

PROPERTIES OF A MATRIX GROUP ASSOCIATED TO A $\{K, S + 1\}$ -POTENT MATRIX*

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Abstract. In a previous paper, the authors introduced and characterized a new kind of matrices called $\{K, s + 1\}$ -potent. In this paper, an associated group to a $\{K, s + 1\}$ -potent matrix is explicitly constructed and its properties are studied. Moreover, it is shown that the group is a semidirect product of \mathbb{Z}_2 acting on $\mathbb{Z}_{(s+1)^2-1}$. For some values of s , more specifications on the group are derived. In addition, some illustrative examples are given.

Key words. Involutionary matrix, $\{K, s + 1\}$ -potent matrix, Group.

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1. Introduction. Let $K \in \mathbb{C}^{n \times n}$ be an involutory matrix, that is $K^2 = I_n$, where I_n denotes the $n \times n$ identity. In [5], the authors introduced and characterized a new kind of matrices called $\{K, s + 1\}$ -potent matrices where K is involutory. We recall that for an involutory matrix $K \in \mathbb{C}^{n \times n}$ and $s \in \{0, 1, 2, 3, \dots\}$, a matrix $A \in \mathbb{C}^{n \times n}$ is called $\{K, s + 1\}$ -potent if

$$(1.1) \quad KA^{s+1}K = A.$$

These matrices generalize all the following classes of matrices: k -potent matrices, idempotent matrices, periodic matrices, involutory matrices, centrosymmetric matrices, mirror symmetric matrices, circulant matrices, etc. Several applications of these matrices can be found in the literature [1, 9, 13]. The class of $\{K, s + 1\}$ -potent matrices was linked to other kind of matrices (as $\{s + 1\}$ -generalized projectors, $\{K\}$ -Hermitian matrices, normal matrices, etc.) in [6]. Throughout this paper, we consider $K \in \mathbb{C}^{n \times n}$ to be an involutory matrix.

Some results on a similar class of 2×2 matrices and $n \times n$ invertible matrices have been presented in [2]. On the other hand, matrices commuting with a permutation and $\{K\}$ -centrosymmetric matrices (that correspond to $s = 0$) have received increasing

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interest in the last twenty years. Some of their properties can be found in [1, 8, 9, 13]. Furthermore, matrices with k -involutory symmetries have been studied in [11, 12]. Moreover, some spectral properties related to similar classes of matrices are given in [3, 5, 10].

Related to the group theory, we recall that if G is a finite group with identity element e and $a \in G$ then $a^m = e$ implies that the order of a divides to m for any natural power m .

Motivated by the fact that the definition of $\{K, s + 1\}$ -potent matrices involves products of the two matrices A and K , we wonder if there are any other relationships between products where these matrices appear. As a particular case, when s is the smallest positive integer such that $A^{s+1} = A$, it is clear that $\{A, A^2, A^3, \dots, A^s\}$ is a cyclic group (and, therefore, commutative and normal) of order s . This leads to our main aim, which is to extend these results to $\{K, s + 1\}$ -potent matrices.

This paper is organized as follows. First, properties of a $\{K, s + 1\}$ -potent matrix A are studied in Section 2 involving products and powers of A and K . These properties are necessary to construct, in Section 3, a finite group G from a given $\{K, s + 1\}$ -potent matrix. As a consequence, this group is a semidirect product of \mathbb{Z}_2 acting on $\mathbb{Z}_{(s+1)^2-1}$ where \mathbb{Z}_r is the group of integers modulo r . Moreover, the group G is calculated in some simple cases. The case $A^k = A$ for some $k < (s + 1)^2$ is also analyzed in Section 4. Finally, in Section 5, some illustrative examples are given.

2. Basic properties of $\{K, s + 1\}$ -potent matrices. It is clear that for each $n \in \{1, 2, 3, \dots\}$, there exists at least one matrix $A \in \mathbb{C}^{n \times n}$ such that A is $\{K, s + 1\}$ -potent for each involutory matrix K and for each $s \in \{1, 2, 3, \dots\}$. It is also easy to see that such a matrix is not unique [5].

