The diamond partial order in rings

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Abstract

In this paper we introduce a new partial order on a ring, namely the diamond partial order. This order is an extension of a partial order defined in a matrix setting in [J.K. Baksalary and J. Hauke, A further algebraic version of Cochran's theorem and matrix partial orderings, Linear Algebra and its Applications, 127, 157–169, 1990]. We characterize the diamond partial order on rings and study its relationships with other partial orders known in the literature. We also analyze successors, predecessors and maximal elements under the diamond order.

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1 Introduction and Background

Let R be an associative ring with unity 1. For a given $a \in R$, we will denote

$$a\{1\} := \{x \in R : axa = a\}$$

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the set of all $\{1\}$-inverses of a. A particular $\{1\}$-inverse of a will be written as a^-, and the element a is regular if $a\{1\} \neq \emptyset$. As usual, R is a regular ring if all elements of R are regular. A $\{1,2\}$-inverse of a is a $\{1\}$-inverse of a that is a solution of the ring equation $xax = x$, it will be denoted by $x \in a\{1,2\}$. The unique $\{1,2\}$-inverse of a that commutes with a is called the group inverse of a (when it exists) and denoted by $a^\#$. The set of group invertible elements is denoted by $R^\#$.

An involution $*$ in R is an anti-isomorphism of degree 2 in R, that is to say, $(x^*)^* = x$, $(x+y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$, for all $x, y \in R$. We will use the following notation: $aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$ the principal ideals; $^\circ(a) = \{x \in R : xa = 0\}$ and $(a)^\circ = \{x \in R : ax = 0\}$.

We say $a \in R$ is Moore-Penrose invertible (with respect to $*$) if the equations $axa = a$, $xax = x$, $(ax)^* = ax$, $(xa)^* = xa$ have a common solution. If such a solution exists, then it is unique, and denoted by $a^\dagger$. The set of Moore-Penrose invertible elements is denoted by $R^\dagger$.

We recall some well-known partial orders on a regular ring R:

- the minus partial order: $a \leq^- b$ iff $a^-a = a^-b$ and $aa^- = ba^-$.
- the star partial order: $a \leq^* b$ iff $a^*a = a^*b$ and $aa^* = ba^*$, which in turn is equivalent to $a^\dagger a = a^\dagger b$ and $aa^\dagger = ba^\dagger$ in $R^\dagger$.
- the left star partial order: $a* \leq b$ iff $a^*a = a^*b$ and $aR \subseteq bR$.
- the right star partial order: $a \leq *b$ iff $aa^* = ba^*$ and $Ra \subseteq Rb$.
- the sharp partial order in $R^\#$: $a \leq^\# b$ iff $a^\#a = a^\#b$ and $aa^\# = ba^\#$.
- the direct sum partial order: $a \leq^\oplus b$ iff $bR = aR \oplus (b-a)R$.

A detailed analysis of these partial orders has been done in [8] for a matrix approach.

Throughout this paper, R will be a ring with involution and we will assume R is *-regular, i.e., all elements have a Moore-Penrose inverse.

We define (see [3])

$$a \leq_\diamond b \qquad \text{iff} \qquad aR \subseteq bR,\ Ra \subseteq Rb \ \text{ and } \ aa^*a = ab^*a.$$

In Section 3 we are going to prove that the binary relation $\leq_\diamond$ defines a partial order on R and, from now on, it is called the diamond partial order. It should be mentioned that the diamond partial order has not been considered in the literature, as far as we know, in the setting of rings.

We recall some well-known facts.

Lemma 1 *Let $a \in R$ and $a^-, a^= \in a\{1\}$. Then $a^- a a^= \in a\{1,2\}$.*

Lemma 2 *Let $a, b \in R$. Then*

(a) $a \leq^- b$ iff there exists b^- such that $bb^-a = ab^-b = ab^-a = a$.

(b) $aR \subseteq bR$ iff $Ra^* \subseteq Rb^*$ and $Ra \subseteq Rb$ iff $a^*R \subseteq b^*R$.

(c) $a^*R = a^\dagger R$ and $Ra^* = Ra^\dagger$.

