Algorithms for $\{K, s+1\}$-potent matrices constructions

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Abstract

In this paper, we deal with $\{K, s+1\}$-potent matrices. These matrices generalize all the following classes of matrices: k-potent matrices, periodic matrices, idempotent matrices, involutory matrices, centrosymmetric matrices, mirrorsymetric matrices, circulant matrices, etc. Several applications of these classes of matrices can be found in the literature. We develop algorithms in order to compute $\{K, s+1\}$-potent matrices and $\{K, s+1\}$-potent linear combinations of $\{K, s+1\}$-potent matrices. In addition, some examples are presented in order to show the numerical performance of the method.

Keywords: Potent matrices, Involutory matrices, Linear relation, Eigenvalues

1. Introduction

In recent years, real applications for certain classes of matrices have been developed. Specifically, the problem of multiconductor transmission lines has been studied by means of mirror symmetric matrices in [7, 8]. Also, Circulant matrices have been applied to solve problems in several areas such as numerical computation, solid state-physics, image and signal processing, coding theory, mathematical statistics, and molecular vibration [2, 3]. Some applications of centrosymmetric matrices have been given in [1], for example, for solving problems in pattern recognition, antenna theory, mechanical and electrical systems, and quantum physics. In this last case, symmetric and skew-symmetric eigenvectors have been used [9].

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Related to the aforementioned classes of matrices, another type was introduced in [5], namely the $\{K, s+1\}$-potent matrices. For a given involutory matrix $K \in \mathbb{C}^{n\times n}$ ($K^2 = I_n$) and $s \in \{0, 1, 2, 3, \dots\}$, we recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent if it satisfies

$$KA^{s+1}K = A. \tag{1}$$

When $s = 0$, the matrix A is called $\{K\}$-centrosymmetric. It can be seen that $\{K, s+1\}$-potent matrices generalize all the following classes: k-potent matrices, periodic matrices, idempotent matrices, involutory matrices, centrosymmetric matrices, mirrorsymetric matrices, circulant matrices, etc.

In [5], the authors have given characterizations of $\{K, s+1\}$-potent matrices by using spectral theory. Later, in [6] that class of matrices was linked to other kind of matrices (as $\{s+1\}$-generalized projectors, $\{K\}$-Hermitian matrices, normal matrices, etc.). In both papers, a theoretical point of view has been used. Hence, it is interesting to know how to construct members of this class in an effective form. One of the main aims of this paper is to develop a numerical method to construct them.

Throughout this paper, K stands for an involutory matrix. We will denote by Ω_k the set of all k^{th} roots of unity with k a positive integer, that is, if we define $\omega_k = e^{2\pi i/k}$ then $\Omega_k = \{\omega_k, \omega_k^2, \dots, \omega_k^k\}$.

The following function will be necessary. Let $\mathbb{N}_s = \{0, 1, 2, \dots, (s+1)^2-2\}$ for $s \geq 1$, and let

$$\varphi : \mathbb{N}_s \to \mathbb{N}_s$$

be the bijective function given by $\varphi(j) = b_j$ where b_j is the smallest nonnegative integer such that $b_j \equiv j(s+1)$ [mod $((s+1)^2 - 1)$] [5].

This paper is organized as follows. In Section 2, we present an algorithm to compute $\{K, s+1\}$-potent matrices. In Section 3, we obtain $\{K, s+1\}$-potent matrices commuting with a given $\{K, s+1\}$-potent matrix. In Section 4, we develop an algorithm to obtain all of the $\{K, s+1\}$-potent linear combinations of $\{K, s+1\}$-potent matrices. Finally, in Section 5, some numerical examples are presented in order to show the numerical performance of the method.

2. Algorithm for computing $\{K, s+1\}$-potent matrices

We analyze two situations: $s = 0$ and $s \geq 1$.