Throughout this section, we consider $s \in \{1, 2, 3, \dots\}$. It is well-known [5] that a matrix $A \in \mathbb{C}^{n \times n}$ is $\{K, s + 1\}$ -potent if and only if any of the following conditions are (trivially) equivalent: $KAK = A^{s+1}$, $KA = A^{s+1}K$, and $AK = KA^{s+1}$.

We now establish properties regarding $\{K, s + 1\}$ -potent matrices.

LEMMA 2.1. *If $A \in \mathbb{C}^{n \times n}$ is a $\{K, s + 1\}$ -potent matrix then the following properties hold*

- (a) $KA^{s+2} = A^{s+2}K$ and $KA^{s+2}K = A^{s+2}$.
- (b) $A^{s+2} = (KA)^2 = (AK)^2$.
- (c) $A^{(s+1)^2} = A$.
- (d) $(A^{(s+1)^2-1})^k = A^{(s+1)^2-1}$ for every $k \in \{1, 2, 3, \dots\}$.
- (e) $(A^{s+2})^{s+1} = A^{s+2}$.
- (f) $(KA)^{2s+1} = KA$ and $(AK)^{2s+1} = AK$.

- (g) $KA^jK = A^{j(s+1)}$ and $A^jK = KA^{j(s+1)}$ for every $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.
 (h) $(KA^j)^2 = A^{j(s+2)}$ for all $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.
 (i) $A^jA^{(s+1)^2-1} = A^{(s+1)^2-1}A^j = A^j$ and $(KA^j)A^{(s+1)^2-1} = A^{(s+1)^2-1}(KA^j) = KA^j$, for all $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.
 (j) For each $j \in \{1, 2, \dots, (s+1)^2 - 1\}$, one has $(KA^j)(KA^k) = A^{(s+1)^2-1}$, where k is the unique element of $\{1, 2, \dots, (s+1)^2 - 1\}$ such that $k \equiv -j(s+1) \pmod{((s+1)^2 - 1)}$.
 (k) $K(KA^j)^{s+1}K = \begin{cases} KA^{j\frac{s}{2}(s+4)+j} & \text{if } s \text{ is even} \\ (A^{j(s+2)})^{\frac{s+1}{2}} & \text{if } s \text{ is odd} \end{cases}$ for all $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.

Proof. (a) One has $KA^{s+2} = KA^{s+1}A = AK A = AA^{s+1}K = A^{s+2}K$. The second equality can be deduced post-multiplying both sides by K .

(b) From (a) and the definition, we have $A^{s+2} = KA^{s+2}K = KAA^{s+1}K = KAKA = (KA)^2$. The other equality in (b) can be similarly deduced.

(c) By definition we have $A^{(s+1)^2} = (A^{s+1})^{s+1} = (KAK)^{s+1} = KA^{s+1}K = A$.

(d) Using Property (c) we get

$$(A^{(s+1)^2-1})^2 = A^{(s+1)^2}A^{(s+1)^2-2} = AA^{(s+1)^2-2} = A^{(s+1)^2-1},$$

and now Property (d) can be easily shown by induction.

(e) From (c) we get $(A^{s+2})^{s+1} = (A^{s+1}A)^{s+1} = (A^{s+1})^{s+1}A^{s+1} = A^{(s+1)^2}A^{s+1} = AA^{s+1} = A^{s+2}$.

(f) From (b) and (c) the equalities $(KA)^{2s+1} = KA(KA)^{2s} = KA(A^{s+2})^s = KA^{s^2+2s+1} = KA^{(s+1)^2} = KA$ hold, and in a similar way it can be shown the equality $(AK)^{2s+1} = AK$.

(g) We proceed by recurrence. In fact, by definition we have

$$(2.1) \quad KAK = A^{s+1}.$$

Then Equality (2.1) yields $KA^2K = KAAK = A^{s+1}KAK = A^{s+1}A^{s+1} = A^{2(s+1)}$. Following a similar reasoning it can be proven that $KA^jK = A^{j(s+1)}$ for all $j \in \{1, 2, \dots, s\}$. Now, by using the definition and $A^{(s+1)^2} = A$ we get the property for $j = s+1$ as follows: $KA^{s+1}K = A = A^{(s+1)^2} = A^{(s+1)(s+1)}$. From now on, following a similar reasoning as before it can be proven that $KA^jK = A^{j(s+1)}$ for all $j \in \{1, 2, \dots, (s+1)^2 - 1\}$, and the other equality in (g) is easily obtained from $K^2 = I_n$.

