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ON A QUESTION OF BEIDLEMAN AND ROBINSON*

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Abstract

In (1, Theorem A), Beidleman and Robinson proved that if a group satisfies the permutizer condition, it is soluble, its chief factors have order a prime number or 4 and G induces the full group of automorphisms in the chief factors of order 4. In this paper, we show that the converse of this theorem is false by showing some counterexamples. We also find some sufficient conditions for a group satisfying the converse of that theorem to satisfy the permutizer condition.

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1 INTRODUCTION

All groups considered in this paper are finite.

Given a subgroup H of a group G, the *permutizer* $P_G(H)$ of H in G is defined as the subgroup generated by all cyclic subgroups of G that permute with H. Thus $H \leq P_G(H)$ and $H \neq P_G(H)$ if and only if $H\langle x\rangle = \langle x\rangle H$ for some $x \in G \setminus H$. A group G such that $H \neq P_G(H)$ for every proper subgroup H is said to satisfy the *permutizer condition* or to be a **P**-group.

Beidleman and Robinson (1, Theorem A) proved the following result:

Theorem 1. *Let G be a finite group satisfying the permutizer condition. Then G is soluble and each chief factor of G has order 4 or a prime. In addition, if F is a chief factor of order 4, then G induces the full group of automorphisms in F, i. e., $G/C_G(F) \cong \Sigma_3$.*

In the same paper, the authors asked whether the converse is true. We show in this paper that the converse is not true and find sufficient conditions for a group satisfying the converse of that theorem to be a **P**-group.

2 A COUPLE OF EXAMPLES

In this section we present a couple of examples to show that the converse of Theorem 1 is false.

Example 1. Let V be an irreducible and faithful $\langle b\rangle$-module over the field of 2 elements, where $\langle b\rangle$ is a cyclic group of order 3, such that the corresponding semidirect product is isomorphic to A_4, the alternating group of degree 4. Let A_1 and A_2 be two copies of V, and consider $A = A_1 \times A_2$. It is clear that A is a faithful $\langle b\rangle$-module. Denote $B = [A]\langle b\rangle$ the corresponding semidirect product. Then we can choose generators c, d of A_1 and e, f of A_2 such that $c^b = cd$, $d^b = c$, $e^b = ef$ and $f^b = e$. The group B has an automorphism a of order 2 such that $b^a = b^2$, $c^a = ef$, $d^a = f$, $e^a = cd$ and $f^a = d$, so we can consider the semidirect product $G = [B]\langle a\rangle$.

The normal series

$$1 < N = \langle cef, de\rangle < A < B < G$$

is a chief series of G. Hence the chief factors of G have order 2, 3 or 4. On the other hand, $C_G(N) = A$, $C_G(A/N) = A$, $C_G(B/A) = B$ and $C_G(G/B) = G$. Consequently, G satisfies the converse of Theorem 1.

Let $H = [A_2]\langle b\rangle$. We show that $P_G(H) = H$. Suppose there exists an element $x \in G$ of order 3 such that $H\langle x\rangle$ is a subgroup of G. Then there exists a Sylow 3-subgroup H_3 of H such that $H_3\langle x\rangle$ is a Sylow 3-subgroup of $H\langle x\rangle$. Since Sylow 3-subgroups of G have order 3, it follows that $H_3 = \langle x\rangle$ and $x \in H$.

Suppose now that there exists a 2-element $y \in G$ such that $H\langle y\rangle$ is a subgroup of G. It is clear that in this case $A_2\langle y\rangle$ is a Sylow 2-subgroup of $H\langle y\rangle$. If $y \in A$, then $A_2\langle y\rangle$ is an elementary abelian 2-group normalized by b. Hence, if $A_2 \neq A_2\langle y\rangle$, there exists an element $z \in A_2\langle y\rangle$ such that $A_2\langle y\rangle = A_2 \times \langle z\rangle$ and $b \in C_G(\langle z\rangle)$, a contradiction. Therefore $A_2\langle y\rangle = A_2$ and $y \in H$. Assume that $y = y_1 a$ for some $y_1 \in A$. Notice that $y^2 \in A$ and so $A \cap A_2\langle y\rangle = A_2\langle y^2\rangle$. Thus $A_2\langle y^2\rangle \leq A$ is a normal subgroup of $A_2\langle y\rangle$. In particular, $A_1 = A_2^y = A_2^a \leq A_2\langle y^2\rangle$, a contradiction.

Finally, assume that there exists an element $g \in G$ such that $H\langle g\rangle$ is a subgroup of G. Then $\langle g\rangle = \langle g_1\rangle \times \langle g_2\rangle$, where $|\langle g_1\rangle|$ is a 2-number and $|\langle g_2\rangle| \in \{1, 3\}$. We can find a Sylow 3-subgroup H_3 of H such that $H_3\langle g_2\rangle$ is a Sylow 3-subgroup of $H\langle g\rangle$. Therefore $H_3 = \langle g_2\rangle$ and $g_2 \in H$ because Sylow 3-subgroups of G have order 3. Hence $H\langle g\rangle = H\langle g_2\rangle$ and so $g_2 \in H$ by the above case. Consequently $g \in H$ as we want to prove.

Our next example is quite surprising bearing in mind the results of the next section.

Theorem 2. *$G = \Sigma_4 \times \Sigma_4 \times \Sigma_4 \times \Sigma_4$ is not a* $\mathbf{P}$*-group.*

Proof. It is enough to find a proper subgroup H of G such that $P_G(H) = H$. Denote by G_i, $1 \leq i \leq 4$, the factors of G isomorphic to Σ_4 and by $N_i = \langle a_i, b_i\rangle$ the unique minimal normal subgroup of G_i. Let g_i be an element of order 3 in G_i such that $L_i = N_i\langle g_i\rangle$ is isomorphic to A_4.

We consider the set $H = H_1\langle g\rangle$, where

$$H_1 = \langle a_1a_3a_4, a_1^{g_1}a_3^{g_3}a_4^{g_4}, a_2a_3a_4^{g_4}, a_2^{g_2}a_3^{g_3}a_4^{g_4^2}\rangle$$

and $g = g_1g_2g_3g_4$. Since $a_i^{g_i^2} = a_ia_i^{g_i}$ for all i, it follows that $g \in N_G(H_1)$. Hence H is a subgroup of G. We prove that $P_G(H) = H$.

