Lie Algebra on the Transverse Bundle of a Decreasing Family of Foliations

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Abstract

J.Lehmann-Lejeune in [Cohomologies sur le fibré transverse à un feuilletage, C.R.A.S. Paris, 295 (1982), 495-498] defined on the transverse bundle V to a foliation on a manifold M, a zero-deformable structure J such that $J^2 = 0$ and for every pair of vector fields X, Y on M: $[JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] = 0$ .For every open set Ω of V, J.Lehmann-Lejeune studied the Lie Algebra $L_J(\Omega)$ of vector fields X defined on Ω such that the Lie derivative $L(X)J$ is equal to zero i.e., for each vector field Y on Ω: $[X, JY] = J[X, Y]$ and showed that for every vector field X on Ω such that $X \in KerJ$, we can write $X = \sum [Y, Z]$ where $\sum$ is a finite sum and Y, Z belongs to $L_J(\Omega) \cap (KerJ_{|\Omega})$.
In this note, we study a generalization for a decreasing family of foliations.

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1. INTRODUCTION

Let M be a differentiable manifold endowed with a decreasing family F_i of k foliations (particular case of "Multifoliate Structures" of Kodaira Spencer [3]). We define a so-called "order k bundle V^k transverse to the foliations F_i". This note is divided into three sections.

In the first, we define V^k and we show that there exists a (1,1) tensor J of V^k such that $J^k \neq 0$, $J^{k+1} = 0$ and for every pair of vector fields X, Y on V^k:

$$[JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] = 0 .$$

In section two, Ω being an open set of V^k, we denote by $L_J(\Omega)$ the Lie Algebra of vector fields X defined on Ω such that the Lie derivative $L(X)J$ is equal to zero i.e., for each vector field Y on Ω:

$$[X, JY] = J[X, Y]$$

In $L_J(\Omega)$, we find the canonicals lifts in V^k of the infinitesimal automorphisms of the k foliations on M.

In section three, we define L_1, subset of L_J, constituted by the vector field X on V^k such that $X \in KerJ$. We show that for every $X \in L_1(V^k)$, we can write $X = \sum_i [Y_i, Z_i]$ where $\sum_i$ is a finite sum and Y_i, Z_i belongs to $L_1(V^k)$.

2. THE ORDER k BUNDLE TO k FOLIATIONS

Let M be a differentiable manifold of dimension m endowed with k foliations $F_1, F_2, ..., F_k$, $k \geq 1$, of respective codimensions $p_1, p_1 + p_2, ..., p_1 + p_2 + ... + p_k$ such that $F_1 \supset F_2 \supset ... \supset F_k$ ($m = p_1 + p_2 + ... + p_k + p_{k+1}$, $p_1 > 0$, $p_i \geq 0$, $2 \leq i \leq k+1$).

Notation: we set :

$a(h) = p_1 + p_2 + ... + p_h$ for $1 \leq h \leq k+1$,

$a(h) = 0$ for $h \leq 0$,

$c(t) = a(k+1) + a(k) + ... + a(k-t+2)$ for $1 \leq t \leq k+1$,

$c(t) = 0$ for $t \leq 0$

The order k tangent bundle of M is the manifold of dimension $(k+1)m$ of the $k-$jets of origin 0 of differentiable mappings from IR to M denoted $T^k M$ (cf. [1]).

Let s and h be two integers such that $0 \leq s \leq h \leq k$, $h \geq 1$.
On the set of $h-$jets of differentiable mappings of origin 0 from IR to M, we define an equivalence relation. Let φ and ψ be two differentiable mappings from IR to M such that $\varphi(0) = \psi(0)$.

Denote by $(u_1, u_2, ..., u_m)$ the local coordinates of an open set $\hat{U} \subset M$, adapted to the k foliations (i.e. $u_1, u_2, ..., u_{a(h)}$ are constants on the leaves of F_h, $1 \leq h \leq k$), such that $\varphi(0) = \psi(0) = x_0 \in \hat{U}$.

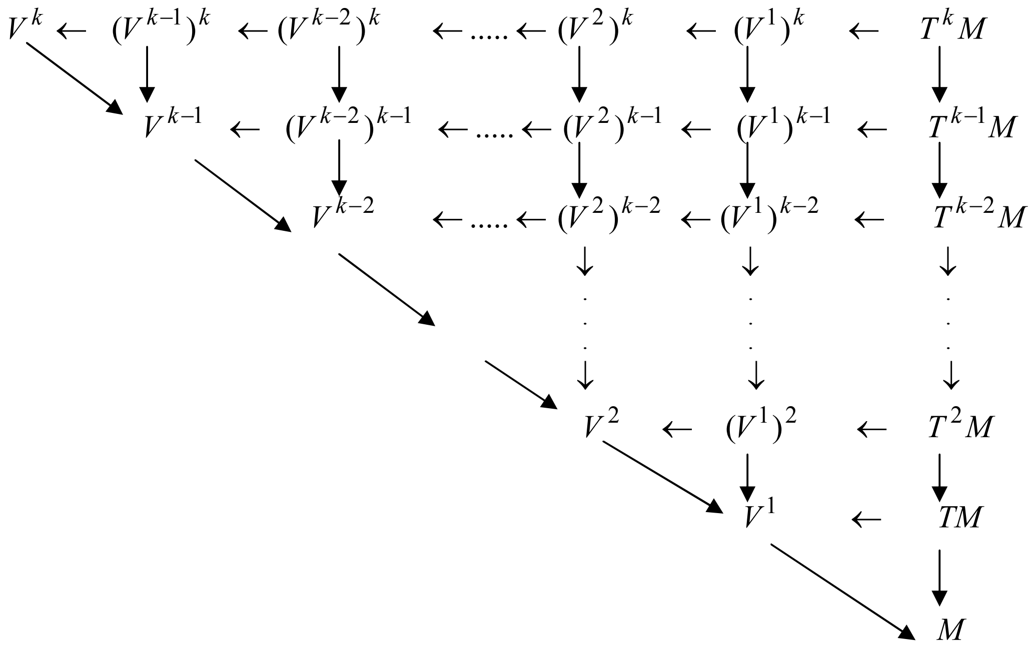
We say that the $h-$jets of φ and ψ are equivalent if: $\dfrac{d^b \varphi_l}{d\rho^b}(0) = \dfrac{d^b \psi_l}{d\rho^b}(0)$, $1 \leq b \leq s$, $1 \leq l \leq a(k+1-b)$ and $s+1 \leq b \leq h$, $1 \leq l \leq a(k+1-s)$.

