On the equivalence of Adiabatic and DeWitt-Schwinger renormalization schemes

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We prove that adiabatic regularization and DeWitt-Schwinger point-splitting provide the same result for the renormalized expectation values of the stress-energy tensor for spin-1/2 fields. This generalizes the equivalence found for scalar fields, which is here recovered in a different way. We also argue that the coincidence limit of the DeWitt-Schwinger proper time expansion of the two-point function exactly agrees with the analogous expansion defined by the adiabatic regularization method at any order (for both scalar and spin-1/2 fields). We also illustrate the power of the adiabatic method to compute higher order DeWitt coefficients in FLRW universes.

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I. INTRODUCTION

The quantization of the gravitational interaction is one of the most important and difficult problems in theoretical physics. Quantum field theory in curved spacetime offers a first step to joint Einstein's theory of general relativity and quantum field theory in Minkowski space within a self-consistent and successful framework [1, 2]. The discovery of particle creation in the expanding universe [3–5] has revealed of fundamental importance. It implies that particles, perturbations and gravitational waves are created out of the vacuum in the very early universe. This effect explains the generation of primordial perturbations in the very early universe [6, 7] and also constitutes the driving mechanism to account for the quantum radiance of black holes [8]. Within this framework the quantum analysis of the expectation values of the stress-energy tensor is of major importance. Since these and other quantities of physical interest are non-linear in the fields and their derivatives at a single point, the corresponding expectation values diverge in the ultraviolet (UV) regime. This requires renormalization procedures to get rid off the UV infinities in a self-consistent way. Even for free fields, a curved space-time background introduces additional divergences that are absent in

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Minkowski space. The renormalization program gets then more involved and a number of methods have been developed to regularize and renormalize expectation values of the stress-energy tensor or other quantities of physical relevance.

The adiabatic subtraction method of regularization is the most efficient method to carry out the renormalization program in homogeneous cosmological spacetimes. It is especially appropriated in studies in which numerical methods has to be used. It was originally conceived as a way to overcome the UV divergences in the expectation value of the particle number operator in Parker's pioneer work on gravitational particle creation [3, 4]. It was later generalized by Parker and Fulling [9] to consistently deal with the UV divergences of the stress-energy tensor of scalar fields in Fridmann-Lemaître-Robertson-Walker (FLRW) space-times. The adiabatic method identifies naturally the UV subtracting terms in momentum space, since it is based on the adiabatic asymptotic expansion of the modes characterized by the comoving momentum $\vec{k}$. It involves a mode-by-mode subtraction process, and in such a way that locality and covariance of the overall renormalization procedure are fully respected. The adiabatic method is also particularly suitable to scrutinize the primordial power spectrum in inflationary cosmology [10] (see also [11]). It has also been used in the low-energy regime of quantum gravity [12], and more recently, in studies on the breaking of electric-magnetic duality symmetry in curved space-time [13].

An alternative asymptotic expansion (for the two-point function) to consistently identify the subtraction terms in a generic spacetime was suggested by DeWitt [15], generalizing the Schwinger proper-time formalism. The DeWitt-Schwinger expansion was implemented with the point-splitting renormalization technique in [16] and it was nicely rederived from the local momentum-space representation introduced by Bunch and Parker [17]. Furthermore, by brute force calculation Birrell [18] (see also the appendix in [19]) checked that point-splitting and adiabatic renormalization give the same renormalized stress-energy tensor when applied to scalar fields in homogeneous universes.

The extension of the adiabatic regularization method to spin-1/2 fields has been achieved very recently [20, 21]. The main difficulty to extend the adiabatic scheme to fermion fields is that the proper asymptotic adiabatic expansion of the spin-1/2 fields modes does not fit the WKB-type expansion, as happens for scalar fields. However, as showed in [20, 21], the method has passed very non-trivial test of consistency. One major goal of this paper is to prove that adiabatic regularization and DeWitt-Schwinger point-splitting will give the same result for the renormalized expectation values of the stress-energy tensor of spin-1/2 fields. We base our proof on the well-known fact that two different methods to compute $\langle T_{\mu\nu} \rangle$ can differ at most by a linear combination of conserved local curvature tensors. This result assumes that the renormalization methods obey locality and covariance [22]. Since $\langle T_{\mu\nu}\rangle$ has dimensions of (length)$^{-4}$ the only candidates are $m^4 g_{\mu\nu}$, $m^2 G_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ (the last two terms can be obtained by functionally differentiating the quadratic curvature Lagrangians R^2 and $R_{\mu\nu}R^{\mu\nu}$). It can be seen that the stress-energy tensor needs only subtraction up to fourth order in the derivatives of the metric [1, 2], so that higher order contributions need not be considered.

Therefore, the potential difference between the expectation values $\langle T_{\mu\nu}\rangle^{Ad}$, computed with adiabatic regularization, and $\langle T_{\mu\nu}\rangle^{DS}$, computed with the (DeWitt-Schwinger) point-splitting method, is parametrized by four dimensionless constants c_i, $i = 1, ...4$.

$$\langle T^{Ad}_{\mu\nu}\rangle - \langle T^{DS}_{\mu\nu}\rangle = c_1 {}^{(1)}H_{\mu\nu} + c_2 {}^{(2)}H_{\mu\nu} + c_3 m^2 G_{\mu\nu} + c_4 m^4 g_{\mu\nu} \ . \tag{1}$$

In our case, the constant c_4 is necessarily zero since both prescriptions lead to a vanishing renormalized stress-energy tensor when restricted to Minkowski spacetime. Moreover, in a FLRW space-time the conserved tensors ${}^{(2)}H_{\mu\nu}$ and ${}^{(1)}H_{\mu\nu}$ are not independent, so we can assume without loss of generality that $c_2 \equiv 0$. Therefore, we are left with

$$\langle T^{Ad}_{\mu\nu}\rangle - \langle T^{DS}_{\mu\nu}\rangle = c_1 {}^{(1)}H_{\mu\nu} + c_3 m^2 G_{\mu\nu} \ . \tag{2}$$

Moreover, taking traces in the above relation we get

$$\langle T^{Ad}\rangle - \langle T^{DS}\rangle = -6c_1 \Box R - c_3 m^2 R \ . \tag{3}$$

In the massless limit, the classical action of the spin-1/2 field is conformally invariant. The trace anomaly calculated with the new adiabatic regularization method has been proved to be in exact agreement with that obtained by other renormalization methods, and in particular with the DeWitt-Schwinger point-splitting method. This implies that $c_1 = 0$. Obviously, the same arguments and conclusions apply for a scalar field. The equivalence between both methods is therefore reduced to check that the remaining parameter c_3 is also zero. This is actually the most subtle point.

The comparison between $\langle T^{Ad}\rangle$ and $\langle T^{DS}\rangle$ can be better studied by taking into account that, for a spin-1/2 fields, $\langle T\rangle = m\langle\bar{\psi}\psi\rangle$. The equivalence is then reduced to proof that

$${}^{(4)}\langle\bar{\psi}\psi\rangle^{Ad} = {}^{(4)}\langle\bar{\psi}\psi\rangle^{DS} \ , \tag{4}$$

where ${}^{(4)}\langle\bar{\psi}\psi\rangle^{Ad,DS}$ stands for the subtraction terms, up to fourth order in the derivatives of the metric, in the adiabatic and DeWitt-Schwinger expansions, respectively. As remarked above, the fourth order is the order required to remove, in general, the UV divergences in the stress-energy tensor. To prove (4) and achieve our goal we will make use of the (Bunch-Parker) local momentum-space representation [17] of the two-point function. A conceptual advantage of our strategy in comparing both renormalization methods is that it offers a better way to spell out their equivalence. In fact, we will also show that the equivalence found at fourth order can be extended to higher order, for both scalar and spin-1/2 fields.

The paper is organized as follows. In section II we consider a similar question for scalar fields. As we have already mentioned, the equivalence of both methods has been checked in [18, 19]. We present here an alternative and simpler route that will allow us to proof the equivalence for spin-1/2 fields. This will be done in section III. In section IV we extend our results to higher-order adiabatic terms. We will argue that the coincidence limit of the DeWitt-Schwinger proper time expansion of the two-point function agrees with the analogous expansion defined by the adiabatic regularization method at any order. Finally, in section V we summarize our conclusions.