First of all, H is not permutable with any subgroup of order 3 which is not contained in H. Assume not and let $q \in G \setminus H$ be an element of order 3 such that $H\langle q\rangle \leq G$. Then H_1 is a Sylow 2-subgroup of $H\langle q\rangle$ which is normalized by q. Consequently, a^q, $b^q \in H_1$, where $a = a_1a_3a_4$ and $b = a_2a_3a_4^{g_4}$. Notice that $H_1 = \langle a, a^g\rangle \times \langle b, b^g\rangle$. Since the component of a^q in G_2 is trivial and the component of b^q in G_1 is also trivial, it follows that $a^q \in \langle a, a^g\rangle$ and $b^q \in \langle b, b^g\rangle$. Hence either $a^q = a$, $a^q = a^g$ or $a^q = aa^g$, and either $b^q = b$, $b^q = b^g$ or $b^q = bb^g$. If $q = q_1q_2q_3q_4$, with $q_i \in G_i$, $1 \leq i \leq 4$, then one of the following cases holds:

Case (a) $a_1^{q_1} = a_1$, $a_2^{q_2} = a_2$, $a_3^{q_3} = a_3$, $a_4^{q_4} = a_4$, $a_4^{g_4 q_4} = a_4^{g_4}$.

Case (b) $a_1^{q_1} = a_1^{g_1}$, $a_2^{q_2} = a_2^{g_2}$, $a_3^{q_3} = a_3^{g_3}$, $a_4^{q_4} = a_4^{g_4}$, $a_4^{g_4 q_4} = a_4 a_4^{g_4}$.

Case (c) $a_1^{q_1} = a_1 a_1^{g_1}$, $a_2^{q_2} = a_2 a_2^{g_2}$, $a_3^{q_3} = a_3 a_3^{g_3}$, $a_4^{q_4} = a_4 a_4^{g_4}$, $a_4^{g_4 q_4} = a_4$.

The case (a) is impossible because the Sylow 3-subgroups of G_i act fixed-point freely on N_i for all i. Suppose that (b) holds. Then $q_i g_i^{-1} \in C_{L_i}(a_i) = N_i$ for all i. Therefore $qg^{-1} \in H\langle q\rangle \cap \mathrm{Soc}(G) = H_1$ and $q \in H_1\langle q\rangle = H$, a contradiction. The case (c) is analogous.

Assume that H is permutable with a subgroup $\langle s\rangle$ contained in $\mathrm{Soc}(G)$. Then, if $s \notin H_1$, we have that $H_1 \times \langle s\rangle$ is a Sylow 2-subgroup of $H\langle s\rangle$ which is normalized by g. This means that there exists a subgroup $\langle z\rangle \leq H_1 \times \langle s\rangle$ such that $g \in C_G(\langle z\rangle)$, a contradiction. Hence $s \in H$.

If H is permutable with a subgroup $\langle x\rangle$ of order 2, with $x \in G \setminus V$, where $V = O_{\{2,3\}}(G)$, then $|H\langle x\rangle : H| = 2$ and so H must be normalized by x. In particular, $x \in N_G(H_1)$. Arguing as above, we have that $a^x \in \{a, a^g, aa^g\}$ and $b^x \in \{b, b^g, bb^g\}$. Consequently one of the following cases holds if $x = x_1x_2x_3x_4$ with $x_i \in G_i$:

Case 1 $a_1^{x_1} = a_1$, $a_2^{x_2} = a_2$, $a_3^{x_3} = a_3$, $a_4^{x_4} = a_4$, $a_4^{g_4 x_4} = a_4^{g_4}$.

Case 2 $a_1^{x_1} = a_1^{g_1}$, $a_2^{x_2} = a_2^{g_2}$, $a_3^{x_3} = a_3^{g_3}$, $a_4^{x_4} = a_4^{g_4}$, $a_4^{g_4 x_4} = a_4 a_4^{g_4}$.

Case 3 $a_1^{x_1} = a_1 a_1^{g_1}$, $a_2^{x_2} = a_2 a_2^{g_2}$, $a_3^{x_3} = a_3 a_3^{g_3}$, $a_4^{x_4} = a_4 a_4^{g_4}$, $a_4^{g_4 x_4} = a_4$.

Suppose that either Case 2 or Case 3 holds. Then the automorphism induced by x_4 on N_4 has order 3, a contradiction. Consequently, Case 1 must hold. In this case, x_4 centralizes N_4. Hence $x_4 = 1$. Since $(a_1a_3a_4)^{gx} = a_1^{g_1x_1} a_3^{g_3x_3} a_4^{g_4}$ is an element of H_1, we have that $a_1^{g_1x_1} a_3^{g_3x_3} a_4^{g_4} = a_1^{g_1} a_3^{g_3} a_4^{g_4}$. Hence x_1 centralizes N_1 and x_3 centralizes N_3. This implies that $x_1 = x_3 = 1$. The same argument applied to $(a_2a_3a_4^{g_4})^{gx}$ shows that $x_2 = 1$. Therefore $x = 1$, a contradiction.

Assume now that H is permutable with a cyclic subgroup $\langle x\rangle$ of order 4 which is not contained in H. It is clear that $x^2 \in \mathrm{Soc}(G)$. If $x^2 \notin H$, then $H \cap \langle x\rangle = 1$ and $|H\langle x\rangle : H| = 4$. Notice that $(x^2)^g$ and $(x^2)^{g^2}$ are not in H. Therefore $H\langle x\rangle$ is equal to the disjoint union $H\langle x\rangle = H \cup Hx^2 \cup H(x^2)^g \cup H(x^2)^{g^2}$. This is a contradiction, because x does not belong to this union.

Consequently, $x^2 \in H \cap \mathrm{Soc}(G)$ and $|H\langle x\rangle : H| = 2$. This implies that x normalizes H and so x also normalizes H_1.

Arguing as in the above paragraph, if $x = x_1x_2x_3x_4$, we have that either $a_4^{x_4} = a_4$ and $a_4^{g_4x_4} = a_4^{g_4}$, or $a_4^{x_4} = a_4^{g_4}$ and $a_4^{x_4g_4} = a_4a_4^{g_4}$, or $a_4^{x_4} = a_4a_4^{g_4}$ and $a_4^{g_4x_4} = a_4$. In the first case, x_4 centralizes N_4, and in the second and third cases, the automorphism induced by x_4 on N_4 is of order 3. The latter possibility gives a contradiction. Hence the first case holds and then $a_1^{x_1} = a_1$, $a_2^{x_2} = a_2$ and $a_3^{x_3} = a_3$. Moreover, $x_4 \in \mathrm{Soc}(G)$. Arguing as above with the elements $(a_1a_3a_4)^{gx}$ and $(a_2a_3a_4^{g_4})^{gx}$, it follows that $x \in \mathrm{Soc}(G)$. This is a contradiction because $\mathrm{Soc}(G)$ has no elements of order 4.

