Some subgroup embeddings in finite groups

A. Ballester-Bolinches* J. C. Beidleman† R. Esteban-Romero‡ M. F. Ragland§

Abstract

In this survey paper several subgroup embedding properties related to some types of permutability are introduced and studied.

2010 Mathematics subject classification: 20D05, 20D10, 20F16

Keywords and phrases: finite group, permutability, S-permutability, semipermutability, primitive subgroup, quasipermutable subgroup.

1 Introduction

All groups in the paper are finite.

The purpose of this survey paper is to show how the embedding of certain types of subgroups of a finite group G can determine the structure of G. The types of subgroup embedding properties we consider include: S-permutability, S-semipermutability, semipermutability, primitivity, and quasipermutability.

A subgroup H of a group G is said to *permute* with a subgroup K of G if HK is a subgroup of G. H is said to be *permutable* in G if H permutes with all subgroups of G. A less restrictive subgroup embedding property is the S-permutability introduced by Kegel and defined in the following way:

*Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain, email: Adolfo.Ballester@uv.es

†Department of Mathematics, University of Kentucky, Lexington KY 40506-0027, USA, email: clark@ms.uky.edu

‡Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, 46022 València, Spain, email: resteban@mat.upv.es. Current address: Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain, email: Ramon.Esteban@uv.es

§Department of Mathematics, Auburn University at Montgomery, P.O. Box 244023, Montgomery, AL 36124-4023, USA, email: mragland@aum.edu

Definition 1. A subgroup H of G is said to be *S-permutable* in G if H permutes with every Sylow p-subgroup of G for every prime p.

In recent years there has been widespread interest in the transitivity of normality, permutability and S-permutability.

Definition 2. 1. A group G is a *T-group* if normality is a transitive relation in G, that is, if every subnormal subgroup of G is normal in G.

2. A group G is a *PT-group* if permutability is a transitive relation in G, that is, if H is permutable in K and K is permutable in G, then H is permutable in G.

3. A group G is a *PST-group* if S-permutability is a transitive relation in G, that is, if H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G.

If H is S-permutable in G, it is known that H must be subnormal in G ([1, Theorem 1.2.14(3)]). Therefore, a group G is a PST-group (respectively a PT-group) if and only if every subnormal subgroup is S-permutable (respectively permutable) in G.

Note that T implies PT and PT implies PST. On the other hand, PT does not imply T (non-Dedekind modular p-groups) and PST does not imply PT (non-modular p-groups). The reader is referred to [1, Chapter 2] for basic results about these classes of groups. Other characterisations based on subgroup embedding properties can be found in [2].

Agrawal ([1, 2.1.8]) characterised soluble PST-groups. He proved that a soluble group G is a PST-group if and only if the nilpotent residual in G is an abelian Hall subgroup of G on which G acts by conjugation as power automorphisms. In particular, the class of soluble PST-groups is subgroup-closed.

Let G be a soluble PST-group with nilpotent residual L. Then G is a PT-group (respectively T-group) if and only if G/L is a modular (respectively Dedekind) group ([1, 2.1.11]).

Definition 3 ([3])**.** A subgroup H of a group G is said to be *semipermutable* (respectively, *S-semipermutable*) provided that it permutes with every subgroup (respectively, Sylow subgroup) K of G such that $\gcd(|H|, |K|) = 1$.

An S-semipermutable subgroup of a group need not be subnormal. For example, a Sylow 2-subgroup of the nonabelian group of order 6 is semipermutable and S-semipermutable, but not subnormal.

Definition 4 (see [4])**.** A group G is called a *BT-group* if semipermutability is a transitive relation in G.

L. Wang, Y. Li, and Y. Wang proved the following theorem which showed that soluble BT-groups are a subclass of PST-groups:

Theorem 5 ([4])**.** *Let G be a group with nilpotent residual L. The following statements are equivalent:*

1. *G is a soluble BT-group;*
2. *every subgroup of G of prime power order is S-semipermutable;*
3. *every subgroup of G of prime power order is semipermutable;*
4. *every subgroup of G is semipermutable;*
5. *G is a soluble PST-group and if p and q are distinct primes not dividing the order of L with G_p a Sylow p-subgroup of G and G_q a Sylow q-subgroup of G, then $[G_p, G_q] = 1$.*

Research papers on BT-groups include [4, 5, 6, 7].

We next present an example of a soluble PST-group which is not a BT-group.

Example 6. Let L be a cyclic group of order 7 and $A = C_3 \times C_2$ be the automorphism group of L. Here C_3 (respectively, C_2) is the cyclic group of order 3 (respectively, 2). Let $G = [L]A$ be the semidirect product of L by A. Let $L = \langle x \rangle$, $C_3 = \langle y \rangle$ and $C_2 = \langle z \rangle$ and note that $[\langle y \rangle^x, \langle z \rangle] \neq 1$. Now G is a PST-group by Agrawal's theorem, but G is not a BT-group by Theorem 5.

