# POLARIZABILITY EFFECTS IN ELECTRONIC AND MUONIC ATOMS

J. Bernabéu[*] and T.E.O. Ericson

CERN -- Geneva

ABSTRACT

The S state polarizability shifts are derived from
the virtual forward Compton scattering in the unre-
tarded dipole approximation. In the non-relativistic
limit $\omega_N/2m \ll 1$, the shift is proportional to the
photonuclear sum rule $\sigma_{-\frac{3}{2}}$, while in the relativistic
limit $\omega_N/2m \gg 1$ it is proportional to a logarithmically
weighted σ_{-2} sum rule. In both cases, the character-
istic momentum transfer is $(2m\omega_N)^{\frac{1}{2}}$. The non-local-
ity from the intermediate lepton propagation removes
the divergence typical of the static limit. Explicit
formulas for the shifts are given for both the rela-
tivistic and non-relativistic limits.

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[*] On leave from Departament de Física Teòrica, Univer-
sitat de València, Spain.
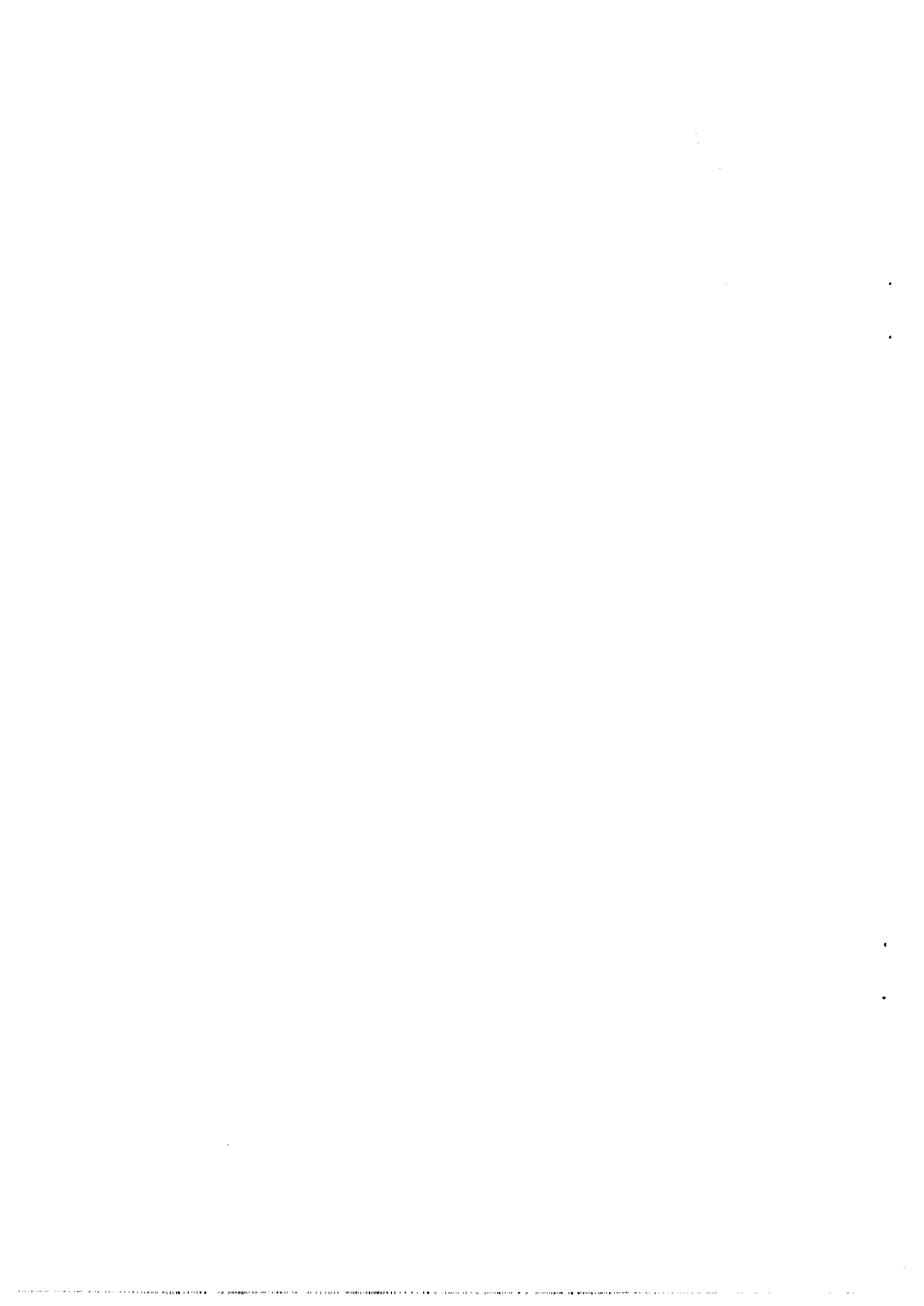
1. - INTRODUCTION