Proof. (a) ($\Longrightarrow$) By hypothesis we have: $a^-a = a^-b$ and $aa^- = ba^-$. Then

$$bb^-a = bb^-aa^-a = bb^-ba^-a = ba^-a = aa^-a = a,$$

$$ab^-b = aa^-ab^-b = aa^-bb^-b = aa^-b = aa^-a = a,$$

and

$$ab^-a = aa^-ab^-aa^-a = aa^-bb^-ba^-a = aa^-ba^-a = aa^-aa^-a = a.$$

($\Longleftarrow$) We assume that there exists b^- such that $bb^-a = ab^-b = ab^-a = a$. First, we notice that b^- is a $\{1\}$-inverse of a since $ab^-a = a$. If we now define $a^- := b^-ab^-$, Lemma 1 assures that $a^- \in a\{1,2\}$. Using the equalities $bb^-a = ab^-b = ab^-a = a$ we obtain:

$$aa^- = ab^-ab^- = ab^- = bb^-ab^- = ba^-$$

and

$$a^-a = b^-ab^-a = b^-a = b^-ab^-b = a^-b$$

Hence, by definition, $a \leq^- b$.

(b) Trivial by definition.

(c) It follows from the properties $a^* = a^\dagger(aa^*) = (a^*a)a^\dagger$ and $a^\dagger = a^*(aa^*)^\dagger = (a^*a)^\dagger a^*$. ■

We remark that

(I) $ab^\dagger a = a \Longrightarrow ab^\dagger$ and $b^\dagger a$ are idempotent.

(II) $bb^\dagger a = a \Longleftrightarrow bb^\dagger aa^\dagger = aa^\dagger$.

(III) $ab^\dagger b = a \Longleftrightarrow a^\dagger ab^\dagger b = a^\dagger a$.

(IV) If the equalities $bb^- a = ab^- b = ab^- a = a$ hold for some b^- then they hold for any choice of b^-. Indeed, the independence of b^- in $bb^- a = ab^- b = a$ follows directly from [9, Lemma 2.1]. For a $\{1\}$-inverse b^- of b, it is well known [10, pp. 26] that all $\{1\}$-inverses of b are of the form $b^= = b^- + (1 - b^- b)h + z(1 - bb^-)$ for some choice of h and z. As $bb^- a = ab^- b = ab^- a = a$ it follows that $ab^= a = a$. To sum up we showed the independence of the equalities $bb^- a = ab^- b = ab^- a = a$ to the choice of b^-.

Lemma 3 *[4, Corollary 4] Let $x, y \in R$ such that y is idempotent. Then $x \leq^- y$ iff $x = x^2 = xy = yx$.*

Proof. For the sake of completeness we include a proof.

($\Longrightarrow$) Since $xx^- = yx^-$ and $x^- x = x^- y$, we get $x = xx^- y = yx^- x$. Then $xy = xx^- y^2 = xx^- y = x$ and $yx = y^2 x^- x = yx^- x = x$. Moreover, $x^2 = xx^- yx = xx^- xy = xy$.

($\Longleftarrow$) From $x^2 = x$ we have that x is group invertible and $x^\# = x$. Taking $x^- = x^\#$ we get $xx^- = x^2 = yx = yx^-$ and $x^- x = x^- y$ is similar. ■

A wide range of properties related to these orders and the generalized inverses involved in each of them can be found in [1, 2, 3, 6, 7, 11, 12, 13].

This paper is organized as follows. In Section 2 we analyze some relationships between the diamond binary relation and the minus, left star, right star, star and sharp partial orders. In Section 3 the diamond partial order on rings is characterized. Section 4 is devoted to the study of successors and predecessors under the diamond order. In addition, maximal elements under the diamond partial order are found.

2 Relations between the diamond order and other partial orders

Firstly, we notice that the equivalence $a \leq_\diamond b \Longleftrightarrow b - a \leq_\diamond b$ does not hold for the diamond partial order (see an example in [3]) whereas it remains valid for the star and minus orders, as stated in the following result.

Lemma 4 *Given regular $x, y \in R$,*

(a) $x \leq^- y$ iff $y - x \leq^- y$.

(b) $x \leq^* y$ iff $y - x \leq^* y$.