II. SCALAR FIELDS

A. Adiabatic regularization

The general wave equation for a scalar field ϕ in a curved space-time is $(\Box + m^2 + \xi R)\phi = 0$, where m is the mass of the field and ξ is the coupling of the field to the scalar curvature R. If the field propagates in a FLRW space-time (for simplicity we shall assume a spatially flat universe with metric $ds^2 = dt^2 - a^2(t)d\vec{x}^2$), it can be naturally expanded in the form

$$\phi(x) = \int d^3k \left[A_{\vec{k}} f_{\vec{k}}(\vec{x}, t) + A^\dagger_{\vec{k}} f^*_{\vec{k}}(\vec{x}, t) \right] \ , \tag{5}$$

where the field modes $f_{\vec{k}}$ are

$$f_{\vec{k}}(t, \vec{x}) = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{2(2\pi)^3 a^3(t)}} h_k(t) \ . \tag{6}$$

These modes are assumed to obey the normalization condition with respect to the conserved Klein-Gordon product. This condition translates to a Wronskian-type condition for the modes: $h^*_k \dot{h}_k - \dot{h}^*_k h_k = -2i$, where the dot means derivative with respect to proper time t. Adiabatic renormalization is based on a generalized WKB-type asymptotic expansion of the modes according to the ansatz

$$h_k(t) \sim \frac{1}{\sqrt{W_k(t)}} e^{-i\int^t W_k(t')dt'} \ , \tag{7}$$

which guarantees the Wronskian condition. One then expands W_k in an adiabatic series, in which each contribution is determined by the number of time derivatives of the expansion factor $a(t)$

$$W_k(t) = \omega^{(0)}(t) + \omega^{(2)}(t) + \omega^{(4)}(t) + ... \ , \tag{8}$$

where the leading term $\omega^{(0)}(t) \equiv \omega(t) = \sqrt{k^2/a^2(t) + m^2}$ is the usual physical frequency. Higher order contributions can be univocally obtained by iteration (for details, see Appendix A), which comes from introducing (7) into the equation of motion for the modes. This adiabatic expansion (8) is basic to identify and remove the UV divergences of the expectation values of the stress-energy tensor.

The adiabatic expansion of the modes can be translated easily to an expansion of the 2-point function $\langle\phi(x)\phi(x')\rangle \equiv G(x, x')$ at coincidence $x = x'$

$$G_{Ad}(x,x) = \frac{1}{2(2\pi)^3 a^3} \int d^3\vec{k}\, [\omega^{-1} + (W^{-1})^{(2)} + (W^{-1})^{(4)} + ...] \ . \tag{9}$$

As remarked above the expansion must be truncated to the minimal adiabatic order necessary to cancel all UV divergences that appear in the formal expression of the vacuum expectation value that one wishes to compute. The calculation of the renormalized variance $\langle\phi^2\rangle$ requires only second adiabatic order, given by

$$(W^{-1})^{(2)} = \frac{m^2\dot{a}^2}{2a^2\omega^5} + \frac{m^2\ddot{a}}{4a\omega^5} - \frac{5m^4\dot{a}^2}{8a^2\omega^7} + \frac{3(\frac{1}{6}-\xi)(\dot{a}^2 + a\ddot{a})}{a^2\omega^3} \ . \tag{10}$$

The renormalization of the vacuum expectation value of the stress-energy tensor needs up to fourth adiabatic order subtraction. The corresponding fourth-order contribution $(W^{-1})^{(4)}$ has 30 terms and can be found in [9]. Therefore, the adiabatic subtraction terms, truncated to fourth adiabatic order, can be rewritten as

$$^{(4)}G_{Ad}(x,x) = \frac{1}{2(2\pi)^3 a^3} \int d^3\vec{k} \left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3} + \frac{m^2\dot{a}^2}{2a^2\omega^5} + \frac{m^2\ddot{a}}{4a\omega^5} - \frac{5m^4\dot{a}^2}{8a^2\omega^7} + (W^{-1})^{(4)}\right] \ , \tag{11}$$

where we have taken into account that $R = 6[\dot{a}^2/a^2 + \ddot{a}/a]$ in FLRW universes.

We note that only the first two terms in (11) are divergent. The remaining terms can be integrated exactly in momenta producing well-defined finite geometric quantities. Taking into account that $\omega = (\vec{k}^2/a^2 + m^2)^{1/2}$, the integration of the second-order adiabatic terms is independent of the mass and gives

$$\frac{1}{2(2\pi)^3 a^3} \int d^3\vec{k} \left[\frac{m^2\dot{a}^2}{2a^2\omega^5} + \frac{m^2\ddot{a}}{4a\omega^5} - \frac{5m^4\dot{a}^2}{8a^2\omega^7}\right] = \frac{R}{288\pi^2} \ . \tag{12}$$

The integration of the fourth-order terms turns out to be also a well-defined geometrical quantity

$$\frac{1}{2(2\pi)^3a^3}\int d^3\vec{k}\,(W^{-1})^{(4)} = \frac{a_2}{16\pi^2m^2}\ , \tag{13}$$

where

$$a_2 = \frac{1}{2}\left[\xi - \frac{1}{6}\right]^2 R^2 - \frac{1}{6}\left[\frac{1}{5} - \xi\right]\Box R - \frac{1}{180}(R_{\mu\nu}R^{\mu\nu} - R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta})\ , \tag{14}$$

is just the coincident point limit $a_2(x) \equiv \lim_{x\to x'} a_2(x,x')$ of the second DeWitt coefficient $a_2(x,x')$ [15]. (We note that, for our conformally flat space-times, we have $R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} = 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2$).

Summarizing, the two-point function for a scalar field at coincidence and at fourth-adiabatic order is given by

$${}^{(4)}G_{Ad}(x,x) = \frac{1}{4\pi^2a^3}\int_0^\infty dkk^2\left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3}\right] + \frac{R}{288\pi^2} + \frac{a_2}{16\pi^2m^2}\ , \tag{15}$$

where the formal divergent term can be understood, for future purposes, as the point-splitting limit

$$\frac{1}{4\pi^2a^3}\int_0^\infty dkk^2\left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3}\right] \equiv \lim_{|\Delta\vec{x}|\to 0}\frac{1}{4\pi^2a^3}\int_0^\infty dkk^2\frac{\sin(k|\Delta\vec{x}|)}{k|\Delta\vec{x}|}\left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3}\right]\ . \tag{16}$$

B. Local momentum-space representation and DeWitt-Schwinger expansion

An alternative asymptotic expansion of the 2-point function in momentum space was introduced by Bunch and Parker in [17]. It was proposed aiming at extending to curved space the standard momentum-space methods of perturbation theory for interacting fields in Minkowski space. This way the standard Minkowskian propagator of a scalar free field in momentum space $(-k^2+m^2)^{-1}$ is replaced by a series expansion. The Fourier transform leading to local-momentum space is crucially performed with respect to Riemann normal coordinates y^μ around a given point x', which constitutes the best possible approximation in curved space to the inertial coordinates of Minkowski space. In contrast to the adiabatic regularization, the method is valid for an arbitrary space-time. It does not serve to (adiabatically) expand the mode functions, which are otherwise highly ambiguous in a general background. The method works directly with the two-point functions, which are regarded as the basic buildings blocks of the renormalization process.

The covariant expansion of the 2-point function $G_{DS}(x,x')$, obeying the equation

$$(\Box_x + m^2 + \xi R)G_{DS}(x,x') = -|g(x)|^{-1/2}\delta(x-x')\ , \tag{17}$$

is defined in the local-momentum space

$$G_{DS}(x,x') = \frac{-i|g(x)|^{-1/4}}{(2\pi)^4}\int d^4k\ e^{iky}\bar{G}(k)\ , \tag{18}$$

where $ky \equiv k_0 y^0 - \vec{k}\vec{y}$ (note that $y^\mu(x')=0$), by the series

$$\bar{G}(k) = \frac{1}{-k^2+m^2} + \frac{(\frac{1}{6}-\xi)R}{(-k^2+m^2)^2} + \frac{i(\frac{1}{6}-\xi)}{2}R_{;\alpha}\frac{\partial}{\partial k_\alpha}\frac{1}{(-k^2+m^2)^2} + \frac{1}{3}a_{\alpha\beta}\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\beta}(-k^2+m^2)^{-2}$$
$$+ \left[\left(\frac{1}{6}-\xi\right)^2 R^2 + \frac{2}{3}a^\alpha_\alpha\right]\frac{1}{(-k^2+m^2)^3} + \ldots\ , \tag{19}$$

where

$$a_{\alpha\beta} = \frac{(\xi-\frac{1}{6})}{2}R_{;\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}\Box R_{\alpha\beta} - \frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} + \frac{1}{60}R_{\kappa\alpha\lambda\beta}R^{\kappa\lambda} + \frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta}\ . \tag{20}$$

To compare this local-momentum expansion with the adiabatic one introduced in section II A we have to convert the momentum-space four-dimensional integrals into three-dimensional integrals. After performing the k^0 integration in the complex plane, where the poles in $\bar{G}(k)$ at $k_0 = \pm\sqrt{\vec{k}^2+m^2}$ has been displaced in the same way as the analogous Green function in Minkowski space-time, one gets tridimensional integrals. Since all Green functions have the same UV divergences we can perform the contour k^0 integration using, for instance, the Feynman prescription for displacing the poles. The result is, up to fourth adiabatic order

$$^{(4)}G_{DS}(x,x') = \frac{|g(x)|^{-1/4}}{2(2\pi)^3}\int d^3k\, e^{-i(\vec{k}\vec{y}-\sqrt{\vec{k}^2+m^2}y^0)}$$
$$\times\left[\frac{a_0}{(\vec{k}^2+m^2)^{1/2}} + \frac{a_1(x,x')(1-iy^0\omega)}{2(\vec{k}^2+m^2)^{3/2}} + \frac{3a_2(x,x')(1-iy^0\omega-(y^0)^2\omega^2)}{4(\vec{k}^2+m^2)^{5/2}}\right] \tag{21}$$
$$= \frac{|g(x)|^{-1/4}}{(2\pi)^2|\vec{y}|}\int_0^\infty dk\, k\sin(k|\vec{y}|)\, e^{i\sqrt{\vec{k}^2+m^2}y^0}$$
$$\times\left[\frac{a_0}{(\vec{k}^2+m^2)^{1/2}} + \frac{a_1(x,x')(1-iy^0\omega)}{2(\vec{k}^2+m^2)^{3/2}} + \frac{3a_2(x,x')(1-iy^0\omega-(y^0)^2\omega^2)}{4(\vec{k}^2+m^2)^{5/2}}\right]\ , \tag{22}$$