2.1. Case $s \geq 1$

Given an involutory matrix $K \in \mathbb{C}^{n\times n}$ and $s \in \{1, 2, 3, \dots\}$, we want to find a $\{K, s+1\}$-potent matrix $A \in \mathbb{C}^{n\times n}$. Since the cases with $K = \pm I_n$ correspond to the well-known relationship $A^{s+1} = A$, we will assume throughout that $K \neq \pm I_n$.

Since K is involutory, there is a nonsingular matrix $T = \begin{bmatrix} t_1 & \dots & t_n \end{bmatrix}$ such that

$$K = T \begin{bmatrix} -I_r & O \\ O & I_{n-r} \end{bmatrix} T^{-1} \tag{2}$$

where the first r eigenvectors of K are associated with the eigenvalue -1. Without loss of generality, we will assume that $r \leq n - r$. Otherwise, we pick $-K$ instead of K obtaining the same solution. It is well-known [5] that the eigenvalues of A are included in the following set

$$\Lambda = \left\{0, \omega^1_{(s+1)^2-1}, \dots, \omega^{(s+1)^2-2}_{(s+1)^2-1}, 1\right\}$$

and A is diagonalizable, i.e.

$$A = S\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\, S^{-1} \quad \text{with} \quad S = \begin{bmatrix} s_1 & \dots & s_n \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix}.$$

It is easy to see that $y_i^T s_j = \delta_{ij}$ because $S^{-1}S = I_n$ where δ_{ij} indicates the Kronecker delta. Then, denoting by $P_i = s_i y_i^T$ we have

$$P_i P_j = \begin{cases} O & \text{if } i \neq j \\ P_i & \text{if } i = j \end{cases}$$

and by using the fact $SS^{-1} = I_n$ we get $\sum_{i=1}^n P_i = I_n$. So, matrix A can be written as

$$A = \sum_{i=1}^{n} \lambda_i P_i. \tag{3}$$

When all the λ_i's are different, expression (3) provides the spectral decomposition of A. Otherwise, in order to obtain such a decomposition it is sufficient to multiply the corresponding eigenvalue by the sum of all its associated P_i's.

Since $KP_iK = P_{\varphi(i)}$ (by Theorem 2 in [5]), we can choose

$$Ks_i = s_{\varphi(i)} \qquad \text{and} \qquad K^T y_i = y_{\varphi(i)} \tag{4}$$

in order to satisfy the equality $A^{s+1} = KAK$.

ALGORITHM 1

Inputs: $K \in \mathbb{C}^{n\times n}$, $s \in \{1, 2, 3, \dots\}$.

Outputs: A $\{K, s+1\}$-potent matrix $A \in \mathbb{C}^{n\times n}$ and the projectors P_i.

Step 1 Diagonalize K as in (2).

Step 2 If $r > n - r$, replace K with $-K$ and rearrange as in Step 1.

Step 3 For $i = 1, \dots, r$, compute $s_{2i-1} = t_i + t_{r+i}$ and $s_{2i} = -t_i + t_{r+i}$.

Step 4 For $i = 2r + 1, \dots, n$, set $s_i = t_i$.

Step 5 Solve the linear systems $Sy_i = e_i$ for $i = 1, \dots, n$.

Step 6 Compute $P_i = s_i y_i^T$ for $i = 1, \dots, n$.

Step 7 For $i = 1, \dots, r$, compute $Q_i = \omega P_{2i-1} + \omega^{\varphi(1)} P_{2i}$.

Step 8 Compute $A = \sum_{i=1}^{r} Q_i + \sum_{j=2r+1}^{n} P_j$.

End

In order to clarify this process, the most representative cases are presented in Table 1, where $\omega := \omega_{(s+1)^2-1}$. We consider the following different involutory matrices $K_i = TD_iT^{-1}$ for $i = 1, 2, 3, 4, 5$ where $D_1 = \mathrm{diag}(-1, 1)$, $D_2 = \mathrm{diag}(-1, 1, 1)$, $D_3 = \mathrm{diag}(-1, 1, 1, 1)$, $D_4 = \mathrm{diag}(-1, -1, 1, 1)$, $D_5 = \mathrm{diag}(-1, -1, 1, 1, 1)$.

