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On Finite Minimal Non-nilpotent Groups*

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Abstract

A *critical group* for a class of groups $\mathfrak{X}$ is a minimal non-$\mathfrak{X}$-group. The critical groups are determined for various classes of finite groups. As a consequence, a classification of the minimal non-nilpotent groups (also called Schmidt groups) is given, together with a complete proof of Gol'fand's theorem on maximal Schmidt groups.

1 Introduction

Given a class of groups $\mathfrak{X}$, we say that a group G is a *minimal non-$\mathfrak{X}$-group*, or an *$\mathfrak{X}$-critical group*, if $G \notin \mathfrak{X}$, but all proper subgroups of G belong to $\mathfrak{X}$. It is clear that detailed knowledge of the structure of minimal non-$\mathfrak{X}$-groups can provide insight into what makes a group belong to $\mathfrak{X}$. All groups considered in this paper are finite

Minimal non-$\mathfrak{X}$-groups have been studied for various classes of groups $\mathfrak{X}$. For instance, minimal non-abelian groups were analysed by Miller and Moreno [10], while Schmidt[14] studied minimal non-nilpotent groups. The latter are now known as *Schmidt groups*. Itô [9] considered the minimal non-p-nilpotent groups for p a prime, which turn out to be just the Schmidt groups. Finally, the third author [12] characterised the minimal non-T-groups (T-*groups* are groups in which normality is a transitive relation). He also

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characterised in [13] the minimal non-PST-groups, where a *PST-group* is a group in which Sylow permutability is a transitive relation.

The aim of this paper is to give more precise information about the structure of Schmidt groups and show how to construct them in an efficient way. As a consequence of our study, a new proof of a classical theorem of Gol'fand is given.

Our approach depends on the classification of critical groups for the class of PST-groups given in [13]. Recall that a subgroup H is said to be *Sylow-permutable*, or S-*permutable*, in a group G if H permutes with every Sylow subgroup of G. We mention a similar class $\mathcal{Y}_p$, which was introduced in [2]. If p is a prime, a group G belongs to *the class* $\mathcal{Y}_p$ if G enjoys the following property: if H and K are p-subgroups of G such that H is contained in K, then H is S-permutable in $\mathrm{N}_G(K)$. Clearly every PST-group is a $\mathcal{Y}_p$-group.

There is a close relation between the class of groups just introduced and p-nilpotence, as in shown by the following result, which was proved in [2; Theorem 5].

Theorem 1. *A group G is a $\mathcal{Y}_p$-group if and only if either it is p-nilpotent or it has an abelian Sylow p-subgroup P and every subgroup of P is normal in $\mathrm{N}_G(P)$.*

Our first main result is:

Theorem 2. *The minimal non-$\mathcal{Y}_p$-groups are just the minimal non-PST-groups with a non-trivial normal Sylow p-subgroup. Such groups are of the types described in I to IV below. Let p and q be distinct primes.*

Type I: *$G = [P]Q$, where $P = \langle a, b\rangle$ is an elementary abelian group of order p^2, $Q = \langle z\rangle$ is cyclic of order q^r, with q a prime such that q^f divides $p-1$, $q^f > 1$ and $r \geq f$, and $a^z = a^i$, $b^z = b^{i^j}$, where i is the least positive primitive q^f-th root of unity modulo p and $j = 1 + kq^{f-1}$, with $0 < k < q$.*

Type II: *$G = [P]Q$, where $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$ and P an irreducible Q-module over the field of p elements with centralizer $\langle z^q\rangle$ in Q.*

Type III: *$G = [P]Q$, where $P = \langle a_0, a_1, \dots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z\rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p-1$ and $r > f$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

Type IV: *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful irreducible Q-module, and z centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

Since a group is a soluble PST-group if and only if it belongs to $\mathcal{Y}_p$ for all primes p [2; Theorem 4], Theorem 2 may be regarded as a local approach to the third author's classification of minimal non-PST-groups [13].

An interesting consequence of Theorem 2 is the following classification of Schmidt groups. In order to describe the classification, we must introduce one further type of group:

Type V: $G = [P]Q$, where $P = \langle a \rangle$ is a normal subgroup of order p, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, and $a^z = a^i$, where i is the least primitive q-th root of unity modulo p.

Our main result can now be stated as:

Theorem 3. *The Schmidt groups are exactly the groups of Type II, Type IV and Type V.*

Our next result shows that p-soluble groups with Sylow p-subgroups isomorphic to a normal subgroup of a minimal non-$\mathcal{Y}_p$-group have a restricted structure.

