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A Tree–Loop Duality Relation at Two Loops and Beyond

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Abstract

The duality relation between one–loop integrals and phase–space integrals, developed in a previous work, is extended to higher–order loops. The duality relation is realized by a modification of the customary $+i0$ prescription of the Feynman propagators, which compensates for the absence of the multiple–cut contributions that appear in the Feynman tree theorem. We rederive the duality theorem at one–loop order in a form that is more suitable for its iterative extension to higher–loop orders. We explicitly show its application to two– and three–loop scalar master integrals, and we discuss the structure of the occurring cuts and the ensuing results in detail.

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1 Introduction

With the start of the LHC, the physics of elementary particles enters a new era, opening a powerful window to discover the Higgs boson and to explore new interactions beyond the Standard Model (SM) at the TeV energy scale. Precision theoretical predictions for background and signal multi–particle hard scattering processes, in the SM and beyond, are mandatory for the phenomenological interpretation of experimental data, and thus to achieve a successful exploitation of the LHC physics programme.

While leading–order (LO) predictions of multi–particle processes at hadron colliders in perturbative Quantum Chromodynamics (pQCD) provide, in general, a rather poor description of experimental data, next–to–leading order (NLO) is the first order at which normalizations, and in some cases the shapes, of cross sections can be considered reliable, [1]. Next–to–next–to leading order (NNLO), besides improving the determination of normalizations and shapes, is also generally accepted to provide the first serious estimate of the theoretical uncertainty in pQCD. Despite the relatively smaller coupling, electroweak (EW) radiative NLO corrections might also be sizable at the LHC, [2, 3].

Computing higher–order corrections in Quantum Field Theories, in particular in QCD or in the EW sector of the SM, is highly challenging and substantially demanding as the complexity increases with the number of external particles, and the order in perturbation theory at which the hard scattering process must be calculated in order to match the experimental precision. In the recent years, important efforts have been devoted to developing efficient methods able to boost forward the calculational capability both at the multi–leg and the multi–loop frontier. Today, $2 \to 4$ processes at NLO, either from Unitarity based methods, [4, 5, 6], or from a more traditional Feynman diagrammatic approach, [7], are affordable and are even becoming standardized. There has also been a lot of progress concerning NNLO calculations [8, 9, 10, 11].

In Ref. [12], a duality relation between one–loop integrals and phase–space integrals has been demonstrated. The duality relation is suitable to numerically compute, [13], multi–leg one–loop cross sections in perturbative field theories (local and unitary). It has analogies with the Feynman tree theorem (FTT), [14, 15], but involves only single cuts of the one–loop Feynman diagrams. The duality theorem requires to properly regularize propagators by a complex Lorentz–covariant prescription, which is different from the customary $+i0$ prescription of the Feynman propagators. The main consequence of this new prescription is that the multiple cuts appearing in the FTT are avoided.

The computation of cross sections at NLO (or NNLO) requires the separate evaluation of real and virtual radiative corrections. Real (virtual) radiative corrections are given by multi–leg tree–level (loop) matrix elements to be integrated over the multi–particle phase–space of the physical process. The loop–tree duality at one–loop presented in Ref. [12], as well as other methods relating one–loop and phase–space integrals [16, 17, 18], have the attractive feature that they recast the virtual radiative corrections in a form that closely parallels the contribution of the real radiative corrections. This close correspondence can help to directly combine real and virtual contributions to NLO cross sections. In this paper, we extend the loop–tree duality theorem derived in Ref. [12] to higher–order loops, as a first attempt towards extending the duality method to the computation of cross sections at NNLO or even higher orders. Preliminary results were presented in Ref. [19].

The outline of the paper is as follows: In Section 2, we reconsider the tree–loop duality theorem at one–loop. This involves the definition of dual propagators in addition to the Feynman, advanced and retarded propagators commonly known. Afterwards, we reformulate the duality theorem in a way which is more appropriate for extending it to higher loop orders. This is done by providing functions of propagators of complete sets of momenta, corresponding to the internal lines of the diagram. As for their single–momenta analogues, it is likewise possible to infer relations amongst these functions. The proof of the main relation which is crucial for the extension to higher loop orders, is given in the Appendix A. In Section 3, we then provide a duality theorem for the two–loop master diagram with N external legs by using the previously defined relations and iteratively applying the duality theorem to the occurring loops. We also discuss some subtleties involved, as well as the structure of the result and occurring cuts. Furthermore, we derive a two–loop representation of the Feynman tree theorem. Having set up the basic method in this way, we will continue with the four basic master topologies at three loops in Section 4, and show that the method is indeed extendible to even higher loop orders in an iterative manner. We end this section with a brief comment on the extension of the duality theorem at the amplitude level. In Section 5, we conclude and provide an outlook. In the Appendix B, we provide a simple example of a two–loop scalar integral calculated from its dual representation.

2 Duality relation at one loop

In this section, we provide the basic quantities, definitions and relations used in the rest of the paper, and sketch the steps for the derivation of the tree–loop duality theorem at one–loop, as presented in Ref. [12]. We will also rederive this tree–loop duality theorem in a form which is more suitable for its extension to higher orders by introducing the main formulas used for the iterative application of the duality.

Let us start by considering a general one–loop N–leg diagram, as shown in Fig. 1, which is represented by the scalar integral:

$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_{\ell_1} \prod_{i=1}^{N} G_F(q_i) \ . \tag{1}$$

The four–momenta of the external legs are denoted p_i, $i \in \{1, 2, \dots N\}$. All are taken as outgoing and ordered clockwise. The loop momentum is ℓ_1, which flows anti–clockwise. The momenta of the internal lines q_i, are defined as

$$q_i = \ell_1 + p_{1,i} \ , \qquad i \in \alpha_1 = \{1, 2, \dots N\} \ . \tag{2}$$

As is commonly used, we define $p_{i,j} = p_i + p_{i+1} + \dots + p_j$. Momentum conservation is equivalent to $p_{1,N} = 0$. We use dimensionally regularized integrals with the number of space–time dimensions equal to d, and introduce the following shorthand notation:

$$\int_{\ell_i} \cdots \equiv -i \int \frac{d^d \ell_i}{(2\pi)^d} \cdots \ . \tag{3}$$

The space–time coordinates of any momentum k_μ are denoted as $k_\mu = (k_0, \mathbf{k})$, where k_0 is the energy (time) component of k_μ. The Feynman propagators $G_F(q_i)$ in Eq. (1) have real internal masses m_i:

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} \ . \tag{4}$$

The derivation of the duality theorem is exactly the same regardless of the internal lines being massive or massless ($m_i = 0$), as long as the masses are real. Non–vanishing internal real masses only account for a

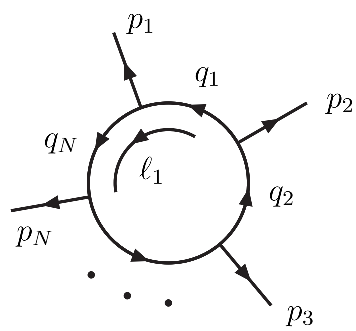

Figure 1: *Momentum configuration of the one–loop N–point scalar integral.*

displacement of the poles of the propagators along the real axis, which does not change the derivation of the duality theorem, as will become obvious in the following. Moreover, they do not alter the relationship between Feynman, advanced, retarded and dual propagators, which is the basis of both, the duality theorem as a duality to the FTT, as well as the extension of the method to higher loop orders. The case of unstable particles with complex masses has been discussed in detail in Ref. [12], and we do not consider this possibility in the current paper.

