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ON TOTAL MUON CAPTURE RATES AND THE AVERAGE NEUTRINO ENERGY

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A B S T R A C T

A method of avoiding the uncertainty associated with the average neutrino energy ν in the usual closure approach to muon capture rates is discussed. Instead of neglecting the kinematic dependence on each particular channel, the partial capture rate is approximated by a first order expansion around ν. After the sum over the final states is performed, the result is quite independent of the specific value of ν. Application to ^{3}He, ^{6}Li and closed shell nuclei is given, as an effective test of the nuclear models used.

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1. - INTRODUCTION

The experimental material accummulated on total muon capture rates in nuclei is rather abundant - see, for example, the results listed by Eckhause et al. [1]. The original motivation of this study was to establish the basic assumptions of the theory of semi-leptonic weak interactions. This tool has been disregarded due to the uncertainty in the nuclear physics aspects, and the possibility of inferring some quantitative information on the weak form factors is restricted to experiments on hydrogen, deuterium and other selected partial transitions [2].

The theory of total muon capture in nuclei has been developed especially by Primakoff [3] and by Luyten, Rood and Tolhoek [4]. In closure approximation, the rate can be written in terms of expected values of two-body operators on the initial state, and one can think of using these processes as an effective tool for studying nuclear structure. However, one has to introduce in the closure approximation an average neutrino energy ν and the result is very sensitive to the choice of ν. The uncertainty on this parameter is so important that conclusions for the adequacy of the nuclear model used cannot be reached.

Recently, Do Dang [5] has indicated a possibility of avoiding this difficulty retaining in the dipole contribution of the matrix element a first order expansion around the parameter ν. This is applied after the factorization of the elastic form factor, and then the ν_{ab} dependence in each channel is just ν_{ab}^4, as corresponds to the dipole term in the limit $\nu_{ab} \to 0$. In this paper it will be shown that the expansion of the <u>effective</u> ν_{ab} dependence - that is, for all possible multipoles and powers - is equivalent to a first order expansion of the <u>capture rate</u> around ν. Under this last form, the result is easily worked out to obtain independence on the specific value of ν, within a large range of plausible values.

In Section 2 our method is clearly stated. This leads to consider the study of two new terms besides the usual one appearing from the closure approximation. It is then pointed out how one of them can be calculated immediately from the "closure term", and the other one from a generalized Thomas-Reiche-Kuhn sum rule [6]. In our case, this sum rule is expected to hold for nuclei in which spin-dependent forces can be ignored, and thus the applications to closed shell nuclei is given in Section 3. In Section 4, the cases of ^3He and ^6Li are studied, where the L-S coupling is rather good. Finally Section 5 gives some discussion of the obtained results and our conclusions.

2. - THEORY

In the usual notation the muon capture rate to a particular final state $a \to b$ can be written in the form

$$\Lambda(a \to b) = [G_V^2 + 2G_A^2 + (G_A - G_P)^2] \frac{m^2 (m\alpha)^3}{2\pi^2} Z_{eff}^4 \Lambda_r(a \to b) \tag{1}$$

where the reduced capture rate Λ_r is given in the impulse approximation by

$$Z \Lambda_r(a \to b) = \left(\frac{\nu_{ab}}{m}\right)^2 \Big\{ A \,|\langle b| \sum_j \tau_j^- e^{i\vec{\nu}_{ab}\cdot\vec{x}_j} |a\rangle|^2$$
$$+ B\,|\langle b| \sum_j \tau_j^- e^{i\vec{\nu}_{ab}\cdot\vec{x}_j} \sigma_{x_j} |a\rangle|^2 + C\,|\langle b| \sum_j \tau_j^- e^{i\vec{\nu}_{ab}\cdot\vec{x}_j} \sigma_{z_j} |a\rangle|^2 \Big\} \tag{2}$$

and the sum and average over the final and initial polarizations are understood. A, B, C can be written in terms of the effective coupling constants G_V, G_A, G_P and their values are [7] A = 0.19, B = 0.69, C = 0.12.

