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Comment on the Lepton Mixing Matrix

by

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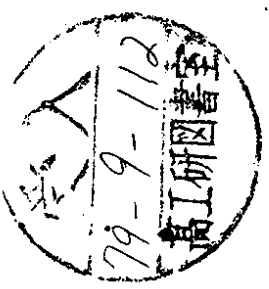
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Abstract

The problem of finding a suitable parametrization for a nontrivial lepton mixing matrix is considered in the framework of the "sequential" Weinberg-Salam theory. It is noted that in the case of  $n$  lepton generations there are in fact  $n+1$  different theories, corresponding to different numbers  $m$  of right handed neutrino fields present in the Lagrangian. These  $(n,m)$  models conserve CP for leptonic couplings naturally only for  $m=0$  or 1 when  $n \geq 3$ .

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At present, experiments on weak interactions are consistent with a theory in which there are  $n$  ( $n \geq 3$ ) "generations" of leptons and quarks, each transforming identically under the gauge group  $SU(2)_L \times U(1)$ . In this theory the "physical" leptons and quarks of each charge and helicity are related to the original fields in the Lagrangian by certain  $n$  dimensional unitary matrices. These in turn determine the generalized Cabibbo or Kobayashi-Maskawa mixing matrices which are of fundamental importance for understanding the experimentally observed pattern of weak decays. Recently, [2] progress has been made in obtaining the numerical values of the four parameters of the  $n=3$  hadron mixing matrix. The  $n=3$  lepton mixing matrix has also been recently investigated. [3] Here one must of course assume that right handed neutrino fields are present so that neutrinos may acquire mass. Otherwise the lepton mixing matrix becomes the unit matrix. Thus the analysis was carried out by analogy to the  $n=3$  hadron case, implicitly introducing a four parameter matrix. There are however other more restrictive logical possibilities for the lepton case which may (eventually) turn out to be interesting. This situation arises because, in contrast to the hadron case where all quarks are expected to have right handed components, it is not yet clear whether any or all of the neutrinos have right handed components (i.e. mass). Thus for an  $n$  generation model there is the possibility that only  $m$  right handed

neutrino fields exist. We would like to stress that each of these  $(n,m)$  models is dynamically different since the original Lagrangians for different  $m$  have different numbers of "degrees of freedom". (Of course for some purposes the  $(n,m)$  theory can be regarded as a special case of the  $(n,m)$  theory). We will now give a brief discussion of this class of intermediate models. Since the mixing matrix for  $(n,m)$  where  $m < n$  generally contains fewer parameters than the  $(n,n)$  matrix, it would appear logical to analyze experiments first with  $m=1$ , then if this turns out to be inconsistent to go to  $m=2$ , etc.

In the  $(n,m)$  model the "unphysical" lepton fields are as follows:

$$\begin{pmatrix} N_1 \\ \vdots \\ N_n \\ E_1 \\ \vdots \\ E_n \end{pmatrix}_L ; E_1 R \dots E_n R ; N_{n-m+1} R \dots N_n R \quad (1)$$

and the charged intermediate boson part of the Lagrangian density is  $\mathcal{L} = ig_2^{-1}/2 W_{\mu\nu}^\dagger F_{\mu\nu}^\dagger N_{\mu a}^\dagger h.c.$  in a standard notation. Diagonalizing the mass matrices of the theory gives the physical left handed neutrino fields  $\nu_{aL}$  and the physical left handed electron fields  $e_{aL}$  as  $N_{aL} = \sum_b \Omega_{ab} \nu_{bL}$  and  $E_{aL} = \sum_b \Omega_{ab} e_{bL}$ . The mixing matrix is then  $U = \Omega^{-1}$  and the Lagrangian term is in matrix notation

$$\mathcal{L} = ig_2^{-1}/2 W_{\mu\nu}^\dagger U_{\mu L}^\dagger \nu_L + h.c. \quad (2)$$

