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On a class of p-soluble groups*

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Abstract

Let p be a prime. The class of all p-soluble groups G such that every p-chief factor of G is cyclic and all p-chief factors of G are G-isomorphic is studied in this paper. Some results on T-, PT-, and PST-groups are also obtained.

Keywords: finite groups, permutability, subnormality.

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1 Introduction

All groups considered in the paper will be finite.

Let p be a prime number. Denote by $\mathcal{U}_p^*$ the class composed of all p-soluble groups G such that every p-chief factor of G is cyclic (G is p-supersoluble) and all p-chief factors of G are G-isomorphic.

For a p-soluble group G, the following statements are pairwise equivalent:

1. G belongs to $\mathcal{U}_p^*$.

2. Every p'-perfect subnormal subgroup of G permutes with every Hall p'-subgroup of G ([2, Theorem 6]).

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3. If $H \leq K$ are p-subgroups of G, then H permutes with all Sylow subgroups of $N_G(K)$ ([5, Theorem 9]).

4. G is either p-nilpotent, or G has an abelian Sylow p-subgroup P and every subgroup of P is normal in $N_G(P)$ ([5, Theorem 5]).

5. Either G is p-nilpotent or $G(p)/O_{p'}\big(G(p)\big)$ is an abelian normal Sylow p-subgroup of $G/O_{p'}\big(G(p)\big)$ such that the elements of $G/O_{p'}\big(G(p)\big)$ induce power automorphisms in $G(p)/O_{p'}\big(G(p)\big)$, where $G(p)$ denotes the p-nilpotent residual of G, that is, the smallest normal subgroup of G such that $G/G(p)$ is p-nilpotent ([4, Theorem A]).

These characterisations let the class $\mathcal{U}_p^*$ play a major role in the study of three interesting classes of groups: T-groups (or groups in which normality is transitive), PT-groups (or groups in which permutability is transitive), and PST-groups (or groups in which permutability with Sylow subgroups is transitive). These classes have been widely studied (see [1, 2, 4, 5, 7, 8, 9, 10, 13]).

The main goal of this paper is to study the behaviour of $\mathcal{U}_p^*$ as a class of groups and apply the results to get information about the classes of T-, PT-, and PST-groups.

It is clear that $\mathcal{U}_p^*$ is a subgroup-closed homomorph. However, the direct product of a symmetric group of degree 3 with a cyclic group of order 3 shows that it is not closed under taking direct products. In particular, $\mathcal{U}_p^*$ is not a formation. More precisely, we have:

Theorem A. *The class of all p-nilpotent groups is the largest formation contained in $\mathcal{U}_p^*$.*

Since a soluble PST-group is a $\mathcal{U}_p^*$-group for all primes p ([2, Theorems 6 and 8]), we have:

Corollary 1. *The class of all nilpotent groups is the largest formation contained in the class of all soluble PST-groups.*

The class $\mathcal{U}_p^*$ is not saturated in general (see [3] or [12]). However we have:

Theorem B. *Let G be a group. The following statements are equivalent:*

1. for every subgroup H of G, $H/\Phi(H)$ is a $\mathcal{U}_p^*$-group, and

2. G is a $\mathcal{U}_p^*$-group.

This theorem is a consequence of the following result, which is proved in [6]:

Theorem C. *Let G be a p-supersoluble group. The following statements are equivalent:*

1. *G belongs to $\mathcal{U}_p^*$.*

2. *G does not have any subgroup of the form $X = [P]Q$, where p and q are primes such that $q^f \mid p-1$, with $f \geq 1$, i is a primitive root of unity modulo p, $j = 1 + kq^{f-1}$, with $0 < k < q$, $P = \langle a, b\rangle$ is an elementary abelian group of order p^2, $Q = \langle z\rangle$ is a cyclic q-group of order q^r with $r \geq f$ such that $a^z = a^i$, $b^z = b^{i^j}$.*

Following Van der Waall and Fransman [12], we say that a group G is a T_0-group (respectively, a PT_0-group, a PST_0-group) if $G/\Phi(G)$ is a T-group (respectively, a PT-group, a PST-group).

As a consequence of Theorem B, [2, Theorems 6 and 8] and the fact that the class of soluble PST-groups is subgroup-closed , we have the following result.

