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On minimal non-supersoluble groups*

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Dedicated to the memory of Klaus Doerk (1939–2004)

Abstract

The aim of this paper is to classify the finite minimal non-p-supersoluble groups, p a prime number, in the p-soluble universe.
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1 Introduction

All groups considered in this paper are finite.

Given a class $\mathfrak{X}$ of groups, we say that a group G is a *minimal non-$\mathfrak{X}$-group* or an *$\mathfrak{X}$-critical group* if $G \notin \mathfrak{X}$, but all proper subgroups of G belong to $\mathfrak{X}$. It is rather clear that detailed knowledge of the structure of $\mathfrak{X}$-critical groups could help to give information about what makes a group belong to $\mathfrak{X}$.

Minimal non-$\mathfrak{X}$-groups have been studied for various classes of groups $\mathfrak{X}$. For instance, Miller and Moreno [10] analysed minimal non-abelian groups, while Schmidt [14] studied minimal non-nilpotent groups. These groups are now known as *Schmidt groups*. Rédei classified completely the minimal non-abelian groups in [12] and the Schmidt groups in [13]. More precisely,

Theorem 1 ([12])**.** *The minimal non-abelian groups are of one of the following types:*

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1. *$G = [V_q]C_{r^s}$, where q and r are different prime numbers, s is a positive integer, and V_q is an irreducible C_{r^s}-module over the field of q elements with kernel the maximal subgroup of C_{r^s},*

2. *the quaternion group of order 8,*

3. *$G_{II}(q, m, n) = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}} \rangle$, where q is a prime number, $m \geq 2$, $n \geq 1$, of order q^{m+n}, and*

4. *$G_{III}(q, m, n) = \langle a, b \mid a^{q^m} = b^{q^n} = [a, b]^q = [a, b, a] = [a, b, b] = 1 \rangle$, where q is a prime number, $m \geq n \geq 1$, of order q^{m+n+1}.*

We must note that there is a misprint in the presentation of the last type of groups in Huppert's book [7; Aufgabe III.22].

Theorem 2 ([13], see also [2]). *Schmidt groups fall into the following classes:*

1. *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p - 1$ and P an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

2. *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

3. *$G = [P]Q$, where $P = \langle a \rangle$ is a normal subgroup of order p, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q dividing $p - 1$, and $a^z = a^i$, where i is the least primitive q-th root of unity modulo p.*

Here $[K]H$ denotes the semidirect product of K with H, where H acts on K.

Itô [8] considered the minimal non-p-nilpotent groups for a prime p, which turn out to be Schmidt groups.

Doerk [5] was the first author in studying the minimal non-supersoluble groups. Later, Nagrebeckiĭ [11] classified them.

Let p be a prime number. A group G is said to be *p-supersoluble* whenever G is p-soluble and all p-chief factors of G are cyclic groups of order p.

Kontorovič and Nagrebeckiĭ [9] studied the minimal non-p-supersoluble groups for a prime p with trivial Frattini subgroup. Tuccillo [15] tried to classify all minimal non-p-supersoluble groups in the soluble case, and gave results about non-soluble minimal non-p-supersoluble groups. Unfortunately, there is a gap in his paper and some groups are missing from his classification.

Example 3. The extraspecial group $N = \langle a, b\rangle$ of order 41^3 and exponent 41 has automorphisms y of order 5 and z of order 8, given by $a^y = a^{10}$, $b^y = b^{37}$, and $a^z = b^{19}$, $b^z = a^{35}$, satisfying $y^z = y^{-1}$. The semidirect product G of N by $\langle x, y\rangle$ is a minimal non-supersoluble group such that the Frattini subgroup $\Phi(N)$ of N is not a central subgroup of G. This is a minimal non-41-supersoluble group not appearing in any type of Tuccillo's result.

Example 4. The extraspecial group $N = \langle a, b\rangle$ of order 17^3 and exponent 17 has an automorphism z of order 32 given by $a^z = b$, $b^z = a^3$. The semidirect product $G = [N]\langle z\rangle$ is a minimal non-17-supersoluble group. It is clear that $[a, b]^z = [a, b]^{14}$ and so $[a, b]$ does not belong to the centre of G. This is another group missing in Tuccillo's work.

Example 5. The automorphism group of the extraspecial group of order 7^3 and exponent 7 has a subgroup isomorphic to the symmetric group Σ_3 of degree 3. The corresponding semidirect product is a minimal non-7-supersoluble group not corresponding to any case of Tuccillo's work.

Example 6. Let $E = \langle x_1, x_2\rangle$ be an extraspecial group of order 125 and exponent 5. This group has two automorphisms α and β given by $x_1^\alpha = x_2^4$, $x_2^\alpha = x_1$, $x_1^\beta = x_1^2$, and $x_2^\beta = x_2^3$ generating a quaternion group H of order 8 such that the corresponding semidirect product $[E]H$ is a minimal non-5-supersoluble group. This group is also missing in [15].

Example 7. With the same notation as in Example 6, the automorphisms β and γ defined by $x_1^\gamma = x_2$, $x_2^\gamma = x_1$ generate a dihedral group D of order 8. The corresponding semidirect product $[E]D$ is a minimal non-5-supersoluble group not appearing in [15].