In addition to the electromagnetic level shifts produced by the spread out charge distribution of the nucleus in electronic and muonic atoms, there are also additional level shifts associated with the virtual excitation of the nucleus [1]. The physics of these shifts is well understood for orbits of high angular momentum, for which they simply describe the nuclear dipole response to the external electric field at the nucleus produced by the lepton, since this has the longest range. One gets [2]

$$\Delta E_{n\ell} = - \frac{1}{2} \alpha \langle \vec{E}^2 \rangle_{n\ell} = - \frac{1}{2} \alpha \left(\frac{e^2}{4\pi} \right) \langle \frac{1}{r^4} \rangle_{n\ell} \quad (1)$$

The nuclear polarizability α has its strength determined by the photo-nuclear sum rule

$$\alpha = \frac{1}{2\pi^2} \sigma_{-2} \quad (2)$$

For atomic orbits of $\ell = 0$, the simple expression (1) is invalid and divergent. Consequently, the usual approaches to the polarizability shift for S states are quite different in their formulation, even when the basic nuclear ingredients are very similar to those in Eq. (2). In heavy muonic atoms, most approaches [3] separately sum over the various nuclear multipole excitations, while the muonic contributions are treated non-statically. The actual results are dominated by dipole excitations with non-negligible additional monopole and quadrupole contributions. In light elements, the $2S_{\frac{1}{2}}$ polarizability shift for the $(\mu^4He)^+$ system has been deduced to be -3.1 meV $\pm$ 20% using a dispersive approach [4] and by time-ordered perturbation theory [5]. This shift is relevant to the precision measurement [6] of the energy difference $\Delta E(2P_{\frac{3}{2}} - 2S_{\frac{1}{2}}) = 1527.4 \pm 0.9$ meV. Our objective in the present work is to establish the dominant physical effects and the parameters which govern the polarizability shifts in atomic S states. This is particularly interesting and important since the S states have the largest polarizability corrections.

A qualitative insight into the quantities which govern the polarizability shifts in muonic atoms is obtained by considering the non-locality in the leptonic co-ordinate for the polarizability operator. In the intermediate state the muon will give rise to a Yukawa behaviour $\exp\{-\kappa|\underline{r}-\underline{r}'|\}/|\underline{r}-\underline{r}'|$, with a characteristic inverse range $\kappa = (2m_\mu\omega)^{\frac{1}{2}}$ corresponding to a nuclear excitation ω. For excitation of the nuclear giant dipole resonance, this characteristic non-locality is

$\kappa_D^{-1} \simeq 1.5 \; A^{\frac{1}{6}}$ fm. This non-locality is small on the level of atomic scales, apart from the heaviest elements. Consequently, the approximation (1) for $\ell \neq 0$ states is a good one. For S states, the non-locality will provide the appropriate regularization of the behaviour r^{-4}, together with finite size of the nucleus which is of similar magnitude.

The polarizability behaviour of the electronic atom is totally different and it is dominated by relativistic effects in the intermediate propagation. For relative distances inside the electron Compton wavelength $m_e^{-1} \simeq 400$ fm, the problem must be treated fully relativistically. In the range $\omega^{-1} \simeq 2.5 \; A^{\frac{1}{3}}$ fm $< r < m_e^{-1} \simeq$ $\simeq 400$ fm, closure in the electron variable can be made, while inside this region the electron behaves as a free massless object. These regions are relevant for atomic S states. For $\ell \neq 0$ states, relevant atomic distances are very much larger than these, so the approximation (1) is a good one. In all cases, the non-locality extends over a far wider region in space than the nuclear size, so the dependence on the details of the nuclear shape is expected to be very weak. In the relativistic case, ω^{-1} provides the regularization for the inner region and m_e^{-1} for the outer region.