This equivalence relation is independent of the open set $\hat{U}$ of coordinates adapted to the k foliations containing x_0.

We denote by $(V^s)^h$ the quotient space of the $h-$jets of differentiable mappings from IR to M endowed with this equivalence relation.
This is a manifold of dimension $\sum_{0 \leq t \leq s} a(k+1-t) + (h-s)a(k+1-s)$.

For $s = h$, $(V^s)^s$ will be denoted, for simplicity, by V^s.

We have the following diagram, where the arrows are the natural projections:



V^k is called order k bundle transverse to the k foliations $F_1, F_2, ..., F_k$.
The dimension of V^k is $n = \sum_{0 \le t \le k} (t+1) p_{k+1-t} = \sum_{0 \le t \le k} a(k+1-t)$.

Remark 1: $(V^s)^k$, $0 \le s \le k$, can be considered as a V'^k.
In fact, it is sufficient to set: $p_1' = a(k+1-s)$; $p_i' = 0$, $2 \le i \le k+1-s$; $p_j' = p_j$, $k+2-s \le j \le k+1$.
Thus V'^k is the order k bundle transverse to the k foliations $F_1', F_2', ..., F_k'$ of codimensions respectively p_1', $p_1' + p_2'$, ..., $p_1' + p_2' + ... + p_{k+1}'$ such that $F_1' \supset F_2' \supset ... \supset F_k'$, where $F_i' = F_1'$ for $2 \le i \le k+1-s$.

An element of V^k, i.e. an equivalence class of $k-$jets $j_0^k \varphi$ is uniquely expressed by the set $(u_l, u_{c(h)+r_h})$, $l = 1, ..., m$, $1 \le h \le k$, $1 \le r_h \le a(k-h+1)$, u_l being the coordinates of x_0 in $\hat{U}$ and $u_{c(h)+r_h}$ being defined by : $u_{c(h)+r_h} = \frac{1}{h!} \frac{d^h \varphi_{r_h}}{d\rho^h}(0)$.
Thus, to every open set $\hat{U}$ of M is associated an open set $U = \pi^{-1}(\hat{U})$ in V^k, where π is the canonical projection from V^k to M.

On M, let $\{\hat{U}, u_i\}$ and $\{\hat{U}', u_i'\}$, $i = 1, ..., a(k+1)$, be two local coordinates charts adapted to the k foliations such that $\hat{U} \cap \hat{U}' \ne \varnothing$. We have, in $\hat{U} \cap \hat{U}'$, for $1 \le h \le k+1$, $a(h-1)+1 \le i \le a(h)$:

$$u'_i = f_i(u_1, ..., u_{a(h)})$$

$$\partial_i = \sum_{a(h-1)+1 \le r \le a(k+1)} \partial_i f_r \partial'_r \quad \text{(we set } \partial_i = \frac{\partial}{\partial u_i},\ \partial'_r = \frac{\partial}{\partial u'_r}\text{)}$$

Let φ and ψ be two mappings from IR to M such that $\varphi(0)$ and $\psi(0) \in \hat{U} \cap \hat{U}'$.
For ρ close enough to zero, $\varphi(\rho)$ (resp. $\psi(\rho)$) can be written with the local coordinates u_i (resp. $u_i^{'}$): $(\varphi_1(\rho),....,\varphi_{a(k+1)}(\rho))$ (resp. $(\psi_1(\rho),....,\psi_{a(k+1)}(\rho))$. We have:

$$\psi_i(\rho) = f_i(\varphi_1(\rho),....,\varphi_{a(h)}(\rho)) \quad \text{where} \quad a(h-1)+1 \le i \le a(h),\ 1 \le h \le k+1, \quad \text{and}$$

$$\frac{1}{j!}\frac{d^j \psi_{r_j}(\rho)}{d\rho^j} = \frac{1}{j!}\frac{d^j[f_{r_j}(\varphi_1(\rho),....,\varphi_{a(k+1-j)}(\rho))]}{d\rho^j} \quad \text{where } 1 \le j \le k, \quad 1 \le r_j \le a(k+1-j).$$

We set $u_i = \varphi_i(0)$, $u_i^{'} = \psi_i(0)$ and $u_{c(j)+r_j} = \dfrac{1}{j!}\dfrac{d^j \varphi_{r_j}}{d\rho^j}(0)$, $u'_{c(j)+r_j} = \dfrac{1}{j!}\dfrac{d^j \psi_{r_j}}{d\rho^j}(0)$.

We have, for $1 \le h \le k$, $a(h-1)+1 \le i \le a(h)$: $\quad u'_{c(1)+i} = \sum_{1 \le j \le a(h)} \partial_j f_i \ u_{c(1)+j}$.

We verify that for $1 \le q \le k$, $1 \le h \le k-q+1$, $a(h-1)+1 \le i \le a(h)$:

$$\frac{1}{q!}\frac{d^q \psi_i(t)}{d\rho^q} = \sum \frac{\partial^t f_i}{\partial u_1^{i_1} ... \partial u_j^{i_j} ... \partial u_{a(h)}^{i_{a(h)}}} \prod_{1 \le j \le a(h)} \left[\prod_{1 \le r \le q} \frac{1}{b_j^r !} \left(\frac{1}{r!} \frac{d^r \varphi_j}{d\rho^r} \right)^{b_j^r} \right]$$

where $\sum$ is taken on all the possible families of integers ≥ 0, i_j, b_j^r, $1 \le j \le a(h)$ such that $i_1 + ... + i_j + ... + i_{a(h)} = t$, $1 \le t \le q$,

$$\sum_{1 \le r \le q} b_j^r = i_j \quad , \quad \sum_{1 \le j \le a(h)} \left(\sum_{1 \le r \le q} r b_j^r \right) = q \, .$$

