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CP PROPERTIES OF THE LEPTONIC SECTOR FOR MAJORANA NEUTRINOS

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A B S T R A C T

The leptonic sector of the electroweak theory is analyzed for massive Majorana neutrinos. For n generations, the Majorana mass Lagrangian is diagonalized using the polar reduction to guarantee physical positive masses independently of the CP properties or the choice of the phases of the fields. When CP invariance holds, the CP eigenvalues of the definite mass neutrino fields are determined without commitment to a particular phase choice. For charged current interactions, we find that the observable CP violating phases can be parametrized à la Kobayashi-Maskawa for the vertex. Extra (n-1) relative phases of the massive neutrino fields are significant. The extra phases are observable only in processes mediated by "neutrino-antineutrino" propagation and therefore proportional to neutrino masses. We discuss the informational content of the relevant processes.

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1. - INTRODUCTION

There is at present a big effort to look for non-vanishing neutrino masses [1]. From the point of view of the fermion content, the most economical way to generate neutrino masses is à la Majorana. The standard electroweak theory, with the minimal Higgs doublet, cannot accommodate massive Majorana neutrinos. A simple model that gives a Majorana mass to neutrinos is the standard model with a Higgs triplet in addition to the usual doublet [2]. Most of the grand unification schemes lead to $SU(2)\times U(1)$ at intermediate energies, so it is interesting to investigate the properties of the leptonic sector of the theory within the standard scheme, only modified by Majorana neutrino masses.

Several authors [3)-5)] have noticed that, after the diagonalization of the Majorana-type mass terms, the leptonic charged current contains more CP violating phases than the ones discussed by Kobayashi-Maskawa [6] for three generations in the quark sector. These additional phases are characteristic of Majorana neutrinos and stem from the fact that the phase of the massive Majorana field has been fixed to satisfy the condition $\nu(x) = \nu(x)^c \equiv \mathcal{C}\overline{\nu(x)}^T$, i.e., a real field in the Majorana representation. Wolfenstein [7] has pointed out, however, that in theories with CP invariance the relative signs of the CP eigenvalues of the massive Majorana neutrinos are significant. For different CP eigenvalues, the corresponding Cabibbo matrix for charged currents would have relative phases $\pi/2$ in the above treatments, so there will be relative imaginary couplings for the charged current weak interactions. Instead of being a maximally CP violating interaction, as sometimes referred to in the literature, this situation corresponds to a CP invariant theory. Therefore, with the choice $\nu(x) = \nu(x)^c$ one has to analyze whether the extra phases of the charged current interaction are a true signal of CP violation.

Usually, the Majorana mass matrix is diagonalized using a congruent transformation with a unitary matrix. In this case all phases of the unitary matrix, and therefore the phases of the physical neutrino fields, are fixed by the requirement that the mass eigenvalues be positive. This immediately leads to situations, such as the one discussed by Wolfenstein, in which the phases appearing in the couplings of the leptonic fields to $W^{\pm}$ have nothing to do with CP violation. In this paper we propose a different scheme which guarantees diagonal positive masses without commitment to a particular choice of phases for the physical neutrinos. In Section 2, we diagonalize the Majorana mass matrix using the polar reduction and a similarity transformation with a unitary matrix and we obtain the physical neutrino fields in terms of the weak interaction current fields. The phases of these massive fields are linked to the arbitrary phases of the unitary matrix of the similarity transformation. The transformation properties of these Majorana fields under discrete symmetries are then discussed for arbitrary phases. Section 3 particularizes to the CP invariant case, showing that the obtained physical fields have CP eigenvalues determined univocally by the same procedure that diagonalizes the mass matrix. We then have physical neutrino fields with definite positive mass and definite CP eigenvalues without using a particular choice of the field phases. This freedom is used in Section 4 to reduce the CP violating phases of the vertex for the leptonic charged current as is conventionally done in the hadronic sector. The resulting (n-1) relative phases of the physical neutrino fields are, however, observables in those processes of neutrino-antineutrino propagation mediated by the mass term in the Lagrangian. Furthermore, we explicitly build the relevant propagators and the amplitudes associated with the physical observables. In Section 5, some discussion and the informational content of the corresponding processes such as $\beta\beta$ decay, $\mu^- \to e^+$ conversion or neutrino-antineutrino oscillations are given. Some necessary properties of matrices are included in the Appendix.