Besides the customary Feynman propagators $G_F(q_i)$, we also encounter advanced, $G_A(q_i)$, and retarded, $G_R(q_i)$, propagators, defined by:

$$G_A(q_i) = \frac{1}{q_i^2 - m_i^2 - i0\, q_{i,0}} \ , \qquad G_R(q_i) = \frac{1}{q_i^2 - m_i^2 + i0\, q_{i,0}} \ . \tag{5}$$

The Feynman, advanced, and retarded propagators only differ in the position of the particle poles in the complex plane. Using $q_i^2 = q_{i,0}^2 - \mathbf{q}_i^2$, we therefore find the poles of the Feynman and advanced propagators in the complex plane of the variable $q_{i,0}$ at:

$$[G_F(q_i)]^{-1} = 0 \Longrightarrow q_{i,0} = \pm\sqrt{\mathbf{q}_i^2 - m_i^2 - i0} \ \text{ and } [G_A(q_i)]^{-1} = 0 \Longrightarrow q_{i,0} \simeq \pm\sqrt{\mathbf{q}_i^2 - m_i^2} + i0 \, . \tag{6}$$

Thus, the pole with positive/negative energy of the Feynman propagator is slightly displaced below/above the real axis, while both poles of the advanced/retarded propagator, independently of the sign of the energy, are slightly displaced above/below the real axis (cf. Fig. 2). We further define

$$\widetilde{\delta}\,(q_i) \equiv 2\pi\, i\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2) = 2\pi\, i\, \delta_+(q_i^2 - m_i^2) \ , \tag{7}$$

where the subscript $+$ of δ_+ refers to the on–shell mode with positive definite energy, $q_{i,0} \geq 0$. Hence, the phase–space integral of a physical particle with momentum q_i, i.e., an on–shell particle with positive–definite energy, $q_i^2 = m_i^2$, $q_{i,0} \geq 0$, reads:

$$\int \frac{d^d q_i}{(2\pi)^{d-1}}\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2) \ \cdots \equiv \int_{q_i} \widetilde{\delta}\,(q_i) \ \cdots \ . \tag{8}$$

We continue by shortly recalling the duality theorem at one–loop order, which was derived in Ref. [12]. For a detailed discussion of all definitions and steps, as well as subtleties related to them, we refer the reader to this paper. In order to derive the duality theorem, one directly applies the residue theorem to the computation of $L^{(1)}(p_1, p_2, \ldots, p_N)$ in Eq. (1): Each of the Feynman propagators $G_F(q_i)$ has single poles

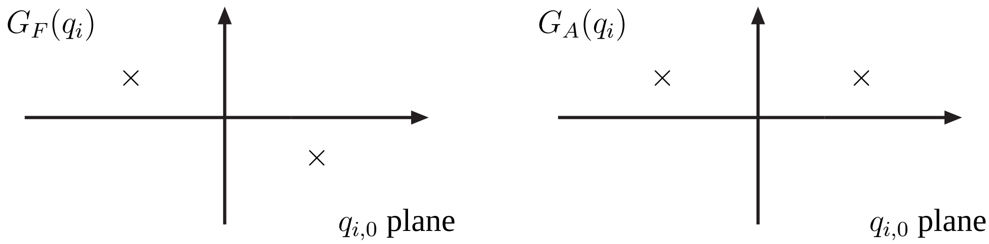

Figure 2: *Location of the particle poles of the Feynman (left) and advanced (right) propagators* $G_F(q_i)$ *and* $G_A(q_i)$ *in the complex plane of the variable* $q_{i,0}$.

in both the upper and lower half–planes of the complex variable $\ell_{1,0}$. Since the integrand is convergent when $\ell_{1,0} \to \infty$, by closing the contour at ∞ in the lower half–plane and applying the Cauchy theorem, the one–loop integral becomes the sum of N contributions, each of them obtained by evaluating the loop integral at the residues of the poles with negative imaginary part belonging to the propagators $G_F(q_i)$. The calculation of the residue of $G_F(q_i)$ gives

$$\mathrm{Res}[G_F(q_i)]_{\mathrm{Im}(q_{i,0})<0} = \int d\ell_{1,0}\, \delta_+(q_i^2 - m_i^2) \ , \tag{9}$$

with $\delta_+(q_i^2 - m_i^2)$ defined in Eq. (7). This result shows that considering the residue of the Feynman propagator of the internal line with momentum q_i is equivalent to cutting that line by including the corresponding on–shell propagator $\delta_+(q_i^2 - m_i^2)$. The propagators $G_F(q_j)$, with $j \neq i$, are not singular at the value of the pole of $G_F(q_i)$ and can therefore be directly evaluated at this point, yielding to

$$\prod_{j\neq i} G_F(q_j)\Big|_{\substack{G_F(q_i)^{-1}=0 \\ \mathrm{Im}(q_{i,0})<0}} = \prod_{j\neq i} G_D(q_i;q_j) \ , \tag{10}$$

where

$$G_D(q_i;q_j) = \frac{1}{q_j^2 - m_j^2 - i0\,\eta(q_j - q_i)} \ , \tag{11}$$

is the so–called dual propagator, as defined in Ref. [12], with η a *future–like* vector,

$$\eta_\mu = (\eta_0, \eta) \ , \qquad \eta_0 \geq 0, \ \eta^2 = \eta_\mu \eta^\mu \geq 0 \ , \tag{12}$$

i.e., a d–dimensional vector that can be either light–like $(\eta^2 = 0)$ or time–like $(\eta^2 > 0)$ with positive definite energy η_0.

Collecting the results from Eq. (9) and Eq. (10), the tree–loop duality theorem at one–loop, [12], takes the final form

$$L^{(1)}(p_1, p_2, \dots, p_N) \ = \ -\sum \int_{\ell_1} \widetilde{\delta}\,(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_i;q_j) \ . \tag{13}$$

Contrary to the FTT, [14, 15], Eq. (13) contains only single–cut integrals. Multiple–cut integrals, like those that appear in the FTT, are absent thanks to modifying the original $+i0$ prescription of the uncut

Feynman propagators in Eq. (10) by the new prescription $-i0\,\eta(q_j - q_i)$, which is named the 'dual' $i0$ prescription or, briefly, the η prescription. This is the main result of Ref. [12]. The dual $i0$ prescription arises from the fact that the original Feynman propagator $G_F(q_j)$ is evaluated at the *complex* value of the loop momentum ℓ_1, which is determined by the location of the pole at $q_i^2 - m_i^2 + i0 = 0$. The $i0$ dependence of the pole of $G_F(q_i)$ modifies the $i0$ dependence in the Feynman propagator $G_F(q_j)$, leading to the total dependence as given by the dual $i0$ prescription. The presence of the vector η_μ is a consequence of using the residue theorem and the fact that the residues at each of the poles are not Lorentz–invariant quantities. The Lorentz–invariance of the loop integral is recovered after summing over all the residues. Furthermore, in the one–loop case, the momentum difference $\eta(q_j - q_i)$ is independent of the integration momentum ℓ_1, and only depends on the momenta of the external legs (cf. Eq. (2)).

We now rederive the one–loop duality theorem Eq. (13) by exploiting the relationship between Feynman, advanced and dual propagators. This will prove to be useful when extending the duality theorem to higher orders. Using the elementary identity

$$\frac{1}{x \pm i0} = \mathrm{PV}\left(\frac{1}{x}\right) \mp i\pi\,\delta(x) \quad , \tag{14}$$

where PV denotes the principal–value prescription, we can transform one kind of propagators into the other:

$$G_A(q_i) = G_F(q_i) + \widetilde{\delta}\,(q_i) \quad , \qquad G_R(q_i) = G_F(q_i) + \widetilde{\delta}\,(-q_i) \quad , \qquad G_A(-q_i) = G_R(q_i)\;. \tag{15}$$

Dual and Feynman propagators are related through, [12],

$$\widetilde{\delta}\,(q_i)\;\; G_D(q_i; q_j) = \widetilde{\delta}\,(q_i)\;\; \Big[G_F(q_j) + \tilde{\theta}\,(q_j - q_i)\;\; \widetilde{\delta}\,(q_j)\Big]\;, \tag{16}$$

with $\tilde{\theta}\,(q) = \theta(\eta q)$.

In the following, we extend the definition of propagators of single momenta to combinations of propagators of sets of internal momenta: Let α_k be any set of internal momenta with $q_i, q_j \in \alpha_k$. We then define Feynman, advanced, retarded and dual propagator functions of this set α_k in the following way:

$$G_{F(A,R)}(\alpha_k) = \prod_{i\in\alpha_k} G_{F(A,R)}(q_i)\;, \qquad G_D(\alpha_k) = \sum_{i\in\alpha_k} \widetilde{\delta}\,(q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i; q_j)\;. \tag{17}$$

By definition, $G_D(\alpha_k) = \widetilde{\delta}\,(q_i)$, when $\alpha_k = \{i\}$ and thus consists of a single four momentum. At one–loop order, α_k is naturally given by all internal momenta of the diagram which depend on the single integration loop momentum ℓ_1, $\alpha_k = \{1, 2, \ldots, N\}$. However, let us stress that α_k can in principle be any set of internal momenta. At higher order loops, e.g., several integration loop momenta are needed, and we can define several loop lines α_k to label all the internal momenta (cf. Eq. (27)) where Eq. (17) will be used for these loop lines or unifications of these. To simplify the notation, we also introduce

$$G_D(-\alpha_k) = \sum_{i\in\alpha_k} \widetilde{\delta}\,(-q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(-q_i; -q_j)\;, \qquad G_A(-\alpha_k) = \prod_{i\in\alpha_k} G_A(-q_i) = G_R(\alpha_k)\;, \tag{18}$$

where the sign in front of α_k indicates that we have reversed the momentum flow of all the internal lines in α_k. For Feynman propagators, moreover, $G_F(-\alpha_k) = G_F(\alpha_k)$.