where $a_0 \equiv 1$ and, to fourth adiabatic order,

$$a_1(x,x') = \left[\frac{1}{6}-\xi\right]R(x') + \frac{1}{2}\left[\frac{1}{6}-\xi\right]R_{;\alpha}(x')y^\alpha - \frac{1}{3}a_{\alpha\beta}(x')y^\alpha y^\beta\ ,$$
$$a_2(x,x') = \frac{1}{2}\left[\frac{1}{6}-\xi\right]^2 R^2(x') + \frac{1}{3}a^\alpha_\alpha(x')\ , \tag{23}$$

turn out to be the first DeWitt coefficients. The integrals can be worked out analytically and (21) gives the first three terms in the DeWitt-Schwinger expansion of the two-point function [2]

$$^{(4)}G_{DS}(x,x') = \frac{|g(x)|^{-1/4}}{4\pi^2}\left[\frac{m}{\sqrt{-2\sigma}}K_1(m\sqrt{-2\sigma}) + \frac{a_1(x,x')}{2}K_0(m\sqrt{-2\sigma}) + \frac{a_2(x,x')}{4m}\sqrt{-2\sigma}K_1(m\sqrt{-2\sigma})\right]\ , \tag{24}$$

where $\sigma(x,x')$ is half the square of the geodesic distance between x and x', i.e., $\sigma(x,x') = \frac{1}{2}y_\mu y^\mu = ((y^0)^2 - \vec{y}^2)/2$, and K are the modified Bessel functions of second kind.

It is also important to note that the factor $|g(x)|^{-1/4}$ in the above expressions is evaluated in Riemann normal coordinates with origin at x'. The biscalar that reduces to $|g(x)|^{-1/4}$ in arbitrary coordinates is $\Delta^{1/2}(x,x')$, where $\Delta(x,x')$ is the Van Vleck-Morette determinant, defined as

$$\Delta(x,x') = -|g(x)|^{-1/2}\det[-\partial_\mu\partial_{\nu'}\sigma(x,x')]|g(x')|^{-1/2} \ , \tag{25}$$

These expressions fit identically with the conventional definition of the DeWitt-Schwinger expansion, as first stressed in [17], which is usually written as

$$G_{DS}(x,x') \equiv \frac{\Delta^{1/2}(x,x')}{16\pi^2}\int_0^\infty \frac{ids}{(is)^2}\exp\left(-im^2 s + \frac{\sigma}{2is}\right)F(x,x';is) \ , \tag{26}$$

with

$$F(x,x';is) = a_0 + a_1(x,x')is + a_2(x,x')(is)^2 + ... \ , \tag{27}$$

where $a_0 = 1, a_1, a_2, ...$ are the DeWitt coefficients. To sum up, the Bunch-Parker local momentum-space expansion turns out to be the momentum-space version of the DeWitt-Schwinger expansion of the two-point function.

C. Comparison between ${}^{(4)}G_{Ad}(x,x)$ and ${}^{(4)}G_{DS}(x,x)$

To compare the expression (24) for $G_{DS}(x,x')$ with the result of adiabatic regularization we have to take the coincident limit $x = x'$ and restrict our analysis to a spatially flat FLRW universe $ds^2 = dt^2 - a^2(t)d\vec{x}^2$. The comparison is not trivial since in the DeWitt-Schwinger formalism the point-splitting is studied in terms of the geodesic distance σ. As a first approximation, the normal Riemann coordinates in our FLRW space-time are $\vec{y} \approx a\Delta\vec{x}$. To rigorously compare with the adiabatic expansion we need the higher order relations between the physical coordinates $(t,\vec{x})$ and the normal Riemann coordinates $(y^0,\vec{y})$. The following relations (with $H = \dot{a}/a$) hold [23]

$$y^0 = \Delta t + \frac{1}{2}a^2\Delta\vec{x}^2 H + \frac{1}{3}a^2\Delta\vec{x}^2\Delta t\left(\frac{R}{12} + H^2\right) + \dots \ , \tag{28}$$

$$y^i = a\Delta x^i\left[1 + H\Delta t + \frac{1}{6}a^2\Delta\vec{x}^2 H^2 + \frac{\Delta t^2}{3}\left(\frac{R}{6} - H^2\right) + \dots\right] \ . \tag{29}$$

Moreover,

$$-2\sigma = -\Delta t^2 + a^2\Delta\vec{x}^2 + a^2\Delta\vec{x}^2 H\Delta t + \frac{1}{3}a^2\Delta\vec{x}^2\Delta t^2\left(\frac{R}{6} - H^2\right) + \frac{a^4\Delta\vec{x}^4}{12}H^2 + \dots \ , \tag{30}$$

where, in order to compare to our previous result using the adiabatic regularization, we can just take $\Delta t = 0$ without loss of generality and remain the point splitting in $\Delta\vec{x}$.

A useful identity for our purposes, using (30) at temporal coincidence $\Delta t = 0$, is

$$\frac{1}{-2\sigma} = \frac{1}{a^2\Delta\vec{x}^2} - \frac{H^2}{12} + O(\Delta\vec{x}^2)\ . \tag{31}$$

Note also that the factor $|g(x)|^{-1/4}$ in (24) is evaluated in Riemann normal coordinates with origin at x' so we can expand $|g(x,x')|^{-1/4} = \Delta^{1/2}(x,x') = 1 - \frac{1}{12}R_{\mu\nu}y^\mu y^\nu + ...$. Another useful relation can be derived using this with formulas (28)-(29) [note also that $R_{00} = 3\frac{\ddot{a}}{a}; R_{ii} = -a^2\left(\frac{\ddot{a}}{a} + 2H^2\right)$],

$$|g(x)|^{-1/4} = 1 - \left[2H^2 + \frac{\ddot{a}}{a}\right]\frac{\sigma}{6} + O(\sigma^{3/2})\ . \tag{32}$$

Taking into account (31) and (32), the zeroth order contribution to ${}^{(4)}G_{DS}(x,x)$ can be re-expressed as

$$\lim_{x\to x'}\frac{|g(x)|^{-1/4}m}{(2\pi)^2\sqrt{-2\sigma}}K_1(m\sqrt{-2\sigma}) = \lim_{x\to x'}|g(x)|^{-1/4}\left[-\frac{1}{8\pi^2\sigma} + O(\log(-\sigma))\right] \tag{33}$$

$$= \frac{R}{288\pi^2} + \lim_{\Delta\vec{x}\to 0}\frac{m}{4\pi^2 a|\Delta\vec{x}|}K_1(m\,a|\Delta\vec{x}|) \tag{34}$$

$$= \frac{R}{288\pi^2} + \lim_{\Delta\vec{x}\to 0}\frac{1}{4\pi^2a^3}\int_0^\infty dkk^2\frac{\sin(k|\Delta\vec{x}|)}{k|\Delta\vec{x}|}\frac{1}{\omega}\ . \tag{35}$$

Furthermore, the second order contribution is

$$\lim_{x\to x'}\frac{|g(x)|^{-1/4}}{4\pi^2}\frac{a_1(x,x')}{2}K_0(m\sqrt{-2\sigma}) = \lim_{\Delta\vec{x}\to 0}|g(x)|^{-1/4}\times O(\log(-\sigma)) \tag{36}$$

$$= \lim_{\Delta\vec{x}\to 0}\frac{1}{4\pi^2}\frac{\left(\frac{1}{6}-\xi\right)R}{2}K_0(m\,a|\Delta\vec{x}|) \tag{37}$$

$$= \lim_{\Delta\vec{x}\to 0}\frac{1}{4\pi^2a^3}\int_0^\infty dkk^2\frac{\sin(k|\Delta\vec{x}|)}{k|\Delta\vec{x}|}\frac{\left(\frac{1}{6}-\xi\right)R}{2\omega^3}\ , \tag{38}$$

while the fourth adiabatic term is given by

$$\lim_{x\to x'}\frac{|g(x)|^{-1/4}}{4\pi^2}\frac{a_2(x,x')}{4m}\sqrt{-2\sigma}K_1(m\sqrt{-2\sigma}) = \frac{a_2(x)}{16\pi^2m^2}\ . \tag{39}$$

To sum up, we finally get

$${}^{(4)}G_{DS}(x,x) = \lim_{|\Delta\vec{x}|\to 0}\frac{1}{4\pi^2a^3}\int_0^\infty dkk^2\frac{\sin(k|\Delta\vec{x}|)}{k|\Delta\vec{x}|}\left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3}\right] + \frac{R}{288\pi^2} + \frac{a_2(x)}{16\pi^2m^2}\ . \tag{40}$$

