THE ELECTROMAGNETIC POLARIZABILITIES OF THE PROTON

AND THE SCALAR-ISOSCALAR $\gamma\gamma \to \pi\pi$ AMPLITUDE

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A B S T R A C T

In a dispersive approach to the calculation of the electromagnetic polarizabilities of the proton, the sensitivity to the annihilation channel input is analyzed. It is shown that the isoscalar s wave absorptive part gives the overwhelming contribution. The connection with the poorly known $\gamma\gamma \to \pi\pi$ amplitude is established and some numerical estimates are presented.

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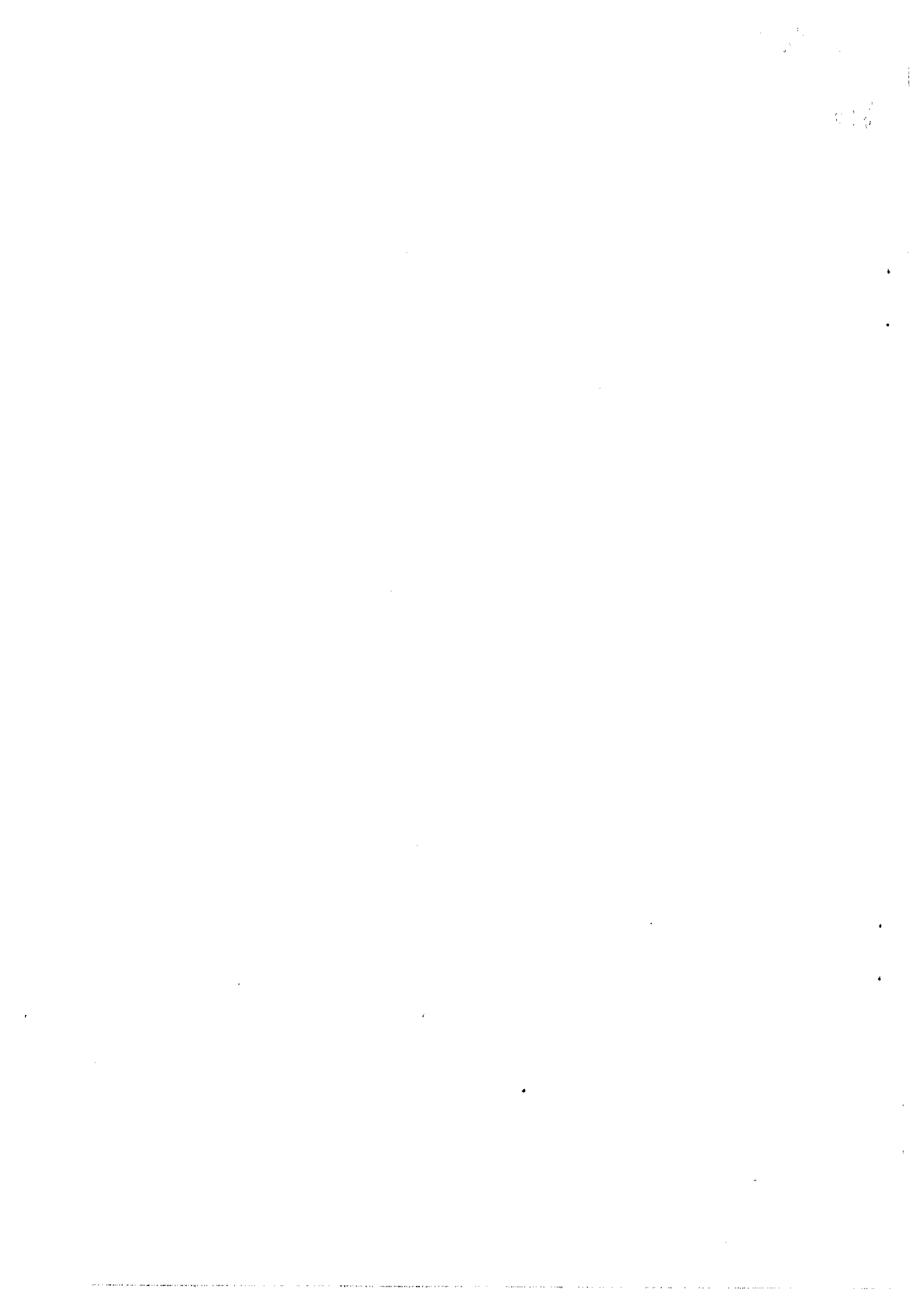
This letter is concerned with a theoretical derivation of the electromagnetic polarizabilities of the nucleon by means of a dispersion relation technique. These two structure constants determine (apart from mass, charge and magnetic moment) the Compton scattering amplitude [1] up to second order in the energy of the photon and the differential cross-section up to third order. They have been shown [2] to determine the longest range interaction between two objects, one of which at least is neutral. The knowledge of these parameters, of undoubted interest for hadron physics, is also relevant for certain implications in nuclear and astrophysical studies [3].