We would like to mention at this point that what is proved in the above cases is that if x is either an element of order 2 in $G \setminus V$ or x is an element of order 4, then $\langle x\rangle$ does not permute with H_1.

Finally, suppose that $\langle x\rangle = \langle x\rangle_2 \times \langle x\rangle_3$ is a subgroup of G such that $H\langle x\rangle$ is a subgroup of G. Then $H_1\langle x\rangle_2$ is a Sylow 2-subgroup of $H\langle x\rangle$. Suppose that $\langle x\rangle_2$ is not contained in H. By the above remark, $\langle x\rangle_2$ is a subgroup of order 2 contained in $\mathrm{Soc}(G)$. Therefore $H\langle x\rangle \le L_1 \times L_2 \times L_3 \times L_4$, and then $g \in N_G(H_1\langle x\rangle_2)$. This means that there exists an element $y \in H_1\langle x\rangle_2$ such that $H_1\langle x\rangle_2 = H_1 \times \langle y\rangle$ and $g \in C_G(\langle y\rangle)$, a contradiction. Thus $\langle x\rangle_2 \le H$ and $H\langle x\rangle = H\langle x\rangle_3$. Bearing in mind the first case, it follows that $\langle x\rangle_3 \le H$ and $\langle x\rangle \le H$.

Consequently, $P_G(H) = H$ and the theorem is proved. □

3 QP-GROUPS

We say that a group G is a **QP**-*group* (or G satisfies the property **QP**) if G is soluble, each chief factor of G has order 4 or a prime, and if A/B is a chief factor of G of order 4, then G induces the full group of automorphisms in A/B, i. e., $G/C_G(A/B) \cong \Sigma_3$.

Recall that in any group G, there is a unique maximum normal supersolubly embedded subgroup, denoted here by $Z_{\mathfrak{U}}(G)$. It is known that there is a G-invariant series in $Z_{\mathfrak{U}}(G)$ with cyclic factors, while $G/Z_{\mathfrak{U}}(G)$ has no nontrivial normal cyclic subgroups. $Z_{\mathfrak{U}}(G)$ is the $\mathfrak{U}$-hypercentre of G, where $\mathfrak{U}$ is the formation of all supersoluble groups (see (2, Section IV.6)).

Beidleman and Robinson proved in (1, 3.1) the following result:

Lemma 1. *A group G is a* **P**-*group if and only if $G/Z_{\mathfrak{U}}(G)$ is a* **P**-*group.*

In order to prove that a **QP**-group is a **P**-group, one often can assume that the $\mathfrak{U}$-hypercentre is trivial. Hence the following result applies.

Lemma 2. *Let G be a* **QP**-*group such that $Z_{\mathfrak{U}}(G) = 1$. Then:*

1. *$O_{2'}(G) = 1$ and G is a $\{2,3\}$-group.*

2. *$G/F(G)$ is isomorphic to a subgroup of a direct product of Σ_3 and C_2. Consequently, $G/F(G)$ is an extension of a 3-group $V/F(G)$ by a 2-group G/V, where $V = O_{\{2,3\}}(G)$.*

3. *The supersoluble normalizers of G are exactly the normalizers of the Sylow 3-subgroups of G.*

Proof. 1. Since every chief factor of G of odd order is cyclic, it follows that $O_{2'}(G)$ is supersolubly embedded in G. Hence $O_{2'}(G) \leq Z_{\mathfrak{U}}(G) = 1$. Applying (1, (3.5)), G is a $\{2,3\}$-group.

2. follows from the fact that $F(G)$ is the intersection of the centralizers of the chief factors of G.

3. Let D be a supersoluble normalizer of G (for properties of normalizers see, for example, (2, Chapter V)). Then there exists a Hall system $\Sigma = \{1, G_2, G_3, G\}$ of G, where G_2 is a Sylow 2-subgroup of G and G_3 is a Sylow 3-subgroup of G, such that D is the supersoluble normalizer associated to Σ. For each prime p, we denote $f(p)$ the formation of all abelian groups whose exponent divides $p-1$ and $v(p) = G^{f(p)}$, the $f(p)$-residual of G. It is known ((2, IV.3.4)) that f is an integrated local definition of $\mathfrak{U}$.

Then, according to (2, V.1.1),

$$D = \bigcap_{p \in \{2,3\}} N_G\big(G_{p'} \cap v(p)\big) = N_G\big(G_{2'} \cap v(2)\big) \cap N_G\big(G_{3'} \cap v(3)\big).$$

Since $f(2) = 1$, it follows that $v(2) = G$. Moreover, $v(3)$ is contained in V because G/V is an elementary abelian 2-group. On the other hand, $G_2 \cap V = F(G)$. Hence $G_{3'} \cap v(3) = G_2 \cap v(3) \leq F(G) \cap v(3) \leq G_2 \cap v(3)$ and $N_G\big(G_{3'} \cap v(3)\big) = N_G\big(F(G) \cap v(3)\big) = G$. This implies that there exists a Sylow 3-subgroup G_3 of G such that $D = N_G(G_3)$. The result now follows from the fact that the supersoluble normalizers are conjugate (see (2, V.2.3)).

□

The following results turn out to be crucial in the proof of our main theorems.

Lemma 3. *Let G be a* **QP**-*group contained in a direct product $S_1 \times \cdots \times S_r$ of r copies of Σ_4 and containing the corresponding direct product $A_1 \times \cdots \times A_r$ of r copies of A_4. Assume that H is a subgroup of G such that $P_G(H) = H$ and let H_2 and H_3 be, respectively, a Sylow 2-subgroup and a Sylow 3-subgroup of H. If N_i is the minimal normal subgroup of G contained in A_i, then:*

1. *If $N_i \cap H \neq 1$, then $A_i \leq H$.*

2. *If $N_i \cap H = 1$, then $H_2 \leq C_G(N_i)$ and $H_3 \not\leq C_G(N_i)$.*

Proof. 1. Suppose that $N_i \cap H \neq 1$ and let $1 \neq a_i \in N_i \cap H$. Since N_i is an elementary abelian 2-group of order 4, we can find an element $1 \neq b_i \in N_i$ such that $N_i = \langle a_i, b_i \rangle$. Then $H\langle b_i \rangle = HN_i$ and so $b_i \in P_G(H) = H$. Hence N_i is contained in H. Let now $\langle g_i \rangle$ be a Sylow 3-subgroup of A_i. Then $A_i = N_i\langle g_i \rangle$ and so $HA_i = H\langle g_i \rangle$. This means that $g_i \in P_G(H) = H$. Consequently A_i is contained in H.