Thus if $U = \pi^{-1}(\hat{U})$ and $U' = \pi^{-1}(\hat{U}')$, we have in $U \cap U'$, for $1 \le h' \le k+1$, $a(h'-1)+1 \le i' \le a(h')$, $1 \le q \le k$, $1 \le h \le k-q+1$, $a(h-1)+1 \le i \le a(h)$:

$$(1) \begin{cases} u_{i'}^{'} = f_{i'}(u_1,....,u_{a(h')}) \\ u_{c(q)+i}^{'} = \sum \dfrac{\partial^t f_i}{\partial u_1^{i_1} ... \partial u_j^{i_j} ... \partial u_{a(h)}^{i_{a(h)}}} \prod_{1 \le j \le a(h)} \left[\prod_{1 \le r \le q} \dfrac{\left(u_{c(r)+j}\right)^{b_j^r}}{b_j^r !} \right], \\ \partial_{i'} = \sum_{a(h'-1)+1 \le r' \le a(k+1)} \partial_{i'} f_{r'} \partial_{r'}^{'} \\ \partial_{c(q)+j} = \sum_{\substack{0 \le s \le k-q \\ 1 \le h \le k-s \\ a(h-1)+1 \le i \le a(h)}} \sum \dfrac{\partial^t f_i}{\partial u_1^{i_1} ... \partial u_j^{i_j} ... \partial u_{a(h)}^{i_{a(h)}}} \prod_{1 \le j \le a(h)} \left[\prod_{\substack{1 \le r \le q+s \\ r \ne q}} \dfrac{\left(u_{c(r)+j}\right)^{b_j^r}}{b_j^r !} \dfrac{\left(u_{c(q)+j}\right)^{b_j^q - 1}}{(b_j^q - 1)!} \right] \partial_{c(q+s)+i}^{'} \end{cases}$$

$T^k M$ (which can be considered as a $(V^s)^k$ with $s = 0$) is equipped with an order k nearly tangent structure J_0 of constant range km (cf. [1]). We define a vector field Z and a (1,1) tensor J on V^k in the following way:

Let U and U' be two open sets of adapted local coordinates $(u_1,....,u_n)$, $(u_1^{'},....,u_n^{'})$, respectively. We set:

$$Z^U=\sum_{1\le h\le k} h\left(\sum_{1\le j\le a(k+1-h)} u_{c(h)+j}\partial_{c(h)+j}\right),\qquad Z^{U'}=\sum_{1\le h\le k} h\left(\sum_{1\le j\le a(k+1-h)} u^{'}_{c(h)+j}\partial^{'}_{c(h)+j}\right),$$

$$(2)\quad\begin{cases}\text{for each h,}\quad 0\le h\le k: \\ J^U\partial_{c(h)+a(k-h)+i}=0,\qquad J^{U'}\partial^{'}_{c(h)+a(k-h)+i}=0,\qquad 1\le i\le p_{k+1-h} \\ \text{for each h,}\quad 0\le h\le k-1: \\ J^U\partial_{c(h)+a(r-1)+i} = \partial_{c(h+1)+a(r-1)+i} \\ J^{U'}\partial^{'}_{c(h)+a(r-1)+i} = \partial^{'}_{c(h+1)+a(r-1)+i}\qquad 1\le r\le k-h,\quad 1\le i\le p_r\end{cases}$$

Using (1), we verify that we have, in $U\cap U'$, if $U\cap U'\neq\varnothing$:

$$Z^U_{|U\cap U'}=Z^{U'}_{|U\cap U'},\quad J^U_{|U\cap U'}=J^{U'}_{|U\cap U'}.$$

Hence Z and J are, in fact, globally defined. Z is called the "field of the homotheties" on V^k. J is the projection on V^k of the nearly tangent operator J_0 of order k on T^kM. Its rank is constant and equal to $\sum_{1\le t\le k} a(k+1-t)$: it verifies $J^{k+1}=0$ and for every pair of vector fields X, Y on V^k:

$$[JX,JY]-J[JX,Y]-J[X,JY]+J^2[X,Y]=0.$$

According to the remark1, for $0\le s\le k$, we have also defined a vector field Z_s and a $(1,1)$ tensor J_s on $(V^s)^k$. In U (open set of adapted local coordinates $(u_1,....,u_n)$ of $(V^s)^k$, where n is the $(V^s)^k$ dimension), we have:

$$Z^U_s=\sum_{1\le h\le s} h\left(\sum_{1\le j\le a(k+1-h)} u_{c(h)+j}\partial_{c(h)+j}\right)+\sum_{1\le t\le k-s} t\left(\sum_{1\le i\le a(k+1-r)} u_{c(r)+ta(k+1-r)+i}\partial_{c(r)+ta(k+1-r)+i}\right)$$

For each h, $0\le h\le s-1$:

$J^U_s\partial_{c(h)+a(k-h)+i}=0,\quad 1\le i\le p_{k-h+1}$,

$J^U_s\partial_{c(h)+a(r-1)+i}=\partial_{c(h+1)+a(r-1)+i}$, $1\le r\le k-h$, $1\le i\le p_r$ and

$J^U_s\partial_{c(s)+ta(k+1-s)+i}=\partial_{c(s)+(t+1)a(k+1-s)+i}$, $0\le t\le k-s-1$, $1\le i\le a(k+1-s)$,

$J^U_s\partial_{c(s)+(k-s)a(k+1-s)+i}=0$, $1\le i\le a(k+1-s)$.

For $s=k$, $Z_s=Z$ and $J_s=J$.

3. THE LIE ALGEBRA L_J

Let Ω be an open set of V^k. We denote by $L_J(\Omega)$ the Lie Algebra of vector fields X defined on Ω such that the Lie derivative $L(X)J$ is equal to zero (i.e. the infinitesimal automorphisms of the structure), which means that, for each vector field Y on Ω: $[X, JY] = J[X, Y]$

Let U be an open set of adapted local coordinates $(u_1, ..., u_n)$ and X a vector field on U. We easily verify that $X \in L_J(U)$ if and only if

$$[X, J\partial_i] = J[X, \partial_i] \quad \text{for each i, } 1 \le i \le n.$$

In particular, for each i, $1 \le i \le n$, $\partial_i \in L_J(U)$. Finally, a vector field X on U belongs to $L_J(U)$ if and only if, for each h, $1 \le h \le k$, we have :

$$[X, J^h \partial_i] = J^h[X, \partial_i] = [\ J^h X, \partial_i\ ], \qquad 1 \le i \le n.$$

Let $X \in L_J(U) \cap (KerJ^s_{|U})$ be a vector field on U, $1 \le s \le k+1$. We set:

$$X = \sum_{\substack{0 \le h \le k+1-s \\ 1 \le l \le p_{k-h-s+2}}} X_{c(h)+a(k-s-h+1)+l}\,\partial_{c(h)+a(k-s-h+1)+l} + Y \quad \text{where } Y \in KerJ^{s-1}_{|U} \text{ (for } s=1,\ Y=0\text{)}.$$

For each i, $1 \le i \le n$, we have : $[\ J^s X, \partial_i\ ] = 0 = J^s[X, \partial_i] = J^{s-1}[X, J\partial_i] = [\ J^{s-1}X, J\partial_i\ ]$, thus $X_{c(h)+a(k-s-h+1)+l}$, $0 \le h \le k+1-s$, $1 \le l \le p_{k-h-s+2}$, only depends on $(u_1, ..., u_{c(1)})$.