In analogy to Eq. (15), the following relation holds for any set of internal momenta α_k:

$$G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k) \ . \tag{19}$$

This is a non–trivial relation, considering Eq. (16). Note that individual terms in $G_D(\alpha_k)$ depend on the dual vector η, but the sum over all terms contributing to $G_D(\alpha_k)$ is independent of it. We leave the detailed proof by induction of Eq. (19) for the Appendix A. Eq. (19) is our main result for a straightforward derivation of the duality theorem.

Another crucial relation for the following is given by a formula that allows to express the dual function of a set of momenta in terms of chosen subsets. Consider the following set $\beta_N \equiv \alpha_1 \cup ... \cup \alpha_N$, where β_N is the unification of various subsets α_i. Solving for the dual part, Eq. (19) then has the following form:

$$G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) = G_A(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) - G_F(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \ . \tag{20}$$

We continue by using the multiplicativity of $G_A(\beta_N)$ and $G_F(\beta_N)$, as defined in Eq. (17), to obtain:

$$\begin{aligned} G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) &= \prod_{i=1}^{N} G_A(\alpha_i) - \prod_{i=1}^{N} G_F(\alpha_i) \\ &= \prod_{i=1}^{N} \left[G_F(\alpha_i) + G_D(\alpha_i) \right] - \prod_{i=1}^{N} G_F(\alpha_i) \\ &= \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \ \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2}) \, . \end{aligned} \tag{21}$$

The sum runs over all partitions of β_N into exactly two blocks $\beta_N^{(1)}$ and $\beta_N^{(2)}$ with elements α_i, $i \in \{1, ..., N\}$, where, contrary to the usual definition, we include the case: $\beta_N^{(1)} \equiv \beta_N$, $\beta_N^{(2)} \equiv \emptyset$. This relation will be extensively used in the subsequent calculations. For the case of $N = 2$, e.g., where $\beta_2 \equiv \alpha_1 \cup \alpha_2$, we have:

$$G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1) \, G_D(\alpha_2) + G_D(\alpha_1) \, G_F(\alpha_2) + G_F(\alpha_1) \, G_D(\alpha_2) \ . \tag{22}$$

Since in general relation (21) holds for any constellation of basic elements α_i which are sets of internal momenta, one can look at these expressions in different ways, depending on the given sets and subsets considered. If we define, for example, the basic subsets α_i to be given by single momenta q_i, and since in that case $G_D(q_i) = \widetilde{\delta}\,(q_i)$, Eq. (21) then denotes a sum over all possible differing m–tuple cuts for the momenta in the set β_N, while the uncut propagators are Feynman propagators. These cuts start from single cuts up to the maximal number of cuts given by the term where all the propagators of the considered set are cut.

Let us now return to the one–loop integral: Since advanced propagators have poles with positive imaginary part only, we do not enclose any singularity by closing the integration contour at ∞ in the lower half–plane, and therefore a loop integral over advanced propagators vanishes. Thus, by using Eq. (19), we find

$$0 = \int_{\ell_1} G_A(\alpha_1) = \int_{\ell_1} \left[G_F(\alpha_1) + G_D(\alpha_1) \right] \ , \tag{23}$$

where α_1 as in Eq. (2) labels *all* internal momenta q_i. The first term on the right–hand side of Eq. (23), containing only Feynman propagators, is the original one–loop integral. Therefore,

$$L^{(1)}(p_1, p_2, \ldots, p_N) = - \int_{\ell_1} G_D(\alpha_1) \ . \tag{24}$$

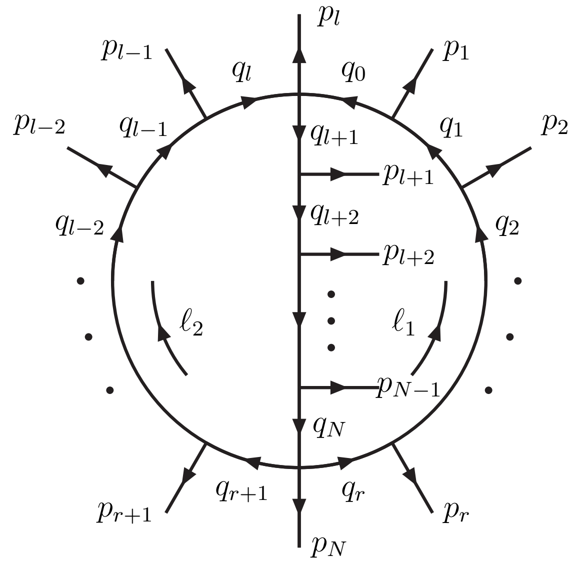

Figure 3: *Momentum configuration of the two–loop N–point scalar integral.*

In this way, we directly obtain the duality relation between one–loop integrals and single–cut phase–space integrals and hence Eq. (24) can be interpreted as the application of the duality theorem to the given set of momenta α_1. It obviously agrees with Eq. (13). Furthermore, by using Eq. (21) in its refined form where the subsets α_i are given by the single momenta q_i of the inner lines of the one–loop integral, we rederive the FTT at one–loop, namely the one–loop integral written in terms of multiple–cut contributions and Feynman propagators:

$$L^{(1)}(p_1, p_2, \ldots, p_N) \;=\; - \sum_{\alpha_1^{(1)} \cup \alpha_1^{(2)} = \alpha_1} \int_{\ell_1} \prod_{i_1 \in \alpha_1^{(1)}} \widetilde{\delta}\,(q_{i_1}) \prod_{i_2 \in \alpha_1^{(2)}} G_F(q_{i_2})\;. \tag{25}$$

The sum runs over all partitions of α_1 as defined in Eq. (21), excluding the possibility to have a term with only Feynman propagators. The m–cut integral of the FTT is given by the sum of the contributions from all partitions of α_1, with $\alpha_1^{(1)}$ containing precisely m elements*.

The extension of the duality theorem and of the FTT from scalar loop integrals to full scattering amplitudes in the case of unitary, local field theories and in the occurrence of real masses is straightforward and has been discussed in detail in Ref. [12].

3 Duality relation at two loops

We now turn to the general two–loop master diagram, as presented in Fig. 3. Again, all external momenta p_i are taken as outgoing, and we have $p_{i,j} = p_i + p_{i+1} + \ldots + p_j$, with momentum conservation $p_{1,N} = 0$. The label i of the external momenta is defined modulo N, i.e., $p_{N+i} \equiv p_i$. Note, however, that one or both of the external momenta attached to the four–leg vertices might be absent: $p_\ell = 0$ and/or $p_N = 0$. In the two–loop case, unlike at the one–loop order, the number of external momenta might differ from the

*If the number of space–time dimensions is d, then m is limited to be $m \leq d$; the terms with larger values of m vanish, since the corresponding number of delta functions in the integrand is larger than the number of integration variables.

number of internal momenta. The loop momenta are ℓ_1 and ℓ_2, which flow anti–clockwise and clockwise respectively. The momenta of the internal lines are denoted by q_i and are explicitly given by

$$q_i = \begin{cases} \ell_1 + p_{1,i} & , i \in \alpha_1 \\ \ell_2 + p_{i,l-1} & , i \in \alpha_2 \\ \ell_1 + \ell_2 + p_{i,l-1} & , i \in \alpha_3 \ , \end{cases} \tag{26}$$

where α_k, with $k = 1, 2, 3$, are defined as the set of lines, propagators respectively, related to the momenta q_i, for the following ranges of i:

$$\alpha_1 \equiv \{0, 1, ..., r\} \ , \qquad \alpha_2 \equiv \{r+1, r+2, ..., l\} \ , \qquad \alpha_3 \equiv \{l+1, l+2, ..., N\} \ . \tag{27}$$

In the following, we will use α_k for denoting a set of indices or the set of the corresponding internal momenta synonymously. Furthermore, we will refer to these lines often simply as the “loop lines”.