A subclass of the class of soluble BT-groups is the class of soluble SST-groups, which has been introduced in [8].

Definition 7 (see [9])**.** A subgroup H of a group G is said to be *SS-permutable* (or *SS-quasinormal*) in G if H has a supplement K in G such that H permutes with every Sylow subgroup of K.

Definition 8 (see [8])**.** We say that a group G is an *SST-group* if SS-permutability is a transitive relation.

SS-permutability can be used to obtain a characterisation of soluble PST-groups.

Theorem 9 ([8])**.** *Let G be a group. Then the following statements are equivalent:*

1. *G is soluble and every subnormal subgroup of G is SS-permutable in G.*

2. *G is a soluble PST-group.*

Theorem 10 ([8])**.** *A soluble SST-group G is a BT-group.*

The following example shows that a soluble BT-group is not necessarily an SST-group.

Example 11 ([8])**.** Let $G = \langle x, y \mid x^5 = y^4 = 1, x^y = x^2 \rangle$. The nilpotent residual of G is the Sylow 5-subgroup $\langle x \rangle$. By Theorem 5, G is a soluble BT-group. Let $H = \langle y \rangle$ and $M = \langle y^2 \rangle$. Suppose that M is SS-permutable in G. Then G is the unique supplement of M in G. It follows that M is S-permutable in G, and thus $M \leq \mathrm{O}_2(G)$. This implies that either $\mathrm{O}_2(G) = H$ or $\mathrm{O}_2(G) = M$. Since $y^x = yx^{-1}$ and $(y^2)^x = y^2x^2$, neither H nor M are normal subgroups of G. This contradiction shows that M is not SS-permutable in G. Since M is SS-permutable in $\langle x, y^2 \rangle$ and this subgroup is SS-permutable in G, we obtain that the soluble group G cannot be an SST-group.

A less restrictive class of groups is the class of T_0-groups which has been studied in [5, 7, 10, 11, 12].

Definition 12. A group G is called a *T_0-group* if the Frattini factor group $G/\Phi(G)$ is a T-group.

Theorem 13 ([11])**.** *Let L be the nilpotent residual of the soluble T_0-group. Then:*

1. *G is supersoluble;*

2. *L is a nilpotent Hall subgroup of G.*

Theorem 14 ([10])**.** *Let G be a soluble T_0-group. If all the subgroups of G are T_0-groups, then G is a PST-group.*

A group G is called an *MS-group* if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable.

Theorem 15 ([13])**.** *If G is an MS-group, then G is supersoluble.*

Theorem 16 ([7])**.** *Let L be the nilpotent residual of an MS-group G. Then:*

1. *L is a nilpotent Hall subgroup of G;*

2. *G is a soluble T_0-group.*

We now provide three examples which illustrate several properties and differences of some of the classes presented in this paper. These examples are from [6, 7].

Example 17. Let $C = \langle x \rangle$ be a cyclic group of order 7 and let $A = \langle y \rangle \times \langle z \rangle$ be a cyclic group of order 6 with y an element of order 3 and z an element of order 2. Then $A = \mathrm{Aut}(C)$. Let $G = [C]A$ be the semidirect product of C by A. Then $[\langle y \rangle^z, z] \neq 1$ and G is not a soluble BT-group. However, G is an MS-group.

Example 18 shows that the classes of MS- and T_0-groups are not subgroup closed.

Example 18. Let $H = \langle x, y \mid x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1 \rangle$ be an extraspecial group of order 27 and exponent 3. Then H has an automorphism a of order 2 given by $x^a = x^{-1}$, $y^a = y^{-1}$ and $[x, y]^a = [x, y]$. Put $G = [H]\langle a \rangle$, the semidirect product of H by $\langle a \rangle$. Let $z = \langle x, y \rangle$. Then $\Phi(G) = \Phi(H) = \langle z \rangle = \mathrm{Z}(G) = \mathrm{Z}(H)$. Note that $G/\Phi(G)$ is a T-group so that G is a T_0-group. The maximal subgroups of H are normal in G and it follows that G is an MS-group. Let $K = \langle x, z, a \rangle$. Then $\langle xz \rangle$ is a maximal subgroup of $\langle x, z \rangle$, the Sylow 3-subgroup of K. However, $\langle xz \rangle$ does not permute with $\langle a \rangle$ and hence $\langle xz \rangle$ is not an S-semipermutable subgroup of K. Therefore, K is not an MS-subgroup of G. Also note that $\Phi(K) = 1$ and so K is not a T-subgroup of G and K is not a T_0-subgroup of G. Hence the class of soluble T_0-groups is not closed under taking subgroups. Note that G is not a soluble PST-group.