By looking at these examples, we see that the classification of minimal non-p-supersoluble groups given in [15] is far from being complete. In our examples, the Frattini subgroup of the Sylow p-subgroup is not a central subgroup, contrary to the claim in [15; 1.7].

The aim of this paper is to give the complete classification of minimal non-p-supersoluble groups in the p-soluble universe. This restriction is motivated by the following result.

Proposition 8. *Let G be a minimal non-p-supersoluble group. Then either $G/\Phi(G)$ is a simple group of order divisible by p, or G is p-soluble.*

Our main theorem is the following:

Theorem 9. *The minimal p-soluble non-p-supersoluble groups for a prime p are exactly the groups of the following types:*

Type 1: *Let q be a prime number such that q divides $p-1$. Let C be a cyclic group of order p^s, with $s \geq 1$, and let M be an irreducible C-module over the field of q elements with kernel the maximal subgroup of C. Consider a group E with a normal q-subgroup F contained in the Frattini subgroup of E and E/F isomorphic to the semidirect product $[M]C$. Let N be an irreducible E-module over the field of p elements with kernel the Frattini subgroup of E. Let $G = [N]E$ be the corresponding semidirect product. In this case, $\Phi(G)_p$, the Sylow p-subgroup of $\Phi(G)$, which coincides with the Frattini subgroup of a Sylow p-subgroup of E, is a central subgroup of G and $\Phi(G)_q$, the Sylow q-subgroup of $\Phi(G)$, is equal to $\Phi(E)$, which coincides with the Frattini subgroup of a Sylow q-subgroup of E and centralises N.*

Type 2: *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

Type 3: *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

Type 4: *$G = [P]Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z \rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

Type 5: *$G = [P]Q$, where $P = \langle a_0, a_1 \rangle$ is an extraspecial group of order p^3 and exponent p, $Q = \langle z \rangle$ is cyclic of order 2^r, with 2^f the largest power of 2 dividing $p-1$ and $r > f \geq 1$. Define $a_1 = a_0^z$ and $a_1^z = a_0^i x$, where $x \in \langle [a_0, a_1] \rangle$ and i is a primitive 2^f-th root of unity modulo p.*

Type 6: *$G = [P]E$, where E is a 2-group with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a quaternion group of order 8 and P is an irreducible module for E with kernel F over the field of p elements of dimension 2, where $4 \mid p-1$.*

Type 7: *$G = [P]E$, where E is a 2-group with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a quaternion group of order 8, P is an extraspecial group of order p^3 and exponent p, where $4 \mid p-1$, and $P/\Phi(P)$ is an irreducible module for E with kernel F over the field of p elements.*

Type 8: *$G = [P]E$, where E is a q-group for a prime q with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a group $G_{II}(q, m, 1)$ of Theorem 1, P is an irreducible E-module of dimension q over the field of p elements with kernel F, and q^m divides $p - 1$.*

Type 9: *$G = [P]E$, where E is a 2-group with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a group $G_{II}(2, m, 1)$ of Theorem 1, P is an extraspecial group of order p^3 and exponent p such that $P/\Phi(P)$ is an irreducible E-module of dimension 2 over the field of p elements with kernel F, and 2^m divides $p - 1$.*

Type 10: *$G = [P]E$, where E is a q-group for a prime q with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to an extraspecial group of order q^3 and exponent q, with q odd, P is an irreducible E-module over the field of p elements with kernel F and dimension q, and q divides $p - 1$.*

Type 11: *$G = [P]MC$, where C is a cyclic subgroup of order r^{s+t}, with r a prime number and s and t integers such that $s \geq 1$ and $t \geq 0$, normalising a Sylow q-subgroup M of G, $M/\Phi(M)$ is an irreducible C-module over the field of q elements, q a prime, with kernel the subgroup D of order r^t of C, and P is an irreducible MC-module over the field of p elements, where q and r^s divide $p - 1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, coincides with $\Phi(M) \times D$ and centralises P.*

Type 12: *$G = [P]MC$, where C is a cyclic subgroup of order 2^{s+t}, with s and t integers such that $s \geq 1$ and $t \geq 0$, normalising a Sylow q-subgroup M of G, q a prime, $M/\Phi(M)$ is an irreducible C-module over the field of q elements with kernel the subgroup D of order 2^t of C, and P is an extraspecial group of order p^3 and exponent p such that $P/\Phi(P)$ is an irreducible MC-module over the field of p elements, where q and 2^s divide $p - 1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, is equal to $\Phi(M) \times D$ and centralises P.*

From Proposition 8 and Theorem 9 we deduce immediately that a minimal non-p-supersoluble group is either a Frattini extension of a non-abelian simple group of order divisible by p, or a soluble group.

As a consequence of Theorem 9, bearing in mind that minimal non-supersoluble groups are soluble by [5] and minimal non-p-supersoluble groups for a prime p, we obtain the classification of minimal non-supersoluble groups:

Theorem 10. *The minimal non-supersoluble groups are exactly the groups of Types 2 to 12 of Theorem 9, with r dividing $q-1$ in the case of groups of Type 11.*

The classification of minimal non-p-supersoluble groups can be applied to get some new criteria for supersolubility. A well-known theorem of Buckley [4] states that if a group G has odd order and all its subgroups of prime order are normal, then G is supersoluble. The following generalisation follows easily from our classification:

Theorem 11. *Let G be a group whose subgroups of prime order permute with all Sylow subgroups of G and no section of G is isomorphic to the quaternion group of order 8. Then G is supersoluble.*

As a final remark, we mention that Tuccillo [15] also gave some partial results for Frattini extensions of non-abelian simple groups of order divisible by p. Looking at the results of Section 4 of that paper, it seems that the classification of minimal non-p-supersoluble groups in the general finite universe is a hard task.

