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ORIGIN OF DISCREPANCY IN THE THEORETICAL VALUE OF THE POLARIZABILITY CORRECTION TO THE $(\mu^4He)^+$ ENERGY LEVELS

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A B S T R A C T

Different approaches in calculating the polarizability correction to the energy levels of the muonic ^4He ion are compared. These calculations disagree with each other by giving results 3 to 10 times larger than the experimental uncertainty. The origin of the major discrepancy is traced to the treatment of the nuclear excitations. It is shown that the experimental value of the electric polarizability of ^4He provides a crucial restriction on model calculations.

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Recently the $2P_{\frac{3}{2}} - 2S_{\frac{1}{2}}$ separation energy in $(\mu\,{}^4\mathrm{He})^+$ ion has been measured [1] with a high degree of accuracy, giving the result

$$\Delta E \equiv E(2P_{3/2}) - E(2S_{1/2}) = 1527.4 \pm .9 \quad meV \tag{1}$$

The experiment is now being repeated aiming at an even higher precision. In the near future one intends [2] to measure the quantity $E(2P_{\frac{1}{2}}) - E(2S_{\frac{1}{2}})$ in muonic ^{4}He ion and to extend the measurements to the case of ^{3}He.

Comparison of the separation energy above with the predictions of the quantum electrodynamics requires the calculation of vacuum polarization to order e^2, nuclear finite size, fine structure, vacuum polarization to order e^4, muonic Lamb shift and nuclear polarizability. All contributions, but the polarizability correction, are now settled [3] after some early small (but relevant) discrepancies [4],[5] between different calculations. The theoretical value of the separation energy is

$$\Delta E\,(meV) = 1811.4 - 103.1 \langle r^2 \rangle + (\Delta E)_{pol} \tag{2}$$

with $\langle r^2 \rangle$ expressed in fm^2. $(\Delta E)_{pol}$ is the polarizability correction. The subscript on this quantity is often dropped for simplicity reasons.

The purpose of this note is to examine the different calculations [6]-[9] of the polarizability correction which disagree among themselves by giving results 3 to 10 times larger than the experimental uncertainty in (1). Hopefully, our comparison will allow us to state a theoretical result for $(\Delta E)_{pol}$, with an estimate of uncertainties.

The polarizability correction corresponds to the contribution of the two-photon exchange interaction between the muon and the nucleus, with virtual inelastic excitations of the nucleus. The different approaches in calculating ΔE may be classified as follows.

1. - Classical estimate [7]

A very simple estimate for the polarizability contribution to the energy levels of the muonic atom is obtained in the limit where the field created by the muon is treated as the one by an external static source. The longest range effective potential is due to the interaction of the induced dipole moment with the electric field. One obtains [7)]

$$\Delta E_{n\ell} = -\frac{1}{2}\left(\frac{e^2}{4\pi}\right) \alpha(^4He) \langle r^{-4} \rangle_{n\ell} \tag{3}$$

with $\alpha(^4He)$ the electric polarizability of ^4He. This parameter may be determined from photodisintegration data on ^4He through the sum rule

$$\alpha = \frac{1}{2\pi^2} \int_{\nu_{th}}^{\infty} \frac{d\nu}{\nu^2}\, \sigma(\nu) \tag{4}$$

where $\sigma(\nu)$ is the total photoabsorption cross-section ; ν is the photon energy and ν_{th} denotes the threshold energy for the inelastic excitations. Equation (4) assumes that the magnetic polarizability, if any, is negligible compared to α. This assumption is expected to be very good for ^4He.

No direct measurement of the total photoabsorption cross-section in ^4He is available, but the partial cross-sections measured up to now indicate [10)] that, in (4), only (γ,p) and (γ,n) channels are important. The analysis of the available data gives [11),12)]

$$\alpha(^4He) = (.073 \pm .004)\ fm^3 \tag{5}$$

In the following, we shall assume α as given in (5). Indeed, the trend of the present data indicates that the value in (5) could not change much. Fortunately, the total cross-section is going to be measured in the near future [*)] and should settle the question of accuracy of α. The relevance of this parameter for realistic calculations will be discussed below.