The well-known result in closure approximation is obtained by summing over the final states b replacing ν_{ab} by an average value ν. In terms of the correlation functions $D_{S,T,L}$ introduced in Ref. 7), the total capture rate is obtained as

$$\Lambda_r(\nu) = \left(\frac{\nu}{m}\right)^2 \Big\{ A[1 - Z^{-1} D_S(\nu)] + B[1 - Z^{-1} D_T(\nu)] + C[1 - Z^{-1} D_L(\nu)] \Big\} \tag{3}$$

where the dependence of A, B, C on ν can be neglected.

We turn our attention to Eq. (2). Unlike the procedure of obtaining (3), the dependence on ν_{ab}, i.e., on each particular channel, is not completely neglected. In terms of a parameter ν we can write the identity $\nu_{ab} \equiv \nu\,[1 + (\nu_{ab} - \nu/\nu)]$ and the expansion of the n power of ν_{ab} is

$$\nu_{ab}^n \simeq \nu^n \left\{ 1 + \frac{n}{\nu} [\nu_{ab} - \nu] \right\} \tag{4}$$

This is analogous to the idea used in Ref. 5), because $\nu_{ab} = m - (E_b - E_a)$ *), for the dependence of the unretarded dipole term, after the approximation of factorizing the elastic charge form factor ; in this way, Eq. (4) was used in the particular case of n = 4. We are going to apply (4) for each contribution of (2) when the exponential $e^{i \vec{\nu}_{ab}\cdot\vec{x}}$ is developed in powers of ν_{ab}. It is obvious, and it can be proved straightforwardly, that this expansion is equivalent to approximate Eq. (2) by the following form

$$\Lambda_r (a \to b) \simeq \Lambda_r (a \to b)\Big|_{\nu_{ab}=\nu} + \frac{d\,\Lambda_r (a\to b)}{d\,\nu_{ab}}\Big|_{\nu_{ab}=\nu} (\nu_{ab} - \nu) \qquad (5)$$

This equation for partial transitions will be our starting point for the following discussion. If in (5) we are doing the sum over the final states b we obtain

$$\Lambda_r = \left\{ 1 + (m-\nu)\frac{d}{d\nu}\right\} \Lambda_r(\nu) - \frac{d}{d\nu}\left\{ \sum_b (E_b - E_a)\, \Lambda_r (a\to b)\Big|_\nu \right\} \qquad (6)$$

because the derivatives are calculated for $\nu_{ab} = \nu$, and identifying $\sum_b \Lambda_r(a\to b)\big|_\nu = \Lambda_r(\nu)$ as given by Eq. (3). The result of Eq. (6) must be quite independent on the parameter ν over a large zone of plausible values of the average neutrino energy. A specific value of ν will not be needed. $\Lambda_r(\nu)$ can be calculated once a nuclear model for the initial state is considered.

The last term in Eq. (6) is an energy-weighted sum rule, and we shall use the following theorem 6) : in a many-body system described by the Hamiltonian H = T + V, an operator $\sum_i f_i$ which commutes with the potential energy V satisfies

$$\sum_b (E_b - E_a) \left|\langle b | \sum_i f_i | a\rangle\right|^2 = \frac{1}{2M} \langle a | \sum_i \left\{ \vec{\nabla}_i f_i^+ \right\}\left\{ \vec{\nabla}_i f_i \right\} | a \rangle \qquad (7)$$

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*) In muon capture we have the transition (A,Z) → (A,Z-1) and then the Coulomb energy difference E_c must be included. This correction, $m \to m + E_c$, may be important for heavy nuclei. We acknowledge T. Ericson for this remark.