2. Suppose that $N_i \cap H = 1$. We prove that H_2 centralizes N_i. Assume that this is not true. We distinguish two cases:

   (a) There exists an element $h \in H \setminus C_G(N_i)$ such that the component h_i of H in S_i has order 2. If h has a component h_j, $j \neq i$, of order 3, then h^3 is an element of $H \setminus C_G(N_i)$ whose component in S_i has order 2. Hence without loss of generality we may assume that $o(h) = 2$ or 4.

   Let $a_i \in N_i \setminus \{1\}$ such that $a_i^h \neq a_i$. Then ha_i is an element of G of order 4 (notice that $(ha_i)^2 = h^2 a_i^h a_i \neq 1$). In this case, $H\langle ha_i \rangle = HN_i$ by order considerations. Consequently $ha_i \in P_G(H) = H$ and so $a_i \in N_i \cap H = 1$, a contradiction.

   (b) No element of $H \setminus C_G(N_i)$ has its component in S_i of order 2. We may assume that there exists $h \in H_2 \setminus C_G(N_i)$. Suppose that H_3 does not centralize N_i and let $g \in H_3 \setminus C_G(N_i)$. We know that $o(h_i) = 4$ and so either hg or hg^2 has its component in S_i of order 2. Moreover $hg \in H \setminus C_G(N_i)$ and $hg^2 \in H \setminus C_G(N_i)$ because $h_i \notin A_i$, a contradiction.

   Therefore $H_3 \leq C_G(N_i) = S_1 \times \cdots \times N_i \times \cdots \times S_r$ and so the projection of H_3 in S_i is 1. Assume that the projection of H_2 in S_i is a Sylow 2-subgroup of S_i. Then there exists an element $x \in H \setminus C_G(N_i)$ such that $o(x_i) = 2$. This is impossible by (2a). Hence the projection of H_2 in S_i is just $\langle h_i \rangle$. In particular $\langle h_i^2 \rangle$ permutes with H. Thus $h_i^2 \in H \cap N_i = 1$, a contradiction. Consequently H_2 centralizes N_i.

If $H_3 \leq C_G(N_i)$, then $N_i \leq H$, a contradiction. Hence H_3 cannot centralize N_i. □

Lemma 4. *Let G be a* **QP***-group such that $Z_{\mathfrak{U}}(G) = 1$. Assume that G has r chief factors of order 4 in a given chief series of G. Suppose that* $\mathrm{Soc}(G) = N_1 \times \cdots \times N_r$, *where N_i is a minimal normal subgroup of G of order 4 for all i. Then the order of a Sylow 3-subgroup of G is at most 3^r.*

Moreover, if the order of a Sylow 3-subgroups of G is exactly 3^r, then G is, up to isomorphism, a subgroup of a direct product of r copies of Σ_4 containing the direct product of the corresponding r copies of the alternating group A_4.

Proof. Denote $T = \mathrm{Soc}(G) = N_1 \times \cdots \times N_r$. The hypothesis on G implies that G/T is supersoluble. Hence, if D is a supersoluble normalizer of G, we have that $G = DT$. But T is abelian, therefore $D \cap T$ is a normal subgroup of G contained in D. Hence $D \cap T \leq \mathrm{Core}_G(D) = Z_{\mathfrak{U}}(G) = 1$ and T complements D in G.

Set $F = F(G)$. It is clear that T is contained in F. Hence $F = T(F \cap D)$. Since $F \cap D \trianglelefteq D$ and $T \leq C_G(F \cap D)$, it follows that $F \cap D \trianglelefteq G$ and so $F \cap D \leq \mathrm{Core}_G(D) = 1$. This implies that $T = F$. Since F is the intersection of the centralizers of all chief factors of G, it is clear that $F \leq C_G(N_1) \cap \cdots \cap C_G(N_r) \leq C_G(F)$. The solubility of G implies that $C_G(F) \leq F$. Consequently, $F = C_G(N_1) \cap \cdots \cap C_G(N_r)$ and G/F can be embedded in a direct product of r copies of Σ_3. In particular, if G_3 is a Sylow 3-subgroup of G, we have that $|G_3| \leq 3^r$.

In what follows we assume that $|G_3| = 3^r$ and $D = N_G(G_3)$ (Lemma 2). Set $M_i = \prod_{j \neq i} N_j$. Then $G = N_i(M_i D)$ and $N_i \cap (M_i D) = 1$. Hence $M_i D$ is a maximal subgroup of G and $M_i D \cap T = M_i(D \cap T) = M_i$. Denote $T_i = \mathrm{Core}_G(M_i T)$, $1 \leq i \leq r$. Assume that $\bigcap_{i=1}^r T_i \neq 1$. Then there exists a minimal normal subgroup N of G such that $N \leq \bigcap_{i=1}^r T_i$. But $N \leq T$, therefore $N \leq T \cap T_i = M_i$ for all i, whence $N \leq \bigcap_{i=1}^r M_i = 1$, a contradiction. Thus $\bigcap_{i=1}^r T_i = 1$ and G/T_i is isomorphic to Σ_4 for all i. As a consequence, G can be embedded in a direct product of r copies of Σ_4.

On the other hand, $G/C_G(M_i) = G/\bigcap_{j \neq i} C_G(N_j)$ is isomorphic to a subgroup of the direct product of $G/C_G(N_j)$, $j \neq i$. Since G is a **QP**-group, we have that $G/C_G(N_j) \cong \Sigma_3$ for all j. Consequently $|G/C_G(M_i)|_3 \leq 3^{r-1}$ (n_3 denotes the 3-part of the number n). Since $|G_3| = 3^r$, it follows that $|C_G(M_i)|_3 \geq 3$. In particular, if $B_i = G_3 \cap C_G(M_i)$, we have that B_i is a non-trivial Sylow 3-subgroup of $C_G(M_i)$ and $|B_i| \geq 3$. Moreover, $B_1 B_2 \cdots B_r$ is a Sylow 3-subgroup of $M_1 \cdots M_r$ and $B_i \cap \left(\prod_{j \neq i} B_j\right) \leq G_3 \cap C_G(M_i) \cap C_G(N_i) \leq G_3 \cap F = 1$. Therefore $|B_i| = 3$ for all i. It is clear that B_i is not contained in $C_G(N_i)$. Therefore $N_i B_i$ is isomorphic to A_4.