Theorem 4. *Let G be a p-soluble group with a Sylow p-subgroup P. If P is isomorphic to a non-trivial normal Sylow subgroup of a minimal non-$\mathcal{Y}_p$-group, then G has p-length 1.*

In [4] Gol'fand stated the following result:

Theorem 5. *Let p and q be distinct primes, let r be a given positive integer, and let a be the order of p modulo q. Then there is a unique minimal non-p-nilpotent group G_0 of order $p^{a_0}q^r$, where $a_0 = a$ if a is odd and $a_0 = 3a/2$ if a is even, such that all minimal non-p-nilpotent groups of order $p^t q^r$ are isomorphic to quotients of G_0 by central subgroups.*

Only a sketch of proof of this theorem is given in Golfand's article. In Section 3, we show how to construct the Schmidt groups of Gol'fand and we also give a complete proof of Theorem 5. We remark that Rédei [11] has given another construction of the Schmidt groups of maximum order.

2 Proofs of Theorems 2, 3 and 4

Proof of Theorem 2. Assume that G is a minimal non-$\mathcal{Y}_p$-group and let P be a Sylow p-subgroup of G. Since G does not belong to $\mathcal{Y}_p$, there exist subgroups H and K of P such that $H \leq K$ and H is not S-permutable in $\mathrm{N}_G(K)$. Consequently there is an element $z \in \mathrm{N}_G(K)$ such that z does not normalise H. Here it can be assumed that z has order q^r for some prime $q \neq p$. Then $G = K\langle z\rangle$ because G is a minimal non-$\mathcal{Y}_p$-group. This implies that $K = P$ is a normal Sylow p-subgroup of G and $Q = \langle z\rangle$ is a cyclic Sylow q-subgroup of G. Then G is not a PST-group, yet every proper subgroup has $\mathcal{Y}_p$ and $\mathcal{Y}_q$, and thus is a PST-group by [2].

Conversely, if G is a minimal non-PST-group, then G does not have $\mathcal{Y}_p$ for some prime p. Since all its proper subgroups satisfy $\mathcal{Y}_p$, the group G is a minimal non-$\mathcal{Y}_p$-group.The classification of minimal non-PST-groups given in [13] completes the proof. (Notice that the groups of Types IV and V of [13] are both of Type IV above). □

Proof of Theorem 3. Let G be a minimal non-nilpotent group. Then G is a minimal non-p-nilpotent group for some prime p. Suppose that G is not a $\mathcal{Y}_p$-group, so that G is a minimal non-$\mathcal{Y}_p$-group. By Theorem 2, the group G is of one of the types I–IV. By examining the group structure, we see that groups of Type I and III are not minimal non-p-nilpotent. Therefore G must be of Type II or IV.

Assume now that G belongs to $\mathcal{Y}_p$. Then by [1; Theorem A] and [3; VII, 6.18], the p-nilpotent residual P of G is an abelian minimal normal Sylow subgroup which is complemented in G by a cyclic Sylow q-subgroup Q. Moreover Q normalises each subgroup of P. This implies that P is cyclic of order p, say $P = \langle a\rangle$. In addition, $a^z = a^i$ for some $0 < i < p$ and z^q centralizes a. This implies that i must be a primitive q-th root of unity modulo p and, by taking a suitable power of z as a generator of Q, we can assume that i is the least such positive integer. Hence G is of Type V. □

Proof of Theorem 4. Assume that G is a p-soluble group with p-length > 1 and G has least order subject to possessing a Sylow p-subgroup P which is isomorphic to a non-trivial normal Sylow subgroup of a Schmidt group. By [6; VI, 6.10], we conclude that P is not abelian. Thus P is a Sylow p-subgroup of a group of Type IV in Theorem 2. By minimality of order $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. In addition, since the class of groups of p-length at most 1 is a saturated formation, we have $\Phi(G) = 1$ and hence G has a unique minimal normal subgroup which is an elementary abelian p-group. Let $D = \mathrm{O}_p(G)$; then D is a non-trivial elementary abelian group and $\mathrm{C}_G(D) = D$. Moreover $\Phi(P) = \mathrm{Z}(P) \leq D$ and so P/D is elementary abelian.

Let T be the subgroup defined by $T/D = \mathrm{O}_{p'}(G/D)$. Since P/D is an elementary abelian p-group, G/D has p-length at most 1 by [6; VI, 6.10]. It follows that $(T/D)(P/D)$ is a normal subgroup of G/D. Therefore TP is a normal subgroup of G. Assume that TP is a proper subgroup of G. Now $\mathrm{O}_{p'}(TP) \leq \mathrm{O}_{p'}(G) = 1$, so P is a normal subgroup of TP and hence of G, a contradiction which shows that $G = TP$.