(h) Using (g) one has

$$(KA^j)^2 = (A^{j(s+1)}K)^2 = A^{j(s+1)}(KA^{j(s+1)}K) = A^{j(s+1)}A^j = A^{j(s+2)}.$$

(i) Follows from (c) and (g).

(j) Let $k \geq 1$. One has $KA^jKA^k = A^{j(s+1)+k}$. The right hand side is $A^{(s+1)^2-1}$ if $k = -j(s+1)[\text{mod } ((s+1)^2 - 1)]$.

(k) Case 1. s is even. Using Properties (g) and (c), we get

$$\begin{aligned} K(KA^j)^{s+1}K &= A^jA^{j(s+1)\frac{s}{2}}A^{j\frac{s}{2}}K = A^{j(s+2)\frac{s}{2}}A^jK = A^{j(s+2)\frac{s}{2}}KA^{j(s+1)} \\ &= KA^{(s+1)(j(s+2)\frac{s}{2})}A^{j(s+1)} = KA^{j(s+1)((s+1)\frac{s}{2}+\frac{s}{2})}A^{j(s+1)} \\ &= K\left(A^{(s+1)^2}\right)^{j\frac{s}{2}}A^{(\frac{s}{2}+1)j(s+1)} = KA^{j\frac{s}{2}(s+4)+j}. \end{aligned}$$

Case 2. s is odd. Using Property (g), we get

$$K(KA^j)^{s+1}K = A^jA^{j(s+1)\frac{s+1}{2}}A^{j\frac{s-1}{2}} = A^{j(s+1)\frac{s+1}{2}+j\frac{s-1}{2}} = \left(A^{j(s+2)}\right)^{\frac{s+1}{2}}. \quad \square$$

3. Construction of a matrix group. Firstly, we note that, from a $\{K, s+1\}$ -potent matrix, Lemma 2.1 allows us to construct a group containing a cyclic subgroup of $\{K, s+1\}$ -potent matrices. Throughout this section we assume that $s \geq 1$.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n \times n}$ be a $\{K, s+1\}$ -potent matrix. If $A^i \neq A^j$ for all distinct $i, j \in \{1, 2, \dots, (s+1)^2 - 1\}$ then the set*

$$G = \{A, A^2, A^3, \dots, A^{(s+1)^2-1}, KA, KA^2, KA^3, \dots, KA^{(s+1)^2-1}\}$$

is a group with respect to the matrix product satisfying the following properties:

(a) A is an element of order $(s+1)^2 - 1$, and then the set

$$(3.1) \quad S_A = \{A, A^2, A^3, \dots, A^{(s+1)^2-1}\}$$

is a cyclic subgroup of G .

(b) KA^s and $KA^{(s+1)^2-1}$ are elements of order 2 of G .

(c) $(KA^s)A(KA^s) = A^{s+1}$.

(d) The set S_A is a normal subgroup of G and all its elements are $\{K, s+1\}$ -potent matrices.

(e) The order of G is:

- $(s+1)^2 - 1$ if $KA = A^j$ for some $j \in \{1, 2, \dots, (s+1)^2 - 1\}$ and, in this case, the group G is commutative.
- $2((s+1)^2 - 1)$ if $KA \neq A^j$ for all $j \in \{1, 2, \dots, (s+1)^2 - 1\}$ and, in this case, the group G is noncommutative.

(f) For every $j \in \{1, 2, \dots, (s+1)^2 - 1\}$, the element KA^j of the set $G \setminus S_A$ (when it is nonempty) is $\{K, s+1\}$ -potent if and only if s is even and one of the following conditions $\{4|s, \frac{s}{2} + 1|j\}$ or $\{4 \nmid s, s+2|j\}$ holds.

Proof. From the properties given in Lemma 2.1, it can be checked that $A^{(s+1)^2-1}$ is the identity element of the group G .

(a) G contains clearly a cyclic subgroup generated by the element A of order $(s + 1)^2 - 1$.

(b) Using Property (h) of Lemma 2.1, one has $(KA^s)^2 = A^{s(s+2)} = A^{(s+1)^2-1}$. Similarly, $(KA^{(s+1)^2-1})^2 = A^{(s+1)^2-1}$.