We shall now extend the duality theorem to the two–loop case, by applying Eq. (23) iteratively. We consider first, in the most general form, a set of several loop lines α_1 to α_N depending on the same integration momentum ℓ_i, and find

$$\int_{\ell_i} G_F(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) = - \int_{\ell_i} G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \ , \tag{28}$$

which states the application of the duality theorem, Eq. (23), to the set of loop lines belonging to the same loop. Eq. (28) is the generalization of the duality theorem found at one–loop to a single loop of a multi–loop diagram. Each subsequent application of the duality theorem to another loop of the same diagram will introduce an extra single cut, and by applying the duality theorem as many times as the number of loops, a given multi–loop diagram will be opened to a tree–level diagram. The duality theorem, Eq. (28), however, applies only to Feynman propagators, and a subset of the loop lines whose propagators are transformed into dual propagators by the application of the duality theorem to the first loop might also be part of the next loop (cf., e.g., the “middle” line belonging to α_3 in Fig. 3). The dual function of the unification of several subsets can be expressed in terms of dual and Feynman functions of the individual subsets by using Eq. (21) (or Eq. (22)) though, and we will use these expressions to transform part of the dual propagators into Feynman propagators, in order to apply the duality theorem to the second loop.

We are now ready for extending the duality relation to two loops. Our starting point is the expression for the two–loop N–leg scalar integral

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \ , \tag{29}$$

where the momenta of the internal lines are specified in Eq. (26) and Eq. (27). As stated before, we will apply the duality theorem sequentially to the two different loops associated with the integration momenta ℓ_1 and ℓ_2. Starting with the first loop related to ℓ_1, and hence to the loop lines α_1 and α_3, and using Eq. (28), we obtain:

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{\ell_1} \int_{\ell_2} G_D(\alpha_1 \cup \alpha_3) \, G_F(\alpha_2) \ . \tag{30}$$

We then use Eq. (22) for $G_D(\alpha_1 \cup \alpha_3)$, leading to

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{\ell_1} \int_{\ell_2} \{ G_D(\alpha_1) \, G_D(\alpha_3) + G_D(\alpha_1) \, G_F(\alpha_3) + G_F(\alpha_1) \, G_D(\alpha_3) \} \ G_F(\alpha_2) \ . \tag{31}$$

The first term of the integrand on the right–hand side of Eq. (31) is the product of two dual functions, and therefore already contains double cuts. We do not modify this term further. The second and third terms of Eq. (31) contain *one* $G_D(\alpha_i)$ and hence single cuts only. We thus apply the duality theorem again, i.e., we use Eq. (28) for ℓ_2 in order to generate one more cut. A subtlety arises at this point since due to our choice of momentum flow, α_1 and α_2, appearing in the third term of Eq. (31), flow in the opposite sense. Hence, in order to apply the duality theorem to the second loop we have to reverse the momentum flow of one of these two loop lines. We choose to change the direction of α_1, namely $q_i \to -q_i$ for $i \in \alpha_1$. This change of momentum flow is denoted by a sign in front of α_1. Thus, applying Eq. (28) to the last two terms of Eq. (31), leads to

$$
\begin{aligned}
&L^{(2)}(p_1, p_2, \dots, p_N) \\
&\quad = \int_{\ell_1}\int_{\ell_2} \{-G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_D(\alpha_2 \cup \alpha_3) + G_D(\alpha_3)\, G_D(-\alpha_1 \cup \alpha_2)\} \ .
\end{aligned}
\tag{32}
$$

This is the dual representation of the two–loop scalar integral as a function of double–cut integrals only, since all the terms of the integrand in Eq. (32) contain exactly two dual functions as defined in Eq. (17). The integrand in Eq. (32) can then be reinterpreted as the sum over tree–level diagrams integrated over a two–body phase–space.

The integrand in Eq. (32), however, contains several dual functions of two different loop lines, and hence dual propagators whose dual $i0$ prescription might still depend on the integration momenta. This is the case for dual propagators $G_D(q_i; q_j)$ where each of the momenta q_i and q_j belong to different loop lines. If both momenta belong to the same loop line the dependence on the integration momenta in $\eta(q_j - q_i)$ obviously cancels, and the complex dual prescription is determined by external momenta only. The dual prescription $\eta(q_j - q_i)$ can thus, in some cases, change sign within the integration volume, therefore moving up or down the position of the poles in the complex plane. To avoid this, we should reexpress the dual representation of the two–loop scalar integral in Eq. (32) in terms of dual functions of single loop lines. This transformation was unnecessary at one–loop because at the lowest order all the internal momenta depend on the same integration loop momenta; in other words, there is only a single loop line.

Inserting Eq. (22) in Eq. (32) and reordering some terms, we arrive at the following representation of the two–loop scalar integral

$$
\begin{aligned}
&L^{(2)}(p_1, p_2, \dots, p_N) \\
&\quad = \int_{\ell_1}\int_{\ell_2} \{G_D(\alpha_1)\, G_D(\alpha_2)\, G_F(\alpha_3) + G_D(-\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G^*(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3)\} \ ,
\end{aligned}
\tag{33}
$$

where

$$
G^*(\alpha_k) \equiv G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k) \ . \tag{34}
$$

This is the second main result of this paper. In Eq. (33), the $i0$ prescription of all the dual propagators depends on external momenta only. Through Eq. (34), however, Eq. (33) contains also triple cuts, given by the contributions with three $G_D(\alpha_k)$. The triple cuts are such that they split the two–loop diagram into two disconnected tree–level diagrams. By definition, however, the triple cuts are such that there is no more than one cut per loop line α_k. Since there is only one loop line at one–loop, it is also clear why we did not generate disconnected graphs at this loop order. For a higher number of loops, we expect to find at least the same number of cuts as the number of loops, and topology dependent disconnected tree diagrams built by cutting up to all the loop lines α_k. We explore this possibility at three loops in the next section.

Note that using Eq. (19), $G^*(\alpha_k)$ can also be expressed as

$$G^*(\alpha_k) = G_A(\alpha_k) + G_R(\alpha_k) - G_F(\alpha_k)\ , \tag{35}$$

which contains no cuts, although the imaginary prescription of the advanced and retarded propagators still depends on the integration loop momenta.

Finally, let us remark that from Eq. (33) we can obtain the FTT representation of the two–loop scalar integral. More precisely, we can write each $G_D(\alpha_k)$, for the lines $k \in \{1, 2, 3\}$, in terms of Feynman propagators and multiple cuts of their constituting momenta, by using Eq. (21) with the basic subsets α_i given by single momenta. After regrouping some terms, we obtain

$$\begin{aligned} L^{(2)}(p_1, \cdots, p_N) = \sum_{\substack{\alpha_k^{(1)} \cup \alpha_k^{(2)} = \alpha_k \\ k \in \{1,2,3\}}} \int_{\ell_1} \int_{\ell_2} \Bigg\{ & G_F(\alpha_1) \prod_{i_1 \in \alpha_2^{(1)}} \widetilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_3^{(1)}} \widetilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3}) \\ & + G_F(\alpha_2) \prod_{i_1 \in \alpha_1^{(1)}} \widetilde{\delta}(-q_{i_1}) \prod_{i_2 \in \alpha_3^{(1)}} \widetilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_1^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3}) \\ & + G_F(\alpha_3) \prod_{i_1 \in \alpha_1^{(1)}} \widetilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_2^{(1)}} \widetilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_1^{(2)} \cup \alpha_2^{(2)}} G_F(q_{i_3}) \\ & + \Bigg(\prod_{i_1 \in \alpha_1^{(1)}} \widetilde{\delta}(q_{i_1}) + \prod_{i_1 \in \alpha_1^{(1)}} \widetilde{\delta}(-q_{i_1}) \Bigg) \prod_{i_2 \in \alpha_2^{(1)}} \widetilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_3^{(1)}} \widetilde{\delta}(q_{i_3}) \prod_{i_4 \in \alpha_1^{(2)} \cup \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_4}) \Bigg\}\ , \end{aligned} \tag{36}$$

where the sums always run over the partitions of the index sets as defined and used in Eq. (21). We see that it consists of at least double cuts up to a multiple cut of all the internal momenta.