Assume now that P/D is a non-cyclic elementary abelian group. By [8; X, 1.9], we have $T/D = \langle \mathrm{C}_{T/D}(xD) \mid xD \in P/D, xD \neq D \rangle$. Let $x \in P\ D$. Since P/D centralizes xD, we have $P/D \leq \mathrm{N}_{G/D}\big(\mathrm{C}_{T/D}(xD)\big)$. Let $T_x/D = \mathrm{C}_{T/D}(xD)$. Assume that $PT_x = G$; then $T_x = T$ is a normal subgroup of G and thus $\mathrm{O}_{p'}(G/D) = T_x/D$. This implies that $\langle x \rangle D/D \leq \mathrm{Z}(G/D)$ and $\langle x \rangle D$ is a normal p-subgroup of G, so that $\langle x \rangle D$ is contained in D, a contradiction. Consequently PT_x is a proper subgroup of G for all $1 \neq xD \in P/D$. Hence PT_x has p-length at most 1 by minimality of G. Since $\mathrm{C}_G(D) = D$ and $\mathrm{O}_{p'}(PT_x)$ centralizes D, we conclude that $\mathrm{O}_{p'}(PT_x) = 1$. Therefore P is a normal subgroup of PT_x, which shows that T normalizes P and thus P is a normal subgroup of G. This contradiction shows that P/D is cyclic.

Since P has class 2, we see from [7; IX, 5.5] that, if $p > 3$, then G has p-length at most 1. Therefore $p \leq 3$. Let X be a minimal non-$\mathcal{Y}_p$-group such that P is a Sylow p-subgroup of X. Note that $P/\Phi(P)$ is an irreducible X-module. In particular D, the subgroup of the previous paragraphs, is not normal in X and so $P = DD^g$ for some $g \in X$. Since D is abelian, $D \cap D^g \leq Z(P) = \Phi(P)$, and it follows that $P/\Phi(P)$ has order p^2. This implies that P is an extra-special group of order p^3. If $p = 2$, then, since $C_G(D) = D$, we see that G must be a symmetric group of degree 4. Hence P is dihedral of order 8, which cannot lead to a group of Type IV since $\mathrm{Aut}(P)$ is a 2-group. Hence $p = 3$. But a non-abelian group of order 3^3 cannot occur as the normal Sylow 3-subgroup of a Schmidt group, because the only prime divisor of $3^2 - 1$ is 2 and the order of 3 modulo 2 is 1. This contradiction completes the proof of the theorem. □

3 The Construction of Gol'fand's Groups and a Proof of Gol'fand's Theorem

We begin by constructing groups of Type IV with a Sylow p-subgroup P of order p^{3m} and $|P/\Phi(P)| = p^{2m}$. These groups were constructed in [13] by a different method, but the present approach is more convenient when $p = 2$. We will use the following result on linear operators.

Lemma 6. *Let p be a prime and let r be a positive integer such that* $\gcd(p, r) = 1$*. Let β be a linear operator of order $p^u r$ on a vector space V over the field of p-elements, where u is a non-negative integer. If β has irreducible minimum polynomial f, then β^{p^u} also has minimum polynomial f.*

Proof. Let g be the minimum polynomial of β^{p^u}. Now $f(\beta^{p^u}) = f(\beta)^{p^u} = 0$, so that g divides f. Since f is irreducible, $f = g$.

□

Construction 7. Let p and q be distinct primes such that the order of p modulo q is $2m$, $m \geq 1$. Let F be the free group with basis $\{f_0, f_1, \dots, f_{2m-1}\}$. Write $R = F'F^p$ and $R^* = [F, R]R^p$. Then F/R is an elementary abelian p-group of order p^{2m} and $H = F/R^*$ is a p-group such that $R/R^* = \Phi(H)$ is an elementary abelian p-group contained in $\mathrm{Z}(H)$. Moreover H is a non-abelian group because an extra-special group of order p^{2m+1} is an epimorphic image of H.