2 Preliminary results

First we gather the main properties of a minimal non-supersoluble group. They appear in Doerk's paper [5].

Theorem 12. *Let G be a minimal non-supersoluble group. We have:*

1. *G is soluble.*
2. *G has a unique normal Sylow subgroup P.*
3. *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
4. *The Frattini subgroup $\Phi(P)$ of P is supersolubly embedded in G, i. e., there exists a series $1 = N_0 \leq N_1 \leq \cdots \leq N_m = \Phi(P)$ such that N_i is a normal subgroup of G and $|N_i/N_{i-1}|$ is prime for $1 \leq i \leq m$.*
5. *$\Phi(P) \leq \mathrm{Z}(P)$; in particular, P has class at most 2.*
6. *The derived subgroup P' of P has at most exponent p, where p is the prime dividing $|P|$.*
7. *For $p > 2$, P has exponent p; for $p = 2$, P has exponent at most 4.*

8. *Let Q be a complement to P in G. Then $Q \cap \mathrm{C}_G\big(P/\Phi(P)\big) = \Phi(G) \cap \Phi(Q) = \Phi(G) \cap Q$.*

9. *If $\overline{Q} = Q/\big(Q \cap \Phi(G)\big)$, then $\overline{Q}$ is a minimal non-abelian group or a cyclic group of prime power order.*

In [6; VII, 6.18], some properties of critical groups for a saturated formation in the soluble universe are given. This result has been extended to the general finite universe by the first author and Pedraza-Aguilera. Recall that if $\mathfrak{F}$ is a formation, the $\mathfrak{F}$-residual of a group G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G such that $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$.

Lemma 13 ([3; Theorem 1 and Proposition 1])**.** *Let $\mathfrak{F}$ be a saturated formation.*

1. *Assume that G is a group such that G does not belong to $\mathfrak{F}$, but all its proper subgroups belong to $\mathfrak{F}$. Then $\mathrm{F}'(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$, where $\mathrm{F}'(G) = \mathrm{Soc}\big(G \bmod \Phi(G)\big)$, and $\mathrm{F}'(G) = G^{\mathfrak{F}}\Phi(G)$. In addition, if the derived subgroup of $G^{\mathfrak{F}}$ is a proper subgroup of $G^{\mathfrak{F}}$, then $G^{\mathfrak{F}}$ is a soluble group. Furthermore, if $G^{\mathfrak{F}}$ is soluble, then $\mathrm{F}'(G) = \mathrm{F}(G)$, the Fitting subgroup of G. Moreover $(G^{\mathfrak{F}})' = T \cap G^{\mathfrak{F}}$ for every maximal subgroup T of G such that $G/\mathrm{Core}_G(T) \notin \mathfrak{F}$ and $\mathrm{F}'(G)T = G$.*

2. *Assume that G is a group such that G does not belong to $\mathfrak{F}$ and there exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = M\,\mathrm{F}(G)$. Then $G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of G, $G^{\mathfrak{F}}$ is a p-group for some prime p, $G^{\mathfrak{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, either $G^{\mathfrak{F}}$ is elementary abelian or $(G^{\mathfrak{F}})' = \mathrm{Z}(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.*

It is clear that the class $\mathfrak{F}$ of all p-supersoluble groups for a given prime p is a saturated formation [7; VI, 8.3]. Thus Lemma 13 applies to this class.

The following series of lemmas is also needed in the proof of Theorem 9.

Lemma 14. *Let N be a non-abelian special normal p-subgroup of a group G, p a prime, such that $N/\Phi(N)$ is a minimal normal subgroup of $G/\Phi(N)$. Assume that there exists a series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = \Phi(N)$ with N_i normal in G for all i and cyclic factors N_i/N_{i-1} of order p for $1 \leq i \leq t$. Then $N/\Phi(N)$ has order p^{2m} for an integer m.*

Proof. The result holds if N is extraspecial by [6; A, 20.4]. Assume that N is not extraspecial. Let $T = N_1$ be a minimal normal subgroup of G contained in $\Phi(P)$, then T has order p. It is clear that $(N/T)' = N'/T$ and

$\Phi(N/T) = \Phi(N)/T$. Consequently $(N/T)' = \Phi(N/T)$. On the other hand, $\Phi(N/T) = \Phi(N)/T = \mathrm{Z}(N)/T \leq \mathrm{Z}(N/T)$. If $\Phi(N/T) \neq \mathrm{Z}(N/T)$, then $\mathrm{Z}(N/T) = N/T$ because $N/\Phi(N)$ is a chief factor of G, but this implies that N/T is abelian, in particular, $T = N'$ and N is extraspecial, a contradiction. Therefore G/T satisfies the hypothesis of the lemma and N/T is non-abelian. By induction, $(N/T)\big/\Phi(N/T) \cong N/\Phi(N)$ has order p^{2m}. □

Lemma 15. *Let G be a group, and let N be a normal subgroup of G contained in $\Phi(G)$. If p is a prime and G is a minimal non-p-supersoluble group, then G/N is a minimal non-p-supersoluble group.*

Conversely, if G/N is a minimal non-p-supersoluble group, $N \leq \Phi(G)$, and there exists a series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = N$ with N_i normal in G for all i and whose factors N_i/N_{i-1} are either cyclic of order p or p'-groups for $1 \leq i \leq t$, then G is a minimal non-p-supersoluble group.