By seeing the structure of our operators - Eq. (2) - we expect to apply our considerations to nuclei in which spin-dependent forces can be ignored. This includes the closed shell nuclei ^{4}He, ^{16}O, ^{40}Ca, and the light nuclei with $A < 9$ [8], especially ^{3}He [9] and ^{6}Li [10], where the coupling to $L = 0$ gives a very good description.

Applying Eq. (7) to the matrix elements (2) our final expression is

$$\Lambda_r = \left\{ 1 + (m-\nu)\frac{d}{d\nu} \right\} \Lambda_r(\nu) - 2\,\frac{m}{M}\left(\frac{\nu}{m}\right)^3 \tag{8}$$

3. - APPLICATION TO CLOSED SHELL NUCLEI

Here we are going to study the cases of ^{4}He, ^{16}O and ^{40}Ca, using a double closed shell description with harmonic oscillator wave functions. The present situation is confuse with regard to the adequacy of this simple model for describing the total muon capture rates. It is known [4] that, in the cases of ^{16}O and ^{40}Ca, the oscillator shell model, without closure approximation, gives much too large capture rates. However, the use of $\nu = 80$ MeV - value of the parameter as originally proposed by Primakoff [3] - in Eq. (3) gives a reasonable agreement with the experimental results [7]. We apply our method to this model in order to clarify the situation.

In these cases $D_S = D_T = D_L$ and the correlation functions have been calculated in Ref. 7). We quote the results in the form

$$\Lambda_r(\nu)\big|_1 = \left(\frac{\nu}{m}\right)^2 \left\{ 1 - e^{-\frac{1}{2} b_1^2 \nu^2} \right\}$$

$$\Lambda_r(\nu)\big|_2 = \left(\frac{\nu}{m}\right)^2 \left\{ 1 - \left[1 + \frac{1}{4}\left(\frac{1}{2} b_2^2 \nu^2\right)^2\right] e^{-\frac{1}{2} b_2^2 \nu^2} \right\}$$

$$\Lambda_r(\nu)\big|_3 = \left(\frac{\nu}{m}\right)^2 \left\{ 1 - \left[1 + \frac{1}{2}\left(\frac{1}{2} b_3^2 \nu^2\right)^2 - \frac{1}{10}\left(\frac{1}{2} b_3^2 \nu^2\right)^3 + \frac{1}{40}\left(\frac{1}{2} b_3^2 \nu^2\right)^4\right] e^{-\frac{1}{2} b_3^2 \nu^2} \right\} \tag{9}$$

where 1, 2, 3 refer to ^{4}He, ^{16}O and ^{40}Ca, respectively, and $b_1 = 1.39$ fm, $b_2 = 1.76$ fm and $b_3 = 1.99$ fm, as determined from the experimental root mean square charge radii.

We obtain for Λ_r [Eq. (8)] the results displayed in Fig. 1 (solid lines) where the experimental results

$$\Lambda_r|_1 = 0.086 \pm 0.007$$
$$\Lambda_r|_2 = 0.111 \pm 0.004 \qquad (10)$$
$$\Lambda_r|_3 = 0.130 \pm 0.001$$

are also given for comparison. These ones are computed from Λ's listed by Eckhause [1] using values of Z_{eff} by Ford and Wills [11]. The broken lines show the ν dependence encountered in the usual closure approximation. Needless to say, Fig. 1 clearly indicates that the description of ^{4}He is sufficient to explain its capture rate, whereas this is not the case for ^{16}O and ^{40}Ca ; we obtain, in a closure approach of the problem, too large results according with Ref. 4) without the closure approximation. The agreement in Ref. 7) is due to the particular value of ν used. We do not claim that ν = 80 MeV is not a good value, but that this value is not consistent with the oscillator model. Our method does not need the specific value of ν to predict Λ_r ; if it is preferred, the method determines, once a model for $\Lambda_r(\nu)$ is chosen, the ν value in which this quantity has to be calculated. This value of ν corresponds to that one in which the two corrective terms in Eq. (8) are cancelled. In this sense we can say that the oscillator shell model gives the results

$$\nu_1 = 81 \text{ MeV} \qquad \Lambda_r|_1 = 0.089$$
$$\nu_2 = 87 \text{ MeV} \qquad \Lambda_r|_2 = 0.165 \qquad (11)$$
$$\nu_3 = 88 \text{ MeV} \qquad \Lambda_r|_3 = 0.192$$

to be compared with the experimental results (10).

