From multileg loops to trees (by-passing Feynman's Tree Theorem) *

Germán Rodrigo [a†], Stefano Catani [b‡], Tanju Gleisberg [c§], Frank Krauss [d¶], and Jan-C. Winter [e‖]

[a] Instituto de Física Corpuscular, CSIC-Universitat de València,
Apartado de Correos 22085, E-46071 Valencia, Spain

[b] INFN, Sezione di Firenze and Dipartimento di Fisica, Università di Firenze,
I-50019 Sesto Fiorentino, Florence, Italy

[c] Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

[d] Institute for Particle Physics Phenomenology, Durham University, Durham DH1 3LE, UK

[e] Fermi National Accelerator Laboratory, Batavia, IL 60510, USA

We illustrate a duality relation between one-loop integrals and single-cut phase-space integrals. The duality relation is realised by a modification of the customary $+i0$ prescription of the Feynman propagators. The new prescription regularizing the propagators, which we write in a Lorentz covariant form, compensates for the absence of multiple-cut contributions that appear in the Feynman Tree Theorem. The duality relation can be extended to generic one-loop quantities, such as Green's functions, in any relativistic, local and unitary field theories.

1. INTRODUCTION

The physics program of LHC requires the evaluation of multi-leg signal and background processes at next-to-leading order (NLO). In the recent years, important efforts have been devoted to the calculation of many $2 \to 3$ processes and some $2 \to 4$ processes (see, e.g., [1]).

We have recently proposed a method [2,3,4] to compute multi-leg one-loop cross sections in perturbative field theories. The method uses combined analytical and numerical techniques. The starting point of the method is a duality relation between one-loop integrals and phase-space integrals. In this respect, the duality relation has analogies with the Feynman's Tree Theorem (FTT) [5]. The key difference with the FTT is that the duality relation involves only single cuts of the one-loop Feynman diagrams. In this talk, we illustrate the duality relation, and discuss its correspondence, similarities, and differences with the FTT.

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†E-mail: german.rodrigo@ific.uv.es
‡E-mail: stefano.catani@fi.infn.it
§E-mail: tanju@slac.stanford.edu
¶E-mail: frank.krauss@durham.ac.uk
‖E-mail: jwinter@fnal.gov

2. NOTATION

We consider a generic one-loop integral $L^{(N)}$ with massless internal lines (Fig. 1):

$$L^{(N)}(p_1, p_2, \dots, p_N) = \int_q \prod_{i=1}^{N} \frac{1}{q_i^2 + i0} \ , \qquad (1)$$

where q^μ is the loop momentum (which flows anticlockwise), and we use the shorthand notation:

$$\int_q \cdots \equiv -i \int \frac{d^d q}{(2\pi)^d} \quad \cdots \ . \qquad (2)$$

The momenta of the external legs, which are taken as outgoing and are clockwise ordered, are denoted by $p_1^\mu, p_2^\mu, \dots, p_N^\mu$, with $\sum_{i=1}^{N} p_i = 0$, and $p_{N+i} \equiv p_i$. The momenta of the internal lines are given by $q_i = q + \sum_{k=1}^{i} p_k$.

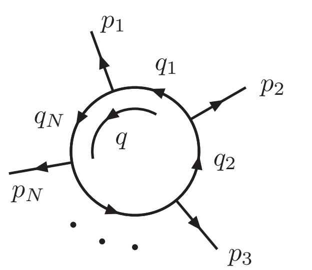

Figure 1. Momentum configuration of the one-loop N-point scalar integral.

3. THE FEYNMAN TREE THEOREM

The FTT [5] applies to any local and unitary quantum field theory in Minkowsky space with an arbitrary number d of space-time dimensions. It relates perturbative scattering amplitudes and Green's functions at the loop level with analogous quantities at the tree level.