The essential difference between the relativistic and non-relativistic limits emphasizes the great importance of the ratio of the nuclear excitation frequency to lepton mass $\omega/2m$ as a measure of the relativistic nature of the system. The leptonic mass dependence of the S state polarization effects in an atom is totally different when this quantity is larger or smaller than unity.

In the following we will investigate in detail the nature of these relations using the unretarded dipole approximation for the nuclear excitations. The results are expressed in terms of the sum rules given by the photoabsorption cross-sections. The calculation will be made in the dispersion approach used previously in Ref. 4), in the limit of a short-ranged interaction on the atomic scale. Since the characteristic momenta transferred to the nucleus are of the order $q^2 \sim 2m\omega$ the unretarded dipole approximation is expected to be excellent for electrons but somewhat worse for muons.

2. - THE POLARIZABILITY SHIFT

To set a formalism valid for both electronic and muonic atoms, we follow the relativistic dispersion approach discussed in Ref. 4). Since the interaction range is small compared to atomic dimensions and the velocity corrections are very

small, the wave function at the origin can be factorized as $|\psi_{no}(0)|^2$. The remainder is proportional to the 2γ contribution to the S wave scattering length (scattering approximation)

$$\Delta E_n^{\ell=0} = - |\psi_{no}(0)|^2 \frac{A_{2\gamma}^{\ell=0}}{2m} \tag{3}$$

Therefore the problem is equivalent to the study of the forward virtual Compton amplitude on the nucleus. The hadronic tensor is decomposed in the usual gauge invariant way and becomes determined by two structure functions $T_{1,2}(q^2,\omega)$. They describe the interaction of transverse and longitudinal photons with the nucleus.

We introduce the longitudinal amplitude and assume that the nuclear $T_2(q^2,\omega)$ and $T_L(q^2,\omega)$ functions satisfy unsubtracted dispersion relations, i.e., validity of second order perturbation theory in the nuclear variables. The corresponding inelastic excitations in the nuclear response will be described in the unretarded electric dipole approximation, in which they satisfy

$$\frac{W_L}{-q^2} \simeq \frac{W_2}{-q^2} \simeq \frac{W_1}{\omega^2} \simeq M \sum_{N \neq 0} |\langle N | D_z | 0 \rangle|^2 \delta(\omega - \omega_N) \tag{4}$$

where D_z is the dipole operator. Retardation in the longitudinal component would imply an additional form factor $F_N(q^2)$. In the transverse component, there are magnetic and higher multipole contributions. We shall neglect all of these and keep the relation (4), which guarantees the correct $q^2 \to 0$ behaviour. The $q^2 \to 0$ limit of the structure function gives the nuclear photo-absorption cross-section, so that

$$\sigma_\gamma(\omega) = \left(\frac{e^2}{4\pi}\right) \frac{4\pi^2}{M} \lim_{q^2 \to 0} \frac{\omega \, W_2(q^2,\omega)}{-q^2} \tag{5}$$

We shall not try an explicit evaluation of the inelastic excitation strength, but instead connect the answer to sum rules involving the photon cross-section. The relevance of the connections (4) and (5) to our problem depends on the important region of the virtual photon mass squared q^2 explored in the polarizability interaction. This study can answer whether the nuclear content is the one associated with real photon physics or rather the one of inelastic electron scattering. As we shall see below, the effective momentum transfer to the nucleus is small $(-q^2) \approx 2m\omega$.