Proof. Assume that G is a minimal non-p-supersoluble group and $N \leq \Phi(G)$. If M/N is a proper subgroup of G/N, then M is a proper subgroup of G. Hence M is p-supersoluble, and so is M/N. If G/N were p-supersoluble, since $N \leq \Phi(G)$, G would be p-supersoluble, a contradiction. Therefore G/N is minimal non-p-supersoluble.

Conversely, assume that G/N is a minimal non-p-supersoluble group, $N \leq \Phi(G)$, and that there exists a series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = N$ with N_i normal in G for all i and factors N_i/N_{i-1} cyclic of order p or p'-groups for $1 \leq i \leq t$. It is clear that G cannot be p-supersoluble. Let M be a maximal subgroup of G. Since $N \leq \Phi(G)$, $N \leq M$. Thus M/N is p-supersoluble. On the other hand, it is clear that every chief factor of M below N is either a p'-group or a cyclic group of order p. Consequently, M is p-supersoluble. □

Lemma 16 ([1])**.** *Let A be a group, and let B be a normal subgroup of A of prime index r dividing $p-1$, p a prime. If M is an irreducible and faithful A-module over* $\mathrm{GF}(p)$ *of dimension greater than* 1 *and the restriction of M to B is a sum of irreducible B-modules of dimension* 1, *then M has dimension r. In this case, M is isomorphic to the induced module of one of the direct summands of M_B from B up to A.*

In the rest of the paper, $\mathfrak{F}$ will denote the formation of all p-supersoluble groups, p a prime.

Lemma 17. *Let G be a minimal non-p-supersoluble group whose p-supersoluble residual $N = G^{\mathfrak{F}}$ is normal Sylow p-subgroup. Then a Hall p'-subgroup $R/\Phi(G)$ of $G/\Phi(G)$ is either cyclic of prime power order or a minimal non-abelian group.*

Proof. By Lemma 15, we can assume without loss of generality that $\Phi(G) = 1$. Then, by Lemma 13, G is a primitive group and $\mathrm{C}_G(N) = N$. In particular, for each subgroup X of G, we have that $\mathrm{O}_{p',p}(XN) = N$. Let M be a maximal subgroup of R. Then MN is a p-supersoluble group and so $MN/\mathrm{O}_{p',p}(MN) = MN/N$ is abelian of exponent dividing $p-1$. Therefore if R is non-abelian, then it is a minimal non-abelian group. Suppose that R is abelian. If R has a unique maximal subgroup, then R is cyclic of prime power order. Assume now that R has at least two different maximal subgroups. Then R is a product of two subgroups of exponent dividing $p-1$. Consequently R has exponent $p-1$ and so N is a cyclic group of order p by [6; B, 9.8], a contradiction. Therefore if R is not cyclic of prime power order, R must be a minimal non-abelian group and the lemma is proved. □

Lemma 18. *Let G be a minimal non-p-supersoluble group with a normal Sylow p-subgroup N such that $G/\Phi(N)$ is a Schmidt group. Then G is a Schmidt group.*

Proof. Let G be a minimal non-p-supersoluble group with a normal Sylow p-subgroup N such that $G/\Phi(N)$ is a Schmidt group. Then $G = NQ$, for a Hall p'-subgroup Q of G. Moreover, since G is not p-supersoluble and $G/\Phi(N)$ is a Schmidt group, we have that Q is a cyclic q-group for a prime q and q does not divide $p-1$ by Theorem 2. Let M be a maximal subgroup of G. If N is not contained in M, then a conjugate of Q is contained in M and so we can assume without loss of generality that $M = \Phi(N)Q$. Since q does not divide $p-1$ and M is p-supersoluble, we have that Q centralises all chief factors of a chief series of M passing through $\Phi(N)$. But by [6; A, 12.4], it follows that Q centralises $\Phi(N)$ by and so M is nilpotent. If N is contained in M, then M is a normal subgroup of G such that $M/\Phi(N)$ is nilpotent. By [7; III, 3.5], it follows that M is nilpotent. This completes the proof. □

3 Proof of the main theorems

Proof of Proposition 8. By Lemma 13, $G/\Phi(G)$ has a unique minimal normal subgroup $T/\Phi(G)$ and $T = G^{\mathfrak{F}}\Phi(G)$. It follows that $T/\Phi(G)$ must have order divisible by p. Assume that $T/\Phi(G)$ is a direct product of non-abelian simple groups. We note that, since $G/\Phi(G)$ is a minimal non-p-supersoluble group by Lemma 15, $T/\Phi(G) = G/\Phi(G)$ and so $G/\Phi(G)$ is a simple non-abelian group.