In S states, (3) has an ultraviolet divergence, reflecting the fact that the potential r^{-4} is modified at short distances. With a cut-off of the order $R \approx 2$ fm, the contribution is [8)]

$$\Delta E^{class} \approx 6\ meV \tag{6}$$

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*) The total photoabsorption cross-section on ^4He will be measured soon. We thank Professor B. Ziegler for this information.

much larger than the experimental uncertainty in (1). Evidently the method is good for an order of magnitude estimate but not accurate enough for a meaningful comparison with experiment.

2. - Calculations with the static Coulomb interaction

The polarizability correction, for low Z atoms, was studied many years ago by Joachain [6]. He assumed a static Coulomb interaction between the muon and the nucleus and calculated the relevant contribution from second order perturbation theory. This goes further, a priori, than the classical limit by taking proper account of the muon propagation and short distance behaviour. However, no relation to experimental parameters [like $\alpha(^4\mathrm{He})$ in (3)] describing the nuclear excitations is established and the simple model used, plane waves for the intermediate nuclear states, is unlikely to be realistic. He found for $^4\mathrm{He}$ [6]

$$\Delta E = 7 \text{ meV} \tag{7}$$

quite similar to (6).

Very recently, Henley, Krejs and Wilets [9] (abbreviated HKW) have recalculated the quantity ΔE. Their approach is quite similar to the old one of Joachain [6], the main difference being that HKW describe the nuclear excitations by a harmonic oscillator model.

The harmonic oscillator model used by HKW has in principle only one parameter, namely the frequency ω. If determined from the r.m.s. radius of $^4\mathrm{He}$, the value obtained is $\omega_0 = 17.2$ MeV, which is manifestly wrong for the description of <u>inelastic</u> excitations, since the threshold for excitation is about 20 MeV. With $\omega = 17.2$ MeV, they find $\Delta E \approx 10$ MeV, a value which they do not trust. Their strategy is then to keep $\gamma \equiv (M\omega)^{\frac{1}{2}}$ ($M =$ nucleon mass) at the "canonical" value and to take "reasonable" values of ω, in the range 21 to 35 MeV (this is equivalent to introducing an effective mass for the nucleon). HKW believe the most appropriate value of the correction is

$$\Delta E = 7.0 \pm 1.5 \text{ meV} \tag{8}$$

which is compatible with (6) and (7). However, it disagrees substantially with the result obtained by Bernabéu and Jarlskog [8] which is discussed below.

3. - The covariant approach [8]

In this approach one calculates the two-photon exchange diagrams (direct contribution and its crossed counter-part) in terms of virtual Compton scattering on ^{4}He. Then, the exchange of transverse photons, besides the longitudinal ones, is automatically included in an explicit gauge invariant and Lorentz covariant framework. In that way, one controls that nothing fundamental of the photon interaction is missing. In Ref. 8), the approximation was made of neglecting "external" velocities of the atomic muon, i.e., terms of order $(|\vec{p}|/m_\mu) \sim e^2 Z$ were dropped in the diagrams, leading to the study of forward virtual Compton scattering on ^{4}He. In that approximation, the modulus squared of the muon wave function at the origin appears as a global factor in ΔE.

In terms of the photon mass squared q^2, it was shown that the dominant contribution (small q^2) could be calculated model independently since all the nuclear physics part was contained in the single parameter $\alpha(^4\mathrm{He})$. For $-q^2 \gtrsim 3\ \mathrm{fm}^{-2}$, nuclear effects disappear and the virtual photon nuclear scattering is completely quasi-elastic, as shown from the experimental results of inelastic electron scattering [13]. The model was only needed to interpolate the region from 0.5 to 3 fm^{-2} for $-q^2$ [see Fig. 4 in Ref. 8)]. The internal consistency of the inputs used for inelastic excitations was explicitly checked by calculating the "reduction factor" f in inelastic electron scattering and comparing it with the experimental results of Ref. 13). The final result for the polarizability correction was [8]

$$\Delta E = 3.1 \ \ \mathrm{meV} \tag{9}$$

The value in Eq. (9) is about a factor two smaller than the classical estimate (6) using the same value for $\alpha(^4\mathrm{He})$. This is compatible with the earlier study [14] of the polarizability correction in lepton-neutron scattering, where it was shown that the effects of the muon propagator tend to decrease the classical result by a factor of about two.