We use the relativistic free lepton propagation, neglecting Coulomb distortion corrections in the intermediate state. These effects have been estimated by Friar [7] to lower the result in muonic atoms by about 10%. With all these ingredients, the energy shift for S states is given by

$$\Delta E_n^{\ell=0} = - \left(\frac{e^2}{4\pi}\right) |\psi_{no}(0)|^2 \frac{4m}{\pi^3} \int_{\omega_{th}}^{\infty} d\omega \ \sigma_\gamma(\omega)$$

$$\int_0^{\infty} \frac{dt}{t} \int_0^{\sqrt{t}} d\xi \sqrt{t-\xi^2} \ \frac{t + 2\xi^4/t}{(\omega^2+\xi^2)(t^2+4m^2\xi^2)} \tag{6}$$

For a fixed value of the nuclear excitation ω, the $t = -q^2$ region of integration is controlled by the lepton mass and by ω. One can check from Eq. (6) that the t integrand is invariant under a change of variable $t \rightarrow (4m^2\omega^2/t')$, in such a way that the central region to be discussed corresponds to $t \approx 2m\omega$, which dominates the intermediate excitations. The other parameters of the problem are $4m^2$ and ω^2, so that one sees that the physics changes completely in going from electrons to muons. For nuclei other than the proton, assuming that the dominant inelastic excitations satisfy $2m_e < \omega < 2m_\mu$, the values of t probed in each case are $4m_e \lesssim t_e \lesssim \omega^2$ for electrons, whereas one has the condition $\omega^2 \lesssim t_\mu \lesssim 4m_\mu^2$ for muonic atoms. We explore the behaviour of the polarizability contribution under these conditions.

Muonic atoms

In the non-relativistic approximation, the t integrand of Eq. (6) for muons becomes

$$\frac{4m_\mu}{\omega\sqrt{t}} \ \frac{1}{t + 2m_\mu\omega} \tag{7}$$

Remarkably, this function is able to reproduce the correct asymptotic limits for $t \rightarrow 0$ and $t \rightarrow \infty$ given by Eq. (6). The $t \rightarrow 0$ limit is the one given by the static approximation, not relevant in our case of S states. When the t values are not restricted by other scales, Eq. (7) indicates that the muon intermediate states have a kinetic energy $t/2m_\mu \sim \omega$. One concludes therefore that the muon does not act as a static external field.

Inserting Eq. (7) into the energy shift, we obtain in the non-relativistic case

$$\Delta E_n^{\ell=0} \simeq - \left(\frac{e^2}{4\pi}\right) |\psi_{no}(0)|^2 (2m_\mu)^{1/2} \frac{1}{\pi} \int_{\omega_{th}}^{\infty} d\omega \ \frac{\sigma_\gamma(\omega)}{\omega^{3/2}} \tag{8}$$

This is the result obtained by Friar [7] and Ericson [8] working directly with non-relativistic muons. The answer is given in terms of the $\sigma_{-\frac{3}{2}}$ sum rule for photons. From our relativistic expression, we can easily make an expansion in $(\omega/2m_\mu)^{\frac{1}{2}}$ to modify the result (8). However, we have realized that this expansion is not appropriate. On the one hand, this expansion converges very slowly even for $\omega/2m_\mu$ significantly below unity. On the other hand, form factor effects become important in the same region. In fact, these results were obtained by means of the unretarded dipole approximation. As the virtual photon mass values $t^{\frac{1}{2}} \sim (2m_\mu\omega)^{\frac{1}{2}}$ are not much smaller than the inverse size of the nuclear system, one expects deviations from the real photon physics for muons. When the virtual photon physics is introduced, form factors will cut off the t region well before $\sim 4m_\mu^2$, so that the energy weighted sum rule $\sigma_{-\frac{3}{2}}$ would become more and more [*] like σ_{-2}.

Electronic atoms

The situation changes drastically here. For $t_e \sim (2m_e\omega)$, electrons are very relativistic. The behaviour obtained from Eq. (6) is in this region

$$\frac{5}{2}\frac{m_e}{\omega^2}\ \frac{1}{t} \tag{9}$$

with a t^{-1} effective dependence of the polarizability interaction. The behaviour (9) is regulated by the changes of regime for large t at $t \sim \omega^2$ and for small t at $t \sim 4m_e^2$. At $t \to \infty$ and $t \to 0$ the behaviour is, respectively, $4m_e/\omega \cdot 1/t^{\frac{3}{2}}$ and $2/\omega^2 \cdot 1/\sqrt{t}$. It is natural to expand in the small parameter $(2m_e/\omega) \ll 1$. In the extreme relativistic limit, our result is

$$\Delta E_n^{\ell=0} \simeq -\left(\frac{e^2}{4\pi}\right)|\psi_{no}(0)|^2\ 5m_e\ \frac{1}{2\pi^2}\int_{\omega_{th}}^{\infty}d\omega\ \frac{\sigma_r(\omega)}{\omega^2}\left[\ln\frac{2\omega}{m_e} + \frac{19}{30} + O\left(\frac{m_e^2}{\omega^2}\ln\frac{\omega}{m_e}\right)\right] \tag{10}$$

