulated, we might be able to “derive” from the first principles such novel

⁹We are grateful to Kurt Langfeld for bringing our attention to this reference.

notions as BR scaling [1, 24], the hidden local symmetry/vector manifestation [2] etc.

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