4. - Comparison and conclusions

Now we compare the covariant calculation which leads to Eq. (9) with the HKW approach [9]. Since HKW claim their calculation to be accurate it is worth while to single out the origin of the discrepancy between the results of Refs. 8) and 9).

4.1. - The sensitivity to the muon wave function

The potential approach of HKW, with only the static field, takes into account the modification of the muon wave function at large distances, something which is not present in Ref. 8) from the very beginning. An upper limit for the error introduced by the factorization of $|\psi_{2S}(0)|^2$ is obtained by comparing, in the HKW calculation, the matrix element of longest range as calculated with and without factorization of the muon wave function at the origin. From Eqs. (7), (A.11) and (A.12) of HKW, that comparison says that, due to this effect, the result of Ref. 8) could be, at most, 12% larger than the one of HKW and never smaller.

4.2. - Contribution of transverse photons

The main difference between the two approaches is due to the automatic presence of transverse photons in Ref. 8) which are not considered in the HKW approach. Their influence can be estimated from Eq. (19) of Bernabéu and Jarlskog [8], which says

$$\frac{\text{Transv. contr.}}{\text{Longit. contr.}} \approx \frac{-3\, \langle -q^2 \rangle^{1/2}}{8\, m_\mu} \approx -20\,\% \tag{10}$$

where $\langle -q^2 \rangle^{\frac{1}{2}} \approx 0.3\ \mathrm{fm}^{-1}$ is a typical mass of the virtual photon [see again Fig. 4 of Ref. 8)]. This result (10) goes in the right direction, but it is clearly insufficient to explain the difference in the results (8) and (9).

4.3. - Test of the model used by HKW

So far we have seen that the different treatment of the muonic wave function and transverse photons is not the origin of the discrepancy between the results (8) and (9). Other features of both calculations are similar. In particular, the static muon contribution (denoted by A in HKW) is reduced by a factor two when the muon propagation (term B) is included.

With this in mind and from 4.1 and 4.2, we conclude that if the nuclear physics part of the problem were similar, the different approaches of Refs. 8) and 9) would lead to results which, at most, could differ by about 20%. As this is not the case, we examine now the nuclear vertices.

In fact, an examination of the nuclear model used by HKW tells us that it is unable to explain even the global features of the inelastic excitations of ^{4}He, although the low q^2 elastic form factor is reproduced (we wish to point out that, for nuclei, an excitation of 30 MeV is not a near-ground state behaviour). To demonstrate our point, we have tested the model in HKW by calculating the electric polarizability of ^{4}He in the model and comparing it with the result (5). Apart from the relevance of this parameter to the contribution we discuss here (to be shown below), the electric polarizability has to do with inelastic excitations and it provides a test. We obtain

$$\alpha(^4\mathrm{He}) = \left(\frac{e^2}{4\pi}\right)\frac{1}{\gamma^2\omega} \qquad [\text{HKW model}] \tag{11}$$

where ω and γ are the parameters of the harmonic oscillator model. Using the "canonical" values given in Table 1 of HKW we find $\alpha(^4\mathrm{He}) = 0.20\ \mathrm{fm}^3$, which is about three times larger than the result (5).