This result is easily understood because, as shown in Ref. 12), the effects of SU(4) breaking are completely negligible in ^{4}He whereas they are very important in ^{16}O and ^{40}Ca. The effective excitation energy is different for the vector and axial terms, and this gives a reduction of the capture rate. Furthermore, we know [13] the existence of deformed 2p - 2h admixtures in the ground state wave function of ^{16}O.

4. - THE LIGHT NUCLEI ^{3}He AND ^{6}Li

The cases of ^{3}He and ^{6}Li are particularly interesting, because the behaviour of the correlation functions is very different from the one appearing for closed shell nuclei. Furthermore, it is well-known that the wave functions can be written in a realistic way using the L-S coupling. Thus we expect the sum rule given in Eq. (7) to work.

We shall describe ^{3}He as the coupling of three nucleons in s wave to a spin $S = \frac{1}{2}$ with a symmetric spatial wave function of Gaussian kind. This has been proved to be very successful in many processes, and particularly for the partial transition $^{3}He \rightarrow {}^{3}H$ in muon capture [9]. In these conditions $D_S(\nu) = D_T(\nu) = D_L(\nu)$ with $D_S(0) = 1$. The reduced capture rate in the closure approximation is then

$$\Lambda_r(\nu) = \left(\frac{\nu}{m}\right)^2 \left\{ 1 - \frac{1}{2} e^{-\nu^2/6\alpha^2} \right\} \tag{12}$$

with $\alpha = 0.356\ fm^{-1}$ as determined from electron scattering [14]. We see that the ν dependence is weaker than the other cases, and $\Lambda_r(\nu) \propto \nu^2$ if $\nu \rightarrow 0$.

^{6}Li is described [10] as a core (^{4}He) adding two nucleons in the 1p shell coupled to L = 0, S = 1. As it is an odd-odd nucleus, the relation $D_S(\nu) = D_T(\nu) = D_L(\nu)$ is no longer true *). This is violated by the contribution to the correlation functions coming from the expected value of the operators on the two external nucleons, in a way very similar to the well-known deuteron case [16]. We have for ^{6}Li

$$\begin{aligned} D_S(\nu) &= 2\, I_0^2(s,s) + I_0^2(p,p) + 2\, I_2^2(p,p) + 2\, I_1^2(s,p) \\ D_T(\nu) = D_L(\nu) &= 2\, I_0^2(s,s) + \frac{1}{3} I_0^2(p,p) + \frac{2}{3} I_2^2(p,p) + 2\, I_1^2(s,p) \end{aligned} \tag{13}$$

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*) On this point, compare with the conclusions reached in Ref. 15). For a N = Z odd-odd nucleus, the partition (P,P',P") in the SU(4) scheme is effectively (1,0,0), but the assignment of "physical" quantum numbers is in this case $P = (K_3 + L_3)_{max}$, $P' = T = |T_3|$ instead of $P \rightleftarrows P'$.

where $I_\ell(x,y)$ refer to the radial integrals

$$I_\ell(x,y) = \int_0^\infty dr \; r^2 \, R_x(r) \, j_\ell(\nu r) \, R_y(r) \tag{14}$$

