SOME LOCAL PROPERTIES DEFINING $\mathcal{T}_0$-GROUPS AND RELATED CLASSES OF GROUPS

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ABSTRACT. We call G a Hall$_{\mathcal{X}}$-group if there exists a normal nilpotent subgroup N of G for which G/N' is an $\mathcal{X}$-group. We call G a $\mathcal{T}_0$-group provided $G/\Phi(G)$ is a $\mathcal{T}$-group, that is, one in which normality is a transitive relation. We present several new local classes of groups which locally define Hall$_{\mathcal{X}}$-groups and $\mathcal{T}_0$-groups where $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$; the classes $\mathcal{PT}$ and $\mathcal{PST}$ denote, respectively, the classes of groups in which permutability and S-permutability are transitive relations.

1. Introduction

All groups considered will be finite. All unexplained notation or terminology can be found in [3] or [8].

There is a well-known theorem of Philip Hall which states that if a group G possesses a normal nilpotent subgroup N such that G/N' is nilpotent, then G is nilpotent. It has been the subject of several papers to consider what can be said about the class of groups G possessing a normal nilpotent subgroup N such that G/N' is an $\mathcal{X}$-group for various classes $\mathcal{X}$. In particular, the fourth author in [6], for the solvable case, answered this question when $\mathcal{X}$ is any one of the classes $\mathcal{T}$, $\mathcal{PT}$, or $\mathcal{PST}$, that is, respectively, the classes where normality, permutability, and S-permutability are transitive relations. Let us call G a Hall$_{\mathcal{X}}$-group if there exists a normal nilpotent subgroup N of G for which G/N' is an $\mathcal{X}$-group. Since the Fitting subgroup of a group G contains all normal nilpotent subgroups one can simply replace N in the definition just given with F, the Fitting subgroup of G.

It is a classical problem to determine the structure of a group G such that $G/\Phi(G)$ is an $\mathcal{X}$-group for various classes $\mathcal{X}$. For a class $\mathcal{X}$, let us call G an $\mathcal{X}_0$-group if $G/\Phi(G)$ is an

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$\mathcal{X}$-group. The fourth author in [6] characterized the solvable $\mathcal{X}_0$-groups when $\mathcal{X}$ is one of the classes $\mathcal{T}$, $\mathcal{PT}$, or $\mathcal{PST}$. In particular, it was shown that within the class of solvable groups, the classes $\mathcal{T}_0$, $\mathcal{PT}_0$, and $\mathcal{PST}_0$ coincide.

In this paper, we turn to the concept of a "local" characterization theorem for solvable $\mathcal{T}_0$-groups and Hall$_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. That is, we are interested in characterizations in terms of the different prime divisors of the orders of such groups.

2. Preliminaries

The classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$, and the theorems that follow, will be critical in giving the desired local characterization theorems.

Definition.

(1) $\mathscr{C}_p$ is the class of groups G for which each subgroup of a Sylow p-subgroup P of G is normal in the normalizer, $\mathrm{N}_G(P)$, of P.

(2) $\mathscr{X}_p$ is the class of groups G for which each subgroup of a Sylow p-subgroup P of G is permutable in the normalizer, $\mathrm{N}_G(P)$, of P.

(3) $\mathscr{Y}_p$ is the class of groups G where $H \leq S \leq P$ with P a Sylow p-subgroup of G, always implies H is S-permutable in the normalizer, $\mathrm{N}_G(S)$, of S.

Theorem 1. *Let G be a group.*

(1) (Robinson [7], see also [3, 2.2.2]) *G is a solvable $\mathcal{T}$-group if and only if G is a $\mathscr{C}_p$-group for all primes p.*

(2) (Beidleman, Brewster, and Robinson [5], see also [3, 2.2.3]) *G is a solvable $\mathcal{PT}$-group if and only if G is an $\mathscr{X}_p$-group for all primes p.*

(3) (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.9]) *G is a solvable $\mathcal{PST}$-group if and only if G is a $\mathscr{Y}_p$-group for all primes p.*

We will need some results on the classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$ to aid in our characterization theorems.

Lemma 1 (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.13])**.** *A group G is a $\mathscr{Y}_p$-group if and only if G is p-nilpotent or G is a $\mathscr{C}_p$-group with abelian Sylow p-subgroups.*

Lemma 2 (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.4])**.** *A group G is an $\mathscr{X}_p$-group (respectively, $\mathscr{C}_p$-group) if and only if G is a $\mathscr{Y}_p$-group and the Sylow p-subgroups of G are Iwasawa (respectively, Dedekind) groups.*

In the previous lemma, Iwasawa groups (respectively, Dedekind groups) are the groups for which every subgroup is permutable (respectively, normal).