(c) Using the definition we get $(KA^s)A(KA^s) = KA^{s+1}KA^s = AA^s = A^{s+1}$.

(d) The set S_A is a subgroup of G of index 2. Then it is normal. As a direct consequence of Property (f) of Lemma 2.1 we obtain the second part of Property (d) since A^j are $\{K, s + 1\}$ -potent matrices for all $j \in \{1, 2, \dots, (s + 1)^2 - 1\}$.

(e) Case 1. There exists $j \in \{1, 2, \dots, (s + 1)^2 - 1\}$ such that $KA = A^j$. Then $KA^i = A^{j+i-1}$, that is, it is also one of the $(s + 1)^2 - 1$ powers of A indicated in (3.1) for all $i \in \{2, 3, \dots, (s + 1)^2 - 1\}$. Thus, the order of G is $(s + 1)^2 - 1$ and then G is commutative.

Case 2. For every $j \in \{1, 2, \dots, (s + 1)^2 - 1\}$ one has $KA \neq A^j$. It is clear that the order of G is $2((s + 1)^2 - 1)$ because if $i, j \in \{1, 2, \dots, (s + 1)^2 - 1\}$ such that $KA^i = KA^j$ with $i \neq j$ exist, then $A^i = K^2A^i = K^2A^j = A^j$, which is impossible. Assume that G is commutative. Then $(KA)(KA^{s+1}) = (KA^{s+1})(KA)$, that implies $A^{2s+1} = A$. We obtain a contradiction.

(f) Assume that $G \setminus S_A \neq \emptyset$ and $KA^j \in G \setminus S_A$. Then the fact that KA^j is $\{K, s + 1\}$ -potent implies that s is even, by Property (k) of Lemma 2.1. This same property assures that, in this case, $K(KA^j)^{s+1}K = KA^{j\frac{s}{2}(s+4)+j}$. Then,

$$KA^j \text{ is } \{K, s + 1\} \text{ - potent} \iff KA^{j\frac{s}{2}(s+4)+j} = KA^j \iff A^{j\frac{s}{2}(s+4)+j} = A^j$$

$$\iff A^{j\frac{s}{2}(s+4)+j} A^{(s+1)^2-1-j} = A^{(s+1)^2-1} \iff A^{j\frac{s}{2}(s+4)} = A^{(s+1)^2-1}$$

and so, this is equivalent to the statement $(s+1)^2-1 = s(s+2)$ divides $j\frac{s}{2}(s+4)$. Now, it can be checked that $s(s+2)|j\frac{s}{2}(s+4)$ is equivalent to $\{4|s, \frac{s}{2}+1|j\}$ or $\{4 \nmid s, s+2|j\}$. In the first case, there are $2s$ such j and in the second one, there are s such j . \square

COROLLARY 3.2. *The group G is a semidirect product of \mathbb{Z}_2 acting on $\mathbb{Z}_{(s+1)^2-1}$ when G has order $2((s + 1)^2 - 1)$.*

Proof. We consider a semidirect product of \mathbb{Z}_2 acting on \mathbb{Z}_r . Then its presentation is in the form $\langle a, b | a^2 = e, b^r = e, aba = b^m \rangle$ where m, r are coprime. Here $r = (s + 1)^2 - 1$, $a = KA^s$, $b = A$, $m = s + 1$. \square

Now, we need the following definition.

DEFINITION 3.3 ([7]).

- (i) The dihedral group, denoted by D_m , is generated by an element a of order m and another element b of order 2 such that $b^{-1}ab = a^{-1}$. It has order $2m$.
- (ii) The quasi-dihedral group, denoted by Q_{2^m} for $m \geq 4$, is generated by an element a of order 2^{m-1} and another element b of order 2 such that $b^{-1}ab = a^{2^{m-2}-1}$. It has order 2^m .

We now classify the group G defined in Theorem 3.1. It is clear that G depends on A , K , and s . That is why, for the following result, we will denote G by G_s and so we stress on the parameter s . We shall see a relation between the group G_s and dihedral and quasi-dihedral groups in the following result.