4 Duality relation beyond two loops

In this section, we will take a first look on the duality relation beyond the two–loop order by considering the master topologies at three loops, as represented in Fig. 4. We obtain explicit representations of the diagrams of Fig. 4 by using the iterative method described in the previous section. Although the dual representations obtained in this section are not unique, as the diagrams can be expressed in different ways in terms of dual and Feynman propagators depending on the choice of lines whose momentum flow is changed in the course of applying the duality theorem to each loop, we have followed a systematic way in order to minimize the number of terms.

Diagrams 4(a) - 4(c) are in a certain sense of the same type, and will be treated in the same way. We first cut these diagrams on the disjoint loops assigned to the lines $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. Considering the basket ball diagram (a) of Fig. 4, for example, this means:

$$L^{(3)}_{\rm basket}(p_1, p_2, \dots, p_N) = \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1 \cup \alpha_2)\ G_D(\alpha_3 \cup \alpha_4)\ . \tag{37}$$

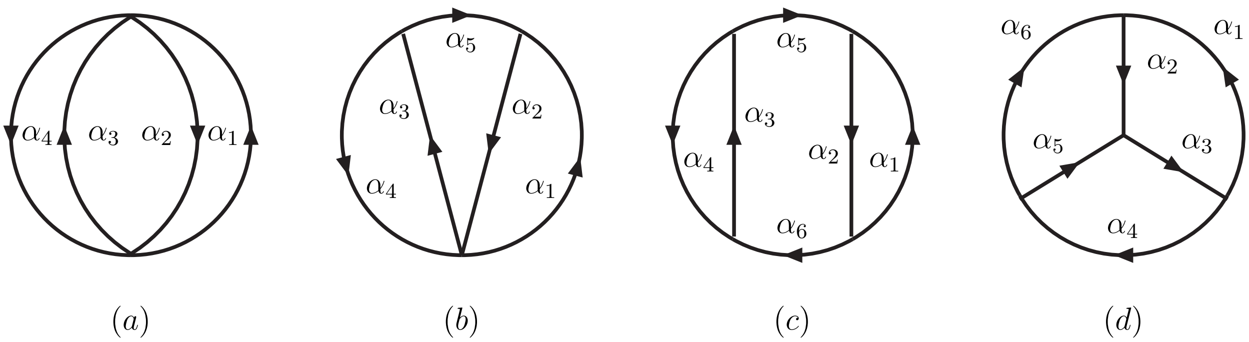

Figure 4: *Master topologies of three–loop scalar integrals. Each internal line α_k can be dressed with an arbitrary number of external lines, which are not shown here.*

Remember that any dual function $G_D(\alpha_k)$ of any set of momenta, and hence any application of the duality to a loop, contains at least one cut. Since we have a product of two expressions of the dual type, all terms in the expansion of this product in Eq. (37) via Eq. (22) contain at least two cuts. Terms with triple and more cuts are already in their final form. They belong to the case where either all lines are dual, or one line is of the type "Feynman". However, the double–cut terms, as, e.g., $G_D(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3)\, G_D(\alpha_4)$, stemming from the combination of one Feynman propagator from each of the two different loops, form a third loop, which still consists only of Feynman propagators and hence still needs one more application of the duality theorem in order to generate the third cut:

$$\int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3)\, G_D(\alpha_4) \;\; \rightarrow \;\; - \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1)\, G_D(\alpha_2 \cup \alpha_3)\, G_D(\alpha_4)\, . \quad (38)$$

In the case of the zigzag diagram, 4(b), or the ladder diagram, 4(c), this third loop consists of one more internal loop line: line α_5 in the former case and the lines α_5 and α_6 in the latter. Hence the third loop now consists of exactly one loop line from the first and second loop, and these additional loop lines. Due to the nature of the application of the duality through Eq. (28) and Eq. (21), we have to sum over all possibilities to build such sets of loop lines fulfilling these properties. Additionally, we have to assure that for each application of the duality the integration momentum runs in the same sense and hence change the momentum–flow direction for some chosen loop lines. Since each application of the duality generates one minus sign, we obtain the following general result for diagrams 4(a) - 4(c):

$$\begin{aligned}
L^{(3)}_{\rm (a),(b),(c)}(p_1, p_2, \dots, p_N) &= \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1 \cup \alpha_2)\; G_D(\alpha_3 \cup \alpha_4)\; G_F(\beta) \\
= \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Bigg\{ \Bigg[& G_D(\alpha_2, \alpha_3, \alpha_4)\; G_F(\alpha_1) + G_D(\alpha_1, \alpha_3, \alpha_4)\; G_F(\alpha_2) + G_D(\alpha_1, \alpha_2, \alpha_4)\; G_F(\alpha_3) \\
& + G_D(\alpha_1, \alpha_2, \alpha_3)\; G_F(\alpha_4) + G_D(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \Bigg] G_F(\beta) \\
& - G_D(\alpha_1, \alpha_3)\; G_D(\alpha_2 \cup -\alpha_4 \cup \beta) - G_D(\alpha_1, \alpha_4)\; G_D(\alpha_2 \cup \alpha_3 \cup \beta) \\
& - G_D(\alpha_2, \alpha_3)\; G_D(-\alpha_1 \cup -\alpha_4 \cup \beta) - G_D(\alpha_2, \alpha_4)\; G_D(-\alpha_1 \cup \alpha_3 \cup \beta) \Bigg\}\, , \qquad (39)
\end{aligned}$$

where $\beta = \emptyset$ in the case of diagram 4(a) (with $G_F(\emptyset) = 1$), $\beta = \alpha_5$ for diagram 4(b) and $\beta = \alpha_5 \cup \alpha_6$ in the case of the ladder diagram 4(c). Note that for the ladder diagram, lines α_5 and α_6 depend on the same integration momentum and can therefore be considered as a *single* loop line as defined in this context. Hence, in this sense, the diagram naturally reduces to the zigzag–case, as long as the relative sense of momentum flow in these lines is correct and stays unchanged. For brevity, we defined the product of dual propagators as $G_D(\alpha_1, ..., \alpha_N) = \prod_{i=1}^N G_D(\alpha_i)$, in contrast to $G_D(\alpha_1 \cup ... \cup \alpha_N)$, given in Eq. (21). The dual representation of the three–loop scalar integral in Eq. (39), contains mostly triple cuts. We have allowed for a single four–cut contribution in order to make this expression more symmetric, but this term can be rewritten in terms of triple–cut contributions with the help of Eq. (22).

If we expand all existing dual functions in Eq. (39) in terms of dual functions of single loop lines by using Eq. (21), we obtain, e.g., for diagram 4(a):

$$
\begin{aligned}
L^{(3)}_{\rm basket}(p_1, p_2, \dots, p_N) = &- \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ G_D(\alpha_2, \alpha_3, -\alpha_4)\; G_F(\alpha_1) + G_D(\alpha_1, \alpha_3, -\alpha_4)\; G_F(\alpha_2) \\
&+ G_D(-\alpha_1, \alpha_2, \alpha_4)\; G_F(\alpha_3) + G_D(-\alpha_1, \alpha_2, \alpha_3)\; G_F(\alpha_4) \\
&+ G_D(-\alpha_1, \alpha_2, \alpha_3, \alpha_4) + G_D(\alpha_1, \alpha_2, \alpha_3, -\alpha_4) + G_D(-\alpha_1, \alpha_2, \alpha_3, -\alpha_4) \Big\} . \qquad (40)
\end{aligned}
$$

In this expression, the complex dual prescription of all the dual propagators depend on external momenta only, although at the price of generating disconnected tree diagrams. Similar results can be obtained for diagrams 4(b) and 4(c). There are up to four cuts for diagram 4(a) (cf. Eq. (40)), five cuts for diagrams 4(b) and six cuts for diagram 4(c), although in this last case five cuts are enough if $\alpha_5 \cup \alpha_6$ is considered as a single loop line.