The relevance of the value of $\alpha(^4\mathrm{He})$ for the polarizability correction is apparent in the classical limit, as it was shown above. In the "realistic" calculations of Refs. 8) and 9) this relevance is somewhat shadowed by the technical machinery. However, it is there too. In the covariant approach [8)], the manifestation of $\alpha(^4\mathrm{He})$ is present in the leading order (in q^2) longitudinal contribution and this constraint was explicitly used. In the potential approach of HKW, it must correspond to the longest range effective potential r^{-4} associated to the iteration of the static Coulomb interaction. Let us check this last statement. From Eqs. (A.11) and (A.12) of HKW, we see that the longest range effective interaction is determined by the intermediate dipole excitation. In their notation

$$A = -\langle n_0 | \sum_{N\neq N_0} \frac{|\langle N|V|N\rangle|^2}{E_N - E_0} | n_0\rangle \equiv \left(\frac{e^2}{4\pi}\right)\langle n_0 | V_{eff}(r) | n_0\rangle \tag{12a}$$

$$V_{eff}(r) \xrightarrow[r\to\infty]{} -\left(\frac{4\pi}{e^2}\right)\frac{|\langle 11,10|V|0\rangle|^2}{\omega} \xrightarrow[r\to\infty]{} -\frac{1}{2}\left(\frac{e^2}{4\pi}\right)\frac{1}{\gamma^2\omega}\,\frac{1}{r^4} \tag{12b}$$

which corresponds to the value of $\alpha(^4\mathrm{He})$ as given in Eq. (11). One should notice that the matrix elements of the (11,10) dipole excitations are by far the dominant ones (see Table III of HKW). We are led to the conclusion that their result 10 meV for the polarizability contribution, when compared with 3.1 meV in Eq. (9), is essentially a manifestation that their "implicit" value of $\alpha(^4\mathrm{He})$ is about three times larger than the one used in Ref. 8). In fact, a naïve scaling of all matrix elements as the one dictated by the value of α gives for the HKW calculation

$$\Delta E = 10 \text{ meV} \times \frac{.07}{.20} = 3.5 \text{ meV} \tag{13}$$

which, within 10-20% allowed by the different "photon" approaches, reproduces the earlier result obtained in Ref. 8), Eq. (9) here.

Since $\omega = 17.2$ MeV is not physically acceptable, HKW kept γ fixed but ω was increased in the range 21 to 35 MeV, then ΔE is reduced to the value given in Eq. (8). The corresponding value of $\alpha(^4\mathrm{He})$ is then $0.13 \pm 0.03\ \mathrm{fm}^3$, which is again satisfying the naïve scaling shown in Eq. (13).

Of course, the discussion here rests on the basis of the present experimental value (5) for the electric polarizability α. This question will be hopefully settled in the near future through its "direct" determination from the total photoabsorption cross-section. If the experimental value of α would change in the future, the arguments in this note allow us to give the following theoretical result for the polarizability correction in the form

$$(\Delta E)_{Pol}\,(\text{meV}) = 2.8 + 40\,(\alpha - .07) \tag{14}$$

with an estimated conservative error of 20%, coming from possible deviations of the used scaling in nuclear matrix elements and from the uncertainties in the muon-photon approach. In Eq. (14), the experimental value of $\alpha(^4\mathrm{He})$ should be used, in units of fm^3. The value 2.8 in Eq. (14) is due to the present knowledge of the muonic wave function correction (see 4.1) to the result [8] given in Eq. (9). Note that the value 2.8 also follows from the scaled result of HKW, Eq. (13), by taking into account the contribution of transverse photons (see 4.2).

HKW have also calculated the polarizability contribution for the case of ^{3}He. In their model, the static polarizability is given by $\alpha(^3\mathrm{He}) = (e^2/4\pi)\ 4/9\gamma^2\omega = 0.28\ \mathrm{fm}^3$, also larger than the existing quoted value [12),15)] $\alpha(^3\mathrm{He}) = 0.16 \pm 0.03\ \mathrm{fm}^3$, obtained from $p+d$, and $p+p+n$ channels in photoabsorption. In ^{3}He, the possible contributions of magnetic transitions to the last figure are less clear than in ^{4}He. Accepting the present experimental value, if the arguments put forward for ^{4}He were also valid for the ^{3}He case, we would scale the HKW result $\Delta E = 12$ meV and would guess $(\Delta E)^{^3\mathrm{He}}_{\mathrm{pol}} \approx 6$ meV.

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