Denote by g_i the image of f_i under the natural epimorphism of F onto $H = F/R^*$, $0 \leq i \leq 2m-1$. Since H has class 2, we know that $\Phi(H)$ is generated by all $[g_i, g_j]$, with $i < j$, and g_i^p. Therefore $\Phi(H)$ has dimension as $\mathrm{GF}(p)$-vector space at most $\frac{1}{2}(2m(2m-1) + 2m = m(2m+1)$. Assume that the dimension is less than $m(2m+1)$. Then there exists an element

$$r = \prod_j (f_j^p)^{\lambda_j} \prod_{j<k} [f_j, f_k]^{\mu_{jk}} \in R^*$$

with some λ_j or μ_{jk} not divisible by p. It is clear that $p \mid \lambda_j$ for all j since $F^p F'/F'$ is a free abelian group with basis $\{f_j^p F' \mid 0 \leq j \leq 2m-1\}$. Suppose that $p \nmid \mu_{ik}$ for some $i < k$ and let ρ_i be the endomorphism of F defined by $f_i^{\rho_i} = f_i^2$, $f_l^{\rho_i} = f_l$ for $l \neq i$. Then $r^{\rho_i} R^* = R^*$ and so $r^{\rho_i} r^{-1} R^* = R^*$. This implies that

$$w = \prod_{j<i} [f_j, f_i]^{\mu_{ji}} \prod_{i<l} [f_i, f_l]^{\mu_{il}} \in R^*.$$

On the other hand, by applying ρ_k we find that

$$w^{\rho_k} w^{-1} R^* = [f_i, f_k]^{\mu_{ik}} R^* = R^*.$$

Since $p \nmid \mu_{ik}$, it follows that μ_{ik} has an inverse modulo p. This means that $[f_i, f_k] \in R^*$. Now since permutations of the generators of F induce endomorphisms in F and R^* is fully invariant, it follows that $F' \leq R^*$ and H is abelian, a contradiction. Therefore $\Phi(H)$ has dimension $m(2m+1)$ and so $|\Phi(H)| = p^{m(2m+1)}$.

Let $f(t) = c_0 + c_1 t + \cdots + c_{2m-1} t^{2m-1} + t^{2m}$ be an irreducible factor of the cyclotomic polynomial of order q over $\mathrm{GF}(p)$ and let α be the endomorphism of F given by $f_i^\alpha = f_{i+1}$ for $0 \le i \le 2m-2$, $f_{2m-1}^\alpha = f_0^{-c_0} f_1^{-c_1} \cdots f_{2m-1}^{-c_{2m-1}}$. Since R^* is a fully invariant subgroup of F, it follows that α induces an endomorphism β on $H = F/R^*$. In turn, β induces an automorphism $\bar{\beta}$ on $H/\Phi(H)$. Since $H/\Phi(H) = \big(H/\Phi(H)\big)^{\bar{\beta}} \le H^\beta \Phi(H)/\Phi(H)$, it follows that $H = H^\beta \Phi(H)$, whence $H = H^\beta$. Consequently β is an automorphism of H.

It is clear that β induces the linear operator $\bar{\beta}$, with minimum polynomial f, on the vector space $H/\Phi(H)$. Now by [6; III, 3.18], we conclude that β^q has order p^u for a some u and hence β has order $p^u q$. By Lemma 6, there is a GF(p)-basis $\{g'_0, g'_1, \ldots, g'_{2m-1}\}$ of $H/\Phi(H)$, where $g'_i = g_i \Phi(H)$, such that ${g'_i}^{\bar{\beta}^{p^u}} = g'_{i+1}$ for $0 \le i \le 2m-2$ and ${g'_{2m-1}}^{\bar{\beta}^{p^u}} = {g'_0}^{-c_0} {g'_1}^{-c_1} \cdots {g'_{2m-1}}^{-c_{2m-1}}$. Hence we can replace β by β^{p^u} and assume without loss of generality that β has order q.

It follows that $\Phi(H)$ is a GF$(p)T$-module, where $T = \langle \beta \rangle$ is a cyclic group of order q. By Maschke's Theorem $\Phi(H)$ is a direct sum of irreducible T-modules. Let N be the sum of all non-trivial irreducible submodules in the direct decomposition and write $P = H/N$. It is clear that N is β-invariant and therefore β induces an automorphism γ of order q in P. Let $Q = \langle z \rangle$ be a cyclic group of order q^r acting on P via $z \mapsto \gamma$. Let $G = [P]Q$ be the corresponding semidirect product.

It is easily checked that G is a Schmidt group. Next we show that P has order p^{3m}. From Theorem 3 we see that $\Phi(P)$ has order at most p^m, where $|P/\Phi(P)| = p^{2m}$. On the other hand, $|\Phi(H)| = p^{m(2m+1)}$, and N has order a power of p^{2m} because every faithful irreducible $< \beta >$-module over GF(p) has dimension $2m$. Therefore $|\Phi(P)| = p^m$.