PROPOSITION 3.4. Assume that G_s has order $2((s+1)^2 - 1)$. One has

- (a) G_s is a dihedral group if and only if $s = 1$. In this case, $G_1 \simeq D_3$.
- (b) Let $s > 1$. Then G_s is a quasi-dihedral group if and only if $s = 2$. In this case, $G_2 \simeq Q_{16}$.

Proof. (a) If $s = 1$, the proof for case (a) can be deduced from the fact that G_s is generated by the element A of order 3, the element KA of order 2, and furthermore, the relation $(KA)A(KA) = A^2$ holds, where A^2 is the inverse of A .

Let $s > 1$. In this case, it is known that the group G_s has an element A of order $(s+1)^2 - 1$. On the other hand, we check that for any $\alpha \in \{1, 2, \dots, (s+1)^2 - 1\}$ the element KA^α has order 2 if and only if α is a multiple of s . Indeed,

$$A^{(s+1)^2-1} = (KA^\alpha)^2 \iff A^{s(s+2)} = KA^\alpha KA^\alpha = A^{\alpha(s+1)} A^\alpha = A^{\alpha(s+2)}$$

$$\iff s(s+2) \text{ divides } \alpha(s+2) \iff s \text{ divides } \alpha,$$

and hence all the elements of order 2 of G_s have the form KA^{st} . However, for any element of the form KA^{st} , where

$$t \in \left\{ \frac{1}{s}, \frac{2}{s}, \dots, \frac{(s+1)^2 - 1}{s} \right\} \cap \mathbb{N} = \{1, 2, 3, \dots, s+2\},$$

the equality $(KA^{st})A(KA^{st}) = A^{(s+1)^2-2}$ is not satisfied since

$$(KA^{st})A(KA^{st}) = (KA^{st+1}K)A^{st} = A^{((s+1)^2-1)t+(s+1)} = (A^{(s+1)^2-1})^t A^{s+1} = A^{s+1}.$$

Finally $s > 1$ implies that $s+1 < (s+1)^2 - 2$, a contradiction.

(b) It is necessary to take into account the fact that the properties of G_s for $s = 2$ coincide with those of Q_{2^m} for $m = 4$. In fact, if $s = 2$ then G_s has an element A of order $8 = 2^{4-1}$, another element KA^2 of order 2 and, moreover, by Property (c)

of Theorem 3.1 we have $(KA^2)A(KA^2) = A^3 = A^{2^{4-2}-1}$ and therefore $G_2 \simeq Q_{2^4}$. Conversely, if $G_s \simeq Q_{2^m}$ then we have $2^{m-1} = (s + 1)^2 - 1 = s(s + 2)$. Thus s is a power of 2 and $s = 2, m = 4$. \square

4. What about the group when $A^k = A$ for $k < (s + 1)^2$? We can ask: what would happen if a power k of A less than $(s + 1)^2$ and such that $A^k = A$ existed?

If $A \in \mathbb{C}^{n \times n}$ is a $\{K, s + 1\}$ -potent matrix then Property (b) of Lemma 2.1 allows us to construct, by Theorem 3.1, the group G considering the subgroup S_A of order $(s + 1)^2 - 1$ when all the powers of A are different. But, it may occur that there exists an integer k such that $A^k = A$ with $2 \leq k < (s + 1)^2$. In this case, it is also possible to consider the group $G_{s,k} = \{A, A^2, \dots, A^{k-1}, KA, KA^2, \dots, KA^{k-1}\}$ associated to the matrix A . Therefore, the subset $S_A^k = \{A, A^2, \dots, A^{k-1}\}$ is a (cyclic) subgroup of the group S_A and then $k - 1$ has to divide $(s + 1)^2 - 1 = s(s + 2)$.

How many groups $G_{s,k}$ can we construct in this way? One only: the group corresponding to the smallest power k such that $A^k = A$ (otherwise, we obtain exactly the same group $G_{s,k}$). Consequently, the only possibilities for the order of the group are: $(s + 1)^2 - 1, 2((s + 1)^2 - 1), k - 1$ or $2(k - 1)$ (if such k exists).

For some values of s and k , more specifications on the group are given in the following result. In order to analyze these special cases we recall the following definition.