2. - MASSIVE MAJORANA NEUTRINOS

The fermion content of the theory is the left-handed doublet $\{\nu', \ell'\}_L$ and the singlet ℓ'_R. Because of CPT invariance, the corresponding conjugate field ν'^*_L also appears. The most general mass Lagrangian which is self-adjoint and invariant under proper Lorentz transformations, is

$$\mathcal{L}_m(x) = -\frac{1}{2}\,[\nu'_L(x)]^T \mathcal{C}\, M'\, \nu'_L(x) - \frac{1}{2}\,[\nu'_L(x)]^+ \mathcal{C}\, M'^+ [\nu'_L(x)]^* \tag{1}$$

where the matrix $\mathcal{C}$ satisfies

$$\mathcal{C}\,\gamma^\mu\,\mathcal{C}^{-1} = -\gamma^{\mu T} \quad , \quad \mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^+ = -\mathcal{C}^* = -\mathcal{C}^T \tag{2}$$

Without any restriction, M' can be taken as a complex symmetric matrix : $M' = M'^T$. Assuming that M' is a regular matrix, we then write it, using the polar reduction, as

$$M' = M\,U \quad , \qquad M \equiv \sqrt{M'\,M'^+} \tag{3}$$

where M is the unique square root of the self-adjoint positive definite matrix $M'M'^{+}$, and U is the unitary matrix given by $U = M^{-1}M'$. In our case, the symmetry of M' implies (see the Appendix)

$$U = U^{T} \tag{4}$$

Let us consider the positive definite matrix M. Its eigenvalues are real and positive, and it is diagonalized by a similarity transformation

$$M = S^{+} D S \tag{5}$$

where the unitary diagonalizing matrix is built as $[S^{+}]_{ik} = \eta^{(k)} x_i^{(k)}$, where $x_i^{(k)}$ is the i component of the eigenvector associated to the (k) eigenvalue, which for simplicity sake, we will assume to be non-degenerate. Notice that the $\eta^{(k)}$ are arbitrary phases of the diagonalizing matrix S, once a choice of $x_i^{(k)}$ is made, whereas U is completely fixed when $\det(M') \neq 0$.

Using this procedure, the mass Lagrangian of Eq. (1) becomes

$$\mathcal{L}_m(x) = -\frac{1}{2}\,[\nu'_L(x)]^T \mathcal{C}\, S^{+} D S U\, \nu'_L(x) - \frac{1}{2}\,[\nu'_L(x)]^{+} \mathcal{C}\, U^{+} S^{+} D S\, [\nu'_L(x)]^{*} \tag{6}$$

Let us define the physical Majorana neutrino field as

$$\begin{aligned} \nu(x) &\equiv S U\, \nu'_L(x) + S\gamma^0 \mathcal{C}\, [\nu'_L(x)]^{*} \\ &= S U\, \nu'_L(x) - S\, \mathcal{C}\, \overline{\nu'_L(x)}^{T} \end{aligned} \tag{7}$$

with the desired result

$$\mathcal{L}_m(x) = -\frac{1}{2}\, \overline{\nu(x)}\, D\, \nu(x) \tag{8}$$

As seen in Eq. (7), the phase of the massive field is linked to the (arbitrary) phase $\eta^{(k)}$ of the diagonalizing matrix S. In general, $\nu(x)$ and $\nu(x)^c \equiv \mathcal{C}\,\overline{\nu(x)}^T$ are related by