The matrix  $U$  is arbitrary and can be chosen to satisfy  $\det U = 1$  if an overall electron-neutrino relative phase is appropriately chosen. We parameterize  $U$  following the idea of Maiani [4] in

the  $n=3$  case but we use instead Okubos'  $U(n)$  tensor notation [5] to permit immediate generalization to  $n \neq 3$ . We introduce the  $n^2$  generators  $A_a^b$  whose  $ij$  matrix elements are

$$(A_a^b)_{ij} = \delta_{bi} \delta_{aj} \quad (3)$$

These satisfy the commutation relations  $[A_a^b, A_c^d] = \delta_a^d A_c^b - \delta_a^c A_b^d$ . The following matrices are convenient

$$\omega(\eta_{ab}) \equiv \exp(\eta_{ab} A_a^b - \eta_{ab} A_b^a), \quad (a \neq b) \quad (4)$$

$$\omega_0(\alpha) \equiv \exp i \sum_{a=1}^n \alpha_a A_a^a, \quad (5)$$

with  $\alpha_a = \alpha_a^*$  and  $\sum_a \alpha_a = 0$ . For example with  $\eta_{ab} = |\eta_{ab}| e^{i\theta_{ab}}$  we have

$$\omega(\eta_{12}) = \begin{pmatrix} \cos|\eta_{12}| & \sin|\eta_{12}| e^{i\theta_{12}} & 0 \\ -\sin|\eta_{12}| e^{-i\theta_{12}} & \cos|\eta_{12}| & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ etc.} \quad (6)$$

The following identity is useful

$$\omega_0(\alpha) \omega(|\eta_{ab}| e^{i\theta_{ab}}) \omega_0^\dagger(\alpha) = \omega(|\eta_{ab}| e^{i(\alpha_a + \theta_{ab} - \alpha_b)}) \quad (7)$$

Now we choose to write  $U$ , which in general has  $n^2-1$  real parameters, as

$$U = \omega_0(\gamma) \tilde{U} V \quad (8)$$

$$V = \prod_{a < b \leq n-m} \omega(\eta_{ab}) \quad (8a)$$

$$\tilde{U} = \prod_{a < b} \omega(\eta_{ab}), \quad (8b)$$

where the prime in (8b) means that the  $\eta_{ab}$ 's in  $V$  have been deleted.  $\omega_0(\gamma)$  contains  $n-1$  real parameters, and  $V$  contains  $(n-m)(n-m-1)$  real parameters. In (8) definite but unspecified orders for the factors in  $V$  and  $\tilde{U}$  are assumed. In general we have the freedom to redefine the phases of the physical fields according to  $v_{L,R} = \omega_0(\alpha) v_{L,R}'$ ,  $e_{L,R} = \omega_0(\beta) e_{L,R}'$  without changing any other terms in the complete Lagrangian. Now because there must be  $(n-m)$  massless physical neutrino fields in the theory we have the additional freedom to transform  $v_{L,R} = T v_{L,R}'$  where  $T$  is an appropriate  $SU(n-m)$  transformation without changing the rest of  $\mathcal{L}$ . Thus we redefine the physical fields

$$\begin{aligned} v_{L,R} &= v_{L,R}' \omega_0^\dagger(\alpha) v_{L,R}' \\ e_{L,R} &= \omega_0^\dagger(\alpha - \gamma) e_{L,R}' \end{aligned} \quad (9)$$

Putting (8) and (9) into (2) and using the identity (7) gives

$$\begin{aligned} \mathcal{L} &= ig 2^{-1/2} W_\mu^- e_{L,R}^\dagger \gamma_\mu \omega_0^\dagger(\gamma) U V^\dagger \omega_0^\dagger(\alpha) v_{L,R}' + \text{h.c.} \\ &= ig 2^{-1/2} W_\mu^- e_{L,R}^\dagger \gamma_\mu \omega_0^\dagger(|\eta_{ab}| e^{i(\alpha_a + \theta_{ab} - \alpha_b)}) v_{L,R}' + \text{h.c.}, \end{aligned} \quad (10)$$

where the  $\alpha_a$  (satisfying  $\sum_a \alpha_a = 0$ ) are at our disposal. We may evidently choose then to eliminate any of the  $\theta_{ab}$ 's. Eq. (10) is then the final result. The total number of angles  $\theta_{ab}$  in  $\tilde{U}$  is  $\frac{1}{2}[n(n-1) - (n-m)(n-m-1)]$  and the total number of  $|\eta_{ab}|$ 's is the same. Since  $(n-1)$  of the  $\theta_{ab}$ 's can be eliminated, the net number  $p(n,m)$  of phase-angles for the  $(n,m)$  theory, is [6]

$$p(n,m) = 1 + n(m-1) - \frac{m(m+1)}{2}, \quad (m \neq 0) \quad (11)$$

Noting that the condition for CP invariance can be taken to be  $U_{ab} = U_{ab}^*$  we see that  $p(n,m)$  gives the number of CP violating phases in the mixing matrix. The total number of real parameters in the mixing matrix is [6]

$$r(n,m) = 1 + n(2m-1) - m(m+1), \quad (m \neq 0). \quad (12)$$

From (11) we see that

$$p(n,1) = 0, \quad (\text{all } n) \quad (13)$$

but that  $p(n,m)$  with  $m \geq 2$  and  $n \geq 3$  is always positive. Thus the only ways to naturally avoid CP violation in the leptonic mixing matrix are to have either no massive neutrino or just one massive neutrino.