Assume now that $T/\Phi(G)$ is a p-group. By Lemma 13, we have that $G^{\mathfrak{F}}$ is a p-group. In this case, $T/\Phi(G)$ is complemented by a maximal subgroup

$M/\Phi(G)$ of $G/\Phi(G)$. Since M is p-supersoluble, so is $M/\Phi(G)$. Therefore $G/\Phi(G)$ is p-soluble. It follows that G is p-soluble. □

Proof of Theorem 9. Assume that G is a p-soluble minimal non-p-supersoluble group. By Lemma 13 and Proposition 8, $N = G^{\mathfrak{F}}$ is a p-group.

Assume first that N is not a Sylow subgroup of G. By Lemma 13, $N/\Phi(N)$ is non-cyclic.

Assume that $\Phi(G) = 1$. Then N is the unique minimal normal subgroup of G, which is an elementary abelian p-group, and it is complemented by a subgroup, R say. Moreover, N is self-centralising in G. This implies that $\mathrm{O}_{p',p}(G) = N = \mathrm{O}_p(G)$. Since N is not a Sylow p-subgroup of G, we have that p divides the order of R. Consider a maximal normal subgroup M of R. Observe that NM is a p-supersoluble group and $\mathrm{O}_{p',p}(NM) = \mathrm{O}_p(NM) = N$ because $\mathrm{O}_p(M)$ is contained in $\mathrm{O}_p(R) = 1$. Therefore $M \cong MN/\mathrm{O}_{p',p}(MN)$ is abelian of exponent dividing $p - 1$. It follows that M is a normal Hall p'-subgroup of R and $|R : M| = p$ because p divides $|R|$. In particular, M is the only maximal normal subgroup of R. Moreover, if C is a Sylow p-subgroup of R, then C is a cyclic group of order p.

Let M_0 be a normal subgroup of R such that M/M_0 is a chief factor of R. Let $X = NM_0C$. Since X is a proper subgroup of G, we have that X is p-supersoluble. Hence $X/\mathrm{O}_{p',p}(X)$ is an abelian group of exponent dividing $p - 1$. It follows that $C \leq \mathrm{O}_{p',p}(X)$. In particular, $C = M_0C \cap \mathrm{O}_{p',p}(X)$ is a normal subgroup of M_0C which intersects trivially M_0. We conclude that C centralises M_0. If M_1 is another normal subgroup of R such that M/M_1 is a chief factor of R, then $M = M_0M_1$. The same argument shows that C centralises M_1 and so C centralises M as well, a contradiction because in this case $C \leq \mathrm{Z}(R)$ and then $C \leq \mathrm{O}_p(R) = 1$. Consequently M_0 is the unique such normal subgroup. Since M is abelian, we have that $M_0 \leq \mathrm{Z}(R)$.

Now R has an irreducible and faithful module N over $\mathrm{GF}(p)$. By [6; B, 9.4], $\mathrm{Z}(R)$ is cyclic. In particular, M_0 is cyclic. We will prove next that $M_0 = 1$. In order to do so, assume, by way of contradiction, that M is not a minimal normal subgroup of R. First of all, if M is not a q-group for a prime q, then M is a direct product of its Sylow subgroups, but all of them should be contained in M_0, a contradiction. Therefore, M is a q-group for a prime q. Since M has exponent dividing $p - 1$, we have that q divides $p - 1$. If $\mathrm{Soc}(M)$ is a proper normal subgroup of M, then $\mathrm{Soc}(M) \leq M_0$. Since M_0 is cyclic, we have that M is an abelian group with a cyclic socle. Therefore M is cyclic. But since q divides $p - 1$, we have that C centralises M and so $C \leq \mathrm{O}_p(R) = 1$, a contradiction. Consequently $M = \mathrm{Soc}(M)$, and M is a C-module over $\mathrm{GF}(q)$. If M is not irreducible as C-module, then M can be expressed as a direct sum of proper C-modules over $\mathrm{GF}(q)$. Hence M has at least two maximal C-submodules, which yield two different chief factors M/M_1 and M/M_2 of R, a contradiction. Therefore M is a minimal normal subgroup of R, $R = MC$, and $\mathrm{C}_R(M) = M$. On the other hand, N is a faithful and irreducible R-module over $\mathrm{GF}(p)$. By Clifford's theorem [6; B, 7.3], the restriction of N to M is a direct sum of $|R : T|$ homogeneous components, where T is the inertia subgroup of one of the irreducible components of N when regarded as an M-module. Moreover, by [6; B, 8.3], we have that each of these homogeneous components N_i is irreducible. Therefore they have dimension 1 because N_iM is supersoluble for every i. Since N is not cyclic, we have that $|R : T| > 1$. Since $M \le T \le R$, we have that $M = T$ and so N has order p^p.