Remark 8. In the group of Construction 7, we may assume that $\bar{g}_{2m-1}^z = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}$, where $\bar{g}_i = g_i N$.

Proof. We know that $\bar{g}_{2m-1}^z = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}} \bar{w}$ where $\bar{w} \in \Phi(P)$. Since $f(t)$ is irreducible, 1 is not a root of $f(t)$ and it follows that $c = c_0 + c_1 + \cdots + c_{2m-1} + 1 \not\equiv 0 \pmod{p}$. Consequently there exists an integer d such that $cd \equiv -1 \pmod{p}$. Put $\bar{w}_0 = \bar{w}^d$ and consider the automorphism δ of P defined by $\bar{g}_i^\delta = \bar{g}_i \bar{w}_0$ for$0 \le i \le 2m-1$. If we write $\gamma_0 = \delta\gamma\delta^{-1}$, it is easily checked by an elementary calculation that $\bar{g}_i^{\gamma_0} = \bar{g}_{i+1}$ for $0 \le i \le 2m-2$, and $\bar{g}_{2m-1}^{\gamma_0} = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}$. Let $\langle z_0 \rangle$ be a cyclic group of order q^r, with z_0 acting on P via $z_0 \mapsto \gamma_0$. Since $\langle z_0 \rangle$ and $\langle z \rangle$ are conjugate in Aut(P), it follows by [3; B, 12.1] that the groups $P\langle z \rangle$ and $P\langle z_0 \rangle$ are isomorphic. □

Remark 9. The group in Construction 7 does not depend on the choice of irreducible factor $f(t)$.

Proof. Assume that the group $G_1 = [P_1]\langle z_1\rangle$ has been constructed by using another irreducible factor $g(t)$ of the cyclotomic polynomial of order q over $\mathrm{GF}(p)$. Since G and G_1 have the same order, it will be enough to find a set of generators of G_1 for which the relations of G hold. Since z centralizes $\Phi(P)$ and z_1 centralizes $\Phi(P_1)$, we have $G/\Phi(P) \cong [P/\Phi(P)]\langle z\rangle$ and $G_1/\Phi(P_1) \cong [P_1/\Phi(P_1)]\langle z_1\rangle$. But $P/\Phi(P)$ and $P_1/\Phi(P_1)$ are faithful irreducible modules for a cyclic group of order q. Therefore $[P/\Phi(P)](\langle z\rangle/\langle z^q\rangle)$ is isomorphic to $[P_1/\Phi(P_1)](\langle z_1\rangle/\langle z_1^q\rangle)$ by [3; B, 12.4]. Let ϕ be an isomorphism between these groups. Then it is clear that ϕ induces an isomorphism ψ between $G/\Phi(P)$ and $G_1/\Phi(P_1)$.

Let $\bar{h}_i = h_i\Phi(P)$, $0 \leq i \leq 2m-1$. Put $\bar{k}_i = \bar{h}_i^{\psi}$ and $\bar{u} = \bar{z}^{\psi}$. We show how to extend the isomorphism ψ to an isomorphism between G and G_1. In order to do so, we choose representatives k_i of $\bar{k}_i$ and u of $\bar{u}$ such that the order of u is q^r. There is no loss of generality in assuming that $k_i^u = k_{i+1}$ for $0 \leq i \leq 2m-2$: indeed, if $k_i^u = k_{i+1}w_{i+1}$, with $w_{i+1} \in \Phi(P_1)$, then $k_i' = k_i w_i \cdots w_1$ for $1 \leq i \leq 2m-1$, $k_0' = k_0$ are representatives of $\bar{k}_i$ and $k_i'^u = k_{i+1}'$ for $1 \leq i \leq 2m-1$ because u centralizes $\Phi(P_1)$. By using the same argument as in Remark 8, we may also assume $k_{2m-1}^u = k_0^{-c_0}k_1^{-c_1}\cdots k_{2m-1}^{-c_{2m-1}}$. Therefore G and G_1 satisfy the same relations and by Von Dyck's theorem they are isomorphic. $\square$

Remark 10. In Construction 7, it is not necessary to assume that β has order q. Indeed, it can be proved that β^q fixes all elements of $\Phi(H)$ and that the automorphism γ induced by β in H/N has order q.

Gol'fand's result (Theorem 5) can be recovered with the help of Construction 7 and Theorem 3.