There are some results of the available experiments [4],[5] of Compton scattering on protons at energies ν between 50 and 110 MeV that can be analyzed in terms of α and β, electric and magnetic polarizabilities, respectively. Such a fit has been carried out [6] by imposing the constraint resulting from the forward dispersion relation [7] for the sum :

$$\alpha + \beta = (14.2 \pm 0.3) \times 10^{-4} \, fm^3 \qquad (1)$$

The result is the following

$$\alpha = (12.4 \pm 0.6) \times 10^{-4} \, fm^3 \quad , \quad \beta = (1.8 \pm 0.9) \times 10^{-4} \, fm^3 \qquad (2)$$

where the errors of α and β are strongly correlated. Such an analysis is not free of problems ; in fact, it has been realized that for energies around 100 MeV the contribution of the π^o meson pole (which enters with an energy dependence starting with ν^4) to the differential cross-section amounts to ~10% at backward angles. This is essential for the extraction of the proton polarizabilities and can change considerably [8] the reported values. Therefore, our attitude will be to consider the result (2) as an indication rather than a settled question.

A separate theoretical determination of α and β needs more ingredients than the ones present in the forward sum rule [7]. Bernabéu, Ericson and Ferro Fontan [9] investigated this problem by the use of a backward dispersion relation for the physical spin averaged amplitude at 180°. In terms of the spin averaged amplitudes $A_1(s,u)$ and $A_2(s,u)$ introduced by Bardeen and Tung [10] which are free from kinematical problems and symmetric under crossing, $A_i(s,u) = A_i(u,s)$, the following relation was found [9]

$$\frac{1}{2m} \left[P^2 A_2 \right]^C = \alpha + \beta \tag{3a}$$

$$\frac{1}{2m} \left[2 A_1 + P^2 A_2 \right]^C = \alpha - \beta + \left(\frac{e^2}{4\pi}\right) \frac{2 Z \varkappa + \varkappa^2}{2 m^3} \tag{3b}$$

where the superscript C means the limit $s \to m^2$ of the continuum contributions to these amplitudes, at fixed scattering angle, Z and $\varkappa$ are the charge and anomalous magnetic moment in natural units and $P = \frac{1}{2}(p+p')$ is the average of initial and final nucleon momenta. One gets Eq. (1) from a forward dispersion relation for the amplitude of (3a). In order to obtain good convergence properties, a backward dispersion relation was written [9] for the amplitude of (3b). Explicit knowledge of the asymptotic value $t \to \infty$ at $180°$ of the pole contribution to this amplitude in a dispersive approach allows one to arrive at the following sum rule [*]

$$\alpha - \beta = \frac{1}{2\pi^2} \int_{\nu_{th}}^{\infty} \frac{d\nu}{\nu^2} \left(1 + 2 \frac{\nu}{m}\right)^{1/2} \left[\sigma(\Delta\pi = yes) - \sigma(\Delta\pi = no) \right]$$

$$+ \frac{1}{2\pi m} \int_{4\mu^2}^{\infty} \frac{dt}{t} \, Abs \left(2 A_1 + P^2 A_2\right) \tag{4}$$

where the backward direction must be followed in the last integration, i.e., one needs the amplitude at the point

$$s(u) = m^2 - \frac{t}{2} + (-) \frac{1}{2} \sqrt{t(t - 4m^2)} \tag{5}$$

As seen in Eq. (4), the absorptive part in the direct channel contribution is obtained from that of the forward physical amplitude by changing the sign of the non-parity flip multipoles ($\Delta\pi = no$). A reliable evaluation of this integrand gives [9]

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[*] The first term present in the right-hand side of Eq. (7) of Ref. 9) should not be there. It amounted to a negative contribution to $\alpha - \beta$ of 2.4×10^{-4} fm^3.

$$[\alpha - \beta]_s = -4 \times 10^{-4} \ fm^3 \tag{6}$$

The second term in the sum rule (4) comes from the physical and pseudo-physical regions in the $\gamma\gamma \to N\bar{N}$ annihilation channel. Comparison of Eqs. (2) and (6) indicates that the contribution of this second term should be very important. Its significance and the need for its inclusion has repeatedly been pointed out in the literature [11],[12],[6]. Even with a fairly small radiative coupling of an effective scalar ϵ meson, one obtains larger absolute values [11] than the result (6).