Assume now that $\Phi(G) \neq 1$. In this case, $\overline{G} = G/\Phi(G)$ is a minimal non-p-supersoluble group by Lemma 15 and $\Phi(\overline{G}) = 1$. We observe that $N\Phi(G)/\Phi(G)$ cannot be a Sylow p-subgroup of $G/\Phi(G)$, because otherwise NH, where H is a Hall p'-subgroup of G, would be a proper supplement to $\Phi(G)$ in G, which is impossible. In particular, if T is a normal subgroup of G contained in $\Phi(G)$, then the p-supersoluble residual NT/T of G/T is not a Sylow p-subgroup of G/T. Therefore $\overline{G}$ has the above structure. Since $N\Phi(G) = \mathrm{F}(G)$, $\mathrm{F}\bigl(G/\Phi(G)\bigr) = \mathrm{F}(G)/\Phi(G)$, and $\Phi\bigl(\mathrm{F}(G)/\Phi(G)\bigr) = 1$, we have that $\overline{N} = (\overline{G})^{\mathfrak{F}} = N\Phi(G)/\Phi(G)$ satisfies

$$\begin{aligned}\overline{N}/\Phi(\overline{N}) &= \bigl(N\Phi(G)/\Phi(G)\bigr)/\Phi\bigl(N\Phi(G)/\Phi(G)\bigr)\\ &= \bigl(\mathrm{F}(G)/\Phi(G)\bigr)/\Phi\bigl(\mathrm{F}(G)/\Phi(G)\bigr),\end{aligned}$$

which is isomorphic to $\mathrm{F}(G)/\Phi(G) = N\Phi(G)/\Phi(G)$, and the latter is G-isomorphic to $N/\bigl(N \cap \Phi(G)\bigr) = N/\Phi(N)$ by Lemma 13. Assume that $\Phi(N) \neq 1$. By Lemma 14, we have that $N/\Phi(N)$ has square order. But this order is equal to $|\overline{N}/\Phi(\overline{N})| = p^p$, which implies that $p = 2$. This contradicts the fact that q divides $p - 1$. Therefore $\Phi(N) = 1$. Now we will prove that $\Phi(G)_p$, the Sylow p-subgroup of $\Phi(G)$, is a central cyclic subgroup of G. Assume first that $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, is trivial. We have that $G/\Phi(G) = \overline{N}\,\overline{M}\,\overline{C}$, where $\overline{C}$ is a cyclic group of order p, $\overline{M}$ is an irreducible and faithful module for $\overline{C}$ over $\mathrm{GF}(q)$, q a prime dividing $p - 1$, and $\overline{N}$ is an irreducible and faithful module for $\overline{M}\,\overline{C}$ over $\mathrm{GF}(p)$ of dimension p. Let N, M, and C be, respectively, preimages of $\overline{N}$, $\overline{M}$, and $\overline{C}$ by the canonical epimorphism from G to G/T. We can assume that $N = G^{\mathfrak{F}}$ and M is a Sylow q-subgroup of G. Since $\overline{C}$ is cyclic of order p, we can find a cyclic subgroup C of G such that $\overline{C} = C\Phi(G)/\Phi(G)$. Consider now a chief factor H/K of G contained in $\Phi(G)_p$. Then $G/\mathrm{C}_G(H/K)$ is an abelian group of exponent dividing $p-1$ and H/K is centralised by a Sylow p-subgroup of G/K; in particular, $G/\mathrm{C}_G(H/K)$ is isomorphic to a factor group of a group with a unique normal subgroup of index p. It follows that

$\mathrm{C}_G(H/K) = G$, that is, H/K is a central factor of G. Now N centralises $\Phi(G)$ because $\Phi(N) = 1 = N \cap \Phi(G)$ and M is a q-group stabilising a series of $\Phi(G)$. By [6; A, 12.4], M centralises $\Phi(G)$. Moreover C normalises M because $M\Phi(G) = M \times \Phi(G)$ is normalised by C. In particular, MC is a subgroup of G. Since $G = N(MC)$ and N is a minimal normal subgroup of G, it follows that MC is a maximal subgroup of G. Hence $\Phi(G)$ is contained in MC and so in C. This implies that $\Phi(C) \leq \mathrm{Z}(G)$. In the general case, we have that $\Phi(G)/\Phi(G)_{p'} \leq \mathrm{Z}\big(G/\Phi(G)_{p'}\big)$. Then $[G, \Phi(G)_p] \leq \Phi(G)_{p'}$. Therefore $\Phi(G)_p \leq \mathrm{Z}(G)$. On the other hand, it is clear that $\Phi(G)_p$ is a proper subgroup of C. Thus $\Phi(G)_p \leq \Phi(C)$ and so $\Phi(G)_p \leq \Phi(MC)$. Now $\Phi(G)_{p'} = \Phi(G)_q$, the Sylow q-subgroup of $\Phi(G)$, is contained in M and $M/\Phi(G)_{p'}$ is elementary abelian. Hence $\Phi(M) \leq \Phi(G)_{p'}$. Moreover, by Maschke's theorem [6; A, 11.4], the elementary abelian group $M/\Phi(M)$ admits a decomposition $M/\Phi(M) = \Phi(G)_{p'}/\Phi(M) \times A/\Phi(M)$, where A is normalised by C. In this case, $R = MC = A\big(C\Phi(G)_{p'}\big)$. Since C normalises A, we have that AC is a subgroup of G. Therefore $N(AC)$ is a subgroup of G and so $G = (NAC)\Phi(G)_{p'}$. We conclude that $G = NAC$. By order considerations, we have that $M = A$ and so $\Phi(M) = \Phi(G)_{p'}$.