Let $S_i = \bigcap_{j \neq i} T_j$. We show that if $V = O_{\{2,3\}}(G)$, then $V \cap S_i = N_i B_i$. Since $N_j \cap T_j = 1$ for all j, we have that $S_i = \bigcap_{j \neq i} T_j \leq \bigcap_{j \neq i} C_G(N_j) = C_G(M_i)$, so that there exists a Sylow 3-subgroup $(S_i)_3$ of S_i such that $(S_i)_3 \leq B_i$. If $(S_i)_3 = 1$, then S_i is a 2-group. Hence a Sylow 3-subgroup of G/S_i has order 3^r. This is not possible because G/S_i is isomorphic to a direct product of $r-1$ copies of Σ_4. Therefore $(S_i)_3 = B_i$, and hence $N_i B_i \leq S_i \cap V$. Let us prove now that $S_i \leq N_i D$. We may assume, without loss of generality, that $i = 1$. We have that $T_2 \cap \cdots \cap T_r \leq M_2D \cap M_3D \cap \cdots \cap M_rD = N_1D(N_3 \cdots N_r \cap M_3D \cap \cdots \cap M_rD)$. Since $N_3 \cdots N_r \leq T$, it follows that $(N_3 \cdots N_r) \cap M_3D \cap \cdots \cap M_rD = (N_3 \cdots N_r) \cap M_3D \cap \cdots \cap M_rD \cap T = (N_3 \cdots N_r) \cap M_3 \cap \cdots \cap M_r = 1$. Hence $S_1 \leq N_1D$ and, analogously, $S_i \leq N_iD$ for all i. This implies that $S_i \cap V \leq N_iD \cap V = N_i(D \cap V) = N_iG_3$ (notice that $D \cap V = G_3$). Consequently $S_i \cap V \leq N_iG_3 \cap C_G(M_i) = N_i\big(G_3 \cap C_G(M_i)\big) = N_iB_i$ and $S_i \cap V = N_iB_i$. Notice that $G_3 = B_1 \times \cdots \times B_r$. Hence $V = N_1B_1 \cdots N_rB_r$ by order considerations. Moreover, $N_iB_i \leq \bigcap_{j \neq i} T_j$ and $\langle N_1B_1, \ldots, N_{i-1}B_{i-1}, N_{i+1}B_{i+1}, \ldots, N_rB_r \rangle$ is contained in T_i. Therefore $N_iB_i \cap \langle N_1B_1, \ldots, N_{i-1}B_{i-1}, N_{i+1}B_{i+1}, \ldots, N_rB_r \rangle \leq T_1 \cap \cdots \cap T_r = 1$ and $V = N_1B_1 \times \cdots \times N_rB_r$. Consequently G is isomorphic to a subgroup of the direct product of r copies of Σ_4 containing the corresponding direct product of copies of A_4. □

4 SUFFICIENT CONDITIONS FOR A QP-GROUP TO BE A P-GROUP

We begin with the following elementary lemma.

Lemma 5. *Let H be a subgroup of G and let N be a normal subgroup of G contained in H. Then:*

1. *If $xN \in P_{G/N}(H/N)$, then $x \in P_G(H)$.*
2. *If $P_G(H) = H$, then $P_{G/N}(H/N) = H/N$.*

Proof. It is clear that 2 is a consequence of 1. Hence only 1 must be proved. Let $xN \in P_{G/N}(H/N)$. Then $(H/N)\langle xN \rangle = \langle xN \rangle (H/N)$. Let hx^a be an element of $H\langle x \rangle$. Then $(hN)(xN)^a \in \langle xN \rangle (H/N)$, whence $(hN)(xN)^a = (xN)^b(h'N)$ for some $h' \in H$ and there exists an element $n \in N$ such that $hx^a = x^bh'n$. Since $N \leq H$, it follows that $hx^a \in \langle x \rangle H$. Consequently $H\langle x \rangle \subseteq \langle x \rangle H$. The other inclusion is analogous. □

Our aim in this section is to find sufficient conditions for a **QP**-group to be a **P**-group in terms of the number of the chief factors of order 4.

Theorem 3. *Let G be a* **QP***-group with just one chief factor of order* 4 *in a given chief series of G. Then G is a* **P***-group.*

Proof. Let $D \neq G$ be a supersoluble normalizer of G. Then, by (2, V.3.8), D can be joined with a chain of subgroups:

$$D = D_0 <\cdot\, D_1 <\cdot \cdots <\cdot\, D_r = G$$

such that D_{i-1} is a maximal subgroup of D_i, $D_i/\operatorname{Core}_{D_i}(D_{i-1})$ is not supersoluble and $D_i = D_{i-1}F(D_i)$ for all i. Moreover, D covers all the chief factors of G of prime order and avoids the chief factors of G of order 4. Since G has only one chief factor of order 4, we conclude that D is a maximal subgroup of G. Since $G/\operatorname{Core}_G(D)$ is not supersoluble, it follows that $G/\operatorname{Core}_G(D) \cong \Sigma_4$. By (2, V.2.4), $Z_{\mathfrak{U}}(G) = \operatorname{Core}_G(D)$. Hence $G/Z_{\mathfrak{U}}(G)$ is a **P**-group. By Lemma 1, G is a **P**-group. If $D = G$, then G is supersoluble and so G is a **P**-group. □

To find sufficient conditions for a **QP**-group G with more than one chief factor in a given chief series to be **P**-group seems to be quite difficult. However we have obtained some interesting results when $\operatorname{Soc}(G)$ is a direct product of minimal normal subgroups of G of order 4.

Theorem 4. *Assume that G is a* **QP**-*group contained in a direct product $S_1 \times S_2$ of two copies of Σ_4 and containing the direct product $A_1 \times A_2$ of the copies of A_4. Then G is a* **P**-*group.*

Proof. Denote by N_i, $i = 1, 2$, the minimal normal subgroups of G contained in $A_1 \times A_2$.

Assume that G is not a **P**-group and let H be a proper subgroup of G such that $H = P_G(H)$. If $H \cap N_i \neq 1$ for some i, then $A_i \leq H$ by Lemma 3. It is clear that G/A_i is isomorphic to either Σ_4 or $\Sigma_4 \times C_2$, which are both **P**-groups. By Lemma 5, it follows that $H = G$, a contradiction. Hence $H \cap N_1 = 1 = H \cap N_2$. Bearing in mind the notation of Lemma 3, we have that $H_2 \leq C_G(N_i)$ and $H_3 \neq C_G(N_i)$ for $i = 1, 2$.