Table 1. The most representative cases.

	Construction of s_i's	Construction of A_i's
D_1	$s_1 = t_1 + t_2$ $s_2 = -t_1 + t_2$	$A = \omega P_1 + \omega^{\varphi(1)} P_2$
D_2	$s_1 = t_1 + t_2$ $s_2 = -t_1 + t_2$ $s_3 = t_3$	$A = \omega P_1 + \omega^{\varphi(1)} P_2 + P_3$
D_3	$s_1 = t_1 + t_2$ $s_2 = -t_1 + t_2$ $s_3 = t_3$ $s_4 = t_4$	$A = \omega P_1 + \omega^{\varphi(1)} P_2 + P_3 + P_4$
D_4	$s_1 = t_1 + t_3$ $s_2 = -t_1 + t_3$ $s_3 = t_2 + t_4$ $s_4 = -t_2 + t_4$	$A = \omega(P_1 + P_3) + \omega^{\varphi(1)}(P_2 + P_4)$
D_5	$s_1 = t_1 + t_3$ $s_2 = -t_1 + t_3$ $s_3 = t_2 + t_4$ $s_4 = -t_2 + t_4$ $s_5 = t_5$	$A = \omega(P_1 + P_3) + \omega^{\varphi(1)}(P_2 + P_4) + P_5$

The only step in the algorithm that needs to be justified is Step 8. In fact,

$$\begin{aligned} A^{s+1} &= \left(\sum_{i=1}^{r} (\omega P_{2i-1} + \omega^{\varphi(1)} P_{2i}) + \sum_{j=2r+1}^{n} P_j \right)^{s+1} \\ &= \sum_{i=1}^{r} (\omega^{\varphi(1)} P_{2i-1} + \omega P_{2i}) + \sum_{j=2r+1}^{n} P_j \end{aligned}$$

and by using expression (4) we get

$$\begin{aligned} KAK &= \sum_{i=1}^{r} (\omega K P_{2i-1} K + \omega^{\varphi(1)} K P_{2i} K) + \sum_{j=2r+1}^{n} K P_j K \\ &= \sum_{i=1}^{r} (\omega P_{2i} + \omega^{\varphi(1)} P_{2i-1}) + \sum_{j=2r+1}^{n} P_j. \end{aligned}$$

Although we have constructed only one $\{K, s+1\}$-potent matrix A, it is clear that this method allows us to construct more of them (e.g., by changing adequately the ω's in $\Omega_{(s+1)^2-1}$).

2.2. Case $s = 0$

This case corresponds to matrices $A \in \mathbb{C}^{n \times n}$ commuting with K. The spectral theorem allows us to state that a diagonalizable matrix $A \in \mathbb{C}^{n \times n}$ is $\{K\}$-centrosymmetric if and only if $KP_i = P_i K$ where P_i are the projectors appearing in that decomposition. We could try a similar algorithm to the previous one. However, by using a block decomposition it is easy to see that a more straightforward method can be developed which gives an immediate result. In fact, if matrix K is diagonalized as in (2), an easy computation provides us the required $\{K\}$-centrosymmetric matrices:

$$A = T \begin{bmatrix} X_A & O \\ O & Y_A \end{bmatrix} T^{-1} \tag{5}$$

where $X_A \in \mathbb{C}^{r \times r}$ and $Y_A \in \mathbb{C}^{(n-r)\times(n-r)}$ are arbitrary matrices.

3. Obtaining $\{K, s+1\}$-potent matrices commuting with a given $\{K, s+1\}$-potent matrix

Let $s \geq 1$. Our next objective is to find a $\{K, s+1\}$-potent matrix $B \in \mathbb{C}^{n \times n}$ such that $AB = BA$ for the $\{K, s+1\}$-potent matrix $A \in \mathbb{C}^{n \times n}$ obtained by means of Algorithm 1. Since B is $\{K, s+1\}$-potent, it must be diagonalizable. Then, in order to satisfy the condition $AB = BA$ both matrices A and B have to be simultaneously diagonalizable [4], that is, $A = S\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,S^{-1}$ and $B = S\,\mathrm{diag}(\mu_1, \ldots, \mu_n)\,S^{-1}$, where the μ_i, which are in Λ, remain to be determined.