Let us introduce the customary Feynman (G) and advanced (G_A) propagators:

$$G(q) \equiv \frac{1}{q^2 + i0} \ , \qquad G_A(q) \equiv \frac{1}{q^2 - i0\, q_0} \ , \qquad (3)$$

where q_0 is the energy component of the d-dimensional momentum q_μ. They are related by

$$G_A(q) = G(q) + \widetilde{\delta}(q) \ , \qquad (4)$$

with $\widetilde{\delta}(q) \equiv 2\pi i\, \theta(q_0)\, \delta(q^2) = 2\pi i\, \delta_+(q^2)$. In the complex plane of q_0 both propagators have two poles. The pole of the Feynman propagator with positive (negative) energy is slightly displaced below (above) the real axis, while both poles of the advanced propagator are located above the real axis. Hence, by using the Cauchy residue theorem in the q_0 complex plane, with the integration contour closed at ∞ in the lower half-plane, the one-loop integral of advanced propagators vanishes:

$$\int_q \prod_{i=1}^{N} G_A(q_i) = 0 \ . \qquad (5)$$

Inserting Eq. (4) in Eq. (5), and collecting all the terms with an equal number of delta functions, we obtain

$$0 = \int_q \prod_{i=1}^{N} \left[G(q_i) + \widetilde{\delta}(q_i) \right] = L^{(N)} + \sum_{m=1}^{N} L^{(N)}_{\text{m-cut}} \ . \qquad (6)$$

The m-cut integrals $L^{(N)}_{\text{m-cut}}$ are the contributions with precisely m delta functions:

$$L^{(N)}_{\text{m-cut}}(p_1, p_2, \dots, p_N) = \int_q \Big\{ \widetilde{\delta}(q_1) \cdots \widetilde{\delta}(q_m) \times G(q_{m+1}) \cdots G(q_N) + \text{uneq. perms.} \Big\} \ , \qquad (7)$$

where the sum in the curly bracket includes all the permutations of $q_1, \dots, q_N$ that give unequal terms in the integrand. From Eq. (6), we thus derive the FTT:

$$L^{(N)} = - \sum_{m=1}^{N} L^{(N)}_{\text{m-cut}} \ . \qquad (8)$$

The FTT relates the one-loop integral $L^{(N)}$ to the multiple-cut[1] integrals $L^{(N)}_{\text{m-cut}}$. Each delta function $\widetilde{\delta}(q_i)$ in $L^{(N)}_{\text{m-cut}}$ replaces the corresponding Feynman propagator in $L^{(N)}$ by 'cutting' the internal line with momentum q_i. Here, 'cutting' is synonymous to setting the respective particle on shell. An m-particle cut decomposes the one-loop diagram in m tree diagrams: in this sense, the FTT allows us to calculate loop diagrams from tree-level diagrams.

The extension of the FTT from one-loop integrals $L^{(N)}$ to one-loop scattering amplitudes $\mathcal{A}^{(1-\text{loop})}$ (or Green's functions) in perturbative field theories is straightforward, provided the corresponding field theory is unitary and local. The generalization of Eq. (8) to arbitrary scattering amplitudes is [5]:

$$\mathcal{A}^{(1-\text{loop})} = - \sum_{m=1}^{N} \mathcal{A}^{(1-\text{loop})}_{\text{m-cut}} \ , \qquad (9)$$

where $\mathcal{A}^{(1-\text{loop})}_{\text{m-cut}}$ is obtained by considering all possible replacements of m Feynman propagators $G(q_i)$ of internal loop lines in $\mathcal{A}^{(1-\text{loop})}$ with m on-shell propagators $\widetilde{\delta}(q_i)$.

[1] If the number of space-time dimensions is d, the right-hand side of Eq. (8) receives contributions only from the terms with $m \leq d$.

4. A DUALITY THEOREM

In this Section we present and illustrate the duality relation between one-loop integrals and single-cut phase-space integrals [2].