Suppose first that $H_3 = \langle g_1 g_2 \rangle$, where $g_1 \in A_1$, $g_2 \in A_2$ are elements of order 3. Assume that $H \cap \operatorname{Soc}(G) = 1$, then $H_2 = 1$ because $H_2 \leq C_G(\operatorname{Soc}(G)) = \operatorname{Soc}(G)$. Hence $H = H_3$. Let $h_i \in S_i$ be an element of order 2 such that $g_i^{h_i} = g_i^{-1}$, $i = 1, 2$. Then $h_1 h_2 \in G$ because G is a **QP**-group. It is clear that $h_1 h_2 \in N_G(H) = H$ and so $H_2 \neq 1$, a contradiction. Therefore $H \cap \operatorname{Soc}(G) \neq 1$.

Since $H_2 \leq \operatorname{Soc}(G)$, it follows that $H \cap \operatorname{Soc}(G) = H_2$. Hence $H_2 \trianglelefteq H$ and H_2 is an H_3-module over the field of 2 elements. Let $a_1 a_2 \in H_2$, with $1 \neq a_i \in N_i$, $i = 1, 2$. Then $a_1^{g_1} a_2^{g_2} \in H_2$ and $a_1 a_2 \neq a_1^{g_1} a_2^{g_2}$. Consequently the subgroup $T = \langle a_1 a_2, a_1^{g_1} a_2^{g_2} \rangle$ has order 4. Suppose that $T < H_2$. Since H_2 is a completely reducible H_3-module, there exists a normal subgroup T_0 of H such that $H_2 = T \times T_0$. Let $c_1 c_2$ be an element belonging to T_0, where $1 \neq c_i \in N_i$, $i = 1, 2$. Then $c_1^{g_1} c_2^{g_2} \in T_0$. This implies that H_2 has order 16 and so $\operatorname{Soc}(G) \leq H$, a contradiction. Therefore $H_2 = \langle a_1 a_2, a_1^{g_1} a_2^{g_2} \rangle$. Let $h_1 h_2$ be an element of $S_1 \times S_2$ such that $o(h_i) = 2$, $g_i^{h_i} = g_i^{-1}$ and $a_i^{h_i} = a_i^{g_i}$

for $i = 1, 2$. Since G is a **QP**-group, it follows that $h_1h_2 \in G$. It is clear that $h_1h_2 \in N_G(H) = H$. This is a contradiction.

Assume that $H_3 = \langle g_1, g_2 \rangle$, with $g_i \in A_i$, $i = 1, 2$, of order 3. Let $a_i a_2$ be an element of $H_2 = \mathrm{Soc}(G) \cap H$. Then $a_1^{g_1} a_2 \in H$ and so $a_1 a_1^{g_1} \in H \cap N_1 = 1$. Therefore $a_1^{g_1} = a_1$ and then $a_1 = 1$. Analogously $a_2 = 1$. Hence $H_2 = 1$. Let $h_1h_2 \in G$ be an element of order 2 such that $g_i^{h_i} = g_i^{-1}$, $i = 1, 2$. Then $h_1h_2 \in N_G(H) = H$. This contradicts the fact $H_2 = 1$.

Consequently G is a **P**-group. □

Theorem 5. *Let G be a subgroup of the direct product $\Sigma_4 \times \Sigma_4 \times \Sigma_4$ of three copies of the symmetric group of degree 4. Assume that G contains $A_4 \times A_4 \times A_4$, the corresponding direct product of the alternating groups. If G is a* **QP***-group and G contains an element of order 2 inverting all the elements of a Sylow 3-subgroup of G, then G is a* **P***-group.*

Proof. Denote by G_i, $1 \leq i \leq 3$, the factors of the direct product $\Sigma_4 \times \Sigma_4 \times \Sigma_4$. Let N_i, $1 \leq i \leq 3$, be the Klein four-group in G_i. Then N_i is a minimal normal subgroup of G for all i.

Suppose there exists a proper subgroup H of G such that $P_G(H) = H$. If $H \cap N_i \neq 1$, then A_i, the alternating group in G_i, is contained in H. Moreover, G/A_i is isomorphic to one of the groups of Theorem 4 or to a direct product of one of these groups with C_2. Then G/A_i is a **P**-group, a contradiction. Therefore $H \cap N_1 = H \cap N_2 = H \cap N_3 = 1$ and, by Lemma 3, every Sylow 2-subgroup H_2 of H centralizes N_i for all i and none Sylow 3-subgroup H_3 of H centralizes N_i for all i. Consequently every Sylow 2-subgroup H_2 of H is contained in $C_G(N_1 \times N_2 \times N_3) = N_1 \times N_2 \times N_3 = \mathrm{Soc}(G)$ and $H_2 = H \cap \mathrm{Soc}(G)$ is the unique Sylow 2-subgroup of H.

Suppose that H_3 is of order 3. Then, since $H_3 \not\leq C_G(N_i)$ for all i, it follows that $H_3 = \langle g_1g_2g_3 \rangle$, where $g_i \in G_i$ is an element of order 3 for all i. If $H_3 = H$, then we consider an element $h_i \in G_i$ of order 2 such that $g_i^{h_i} = g_i^{-1}$ for all i and $h_1h_2h_3 \in G$. Such an element exists because G has an element of order 2 inverting all the elements of a Sylow 3-subgroup of G and G contains the direct product of the alternating groups. Then $h_1h_2h_3 \in N_G(H) = H$, a contradiction. Consequently $H \cap \mathrm{Soc}(G) \neq 1$. Let $1 \neq a_1a_2a_3 \in H \cap \mathrm{Soc}(G)$, with $a_i \in N_i$, $i \in \{1, 2, 3\}$. Notice that $a_1^{g_1} a_2^{g_2} a_3^{g_3} \in H \cap \mathrm{Soc}(G)$. This implies that the rank of $H \cap \mathrm{Soc}(G)$ is a multiple of 2.

Assume that $H \cap \mathrm{Soc}(G) = \langle a_1a_2a_3, a_1^{g_1} a_2^{g_2} a_3^{g_3} \rangle$. Let $h_i \in G_i$ be an element of order 2 such that $g_i^{h_i} = g_i^{-1}$ and $a_i^{h_i} = a_i^{g_i}$ for $i \in \{1, 2, 3\}$. Then $h_1h_2h_3 \in G$ and $h_1h_2h_3 \in N_G(H_2) \cap N_G(H_3) \leq N_G(H) = H$. In particular, $h_1h_2h_3 \in H_2 \leq C_G(N_i)$ for all i, a contradiction.