The t channel contribution to $\alpha-\beta$ coming from two-pion intermediate states was evaluated [12] in the Born approximation. That means that only the Born contribution is taken for both the $N\bar{N} \to \pi\pi$ and the $\gamma\gamma \to \pi\pi$ in terms of which the t channel absorptive part of Eq. (4) is written. One should recall that the use of the Born approximation for the $N\bar{N} \to \pi\pi$ amplitude relevant for this problem is meaningless. These amplitudes are by now well studied[13]-[16] because of their relevance for nucleon-nucleon forces. It is well known that for the Frazer-Fulco [17] $f_+^0(t)$ partial wave helicity amplitude (the most relevant one for us) the Born approximation is not only misleading but totally wrong, even in sign [*]. Since $N\bar{N} \to \pi\pi$ amplitudes are now known to a $\pm 20\%$ level [16], one should make use of them as an input in the calculation of the absorptive part of the t channel needed for Eq. (4). Unfortunately, no statement of this kind would apply to the $\gamma\gamma \to \pi\pi$ amplitudes, on which the information is scarce [18],[19] and not free from ambiguities. The reason why $N\bar{N} \to \pi\pi$ amplitudes are well known in the pseudo-physical region above $4\mu^2$ lies in the fact that there is a bounty of low energy πN experimental data. In the absence of pion Compton scattering data, nucleon Compton scattering may play an important rôle in helping to determine reliable physical $\gamma\gamma \to \pi\pi$ amplitudes.

The purpose of this work is precisely to provide the connection of $\alpha-\beta$ to the $\gamma\gamma \to \pi\pi$ amplitudes. As we are going to see, the sensitivity to different solutions for the $\gamma\gamma \to \pi\pi$ amplitudes is very strong.

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[*] This, together with an error in sign in the expressions of Ref. 12) for the absorptive part made their result go in the right direction. The sign error has been confirmed to us by the authors (private communication from E.E. Radescu).

Therefore this connection gives a supplementary criterion which must be satisfied by those amplitudes. Of course, if they were determined from $e^+e^- \to e^+e^-\pi\pi$ experiments then a reliable value of $\alpha-\beta$ would be obtained. It should be mentioned also that those amplitudes are crucial for the study of the radiative decays of scalar and tensor mesons.

The absorptive part of the amplitude $2A_1 + P^2A_2$ needed in Eq. (4) will be obtained from unitarity taking only two-pion intermediate states. The sum over the three possible two-pion charge combinations projects out the isoscalar combination, which for two pions implies even partial waves only. We obtain

$$Abs\left[2A_1(t,\cos\theta_t=-1) + (P^2A_2)(t,\cos\theta_t=-1)\right]$$

$$= \frac{1}{32\pi^3}\frac{m}{t}\sqrt{\frac{t-4\mu^2}{t}}\int d\Omega\left[A^{(+)}(t,\cos\theta)+im\sqrt{\frac{t-4\mu^2}{4m^2-t}}\cos\theta\,B^{(+)}(t,\cos\theta)\right]F_o^*(t,\cos\theta)$$

$$(7)$$

where $A^{(+)}$ and $B^{(+)}$ are the usual CGLN isospin even πN amplitudes and $F_o(t,\cos\theta)$ is the isospin zero Gourdin-Martin[20] $\gamma\gamma\to\pi\pi$ amplitude. Since the dispersion relation has been written for a spin averaged amplitude the $N\bar{N}\to\pi\pi$ and $\gamma\gamma\to\pi\pi$ amplitudes which appear in Eq. (7) are precisely the $++$ helicity amplitudes. This is directly seen from the particular combination of $A^{(+)}$ and $B^{(+)}$ for the first one and from the absence of the G_o amplitude[19] for the second one. A partial wave expansion keeping the first two even waves and the integration over the solid angle Ω in Eq. (7) leads to

$$Abs\left(2A_1 + P^2A_2\right) = \frac{16\,m}{4m^2-t}\sqrt{\frac{t-4\mu^2}{t}}\frac{1}{t}$$