$$\nu(x) = -\, S U S^{T}\, \nu(x)^{c} \tag{9}$$

We show in the Appendix that SUS^T is a unitary diagonal matrix of phases, so that the n relations (9) are decoupled for each massive neutrino. Of course, one could adjust the phases so that the usual convention for Majorana fields is obtained, corresponding here to the condition $SU = -S^*$. This is, however, unfortunate because the matrix S will appear in the generalized Cabibbo matrix for charged currents (see below) and it is uncontrollable whether these phases are genuine observable CP violating phases. We prefer to keep the Majorana field with phases, as given by Eq. (9). In fact, it is immediate to see that the CPT transformed field of $\nu(x)$ is precisely

$$(CTP)\,\nu(x)\,(CTP)^{-1} = i\,\mathcal{T}\,\mathcal{C}\,[\nu(-x)]^* = [SUS^T]^+\, i\,\mathcal{T}\,\gamma^0\,\nu(-x) \tag{10}$$

where $\mathcal{T}$ satisfies

$$\mathcal{T}\,\gamma^\mu\,\mathcal{T}^{-1} = \gamma^{\mu T}\ , \qquad \mathcal{T} = \mathcal{T}^{-1} = \mathcal{T}^+ = -\mathcal{T}^* = -\mathcal{T}^T \tag{11}$$

<u>Equation (10) defines the Majorana field for physical interacting neutrinos</u>. The phase relation between $\nu(x)$ and $\nu(x)^c$ that we impose for the Majorana field is the one coming out when the free field is explicitly written as

$$\psi(x) = \frac{\phi}{(2\pi)^3}\int\frac{d^3p}{2E(\vec{p})}\sum_\lambda\left\{u(\vec{p},\lambda)\,a(\vec{p},\lambda)\,e^{-ipx} + \varphi\,u^c(\vec{p},\lambda)\,a^+(\vec{p},\lambda)\,e^{ipx}\right\} \tag{12}$$

with $u^c(\vec{p},\lambda) \equiv \mathcal{C}\bar{u}^T(\vec{p},\lambda)$ and we allow for arbitrary phases ϕ and φ. The field (12) satisfies $\psi(x) = (\varphi\phi^2)\psi(x)^c$, so we see the meaning of the diagonal matrix SUS^T appearing in Eq. (9).

We are interested in the Green's functions associated with the Majorana neutrino propagation. Using conventional terminology, the "neutrino-neutrino" propagation is given by

$$S(p) \equiv -i\int d^4x\, e^{ipx}\langle 0|T(\psi(x)\,\overline{\psi(0)})|0\rangle = \frac{\not p + m}{p^2 - m^2 + i\epsilon} \tag{13}$$

as for Dirac particles. The new ingredient for Majorana particles is the "neutrino-antineutrino" propagation given by

$$\tilde{S}(p) \equiv -i\int d^4x\, e^{ipx}\langle 0|T(\psi(x)\,\psi(0)^T)|0\rangle = -(\varphi\phi^2)\,\frac{m\,\mathcal{C}}{p^2 - m^2 + i\epsilon} \tag{14}$$

proportional to the neutrino mass m. Unlike (13), the propagation (14) contains the product of phases ϕ times $\varphi\phi$ of the phases present in the field (12). For our massive neutrino fields given in Eq. (7) for n generations, the set of phases $-(\varphi\phi^2)$ are the ones of SUS^T.

3. - CP INVARIANCE

If we assume CP invariance, Majorana neutrinos have definite CP eigenvalues. How do they appear through the diagonalization? Which eigenvalue corresponds to each generation? Given the current neutrino field $\nu'_L(x)$, we know from CPT invariance that the CP transformed field exists and we can write

$$(CP)\ \nu'_L(x)\ (CP)^{-1} = -i\ B\ \mathcal{C}\,[\nu'_L(x^0,-\vec{x})]^* \tag{15}$$

where B is a diagonal matrix of phases. Under the transformation (15) of the current fields all other terms of the Lagrangian, including the interaction, are invariant. The condition that the mass Lagrangian (1) is also CP invariant implies that the mass matrix M' has to satisfy the hermiticity condition

$$(B\,M')^+ = B\,M' \tag{16}$$

The condition (16) translates equally into the unitary matrix defined by (3), so that

$$(B\,U)^+ = B\,U \tag{17}$$

Using Eq. (15), the massive neutrino fields (7) are CP transformed to

$$(CP)\ \nu(x)\ (CP)^{-1} = -i\ \gamma^0\ S\,U\,B\,S^+\ \nu(x^0,-\vec{x}) \tag{18}$$