Proof of Theorem 5. Let p and q be distinct primes and let a be the order of p modulo q. Then a is the dimension of each non-trivial irreducible module for a cyclic group of order q over $\mathrm{GF}(p)$. Assume that a is odd. Then every Schmidt group G with a normal Sylow p-subgroup P such that $|P/\Phi(P)| = p^a$ is of Type II or Type V. Then the theorem holds in this case because all Schmidt groups of the same type with isomorphic Sylow q-subgroups are actually isomorphic.

Assume now that a is even, with say $a = 2m$. Then we are dealing with Schmidt groups of Type II or Type IV. Let G_0 be the group of Construction 7. Then $|G_0| = p^{3m}q^r$ and $|P_0/\Phi(P_0)| = p^{2m}$, where P_0 is a normal Sylow p-subgroup of G_0. It is clear that $G_0/\Phi(P_0)$ is a Schmidt group of Type II. Therefore, if G is a Schmidt group of Type II with order p^tq^r and a normal Sylow p-subgroup, then $G \cong G_0/\Phi(P_0)$ and $\Phi(P_0) \leq \mathrm{Z}(G_0)$. Consequently, we need only show that all Schmidt groups of Type IV and order p^tq^r, $t \leq 3m$, which have a normal Sylow$p$-subgroup are isomorphic to quotients of G_0 by central subgroups.

Let $\overline{G}$ be a Schmidt group of Type IV and order $p^t q^r$ with a normal Sylow p-subgroup $\bar{P}$. Then $G_0/\Phi(P_0)$ and $\overline{G}/\Phi(\overline{P})$ are isomorphic. Let us choose generators z and $\bar{z}$ of Sylow q-subgroups Q of G_0 and $\overline{Q}$ of $\overline{G}$ such that the minimum polynomial of the actions of z on $P_0/\Phi(P_0)$ and $\bar{z}$ on $\overline{P}/\Phi(\overline{P})$ coincide. Also choose generators $g_0, g_1, \dots, g_{2m-1}$ of the Sylow p-subgroup P_0 of G_0 and generators $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_{2m-1}$ of the Sylow p-subgroup $\overline{P}$ of $\overline{G}$ such that $g_j^z = g_{j+1}$ and $\bar{g}_j^{\bar{z}} = \bar{g}_{j+1}$ for $0 \leq j \leq 2m-2$. Since $\Phi(P_0) = P_0'$ and $\Phi(\overline{P}) = \overline{P}'$, and both P_0 and $\overline{P}$ have class 2, the subgroup $\Phi(P_0)$ can be generated by the commutators $[g_i, g_j]$, while $\Phi(\overline{P})$ is generated by the commutators $[\bar{g}_i, \bar{g}_j]$. On the other hand, if $u_i = [g_0, g_0^{z^i}]$, we have $u_i = u_i^{z^k} = [g_k, g_k^{z^i}]$. It is easy to see that $u_i = [g_0, g_0^{z^i}] = [g_0^{z^q}, g_0^{z^i}] = u_{q-i}^{-1}$.

Observe that q is odd since $2m$ divides $q-1$: write $q = 2s+1$. By definition of the g_i and u_i, and use of the minimum polynomial of the action of z on $P_0/\Phi(P_0)$, it may be shown that for $l \geq 1$

$$u_{s+m+l} = u_{s-m+l}^{-c_0} u_{s-m+l+1}^{-c_1} \cdots u_{s+m+l-2}^{-c_{2m-2}} u_{s+m+l-1}^{-c_{2m-1}}.$$

Now this formula and the relations $u_i = u_{q-i}^{-1}$ allow us to show by induction that each u_{s+m+l} can be expressed in terms of elements of the set $B = \{u_{s-m+l}, u_{s-m+2}, \dots, u_s\}$. Since $\Phi(P_0)$ has dimension m over $\mathrm{GF}(p)$, this expression is unique. It follows that each u_j can be uniquely expressed in terms of the elements of B, and so this is also true for each generator of $\Phi(P_0)$. The same argument shows that the generators of $\Phi(\overline{P})$ have a similar unique expression subject to the same relations.

The arguments of Remark 9 allow us to assume that

$$g_{2m-1}^z = g_0^{-c_0} g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}} \text{ and } \bar{g}_{2m-1}^{\bar{z}} = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}.$$

Consequently, all relations of G_0 are satisfied by $\overline{G}$. By Von Dyck's theorem, it follows that $\overline{G}$ is an epimorphic image of G_0 by a central subgroup of G_0. □

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