By direct comparison with (15) and (16) we obtain

$${}^{(4)}G_{Ad}(x,x) = {}^{(4)}G_{DS}(x,x)\ . \tag{41}$$

D. Equivalence for $\langle T_{\mu\nu}\rangle$

For the sake of simplicity it is now convenient to restrict ourselves to the case $\xi = 1/6$. The reason for which we focus on this particular case is because the spin 1/2 case turns out to be completely analogous, so that it is an illustrative example. In this situation the trace of the stress-energy tensor can be expressed as $\langle T\rangle = m^2\langle\phi^2\rangle$. The equivalence $\langle T\rangle^{Ad} = \langle T\rangle^{DS}$, and hence $\langle T_{\mu\nu}\rangle^{Ad} = \langle T_{\mu\nu}\rangle^{DS}$ (i.e., $c_3 = 0$, according to the definitions and arguments given in section I), comes directly from the equivalence ${}^{(4)}G_{Ad}(x,x) = {}^{(4)}G_{DS}(x,x)$, since

$$\langle T^{Ad}\rangle - \langle T^{DS}\rangle = m^2\left[{}^{(4)}G_{DS}(x,x) - {}^{(4)}G_{Ad}(x,x)\right] = 0 \ . \tag{42}$$

For a general ξ, one can compute the stress-energy tensor by acting on the symmetric part of $G(x,x') - {}^{(4)}G(x,x')$ with a certain non-local operator, $\langle T_{\mu\nu}(x)\rangle = \lim_{x'\to x} D_{\mu\nu}(x,x')[G(x,x') - {}^{(4)}G(x,x')]$ [2, 15, 16]. In section V we have shown the equivalence ${}^{(4)}G_{Ad}(x,x') = {}^{(4)}G_{DS}(x,x')$, which immediately implies $\langle T_{\mu\nu}\rangle^{Ad} = \langle T_{\mu\nu}\rangle^{DS}$ for a general ξ.

III. SPIN-1/2 FIELDS

A. Adiabatic regularization

The first step in the adiabatic regularization is to define an asymptotic expansion of the field modes. The expansion can be regarded as definitions of approximate particle states in an expanding universe in the limit of infinitely slow expansion. Spin-1/2 fields obey the Dirac equation. In a general background it is given by (see, for instance [1, 2])

$$\left(i\underline{\gamma}^{\mu}\nabla_{\mu} - m\right)\psi = 0 \ , \tag{43}$$

where $\underline{\gamma}^{\mu}(x)$ are the spacetime dependent $\underline{\gamma}$-matrices satisfying the condition $\{\underline{\gamma}^{\mu},\underline{\gamma}^{\nu}\} = 2g^{\mu\nu}$ and $\nabla_{\mu} = \partial_{\mu} - \Gamma_{\mu}$ is the covariant derivative associated to the spin connection Γ_{μ}. Let us assume a spatially flat FLRW space-time, with line element $ds^2 = dt^2 - a^2(t)d\vec{x}^2$. The $\underline{\gamma}$-matrices are related with the constant Dirac γ-matrices in Minkowski spacetime by the simple relations: $\underline{\gamma}^0 = \gamma^0$, $\underline{\gamma}^i(t) = \gamma^i/a(t)$. The Dirac equation takes the form

$$\left[i\gamma^0\partial_0 + \frac{3i}{2}\frac{\dot{a}}{a}\gamma^0 + \frac{i}{a}\vec{\gamma}\cdot\vec{\nabla} - m\right]\psi = 0 \ . \tag{44}$$

For our purposes it is convenient to work with the Dirac-Pauli representation for the Minkowskian Dirac matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} , \tag{45}$$

where the components of $\vec{\sigma}$ are the usual Pauli matrices. For a given comoving momentum $\vec{k}$, the basic independent (normalized) spinor solutions are

$$u_{\vec{k}\lambda}(x) = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3 a^3}} \begin{pmatrix} h_k^I(t)\xi_\lambda(\vec{k}) \\ h_k^{II}(t)\frac{\vec{\sigma}\cdot\vec{k}}{k}\xi_\lambda(\vec{k}) \end{pmatrix} , \tag{46}$$

$$v_{\vec{k}\lambda}(x) = \frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3 a^3}} \begin{pmatrix} -h_k^{II*}(t)\xi_{-\lambda}(\vec{k}) \\ -h_k^{I*}(t)\frac{\vec{\sigma}\cdot\vec{k}}{k}\xi_{-\lambda}(\vec{k}) \end{pmatrix} , \tag{47}$$

where $k \equiv |\vec{k}|$ and ξ_λ are constant and normalized two-component spinor $\xi_\lambda^\dagger \xi_{\lambda'} = \delta_{\lambda'\lambda}$. They are chosen to be helicity eigenstates $\frac{\vec{\sigma}\cdot\vec{k}}{2k}\xi_\lambda(\vec{k}) = (\lambda/2)\xi_\lambda(\vec{k})$, where $\lambda/2 = \pm 1/2$. In this decomposition, h_k^I and h_k^{II} are two particular time-dependent functions obeying the following coupled differential equations

$$h_k^{II} = \frac{ia}{k}(\partial_t + im)h_k^I , \qquad h_k^I = \frac{ia}{k}(\partial_t - im)h_k^{II} . \tag{48}$$

The following self-consistent expansion for the field modes was found in [20]

$$h_k^I(t) \sim \sqrt{\frac{\omega + m}{2\omega}} e^{-i\int^{t'}\Omega(t')dt'} F(t) \quad , \quad h_k^{II}(t) \sim \sqrt{\frac{\omega - m}{2\omega}} e^{-i\int^{t'}\Omega(t')dt'} G(t) , \tag{49}$$

where $\omega \equiv \omega^0 \equiv \sqrt{(k/a(t))^2 + m^2}$ is the frequency of the mode and the time-dependent functions $\Omega(t)$, $F(t)$ and $G(t)$ are expanded adiabatically as

$$\Omega(t) = \sum_{n=0}^{\infty} \omega^{(n)}(t) , \quad F(t) = \sum_{n=0}^{\infty} F^{(n)}(t) , \quad G(t) = \sum_{n=0}^{\infty} G^{(n)}(t) . \tag{50}$$

$\omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$ are functions of adiabatic order n, which means that they contain n derivatives of the scale factor $a(t)$. We impose $F^{(0)} = G^{(0)} \equiv 1$ at the zeroth order to recover the Minkowskian solutions for $a(t) = 1$. We can solve $\omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$ for $n > 1$ by direct substitution of the ansatz (49) into (48) and solving the system of equations order by order. We also have to impose, as an additional order by order requirement, the normalization condition $|h_k^I(t)|^2 + |h_k^{II}(t)|^2 = 1$. For details, see [20, 21]. The adiabatic series obtained in this way contain an ambiguity, which do disappear in the adiabatic expansion of physical vacuum expectation values. It is very convenient, for the sake of simplicity, to impose at all adiabatic orders the additional condition $ImG^{(n)}(m) = -ImF^{(n)}(m)$. It implies that $F^{(n)}(-m) = G^{(n)}(m)$ and removes all the ambiguities. Explicit expressions for the series expansion up to fourth adiabatic order are displayed in [20, 21]. The algorithm to obtain systematically $\omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$ for any nth adiabatic order is shown in Appendix B.

In parallel to the scalar field, the adiabatic expansion of the spin-1/2 field modes can be translated to an expansion of the two-point function $\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle \equiv S_{\alpha\beta}(x,x')$ at coincidence $x = x'$. Moreover, since we are mainly interested to study the stress-energy tensor we will restrict our analysis to the trace of the two-point function $\langle\bar{\psi}(x')\psi(x)\rangle = trS(x,x')$. Evaluating this at coincidence, the adiabatic expansion up to fourth order is

$$tr^{(4)}S_{Ad}(x,x) = \frac{-2}{(2\pi)^3a^3}\int d^3k\,[|g_k^{I(4)}|^2 - |g_k^{II(4)}|^2]\ , \qquad (51)$$

where

$$g_k^{I(4)}(t) \equiv \sqrt{\frac{\omega+m}{2\omega}}\sum_{n=0}^{4}F^{(n)}(t)\exp\left[-i\int^{t'}\sum_{n=0}^{4}\omega^{(n)}(t')dt'\right]\ ,$$
$$g_k^{II(4)}(t) \equiv \sqrt{\frac{\omega-m}{2\omega}}\sum_{n=0}^{4}G^{(n)}(t)\exp\left[-i\int^{t'}\sum_{n=0}^{4}\omega^{(n)}(t')dt'\right]\ . \qquad (52)$$

Taking into account that the trace of the stress-energy tensor can be expressed as $\langle T\rangle = m\langle\bar{\psi}(x)\psi(x)\rangle$, it is very convenient for our purposes to rewrite (51) in terms of the expansion for the energy density and pressure [21]

$$tr^{(4)}S_{Ad}(x,x) = \frac{1}{(2\pi)^3a^3m}\int d^3k\,\sum_{i=0}^{2}[\rho_k^{(2i)} - 3p_k^{(2i)}]\ , \qquad (53)$$

where,

$$\rho_k^{(0)} = -2\omega\ , \qquad (54)$$
$$\rho_k^{(2)} = -\frac{m^4\dot{a}^2}{4\omega^5a^2} + \frac{m^2\dot{a}^2}{4\omega^3a^2}\ , \qquad (55)$$
$$p_k^{(0)} = -\frac{2\omega}{3} + \frac{2m^2}{3\omega}\ , \qquad (56)$$
$$p_k^{(2)} = -\frac{m^2\dot{a}^2}{12w^3a^2} - \frac{m^2\ddot{a}}{6w^3a} + \frac{m^4\ddot{a}}{6w^5a} + \frac{m^4\dot{a}^2}{2w^5a^2} - \frac{5m^6\dot{a}^2}{12w^7a^2}\ , \qquad (57)$$

and the contribution of the fourth adiabatic order is itself finite and gives

$$\frac{1}{(2\pi)^3a^3m}\int d^3k\,[\rho^{(4)} - 3p^{(4)}] = \frac{trA_2}{16\pi^2m}\ , \qquad (58)$$

where A_2 turns out to be one of the DeWitt coefficients for spin 1/2 fields at coincidence [1, 17] (see next subsection)

$$A_2(x) = a_2(\xi = 1/4)\mathbb{I} + \frac{1}{48}\Sigma_{[\alpha\beta]}\Sigma_{[\gamma\delta]}R^{\alpha\beta\lambda\xi}R^{\gamma\delta}{}_{\lambda\xi} \ . \tag{59}$$