For $s = 1$, that completely determines $X \in L_J(U) \cap (KerJ_{|U})$.

Assume now $2 \le s \le k+1$: $\partial_{a(k-s+2)+j}$, where $1 \le j \le p_{k-s+3} + ... + p_{k+1}$, belongs to $KerJ^{s-1}_{|U}$ thus $[X, J^{s-1}\partial_{a(k-s+2)+j}] = 0 = [\ J^{s-1}X, \partial_{a(k-s+2)+j}\ ]$, hence we deduce that $X_{c(h)+a(k-s-h+1)+l}$, $0 \le h \le k+1-s$, $1 \le l \le p_{k-h-s+2}$, only depends on $(u_1, ..., u_{a(k-s+2)})$.

We set:

$$(3)\quad \begin{cases} \tilde{X} = \displaystyle\sum_{\substack{0 \le q \le s-1 \\ 0 \le h \le k+1-s \\ 1 \le l \le p_{k-h-s+2}}} X_{c(h+q)+a(k+1-s-h)+l}\ \partial_{c(h+q)+a(k+1-s-h)+l} \\ where, for\ \ q, \quad 1 \le q \le s-1 \\ X_{c(h+q)+a(k+1-s-h)+l} = \displaystyle\sum \frac{\partial^i X_{c(h)+a(k+1-s-h)+l}}{\partial u_1^{i_1} ... \partial u_j^{i_j} ... \partial u_r^{i_r}} \prod_{1 \le j \le r} \left[\prod_{1 \le t \le q} \frac{\left(u_{c(t)+j}\right)^{b_j^t}}{b_j^t\,!} \right] \end{cases}$$

where $r = a(k-s+2)$ and $\sum$ is taken on all the possible families of integers ≥ 0, i_j, b_j^t, $1 \le j \le r$, such that :

$$i_1 + ... + i_j + ... + i_r = i, \quad \sum_{1 \le t \le q} b_j^t = i_j\ , \quad 1 \le i \le q, \ , \quad \sum_{1 \le j \le r} \left(\sum_{1 \le t \le q} t b_j^t \right) = q\,.$$

In particular, $X_{c(h+q)+a(k+1-s-h)+l}$ is independent of $u_{c(q+1)+i}$, $1 \le i \le n - c(q+1)$.

We verify that, for $1 \le t \le q \le s-1$, $1 \le j \le a(k-s+2)$,

$$\partial_{c(t)+j} X_{c(h+q)+a(k+1-s-h)+l} = \partial_j X_{c(h+q-t)+a(k+1-s-h)+l} .$$

For $a(k-s+2)+1 \le i \le a(k+1)$, $1 \le t \le k$: $\left[\tilde{X}, J^t \partial_i\right] = 0 = J^t \left[\tilde{X}, \partial_i\right]$

For $1 \le i \le a(k-s+2)$, $1 \le t \le s-1$: $\left[\tilde{X}, J^t \partial_i\right] = J^t \left[\tilde{X}, \partial_i\right]$.

Indeed, $J^t \partial_i = \partial_{c(t)+i}$ then

$$\begin{aligned}
\left[\tilde{X}, J^t \partial_i\right] = \left[\tilde{X}, \partial_{c(t)+i}\right] &= -\sum_{0 \le q \le s-1} \partial_{c(t)+i} X_{c(h+q)+a(k+1-s-h)+l} \ \partial_{c(h+q)+a(k+1-s-h)+l} \\
&= -\sum_{0 \le q \le s-1} \partial_i X_{c(h+q-t)+a(k+1-s-h)+l} \ J^t \partial_{c(h+q-t)+a(k+1-s-h)+l} \\
&= J^t \left(-\sum_{0 \le q \le s-1} \partial_i X_{c(h+q-t)+a(k+1-s-h)+l} \ \partial_{c(h+q-t)+a(k+1-s-h)+l} \right) \\
&= J^t \left[\tilde{X}, \partial_i\right].
\end{aligned}$$

For $1 \le i \le a(k-s+2)$, $s \le t \le k$: $\left[\tilde{X}, J^t \partial_i\right] = 0$, and $J^t \left[\tilde{X}, \partial_i\right] = \left[J^t \tilde{X}, \partial_i\right] = 0$ because $J^t \tilde{X} = 0$ for $s \le t \le k$.

Thus $\tilde{X} \in L_J(U) \cap (Ker J^s_{|U})$, and so does $X - \tilde{X}$.

But $X - \tilde{X} = Y - \sum_{1 \le q \le s-1} X_{c(h+q)+a(k+1-s-h)+l} \ \partial_{c(h+q)+a(k+1-s-h)+l}$, thus $X - \tilde{X} \in Ker J^{s-1}_{|U}$. This completely determines $L_J(U)$ by induction.

We deduce that, for every open set Ω of V^k :

Lemma 1: *X belongs to $L_J(\Omega)$ if and only if, for every open set U of adapted local coordinates $(u_1, ..., u_n)$ such that $\Omega \cap U \ne \emptyset$, $X_{|\Omega \cap U}$ is a vector field finite sum $A(s,h,l) = \sum_{0 \le q \le s-1} X_{c(h+q)+a(k+1-s-h)+l} \ \partial_{c(h+q)+a(k+1-s-h)+l}$, where $1 \le s \le k+1$, $0 \le h \le k+1-s$, $1 \le l \le p_{k-h-s+2}$, $X_{c(h)+a(k-s-h+1)+l}$ only depends on $(u_1, ..., u_{a(k-s+2)})$ and for $1 \le q \le s-1$, $X_{c(h+q)+a(k+1-s-h)+l}$ is given by (3).*

$A(s,h,l)$ is hence completely determined by its non zero first component $X_{c(h)+a(k-s-h+1)+l}$;
if $s = 1$, it will be its only one non zero component.

(4) We set: $A^h_s(U) = \sum_{1 \le l \le p_{k-h-s+2}} A(s,h,l)$ where $1 \le s \le k+1$, $0 \le h \le k+1-s$.