$$\times\left\{f_+^o(t)\frac{F_o^{o*}(t)}{8\pi} - \left(m^2-\frac{t}{4}\right)\left(\frac{t}{4}-\mu^2\right)f_+^2(t)\frac{F_o^{2*}(t)}{8\pi}\right\}$$

$$(8)$$

where the partial wave helicity amplitudes $f_+^o(t)$ and $f_+^2(t)$ for $N\bar{N} \to \pi\pi$ are Frazer and Fulco's [17] and the partial wave helicity amplitudes $F_o^o(t)$ and $F_o^2(t)$ for $\gamma\gamma \to \pi\pi$ are defined as in Ref. 19). Equation (8) feeded into Eq. (4) will thus give the t channel contribution to $\alpha-\beta$ coming from the four lowest waves of the two-pion intermediate state. A good knowledge of the helicity amplitudes of Eq. (8) would thus determine the value of $\alpha-\beta$, since the s channel contribution, Eq. (6), is well established.

It is known from unitarity that the phases of f_+^o and f_+^2 are given by δ_o^o and δ_o^2, the elastic $\pi\pi$ $I = 0$, $J = 0$ and two phase shifts, at least in the region $4\mu^2 \le t \le 16\mu^2$ and up to multiples of π. The same is of course true for F_o^o and F_o^2. Therefore the products of helicity amplitudes which appear in Eq. (8) are the products of their moduli, where the moduli might take negative values if the phase of the corresponding amplitude differs from the $\pi\pi$ phase shift in an odd number of π's. This gives the following expression for the t channel contribution to $\alpha-\beta$

$$\left[\alpha - \beta\right]_t = \frac{8}{\pi} \int_{4\mu^2}^{t_o} \frac{dt}{4m^2 - t} \sqrt{\frac{t - 4\mu^2}{t}} \frac{1}{t^2}$$

$$\times \left\{ \left| f_+^o(t) \right| \left| \frac{F_o^o(t)}{8\pi} \right| - \left(m^2 - \frac{t}{4}\right)\left(\frac{t}{4} - \mu^2\right)\left| f_+^2(t) \right| \left| \frac{F_o^2(t)}{8\pi} \right| \right\}$$

$$(9)$$

From the factor outside the bracket in Eq. (9) one sees the strong dominance of the low t region in the integration. The upper limit t_o will be taken to be $t_o = 50 \mu^2$, and this because of three good reasons. First, the $I = J = 0$ $\pi\pi$ state scatters elastically up to $t \simeq 50 \mu^2$ [21], so that in the two-pion intermediate state approximation the dominant s wave contribution to Eq. (9) is correct in this whole region. Second, the best values for $\left| f_+^o(t) \right|$ are known up to $t = 50 \mu^2$ [15],16). Third, we have explicitly checked that the contribution coming from above $t = 50 \mu^2$ cannot give more than about 10% of the one coming from $4\pi^2 < t < 50\mu^2$.

We shall now consider the f amplitudes on a different footing than the F amplitudes since our semi-phenomenological knowledge on them lies at very different levels. The Frazer-Fulco f amplitudes are by now known within $\pm 20\%$ errors [16] and we shall take them as a fixed input. The Gourdin-Martin F amplitudes are on the contrary poorly known. To our

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knowledge, the latest and most complete study on them is that of Ref. 19).
Besides the approximations needed to construct the amplitude for $t < 0$,
and for the choice of the strong interaction amplitude, the result is non
unique due to the polynomial ambiguities. In order to fix the coefficients
of the polynomials one would need further theoretical or experimental in-
formation, which is missing. The F amplitudes will therefore be consi-
dered as the variable input in Eq. (9) to show the sensitivity of the
result for $\alpha-\beta$ to the different choices.

Let us now turn to the numerical calculations. We shall start
with the d wave, since there the situation seems to be more settled.
$|f_+^2(t)|$ is given until $t = 24\mu^2$ in Ref. 16) and we extrapolate it softly
until $t = 50\mu^2$. This extrapolation is of little importance, since it con-
tributes only about a 20% of what comes from the lower region $4\mu^2 < t < 24\mu^2$.
This is understood from the slow variation of both $(t/4 - \mu^2)|f_+^2(t)|$ and
$|F_0^2(t)|$ with t, which makes the integrand in Eq. (9) decrease essentially
as $\sim t^{-2}$. Anyhow, as we shall see, the d wave contributes less than the
s wave. The amplitude $|F_0^2(t)|$ is taken from Ref. 19) and it seems to be
quite stable with respect to ambiguities in the region which interests us.
Furthermore, it differs very little from the Born contribution in the region
of integration. We obtain in this way a contribution to $\alpha-\beta$ which is

$$[\alpha - \beta]_t^d \simeq -6 \times 10^{-4} \, fm^3 \tag{10}$$

Since both $|f_+^2(t)|$ and $|F_0^2(t)|$ do not differ very much from their Born
contributions, the d wave may be calculated in the Born-Born approximation
leading then to about $\simeq -5 \times 10^{-4} \, fm^3$.