In this equation $a_2(\xi = 1/4)$ is the DeWitt coefficient for a scalar field with curvature coupling $\xi = 1/4$, and

$$\Sigma_{[\alpha\beta]} \equiv \frac{1}{4}\left[\underline{\gamma}_\alpha\underline{\gamma}_\beta - \underline{\gamma}_\beta\underline{\gamma}_\alpha\right] \ . \tag{60}$$

Taking into account that

$$tr\left\{\Sigma_{[\alpha\beta]}\Sigma_{[\gamma\delta]}\right\} = g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta} \ , \tag{61}$$

the term (58) accounts for the trace anomaly in the massless limit

$$-\frac{trA_2}{16\pi^2} = \frac{2}{2880\pi^2 m}\left[-\frac{11}{2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + 3\Box R\right] \ . \tag{62}$$

Let us analyze with detail the lower orders. The zeroth order contribution is easy to handle

$$\frac{1}{(2\pi)^3a^3m}\int d^3k\,[\rho^{(0)} - 3p^{(0)}] = \frac{-m}{\pi^2a^3}\int_0^\infty dk\,k^2\frac{1}{\omega} \ . \tag{63}$$

However, the second adiabatic order is more subtle. Using the stress-energy tensor conservation [which is equivalent to impose the condition $\dot{\rho}_k^{(n)} + 3Hp_k^{(n)} = 0$], and dimensional regularization, one can eventually arrive at the following expression

$$\frac{1}{(2\pi)^3a^3m}\int d^3k\,[\rho^{(2)} - 3p^{(2)}] = \lim_{n\to 4}\frac{-mR}{24\pi^2}\left[\frac{1}{n-4} + \frac{4}{3} - \log 2\right] \ . \tag{64}$$

Using now the identity

$$\frac{4m}{4\pi^2a^3}\int_0^\infty dk\,k^2\frac{R}{24\omega^3} = \lim_{n\to 4}\frac{-mR}{24\pi^2}\left[\frac{1}{n-4} + 1 - \log 2\right] \ , \tag{65}$$

(64) can be finally expressed as

$$\frac{1}{(2\pi)^3a^3m}\int d^3k\,[\rho^{(2)} - 3p^{(2)}] = -4\left[\frac{mR}{288\pi^2} - \frac{m}{4\pi^2a^3}\int_0^\infty dk\,k^2\frac{R}{24\omega^3}\right] \ . \tag{66}$$

Summing up we have

$$\begin{aligned} tr^{(4)}S_{Ad}(x,x) &= \frac{-m}{\pi^2a^3}\int_0^\infty dk\,k^2\left[\frac{1}{\omega} - \frac{R}{24\omega^3}\right] - \frac{4mR}{288\pi^2} + \frac{trA_2}{16\pi^2 m} \\ &= -4m^{(2)}G_{Ad}(x,x)|_{\xi=1/4} + \frac{trA_2}{16\pi^2 m} \ . \end{aligned} \tag{67}$$

B. Local momentum-space representation and DeWitt-Schwinger expansion

Following [1, 17], one can construct an asymptotic expansion for the two-point function $\langle\psi(x)\bar{\psi}(x')\rangle \equiv S(x,x')$ as follows. Introduce the bispinor $\mathcal{G}(x,x')$ as

$$S(x,x') \equiv (i\underline{\gamma}^{\mu}\nabla_{\mu} + m)\mathcal{G}(x,x') \ . \tag{68}$$

This way we have, as desired,

$$\left(i\underline{\gamma}^{\mu}\nabla_{\mu} - m\right) S(x,x') = \left(\Box + m^2 + \frac{1}{4}R\right)[-\mathcal{G}(x,x')] = |g(x)|^{-1/2}\delta(x-x') \ , \tag{69}$$

where we used the identity, $\left(\underline{\gamma}^{\mu}\nabla_{\mu}\right)^2 = \Box + \frac{1}{4}R$ [1]. We can perform a Fourier expansion in Riemann normal coordinates around x', as in the scalar case,

$$\mathcal{G}(x,x') = \frac{-i|g(x)|^{-1/4}}{(2\pi)^4}\int d^4k \ e^{iky}\bar{\mathcal{G}}(k) \ . \tag{70}$$

The local-momentum expansion for spin 1/2 fields is basically that one for spin 0 fields taking $\xi = 1/4$, except for additional spinorial contributions. The detailed expansion can be looked up in [1, 17], and up to fourth adiabatic order reads

$$\begin{aligned}\bar{\mathcal{G}}(k) = -\Bigg\{ & \frac{\mathbb{I}}{-k^2+m^2} - \frac{R\,\mathbb{I}}{12(-k^2+m^2)^2} - i\left[\frac{\mathbb{I}}{24}R_{;\alpha} + \frac{1}{12}\Sigma_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu}{}^{\lambda}{}_{;\lambda}\right]\frac{\partial}{\partial k_{\alpha}}\frac{1}{(-k^2+m^2)^2} \\ & + \left[\frac{\mathbb{I}}{3}a_{\alpha\beta}(\xi=1/4) - \frac{1}{48}\Sigma_{[\alpha\beta]}(RR^{\alpha\beta}{}_{\mu\nu} + R^{\alpha\beta\lambda}{}_{\mu;\lambda\nu} + R^{\alpha\beta\lambda}{}_{\nu;\lambda\mu}) + \frac{1}{96}\Sigma_{[\alpha\beta]}\Sigma_{[\gamma\delta]}(R^{\alpha\beta\lambda}{}_{\mu}R^{\gamma\delta}{}_{\lambda\nu} + R^{\alpha\beta\lambda}{}_{\nu}R^{\gamma\delta}{}_{\lambda\mu})\right] \\ & \times \frac{\partial}{\partial k_{\alpha}}\frac{\partial}{\partial k_{\beta}}(-k^2+m^2)^{-2} \\ & + \left[\left(\frac{R^2}{288} + \frac{1}{3}a^{\alpha}_{\alpha}(\xi=1/4)\right)\mathbb{I} + \frac{1}{48}\Sigma_{[\alpha\beta]}\Sigma_{[\gamma\delta]}R^{\alpha\beta\lambda\xi}R^{\gamma\delta}{}_{\lambda\xi}\right]\frac{2}{(-k^2+m^2)^3} + \ldots\Bigg\} \ ,\end{aligned} \tag{71}$$

The above expression for the spinor matrix $S(x,x')$ provides an asymptotic expansion of the two-point function $\langle\psi(x)\bar{\psi}(x')\rangle$, which also turns out to be equivalent to the DeWitt-Schwinger expansion [17]. Since we are mainly interested in $\langle\bar{\psi}(x)\psi(x)\rangle DS$ we take the trace of $S(x,x')$ in the above formulas. Taking into account that $tr\,(\gamma^{\mu_1}\ldots\gamma^{\mu_{2k+1}}) = 0$, and after performing the contour k^0 integration, as in the scalar case, we obtain

$$tr\ {}^{(4)}S_{DS}(x,x') = -4m\frac{|g(x)|^{-1/4}}{2(2\pi)^3}\int d^3k e^{-i(\vec{k}\vec{y}-\sqrt{\vec{k}^2+m^2}y^0)}\left[\frac{1}{(\vec{k}^2+m^2)^{1/2}} - \frac{R(1-iy^0\omega)}{24(\vec{k}^2+m^2)^{3/2}} + \ldots\right] . \tag{72}$$

Restricting now the analysis to a spatially flat FLRW spacetime with metric $ds^2 = dt^2 - a^2(t)d\vec{x}^2$ and proceeding in parallel to the scalar case we get, at coincidence $x = x'$,

$$tr\ {}^{(4)}S_{DS}(x,x) = -4m\ {}^{(2)}G_{DS}(x,x)|_{\xi=1/4} + \frac{trA_2(x)}{16\pi^2 m} \ . \tag{73}$$

C. Comparison between $tr\ {}^{(4)}S(x,x)_{DS}$ and $tr\ {}^{(4)}S(x,x)_{Ad}$ and equivalence of $\langle T_{\mu\nu}\rangle$

It is clear from our previous results that we have a complete agreement between $tr\ {}^{(4)}S(x,x)_{DS}$ and $tr\ {}^{(4)}S(x,x)_{Ad}$

$$tr\ {}^{(4)}S(x,x)_{DS} = tr\ {}^{(4)}S(x,x)_{Ad} = -\frac{m}{\pi^2 a^3}\int_0^\infty dk\, k^2 \left[\frac{1}{\omega} - \frac{R}{24\omega^3}\right] - \frac{4mR}{288\pi^2} + \frac{trA_2(x)}{16\pi^2 m} \ . \qquad (74)$$

As argued in section I, and taking into account that $\langle T\rangle = m\langle\bar\psi\psi\rangle$, the equivalence $\langle T\rangle^{Ad} = \langle T\rangle^{DS}$ for spin-1/2 fields, and hence $\langle T_{\mu\nu}\rangle^{Ad} = \langle T_{\mu\nu}\rangle^{DS}$, can be simply derived from (74).