Lemma 3. *Let G be a group with N a normal p'-subgroup of G and M a normal p-subgroup of G.*

(1) *If G is a $\mathscr{Y}_p$-group, then G/M is a $\mathscr{Y}_p$-group.*

(2) *If G is an $\mathscr{X}_p$-group, then G/M is an $\mathscr{X}_p$-group.*

(3) *If G is a $\mathscr{C}_p$-group, then G/M is a $\mathscr{C}_p$-group.*

(4) *G is a $\mathscr{Y}_p$-group if and only if G/N is a $\mathscr{Y}_p$-group.*

(5) *G is an $\mathscr{X}_p$-group if and only if G/N is an $\mathscr{X}_p$-group.*

(6) *G is a $\mathscr{C}_p$-group if and only if G/N is a $\mathscr{C}_p$-group.*

Proof. (1), (2), and (3) follow directly from the definitions of $\mathscr{Y}_p$, $\mathscr{X}_p$, and $\mathscr{C}_p$.

(4). This can be found in [4] as Lemma 6.

(5). This can be found in [1] as Lemma 4.1.

(6). Let G be a $\mathscr{C}_p$-group. By Lemma 2, this is equivalent to affirming that G is a $\mathscr{Y}_p$-group with Dedekind Sylow p-subgroups. Since the Sylow p-subgroups of G and G/N are isomorphic, by (4) we have that this is equivalent to stating that G/N is a $\mathscr{Y}_p$-group with Dedekind Sylow p-subgroups. By Lemma 2, this is to say that G/N is a $\mathscr{C}_p$-group. This completes the proof. □

Remark. In fact, the classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$ are quotient closed in general, however, this is a much more difficult fact to deduce. For our purposes, quotient closure for normal p'-subgroups and p-subgroups is sufficient.

Lastly, we record some results on $\mathcal{T}_0$-groups which will be needed.

Theorem 2.

(1) (Van der Wall and Fransman [9]) *The class of solvable $\mathcal{T}_0$-groups is a quotient closed class of groups.*

(2) (Van der Wall and Fransman [9]) *A solvable $\mathcal{T}_0$-group is supersolvable.*

(3) (Ragland [6]) *A group G is a solvable* $\text{Hall}_{\mathcal{PST}}$*-group if and only if G is a solvable $\mathcal{T}_0$-group.*

(4) (Ragland [6]) *A group G is a solvable $\mathcal{PST}_0$-group if and only if G is a solvable $\mathcal{T}_0$-group.*

(5) (Ragland [6]) *A group G is a solvable $\mathcal{T}_0$-group with abelian nilpotent residual if and only if G is a solvable $\mathcal{PST}$-group.*

3. Main Results

In this section we provide local characterization theorems for the solvable $\mathcal{T}_0$-groups, $\text{Hall}_{\mathcal{T}}$-groups, $\text{Hall}_{\mathcal{PT}}$-groups, and $\text{Hall}_{\mathcal{PST}}$-groups. We will need the following definitions:

Definition. Let p be a prime. For a group G, let $\Phi(G)_p$ denote the Sylow p-subgroup of the Frattini subgroup of G. Let F_p' denote the derived subgroup of the Sylow p-subgroup of the Fitting subgroup, F, of G. Let L_p' denote the derived subgroup of a Sylow p-subgroup of the nilpotent residual, L, of G. If p does not divide the order of $\Phi(G)$ then let us denote by $\Phi(G)_p$ the identity subgroup. Likewise, $F_p' = 1$ if p does not divide the order of F' and $L_p' = 1$ if p does not divide the order of L'. Let $\mathscr{X}$ be one of the classes $\mathscr{C}_p$, $\mathscr{X}_p$, or $\mathscr{Y}_p$.

(1) Define $\Phi_{\mathscr{X}}$ to be the class of groups G for which $G/\Phi(G)_p$ is a $\mathscr{X}$-group.

(2) Define $\mathscr{F}_{\mathscr{X}}$ to be the class of groups G for which G/F_p' is a $\mathscr{X}$-group.

(3) Define $\mathscr{L}_{\mathscr{X}}$ to be the class of groups G for which L is p'-nilpotent and G/L_p' is a $\mathscr{X}$-group.