Using the harmonic oscillator potential well with parameter $b = 1.98$ fm to reproduce the experimental root mean square charge radius, (14) can be solved explicitly and we obtain

$$\begin{aligned} D_S(\nu) &= e^{-\frac{1}{2} b^2 \nu^2} \left\{ 3 + \frac{1}{3} \left(\frac{1}{2} b^2 \nu^2 \right)^2 \right\} \\ D_T(\nu) = D_L(\nu) &= e^{-\frac{1}{2} b^2 \nu^2} \left\{ \frac{7}{3} + \frac{4}{9} \left(\frac{1}{2} b^2 \nu^2 \right) + \frac{1}{9} \left(\frac{1}{2} b^2 \nu^2 \right)^2 \right\} \end{aligned} \tag{15}$$

Then the reduced capture rate in closure approximation is

$$\Lambda_r(\nu) = \left(\frac{\nu}{m} \right)^2 \left\{ 1 - e^{-\frac{1}{2} b^2 \nu^2} \left[0.82 + 0.12 \left(\frac{1}{2} b^2 \nu^2 \right) + 0.05 \left(\frac{1}{2} b^2 \nu^2 \right)^2 \right] \right\} \tag{16}$$

and we observe that $\Lambda_r(\nu) \propto \nu^2$ with $\nu \to 0$. However, the reason for this dependence is completely different from the case of ^{3}He ; for the latter, the difference follows since $N < Z$ while for ^{6}Li it is due to the difference in the functions D_S and D_T.

In Fig. 2, our results for ^{3}He and ^{6}Li are given (solid lines) where the broken lines show the ν dependence given by (12) and (16). The experimental results

$$\begin{aligned} \Lambda_r(^3\mathrm{He}) &= 0.501 \begin{smallmatrix} +0.026 \\ -0.040 \end{smallmatrix} \\ \Lambda_r(^6\mathrm{Li}) &= 0.273 \pm 0.057 \end{aligned} \tag{17}$$

are obtained from Λ's quoted by Eckhause [1] using values of Z_{eff} given by Kim and Primakoff [17]. The agreement is satisfactory, showing that the description of the nuclear states is adequate. The point in which the two corrective terms in (8) are cancelled is localized in

$$\begin{aligned} \nu(^3\mathrm{He}) &= 93 \text{ MeV} \qquad & \Lambda_r(^3\mathrm{He}) &= 0.49 \\ \nu(^6\mathrm{Li}) &= 91 \text{ MeV} \qquad & \Lambda_r(^6\mathrm{Li}) &= 0.31 \end{aligned} \tag{18}$$

to be compared with the experimental results (17).

5. - CONCLUSION

In this work we have discussed a method of avoiding the uncertainty associated with the average neutrino energy in the usual closure approach to muon capture rates. The crucial steps are summarized. Instead of neglecting the kinematic dependence on each particular channel, the capture rate is approximated by an expansion around a parameter ν in the form given in Eq. (5). The sum over the final states can be performed straightforwardly with the help of theorem (7). Our final expression (8) allows us to calculate the total reduced capture rate in terms of a nuclear model for the initial state which gives the usual function (3) in closure approximation. The results we obtain using Eq. (8) are quite independent on the specific value of ν over a large range of plausible values. As we observe in Figs. 1 and 2, it is interesting to notice that the point for which the two corrective terms are cancelled is in a zone of high stability. Thus the study of these processes rests as an effective test of the nuclear model used.

Our comparison with ^3He, ^6Li cases allows us to conclude that the method is successful, confirming that the L-S coupling gives a good description of the capture rates. For the closed shell nuclei, it is clear that ^{16}O and ^{40}Ca present terms of SU(4) breaking and we rule out the simple harmonic oscillator shell model to explain the muon capture. For ^4He, according with Cannata et al. [12], the SU(4) breaking effect must be completely negligible.

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FIGURE CAPTIONS

Figure 1   Reduced capture rates Λ_r for the closed shell nuclei ^4He, ^{16}O and ^{40}Ca. Our predictions (solid lines) are compared to the ν dependence in the usual closure approach (broken lines) and to the experimental results.

Figure 2   Same as Fig. 1, but referred to ^6Li and ^3He.