Proof. (a) If $x \leq^- y$ then by [5, Proposition 3 (i)]

$$y = x + (1 - xx^+)s(1 - x^+x)$$

for some $\{1, 2\}$-inverse x^+ of x and an arbitrary $s \in R$. Setting the idempotents $e = 1 - xx^+$ and $f = 1 - x^+x$ there exists $(y - x)^+ = fwe$ for some $w \in R$. For this choice, $(y-x)(y-x)^+ = esfwe = (x+esf)fwe = y(y-x)^+$. Similarly, $(y - x)^+y = fwe(x + esf) = fwesf = (y - x)^+(y - x)$.

Conversely, if $y - x \leq^- y$ then by the previous implication $x = y - (y - x) \leq^- y$.

(b) From $x \leq^* y$ we obtain the equalities $(y - x)x^* = x^*(y - x) = 0$, from which $(y - x)(y - x)^* = (y - x)y^*$ and $(y - x)^*(y - x) = y^*(y - x)$. Since these are hermitian, the equalities $(y - x)(y - x)^* = y(y - x)^*$ and $(y - x)^*(y - x) = (y - x)^*y$ hold.

Conversely, if $y - x \leq^* y$ then by the previous implication $x = y - (y - x) \leq^* y$. ■

We also observe that neither of the implications $aa^*a = ab^*a \Longrightarrow aa^\dagger a = ab^\dagger a$ (in $\mathbb{M}_2(\mathbb{C})$ take $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$) nor $aa^\dagger a = ab^\dagger a \Longrightarrow aa^*a = ab^*a$ (in $\mathbb{M}_2(\mathbb{C})$ take $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$) is valid in general.

We remark that $*$ is isotone with respect to the diamond partial order. That is to say $a \leq_\diamond b$ exactly when $a^* \leq_\diamond b^*$. This follows from Lemma 2. As a consequence we have the following proposition.

Proposition 1 *$a \leq_\diamond (a^\dagger)^*$ iff a is a partial isometry (i.e., $a^\dagger = a^*$).*

Proof. If $a \leq_\diamond (a^\dagger)^*$ then $aa^*a = aa^\dagger a = a$, that is $a^* \in a\{1\}$. Since aa^* and a^*a are hermitian and $a^*aa^* = a^*$, we get $a^* = a^\dagger$. The converse is trivial. ■

Some equivalent conditions to $aa^*a = ab^*a$ are given in the following result.

Lemma 5 *Let $a, b \in R$. Then the following conditions are equivalent:*

(a) $aa^*a = ab^*a$.

(b) $a^\dagger ba^\dagger \in a\{1\}$.

(c) $a^\dagger ba^\dagger \in a\{1,2\}$.

(d) $a^\dagger ba^\dagger = a^\dagger$.

Proof. (a) $\Longrightarrow$ (b) Multiplying $aa^*a = ab^*a$ on the left and right sides by $a^\dagger$ we get $a^* = a^\dagger aa^*aa^\dagger = a^\dagger ab^*aa^\dagger = a^*(a^\dagger ba^\dagger)^*a^*$. Hence, $a^\dagger ba^\dagger \in a\{1\}$.

(b) $\Longrightarrow$ (a) Multiplying $a = a(a^\dagger ba^\dagger)a$ on the left and right sides by a^* we get $a^*aa^* = a^*aa^\dagger ba^\dagger aa^* = a^*(aa^\dagger)^*b(a^\dagger a)^*a^* = (aa^\dagger a)^*b(aa^\dagger a)^* = a^*ba^*$. Thus $aa^*a = ab^*a$.

(b) $\Longrightarrow$ (c) It follows applying Lemma 1 with $a^- = a^\dagger ba^\dagger$ and $a^= = a^\dagger$.

(c) $\Longrightarrow$ (b) is trivial.

(a) $\Longrightarrow$ (d) Multiplying firstly both sides of $a^*aa^* = a^*ba^*$ by $(a^\dagger)^*$ we have $(a^\dagger)^*a^*aa^*(a^\dagger)^* = (a^\dagger)^*a^*ba^*(a^\dagger)^*$, that is $(aa^\dagger)^*a(a^\dagger a)^* = (aa^\dagger)^*b(a^\dagger a)^*$. Then $a = aa^\dagger ba^\dagger a$. Now multiplying both sides by $a^\dagger$ we get $a^\dagger = a^\dagger ba^\dagger$.