Suppose now that $H \cap \mathrm{Soc}(G)$ has rank 4. Then

$$H \cap \mathrm{Soc}(G) = \langle a_1a_2a_3, a_1^{g_1} a_2^{g_2} a_3^{g_3}, c_1c_2c_3, c_1^{g_1} c_2^{g_2} c_3^{g_3} \rangle.$$

Operating with the generators of $H \cap \mathrm{Soc}(G)$, we can conclude that $H \cap \mathrm{Soc}(G) = \langle b_1b_3, b_1^{g_1}b_3^{g_3}, d_2d_3, d_2^{g_2}d_3^{g_3}\rangle$ and, by taking a suitable conjugate of d_2d_3, we can also assume that either $d_3 = b_3$ or $b_3 = 1$. Suppose that $d_3 = b_3$ (the case $b_3 = 1$ is similar). Then we can take $h_1 \in G_i$ of order 2 such that $g_1^{h_1} = g_1^{-1}$ and $b_1^{h_1} = b_1^{g_1}$, an element $h_2 \in G_2$ of order 2 such that $g_2^{h_2} = g_2^{-1}$ and $d_2^{h_2} = d_2^{g_2}$, and an element $h_3 \in G_3$ of order 2 such that $g_3^{h_3} = g_3^{-1}$ and $b_3^{h_3} = b_3^{g_3}$. Then the element $h_1h_2h_3 \in G$ and it normalizes H, a contradiction.

If $H \cap \mathrm{Soc}(G)$ has rank 6, then $\mathrm{Soc}(G) \leq H$, a contradiction.

Assume that H_3 is of order 9. We can suppose, by reordering the suffices, that either $H_3 = \langle g_1, g_2g_3\rangle$ or $H_3 = \langle g_1g_3, g_2g_3\rangle$.

Suppose that $H_3 = \langle g_1, g_2g_3\rangle$. If $1 \neq a_1a_2a_3 \in H \cap \mathrm{Soc}(G)$, with $a_i \in N_i$, $i \in \{1, 2, 3\}$, we have that $(a_1a_2a_3)^{g_1} = a_1^{g_1}a_2a_3 \in H$, so their product $a_1a_1^{g_1} \in H \cap N_1 = 1$. This implies that $a_1 = 1$. If $H \cap \mathrm{Soc}(G)$ has rank 4, it follows that $N_i \leq H$ for $i \in \{2, 3\}$, a contradiction. Hence $H \cap \mathrm{Soc}(G)$ has rank 2. Then $H \cap \mathrm{Soc}(G) = \langle a_2a_3, a_2^{g_2}a_3^{g_3}\rangle$. We consider an element $h_1 \in G_1$ of order 2 such that $g_1^{h_1} = g_1^{-1}$, an element $h_2 \in G_2$ of order 2 such that $g_2^{h_2} = g_2^{-1}$ and $a_2^{h_2} = a_2^{g_2}$, an an element $h_3 \in G_3$ of order 2 such that $g_3^{h_3} = g_3^{-1}$ and $a_3^{h_3} = a_3^{g_3}$. Then $h_1h_2h_3 \in G$ and $h_1h_2h_3 \in N_G(H) = H$, a contradiction.

Suppose that $H_3 = \langle g_1g_3, g_2g_3\rangle$. If $1 \neq a_1a_2a_3 \in H \cap \mathrm{Soc}(G)$, with $a_i \in N_i$, $i \in \{1, 2, 3\}$, we have that $(a_1a_2a_3)^{g_1g_3} = a_1^{g_1}a_2a_3^{g_3} \in H$. Thus $(a_1a_2a_3)(a_1^{g_1}a_2a_3^{g_3}) = a_1a_1^{g_1}a_3a_3^{g_3} \in H$. Then $a_1a_3 \in H$. Therefore $(a_1a_3)^{g_2g_3} = a_1a_3^{g_3} \in H$. This implies that $a_3a_3^{g_3} \in H \cap N_3 = 1$, whence $a_3 = 1$ and $a_1 \in H \cap N_1 = 1$, a contradiction.

If $H_3 = \langle g_1, g_2, g_3\rangle$ and $1 \neq a_1a_2a_3 \in H \cap \mathrm{Soc}(G)$, with $a_i \in N_i$, $1 \leq i \leq 3$, then $(a_1a_2a_3)^{g_1} = a_1^{g_1}a_2a_3 \in H$. Hence $a_1a_1^{g_1} \in H \cap N_1 = 1$. This implies $a_1 = 1$. Analogously $a_2 = 1 = a_3$. This is a contradiction.

Therefore G is a **P**-group. □

If G is a **QP**-group such that $A_4 \times A_4 \times A_4 \leq G \leq \Sigma_4 \times \Sigma_4 \times \Sigma_4$, then G is not a **P**-group in general, as the next example shows.

Example 2. The direct product $\Sigma_4 \times \Sigma_4 \times \Sigma_4$ can be considered as a subgroup of Σ_{12} and, so viewed, we consider the group G of all even permutations of $\Sigma_4 \times \Sigma_4 \times \Sigma_4$. Let $a = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$ and $g = (1, 2, 3)(5, 6, 7)(9, 10, 11)$. Then $H = \langle a, g\rangle$ is a subgroup of G isomorphic to A_4. With similar arguments to those used in the proof of Theorem 2 we have that $P_G(H) = H$. Therefore G is not a **P**-group.

Combining Lemma 4 and Theorems 4 and 5 we have:

Theorem 6. *Let G be a* **QP***-group such that $Z_{\mathfrak{U}}(G) = 1$. Suppose that G has a chief series with exactly r chief factors of order 4 and* $\mathrm{Soc}(G) = N_1 \times \cdots \times N_r$*, where N_i is a minimal normal subgroup of G of order 4. If* *either $r = 2$ and Sylow 3-subgroups of G have order 9 or $r = 3$, Sylow 3-subgroups of G have order 3^3 and there exists an element of G of order 2 inverting the elements of a Sylow 3-subgroup of G, then G is a $\mathbf{P}$-group.*

Theorem 7. *Suppose that G is a subgroup of $\Sigma_4 \times \Sigma_4$ containing the product $N_1 \times N_2$ of both minimal normal subgroups of $\Sigma_4 \times \Sigma_4$, and that G is a $\mathbf{P}$-group with $Z_{\mathfrak{U}}(G) = 1$. Then $|G|_3 = 9$.*

Proof. Let S_1, S_2 be the copies of Σ_4 in $\Sigma_4 \times \Sigma_4$ and A_1, A_2 be the corresponding alternating subgroups.