Remark 2 : To every vector field $\hat{X}$ on M generating a one parameter local subgroup corresponds a one parameter subgroup on $T^k M$; let $R\hat{X}$ be the associated vector field on $T^k M$; $R\hat{X}$ is the " lift of $\hat{X}$ in $T^k M$ ".

We immediately verify that:

Lemma 2: *Let s be $1 \le s \le k$. The following conditions are equivalent:*

i) the vector field $R\hat{X}$ on $T^k M$ is "projectable" on $(V^s)^k$,

ii) $\hat{X}$ is an automorphism of the foliations F_{k+1-t}, $1 \le t \le s$.

For $\hat{X}$, automorphisms of the foliations F_{k+1-t}, $1 \le t \le s$, we have: $P_s J_0(R\hat{X}) = J_s P_s(R\hat{X})$ where P_s is the natural projection from $T^k M$ to $(V^s)^k$.

Then, the vector field on $\hat{U} = \pi(U)$, (see Lemma 1)

$$\hat{X} = X_{c(h)+a(k+1-s-h)+l}(u_1,....,u_{a(k-s+2)})\partial_{a(k+1-s-h)+l}$$

where $1 \le s \le k+1$, $0 \le h \le k+1-s$, $1 \le l \le p_{k-h-s+2}$ is an automorphism of the foliations F_{k+1-t}, $1 \le t \le s-1$, and we have :

$$A(s,h,l) = P_k\left(J_0^h(R\hat{X})\right).$$

4. L_1, SUBSPACE OF L_J

Let Ω be an open set of V^k. We set : $L_1(\Omega) = L_J(\Omega) \cap (KerJ_{|\Omega})$.

For $0 \le h \le k$, let $L_1^h(\Omega)$ be the set of the vector fields $X \in L_J(\Omega)$ such that, for every open set U of adapted local coordinates of V^k, $X_{|\Omega \cap U} \in A_1^h(U)$. We have a direct sum decomposition of $L_1(\Omega)$: $L_1(\Omega) = \underset{0 \le h \le k}{\oplus} L_1^h(\Omega)$.

For $s \ge 2$, if we have two open sets U and U' of adapted local coordinates, $A_s^h(U)_{|U \cap U'}$ is different from $A_s^h(U')_{|U \cap U'}$ in general. To avoid this problem, we consider a metric g on M.

Let $\hat{\Omega}$ be an open set of M and Ω an open set of V^k: $\hat{\Omega} = \pi(\Omega)$. For $2 \le s \le k+1$, $0 \le h \le k+1-s$, we denote by $L_s^h(\hat{\Omega})$ the set of the automorphisms of the foliation F_{k-s+2}, tangent to the leaves of $F_{k+1-s-h}$ and orthogonal to the leaves of $F_{k-s-h+2}$.

For $2 \le s \le k+1$, $0 \le h \le k+1-s$, J_0^h limited to $RL_s^h(\hat{\Omega})$ is injective.

We set :

$$(5) \qquad L_s^h(\Omega) = P_k\left(J_0^h\left(RL_s^h(\hat{\Omega})\right)\right)_{|\Omega}, \quad 2 \le s \le k+1,\ 0 \le h \le k+1-s.$$

We have:

$$L_J(\Omega) = \underset{1 \le s \le k+1}{\oplus}\left(\underset{0 \le h \le k+1-s}{\oplus} L_s^h(\Omega)\right).$$

Moreover, setting $L_s(\Omega) = \underset{1 \le t \le s}{\oplus}\left(\underset{0 \le h \le k+1-t}{\oplus} L_t^h(\Omega)\right)$ for $2 \le s \le k+1$ then

$$L_s(\Omega) = L_J(\Omega) \cap (KerJ^s_{|\Omega}).$$

We easily verify that:

Lemma 3 : *For every open set* Ω *of* V^k, *every* $X \in L_s(\Omega)$ *is a restriction of an element of* $L_s\left(\pi^{-1}(\pi(\Omega))\right)$.

Lemma 4 : *For every open set* Ω *of* V^k, *and for each* $X \in L_s(\Omega)$, $1 \le s \le k+1$ *and* $Y \in L_J(\Omega)$, *the bracket* $[X,Y]$ *belongs to* $L_s(\Omega)$. $L_s(\Omega)$ *is an ideal of* $L_J(\Omega)$.

Proof: For every $X \in L_s(\Omega)$ and $Y \in L_J(\Omega)$, we have: $J^s_{|\Omega}[X,Y] = [\ J^s_{|\Omega} X, Y] = 0$.
This completes the proof.□

Lemma 5 : *Let* Ω *be an open set of* V^k, *and* $X \in L_1(\Omega)$ *be a vector field on* Ω. *For each* $x \in \Omega$, *the germ at* x *of* X *is the germ at* x *of an* $X' \in L_1(V^k)$.

Proof: Let $x \in \Omega$, $\hat{U}$ be an open set of M diffeomorphic to a Cartesian product of m open intervals of IR such that $y = \pi(x) \in \hat{U}$ and $\hat{U} \subset \pi(\Omega)$, and $\hat{H}$ a function on M with compact support contained in $\hat{U}$, equal to 1 in a neighbourhood of y; to $\hat{H}$ corresponds on V^k a function $H = \hat{H} \circ \pi$ with a support (no compact) contained in the open set of adapted local coordinates $U = \pi^{-1}(\hat{U})$, equal to 1 in a neighbourhood of x.
Let $X \in L_1^h(\Omega)$ be a vector field on Ω, $0 \le h \le k : X$ spread over $\pi^{-1}\left(\pi(\Omega)\right) \supset U$; in U, we can write X as: $X = \sum_{1 \le i \le p_{k+1-h}} X_{c(h)+a(k-h)+i}(u_1,...,u_m)\ \partial_{c(h)+a(k-h)+i}$;

$X' = H\left(\sum_{1 \le i \le p_{k+1-h}} X_{c(h)+a(k-h)+i}\ \ \partial_{c(h)+a(k-h)+i}\right)$ belongs to $L_1^h(V^k)$, $0 \le h \le k$, and coincides with X in a neighbourhood of x. This completes the proof. □

Remark: In general, this property is not true for $X \in L_J(\Omega)$ and $X \notin L_1(\Omega)$.

Let $X \in L_1(V^k)$ be a vector field on V^k; its support ϕ is of the form $\pi^{-1}(\hat{\phi})$ where $\hat{\phi}$ is a closed set of M. We denote by ${}^cL_1(V^k)$ the set of the vector fields $X \in L_1(V^k)$ whose support $\phi = \pi^{-1}(\hat{\phi})$ is such that $\hat{\phi}$ is compact.