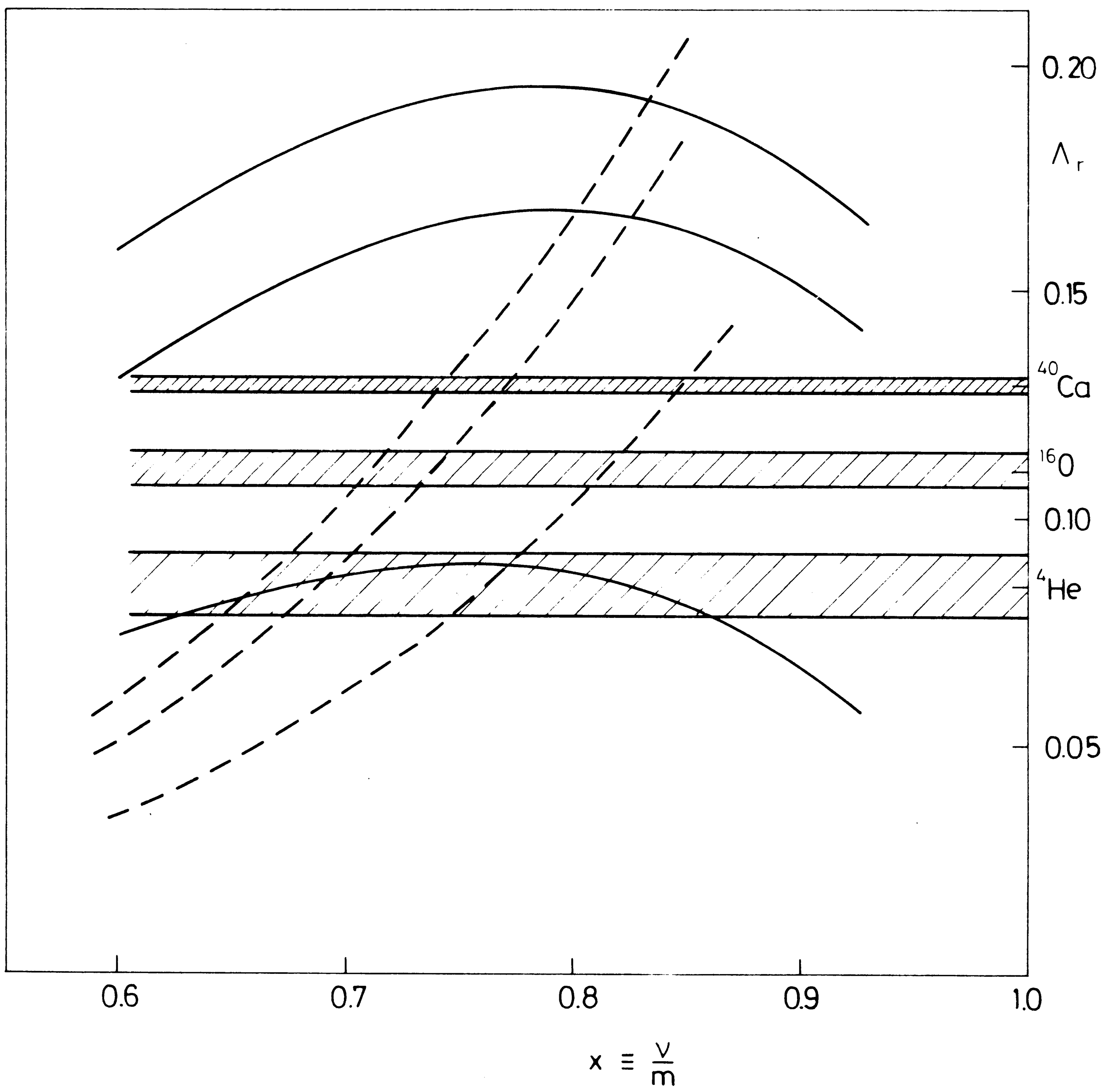

FIG.1