(d) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (d) Multiplying $a(a^\dagger ba^\dagger)a = a$ on the left and right sides by $a^\dagger$ we get $a^\dagger a(a^\dagger ba^\dagger)aa^\dagger = a^\dagger aa^\dagger$, that is $a^\dagger ba^\dagger = a^\dagger$. ■

Theorem 1 *Let $a, b \in R$. Then the following conditions are equivalent:*

(a) $a \leq_\diamond b$.

(b) $aR \subseteq bR$, $Ra \subseteq Rb$ *and* $a^\dagger ba^\dagger \in a\{1\}$.

(c) $aR \subseteq bR$, $Ra \subseteq Rb$ *and* $a^\dagger ba^\dagger \in a\{1,2\}$.

(d) $aR \subseteq bR$, $Ra \subseteq Rb$ *and* $a^\dagger ba^\dagger = a^\dagger$.

Proof. It follows by the definition of the diamond partial order and Lemma 5. ■

The implications $a* \leq b \Rightarrow a \leq_\diamond b$ and $a \leq_\diamond b \Rightarrow a* \leq b$ are not valid in general. Similarly, for $\leq *$ instead of $* \leq$.

Even for matrices over a field, the implication $a \le *b \Rightarrow a \le_\diamond b$ might not hold. Take, over the field $\mathbb{Z}_{13}$, the matrices $A = \begin{bmatrix} 9 & 7 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 10 & 2 \end{bmatrix}$, and the transposition as the involution. Then $\begin{bmatrix} 9 & 7 \end{bmatrix} = 10 \begin{bmatrix} 10 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \end{bmatrix} = 8 \begin{bmatrix} 10 & 2 \end{bmatrix}$ and row space of A is a subspace of the row space of B. As $AA^* = BA^*$ then $A \le *B$. Nevertheless, $\begin{bmatrix} 9 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ has no solutions in $\mathbb{Z}_{13}$, and hence the column space of A is not a subspace of the column space of B.

Needless to say a similar conclusion can be drawn for $* \le$, as $A^* * \le B^*$ and yet $A^* \not\le_\diamond B^*$ since the row space of A^* is not a subspace of the row space of B^*, where A and B are as the previous example.

Lemma 6 *Let us consider the following statements:*

(a) $a \le_\diamond b$.

(b) $a* \le b$ *and* $Ra \subseteq Rb$.

(c) $a \le *b$ *and* $aR \subseteq bR$.

Then (b) $\Longrightarrow$ *(a) and (c)* $\Longrightarrow$ *(a).*

Proof. (b) $\Longrightarrow$ (a) By definition $a* \le b$ if and only if $a^*a = a^*b$ and $aR \subseteq bR$. Multiplying the equality on the right side by a^* we get $a^*aa^* = a^*ba^*$, that is $aa^*a = ab^*a$. Thus $a \le_\diamond b$.

(c) $\Longrightarrow$ (a) The proof is similar to the previous one. ∎

Remark 1 *Observe that in Lemma 6 neither (a)* $\Longrightarrow$ *(b) nor (a)* $\Longrightarrow$ *(c) as the following example allows us to check in* $\mathbb{M}_2(\mathbb{C})$*:*

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proposition 2 *Let* $a, b \in R$. *Then*

(a) $a \le^* b \Longrightarrow a \le_\diamond b$.

(b) $a \le^\# b \Longrightarrow a^\dagger \le_\diamond b^\dagger \Longleftrightarrow a \le^- b$.

Proof. (a) Suppose $a \leq^* b$, that is, $aa^* = ba^*$ and $a^*a = a^*b$. Then $a^*(b-a)a^* = 0$, from which $aa^*a = ab^*a$. Furthermore, post-multiplying $aa^* = ba^*$ and pre-multiplying $a^*a = a^*b$ by $(a^\dagger)^*$, we obtain $a = ba^*(a^\dagger)^* = (a^\dagger)^*a^*b \in bR \cap Rb$ and as such $aR \subseteq bR$ and $Ra \subseteq Rb$.

(b) It is well known that $a \leq^\# b \Longrightarrow a \leq^- b$ and this last expression is equivalent to $a^\dagger \leq_\diamond b^\dagger$ (as we will see in Theorem 2). ■

Note that, despite Theorem 2 has been not proved yet, we have included Proposition 2 in this section to collect all the relationships between the diamond partial order and the other ones.