Now let G be a minimal non-p-supersoluble group such that N is a Sylow p-subgroup of G. Let Q be a Hall p'-subgroup of G. Then $G = NQ$. Denote with bars the images in $\overline{G} = G/\Phi(G)$. By Lemma 13, $\overline{N} = N\Phi(G)/\Phi(G)$ is a minimal normal subgroup of $\overline{G} = G/\Phi(G)$ and either N is elementary abelian, or $N' = \mathrm{Z}(N) = \Phi(N)$. Note that $\Phi(N) = \Phi(G)_p$, the Sylow p-subgroup of $\Phi(G)$, because $\Phi(N)$ is contained in $\Phi(G)_p$ and $\overline{N}$ is a chief factor of G. Assume that $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, is not contained in $\Phi(Q)$. Then there exists a maximal subgroup A of Q such that $Q = A\Phi(G)_{p'}$. In this case, $G = NQ = NA\Phi(G)_{p'}$ and so $G = NA$. It follows that $A = Q$ by order considerations, a contradiction. Therefore $\Phi(G)_{p'} \leq \Phi(Q)$. We also note that since $\overline{Q} = Q\Phi(G)/\Phi(G) \cong Q/\Phi(G)_q$, where $\Phi(G)_q$ is the Sylow q-subgroup of $\Phi(G)$, has an irreducible and faithful module $\overline{N} = N/\Phi(N)$ over $\mathrm{GF}(p)$, we have that $\mathrm{Z}(\overline{Q})$ is cyclic by [6; B, 9.4].

By Lemma 17 we have that the Hall p'-subgroup $\overline{Q}$ of $\overline{G}$ is either a cyclic group of prime power order or a minimal non-abelian group.

Suppose that $\overline{Q} = \langle \bar{z} \rangle$ is a cyclic group of order a power of a prime number, q say. Since this group is isomorphic to $Q/\Phi(G)_q$ and $\Phi(G)_q \leq \Phi(Q)$, we have that Q is a cyclic group of q-power order, $Q = \langle z \rangle$ say.

Suppose that the order of $\bar{z}$ is q^f. Then q^{f-1} divides $p-1$. If $\bar{z}^q = 1$, then $\overline{G}$ is a Schmidt group. By Lemma 18, G is a Schmidt group. By Theorem 2, G is a group of Type 2 if $\Phi(N) = 1$, or 3 if $\Phi(N) \neq 1$.

Assume now that $f \geq 2$. In this case, q divides $p - 1$ and, by Lemma 16, we have that $\overline{N}$ has order p^q. Let $a_0 \in \overline{N} \setminus 1$. Let $a_i = a_0^{z^i}$ for $1 \leq i \leq q-1$, then $a_0^{z^q} = a_0^i$, where i is a q^{f-1}-root of unity modulo p. It follows that $(a_0^{z^{q^{f-1}}}) = a_0^{i^{q^{f-2}}}$. If i is not a primitive q^{f-1}-th root of unity modulo p, we have that $i^{q^{f-2}} \equiv 1 \pmod{p}$. In particular, $a_0^{z^{q^{f-1}}} = a_0$, which contradicts the fact that the order of $\bar{z}$ is q^f. If $\Phi(N) = 1$, then we obtain a group of Type 4. If $\Phi(N) \neq 1$, then $\overline{N}$ has square order by Lemma 14 and so $q = 2$. Hence N is an extraspecial group of order p^3 and exponent 3, and G is a group of Type 5.

Assume now that Q is not cyclic. In this case, $\overline{Q}$ is a minimal non-abelian group by Lemma 17. Let x be an element of $\overline{Q}$. Since $\overline{N}\langle x\rangle$ is a p-supersoluble group, we have that the order of x divides $p-1$. It follows that the exponent of $\overline{Q}$ divides $p-1$. Since $\overline{N} = N/\Phi(N)$ is an irreducible and faithful $\overline{Q}$-module over $\mathrm{GF}(p)$ of dimension greater than 1 and the restriction of $\overline{N}$ to every maximal subgroup of $\overline{Q}$ is a sum of irreducible modules of dimension 1, we have that $\overline{N}$ has order p^q by Lemma 16.

Suppose that $\overline{Q}$ is a q-group for a prime q. By Theorem 1, either $\overline{Q} \cong Q_8$, or $\overline{Q} \cong G_{\mathrm{II}}(q,m,n)$, or $\overline{Q} \cong G_{\mathrm{III}}(q,m,n)$.

Suppose that $\overline{Q}$ is isomorphic to a quaternion group Q_8 of order 8. In this case, $q = 2$, $|\overline{N}| = p^2$ and $\exp(\overline{Q}) = 4$ divides $p-1$. If $\Phi(N) = 1$, then we have a group of Type 6. Assume that $\Phi(N) \neq 1$. In this case, N is an extraspecial group of order p^3 and exponent p and so G is a group of Type 7.

Suppose that $\overline{Q}$ is isomorphic to $G_{\mathrm{II}}(q,m,n) = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}}\rangle$, where $m \geq 2$, $n \geq 1$, of order q^{m+n}. Since $\overline{Q}$ has an irreducible and faithful module $\overline{N}$, we have that $\mathrm{Z}(\overline{Q})$ is cyclic by [6; B, 9.4]. Since $\langle a^p, b^p\rangle \leq \mathrm{Z}(\overline{Q})$ and $m \geq 2$, we have that $b^p = 1$ and so $n = 1$. Hence q^m divides $p-1$. If $\Phi(N) = 1$, then we obtain a group of Type 8. If $\Phi(N) \neq 1$, then N is non-abelian and so $|\overline{N}|$ is a square by Lemma 14. It follows that $q = 2$ and G is a group of Type 9.