Because $SUBS^+$ is a unitary and hermitian matrix [see Eq. (17)] we are able to prove in the Appendix that it is a real diagonal matrix of elements $+1$ and/or -1. Thus the diagonalized fields $\nu(x)$ have definite CP eigenvalues given by $-SUBS^+$. In terms of these CP values, the massive neutrino fields (7) can be written, using (17), as

$$\nu(x) = (SU)\ \nu'_L(x) + (SUBS^+)(SU)\,B\,\gamma^0\,\mathcal{C}\,[\nu'_L(x)]^* \tag{19}$$

whose meaning is immediate as a combination of one field and its CP transformed with relative sign as given by the resulting CP eigenvalue in $(SUBS^+)$. Notice that these CP values are independent (as it should be!) of the choice of phases made in the diagonalizing matrix S or, equivalently, of the phases choice $(\varphi\phi^2)$ of the massive Majorana fields. An analysis of the phases present in the eigenvectors $x_i^{(k)}$ (see above) shows that

$$(S\,B^+ S^T)_{ij} = -\delta_{ij}\,\eta^{(i)*}\,\eta^{(i)*} \tag{20}$$

where the $\eta^{(i)}$ are the arbitrary phases. Consequently, $SUS^T \equiv (SUBS^+)(SB^+S^T)$ implies the connection for each massive field

$$(\varphi\phi^2)_k = (\eta^{(k)*})^2\,(\epsilon_{CP})_k \tag{21}$$

4. - CHARGED CURRENT INTERACTION

Let us now consider the charged current interaction for the leptonic sector

$$\mathcal{L}_{cc}(x) = \frac{g}{\sqrt{2}}\left\{\overline{\nu'_L(x)}\,\gamma^\mu\,\ell'_L(x)\,W_\mu^+(x) + \overline{\ell'_L(x)}\,\gamma^\mu\,\nu'_L(x)\,W_\mu(x)\right\} \tag{22}$$

Assuming that the charged leptons have already been diagonalized, with all the mixing coming from the neutrino fields, the above Lagrangian is now written in terms of the definite mass neutrino fields as

$$\mathcal{L}_{cc}(x) = \frac{g}{2\sqrt{2}}\left\{\overline{\nu(x)}\,U^{(L)}\gamma^\mu(1+\gamma_5)\,\ell(x)\,W_\mu^+(x) + \overline{\ell(x)}\,\gamma^\mu(1+\gamma_5)\,U^{(L)+}\nu(x)\,W_\mu(x)\right\} \tag{23}$$

where $U^{(L)} \equiv SU$ is the generalized Cabibbo mixing matrix for the n generations of the leptonic sector. How many parameters appear? The unitary matrix $U^{(L)}$ has in general $(n-1)n/2$ moduli and $n(n+1)/2$ phases whose number is, by redefinition of the unobservable phases of the charged lepton fields, reduced to

$$\frac{1}{2}(n-1)\,n \quad \text{moduli} \quad , \qquad \frac{1}{2}(n-1)\,n \quad \text{phases} \tag{24}$$