Example 19 presents an example of a soluble PST-group which is not an MS-group.

Example 19. Let $C = \langle x \rangle$ be a cyclic group of order 19^2, $D = \langle y \rangle$ a cyclic group of order 3^2, and $E = \langle z \rangle$ is a cyclic group of order 2 such that $D \times E \leq \mathrm{Aut}(C)$. Then $G = [C](D \times E)$ is a soluble PST-group and G is not an MS-group since $[\langle y^2 \rangle^x, z] \neq 1$.

The following notation is needed in the presentation of the next theorem which characterises MS-groups. Let G be a group whose nilpotent residual L is a Hall subgroup of G. Let $\pi = \pi(L)$ and let $\theta = \pi'$, the complement of π in the set of all prime numbers. Let θ_N denote the set of all primes p in θ such that if P is a Sylow p-subgroup of G, then P has at least two maximal subgroups. Further, let θ_C denote the set of all primes q in θ such that if Q is a Sylow q-subgroup of G, then Q has only one maximal subgroup, or, equivalently, Q is cyclic.

Theorem 20 ([6]). *Let G be a group with nilpotent residual L. Then G is an MS-group if and only if G satisfies the following:*

1. *G is a T_0-group.*

2. *L is a nilpotent Hall subgroup of G.*

3. *If $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, then a maximal subgroup of P is normal in G.*

4. *Let p and q be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then $[P, Q] = 1$.*

5. *Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ and M is the maximal subgroup of P, then $QM = MQ$ is a nilpotent subgroup of G.*

Theorem 21 ([6]). *Let G be a soluble PST-group. Then G is an MS-group if and only if G satisfies 4 and 5 of Theorem 20.*

Theorem 22 ([6]). *Let G be a soluble PST-group which is also an MS-group. If θ_C is the empty set, then G is a BT-group.*

Definition 23 ([14]). A subgroup H of a group G is called *primitive* if it is a proper subgroup in the intersection of all subgroups containing H as a proper subgroup.

All maximal subgroups of G are primitive. Some basic properties of primitive subgroups include:

Proposition 24.
1. Every proper subgroup of G is the intersection of a set of primitive subgroups of G.

2. If X is a primitive subgroup of a subgroup T of G, then there exists a primitive subgroup Y of G such that $X = Y \cap T$.

Johnson [14] proved that a group G is supersoluble if every primitive subgroup of G has prime power index in G.

The next results on primitive subgroups of a group G indicate how such subgroups give information about the structure of G.

Theorem 25 ([15]). *Let G be a group. The following statements are equivalent:*

1. *every primitive subgroup of G containing $\Phi(G)$ has prime power index;*

2. *$G/\Phi(G)$ is a soluble PST-group.*

Theorem 26 ([16])**.** *Let G be a group. The following statements are equivalent:*

1. *every primitive subgroup of G has prime power index;*
2. *$G = [L]M$ is a supersoluble group, where L and M are nilpotent Hall subgroups of G, L is the nilpotent residual of G and $G = L\mathrm{N}_G(L \cap X)$ for every primitive subgroup X of G. In particular, every maximal subgroup of L is normal in G.*

Let $\mathfrak{X}$ denote the class of groups G such that the primitive subgroups of G have prime power index. By Proposition 24 (1), it is clear that $\mathfrak{X}$ consists of those groups whose subgroups are intersections of subgroups of prime power indices.

The next example shows that the class $\mathfrak{X}$ is not subgroup closed.

Example 27. Let $P = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1\rangle$ be an extraspecial group of order 125 and exponent 5. Let $z = [x, y]$ and note that $\mathrm{Z}(P) = \Phi(P) = \langle z\rangle$. Then P has an automorphism a of order four given by $x^a = x^2$, $y^a = y^2$, and $z^a = z^4 = z^{-1}$. Put $G = [P]\langle a\rangle$ and note that $\mathrm{Z}(G) = 1$, $\Phi(G) = \langle z\rangle$, and $G/\Phi(G)$ is a T-group. Thus G is a soluble T_0-group. Let $H = \langle y, z, a\rangle$ and notice that $\Phi(H) = 1$. Then H is not a T-group since the nilpotent residual L of H is $\langle y, z\rangle$ and a does not act on L as a power automorphism. Thus H is not a T_0-group, and hence not a soluble PST-group. By Theorem 25, G is an $\mathfrak{X}$-group and H is not an $\mathfrak{X}$-group.

Theorem 28 ([17])**.** *Let G be a group. The following statements are equivalent:*

1. *G is a soluble PST-group;*
2. *every subgroup of G is an $\mathfrak{X}$-group.*

We bring the paper to a close with the quasipermutable embedding which is defined in the following way.