DEFINITION 4.1. $[4]$ *The quaternion group, denoted by Q , is generated by three elements a, b, c of order 4 such that $a^2 = b^2 = c^2$ and $bab^{-1} = a^{-1}$. It has order 8.*

PROPOSITION 4.2. *Assuming that $G_{s,k}$ has order $2(k - 1)$, the following statements hold.*

- (a) *Let $s = 1$. Then $k = 2$. In this case, $G_{1,2} \simeq \mathbb{Z}_2$.*
- (b) *Let $s = 2$. Then one of the following statements hold:*
 - (i) *$k = 2$. In this case, $G_{2,2} \simeq \mathbb{Z}_2$.*
 - (ii) *$k = 3$. In this case, $G_{2,3} \simeq D_2$.*
 - (iii) *$k = 5$. In this case, $G_{2,5} \simeq D_4$ or $G_{2,5} \simeq Q$.*
- (c) *Let $s > 2$. Then*
 - (i) *$G_{s,s+1} \simeq \mathbb{Z}_{2s}$ when s is prime.*
 - (ii) *$G_{s,s+3} \simeq D_{s+2}$.*

Proof. (a) Let $s = 1$. If k is an integer such that $2 \leq k \leq 3$ and $A^k = A$, it must be $k = 2$ because $k - 1$ divides 3. Thus, $KA^k = A^2 = A$ and then the group $G_{1,2} = \{KA, (KA)^2 = A\}$ is generated by the element KA of order 2. Hence, $G_{1,2} \simeq \mathbb{Z}_2$.

(b) Let $s = 2$. If k is an integer such that $2 \leq k \leq 8$ and $A^k = A$, it must be $k = 2$, $k = 3$ or $k = 5$ because $k - 1$ divides 8. We now analyze these three cases:

- $k = 2$: the same reasoning as in (a).
- $k = 3$: in this case, $G_{2,3} = \{A, A^2, KA, KA^2\}$ where A and KA^2 have order 2 and $KA^2AKA^2 = A$. Hence, $G_{2,3} \simeq D_2$.
- $k = 5$: in this case, $G_{2,5} = \{A, A^2, A^3, A^4, KA, KA^2, KA^3, KA^4\}$ is a non-commutative group (for instance, $A(KA^2) \neq (KA^2)A$). Then, Proposition 6.3 in [4] assures that $G_{2,5} \simeq D_4$ or $G_{2,5} \simeq Q$.

(c) Let $s > 2$. If k is an integer such that $2 \leq k \leq s(s + 2)$ and $A^k = A$, we get $k = s + 1$ or $k = s + 3$ as particular values of k because $k - 1$ divides $s(s + 2)$. Now, we analyze these two cases:

- $k = s + 1$: in this case, $G_{s,s+1} = \{A, A^2, \dots, A^s, KA, KA^2, \dots, KA^s\}$ is a commutative group of order $2s$. Then, Corollary 6.2 in [4] assures that $G_{s,s+1} \simeq \mathbb{Z}_{2s}$.
 - $k = s + 3$: in this case, $G_{s,s+3} = \{A, A^2, \dots, A^{s+2}, KA, KA^2, \dots, KA^{s+2}\}$ is a non-commutative group (for example, $A(KA^{s+1}) \neq (KA^{s+1})A$). A is of order $s + 2$, KA^s of order 2 and $KA^sAKA^s = A^{s+1}$. Hence, $G_{s,s+3} \simeq D_{s+2}$.
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REMARK 4.3. If $G_{s,k}$ has order $2(k - 1)$ then $G_{s,k}$ is a semidirect product of \mathbb{Z}_2 acting on \mathbb{Z}_{k-1} , when $k - 1, s + 1$ are coprime. Its proof is similar to the proof of Corollary 3.2 where $a = KA^{k-1}$, $b = A$, $r = k - 1$, $m = s + 1$. The Property (h) of Lemma 2.1 allows us to show that KA^{k-1} has order 2. In fact, $a^2 = (KA^{k-1})^2 = A^{(k-1)(s+2)} = A^{k-1} = e$. Moreover, $aba = (KA^{k-1})A(KA^{k-1}) = KA^kKA^{k-1} = A^{s+1}A^{k-1} = A^{s+1} = b^m$.

REMARK 4.4. We observe that $G_{2,5}$ is isomorphic to D_4 or Q because there are (up to isomorphism) exactly two distinct non-commutative groups of order 8. We will see, in Example 2, that these two possibilities can be realized.