Now, we remark that $a \leq_\diamond b$ does not imply $a \leq^* b$. A counterexample can be found by taking the real matrices

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We can also observe that $a \leq^\# b$ does not imply $a \leq_\diamond b$ as the following real matrices show:

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We close this section with the following remark.

Remark 2 The condition on the Moore-Penrose invertibility of a in (a) of the previous Proposition cannot be dropped. We will present an example using matrices over $\mathbb{Z}_4$ with the involution $*$ as transposition. Take $A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ which satisfy $A \leq^* B$, since $AA^* = BA^*$, $A^*A = A^*B$. Yet, $A = B \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ would imply $z = 0$ and $0 = 2$, and therefore $A \leq_\diamond B$ does not hold. Note that $A^\dagger$ does not exist as the (free) $\mathbb{Z}_4$-module generated by the columns of A is not a submodule of the generated by the columns of AA^*. Indeed, there are no solutions in $\mathbb{Z}_4$ for $\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

3 Characterizations of the diamond partial order

Now, we characterize the diamond partial order in terms of the minus partial order.

Theorem 2 *Let $a, b \in R$. Then the following statements are equivalent:*

(a) $a \leq_\diamond b$.

(b) $a^\dagger \leq^- b^\dagger$.

(c) $aa^\dagger bb^\dagger = aa^\dagger$, $b^\dagger ba^\dagger a = a^\dagger a$, $a^\dagger ba^\dagger = a^\dagger$.

Proof. (a) $\Longrightarrow$ (b) and (c) By hypothesis and Lemma 2 we have: $Ra^\dagger \subseteq Rb^\dagger$ and $a^\dagger R \subseteq b^\dagger R$. Since $a^\dagger = a^\dagger aa^\dagger \in a^\dagger R \cap Ra^\dagger \subseteq b^\dagger R \cap Rb^\dagger$.

Since $a^\dagger = b^\dagger x$ for some $x \in R$, $b^\dagger ba^\dagger = b^\dagger bb^\dagger x = b^\dagger x = a^\dagger$. Hence, $b^\dagger ba^\dagger = a^\dagger$ and so $b^\dagger ba^\dagger a = a^\dagger a$.

Since $a^\dagger = yb^\dagger$ for some $y \in R$, $a^\dagger bb^\dagger = yb^\dagger bb^\dagger = yb^\dagger = a^\dagger$. Hence, $a^\dagger bb^\dagger = a^\dagger$ and so $aa^\dagger bb^\dagger = aa^\dagger$.

Lemma 5 assures that the condition $a^*aa^* = a^*ba^*$ is equivalent to $a^\dagger = a^\dagger ba^\dagger$. Finally, Lemma 2 implies that $a^\dagger \leq^- b^\dagger$ holds.

(b) $\Longrightarrow$ (a) From $a^\dagger a = b^\dagger a$ and $aa^\dagger = ab^\dagger$ we get $a^\dagger = b^\dagger aa^\dagger \in b^\dagger R$ and $a^\dagger = a^\dagger ab^\dagger \in Rb^\dagger$. Then $a^\dagger R \subseteq b^\dagger R$ and $Ra^\dagger \subseteq Rb^\dagger$. Moreover, Lemma 2 (a) assures that $a^\dagger = a^\dagger ba^\dagger$. Finally, Lemma 5 implies that $a^*aa^* = a^*ba^*$.

(c) $\Longrightarrow$ (a) From $aa^\dagger bb^\dagger = aa^\dagger$ we get $a^\dagger bb^\dagger = a^\dagger$, that is $a^\dagger \in Rb^\dagger$. Thus, $Ra^* = Ra^\dagger \subseteq Rb^\dagger = Rb^*$ and this implies $aR \subseteq bR$.

Similarly, $b^\dagger ba^\dagger a = a^\dagger a$ yields $Ra \subseteq Rb$. Moreover, the equivalence between $a^\dagger ba^\dagger = a^\dagger$ and $aa^*a = ab^*a$ has been shown in Lemma 5. Hence, $a \leq_\diamond b$. ∎

We remark that in Theorem 2 (c) the hypothesis $a^\dagger ba^\dagger = a^\dagger$ cannot be dropped. Take the real matrices, with transposition as involution, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = I_2$, for which $A \not\leq_\diamond B$ since $AA^TA \neq A^2 = A$, and yet $AA^\dagger BB^\dagger = AA^\dagger$, $B^\dagger BA^\dagger A = A^\dagger A$, but $A^\dagger BA^\dagger \neq A^\dagger$.