Theorem 1: $L_1^0(V^k) = [\ L_1^0(V^k), L_1^0(V^k)\ ]$, *and if* $p_{k+1} \neq 0$, *then* $L_1^h(V^k) = [\ L_1^0(V^k), L_1^h(V^k)\ ]$, $1 \le h \le k$, $L_1(V^k) = [\ L_1(V^k), L_1(V^k)\ ]$ *and* ${}^cL_1(V^k) = [\ {}^cL_1(V^k), {}^cL_1(V^k)\ ]$.

Proof: Let $\left(\hat{U}_\alpha\right)_{\alpha \in I}$ be a covering of M by local coordinates open sets whose closure is compact and diffeomorphic to a Cartesian product of m open intervals of IR, which we suppose centred in the origin; according to $[2]$, th. I. p. 17, it exists an open covering of M, locally finite, finer, $\left(\hat{U}_\nu\right)_{\nu \in I}$ and a partition of I in a finite collection of subsets I_μ ($\mu = 1,....,r$) such that, for each μ, the open sets I_ν where $\nu \in I_\mu$ are pairwise disjoint.

Let $\left(\hat{\theta}_\nu\right)_{\nu\in I}$ be a partition of unity subordinate to the covering $\left(\hat{U}_\nu\right)$ and $\theta_\nu = \hat{\theta}_\nu \circ \pi$ the partition of unity associated to V^k.

Let $X \in L_1^0(V^k)$ (respectively $X \in L_1^h(V^k)$, $1 \le h \le k$) be a vector field on V^k;
We set, for each $\nu \in I$, $X_\nu = \theta_\nu X$; from lemma 1, $X_\nu \in L_1^0(V^k)$ (respectively $X_\nu \in L_1^h(V^k)$, $1 \le h \le k$). The support of X_ν is of the form $\pi^{-1}(\hat{\phi}_\nu)$ where $\hat{\phi}_\nu$ is a compact set of M included in $\hat{U}_\nu$; there exists an open set $\hat{U}_\nu^{'}$ of M, with compact closure, such that $\hat{\phi}_\nu \subset \hat{U}_\nu^{'} \subset \overline{\hat{U}_\nu^{'}} \subset \hat{U}_\nu$ and C^∞ functions on M, $\hat{\beta}_\nu$ and $\hat{\gamma}_\nu$ such that $\hat{\beta}_{\nu}\big|_{\hat{\phi}_\nu} = 1$, $supp\, \hat{\beta}_\nu \subset \hat{U}_\nu^{'}$, $\hat{\gamma}_{\nu}\big|_{\bar{\hat{U}}_\nu^{'}} = 1$, $supp\, \hat{\gamma}_\nu \subset \hat{U}_\nu$; We set $\beta_\nu = \hat{\beta}_\nu \circ \pi$, $\gamma_\nu = \hat{\gamma}_\nu \circ \pi$. Since $\hat{U}_\nu$ is contained in an $\hat{U}_\alpha$, $U_\nu = \pi^{-1}(\hat{U}_\nu)$ is contained in the open set of adapted local coordinates $U_\alpha = \pi^{-1}(\hat{U}_\alpha)$. Let us then set : $X_\nu = \sum_{1\le j\le p_{k+1}} X_{\nu,a(k)+j}(u_1,...,u_m)\ \partial_{a(k)+j}$
(respectively $X_\nu = \sum_{1\le i\le p_{k+1-h}} X_{\nu,c(h)+a(k-h)+i}(u_1,...,u_m)\ \partial_{c(h)+a(k-h)+i}$, $1 \le h \le k$)

We set for $1 \le j \le p_{k+1}$: $T_{\nu,a(k)+j} = \beta_\nu\ \ \partial_{a(k)+j}$,

$$Y_{\nu,a(k)+j} = \beta_\nu \left(\int_0^{u_{a(k)+j}} X_{\nu,a(k)+j}(u_1,...,u_{a(k)+j-1},t,u_{a(k)+j+1},...,u_{a(k+1)})dt\right)\partial_{a(k)+j}$$

(respectively for $1 \le i \le p_{k+1-h}$, $1 \le h \le k$,

$$Y_{\nu,c(h)+a(k-h)+i} = \gamma_\nu \left(\int_0^{u_{a(k)+1}} X_{\nu,c(h)+a(k-h)+i}(u_1,...,u_{a(k)},t,u_{a(k)+2},...,u_{a(k+1)})dt\right)\partial_{c(h)a(k-h)+i}).$$

The $2p_{k+1}$ (respectively, p_{k+1-h}, $1 \le h \le k$) vector fields on V^k, with supports contained in U_ν, $T_{a(k)+j}$, $Y_{a(k)+j}$ where $1 \le j \le p_{k+1}$ (respectively $Y_{c(h)+a(k-h)+i}$ where $1 \le i \le p_{k+1-h}$, $1 \le h \le k$) belongs to $L_1^0(V^k)$ (respectively to $L_1^h(V^k)$, $1 \le h \le k$).
We obtain: $\sum_{1\le j\le p_{k+1}} \left[T_{\nu,a(k)+j}, Y_{\nu,a(k)+j}\right] = \sum_{1\le j\le p_{k+1}} X_{\nu,a(k)+j}\partial_{a(k)+j} = X_\nu$
(respectively , for $1 \le h \le k$,

$$\sum_{1\le i\le p_{k+1-h}} \left[T_{\nu,a(k)+1}, Y_{\nu,c(h)+a(k-h)+i}\right] = \sum_{1\le i\le p_{k+1-h}} X_{\nu,c(h)+a(k-h)+i}\partial_{c(h)+a(k-h)+i} = X_\nu).$$

Since, if ν and $\nu' \in I_\mu$, $U_\nu \cap U_\nu^{'} = \varnothing$, we can set :
$X_\mu = \sum_{\nu\in I_\mu} X_\nu$; for $1 \le j \le p_{k+1}$, $T_{\mu,a(k)+j} = \sum_{\nu\in I_\mu} T_{\nu,a(k)+j}$, $Y_{\mu,a(k)+j} = \sum_{\nu\in I_\mu} Y_{\nu,a(k)+j}$,
(respectively for $1 \le i \le p_{k+1-h}$, $1 \le h \le k$, $Y_{\mu,c(h)+a(k-h)+i} = \sum_{\nu\in I_\mu} Y_{\nu,c(h)+a(k-h)+i}$).