Suppose now that $\overline{Q}$ is isomorphic to $G_{\mathrm{III}}(q,m,n) = \langle a, b \mid a^{q^m} = b^{q^n} = [a,b]^q = [a,b,a] = [a,b,b] = 1\rangle$, where $m \geq n \geq 1$, of order q^{m+n+1}. Since $G_{\mathrm{III}}(2,1,1) \cong G_{\mathrm{II}}(2,2,1)$, we can assume that $(q,m,n) \neq (2,1,1)$.

As before, $\mathrm{Z}(\overline{Q})$ is cyclic. Consider $\langle a^q, b^q, [a,b]\rangle$, which is contained in $\mathrm{Z}(\overline{Q})$. If $m \geq 2$, then $\langle a^q, [a,b]\rangle$ is cyclic. Since $[a,b]$ has order p, we have that $[a,b] = a^{qt}$ for a natural number t. But hence $a^b = a^{1+qt}$ and so $\langle a\rangle$ is a normal subgroup of G. Therefore $|\overline{Q}| = |\langle a,b\rangle| = |\langle a\rangle\langle b\rangle| \leq q^{m+n}$, a contradiction. Consequently $m = 1$. It follows that $\overline{Q}$ is an extraspecial group of order q^3 and exponent q. If $\Phi(N) \neq 1$, then $\overline{N}$ has square order, but this implies that $q = 2$, a contradiction. Consequently, $\Phi(N) = 1$ and we have a group of Type 10.

Assume now that $\overline{Q}$ is a minimal non-abelian group which is not a q-group for any prime q. Then $\overline{Q}$ is isomorphic to $[V_q]C_{r^s}$, where q and r are different primes numbers, s is a positive integer, and V_q is an irreducible C_{r^s}-module over the field of q elements with kernel the maximal subgroup of C_{r^s}. Since $\overline{N}V_q$ is a p-supersoluble subgroup, it follows that the restriction of $\overline{N}$ to V_q can be expressed as a direct sum of irreducible modules of dimension 1. By Lemma 16, we have that $\overline{N}$ has dimension r. We know that $\Phi(G)_{p'} \leq \Phi(Q)$ and $\Phi(G)_p = \Phi(N)$. Since $\overline{Q}$ is isomorphic to $Q/\Phi(G)_{p'}$, and this group is r-nilpotent, Q is r-nilpotent. Consequently Q has a normal Sylow q-subgroup M. On the other hand, $\Phi(G)_q$, the Sylow q-subgroup of $\Phi(G)$, is contained in M and $M/\Phi(G)_q$ is elementary abelian. This implies that $\Phi(M)$ is contained in $\Phi(G)_q$. Let C be a Sylow r-subgroup of G. Then, by Maschke's theorem [6; A, 11.4], $M/\Phi(M) = \Phi(G)_q/\Phi(M) \times A/\Phi(M)$ for a subgroup A of M normalised by C. Then $Q = (AC)\Phi(G)_q = AC$ and so $A = M$. Consequently $\Phi(M) = \Phi(G)_q$. Now the Sylow r-subgroup $\Phi(G)_r$ of $\Phi(G)$ is contained in C. If $\Phi(G)_r$ were not contained in $\Phi(C)$, there would exist a maximal subgroup T of C such that $C = T\Phi(G)_r$. This would imply $Q = MT$ and $T = C$, a contradiction. Hence $\Phi(G)_r$ is contained in $\Phi(C)$ and C is cyclic. Moreover $\Phi(G)_r$ centralises M.

If $\Phi(N) = 1$, then we have a group of Type 11. If $\Phi(N) \neq 1$, then $r = 2$ and N is an extraspecial group of order p^3 and exponent p. This is a group of Type 12.

Conversely, it is clear that the groups of Types 1 to 12 are minimal non-p-supersoluble. □

Proof of Theorem 10. It is clear that all groups of the statement of the theorem are minimal non-supersoluble. Conversely, assume that a group is minimal non-supersoluble. Hence it is soluble, and so its p-supersoluble residual is a p-group by Proposition 8. Note that groups of Type 1 in Theorem 9 are not minimal non-supersoluble. On the other hand, groups of Type 11 are not minimal non-supersoluble when r does not divide $q - 1$, because in this case the subgroup MC is not supersoluble. □

Proof of Theorem 11. Assume that the result is false. Choose for G a counterexample of least order. Since the property of the statement is inherited by subgroups, it is clear that G must be a minimal non-supersoluble group, and so a minimal non-p-supersoluble group for a prime p. In particular, the p-supersoluble residual $N = G^{\mathfrak{F}}$ of G is a p-group. Suppose that N has exponent p. The hypothesis implies that every subgroup of N is normalised by $\mathrm{O}^p(G)$. In particular, $N/\Phi(N)$ is cyclic, a contradiction. Consequently $p = 2$ and the exponent of N is 4. By Theorem 9, the only group with $\mathfrak{F}$-residual of exponent 4 is a group of Type 3. But in this case either $N/\Phi(N)$ has order 4 and N must be isomorphic to the quaternion group of order 8, because the dihedral group of order 8 does not have any automorphism of odd order, or $N/\Phi(N)$ has order greater than 4. In the last case, N has an extraspecial quotient, which has a section isomorphic to a quaternion group of order 8, final contradiction. □

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