Applying the residue theorem directly to $L^{(N)}$, we obtain

$$L^{(N)}(p_1, p_2, \dots, p_N) \\ = -2\pi i \int_{\mathbf{q}} \sum \mathrm{Res}_{\{\mathrm{Im}\ q_0<0\}} \left[\prod_{i=1}^{N} G(q_i) \right] . \tag{10}$$

The integral does not vanish (unlike the case of advanced propagators) since the Feynman propagators produces N poles in the lower half-plane that contribute to the residues in Eq. (10). The calculation of these residues is elementary, but it involves several subtleties [2]. We get

$$\mathrm{Res}_{\{i^{\mathrm{th}}\mathrm{pole}\}} \frac{1}{q_i^2 + i0} = \int dq_0 \; \delta_+(q_i^2) . \tag{11}$$

This result shows that considering the residue of the Feynman propagator of the internal line with momentum q_i is equivalent to cutting that line by including the corresponding on-shell propagator $\delta_+(q_i^2)$. The other propagators $G(q_j)$, with $j \neq i$, which are not singular at the value of the pole of $G(q_i)$, contribute as follows [2]:

$$\prod_{j\neq i} \frac{1}{q_j^2 + i0} \Bigg|_{q_i^2=-i0} = \prod_{j\neq i} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)} , \tag{12}$$

where η is a future-like vector,

$$\eta_\mu = (\eta_0, \boldsymbol{\eta}) , \quad \eta_0 \geq 0 , \quad \eta^2 = \eta_\mu \eta^\mu \geq 0 , \tag{13}$$

i.e. a d-dimensional vector that can be either light-like ($\eta^2 = 0$) or time-like ($\eta^2 > 0$) with positive definite energy η_0.

We see from Eq.(12) that the calculation of the residue at the pole of the i^{th} internal line modifies the $i0$ prescription of the propagators of the other internal lines of the loop. This modified regularization is named 'dual' $i0$ prescription, and the corresponding propagators are named 'dual' propagators. The dual prescription arises from the fact that the original Feynman propagator $1/(q_j^2 + i0)$ is evaluated at the *complex* value of the loop momentum q, which is determined by the location of the pole at $q_i^2 + i0 = 0$.

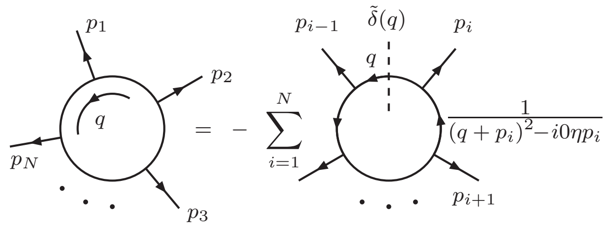

Figure 2. The duality relation for the one-loop N-point scalar integral: graphical representation as a sum of N basic dual integrals.

The presence of η is a consequence of the fact that the residue at each of the poles is not a Lorentz-invariant quantity. A given system of coordinates has to be specified to apply the residue theorem. Different choices of the future-like vector η are equivalent to different choices of the coordinate system. The Lorentz-invariance of the loop integral is recovered after summing over all the residues.

The insertion of the results of Eqs. (11)-(12) in Eq. (10) gives us the duality relation between one-loop integrals and single-cut phase-space integrals [2]:

$$L^{(N)} = -\,\widetilde{L}^{(N)} , \tag{14}$$

where the explicit expression of the phase-space integral $\widetilde{L}^{(N)}$ is (cf. Fig. 2)

$$\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) \\ = \int_q \sum_{i=1}^{N} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)} . \tag{15}$$

Each of the $N-1$ propagators in the integrand is regularized by the dual $i0$ prescription.

Using the invariance of the integration measure under momentum translations in arbitrary d dimensions, we can also rewrite Eq. (15) as a sum of N basic phase-space integrals:

$$\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N)$$

$$= \sum_{i=1}^{N} I^{(N-1)}(p_i, p_{i,i+1}, \ldots, p_{i,i+N-2}) \ , \quad (16)$$

where $p_{i,j} = p_i + p_{i+1} + \ldots + p_j$. The basic one-particle phase-space integrals with n dual propagators are denoted by $I^{(n)}$, and are defined as follows:

$$I^{(n)}(k_1, k_2, \ldots, k_n) = \int_q \widetilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 - i0\,\eta k_j} \ . \quad (17)$$

Summarizing our results, we find that:

- The multiple-cut contributions $L^{(N)}_{\mathrm{m-cut}}$, with $m \geq 2$, of the FTT are completely absent from the duality relation, which involves single-cut contributions only.
- The Feynman propagators in $L^{(N)}$ are replaced by dual propagators in $\widetilde{L}^{(N)}$.
- The dual $i0$ prescription and the basic dual integrals $I^{(n)}$ depend on the auxiliary vector η. However, $\widetilde{L}^{(N)}$ does not depend on η, provided it is fixed to be the same in all its contributing single-cut terms (dual integrals).