Corollary 2. *Let G be a group. The following statements are equivalent:*

1. *Every subgroup of G is a PST_0-group.*

2. *Every subgroup of G is a PST-group.*

3. *G is a soluble PST-group.*

Assume now that every subgroup of G is T_0-group. By Corollary 2, G is a soluble PST-group. Applying Agrawal's theorem [1], G has an abelian normal Hall subgroup D of odd order complemented by a nilpotent subgroup B such that every subgroup of D is normal in G. Consequently, B' centralises every subgroup of D.

In fact we have:

Theorem D. *Let G be a group. The following statements are pairwise equivalent:*

1. *G is a soluble PST-group,*

2. *G is supersoluble and has a normal abelian subgroup D of odd order and a nilpotent subgroup B such that $G = DB$, with $\gcd(|D|, |B|) = 1$, $B' \trianglelefteq G$ and G/B' is a T-group, and*

3. Every subgroup of G is a T_0-group.

4. Every subgroup of G is a PT_0-group.

Note that Van der Waall and Fransman's theorem [12, Theorem 3.10] is the equivalence between 2 and 3.

Applying these results, we are able to give an alternative proof of the main result of [3]. We will also use the following result, which is a particular case of a theorem proved in [6].

Theorem E. *Let G be a p-soluble group such that all proper subgroups of G belong to $\mathcal{U}_p^*$, but G itself does not belong to $\mathcal{U}_p^*$. Then G has a normal Sylow p-subgroup P which is complemented by a non-normal cyclic subgroup whose order is a power of a prime $q \neq p$.*

Theorem F. *Assume that every proper subgroup of a group G is a T_0-group, but G itself is not a T_0-group. Then:*

1. $G = PQ$, where P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G for some distinct primes p and q;

2. $P \lhd G$ and Q is a non-normal cyclic subgroup of G;

3. $G/\Phi(G)$ is a minimal non-T-group.

2 Proofs

Proof of Theorem A. Let $\mathfrak{F}$ be a formation contained in the class $\mathcal{U}_p^*$. Assume that G is a group such that $G \in \mathfrak{F}$, but G is not p-nilpotent. Given a group X, let us denote by $X(p)$ the p-nilpotent residual of X. Since $\mathfrak{F}$ is a formation, we have that $H = G/O_{p'}\big(G(p)\big)$ belongs to $\mathfrak{F}$. By [4, Theorem A], $H(p)$ is an abelian normal Sylow p-subgroup of H such that the elements of H induce power automorphisms in $H(p)$. Since H is not p-nilpotent, there exists a p'-element $x \in H$ such that x does not centralise $H(p)$. On the other hand, since $\mathfrak{F}$ is a formation, we have that $H \times H \in \mathfrak{F}$. Moreover $H \times H$ does not belong to $\mathcal{U}_p^*$, because $(x, 1)$ centralises the p-chief factors of the second factor of the direct product, but does not centralise the p-chief factors of the first factor, a contradiction. □

Proof of Theorem B. It is clear that if G is a $\mathcal{U}_p^*$-group, then for every $H \leq G$, H is a $\mathcal{U}_p^*$-group and hence $H/\Phi(H)$ is a $\mathcal{U}_p^*$-group.

Assume that the converse is false. Let G be a group of minimal order such that for every subgroup H of G, $H/\Phi(H)$ is a $\mathcal{U}_p^*$-group, but G itself is not a $\mathcal{U}_p^*$-group. By minimality of G, we have that all proper subgroups H of G belong to $\mathcal{U}_p^*$, but G itself does not. By Theorem C, it follows that G has the form $G = [P]Q$, where p and q are primes such that $q^f \mid p-1$, with $f \geq 1$, i is a primitive root of unity modulo p, $j = 1 + kq^{f-1}$, with $0 < k < q$, $P = \langle a, b\rangle$ is an elementary abelian group of order p^2, $Q = \langle z\rangle$ is a cyclic q-group of order q^r with $r \geq f$ such that $a^z = a^i$, $b^z = b^{i^j}$. Since $\langle a\rangle$ and $\langle b\rangle$ are normal subgroups of G, it follows that $\langle a\rangle Q$ and $\langle b\rangle Q$ are maximal subgroups of G. Hence $\Phi(G) \leq \langle a\rangle Q \cap \langle b\rangle Q = Q$. Therefore $G/\Phi(G)$ can be expressed as a semidirect product of an elementary abelian normal subgroup $\langle \bar{a}, \bar{b}\rangle$ of order p^2 by a cyclic subgroup $\langle \bar{z}\rangle$ such that $\bar{a}^{\bar{z}} = \bar{a}^i$ and $\bar{b}^{\bar{z}} = \bar{a}^{i^j}$. Since $G/\Phi(G)$ satisfies $\mathcal{U}_p^*$, it follows that $\langle ab\rangle$ is normalised by z. Hence $a^i b^{i^j}$ belongs to $\langle ab\rangle$, which implies that $i^j \equiv i \pmod{q^f}$. Thus $j \equiv 1 \pmod{q^f}$, but $j = 1 + kq^{f-1}$ with $0 < k < q$, a contradiction. Hence G belongs to $\mathcal{U}_p^*$. □