Theorem 2 allows us to assure that the diamond relation is a partial order.

Corollary 1 *The binary relation* $\leq_\diamond$ *is a partial order on the ring* R.

Notice that neither of the implications $a \leq_\diamond b \Longrightarrow a \leq^- b$ nor $a \leq^- b \Longrightarrow a \leq_\diamond b$ are valid in general (see examples in [3, pp. 165]).

We recall that $a \leq^- b$ iff there exist idempotents $e, f \in R$ such that $a = eb = bf$. This will lead to the following result.

Theorem 3 *Let* $a, b \in R$. *Then the following conditions are equivalent:*

(a) $a \leq_\diamond b$.

(b) $(b^\dagger - a^\dagger)^\dagger \leq_\diamond b$.

(c) There exist idempotents $e, f \in R$ *such that* $a = (eb^\dagger)^\dagger = (b^\dagger f)^\dagger$.

Proof. (a) $\Longleftrightarrow$ (b) By Theorem 2, $a \leq_\diamond b$ iff $a^\dagger \leq^- b^\dagger$ and by Lemma 4 $a^\dagger \leq^- b^\dagger$ iff $b^\dagger - a^\dagger \leq^- b^\dagger$ which is equivalent to $(b^\dagger - a^\dagger)^\dagger \leq_\diamond b$ by Theorem 2.

(a) $\Longleftrightarrow$ (c) $a \leq_\diamond b$ iff $a^\dagger \leq^- b^\dagger$ iff there exist idempotents $e, f \in R$ such that $a^\dagger = eb^\dagger = b^\dagger f$. That is $a = (eb^\dagger)^\dagger = (b^\dagger f)^\dagger$. ∎

We remark that $a \leq_\diamond b$ does not imply $a^\dagger \leq_\diamond b^\dagger$, even though $aR \subseteq bR$ and $Ra \subseteq Rb$ imply $Ra^\dagger = Ra^* \subseteq Rb^* = Rb^\dagger$ and $a^\dagger R = a^* R \subseteq b^* R = b^\dagger R$. The implication is not valid as $aa^*a = ab^*a$ is not sufficient to $a^\dagger (a^\dagger)^* a^\dagger = a^\dagger (b^\dagger)^* a^\dagger$. Take the rational matrices, with the transposition as involution, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with $A^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ with $B^\dagger = B^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$. Then $A \leq_\diamond B$ and yet $A^\dagger \not\leq_\diamond B^{-1}$.

Theorem 4 *Let* $a, b \in R$. *Then* $a \leq_\diamond b \Longleftrightarrow a^\dagger \leq^\oplus b^\dagger$.

Proof. It is well known that $x \leq^- y$ iff $x \leq^\oplus y$ [4, Lemma 3]. Then, this item follows directly applying Theorem 2 with $x = a^\dagger$, $y = b^\dagger$. ∎

Neither the implication $a \leq_\diamond b \Longrightarrow a \leq^\oplus b$ nor $a \leq^\oplus b \Longrightarrow a \leq_\diamond b$ are valid in general. Indeed, it follows from the fact that $x \leq^- y$ iff $x \leq^\oplus y$.

Moreover, for the matrices

$$a = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

it is easy to see that $a \leq_\diamond b$ and however $a \not\leq^\# b$.

Defining $a^{\pi,r} = 1 - aa^\dagger$ and $a^{\pi,l} = 1 - a^\dagger a$ we obtain:

Lemma 7 *Let R be a ring with unity and $a, b \in R$. Then the following conditions are equivalent:*

(a) $a \leq_\diamond b$.

(b) $b^{\pi,r} \leq_\diamond a^{\pi,r}$, $b^{\pi,l} \leq_\diamond a^{\pi,l}$ and $(1 - a^{\pi,r})(1 - ba^\dagger) = 0$.

(c) $b^{\pi,r} \leq^- a^{\pi,r}$, $b^{\pi,l} \leq^- a^{\pi,l}$ and $(1 - a^{\pi,r})(1 - ba^\dagger) = 0$.