Since the Frattini factor group of a solvable $\mathcal{T}_0$-group is a $\mathcal{T}$-group, and $\mathcal{T}$-groups can be locally characterized using the class $\mathscr{C}_p$, it stands to reason that $\mathcal{T}_0$-groups can be locally characterized using the class $\mathscr{C}_p$ with certain conditions involving the Frattini subgroup. This is indeed the case, however, recall that $\mathcal{T}_0 = \mathcal{PT}_0 = \mathcal{PST}_0$ in the class of solvable groups. So it also stands to reason that one could replace $\mathscr{C}_p$ with $\mathscr{X}_p$ or $\mathscr{Y}_p$. In view of Theorem 2, part (3), one should be able to characterize the solvable $\mathcal{T}_0$-groups using $\mathscr{Y}_p$ with certain conditions placed upon the Fitting subgroup. The details can be found in the following theorem:

Theorem 3. *The following are equivalent for a group G.*

(1) *G is an $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes p.*

(2) *G is an $\mathscr{F}_{\mathscr{Y}_p}$-group for all primes p.*

(3) *G is a solvable* $\mathrm{Hall}_{\mathcal{PST}}$*-group.*

(4) *G is a $\Phi_{\mathscr{C}_p}$-group for all primes p.*

(5) *G is a $\Phi_{\mathscr{X}_p}$-group for all primes p.*

(6) *G is a $\Phi_{\mathscr{Y}_p}$-group for all primes p.*

(7) *G is a solvable $\mathcal{T}_0$-group.*

Proof. Throughout the proof let us denote the nilpotent residual of G by L and the Fitting subgroup of G by F. Also, for any group H, H_p will denote a Sylow p-subgroup of H.

(1) $\Rightarrow$ (2). If G is an $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes p then L is p'-nilpotent for all primes p and hence nilpotent. Thus $L \leq F$ so that for any prime p we have $L_p' \leq F_p'$. Now since G/L_p' is a $\mathscr{Y}_p$-group for all primes p, we have, by Lemma 3, part (1), that $(G/L_p')/(F_p'/L_p') \simeq G/F_p'$ is a $\mathscr{Y}_p$-group for all primes p.

(2) $\Rightarrow$ (3). If G is an $\mathscr{F}_{\mathscr{Y}_p}$-group for all primes p, then G/F_p' is a $\mathscr{Y}_p$-group for all primes p. Using Lemma 3, part (4), we have $(G/F_p')/(F'/F_p') \simeq G/F'$ is a $\mathscr{Y}_p$-group for all primes p. Hence by Theorem 1, part (3), we have that G/F' is a solvable $\mathcal{PST}$-group so that G is a solvable $\mathrm{Hall}_{\mathcal{PST}}$-group.

(3) $\Rightarrow$ (4). If G is a solvable $\mathrm{Hall}_{\mathcal{PST}}$-group, then, by Theorem 2, part (3), we have that G is a solvable $\mathcal{T}_0$-group. Let $\bar{G} = G/\Phi(G)_p$. Then $\bar{G}$ is a solvable $\mathcal{T}_0$-group by Theorem 2, part (1). By induction, if $\Phi(G)_p \neq 1$, we have $\bar{G}/\Phi(\bar{G})_p$ is a $\mathscr{C}_p$-group. But $\Phi(\bar{G})_p = 1$ and so

$\bar{G}$ is a $\mathscr{C}_p$-group. If $\Phi(G)_p = 1$ then one only needs G itself to be a $\mathscr{C}_p$ group. Now $G/\Phi(G)$ is a $\mathcal{T}$-group and hence a $\mathscr{C}_p$-group by Theorem 1, part (1). Lemma 3, part (6), gives us that G is a $\mathscr{C}_p$-group since $\Phi(G)$ is a p'-group.

Both (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) follow directly from the definitions of $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$.

(6) $\Rightarrow$ (7). If G is a $\Phi_{\mathscr{Y}_p}$-group for all primes p, then $G/\Phi(G)_p$ is a $\mathscr{Y}_p$-group for all primes p. Since $\Phi(G)/\Phi(G)_p$ is a p'-group for all primes p, we have, by Lemma 3, part (4), that $\big(G/\Phi(G)_p\big)/\big(\Phi(G)/\Phi(G)_p\big) \simeq G/\Phi(G)$ is a $\mathscr{Y}_p$-group for all primes p. Thus $G/\Phi(G)$ is a $\mathcal{PST}$-group by Theorem 1, part (3). Hence G is a solvable $\mathcal{PST}_0$-group and thus a solvable $\mathcal{T}_0$-group by Theorem 2, part (4).

(7) $\Rightarrow$ (1). If G is a solvable $\mathcal{T}_0$-group, then G is supersolvable by Theorem 2, part (2). Thus G' is nilpotent in which case so is L. Hence L is p'-nilpotent for all primes p. Note that if L is abelian then, by Theorem 2, part (5), G is a $\mathcal{PST}$-group and hence a $\mathscr{Y}_p$-group for all primes p by Theorem 1, part (3). From Lemma 3, part (1), it would follow that G is a $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes p if L is abelian. So let us assume L is not abelian.