Are all the remaining phases observable CP violating phases? If the phases of the Majorana neutrino fields were all fixed by the condition $-SUS^T = 1$, the $\eta^{(k)}$ phases of S would be fixed automatically with no freedom left in the vertex matrix $U^{(L)}$. The result would be given by Eq. (24), with a parametrization as presented by other authors [4),5)]. With this strategy, the problem which remains is to understand whether phases different from 0 or π correspond to CP violation. Our equation (21) for a CP conserving situation shows that this is <u>not</u> the case. In fact, if $-SUS^T = I$, whenever one has CP eigenvalues with opposite sign for the massive neutrinos, there will be relative phases i among the $\eta^{(k)}$ already fixed and they are translated into the Cabibbo matrix $U^{(L)}$. We prefer a different approach in which the phases $-SUS^T$ are not fixed from the beginning to be $+1$. This method will allow a clear-cut distinction between the CP invariant and CP violating situations.

The natural arbitrariness in our method to diagonalize the mass matrix corresponds to the n phases $\eta^{(k)}$ of S. They are used to absorb phases of $U^{(L)}$ by an appropriate choice. In this way, the different Majorana fields have different phase choices, but the vertex for charged currents is given by the Kobayashi-Maskawa description for three generations. Our parametrization in order to consider a CP violation scenario is thus the following :

$$U^{(L)} \longrightarrow \begin{cases} \frac{1}{2}(n-1)\,n & \text{moduli} \\ \frac{1}{2}(n-2)(n-1) & \text{phases} \end{cases} \tag{25}$$

$$\oplus$$

$$(n-1) \ \text{phases of the Majorana fields}$$

What distinguishes Majorana fields from Dirac fields is that the last $(n-1)$ phases of Eq. (25) are, a priori, observable. Notice that the "neutrino-antineutrino" propagator (14) contains the phases given by $S\bar{U}S^T$ and they cannot be adjusted anymore. On the other hand, the recipe (25) emphasises the fact that the additional $(n-1)$ phases are only observable for these processes which need this characteristic propagator for Majoranas, the amplitudes of which all vanish with the neutrino mass.

Two comments are in order. First, the $(n-1)$ relative phases of the Majorana fields can be complex and a signal of CP violation. This implies the possibility of having CP violation for two generations. The amplitude of a process in which the lepton ℓ_j^+ converts into a charged lepton ℓ_i^- is determined from Eqs. (23) and (14) to be

$$X(\ell_j^+ \to \ell_i^-) = -\sum_k U^{(L)*}_{ki} \frac{m_k (\varphi\phi^2)_k}{p^2 - m_k^2 + i\epsilon} U^{(L)*}_{kj} \tag{26}$$

so that the complete set of parameters (25) is relevant. This is in contrast with the processes in which a charged lepton ℓ_j^- finishes as a charged lepton ℓ_i^- for which the amplitude is given by

$$y(\ell_j^- \rightarrow \ell_i^-) = \sum_k U^{(L)*}_{ki} \frac{\not p + m_k}{p^2 - m_k^2 + i\epsilon} U^{(L)}_{kj} \qquad (27)$$

so that the parameters of $U^{(L)}$ enter, but not the additional $(n-1)$ phases of the fields. Processes governed by Eq. (27) are the ones allowed for massive Dirac neutrinos. The CP violating information contained in these neutrino-neutrino oscillations has been discussed by Cabibbo [8].

Second, the set of phases (25) describes CP violation in the sense that values different from 0 or π are not reproducible in a CP invariant theory. It is a simple exercise, using Section 3 of this paper, to see that the choice of phases $\eta^{(k)}$ of S that makes $U^{(L)}$ a real matrix leads to phases of the Majorana fields that are relatively real and given by their relative CP eigenvalues. Equation (26) for the $\ell_j^+ \rightarrow \ell_i^-$ amplitude becomes

$$X(\ell_j^+ \rightarrow \ell_i^-) = \sum_k U^{(L)}_{ki} \frac{m_k (\epsilon_{CP})_k}{p^2 - m_k^2 + i\epsilon} U^{(L)}_{kj} \qquad (28)$$

only depending on the $(n-1)n/2$ angles of the orthogonal Cabibbo matrix $U^{(L)}$ and the $(n-1)$ relative signs of the CP eigenvalues of the Majorana neutrinos.