Proof. We first observe that $(1 - xx^\dagger)^\dagger = 1 - xx^\dagger$ and $(1 - x^\dagger x)^\dagger = 1 - x^\dagger x$ for $x \in \{a, b\}$. Now, we apply Theorem 2. The following equivalences are valid:

$$aa^\dagger bb^\dagger = aa^\dagger = bb^\dagger aa^\dagger \iff$$

$$\iff 1 - bb^\dagger = (1 - aa^\dagger)(1 - bb^\dagger) = (1 - bb^\dagger)(1 - aa^\dagger) \iff 1 - bb^\dagger \leq^- 1 - aa^\dagger.$$

Similarly, it can be shown that $b^\dagger ba^\dagger a = a^\dagger a$ is equivalent to $b^{\pi,l} \leq_\diamond a^{\pi,l}$ and $a^\dagger ba^\dagger = a^\dagger$ is equivalent to $(1 - a^{\pi,r})(1 - ba^\dagger) = 0$. Hence, (a) $\iff$ (b). The equivalence between (b) and (c) follows directly from Theorem 2. ■

4 Successors and predecessors under the diamond partial order

Let us start this section with a result valid for the minus partial order.

Lemma 8 *Let $x, y \in R$. Then the following conditions are equivalent:*

(a) $x \leq^- y$.

(b) There exists $x^= \in x\{1, 2\}$ such that $y - x \in {}^\circ(x^=) \cap (x^=)^\circ$.

Proof. If $x \leq^- y$ then $x^- x = x^- y$ and $xx^- = yx^-$ for some $x^- \in x\{1\}$. Taking $x^= = x^- xx^-$ we have that $x^= \in x\{1, 2\}$ with $x^= x = x^= y$ and $xx^= = yx^=$. So, $(y - x)x^= = 0$ and $x^=(y - x) = 0$ and this last two equalities are equivalent to $y - x \in {}^\circ(x^=) \cap (x^=)^\circ$.

The converse is trivial. ■

Given $a \in R$, in the following we find all the elements $b \in R$ such that $a \leq_\diamond b$. Such elements b are called the successors of a.

Theorem 5 *Let $a \in R$. Then the following conditions are equivalent:*

(a) There exists $b \in R$ such that $a \leq_\diamond b$.

(b) There exists $h \in {}^\circ((a^\dagger)^=) \cap ((a^\dagger)^=)^\circ$ such that $b = (a^\dagger + h)^\dagger$.

Proof. (a) $\Longrightarrow$ (b) If $a \leq_\diamond b$ then $a^\dagger \leq^- b^\dagger$. Taking $x = a^\dagger$, $y = b^\dagger$ in Lemma 8 we have that there exists $(a^\dagger)^= \in a^\dagger\{1,2\}$ such that $b^\dagger - a^\dagger \in {}^\circ((a^\dagger)^=) \cap ((a^\dagger)^=)^\circ$. Setting $h = b^\dagger - a^\dagger$, we get $b = (a^\dagger + h)^\dagger$.

(b) $\Longrightarrow$ (a) Applying first Lemma 8 and then Theorem 2, we get the result. ■

Given $b \in R$, all the elements $a \in R$ such that $a \leq_\diamond b$ are called the predecessors of b. A partial solution of the problem of finding all the predecessors of a fixed element is given in the following result.

Theorem 6 *Let $b, h \in R$. If $(b-h)^\dagger = b^\dagger - h^\dagger$ and $h \leq^* b$ then $a = b - h$ satisfies $a \leq_\diamond b$.*

Proof. Since $h \leq^* b$, we have $bh^\dagger h = h = hh^\dagger b$. Hence, we get $aa^\dagger = ab^\dagger$ and $a^\dagger a = b^\dagger a$ since

$$aa^\dagger = (b-h)(b-h)^\dagger = (b-h)b^\dagger - bh^\dagger + hh^\dagger = ab^\dagger,$$

$$a^\dagger a = (b-h)^\dagger(b-h) = b^\dagger(b-h) - h^\dagger b + h^\dagger h = b^\dagger a.$$

Now, Theorem 2 finishes the proof. ■

Other method to find predecessors of a given element $b \in R$ has been stated in Theorem 3 where the idempotents in the ring have to be previously found.

Theorem 6 allows us to state a similar result to that in Lemma 4 for the diamond partial order as follows.