REMARK 4.5. We observe that if $s > 2$ and $k = s + 1$, then A satisfies $KA = AK$ since $A^{s+1} = A$. Such a matrix is said to be $\{K\}$ -centrosymmetric [9]. In this case, the group associated to such a matrix A has order $2s$, commutative and then isomorphic to \mathbb{Z}_{2s} if s is prime.

We close this section with the following remark.

REMARK 4.6. (The case $s = 0$). It corresponds to $\{K\}$ -centrosymmetric matrices. It is sometimes also possible to construct a similar group as before. Observe that the condition $A^{(s+1)^2} = A$ does not give any information for $s = 0$. So, if we assume that $A^t = A$ for some positive integer t (where $A^l \neq A^m$ for all $l, m < t$, $l \neq m$),

then $G = \{A, A^2, \dots, A^{t-1}, KA, KA^2, \dots, KA^{t-1}\}$ is a group with the same features mentioned before.

However, when the assumption is not fulfilled, since the powers of A cannot be the identity element, the group does not exist. An example will clarify this situation. For the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

it is impossible to construct such a group because

$$A^{m+1} = \begin{bmatrix} 2^m & 0 & 2^m \\ 0 & 1 & 0 \\ 2^m & 0 & 2^m \end{bmatrix} \quad \text{for every } m \geq 1.$$

5. Examples. Now we present some more examples illustrating the results we have obtained.

EXAMPLE 5.1. Let

$$A = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A is $\{K, 2\}$ -potent as the authors showed in [5]. $G = \{A, A^2, A^3, KA, KA^2, KA^3\}$ is a group of order 6 because $A^3 = I_3$ and, in this case, $G \simeq D_3$ by Proposition 3.4.

EXAMPLE 5.2.

(1) Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A is $\{K, 3\}$ -potent and $A^2 = -I_2$. This example shows a matrix of the class considered in Section 4. For this matrix, $A^5 = A$ holds and so it is possible to construct a group with similar features as in Proposition 4.2. Since $s = 2$, $k = 5$, $G_{2,5}$ has order 4 or 8. In this example, the group is $G_{2,5} = \{\pm I_2, \pm A, \pm K, \pm KA\}$ and, in this case, A is an element of order 4, $KA^2 = -K$ is an element of order 2, and $(KA^2)A(KA^2) = A^3$. So $G_{2,5}$ is isomorphic to D_4 .

(2) Let A be a $\{K, 3\}$ -potent matrix given by

$$A = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1+i & 1-i \\ 0 & 0 & 1-i & 1+i \\ 1+i & 1-i & 0 & 0 \\ 1-i & 1+i & 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \frac{1}{8} \begin{bmatrix} 9 & -1 & 3 & -3 \\ -1 & 9 & -3 & 3 \\ -3 & 3 & -1 & 9 \\ 3 & -3 & 9 & -1 \end{bmatrix}.$$

$A^4 = I_4$ and the associated matrix group (of order 8) is

$$G_{2,5} = \{A, A^2, A^3, I_4, KA, KA^2, KA^3, K\}.$$

$G_{2,5}$ is generated by the three elements A^2, KA, KA^3 of order 4 such that $(A^2)^2 = (KA)^2 = (KA^3)^2$ and $(KA)(A^2)(KA)^3 = A^2$. So $G_{2,5}$ is isomorphic to Q .

EXAMPLE 5.3. Let

$$A = \begin{bmatrix} \frac{\sqrt{5}-1}{2} & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ 0 & -1 \end{bmatrix}.$$

A is $\{K, 4\}$ -potent and $A^5 = I_2$. The associated group (of order 10) is

$$G = \{A, A^2, A^3, A^4, I_2, KA, KA^2, KA^3, KA^4, K\}.$$

G is generated by A of order 5, KA^2 of order 2, and $(KA^2)A(KA^2) = A^4$. Then G is isomorphic to D_5 .

EXAMPLE 5.4. Let A be a $\{K, 5\}$ -potent matrix given by

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

The associated group (of order 6) is $G = \{A, A^2, I_2, KA, KA^2, K\}$ since $A^3 = I_2$. As G is generated by A of order 3, KA^2 of order 2 and $(KA^2)A(KA^2) = A^2$, then G is isomorphic to D_3 .

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