We have: $X_\mu = \sum_{1\le j\le p_{k+1}} \left[T_{\mu,a(k)+j}, Y_{\mu,a(k)+j}\right]$ (respectively
$X_\mu = \sum_{1\le i\le p_{k+1-h}} \left[T_{\mu,a(k)+1}, Y_{\mu,c(h)+a(k-h)+i}\right]$, $1 \le h \le k$) where $T_{\mu,a(k)+j}$, $Y_{\mu,a(k)+j}$,

$1 \le j \le p_{k+1}$ (respectively $Y_{\mu,c(h)+a(k-h)+i}$, $1 \le i \le p_{k+1-h}$, $1 \le h \le k$) belongs to $L_1^0(V^k)$ (respectively to $L_1^h(V^k)$, $1 \le h \le k$); hence the result, since $X = \sum_{1 \le \mu \le r} X_\mu$.

When $X \in {}^c L_1(V^k)$, it is enough to remark that the support of X only meets with a finite number of U_ν; denote by B the finite part of I such that, if $\nu \in I - B$, $supp\, X \cap U_\nu = \varnothing$, thus: $X = \sum_{\substack{1 \le j \le p_{k+1} \\ \nu \in B}} \left[T_{\nu,a(k)+j}, Y_{\nu,a(k)+j} \right]$ (respectively $\sum_{\substack{1 \le i \le p_{k+1-h} \\ \nu \in B}} \left[T_{\nu,a(k)+1}, Y_{\nu,c(h)+a(k-h)+i} \right]$, $1 \le h \le k$).

This completes the proof. □

Theorem 2 : *Let* $X \in L_1^0(V^k)$ *(respectively,* $X \in L_1^h(V^k)$, $1 \le h \le k$ *if* $p_{k+1} \ne 0$*) be a vector field on* V^k *such that the support of* X *is contained in an open set* Ω *verifying* $\Omega = \pi^{-1}(\pi(\Omega))$. *Then* $X = \sum_i [T_i, Y_i]$ *where* $\sum_i$ *is a finite sum,* $T_i \in L_1^0(V^k)$, $Y_i \in L_1^0(V^k)$ *(respectively* $L_1^h(V^k)$, $1 \le h \le k$*) and whose supports are included in* Ω.

Proof : With the same notations as in theorem 1, we have $\hat{\phi}_\nu \subset \hat{U}_\nu \cap \hat{\Omega}$ where $\hat{\Omega} = \pi(\Omega)$; so we can impose the restriction that $\overline{\hat{U}'_\nu} \subset \hat{U}_\nu \cap \hat{\Omega}$; then, the support of $T_{\nu,a(k)+j}$ and $Y_{\nu,a(k)+j}$, for $1 \le j \le p_{k+1}$, (respectively $Y_{\nu,c(h)+a(k-h)+i}$, for $1 \le i \le p_{k+1-h}$, $1 \le h \le k$) is in $U_\nu \cap \Omega$.

As the covering (U_ν) is locally finite, for $1 \le j \le p_{k+1}$,

$supp\, T_{\mu,a(k)+j} = \bigcup_{\nu \in I_\mu} supp\, T_{\nu,a(k)+j} \subset \Omega$, $\quad supp\, Y_{\mu,a(k)+j} = \bigcup_{\nu \in I_\mu} supp\, Y_{\nu,a(k)+j} \subset \Omega$

(respectively, for $1 \le i \le p_{k+1-h}$, $1 \le h \le k$,

$supp\, Y_{\mu,c(h)+a(k-h)+i} = \bigcup_{\nu \in I_\mu} supp\ Y_{\nu,c(h)+a(k-h)+i} \subset \Omega$), which proves the theorem. □

Lemma 6 : *Let U be an open set of adapted local coordinates of* V^k *and s an integer such that* $2 \le s \le k+1$. *Suppose* $p_{k-s+2} \ne 0$. *Every element of* $L_s(U)$ *is a bracket finite sum of elements of* $L_s(U)$ *which means that:* $[L_s(U), L_s(U)] = L_s(U)$.

Proof : Let $X \in A_s^h(U)$, $2 \le s \le k+1$, $0 \le h \le k+1-s$ be a vector field on U (see (4)).

We set, for $1 \le j \le p_{k-s-h+2}$:

$Y_{c(h)+a(k+1-s-h)+j} = \int_{\hat{\hat{u}}_{a(k-s+2)}}^{u_{a(k-s+2)}} X_{c(h)+a(k+1-s-h)+j}(u_1,....,u_{a(k-s+2)-1},t)dt$, where $(u_1,....,\hat{\hat{u}}_{a(k-s+2)},...,u_n)$ and $(u_1,....,u_n)$ belongs to U (U is supposed diffeomorphic to a Cartesian product of n open intervals of IR), and denote by Y the vector field on $U \subset V^k$ determined by its non zero first component : $Y_{c(h)+a(k+1-s-h)+j}$. $Y \in A_s^h(U)$.

We have in U: $[\ \partial_{a(k-s+2)}, Y\ ] = X$ so $[\ A_s^h(U), A_s^0(U)\ ] = A_s^h(U)$.

This completes the proof. □

Lemma 7 : *Let U be an open set of adapted local coordinates* $(u_1,....,u_n)$, $x \in U$ *and s an integer such that* $1 \le s \le k+1$. *Suppose* $p_{k-s+2} \ne 0$. *For every* $X \in A_s^h(U)$, $0 \le h \le k+1-s$, *such that* $j^3(X)(x) = 0$ *(i.e. the* $3-$*jet of each of the component functions of X is zero in* x*), there exists* $Y_1,...,Y_r \in A_s^0(U)$, $T_1,...,T_r \in A_s^h(U)$ *such that :*

$$X = \sum_{1 \le i \le r} [Y_i, T_i] \quad \textit{and} \quad j^1(Y_i)(x) = j^1(T_i)(x) = 0 .$$

Proof : From Lemma 1, it is sufficient to prove the result for

$$X = X_{c(h)+a(k+1-s-h)+l}(u_1,...., u_{a(k-s+2)})\partial_{c(h)+a(k+1-s-h)+l} + \sum_{1 \le q \le s-1} X_{c(h+q)+a(k+1-s-h)+l} \; \partial_{c(h+q)+a(k+1-s-h)+l}$$ where $1 \le s \le k+1$, $0 \le h \le k+1-s$, $1 \le l \le p_{k-h-s+2}$ and for $1 \le q \le s-1$, $X_{c(h+q)+a(k+1-s-h)+l}$ is given by (3).