If $|G|_3 = 1$, then G is a 2-group, whence $Z_{\mathfrak{U}}(G) = G$, a contradiction.

If $|G|_3 = 3$, the projections of a Sylow 3-subgroup G_3 of G in S_i are nontrivial (otherwise $Z_{\mathfrak{U}}(G) \neq 1$). Therefore there exist $g_i \in S_i$, $i = 1, 2$, such that $g_1 g_2 \in G$. There exist $h_i \in S_i \setminus A_i$, $i = 1, 2$, such that $g_i^{h_i} = g_i^{-1}$ and $h_1 h_2 \in G$. If $h_1 \in G$, then $(g_1 g_2)^{h_1} = g_1^{-1} g_2 \in G$, whence $g_1 \in G$ and $g_2 \in G$, a contradiction. Hence $h_1 \notin G$ and, analogously, $h_2 \notin G$. There exists $a_i \in N_i \setminus \{1\}$ such that $a_i^{h_i} = a_i^{g_i}$, $i = 1, 2$. We construct $H = \langle a_1 a_2^{g_2}, a_1^{g_1} a_2^{g_2^2}, g_1 g_2 \rangle$. By arguing like in Example 1, we obtain that H is a proper subgroup of G such that $H = P_G(H)$, a contradiction. □

Theorem 8. *Suppose that G is a subgroup of $\Sigma_4 \times \Sigma_4 \times \Sigma_4$ containing the direct product of the three minimal normal subgroups of $\Sigma_4 \times \Sigma_4 \times \Sigma_4$ and that G is a $\mathbf{P}$-group with $Z_{\mathfrak{U}}(G) = 1$. Then $|G|_3 = 27$.*

Proof. Let S_i, $i \in \{1, 2, 3\}$, be the copies of Σ_4 in $\Sigma_4 \times \Sigma_4 \times \Sigma_4$ and let A_i be the copies of the corresponding alternating subgroups.

If $|G|_3 = 1$, then G is a 2-group, a contradiction with $Z_{\mathfrak{U}}(G) = 1$.

If $|G|_3 = 3$, then the projections of a Sylow 3-subgroup G_3 of G in S_i are nontrivial (otherwise, $Z_{\mathfrak{U}}(G) \neq 1$). Hence there exist $g_i \in S_i$ of order 3, $i \in \{1, 2, 3\}$, such that $g_1 g_2 g_3 \in G$. There exist $h_i \in S_i \setminus A_i$ of order 2 such that $g_i^{h_i} = g_i^{-1}$, $i \in \{1, 2, 3\}$, and $h_1 h_2 h_3 \in G$. If $h_1 \in G$, then $(g_1 g_2 g_3)^{h_1} = g_1^{-1} g_2 g_3 \in G$, whence $g_1 \in G$, a contradiction. Thus $h_1 \notin G$ and, analogously, $h_2 \notin G$, $h_3 \notin G$. Consequently $G = [N_1 \times N_2 \times N_3]\langle g_1 g_2 g_3, h_1 h_2 h_3 \rangle$. Let $a_i \in N_i \setminus \{1\}$ such that $a_i^{h_i} = a_i^{g_i}$ for $i \in \{1, 2, 3\}$. The subgroup

$$H = \langle a_1 a_2 a_3^{g_3}, a_1^{g_1} a_2^{g_2} a_3^{g_3^2}, g_1 g_2 g_3 \rangle$$

is a proper subgroup of G such that $P_G(H) = H$, as we can prove like in Example 1, a contradiction.

If $|G|_3 = 9$, then the projections of a Sylow 3-subgroup G_3 of G in S_i are again nontrivial (otherwise, $Z_{\mathfrak{U}}(G) \neq 1$). By reordering the suffices, we can suppose that either $G_3 = \langle g_1 g_2, g_3 \rangle$ or $G_3 = \langle g_1 g_2, g_1 g_3 \rangle$.

Suppose that $G_3 = \langle g_1g_2, g_3\rangle$. There exist elements $h_i \in S_i \setminus A_i$, $i \in \{1,2,3\}$, of order 2, such that $g_i^{h_i} = g_i^{-1}$ and $h_1h_2h_3 \in G$. If $h_1 \in G$, then $(g_1g_2)^{h_1} = g_1^{-1}g_2$, whence $g_2 \in G$, a contradiction. Consequently $h_1 \notin G$ and, analogously, $h_2 \notin G$. Let $a_i \in N_i \setminus \{1\}$ such that $a_i^{h_i} = a_i^{g_i}$, $i \in \{1,2,3\}$. If $h_3 \in G$, then we consider the subgroup

$$H = \langle a_1a_2^{g_2}, a_1^{g_1}a_2^{g_2^2}, a_3, a_3^{g_3}, g_1g_2, g_3, h_3\rangle,$$

and if $h_3 \notin G$, we take the subgroup

$$H = \langle a_1a_2^{g_2}, a_1^{g_1}a_2^{g_2^2}, a_3, a_3^{g_3}, g_1g_2, g_3\rangle.$$

Like in Example 1 we have that H is a proper subgroup of G such that $P_G(H) = H$.

Suppose that $G_3 = \langle g_1g_2, g_1g_3\rangle$. There exist elements $h_i \in S_i \setminus A_i$, $i \in \{1,2,3\}$, of order 2, such that $g_i^{h_i} = g_i^{-1}$ and $h_1h_2h_3 \in G$. If $h_1 \in G$, then $(g_1g_2)^{h_1} = g_1^{-1}g_2 \in G$, whence $g_1 \in G$, a contradiction. Consequently, $h_1 \notin G$. Analogously, $h_2 \notin G$ and $h_3 \notin G$. Consider an element $a_i \in S_i \setminus A_i$ such that $a_i^{h_i} = a_i^{g_i}$ for $1 \le i \le 3$. Notice that $(g_1g_2)^2(g_1g_3)^2 = g_1g_2^2g_3^2$. The subgroup $H = \langle a_1a_2a_3, a_1^{g_1}a_2^{g_2^2}a_3^{g_3^2}, g_1g_2^2g_3^2\rangle$ is a proper subgroup of G such that $P_G(H) = H$. □

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