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[*] When the t dependence of Eq. (7) is modified by a factor $t_0/(t+t_0)$, with $t_0 \equiv (3/\langle r^2\rangle)$, to approximately account for the dipole retardation, the sum rule (8) is modified as

$$\sigma_{-3/2} \longrightarrow \int_{\omega_{th}}^{\infty}d\omega\ \frac{\sigma_\gamma(\omega)}{\omega^{3/2}}\left(1 + \sqrt{\frac{2m\omega\langle r^2\rangle}{3}}\right)^{-1}$$

Apart from the modulation factor introduced by the logarithm, the sum rule present in Eq. (10) is just the one associated with the polarizability as the physical parameter. This does not mean that the static limit has any relevance. On the contrary, as we have seen, the lepton propagation is relativistic. What happens is that $t_e \sim 2m_e\omega$ is still very small with respect to nuclear scale and to ω^2, so that the nuclear physics involved is that of real photons. In view of this, we expect the result (10) to be excellent for electronic atoms. It is remarkable that hadronic corrections to a higher QED contribution gives such a simple and reliable theoretical answer for a relativistically propagating system.

If the polarizability shift (6) is written as

$$\Delta E_n^{\ell=0} \propto \frac{|\psi_{no}(o)|^2}{m} \int_{\omega_{th}}^{\infty} d\omega \, \sigma_{\gamma}(\omega) \, F(m,\omega) \qquad (11)$$

the function $F(m,\omega)$ is dimensionless and only depends on the ratio $(2m)/\omega$. Although its analytic form is so different for the two extreme situations, given by

$$5 \, \frac{m^2}{\omega^2} \left(\ln \frac{2\omega}{m} + \frac{19}{30} \right) \qquad \text{for} \qquad \frac{2m}{\omega} \to 0$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12)$

$$\pi \left(\frac{2m}{\omega} \right)^{3/2} \qquad \text{for} \qquad \frac{2m}{\omega} \to \infty$$

the connection between the two regimes for $F(m,\omega)$ is very smooth. For the ratio $(2m/\omega) = 1$ the exact value from Eq. (6) is 2.71. The extrapolated value using the relativistic expression is 2.62 while the non-relativistic extrapolation gives 3.14, i.e., they are good to -3% and +16%, respectively, at this interpolating point.

3. - RESULTS AND DISCUSSION

Our results (8) and (10) are the answer for the energy shift in atomic S states due to dipole polarizability corrections. The great difference for muons as compared to electrons comes from the regimes of lepton propagation which is non-relativistic (but not static) for muons, whereas it is extreme relativistic for electrons. In both cases, the relevant virtual photon mass probing the nuclear excitation is given by $t \sim 2m\omega$, where ω is the nuclear excitation frequency. This is a very small quantity for electrons, much smaller than the inverse size squared of the nucleus. Thus, the most important nuclear ingredient is the

dynamics for real photons. As Eq. (4) guarantees the correct low t behaviour we expect the result (10), which used the unretarded dipole approximation as an intermediate step, to be excellent for electrons. The accuracy of the result (8) for muons is less conclusive. Relativistic corrections are not the problem and they can be automatically included in our approach. But the values of $t_\mu \sim 2m_\mu \omega$ are not so small compared to characteristic momentum transfers for nuclear structure. The unretarded dipole approximation result should be taken as only an estimate of the answer. Retardation and the physics of electron scattering become relevant in a more accurate discussion.