The expression (16) of $\widetilde{L}^{(N)}$ as a sum of basic dual integrals is actually a single phase-space integral, whose integrand is the sum of the terms obtained by cutting each of the internal loop lines. The duality relation, therefore, directly expresses the one-loop integral as the phase-space integral of a tree-level quantity. In the case of the FTT, the relation between loop and tree-level quantities is more involved, since the multiple-cut contributions $L^{(N)}_{\mathrm{m-cut}}$ (with $m \geq 2$) contain integrals of expressions that correspond to the product of m tree-level diagrams over the phase-space for different number of particles.

5. RELATING THE FTT WITH THE DUALITY THEOREM

The FTT and the duality theorem can be related in a direct way starting from a basic identity between dual and Feynman propagators [2]:

$$\widetilde{\delta}(q)\, \frac{1}{2qk + k^2 - i0\,\eta k} = \widetilde{\delta}(q) \left[G(q+k) + \theta(\eta k)\, \widetilde{\delta}(q+k) \right] . \quad (18)$$

This identity applies to the dual propagators when they are inserted in a single-cut integral. The proof of the equivalence of the FTT and the duality theorem is purely algebraic [2]. We explicitly illustrate it by considering the massless two-point function $L^{(2)}(p_1, p_2)$. Its dual representation is

$$\widetilde{L}^{(2)}(p_1, p_2) = I^{(1)}(p_1) + I^{(1)}(p_2) = \int_q \widetilde{\delta}(q) \left(\frac{1}{2qp_1 + p_1^2 - i0\,\eta p_1} + (p_1 \leftrightarrow p_2) \right) . \quad (19)$$

Inserting Eq. (18) in Eq. (19), we obtain

$$\widetilde{L}^{(2)}(p_1, p_2) = L^{(2)}_{1-\mathrm{cut}}(p_1, p_2) + [\theta(\eta p_1) + \theta(\eta p_2)]\ L^{(2)}_{2-\mathrm{cut}}(p_1, p_2) \ . \quad (20)$$

Owing to momentum conservation (namely, $p_1 + p_2 = 0$) $\theta(\eta p_1) + \theta(\eta p_2) = 1$, and then the dual and the FTT representations of the two-point function are equivalent.

The proof of the equivalence in the case of higher N-point functions proceeds in a similar way [2], the key ingredient simply being the constraint of *momentum conservation*.

6. MASSIVE PARTICLES, UNSTABLE PARTICLES

We have so far considered massless propagators. The extension to include propagators with finite mass M_i is straightforward, as long as M_i is real. The massless on-shell delta function $\widetilde{\delta}(q_i)$ is replaced by

$$\widetilde{\delta}(q_i; M_i) = 2\pi\, i\ \delta_+(q_i^2 - M_i^2) \ , \quad (21)$$

when the i^{th} line of the loop is cut to obtain the dual representation. The $i0$ prescription of the dual propagators is not affected by real masses. The corresponding dual propagator is

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)} \ . \quad (22)$$

In field theories with unstable particles, a Dyson summation of self-energy insertions is required to properly treat the propagator G_C of those particles. This produces finite-width effects, introducing finite imaginary contributions in those propagators. A typical form of the propagator G_C (such as in the complex-mass scheme [6]) is

$$G_C(q;s) = \frac{1}{q^2 - s} \ , \tag{23}$$

where s denotes the complex mass of the unstable particle ($s = \text{Re}\ s + i\,\text{Im}\ s$, $\text{Re}\ s > 0 > \text{Im}\ s$).

The complex-mass propagators produce poles that are located far off the real axis. Thus, when using the Cauchy theorem, as in Eq. (10), the duality relation is built up from two contributions

$$L^{(N)} = -\left(\widetilde{L}^{(N)} + \widetilde{L}_C^{(N)}\right) \ . \tag{24}$$

Here, $\widetilde{L}^{(N)}$ denotes the terms that correspond to the residues at the poles of the Feynman propagators of the loop integral, while $\widetilde{L}_C^{(N)}$ denotes those from the poles of the complex-mass propagators.