Definition 29. A subgroup H is called *quasipermutable* in G provided there is a subgroup B of G such that $G = \mathrm{N}_G(H)B$ and H permutes with B and with every subgroup (respectively, with every Sylow subgroup) A of B such that $\gcd(|H|, |A|) = 1$.

Theorem 30 contains new characterisations of soluble PST-groups with certain Hall subgroups.

Theorem 30 ([18]). *Let $D = G^{\mathfrak{N}}$ be the nilpotent residual of the group G and let $\pi = \pi(D)$. Then the following statements are equivalent:*

1. *D is a Hall subgroup of G and every Hall subgroup of G is quasipermutable in G;*
2. *G is a soluble PST-group;*
3. *every subgroup of G is quasipermutable in G;*
4. *every π-subgroup of G and some minimal supplement of D in G are quasipermutable in G.*

Acknowledgements

The work of the first and the third authors has been supported by the grant MTM2010-19938-C03-03 from the *Ministerio de Economía y Competitividad*, Spain. The first author has also been supported by the grant 11271085 from the National Natural Science Foundation of China.

References

[1] Ballester-Bolinches A, Esteban-Romero R, Asaad M. Products of finite groups. vol. 53 of de Gruyter Expositions in Mathematics. Berlin: Walter de Gruyter; 2010. Available from: `http://dx.doi.org/10.1515/9783110220612`.

[2] Beidleman JC. Weakly normal subgroups and classes of finite groups. Note Mat. 2012;32(2):115–121.

[3] Chen ZM. On a theorem of Srinivasan. Southwest Normal Univ Nat Sci. 1987;12(1):1–4.

[4] Wang L, Li Y, Wang Y. Finite groups in which (S-)semipermutability is a transitive relation. Int J Algebra. 2008;2(1-4):143–152. Corrigendum in *Int. J. Algebra*, 6(13-16):727–728, 2012.

[5] Al-Sharo KA, Beidleman JC, Heineken H, Ragland MF. Some characterizations of finite groups in which semipermutability is a transitive relation. Forum Math. 2010;22(5):855–862. Corrigendum in *Forum Math.*, 24(7):1333–1334, 2012. Available from: `http://dx.doi.org/10.1515/FORUM.2010.045`.

[6] Ballester-Bolinches A, Beidleman JC, Esteban-Romero R, Ragland MF. On a class of supersoluble groups. Bull Austral Math Soc. in press.

[7] Beidleman JC, Ragland MF. Groups with maximal subgroups of Sylow subgroups satisfying certain permutability conditions. Southeast Asian Bull Math. in press.

[8] Chen XY, Guo WB. Finite groups in which SS-permutability is a transitive relation. Acta Math Hungar. in press. Available from: `http://dx.doi.org/10.1007/s10474-014-0398-0`.

[9] Li SR, Shen ZC, Liu JJ, Liu XC. The influence of SS-quasinormality of some subgroups on the structure of finite groups. J Algebra. 2008;319:4275–4287.

[10] Ballester-Bolinches A, Esteban-Romero R, Pedraza-Aguilera MC. On a class of p-soluble groups. Algebra Colloq. 2005;12(2):263–267.

[11] Ragland MF. Generalizations of groups in which normality is transitive. Comm Algebra. 2007;35(10):3242–3252.

[12] van der Waall RW, Fransman A. On products of groups for which normality is a transitive relation on their Frattini factor groups. Quaestiones Math. 1996;19:59–82.

[13] Ren YC. Notes on π-quasi-normal subgroups in finite groups. Proc Amer Math Soc. 1993;117:631–636.

[14] Johnson DL. A note on supersoluble groups. Canad J Math. 1971;23:562–564.

[15] He X, Qiao S, Wang Y. A note on primitive subgroups of finite groups. Commun Korean Math Soc. 2013;28(1):55–62.

[16] Guo W, Shum KP, Skiba AN. On primitive subgroups of finite groups. Indian J Pure Appl Math. 2006;37(6):369–376.

[17] Ballester-Bolinches A, Beidleman JC, Esteban-Romero R. Primitive subgroups and PST-groups. Bull Austral Math Soc. in press.

[18] Yi X, Skiba AN. Some new characterizations of PST-groups. J Algebra. 2014;399:39–54.

[19] Ballester-Bolinches A, Beidleman JC, Feldman AD. Some new characterizations of solvable PST-groups. Ricerche Mat. 2012;61:291–297.

[20] Ballester-Bolinches A, Esteban-Romero R, Ragland M. A note on finite $\mathcal{PST}$-groups. J Group Theory. 2007;10(2):205–210.

[21] Beidleman JC, Ragland MF. Subnormal, permutable, and embedded subgroups in finite groups. Cent Eur J Math. 2011;9(4):915–921. Available from: `http://dx.doi.org/10.2478/s11533-011-0029-8`.

[22] Robinson DJS. A course in the theory of groups. New-York: Springer-Verlag; 1982.