We can always suppose $u_i(x) = 0$, $1 \le i \le n$.

We set $t = a(k+1-s)+l$, $1 \le l \le p_{k-s+2}$.

1) Consider $X \in A_s^0(U)$ and two vector fields Y and T belonging to $A_s^0(U)$:

$$Y = H\,\partial_t + \sum_{\substack{1 \le r \le s-1 \\ 1 \le j \le a(k-s+2)}} \left(\partial_j H \quad u_{c(r)+j} + ...\right) \, \partial_{c(r)+t} \, ,$$

$$T = G\,\partial_t + \sum_{\substack{1 \le r \le s-1 \\ 1 \le j \le a(k-s+2)}} \left(\partial_j G \quad u_{c(r)+j} + ...\right) \, \partial_{c(r)+t} \, .$$

$[Y,T] \in A_s^0(U)$: $[Y,T] = (H\,\partial_t G - G\,\partial_t H)\partial_t + ...$

It is sufficient to take:

If $X_t = u_t^4 \widetilde{X}(u_1,....,u_{a(k-s+2)})$,

$$H = u_t^2 , \quad G = u_t^2 \int_0^{u_t} \widetilde{X}(u_1,....,u_{t-1},x,u_{t+1},....,u_{a(k-s+2)})dx .$$

If $X_t = u_t^3 u_i \widetilde{X}$ with $i \ne t$, $1 \le i \le a(k-s+2)$,

$$H = u_t u_i , \quad G = u_t \int_0^{u_t} x\widetilde{X}(u_1,....,u_{t-1},x,u_{t+1},....,u_{a(k-s+2)})dx .$$

If $X_t = u_t^2 u_i u_j \widetilde{X}$ with $i \ne t$, $j \ne t$ $1 \le i,j \le a(k-s+2)$,

$$H = u_t u_i , \quad G = u_t u_j \int_0^{u_t} \widetilde{X}(u_1,....,u_{t-1},x,u_{t+1},....,u_{a(k-s+2)})dx .$$

If $X_t = u_t u_i u_j u_f \widetilde{X}$ with $i \ne t$, $j \ne t$, $f \ne t$ $1 \le i,j,f \le a(k-s+2)$,

$$H = u_i u_j , \quad G = u_f \int_0^{u_t} x\widetilde{X}(u_1,....,u_{t-1},x,u_{t+1},....,u_{a(k-s+2)})dx .$$

If $X_t = u_i u_j u_f u_g \widetilde{X}$ with $i \ne t$, $j \ne t$, $f \ne t$, $g \ne t$ $1 \le i,j,f,g \le a(k-s+2)$,

$$H = u_i u_j , \quad G = u_f u_g \int_0^{u_t} \widetilde{X}(u_1,....,u_{t-1},x,u_{t+1},....,u_{a(k-s+2)})dx .$$

We can remark that in any case, we even have $j^2(G)(x) = 0$.

2) Consider $X \in A_s^h(U)$, $1 \le h \le k+1-s$ and two vector fields $Y \in A_s^0(U)$ and $T \in A_s^h(U)$, $1 \le h \le k+1-s$:

$$Y = H\,\partial_t + \sum_{\substack{1\le r\le s-1 \\ 1\le j\le a(k-s+2)}} \left(\partial_j H \quad u_{c(r)+j} + \ldots\right)\ \partial_{c(r)+t} \ ,$$

$$T = G\,\partial_{c(h)+a(k+1-s-h)+l} + \sum_{\substack{1\le r\le s-1 \\ 1\le j\le a(k-s+2)}} \left(\partial_j G \quad u_{c(r)+j} + \ldots\right)\ \partial_{c(r+h)+a(k+1-s-h)+l} \ .$$

$$[Y,T] \in A_s^h(U):\ [Y,T] = H\,\partial_t G\,\partial_{c(h)+a(k+1-s-h)+l} + \ldots..$$

It is enough to take:

If $X_{c(h)+a(k+1-s-h)+l} = u_t^4 \widetilde{X}(u_1,\ldots.,u_{a(k-s+2)})$,

$$H = u_t^2\,,\quad G = \int_0^{u_t} x^2\,\widetilde{X}(u_1,\ldots.,u_{t-1},x,u_{t+1},\ldots.,u_{a(k-s+2)})dx\,.$$

If $X_{c(h)+a(k+1-s-h)+l} = u_t^3 u_i \widetilde{X}$ with $i \ne t$, $1 \le i \le a(k-s+2)$,

$$H = u_t^2\,,\quad G = u_i \int_0^{u_t} x\widetilde{X}(u_1,\ldots.,u_{t-1},x,u_{t+1},\ldots.,u_{a(k-s+2)})dx\,.$$

If $X_{c(h)+a(k+1-s-h)+l} = u_t^2 u_i u_j \widetilde{X}$ with $i \ne t$, $j \ne t$ $1 \le i,j \le a(k-s+2)$,

$$H = u_t^2\,,\quad G = u_i\,u_j \int_0^{u_t} \widetilde{X}(u_1,\ldots.,u_{t-1},x,u_{t+1},\ldots.,u_{a(k-s+2)})dx\,.$$

If $X_{c(h)+a(k+1-s-h)+l} = u_t\,u_i u_j u_f \widetilde{X}$ with $i \ne t$, $j \ne t$, $f \ne t$ $1 \le i,j,f \le a(k-s+2)$,

$$H = u_t\,u_i\,,\quad G = u_j u_f \int_0^{u_t} \widetilde{X}(u_1,\ldots.,u_{t-1},x,u_{t+1},\ldots.,u_{a(k-s+2)})dx\,.$$

If $X_{c(h)+a(k+1-s-h)+l} = u_i u_j u_f u_g \widetilde{X}$ with $i \ne t$, $j \ne t$, $f \ne t$, $g \ne t$, $1 \le i,j,f,g \le a(k-s+2)$,

$$H = u_i\,u_j\,,\quad G = u_f u_g \int_0^{u_t} \widetilde{X}(u_1,\ldots.,u_{t-1},x,u_{t+1},\ldots.,u_{a(k-s+2)})dx\,.$$

Hence, in any case:

$j^1(H)(x) = j^1(Y)(x) = 0$, $j^1(G)(x) = j^1(T)(x) = 0$, $j^2(G)(x) = 0$.

This completes the proof. □

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