Proof of Theorem D. (1) implies (2) Assume that G is a soluble PST-group. By Agrawal's theorem [1], there exists an abelian Hall subgroup D of odd order complemented by a nilpotent subgroup B such that every subgroup of D is normal in G.

Let $d \in D$. Since $\langle d\rangle$ is a normal subgroup of G, it follows that $G/C_G(\langle d\rangle)$ is abelian. Hence $B' \leq C_G(\langle d\rangle)$. It follows that $B' \leq C_G(D)$. Consequently, B' is a normal subgroup of G. Since G is a soluble PST-group, we have that G/B' is a PST-group. Moreover, all Sylow subgroups of G/B' are abelian, because they are Sylow subgroups of the abelian group D or isomorphic to Sylow subgroups of the abelian group B/B'. Therefore, G/B' is a T-group by [5, Theorem 2].

(2) implies (3) Assume that G is a supersoluble group with an abelian normal Hall subgroup D of odd order complemented by a nilpotent subgroup B such that B' is normal in G and G/B' is a T-group. Since B is nilpotent, we have that $B' \leq \Phi(B)$. Hence $B' \leq \Phi(G)$, because B' is a normal subgroup of B. In particular, $G/\Phi(G)$ is a soluble T-group.

Let H be a subgroup of G. Since G is a soluble PST-group, we have that H is a soluble PST-group. Hence the nilpotent residual $H^{\mathfrak{N}}$ of H is a normal Hall subgroup of H of odd order complemented by a subgroup H_B which can be assumed to be contained in B. Since $(H_B)' \leq B'$ and $(H_B)' \leq \Phi(H_B)$, because H_B is nilpotent, and $(H_B)'$ is a normal subgroup of H, we obtain that $(H_B)' \leq \Phi(H)$. Hence $H/\Phi(H)$ is again a soluble T-group. Hence G is a T_0-group.

It is clear that (3) implies (4). Assume now that every subgroup of G is a PT_0-group. Then every subgroup of G is a PST_0-group. By Corollary 2, it follows that G is a soluble PST-group. Therefore (4) implies (1) and the circle of implications is complete. □

Proof of Theorem F. Assume that G is a minimal non-T_0-group. Then all proper subgroups of G are T_0-groups. From Theorem D, we have that all proper subgroups of G are soluble PST-groups. In particular, all proper subgroups of G are supersoluble, which implies that G itself is soluble (see [11]).

If G were a PST-group, then G would be a T_0-group by Theorem E, a contradiction. Therefore G is not a PST-group. Hence there exists a prime p such that G is a minimal non-$\mathcal{U}_p^*$-group. By Theorem E, G can be expressed as $G = PQ$ satisfying conditions 1 and 2.

Since $\Phi(P) \leq \Phi(G)$ because P is a normal subgroup of G, it follows that $G/\Phi(G)$ has a normal abelian Sylow p-subgroup $P\Phi(G)/\Phi(G)$. Consequently, $G/\Phi(G)$ has abelian Sylow subgroups. Since every subgroup of G is a soluble PST-group, it follows that every proper subgroup of $G/\Phi(G)$ is a soluble PST-group, and since Sylow subgroups of $G/\Phi(G)$ are abelian, all proper subgroups of $G/\Phi(G)$ are T-groups by [5, Theorem 2]. Hence $G/\Phi(G)$ is a minimal non-T-group, as desired. □

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