5. - APPLICATIONS

The conclusion of this study is that, in the presence of CP violation, there are $(n-1)$ additional phases in the leptonic sector for Majorana neutrinos which are, a priori, observable. They add to the well-known Kobayashi-Maskawa phases. Although in the observable quantities [see Eq. (26)] these additional phases can be moved from one place to another, we find it useful to describe the situation saying that the $(n-1)$ relative phases of the Majorana fields are observable. This peculiarity is associated with the mass Lagrangian of the theory and is a consequence of the mass insertion present in the neutrino-antineutrino propagation. Outside this phenomenon, these phases are irrelevant. Notice that when we talk about the relative phases of Majorana fields we refer to the combination $(\varpi\phi^2)$ of Eq. (12) which connects $\psi(x)$ with $\psi(x)^c$ in each generation.

Assuming that only two of the n generations were coupled, parametrizing $U^{(L)}$ with a Cabibbo angle θ and putting the relative phase α between the two Majorana neutrinos, Eq. (26) gives

$$X = \begin{bmatrix} X_1 \cos^2\theta + e^{i\alpha} X_2 \sin^2\theta & (X_1 - e^{i\alpha} X_2) \sin\theta \cos\theta \\ (X_1 - e^{i\alpha} X_2) \sin\theta\cos\theta & X_1 \sin^2\theta + e^{i\alpha} X_2 \sin^2\theta \end{bmatrix} \tag{29}$$

where $X_i \equiv m_i/(p^2-m_i^2+i\varepsilon)$. When the neutrino masses are neglected in the denominators, with respect to the momenta, the amplitudes for these lepton number violating processes become

$$\begin{aligned} &\beta\beta\text{ - decay} &:&\quad m_1 \cos^2\theta + e^{i\alpha} m_2 \sin^2\theta \\ &\mu^+\text{-}e^-\ \text{conversion} &:&\quad (m_1 - e^{i\alpha} m_2) \sin\theta\cos\theta \end{aligned} \tag{30}$$

which, for $\alpha = 0,\ \pi$, reduce to the CP invariant case. It is worth noticing that, for nearly degenerate massive neutrinos with equal CP eigenvalues, the $\mu^- - e^+$ conversion tends to cancel. When there are two degenerate neutrinos with opposite CP, they can be combined to form a Dirac neutrino for $\theta = \pi/4$. Although this Dirac neutrino does not correspond to the ones associated with each generation, we see that $\beta\beta$ decay would also disappear in this limit. The $\mu^- - e^+$ conversion, however, would remain as a possible process. The same matrix (29) of amplitudes X is responsible for the so-called neutrino-antineutrino oscillations described by the following situation. If, at time $t = 0$, an "antineutrino" is produced from a charged lepton ℓ_j^+ with definite momentum, what is the probability that at time $t > 0$ it would behave as a neutrino capable of producing a charged lepton ℓ_i^- ? The amplitude is proportional to

$$\sum_{k=1}^{n} U^{(L)*}_{ki}\ m_k\ e^{-iE_k t}\ e^{i\alpha_k}\ U^{(L)*}_{kj} \tag{31}$$

which, for two generations, reduces from the parametrization (24) to the following results. The probability that, if at time $t = 0$ it is produced by an e^+, the neutrino is able, at time t, to produce an e^-, is given by

$$P(e^+ \xrightarrow{t} e^-) = \frac{m_1 m_2}{E^2} \left\{ \frac{m_1}{m_2}\cos^4\theta + \frac{m_2}{m_1}\sin^4\theta \right.$$
$$\left. + \frac{1}{2}\sin^2 2\theta \cos\left[(E_1 - E_2)t + \alpha\right] \right\} \tag{32}$$