Let us turn now to the s wave. $|f_+^0(t)|$ is given up to $t = 50\mu^2$
by Bohannon and Signell [15]. Their solution agrees very well over the whole
energy range with the one of Nielsen and Oades [14]. With the availability
of new information on low energy πN phase shifts, Bohannon [16] improved
the values of $|f_+^0(t)|$ corresponding to the lowest energies (until $t = 25 \, \mu^2$).
This combined solution is considered nowadays as a very reliable one. Its
behaviour might be roughly described by a broad bump centered at $t \simeq 25 \, \mu^2$.
The amplitude $|F_0^0(t)|$ was obtained in Ref. 19) with different assignments
for the polynomials. On the basis of an acceptable behaviour around
$t \simeq (1 \, GeV)^2$, they prefer the solution presented in their Fig. 7b. However,
the precise result obtained for low values of t should not be taken too

seriously since there are other possible solutions [*] without a zero near threshold. In fact, the existence of this zero in the modulus makes the dominant part of the amplitude in the integral of Eq. (9) negative. Taken as given, one would find the result

$$[\alpha - \beta]_t^s \simeq -20 \times 10^{-4} \, fm^3 \qquad (11)$$

which, together with the other contributions (6) and (10), would lead to

$$\alpha - \beta \simeq -30 \times 10^{-4} \, fm^3 \qquad (12)$$

to be compared with the experimental indication of Eq. (2) ! The result would be disastrous. The reason for the wrong sign lies, as said before, in the existence of a zero of $|F_o^o(t)|$ near threshold which corresponds to a jump in the phase of π radians. The large absolute value obtained in Eq. (11) is understood by the large contribution from the high t region, owing to the strongly increasing behaviour of $|F_o^o(t)|$ up to $t \sim 50\mu^2$. All the present experimental results of α and β, Eq. (2), are strongly incompatible with any solution of $|F_o^o(t)|$ which has a zero not too far from threshold.

Let us give, just as a numerical illustration of the importance of the zero, the result we would have got with the same $|F_o^o(t)|$ but without the change of sign : $[\alpha-\beta]_t^s \sim 23 \times 10^{-4}$ fm^3, leading to a total value $\alpha-\beta \simeq 13 \times 10^{-4}$ fm^3, which is about the experimental value of $\alpha-\beta$.

We finally present the values obtained when $|F_o^o(t)|$ is approximated by its Born contribution. It should be stressed that there is absolutely no justification for it. We do it for the sake of showing the strong sensitivity of $\alpha-\beta$ to the chosen input for $|F_o^o(t)|$. The Born term dominates at low values of t and it is positive definite everywhere. The result is the following

$$[\alpha - \beta]_t^{s,B} = 13 \times 10^{-4} \, fm^3 \qquad (13)$$

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[*] Private communication from G. Mennessier.

which would give, together with Eqs. (6) and (10),

$$\alpha - \beta = 3 \times 10^{-4} \, fm^3 \tag{14}$$

Comparison of the values obtained in Eqs. (11) and (13) shows the extraordinary importance of the behaviour of $|F_0^0(t)|$ even at values of t above $25\mu^2$.

To summarize, Eq. (9) presents a connection of the electromagnetic polarizabilities α and β with the isoscalar s wave $\gamma\gamma \to \pi\pi$ amplitude $F_0^0(t)$. All the other ingredients are sufficiently well known to allow its use to provide restrictions on the possible solutions for the amplitude. With present information, one would need a solution for $|F_0^0(t)|$ with a behaviour at low t quite similar to the one found in Ref. 19), but without the jump in π radians for the phase near threshold. This is compatible with keeping an acceptable behaviour at $t \simeq (1 \text{ GeV})^2$. Experiments on $e^+e^- \to e^+e^-\pi\pi$ will settle this question.

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