Also the Mercedes star diagram 4(d) can be expressed in terms of only three–cut contributions or in terms of three– up to six– cut contributions. However, due to the non–planar nature of this diagram, the way of obtaining its dual representation is slightly more involved, whereas the general idea as explained before stays the same. We achieve for the Mercedes star diagram the following dual representation:

$$
\begin{aligned}
&L^{(3)}_{\rm Mercedes}(p_1, p_2, \dots, p_N) \\
&\quad = \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ - G_D(\alpha_1, \alpha_2, \alpha_3)\, G_F(\alpha_4, \alpha_5, \alpha_6) + G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)\, G_D(\alpha_1, \alpha_2)\, G_F(\alpha_6) \\
&\quad + G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)\, G_D(\alpha_2, \alpha_3)\, G_F(\alpha_5) + G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)\, G_D(\alpha_1, \alpha_3)\, G_F(\alpha_4) \\
&\quad + G_D(\alpha_1)[G_D(\alpha_3 \cup \alpha_4)\, G_D(\alpha_5)\, G_F(\alpha_2 \cup \alpha_6) - G_D(\alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6)\, G_D(\alpha_5) \\
&\qquad - G_D(\alpha_3 \cup \alpha_4)\, G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)] \\
&\quad + G_D(\alpha_2)[G_D(-\alpha_1 \cup \alpha_6)\, G_D(\alpha_4)\, G_F(\alpha_3 \cup \alpha_5) - G_D(\alpha_1 \cup \alpha_3 \cup \alpha_5 \cup -\alpha_6)\, G_D(\alpha_4) \\
&\qquad - G_D(-\alpha_1 \cup \alpha_6)\, G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)] \\
&\quad + G_D(\alpha_3)[G_D(-\alpha_2 \cup \alpha_5)\, G_D(-\alpha_6)\, G_F(\alpha_1 \cup \alpha_4) - G_D(-\alpha_1 \cup -\alpha_2 \cup \alpha_4 \cup \alpha_5)\, G_D(-\alpha_6) \\
&\qquad - G_D(-\alpha_2 \cup \alpha_5)\, G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)] \Big\} . \qquad (41)
\end{aligned}
$$

From the inspection of the three–loop case and from the general derivation of the method, it seems obvious how to extend it to even higher loop orders. Eq. (21) can also be used to obtain the FTT representation of scalar integrals or scattering amplitudes at any loop order.

The duality relation can be extended to evaluate not only scalar loop integrals, as discussed so far, but also complete Feynman diagrams. The extension of the one-loop duality relation from scalar integrals to Feynman diagrams was discussed in details in Ref. [12]. This extension relies on the simple observation that the duality relation acts only on the Feynman propagators of the loop, leaving unchanged all the other factors in the Feynman diagram. This is valid in any unitary and local field theories. In spontaneously broken gauge theories, it holds in the 't Hooft-Feynman gauge and in the unitary gauge. In unbroken gauge theories, the duality relation is valid in the 't Hooft-Feynman gauge, and in physical gauges where the gauge vector n^ν is orthogonal to the dual vector η^μ, i.e., $n \cdot \eta = 0$. This excludes gauges where n^ν is time-like. At one-loop order, this choice of gauges avoids the appearance of extra unphysical gauge poles, which in other gauges (e.g. the time-like axial gauge) are also poles of the second order. Within the same choice of gauges, additional (unphysical) gauge poles are absent also at higher-loop level, and the duality relation can be straightforwardly extended from scalar integrals to Feynman diagrams.

In Ref. [12], it was also shown how the one-loop duality relation can directly be expressed at the level of full scattering amplitudes (or, more precisely, off-shell Green's functions). The derivation of the duality between one-loop and tree-level scattering amplitudes requires a detailed discussion of some issues related to tadpole and self-energy configurations. These (and related) issues become more delicate at higher-loop levels. We do not pursue further on this point in this paper, and we postpone detailed investigations to further studies.

5 Conclusion and Outlook

We have rederived the tree–loop duality theorem at one–loop order, which was introduced in Ref. [12], in a way which is more suitable for extending it to higher loop orders. By iteratively applying the duality theorem, we have given explicit representations of the two– and three–loop scalar integrals. The method, however, is easily extendible to higher loop orders beyond three loops. In general, the dual representation of the loop integrals can be written as a sum of terms with exactly the same number of cuts as the number of loops, and in such a way that the loop diagram is opened to a tree–level diagram. However, this requires to deal with uncut propagators with complex dual $i0$ prescription depending on the integration momenta, and thus with complex dual prescriptions that might change sign within the integration volume. Dual representations of the loop integrals with complex dual prescription depending only on the external momenta can be obtained at the cost of introducing extra cuts, which break the loop integrals into disconnected diagrams. This is a new feature of the duality theorem beyond one–loop, which does not happen at the lowest order. The number of extra cuts to be taken into account depends on the topology of the loop diagram. The maximal number of cuts agrees with the number of loop lines, and the cuts are such that it does not appear more than a single cut for each internal loop line. These general facts are true for the application of the duality relation to diagrams with an arbitrary number of loops. The results presented in this paper can also be used to obtain the FTT representation of diagrams at higher orders.

The dual representations obtained in this paper are valid as far as only single poles are present when the residue theorem is applied. At one–loop order, the propagators of the gauge bosons might generate unphysical poles, or even higher order poles. Those non–single poles can be avoided by a convenient choice of the gauge or of the dual vector [12]. At two– or higher loop orders, however, higher order poles might appear when diagrams with selfenergy insertions (nested or disjoint) are considered. At two loops this happens when two of the loop lines are made of single propagators, and no external momenta are attached to any of the two four-leg vertices of Fig. 3. At higher order loops, there are many more possible topologies showing this feature. Extending the duality theorem to this kind of diagrams requires to evaluate the contribution of the higher order poles, which depends on the topology of the diagram and on the nature of the internal propagators and on the form of the interaction vertices. Explicity studies of these loop diagrams are left to future investigations.

Note added: After completion of this paper, a work [20] dealing with similar topics appeared. The author uses retarded boundary conditions, and obtains some combinatorial factors weighting the different terms contributing to the loop-tree duality relation. We haved checked explicitly, in the two-loop case, that such combinatorial factors are the result of averaging over the different dual countertparts of the same loop integral obtained by permuting the loop lines α_i. Although many of the terms obtained in this way are either equivalent or can be related to each other after shifting the loop momenta, a larger amount of terms than by using a single dual counterpart are needed to be summed up for the same loop integral or Feynman diagram. By using Eq. (55) or Eq. (57) dual propagators can be expressed in terms of advanced or retarded propagators in a straightforward way, leading to an equivalent loop-tree duality relation as presented in Ref. [20]. In our paper we have not followed this procedure, since a main feature of our duality relation is that the '$i0$' prescription of the dual propagators depends only on the external momenta.

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A Derivation of some algebraic relations

In this Appendix, we prove by induction several algebraic relations that have been used in the text. The basic ingredient of the proof is the following relation:

$$\theta(\lambda_1)\,\theta(\lambda_1+\lambda_2)\,\ldots\,\theta(\lambda_1+\lambda_2+\cdots+\lambda_{n-1}) + \text{cyclic perms.} = 1 \,, \tag{42}$$

that holds for any set of n real variables λ_i, with $i = 1, 2, \ldots, n$, such that

$$\sum_{i=1}^{n} \lambda_i = 0 \,. \tag{43}$$

Relation (42) was proven in Appendix B of Ref. [12]. It applies, in particular, to $\lambda_i = \eta p_i$, and follows from momentum conservation $\sum_{i=1}^{n} p_i = 0$, where p_i are external momenta. In the following, we will use Eq. (42) by setting $\lambda_i = \eta(q_i - q_{i+1})$ for $i \in \{1, ..., n\}$, with $(n+i) \equiv i$ mod n. The four–momenta q_i are any arbitrary set of internal momenta, and the real variables $\lambda_i = \eta(q_i - q_{i+1})$ might still depend on the loop momenta ℓ_1 and ℓ_2. By construction, however, Eq. (43) is automatically fulfilled. Thus, Eq. (42) can also be written as:

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \tilde{\theta}\left(q_i - q_j\right) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \tilde{\theta}\left(q_j - q_i\right) = 1 \; . \tag{44}$$