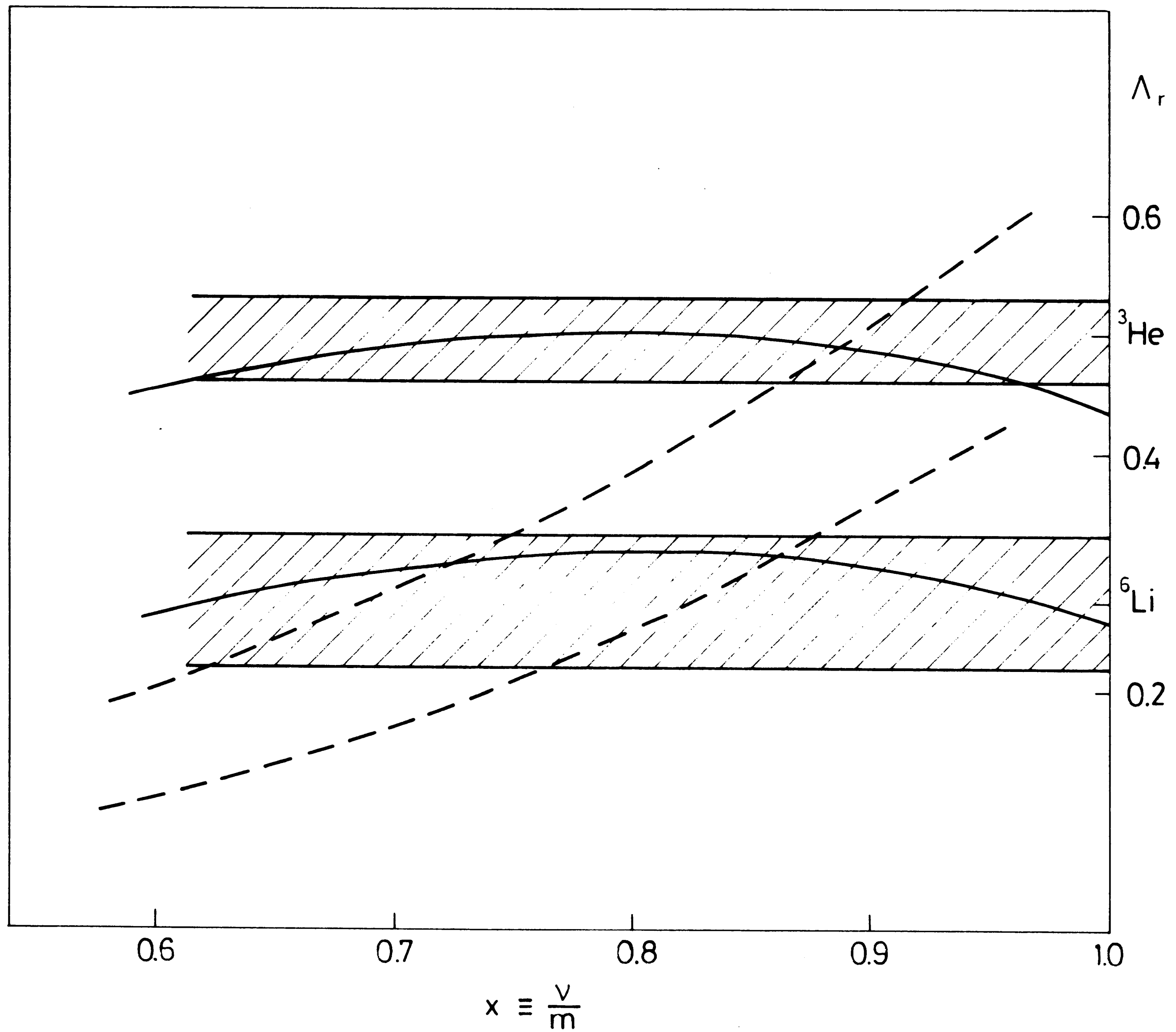

FIG. 2