In other schemes, the propagator of an unstable particle can have a form that differs from Eq. (23). One can introduce, for instance, a complex mass that depends on the momentum q of the propagator, i.e. $s(q^2)$, or even a non-resonant component in addition to the resonant contribution. Independently of its specific form, the propagator G_C of the unstable particle produces singularities that are located at a *finite* imaginary distance from the real axis in the q_0 complex plane. Owing to this finite displacement, the structure of the duality relation (24) is valid in general, although the explicit form of $\widetilde{L}_C^{(N)}$ depends on the actual expression of the propagator G_C.

7. GAUGE POLES

The quantization of gauge theories requires the introduction of a gauge-fixing procedure, which specifies the spin polarization vectors of the gauge bosons and the content of possible compensating fictitious particles (e.g. the Faddeev–Popov ghosts in unbroken non-Abelian gauge theories, or the would-be Goldstone bosons in spontaneously broken gauge theories). The Feynman propagators of the fictitious particles are treated exactly in the same way as those of physical particles when deriving (applying) the duality relation.

The propagators of the gauge particles, however, can introduce extra unphysical poles. The general form of the polarization tensor of a spin-one gauge boson is

$$d^{\mu\nu}(q) = -g^{\mu\nu} + (\zeta - 1)\ \ell^{\mu\nu}(q)\, G_G(q) \ . \tag{25}$$

The second term on the right-hand side is absent only in the 't Hooft–Feynman gauge ($\zeta = 1$). In any other gauge, the tensor $\ell^{\mu\nu}(q)$ propagates longitudinal polarizations. Its specific form is not relevant in the context of the duality relation. Indeed, $\ell^{\mu\nu}(q)$ has a polynomial dependence on the momentum q and, therefore, it does not interfere with the residue theorem. The factor $G_G(q)$ ('gauge-mode' propagator), however, has a potentially dangerous non-polynomial dependence on q, and it can introduce extra (i.e. in addition to the poles of the associated Feynman propagator) poles with respect to the momentum variable q. A typical example of 'gauge poles' are those located at $q \cdot n = 0$ in the case of axial gauges (here n_μ is the axial-gauge vector).

The presence of gauge poles in $G_G(q)$ can modify the form of the duality relation. In general, one can expect that, 'cutting' the loop (i.e. applying the residue theorem to the loop integral), one has to explicitly include the absorptive contribution from the gauge-mode propagators in addition to the customary single-cut contribution from the Feynman propagators. Moreover, this additional contribution would have a different form in different gauges.

The impact of gauge poles on the duality relation is discussed in detail in [2]. The duality relation in the simple form presented here (i.e. with the inclusion of the sole single-cut terms from the Feynman propagators) turns out to be valid [2] in spontaneously-broken gauge theories in the unitary gauge, and in unbroken gauge theories in physical gauges specified by a gauge vector n^ν, *provided* the dual vector η^μ is chosen such that $n \cdot \eta = 0$. This excludes gauges where n^ν is time-like. Note that the validity of the duality relation in this form does not imply that the loop integral receives no extra contributions from the

[2]Of course, the duality relation is obviously valid in the 't Hooft–Feynman gauge, where there are no gauge poles.

gauge poles. It simply implies that these contributions arise after the phase-space integration of the corresponding single-cut integrals.

8. DUALITY AT THE AMPLITUDE LEVEL

The duality relation can be applied to evaluate not only basic one-loop integrals $L^{(N)}$ but also complete one-loop quantities $\mathcal{A}^{(1-\mathrm{loop})}$ (such as Green's functions and scattering amplitudes). The analogue of Eqs. (14) and (15) is the following duality relation [2]:

$$\mathcal{A}^{(1-\mathrm{loop})} = - \, \widetilde{\mathcal{A}}^{(1-\mathrm{loop})} \quad . \tag{26}$$

The expression $\widetilde{\mathcal{A}}^{(1-\mathrm{loop})}$ on the right-hand side is obtained from $\mathcal{A}^{(1-\mathrm{loop})}$ in the same way as $\widetilde{L}^{(N)}$ is obtained from $L^{(N)}$: starting from any Feynman diagram in $\mathcal{A}^{(1-\mathrm{loop})}$, we consider all possible replacements of each Feynman propagator $G(q_i)$ in the loop with the cut propagator $\widetilde{\delta}(q_i; M_i)$, and then we replace the uncut Feynman propagators with dual propagators. All the other factors in the Feynman diagrams are left unchanged in going from $\mathcal{A}^{(1-\mathrm{loop})}$ to $\widetilde{\mathcal{A}}^{(1-\mathrm{loop})}$.