IV. EXTENSION TO HIGHER ORDERS

The results obtained in previous sections suggest that the equivalence may go beyond the fourth adiabatic order, i.e., the order required to prove the equivalence for the renormalized expectation values of the stress-energy tensor. We have checked by computed assisted methods that our fundamental relations ${}^{(4)}G_{Ad}(x,x) = {}^{(4)}G_{DS}(x,x)$ and $tr\ {}^{(4)}S(x,x)_{Ad} = tr\ {}^{(4)}S(x,x)_{DS}$ are also valid at sixth adiabatic order. In the former case we have

$${}^{(6)}G_{Ad}(x,x) = {}^{(6)}G_{DS}(x,x) = \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3}\right] + \frac{R}{288\pi^2} + \frac{a_2}{16\pi^2 m^2} + \frac{a_3}{16\pi^2 m^4} \ , \qquad (75)$$

where the value obtained for the purely sixth adiabatic order contribution matches exactly with the third order DeWitt coefficient a_3. The general expression for the coefficient a_3, which has 28 terms, was first obtained in [24, 25], and can also be found in [1] (see chapter 3, section 3.6). We note that the above agreement is consistent with that found in [26, 27] in terms of the sixth order adiabatic approximation for the renormalized stress-energy tensor for scalar fields.

We have also tested the equivalence at sixth adiabatic order for spin-1/2 fields

$$tr\ {}^{(6)}S(x,x)_{DS} = tr\ {}^{(6)}S(x,x)_{Ad} = -\frac{m}{\pi^2 a^3}\int_0^\infty dk\, k^2 \left[\frac{1}{\omega} - \frac{R}{24\omega^3}\right] - \frac{4mR}{288\pi^2} + \frac{trA_2}{16\pi^2 m} + \frac{trA_3}{16\pi^2 m^3} \ . \qquad (76)$$

The adiabatic method produces the result

$$trA_3 = -\frac{2}{21}\frac{\dot a^4}{a^4}\frac{\ddot a}{a} + \frac{8}{21}\frac{\dot a^2}{a^2}\frac{\ddot a^2}{a^2} - \frac{4\ddot a^3}{45a^3} + \frac{2}{21}\frac{\dot a^3}{a^3}\frac{\dddot a}{a} - \frac{2}{5}\frac{\dot a\ddot a\dddot a}{a^3} - \frac{\dddot a^2}{210a^2} - \frac{\dot a^2\ddddot a}{15a^3} + \frac{\ddot a\ddddot a}{105a^2} + \frac{2\dot a a^{(5)}}{35a^2} + \frac{a^{(6)}}{70a} \ , \qquad (77)$$

where we have used the obvious notation ($a^{(n)} \equiv \frac{d^n}{dt^n}a$). We have checked that (77) agrees with the third order DeWitt coefficient for fermions [28],

$$A_3(x) = a_3(\xi = 1/4)\mathbb{I} - \Sigma^{[ab]}\Sigma^{[cd]}\left[\frac{R}{576}R_{ab\mu\nu}R_{cd}{}^{\mu\nu} + \frac{1}{720}R_{ab\mu\nu}{}^{;\mu}R_{cd\alpha\beta}{}^{;\alpha} + \frac{1}{120}R_{ab\mu\nu}{}^{;\mu}R_{cd\alpha}{}^{\nu;\alpha\mu} \right.$$
$$\left. + \frac{1}{180}R_{ab\mu\nu;\alpha}R_{cd}{}^{\mu\nu;\alpha} + \frac{1}{72}R^{\alpha\beta}R_{ab}{}^{\mu}{}_{\alpha}R_{cd\mu\beta} - \frac{1}{240}R^{\mu\nu\alpha\beta}R_{ab\mu\nu}R_{cd\alpha\beta}\right]$$
$$+ \frac{1}{80}\Sigma^{[ab]}\Sigma^{[cd]}\Sigma^{[e,f]}R_{ab\mu\nu}R_{cd}{}^{\nu}{}_{\gamma}R_{ef}{}^{\gamma\mu} \ . \tag{78}$$

We have also checked that this contribution is consistent to the purely sixth adiabatic order for the renormalized stress-energy tensor that has been reported in [27] (see also [29]).

Taking into account all this, it seems natural to argue that relations (75) and (76) are also valid for an arbitrary nth order, since both adiabatic and DeWitt-Schwinger methods provide a series expansion in which each contribution is univocally derived from some well-defined recursion relations using the first order terms as seeds for iteration. We have explicitly seen that the leading six order contributions agree, so it is very likely that higher order terms will agree as well. The calculation of the fourth and higher order DeWitt coefficients has been an elusive problem for a long time. The formal solution, given by a very involved recursion mechanism, was given in [28]. To show the power of the adiabatic method for cosmological spacetimes, and also as an illustrative example, we have easily worked out the explicit form of the fourth DeWitt-Schwinger coefficient $a_4(x)$ using (A3). It is given in Appendix C.

V. EXTENSION TO SEPARATE POINTS

Finally, we would like to analyze the two-point functions, expanded up to a given adiabatic order, at separate points. The calculations are much more involved. We illustrate here explicitly the equivalence found at fourth adiabatic order for scalar fields. The adiabatic scheme provides the following result

$$^{(4)}G_{Ad}((t,\vec{x}),(t,\vec{x}')) = \frac{1}{2(2\pi)^3a^3}\int d^3\vec{k}\, e^{i\vec{k}\Delta\vec{x}}\left[\frac{1}{\omega} + \frac{(\frac{1}{6}-\xi)R}{2\omega^3} + \frac{m^2\dot{a}^2}{2a^2\omega^5} + \frac{m^2\ddot{a}}{4a\omega^5} - \frac{5m^4\dot{a}^2}{8a^2\omega^7} + (W^{-1})^{(4)}\right]$$
$$= \frac{m}{4\pi^2 a|\Delta\vec{x}|}K_1(am|\Delta\vec{x}|) + \frac{(\frac{1}{6}-\xi)R}{8\pi^2}K_0(am|\Delta\vec{x}|)$$
$$+ \frac{R}{288\pi^2}(ma|\Delta\vec{x}|)K_1(am|\Delta\vec{x}|) - \frac{H^2}{96\pi^2}(am|\Delta\vec{x}|)^2K_0(am|\Delta\vec{x}|)$$
$$+ \frac{1}{2(2\pi)^3a^3}\int d^3\vec{k}\, e^{i\vec{k}\Delta\vec{x}}(W^{-1})^{(4)} \ , \tag{79}$$

where

$$\frac{1}{2(2\pi)^3 a^3}\int d^3\vec{k}\, e^{i\vec{k}\Delta\vec{x}}(W^{-1})^{(4)} =$$
$$\frac{K_0(am|\Delta\vec{x}|)}{\pi^2}\left\{-\frac{7|\Delta\vec{x}|^4a^4H^4}{5760}-\frac{11m^2|\Delta\vec{x}|^4a^4H^2\ddot{a}}{5760a}-\frac{|\Delta\vec{x}|^2\xi aH^2\ddot{a}}{4a}+\frac{43|\Delta\vec{x}|^2a^2H^2\ddot{a}}{960a}\right.$$
$$\left.+\frac{3|\Delta\vec{x}|^2\ddot{a}^2}{320a^2}-\frac{|\Delta\vec{x}|^2\xi\ddot{a}^2}{16a^2}+\frac{7|\Delta\vec{x}|^2}{960}H\frac{\dddot{a}}{a}-\frac{|\Delta\vec{x}|^2\xi H\dddot{a}}{16a}-\frac{|\Delta\vec{x}|^2a^2\ddddot{a}}{960a}\right\}$$
$$+\frac{K_1(am|\Delta\vec{x}|)}{\pi^2}\left\{H^4\left[\frac{|\Delta\vec{x}|a}{32m}-\frac{3\xi|\Delta\vec{x}|a}{8m}+\frac{9\xi^2|\Delta\vec{x}|a}{8m}-\frac{m|\Delta\vec{x}|^3a^3}{180}+\frac{m|\Delta\vec{x}|^3a^3\xi}{32}+\frac{m^3|\Delta\vec{x}|^5a^5}{4608}\right]\right.$$
$$H^2\frac{\ddot{a}}{a}\left[\frac{m|\Delta\vec{x}|^3a^3}{2880}+\frac{m\xi|\Delta\vec{x}|^3a^3}{32}+\frac{29|\Delta\vec{x}|a}{240m}-\frac{17|\Delta\vec{x}|a\xi}{16m}+\frac{9|\Delta\vec{x}|a\xi^2}{4m}\right]$$
$$\frac{\ddot{a}^2}{a^2}\left[\frac{3|\Delta\vec{x}|a}{160m}-\frac{-5\xi|\Delta\vec{x}|a}{16m}+\frac{9|\Delta\vec{x}|a\xi^2}{8m}+\frac{m|\Delta\vec{x}|^3a^3\xi}{640}\right]+\frac{\dddot{a}}{a}H\left[\frac{3|\Delta\vec{x}|a}{80m}+\frac{3|\Delta\vec{x}|a\xi}{16m}+\frac{m|\Delta\vec{x}|^3a^3}{480}\right]$$
$$\left.+\frac{\ddddot{a}}{a}\left[-\frac{|\Delta\vec{x}|}{80m}+\frac{|\Delta\vec{x}|\xi}{16m}\right]\right\}\ .$$