Theorem 7 *Let $a, b \in R$ such that $(b-a)^\dagger = b^\dagger - a^\dagger$. Then $b - a \leq^* b$ iff $a \leq_\diamond b$.*

Proof. By definition of star partial order and the assumption we have that $b-a \leq^* b$ holds iff $(b^\dagger - a^\dagger)(b-a) = (b^\dagger - a^\dagger)b$ and $(b-a)(b^\dagger - a^\dagger) = b(b^\dagger - a^\dagger)$. Some computations leads to $a^\dagger a = b^\dagger a$ and $aa^\dagger = ab^\dagger$. Theorem 2 yields $a \leq_\diamond b$. The converse can be shown in a similar way. ■

Corollary 2 *If* $(b-a)^\dagger = b^\dagger - a^\dagger$ *for all element* $a, b \in R$ *then the star and diamond partial orders coincide, that is* $a \leq^* b$ *iff* $a \leq_\diamond b$.

Theorem 8 *Let* $b \in R$. *The following statements are valid:*

(a) If $b^\dagger$ *is idempotent then* $a \leq_\diamond b$ *iff* $(a^\dagger)^2 = a^\dagger = a^\dagger b^\dagger = b^\dagger a^\dagger$.

(b) If b *is hermitian idempotent then* $a \leq_\diamond b$ *iff* $(a^\dagger)^2 = a^\dagger = a^\dagger b = b a^\dagger$.

Proof. (a) It follows directly from Theorem 2 and Lemma 3 taking $x = a^\dagger$, $y = b^\dagger$.

(b) It is a particular case of (a). ■

Now we characterize the intervals

$$[0, a^{\pi,r}] = \{b \in R :\, 0 \leq_\diamond b \leq_\diamond a^{\pi,r}\} \text{ and } [0, a^{\pi,l}] = \{b \in R :\, 0 \leq_\diamond b \leq_\diamond a^{\pi,l}\}$$

for a fixed element $a \in R$.

Proposition 3 *Let* $a \in R$. *Then*

$$[0, a^{\pi,r}] = \{b \in R : b^\dagger \text{ is idempotent and } b^\dagger \in {}^\circ a \cap (a^\dagger)^\circ\}.$$

and

$$[0, a^{\pi,l}] = \{b \in R : b^\dagger \text{ is idempotent and } b^\dagger \in {}^\circ(a^\dagger) \cap a^\circ\}.$$

Proof. Since $a^{\pi,r}$ is idempotent and hermitian, we apply Theorem 8 and then $b \leq_\diamond a^{\pi,r}$ iff $(b^\dagger)^2 = b^\dagger = b^\dagger a^{\pi,r} = a^{\pi,r} b^\dagger$. Multiplying $b^\dagger = a^{\pi,r} b^\dagger$ on the left side by $a^\dagger$ we get $a^\dagger b^\dagger = a^\dagger(1 - aa^\dagger)b^\dagger = (a^\dagger - a^\dagger a a^\dagger)b^\dagger = 0$. In the same way, multiplying $b^\dagger = b^\dagger a^{\pi,r}$ on the right side by a we have $b^\dagger a = b^\dagger(1 - aa^\dagger)a = b^\dagger a - b^\dagger a a^\dagger a = 0$. Hence $b^\dagger$ is idempotent and $a^\dagger b^\dagger = b^\dagger a = 0$. Similarly for the idempotent $a^{\pi,l}$. ■

Lemma 9 *If* $u \in R$ *is a unit then* u *is maximal under the diamond partial order.*

Proof. Given a unit u and an arbitrary $a \in R$, if $u \leq_\diamond a$ then $u^{-1} \leq^- a^\dagger$. On account of [5, Proposition 3 (i)], $a^\dagger = u^{-1}$ from which $a = u$ is maximal. ■

Theorem 9 *Let $a \in R^{\#}$. Then a is a unit iff a is maximal of $R^{\#}$ under the diamond partial order.*

Proof. ($\Longrightarrow$) It follows directly from Lemma 9.

($\Longleftarrow$) Take $u = a + 1 - aa^{\dagger}$ with $a \in R^{\#}$. Then u is a unit with $u^{-1} = a^{\#} + 1 - aa^{\dagger}$. Therefore, $aR \subseteq uR = R$ and $Ra \subseteq Ru = R$. Also, $aa^*a = au^*a$. These mean $a \leq_{\diamond} u$. Since a is maximal then $a = u$ is a unit and the result follows. ■

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