Equation (26) establishes a correspondence between the one-loop Feynman diagrams contributing to $\mathcal{A}^{(1-\mathrm{loop})}$ and the tree-level Feynman diagrams contributing to the phase-space integral in $\widetilde{\mathcal{A}}^{(1-\mathrm{loop})}$. How are these tree-level Feynman diagrams related to those contributing to the tree-level expression [3] $\mathcal{A}^{(\mathrm{tree})}$? The answer to this question is mainly a matter of combinatorics of Feynman diagrams. If $\mathcal{A}^{(1-\mathrm{loop})}$ is an off-shell Green's function, the phase-space integrand in $\widetilde{\mathcal{A}}^{(1-\mathrm{loop})}$ is directly related to $\mathcal{A}^{(\mathrm{tree})}$ [2]. In a sketchy form, we can write:

$$\mathcal{A}_N^{(1-\mathrm{loop})}(\dots) \sim \int_q \sum_P \tilde{\delta}_+(q; M_P) \, \widetilde{\mathcal{A}}_{N+2}^{(\mathrm{tree})}(q, -q, \dots) \ , \tag{27}$$

where $\sum_P$ denotes the sum over the types of particles and antiparticles that can propagate in the loop internal lines, and $\widetilde{\mathcal{A}}^{(\mathrm{tree})}$ simply differs from $\mathcal{A}^{(\mathrm{tree})}$ by the replacement of dual and Feynman propagators. If the tree-level Green's function $\mathcal{A}_{N+2}^{(\mathrm{tree})}$ with $N+2$ external particles is known, it can be reused in Eq. (27) to calculate the corresponding one-loop Green's function with N external particles.

The extension of Eq. (27) to scattering amplitudes requires a careful treatment of the on-shell limit of the corresponding Green's functions [2].

[3] Here $\mathcal{A}^{(\mathrm{tree})}$ exactly denotes the tree-level counterpart of $\mathcal{A}^{(1-\mathrm{loop})}$.

9. SUMMARY

We have illustrated a duality relation between loops and trees. One-loop integrals are written in terms of single-cut phase-space integrals, with propagators regularized by a new Lorentz-covariant $i0$ prescription. This simple modification of the Feynman propagators compensates for the absence of multiple-cut contributions that appear in the FTT. The duality relation has been extended from Feynman integrals to off-shell Green's functions. Work is in progress [3,4,7] on applications to the computation of one-loop scattering amplitudes and NLO cross sections.

REFERENCES

1. Z. Bern *et al.* [NLO Multileg Working Group], arXiv:0803.0494, proceedings of the *5th Les Houches Workshop on Physics at TeV Colliders*, Jun 2007, Les Houches.
2. S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J.-C. Winter, arXiv:0804.3170.
3. S. Catani, at *HP^2: High Precision for Hard Processes at the LHC*, Sep. 2006, Zurich; G. Rodrigo, at *International Linear Collider (ILC) Workshop (ILC-ECFA and GDE Joint Meeting)*, Nov. 2006, Valencia; T. Gleisberg, at *Heidelberg 07 (DPG Spring Meeting)*, Mar. 2007, Heidelberg.
4. T.Gleisberg, PhD Thesis, TU Dresden (2007).
5. R. P. Feynman, Acta Phys. Polon. **24** (1963) 697; R. P. Feynman, *Closed Loop And Tree Diagrams,* in *Selected papers of Richard Feynman*, ed. L. M. Brown (World Scientific, Singapore, 2000) p. 867.
6. A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Nucl. Phys. B **724** (2005) 247.
7. S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J.-C. Winter, in preparation.