Analogously, the probability that at time t the same neutrino will be able to produce a μ^- is given by

$$P(e^+ \xrightarrow{t} \mu^-) = \frac{m_1 m_2}{E^2} \; \frac{1}{4} \sin^2 2\theta \left\{ \frac{m_1}{m_2} + \frac{m_2}{m_1} - 2 \cos\left[(E_1 - E_2)t + \alpha \right] \right\} \tag{33}$$

Formulas similar to the results (32) and (33) have been given by Schechter and Valle [9], but their CP violating phase would have to take values 0 or $\pi/2$ to reproduce a CP invariant theory. With the CP violating relative phase α given by us, the limit values $0(\pi)$ correspond to equal (opposite) CP eigenvalues of the physical interacting fields, leaving (changing) the sign of the oscillating function with time.

To summarize, the leptonic sector of the electroweak interactions is very rich in new phenomena when neutrinos are massive Majorana particles. With respect to the CP properties of the theory, the relative phases of the physical interacting fields are observable when a neutrino converts into an antineutrino accompanied by flavour violation. This is manifested in Eq. (10) when relating the field to its CPT transformed. When these relative phases of the fields are taken into account, the flavour violating vertex for charged current weak interactions can be parametrized as for the hadronic sector of the theory. The CP invariant limit of the theory is recovered when the relative phases of the physical fields go to signs describing their relative CP eigenvalues.

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APPENDIX

In this Appendix we collect some properties of matrices needed to develop Sections 2 and 3 of the paper.

1) Take $M' = MU$ [Eq. (3)] with $M'^T = M'$. The square root matrix

$$M = \sqrt{M' M'^+} \tag{A.1}$$

is [10] a polynomial in $M'M'^+$ uniquely determined

$$M = \sum_\alpha c_\alpha (M' M'^+)^\alpha \tag{A.2}$$

The unitary matrix $UU^+ = I$ then implies

$$\sum_{\alpha,\beta} c_\alpha c_\beta (M' M'^+)^{\alpha+\beta-1} = I \tag{A,3}$$

If we calculate U^+U^T we get

$$U^+ U^T = \sum_{\alpha,\beta} c_\alpha c_\beta (M' M'^+)^{\alpha+\beta-1} = I \tag{A.4}$$

when using $M'^T = M'$. Then $U^T = U$.

2) The matrix $A \equiv SUS^T$ is the one which connects (see Section 2) the diagonalized field $\nu(x)$ with $\nu(x)^c$ or, equivalently, $\nu(x)$ with its CPT transformed. As $U^T = U$, we see that A is a unitary symmetric matrix. Using the fact that S diagonalizes M by a similarity transformation, we have

$$M' = M U = U M^T = U M^* \tag{A.5}$$

and

$$\left.\begin{aligned} D A &= D S U S^T = S M S^+ S U S^T = S M' S^T \\ A D &= S U S^T D = S U S^T S^* M^* S^T = S M' S^T \end{aligned}\right\} \tag{A.6}$$

so that A is a diagonal matrix, as $SM'S^T$. This completes the proof that A is a diagonal matrix of phases. As we see, it is the matrix $DA = AD$, the one obtained by diagonalizing M' by a congruent transformation $SM'S^T$.

3) The CP eigenvalues of the physical neutrino fields (see Section 3) were determined by

$$C \equiv S\,U\,B\,S^{+} \tag{A.7}$$

which is a unitary self-adjoint matrix, because $(UB)^{+} = UB$. Apart from that, we have

$$\left.\begin{aligned} DC &= S M S^{+} S U B S^{+} = S M' B S^{+} \\ CD &= S B^{+} U^{+} S^{+} S M S^{+} = S B^{+} M'^{+} S^{+} = S M' B S^{+} \end{aligned}\right\} \tag{A.8}$$

so that C is a diagonal matrix, as $S(M'B)S^{+}$. This completes the proof that C is a diagonal matrix of signs. On the other hand, we see that the matrix $DC = CD$ is the one obtained by diagonalizing $(M'B)$ by the similarity transformation $S(M'B)S^{+}$.

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