As an example let us consider the usual case of  $n=3$ . We note that  $p(3,3)=1$  and  $r(3,3)=4$  in agreement with the original Kobayashi Maskawa [7] result for massive quark fields. Since  $p(3,2)=p(3,3)$  and  $r(3,2)=r(3,3)$  [in fact  $p(n,n)=p(n,n-1)$  and  $r(n,n)=r(n,n-1)$  for all  $n$ ] there is no special simplification gotten by looking at the (3,2) theory as compared to the (3,3) theory. On the other hand the (3,1) case is considerably simpler since (13) shows that there is no CP violation and (12) shows that there are only two real parameters. [8] Using (10) and (6) we may write the mixing matrix in this case as, for example

$$\tilde{U} = \omega(|\eta_{23}|) \omega(|\eta_{13}|) \begin{matrix} & \begin{matrix} c_1 & 0 & s_1' \\ s_1 s_1' & c & s c_1' \\ -c s_1' & -s & c c_1' \end{matrix} \\ \omega(|\eta_{13}|) & \end{matrix}, \quad (14)$$

where  $s = \sin|\eta_{23}|$ ,  $c = \cos|\eta_{13}|$ , etc. The relevant experimental information about the lepton mixing matrix comes both from laboratory experiments and a variety of interesting cosmological observations. The most recent investigation has been made by Kolb and Goldman.<sup>5</sup> We would like to point out that their analysis may be significantly sharpened by assuming the (3,1) model to hold since a four parameter fit to the data is reduced to a two parameter fit. The actual results for  $s$  and  $s_1'$  depend very crucially on the assumed value<sup>9</sup> of the  $\tau$  neutrino mass. Cosmological arguments seem<sup>3</sup> in fact to exclude certain mass ranges. To get a rough idea of the bounds on our parameters we may compare eq(14) above with eq(11) of ref. 3. This would give  $|s| \lesssim .01$  and  $|s_1'| \lesssim .10$ . However a separate detailed analysis of the (3,1) model is really called for.

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#### Footnotes and References

1. For generality, we assume that the Yukawa terms in the Lagrangian need not be restricted by any symmetries beyond  $SU(2)_X \times U(1)$ .
  2. V. Barger, W. Long, and S. Pakvasa, Phys. Rev. Letters 42, 1585 (1979); R. Schrock, S. Treiman and L. Wang, Phys. Rev. Letters 42, 1589 (1979).
  3. E. Kolb and T. Goldman, Cal. Tech. preprint 68-700 (1979). See also K. Sato and M. Kobayashi, Prog. Theor. Phys., 58, 1775 (1977) and B. Lee and R. Schrock, Phys. Rev. D16, 1444 (1977).
  4. L. Maiani, Phys. Lett. 62B, 183 (1976). Maiani mentions the possibility of a mass degeneracy for quarks which would reduce the number of parameters in the mixing matrix. For quarks such a degeneracy would probably be accidental whereas in the present lepton case a zero mass degeneracy is guaranteed by our original choice of dynamical variables.
  5. S. Okubo, Prog. Theor. Phys. 28, 24 (1962).
  6. Note that  $p(n,0) = r(n,0) = (n-1)$ . This corresponds to the  $(n-1)$  phases eliminated by  $\omega_0(\alpha)$ . These are trivial in the  $m=0$  case (but not in the others) so should not be counted. For  $m=0$ ,  $U$  is simply 1.
  7. M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).
  8. This statement disagrees with Lee and Schrock, ref. 3 who also consider the  $(3,1)$  model. See also A. Davidson, M. Koca and K.C. Wali, Syracuse preprint SU-4213-137 (1979).
9. Our original motivation in looking at this question arose from a recently proposed test for neutrino masses. See T. Kalogeropoulos, J. Schechter, and J.N.F. Valle, Phys. Letters, to be published.