We start by deriving the following algebraic identity:

$$G_D(\alpha_k) = G_A(\alpha_k) - G_F(\alpha_k) \; , \tag{45}$$

where

$$G_D(\alpha_k) = \sum_{i\in\alpha_k} \widetilde{\delta}\left(q_i\right) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i; q_j) \; , \qquad G_{A(F)}(\alpha_k) = \prod_{i\in\alpha_k} G_{A(F)}(q_i) \; . \tag{46}$$

as defined in Eq. (17). Remember that $G_D(\alpha_k) = \widetilde{\delta}\left(q_i\right)$ when α_k consists of a single four momentum q_i. Using the identity $G_A(q_i) = G_F(q_i) + \widetilde{\delta}\left(q_i\right)$, the right-hand side of Eq. (45), can also be written as

$$G_A(\alpha_k) - G_F(\alpha_k) = \sum_{\alpha_k^{(1)} \cup \alpha_k^{(2)} = \alpha_k} \prod_{i_1 \in \alpha_k^{(1)}} \widetilde{\delta}\left(q_{i_1}\right) \prod_{i_2 \in \alpha_k^{(2)}} G_F(q_{i_2}) \; , \tag{47}$$

where the sum runs over all possible partitions of α_k into exactly two subsets $\alpha_k^{(1)}$ and $\alpha_k^{(2)}$. Additionally, we allow for the subset $\alpha_k^{(2)}$ to be empty, but the subset $\alpha_k^{(1)}$ always contains at least one element, this means that a term with only Feynman propagators is excluded. The first non trivial case for Eq. (45) occurs for $\alpha_k = \{1, 2\}$. By using Eq. (16), and $\tilde{\theta}\left(q_1 - q_2\right) + \tilde{\theta}\left(q_2 - q_1\right) = 1$, it is straightforward to prove that

$$\widetilde{\delta}\left(q_1\right) G_D(q_1; q_2) + \widetilde{\delta}\left(q_2\right) G_D(q_2; q_1) = \widetilde{\delta}\left(q_1\right) G_F(q_2) + \widetilde{\delta}\left(q_2\right) G_F(q_1) + \widetilde{\delta}\left(q_1\right) \widetilde{\delta}\left(q_2\right) \; . \tag{48}$$

Let us now assume that relation (45) is correct for N loop momenta, $\alpha_k = \{1, \ldots, N\}$, and show that it is valid for $\alpha_k^{N+1} = \alpha_k \cup \{N+1\}$. We replace the dual propagators $G_D(q_i; q_{N+1})$ appearing in $G_D(\alpha_k^{N+1})$ by using again Eq. (16). We obtain:

$$\begin{aligned} G_D(\alpha_k^{N+1}) &= \sum_{i=1}^{N+1} \widetilde{\delta}\left(q_i\right) \prod_{\substack{j=1 \\ j\neq i}}^{N+1} G_D(q_i; q_j) = G_D(\alpha_k)\, G_F(q_{N+1}) \\ &+ \widetilde{\delta}\left(q_{N+1}\right) \left(\sum_{i=1}^{N} \tilde{\theta}\left(q_{N+1} - q_i\right) \widetilde{\delta}\left(q_i\right) \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_i; q_j) + \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_{N+1}; q_j) \right) . \end{aligned} \tag{49}$$

Assuming that Eq. (45) and Eq. (47) are valid for N elements, the first term in the right-hand side of Eq. (49), which is proportional to $G_F(q_{N+1})$, becomes

$$G_D(\alpha_k)\, G_F(q_{N+1}) = \left[G_A(\alpha_k) - G_F(\alpha_k)\right] G_F(q_{N+1}) \; . \tag{50}$$

For the remaining terms in Eq. (49), which are proportional to $\widetilde{\delta}\,(q_{N+1})$, we apply again Eq. (16) to all dual propagators. After some algebra, we find:

$$
\begin{aligned}
& G_D(\alpha_k^{N+1}) - G_D(\alpha_k)\, G_F(q_{N+1}) = \\
& \quad \widetilde{\delta}\,(q_{N+1}) \left(G_F(\alpha_k) + \sum_{\alpha_k^{(1)} \cup \alpha_k^{(2)} = \alpha_k} \tilde{\Theta}(\alpha_k^{(1)} \cup \{N+1\}) \prod_{j_1 \in \alpha_k^{(1)}} \widetilde{\delta}\,(q_{j_1}) \prod_{j_2 \in \alpha_k^{(2)}} G_F(q_{j_2}) \right) , \qquad (51)
\end{aligned}
$$

with

$$
\tilde{\Theta}(\alpha_k^{(1)} \cup \{N+1\}) = \sum_{i_1 \in \alpha_k^{(1)} \cup \{N+1\}} \left(\prod_{\substack{i_2 \in \alpha_k^{(1)} \cup \{N+1\} \\ i_2 \neq i_1}} \tilde{\theta}\,(q_{i_1} - q_{i_2}) \right) , \qquad (52)
$$

where the sum runs over all possible products of $\tilde{\theta}$ functions that can be constructed with the four–momenta in the set $\alpha_k^{(1)}$ and q_{N+1}. Obviously, from Eq. (44),

$$
\tilde{\Theta}(\alpha_k^{(1)} \cup \{N+1\}) = 1 , \qquad (53)
$$

for all possible partitions of α_k into $\alpha_k^{(1)}$ and $\alpha_k^{(2)}$. Collecting the results from Eq. (50), Eq. (51), and Eq. (53), we finally obtain

$$
\begin{aligned}
G_D(\alpha_k^{N+1}) &= G_D(\alpha_k)\, G_F(q_{N+1}) + \widetilde{\delta}\,(q_{N+1})\, (G_F(\alpha_k) + G_D(\alpha_k)) \\
&= G_A(\alpha_k)\, G_A(q_{N+1}) - G_F(\alpha_k)\, G_F(q_{N+1}) , \qquad (54)
\end{aligned}
$$

as we wanted to demonstrate.

Another useful relation of advanced and Feynman propagators is the following:

$$
G_D(\alpha_k) = \sum_{i \in \alpha_k} \left(\prod_{\substack{j \in \alpha_k \\ j < i}} G_A(q_j) \right) \widetilde{\delta}\,(q_i) \left(\prod_{\substack{l \in \alpha_k \\ l > i}} G_F(q_l) \right) . \qquad (55)
$$

We do not attempt to present a detailed proof here, as Eq. (55) can be straightforwardly derived by reordering some terms from Eq. (47). Analogously, we also have

$$
G_D(-\alpha_k) = G_R(\alpha_k) - G_F(\alpha_k) , \qquad (56)
$$

and

$$
G_D(-\alpha_k) = \sum_{i \in \alpha_k} \left(\prod_{\substack{j \in \alpha_k \\ j < i}} G_R(q_j) \right) \widetilde{\delta}\,(-q_i) \left(\prod_{\substack{l \in \alpha_k \\ l > i}} G_F(q_l) \right) , \qquad (57)
$$

which also hold straightforward by changing the momentum flow of the four–momenta from α_k in Eq. (45) and Eq. (55) and taking into account that $G_F(-q_i) = G_F(q_i)$ and $G_A(-q_i) = G_R(q_i)$.

B Massless sunrise two–loop two–point function

We consider, as explicit example of the application of the duality relation at two–loops, the massless sunrise two–loop two–point function (Fig. 5). From Eq. (33), the dual representation of the sunrise

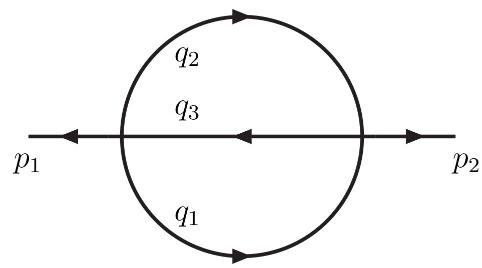

Figure 5: *Sunrise two–loop two–point function.*

two–loop scalar integral is given by

$$L^{(2)}(p_1,p_2) = \int_{\ell_1}\int_{\ell_2} \left\{ \widetilde{\delta}\,(q_1)\;\widetilde{\delta}\,(q_2)\;G_F(q_3) + \widetilde{\delta}\,(-q_1)\;G_F(q_2)\,\widetilde{\delta}\,(q_3) + G^*(q_1)\,\widetilde{\delta}\,(q_2)\;\widetilde{\delta}\,(q_3) \right\} \;. \tag{58}$$

After replacing $G^*(q_1) = G_F(q_1) + \widetilde{\delta}\,(q_1) + \widetilde{\delta}\,(-q_1)$, and shifting some momenta, we obtain

$$\begin{aligned} L^{(2)}(p_1,p_2) \;&=\; \int_{\ell_1}\int_{\ell_2} \widetilde{\delta}\,(\ell_1)\;\widetilde{\delta}\,(\ell_2)\;\Big\{ G_F(\ell_1+\ell_2+p_1) + G_F(\ell_1+\ell_2-p_1) + G_F(\ell_1-\ell_2-p_1) \\ &+ \widetilde{\delta}\,(\ell_1+\ell_2+p_1) + \widetilde{\delta}\,(\ell_1+\ell_2-p_1) \Big\} \;. \end{aligned} \tag{59}$$