On the other hand, the DeWitt-Schwinger calculation provides (24). To compare it with the above result just expand it to fourth adiabatic order [for the following identities we shall use the auxiliary parameter T to denote the number of time-derivatives that are present]. Use

$$g = 1+\frac{1}{3}R_{\alpha\beta}y^\alpha y^\beta+\frac{1}{6}R_{\alpha\beta;\gamma}y^\alpha y^\beta y^\gamma+\left[\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta}-\frac{1}{90}R_{\lambda\alpha\beta}{}^{k}R^{\lambda}{}_{\gamma\delta k}+\frac{1}{20}R_{\alpha\beta;\gamma\delta}\right]y^\alpha y^\beta y^\gamma y^\delta+O(T^{-5})\ , \tag{82}$$

and relations (28)-(30) including the fourth adiabatic contributions [23] (for simplicity we only show here, without loss of generality, the corresponding expressions for $\Delta t = 0$)

$$y^0 = \frac{1}{2}Ha^2\Delta x^2+\frac{1}{144}a^4\Delta x^4HR+O(T^{-5})\ , \tag{83}$$
$$y^i = a\Delta x^i\left\{1+\frac{1}{6}a^2\Delta x^2H^2+\frac{1}{120}a^4\Delta x^4H^2\left[H^2+3\frac{\ddot{a}}{a}\right]\right\}+O(T^{-5})\ , \tag{84}$$
$$-2\sigma = \Delta x^2a^2+\frac{1}{12}a^4\Delta x^4H^2+\frac{1}{360}a^6\Delta x^6H^2\left[H^2+3\frac{\ddot{a}}{a}\right]+O(T^{-5})\ . \tag{85}$$

Using all these auxiliary expressions we can prove that the adiabatic scheme generates the same two-point function as the DeWitt-Schwinger one up to fourth order in the derivatives of the metric,

$$^{(4)}G_{DS}(x,x') = {}^{(4)}G_{Ad}(x,x')\ . \tag{86}$$

We believe that one might extend this identity up to any order by induction. We note in passing that the result (86) implies the equivalence of the renormalized stress-energy tensor for $\xi \neq 1/6$ [see Sec. II. D].

VI. CONCLUSIONS AND FINAL COMMENTS

The main motivation of this paper is to show the equivalence of the renormalized expectation values of the stress-energy tensor for spin-1/2 fields using either adiabatic and DeWitt-Schwinger methods. This is a very natural question since the adiabatic renormalization scheme for Dirac fields has been introduced very recently in the literature. The employed strategy to achieve our goal has led us to show the equivalence for scalar fields as well, in a simpler way to that used in [18, 19]. Moreover, we were naturally led to investigate the equivalence for the two-point function at coincidence for both DeWitt-Schwinger and adiabatic series expansion at any order. We have checked explicitly that the equality hold at sixth adiabatic order and we have argued that the equivalence must hold at an arbitrary order. This way, the adiabatic regularization method will offer a very efficient computational tool to evaluate the higher order DeWitt coefficients in FLRW space-times for both scalar and Dirac fields. This may be relevant to capture non-perturbative aspects of the effective action in cosmological space-times, as those found in [30–32]. Finally, we would like to remark that these results suggest that the equality ${}^{(n)}G_{Ad}(x,x) = {}^{(n)}G_{DS}(x,x)$, $n = 0, 2, 4, 6, ...$ (an the analogue for Dirac fields) could even hold for separate points. This is actually supported by the fact that (41), (74), extended to separate points, coincide at least up to the fourth adiabatic order.

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Appendix A: Adiabatic Expansion for Klein-Gordon fields

In this section we show the generic expression for the nth contribution in the K-G adiabatic expansion given by (8). Introducing the ansatz (7) into the equation of motion for the modes, one finds the following equation [1, 2],

$$W_k^2 = \omega^2 + \sigma + \frac{3}{4}\frac{\dot{W_k}^2}{W_k^2} - \frac{1}{2}\frac{\ddot{W}_k}{W_k} \ , \tag{A1}$$

where

$$\sigma = \left(6\xi - \frac{3}{4}\right)\left(\frac{\dot{a}}{a}\right)^2 + \left(6\xi - \frac{3}{2}\right)\frac{\ddot{a}}{a} \ . \tag{A2}$$

Equation (A1) can be solved *algebraically* by iteration for initial value $\omega^{(0)} \equiv \omega = \sqrt{(k/a)^2 + m^2}$. Performing the calculation up to nth adiabatic order it can be shown,

$$\begin{aligned}
\omega^{(n)} = \frac{1}{2\omega^3}\Bigg\{ & \omega^2\left[(\omega^{(n/2)})^2 + 2\sum_{i=2}^{n/2-1}\omega^{(i)}\omega^{(n-i)}\right] + \sigma\left[(\omega^{(n/2-1)})^2 + 2\sum_{i=0}^{n/2-2}\omega^{(i)}\omega^{(n-2-i)}\right] \\
& + \frac{3}{4}\left[(\dot{\omega}^{(n/2-1)})^2 + 2\sum_{i=0}^{n/2-2}\dot{\omega}^{(i)}\dot{\omega}^{(n-2-i)}\right] - \frac{1}{2}\left[\ddot{\omega}^{(n/2-1)}\omega^{(n/2-1)} + \sum_{i=0}^{n/2-2}\left(\ddot{\omega}^{(i)}\omega^{(n-2-i)} + \omega^{(i)}\ddot{\omega}^{(n-2-i)}\right)\right] \\
& - \Bigg[6\sum_{i=0}^{n/4-1}(\omega^{(i)})^2(\omega^{(n/2-i)})^2 + 4\sum_{k=0}^{n/2-1}(\omega^{(k)})^2\sum_{i=2}^{n/2-k-1}\omega^{(i)}\omega^{(n-i-2k)} + 4\sum_{k=2}^{n/2-1}(\omega^{(k)})^2\sum_{i=0}^{n/2-k-1}\omega^{(i)}\omega^{(n-i-2k)} \\
& + 8\sum_{i=0}^{n/4-2}\omega^{(i)}\sum_{j=i+2}^{n/2-2-i}\omega^{(j)}\sum_{k=j+2}^{n-j-2i-2}\omega^{(k)}\omega^{(n-k-i-j)} + 8\sum_{k=0}^{n/4-3/2}(\omega^{(k)})^2\sum_{i=k+2}^{n/2-k-1}\omega^{(i)}\omega^{(n-i-2k)} + (\omega^{(n/4)})^4\Bigg]\Bigg\} \ ,
\end{aligned} \tag{A3}$$

with $\omega^{(s)} = 0$ for $s < 0$ or s being a fractional number. With this formula we can recover $\omega^{(s)} = 0$, for s being an odd integer, and the corresponding expressions for orders 2 and 4 from [1, 2],

$$\omega^{(2)} = \frac{1}{2}\omega^{-1/2}\frac{d^2}{dt^2}\omega^{-1/2} + \frac{1}{2}\omega^{-1}\sigma \ , \tag{A4}$$

$$\omega^{(4)} = \frac{1}{4}\omega^{(2)}\omega^{-3/2}\frac{d^2}{dt^2}\omega^{-1/2} - \frac{1}{2}\omega^{-1}(\omega^{(2)})^2 - \frac{1}{4}\omega^{-1/2}\frac{d^2}{dt^2}\left[\omega^{-3/2}\omega^{(2)}\right] \ , \tag{A5}$$

as well. In general, (A3) allows us to obtain any $\omega^{(n)}$ in terms of lower order adiabatic terms and its derivatives.