The different behaviour of the lepton propagation for muons and electrons can be understood in configuration space, using time-ordered perturbation theory. The Green's function for the intermediate state lepton propagation is proportional to

$$m \int \frac{d^3q}{E(\vec{q})} e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \left\{ \frac{1}{E(\vec{q})+\omega-m} + \frac{1}{E(\vec{q})+\omega+m} \right\} \quad (13)$$

where $\vec{x}$, $\vec{y}$ are lepton co-ordinates. For distances larger than the Compton wavelength of the lepton, the $\vec{q}$ values are non-relativistic. This is relevant to the muonic atom, because $\omega < 2m_\mu$. The second term in Eq. (13) is negligible and one has

$$G(\omega, \vec{x}-\vec{y}) \propto \int_0^\infty dq\, q\, \frac{\sin q|\vec{x}-\vec{y}|}{|\vec{x}-\vec{y}|} \frac{1}{\omega + q^2/2m} = m \frac{\exp\{-\kappa|\vec{x}-\vec{y}|\}}{|\vec{x}-\vec{y}|} \quad (14)$$

with a Yukawa-type propagation given by $\kappa = (2m\omega)^{\frac{1}{2}}$. Equation (14) is valid for $|\vec{x}-\vec{y}| > m^{-1}$.

For distances smaller than the Compton wavelength of the lepton, one can use $E(\vec{q}) \approx q$. This is relevant to the electronic atom, because $\omega \gg 2m_e$. Then both terms of Eq. (13) are equally important. One gets

$$G(\omega, \vec{x}-\vec{y}) \propto m \int_0^\infty dq\, \frac{\sin q|\vec{x}-\vec{y}|}{|\vec{x}-\vec{y}|} \frac{1}{\omega + q} \quad (15)$$

For $\omega|\vec{x}-\vec{y}| > 1$, one gets a behaviour

$$\frac{m}{\omega} \frac{1}{|\vec{x}-\vec{y}|^2}$$

from Eq. (15), whereas at very short relative distances $|\vec{x}-\vec{y}| < \omega^{-1}$ the nuclear excitation energy is negligible with a propagation

$$\frac{\pi}{2} \; m \; \frac{1}{|\vec{x} - \vec{y}|}$$

The difference between the Yukawa propagation of Eq. (14) and the inverse power behaviour $|\vec{x}-\vec{y}|^{-2}$ in the relativistic case is responsible for the answers we have obtained in muonic and electronic atoms.

We have evaluated the sum rules using Rinker's parametrization [5] of the experimental photo-absorption cross-section in ^{4}He. For muons, this can be compared to the realistic accepted value of -3.1 meV (with uncertainty of about 20%) to elucidate the relevance of the real photon sum rule (8). The absolute value given by Eq. (8) overestimates the detailed predictions by about 50% [*]. Recently, the charge radius of ^{12}C has been determined to high precision independently by electron scattering [9] and muonic X-ray transitions [10]. For the interpretation of the muonic experiment, it is imperative to include the polarizability shift. On the basis of Eq. (8), corrected for finite size effects, we estimate it to be about -1.8 eV.

The polarizability shift in electronic atoms is small due to the proportionality to the electron mass. The logarithmic factor within the polarizability sum rule of Eq. (10) only compensates part of the mass factor. For ^{4}He, we get $\sigma^{log}_{-2}/\sigma_{-2} \approx 5$, where σ^{log}_{-2} is the sum rule appearing in Eq. (10). It is interesting to compare the polarizability shift in electronic atoms to the finite size shift due to the nuclear size. This is given by

$$\Delta E^{\ell=0}_{f.s.} = \left(\frac{e^2}{4\pi}\right) Z \; \frac{2\pi}{3} \langle r^2 \rangle \; |\psi_{no}(0)|^2 \tag{16}$$

and we get $\Delta E^{\ell=0}_{pol}/\Delta E^{\ell=0}_{f.s.} \approx -5\times10^{-4}$ for the $(e^4He)^+$ system. The variation of the relative polarizability shift with A is very weak throughout the periodic system. In heavier nuclei, we have estimated them to be about $-(1-2)\times10^{-4}$ compared to the finite size shift.

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[*] See footnote on p. 5. With the finite size modification, the result of the sum rule (8) would be decreased by about 40%, indicating that this effect is the main remaining correction.

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