For the integration of the first loop momentum ℓ_1, we use the basic integrals already calculated in Ref. [12]:

$$\int_{\ell_1} \widetilde{\delta}\,(\ell_1)\;G_F(\ell_1+k) = d_\Gamma\;\left[k^2+i0\right]^{-\epsilon}\;\left[1+\theta(k^2)\,\theta(-k_0)\;\left(e^{i2\pi\epsilon}-1\right)\right]\;, \tag{60}$$

and

$$\int_{\ell_1} \widetilde{\delta}\,(\ell_1)\;\widetilde{\delta}\,(\ell_1+k) = d_\Gamma\;\left[k^2+i0\right]^{-\epsilon}\theta(-k^2)\;\left(e^{i2\pi\epsilon}-1\right)\;, \tag{61}$$

where

$$d_\Gamma = -\frac{c_\Gamma}{2}\,\frac{1}{\epsilon(1-2\epsilon)}\,\frac{1}{\cos(\pi\epsilon)}\;, \qquad c_\Gamma = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\,\Gamma(1-2\epsilon)}\;, \tag{62}$$

which we have reexpressed in a more suitable way in terms of $[k^2+i0]^{-\epsilon}$. Applying Eq. (60) and Eq. (61) to Eq. (59), we find

$$\begin{aligned} L^{(2)}(p_1,p_2) \;&=\; d_\Gamma \int_{\ell_2} \widetilde{\delta}\,(\ell_2)\;\Big\{\;\left[(\ell_2+p_1)^2+i0\right]^{-\epsilon}\left(e^{i2\pi\epsilon}+1\right) \\ &+ \left[(\ell_2-p_1)^2+i0\right]^{-\epsilon}\left[e^{i2\pi\epsilon} - \theta((\ell_2-p_1)^2)\theta((\ell_2-p_1)_0)\left(e^{i2\pi\epsilon}-1\right)\right]\Big\}\;. \end{aligned} \tag{63}$$

The new phase–space integrals that we have to evaluate now for the integration over the second loop momentum ℓ_2 are then quite similar to those already encountered at one–loop. The calculation is elementary, and we obtain

$$d_\Gamma \int_{\ell_2} \widetilde{\delta}(\ell_2) \left[(\ell_2+k)^2+i0\right]^{-\epsilon} = \\ - G_2 \frac{\sin(\pi\epsilon)\, e^{-i2\pi\epsilon}}{\sin(3\pi\epsilon)} (-k^2-i0)^{1-2\epsilon} \left[1+\theta(k^2)\theta(-k_0)\left(e^{i2\pi\epsilon}-1\right)\right] \;, \tag{64}$$

and

$$d_\Gamma \int_{\ell_2} \widetilde{\delta}(\ell_2) \left[(\ell_2+k)^2+i0\right]^{-\epsilon} \theta((\ell_2+k)^2)\,\theta((\ell_2+k)_0) = \\ G_2 \frac{\sin(\pi\epsilon)}{\sin(3\pi\epsilon)} (-k^2-i0)^{1-2\epsilon} \left[\theta(-k^2)-\theta(k^2)\,\theta(k_0)\, e^{-i2\pi\epsilon}\right] \;, \tag{65}$$

where

$$G_2 = \frac{\Gamma(-1+2\epsilon)\,\Gamma(1-\epsilon)^3}{(4\pi)^{4-2\epsilon}\,\Gamma(3-3\epsilon)} \;. \tag{66}$$

Applying Eq. (64) and Eq. (65) to Eq. (63), and summing up all the theta functions, we finally get

$$L^{(2)}(p_1,p_2) = -\,G_2\,(-p_1^2-i0)^{1-2\epsilon} \;, \tag{67}$$

which is the well–known result for the massless sunrise two–loop two–point function.

References

[1] J. R. Andersen *et al.* [SM and NLO Multileg Working Group], “The SM and NLO multileg working group: Summary report,” arXiv:1003.1241 [hep-ph].

[2] A. Denner, S. Dittmaier, T. Kasprzik and A. Mück, “Electroweak corrections to W + jet hadroproduction including leptonic W-boson decays,” JHEP **0908** (2009) 075 [arXiv:0906.1656 [hep-ph]].

[3] J. H. Kühn, A. Kulesza, S. Pozzorini and M. Schulze, “Electroweak corrections to hadronic production of W bosons at large transverse momenta,” Nucl. Phys. B **797** (2008) 27 [arXiv:0708.0476 [hep-ph]].

[4] C. F. Berger *et al.*, “Precise Predictions for W + 3 Jet Production at Hadron Colliders,” Phys. Rev. Lett. **102** (2009) 222001 [arXiv:0902.2760 [hep-ph]].

[5] K. Melnikov and G. Zanderighi, “W+3 jet production at the LHC as a signal or background,” Phys. Rev. D **81** (2010) 074025 [arXiv:0910.3671 [hep-ph]].

[6] G. Bevilacqua, M. Czakon, C. G. Papadopoulos, R. Pittau and M. Worek, “Assault on the NLO Wishlist: $pp \to t\bar{t}b\bar{b}$,” JHEP **0909** (2009) 109 [arXiv:0907.4723 [hep-ph]].

[7] A. Bredenstein, A. Denner, S. Dittmaier and S. Pozzorini, “NLO QCD corrections to top anti-top bottom anti-bottom production at the LHC: 2. full hadronic results,” JHEP **1003** (2010) 021 [arXiv:1001.4006 [hep-ph]].

[8] P. Bolzoni, F. Maltoni, S. O. Moch and M. Zaro, "Higgs production via vector-boson fusion at NNLO in QCD," arXiv:1003.4451 [hep-ph].

[9] S. Catani, G. Ferrera and M. Grazzini, "W boson production at hadron colliders: the lepton charge asymmetry in NNLO QCD," JHEP **1005** (2010) 006 [arXiv:1002.3115 [hep-ph]].

[10] S. Catani, L. Cieri, G. Ferrera, D. de Florian and M. Grazzini, "Vector boson production at hadron colliders: a fully exclusive QCD calculation at NNLO," Phys. Rev. Lett. **103** (2009) 082001 [arXiv:0903.2120 [hep-ph]].

[11] C. Anastasiou, G. Dissertori and F. Stockli, "NNLO QCD predictions for the $H \to WW \to ll\nu\nu$ signal at the LHC," JHEP **0709** (2007) 018 [arXiv:0707.2373 [hep-ph]].

[12] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J. C. Winter, "From loops to trees by-passing Feynman's theorem," JHEP **0809** (2008) 065 [arXiv:0804.3170 [hep-ph]].

[13] T. Gleisberg, "Automating methods to improve precision in Monte-Carlo event generation for particle colliders," Ph.D. Thesis, University of Dresden, 2007.

[14] R. P. Feynman, "Quantum theory of gravitation," Acta Phys. Polon. **24** (1963) 697.

[15] R. P. Feynman, *Closed Loop And Tree Diagrams*, in *Magic Without Magic*, ed. J. R. Klauder, (Freeman, San Francisco, 1972), p. 355, in *Selected papers of Richard Feynman*, ed. L. M. Brown (World Scientific, Singapore, 2000) p. 867.

[16] D. E. Soper, "QCD calculations by numerical integration," Phys. Rev. Lett. **81** (1998) 2638, "Techniques for QCD calculations by numerical integration," Phys. Rev. D **62** (2000) 014009; M. Krämer and D. E. Soper, "Next-to-leading order numerical calculations in Coulomb gauge," Phys. Rev. D **66** (2002) 054017.

[17] W. Kilian and T. Kleinschmidt, "Numerical Evaluation of Feynman Loop Integrals by Reduction to Tree Graphs," arXiv:0912.3495 [hep-ph].

[18] M. Moretti, F. Piccinini and A. D. Polosa, "A Fully Numerical Approach to One-Loop Amplitudes," arXiv:0802.4171 [hep-ph].

[19] I. Bierenbaum, "Towards a loop–tree duality at two loops and beyond," presented at Loops and Legs in Quantum Field Theory, Wörlitz, 25-30 April 2010, arXiv:1007.0213 [hep-ph].

[20] S. Caron-Huot, arXiv:1007.3224 [hep-ph].