Appendix B: Adiabatic Expansion for Dirac fields

In this section we present the generic expressions for the nth contribution in the Dirac adiabatic expansion given by (49)-(50). Introducing these expressions into the equation of motion for the modes, (48), one gets a set of coupled *algebraic* equations [20]

$$(\omega - m)G = (\Omega - m)F + i\dot{F} - \frac{im\dot{\omega}}{2\omega(\omega+m)}F + (\omega - m)F \ , \tag{B1}$$

$$(\omega + m)F = (\Omega - m)G + i\dot{G} + \frac{im\dot{\omega}}{2\omega(\omega-m)}G + (\omega + m)G \ , \tag{B2}$$

$$2\omega = (\omega + m)FF^* + (\omega - m)GG^* \ , \tag{B3}$$

which can be solved *algebraically* by iteration for initial values $F^{(0)} = G^{(0)} = 1$ and $\omega^{(0)} = \omega$. The general algorithm to compute the three fundamental objects [notice that $G(-m)$ satisfies the same equations as $F(m)$, so we take $G(-m) = F(m)$] is provided by

$$\omega^{(n)} = -\frac{m}{\omega}\left\{\sum_{l=1}^{n-1}\omega^{(l)}F^{(n-l)} + i\dot{F}^{(n-1)} - \frac{im\dot{\omega}}{2\omega(\omega+m)}F^{(n-1)}\right\} \tag{B4}$$

$$+\left(1-\frac{m}{\omega}\right)\left\{-\frac{i}{2}\left[\dot{F}^{(n-1)}+\dot{G}^{(n-1)}\right] - \frac{1}{2}\sum_{l=1}^{n-1}\omega^{(l)}\left[F^{(n-1)}+G^{(n-1)}\right] + \frac{im\dot{\omega}}{4\omega}\left[\frac{F^{(n-1)}}{(\omega+m)} + \frac{G^{(n-1)}}{(\omega-m)}\right]\right\} ,$$

$$Re\, F^{(n)}(m) = \frac{\delta_{n0}}{2} - \frac{1}{4\omega}\sum_{l=1}^{n-1}\left[F^{(l)}F^{*(n-l)}(\omega+m) + G^{(l)}G^{*(n-l)}(\omega-m)\right] + \frac{1}{2\omega}Im\, \dot{F}^{(n-1)}(m)$$

$$-\frac{1}{2\omega}\sum_{l=1}^{n}\omega^{(l)}Re\, F^{(n-l)}(m) - \frac{m\dot{\omega}}{4\omega^2(m+\omega)}Im\, F^{(n-1)}(m) , \tag{B5}$$

$$Im\, F^{(n)}(m) = Im\, G^{(n)}(m) - \frac{1}{\omega-m}\left\{\sum_{l=1}^{n}\omega^{(l)}Im\, F^{(n-l)} + Re\, \dot{F}^{(n-1)} - \frac{m\dot{\omega}}{2\omega(\omega+m)}Re\, F^{(n-1)}\right\} , \tag{B6}$$

with $F = Re\, F(m) + iIm\, F(m)$ and $G = Re\, G(m) + iIm\, G(m)$. Notice that there is an inherent ambiguity in the formalism reflected in the choice for $Im\, G(m)$, but it can be explicitly seen that it does not affect the observables such as $\langle\bar{\psi}\psi\rangle$ or $\langle T_{\mu\nu}\rangle$ [20]. The simplest way to remove the ambiguities is to assume $Im\, G^{(n)}(m) = -Im\, F^{(n)}(m)$. Detailed expressions for the first adiabatic contributions can be found in [20, 21]. In general, (B4)-(B6) allows us to obtain any Dirac adiabatic contribution in terms of lower order adiabatic terms and its derivatives.

Appendix C: a_4 coefficient.

We give here the result for the a_4 DeWitt coefficient for a spatially flat FLRW spacetime, as obtained with the adiabatic regularization method (A3).

$$
\begin{aligned}
a_4(x) = {} & \frac{29\dot{a}^8}{120a^8} - \frac{379\dot{a}^6\ddot{a}}{210a^7} + \frac{899\dot{a}^4\ddot{a}^2}{280a^6} + \frac{83\dot{a}^2\ddot{a}^3}{35a^5} - \frac{13\ddot{a}^4}{21a^4} + \frac{47\dot{a}^5\dddot{a}}{70a^6} + \frac{2\dot{a}^3\ddot{a}\dddot{a}}{3a^5} - \frac{103\dot{a}\ddot{a}^2\dddot{a}}{28a^4} - \frac{647\dot{a}^2\dddot{a}^2}{840a^4} \\
& + \frac{103\ddot{a}^2\ddddot{a}}{210a^3} - \frac{2\dot{a}^4\ddddot{a}}{21a^5} - \frac{93\dot{a}^2\ddot{a}\ddddot{a}}{70a^4} + \frac{34\ddot{a}\dddot{a}^2}{105a^3} + \frac{199\dot{a}\dddot{a}\ddddot{a}}{420a^3} + \frac{11\ddddot{a}^2}{504a^2} - \frac{13\dot{a}^3a^{(5)}}{210a^4} + \frac{41\dot{a}\ddot{a}a^{(5)}}{140a^3} \\
& + \frac{29\dddot{a}a^{(5)}}{1260a^2} + \frac{3\dot{a}^2a^{(6)}}{70a^3} + \frac{13\ddot{a}a^{(6)}}{1260a^2} - \frac{\dot{a}a^{(7)}}{126a^2} - \frac{a^{(8)}}{630a} - \frac{7\xi\dot{a}^8}{5a^8} - \frac{39\xi^2\dot{a}^8}{5a^8} + \frac{36\xi^3\dot{a}^8}{a^8} + \frac{54\xi^4\dot{a}^8}{a^8} \\
& + \frac{383\dot{a}^6\ddot{a}}{20a^7} - \frac{15\xi^2\dot{a}^6\ddot{a}}{a^7} - \frac{234\xi 3\dot{a}^6\ddot{a}}{a^7} + \frac{216\xi^4\dot{a}^6\ddot{a}}{a^7} - \frac{8123\xi\dot{a}^4\ddot{a}^2}{140a^6} + \frac{2859\xi^2\dot{a}^4\ddot{a}^2}{10a^6} \\
& - \frac{432\xi^3\dot{a}^4\ddot{a}^2}{a^6} + \frac{324\xi^4\dot{a}^4\ddot{a}^2}{a^6} - \frac{254\xi\dot{a}^2\ddot{a}^3}{15a^5} + \frac{264\xi^2\dot{a}^2\ddot{a}^3}{5a^5} - \frac{180\xi^3\dot{a}^2\ddot{a}^3}{a^5} + \frac{216\xi^4\dot{a}^2\ddot{a}^3}{a^5} + \frac{523\xi\ddot{a}^4}{105a^4} \\
& - \frac{81\xi^2\ddot{a}^4}{10a^4} - \frac{18\xi^3\ddot{a}^4}{a^4} + \frac{54\xi^4\ddot{a}^4}{a^4} - \frac{211\xi\dot{a}^5\dddot{a}}{20a^6} + \frac{201\xi^2\dot{a}^5\dddot{a}}{5a^6} - \frac{18\xi^3\dot{a}^5\dddot{a}}{a^6} + \frac{53\xi\dot{a}^3\ddot{a}\dddot{a}}{7a^5} - \frac{69\xi^2\dot{a}^3\ddot{a}\dddot{a}}{a^5} \\
& + \frac{72\xi^3\dot{a}^3\ddot{a}\dddot{a}}{a^5} + \frac{439\xi\dot{a}\ddot{a}^2\dddot{a}}{14a^5} - \frac{84\xi^2\dot{a}\ddot{a}^2\dddot{a}}{a^4} + \frac{90\xi^3\dot{a}\ddot{a}^2\dddot{a}}{a^4} + \frac{6\xi\dot{a}^2\dddot{a}^2}{a^4} - \frac{147\xi^2\dot{a}^2\dddot{a}^2}{10a^4} \\
& + \frac{18\xi^3\dot{a}^2\dddot{a}^2}{a^4} - \frac{51\xi^3\ddot{a}\dddot{a}^2}{20a^3} - \frac{3\xi^2\ddot{a}\dddot{a}^2}{a^3} + \frac{18\xi^3\ddot{a}\dddot{a}^2}{a^3} + \frac{11\xi\dot{a}^4\ddddot{a}}{4a^5} - \frac{15\xi^2\dot{a}^4\ddddot{a}}{a^5} + \frac{18\xi^3\dot{a}^4\ddddot{a}}{a^5} \\
& + \frac{157\xi\dot{a}^2\ddot{a}\ddddot{a}}{14a^4} - \frac{153\xi^2\dot{a}^2\ddot{a}\ddddot{a}}{5a^4} + \frac{36\xi^3\dot{a}^2\ddot{a}\ddddot{a}}{a^4} - \frac{19\xi\ddot{a}^2\ddddot{a}}{14a^3} - \frac{24\xi^2\ddot{a}^2\ddddot{a}}{5a^3} + \frac{18\xi^3\ddot{a}^2\ddddot{a}}{a^3} \\
& - \frac{237\xi\dot{a}\dddot{a}\ddddot{a}}{70a^3} + \frac{27\xi^2\dot{a}\dddot{a}\ddddot{a}}{5a^3} - \frac{39\xi\ddddot{a}^2}{140a^2} + \frac{9\xi^2\ddddot{a}^2}{10a^2} + \frac{3\xi\dot{a}^3a^{(5)}}{10a^4} + \frac{41\dot{a}\ddot{a}a^{(5)}}{140a^3} - \frac{15\xi\dot{a}\ddot{a}a^{(5)}}{7a^3} \\
& + \frac{18\xi^2\dot{a}\ddot{a}a^{(5)}}{5a^3} - \frac{12\xi\dddot{a}a^{(5)}}{35a^2} + \frac{6\xi^2\dddot{a}a^{(5)}}{5a^2} + \frac{3\dot{a}^2a^{(6)}}{70a^3} - \frac{23\xi\dot{a}^2a^{(6)}}{70a^3} + \frac{3\xi^2\dot{a}^2a^{(6)}}{5a^3} - \frac{6\xi\ddot{a}a^{(6)}}{35a^2} + \frac{3\xi^2\ddot{a}a^{(6)}}{5a^2} \\
& + \frac{\xi\dot{a}a^{(7)}}{28a^2} + \frac{\xi a^{(8)}}{140a^2}\ . \qquad \text{(C1)}
\end{aligned}
$$

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