Let $\bar{G} = G/L'_p$ and let $\bar{L}$ be the nilpotent residual of $\bar{G}$. Now $\bar{G}$ is a solvable $\mathcal{T}_0$-group by Theorem 2, part (1). By induction, if $L'_p \neq 1$, we have $\bar{G}/\bar{L}'_p$ is a $\mathscr{Y}_p$-group. But $\bar{L}'_p = 1$ and so $\bar{G}$ is a $\mathscr{Y}_p$-group.

Assume that $L'_p = 1$. Since L is not abelian there exists a prime $q \neq p$ for which $L'_q \neq 1$. Now let $\bar{G}$ denote G/L'_q with $\bar{L}$ the nilpotent residual of $\bar{G}$ and note $\bar{G}$ is a $\mathcal{T}_0$-group by Theorem 2, part (1). So, by induction, $\bar{G}/\bar{L}'_p$ is a $\mathscr{Y}_p$-group. Thus $\bar{G}$ is a $\mathscr{Y}_p$-group and hence G is a $\mathscr{Y}_p$-group by Lemma 3, part (4). It now follows that G is a $\mathscr{L}_{\mathscr{Y}_p}$-group for all primes p completing the proof. $\square$

The classes of solvable $\mathrm{Hall}_{\mathcal{T}}$-groups and $\mathrm{Hall}_{\mathcal{PT}}$-groups admit similar local characterizations which we provide now.

Theorem 4. *The following are equivalent for a group G.*

(1) *G is a $\mathscr{F}_{\mathscr{X}_p}$-group for all primes p.*

(2) *G is a solvable* $\mathrm{Hall}_{\mathcal{PT}}$*-group.*

Proof. (1) ⇒ (2). The proof is similar to the proof of (2) ⇒ (3) from Theorem 3. One only needs to use invoke the appropriate results corresponding to $\mathscr{X}_p$ from Lemma 3 and Theorem 1.

(2) ⇒ (1). If G is a solvable Hall$_{\mathcal{PT}}$-group, then it is a solvable Hall$_{\mathcal{PST}}$-group and hence by Theorem 3 we have G/F_p' is a $\mathscr{Y}_p$-group for all primes p.

Now G/F' a solvable $\mathcal{PT}$-group and thus an $\mathscr{X}_p$-group for all primes p by Theorem 1, part (2). Hence the Sylow subgroups of G/F' are Iwasawa. Let P denote a Sylow p-subgroup of G. Then $PF'/F' \simeq P/(P \cap F') = P/F_p'$ and thus P/F_p' is an Iwasawa group.

Now one can conclude from Theorem 2 that G/F_p' is an $\mathscr{X}_p$-group for all primes p so that G is a $\mathscr{F}_{\mathscr{X}_p}$-group for all primes p. □

Theorem 5. *The following are equivalent for a group G.*

(1) *G is a $\mathscr{F}_{\mathscr{C}_p}$-group for all primes p.*

(2) *G is a solvable* Hall$_{\mathcal{T}}$*-group.*

Proof. Using the appropriate theorems and lemmas, the proof is quite similar to that of Theorem 4 and so we omit the proof. □

The next theorem shows that the difference between solvable $\mathcal{T}_0$-groups and Hall$_{\mathcal{PT}}$-groups (Hall$_{\mathcal{T}}$-groups) amounts to the Sylow structure of G/F_p' for all primes p.

Theorem 6. *Let G be a solvable group with $F = \mathrm{Fit}(G)$. Then G is a $\mathcal{T}_0$-group where the Sylow p-subgroups of G/F_p' are Iwasawa (Dedekind) groups for all primes p if and only if G is a* Hall$_{\mathcal{PT}}$*-group (*Hall$_{\mathcal{T}}$*-group).*

Proof. Suppose G is a $\mathcal{T}_0$-group where the Sylow p-subgroups of G/F_p' are Iwasawa (Dedekind) groups. By Theorem 3, we have G/F_p' is a $\mathscr{Y}_p$-group for all primes p. By Lemma 2, we must have G/F_p' is an $\mathscr{X}_p$-group ($\mathscr{C}_p$-group) for all primes p. We can now deduce from Theorem 4 (2) that G is a Hall$_{\mathcal{PT}}$-group (Hall$_{\mathcal{T}}$-group).

Suppose G is a Hall$_{\mathcal{PT}}$-group (Hall$_{\mathcal{T}}$-group). Then Theorem 4 (5) says that G/F_p' is an $\mathscr{X}_p$-group ($\mathscr{C}_p$-group) for all primes p. Using Lemma 2, we can deduce that G/F_p' is a $\mathscr{Y}_p$-group for all primes p and that the Sylow p-subgroups of G/F_p' are Iwasawa (Dedekind) groups. That G is a $\mathcal{T}_0$-group follows from Theorem 3. □

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