Algorithm 2

Inputs: The $\{K, s+1\}$-potent matrix A obtained in Algorithm 1 and the projectors P_i.

Outputs: A $\{K, s+1\}$-potent matrix $B \in \mathbb{C}^{n \times n}$ commuting with A.

Step 1 Set $\mu := \omega^p_{(s+1)^2-1}$, where $p \notin \{1, \varphi(1), \varphi(p)\}$.

Step 2 For $i = 1, \ldots, r$, compute $W_i = \mu P_{2i-1} + \mu^{\frac{\varphi(p)}{p}} P_{2i}$.

Step 3 Compute $B = \sum_{i=1}^{r} W_i + \sum_{i=2r+1}^{n} P_i$.

End

Note that μ is chosen in Λ from among the unused ω's in Algorithm 1. In Step 2 it is clear that $\mu^{\frac{\varphi(p)}{p}} = \omega^{\varphi(p)}$. The remaining construction for $s \geq 1$ follows as in Algorithm 1.

Using Algorithm 2 we have constructed one $\{K, s+1\}$-potent matrix B. It is clear that this method allows us to construct more of them from the same starting matrix A (e.g., by changing adequately the μ's in $\Omega_{(s+1)^2-1}$).

When $s = 0$, and X_A and Y_A are as in (5), then

$$B = T \begin{bmatrix} X_B & O \\ O & Y_B \end{bmatrix} T^{-1}.$$

In this case, it is clear that we obtain all $\{K, 1\}$-potent matrices B that commute with an arbitrarily constructed $\{K, 1\}$-potent matrix A provided that $X_A X_B = X_B X_A$ and $Y_A Y_B = Y_B Y_A$ hold.

When $s \geq 1$, we note that matrix $B = \omega A$ is $\{K, s+1\}$-potent, where ω is a primitive s-root of unity. Similarly, when $s = 0$, $B = \alpha A$ is $\{K, 1\}$-potent for all $\alpha \in \mathbb{C}$. In the next section, in order to obtain non trivial linear combinations of A and B, we will use Algorithm 2.

4. An algorithm for obtaining $\{K, s+1\}$-potent linear combinations

For the matrices A and B obtained by means of Algorithms 1 and 2, we can construct the following linear relationship:

$$C = c_1 A + c_2 B \tag{6}$$

where c_1 and c_2 are nonzero complex numbers to be determined. In this section we find scalars c_1 and c_2 such that C is a $\{K, s+1\}$-potent matrix. The value $s = 0$ does not give any interesting results because all c_1 and c_2 satisfy the equality and so, we will assume that $s \geq 1$.

Since

$$A = S \operatorname{diag}(\lambda_1, \ldots, \lambda_n) S^{-1} \quad \text{and} \quad B = S \operatorname{diag}(\mu_1, \ldots, \mu_n) S^{-1},$$

by using (6), a simple computation yields to solve the following linear system:

$$\begin{bmatrix} \lambda_1 & \mu_1 \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$

where $C = S\,\mathrm{diag}(\gamma_1, \ldots, \gamma_n)\,S^{-1}$ and each γ_i ranges freely over all values in $\{0\} \cup \Omega_{(s+1)^2-1}$.

We now can design an algorithm to compute this class of linear relations.

Algorithm 3

Inputs: The $\{K, s+1\}$-potent matrices A and B obtained in Algorithms 1 and 2, respectively.

Outputs: All values of c_1 and c_2 such that $C = c_1 A + c_2 B$ is a $\{K, s+1\}$-potent matrix.

Step 1 Set $M = \begin{bmatrix} \lambda_1 & \mu_1 \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{bmatrix}$.

Step 2 Choose $\gamma_1, \ldots, \gamma_n \in \{0\} \cup \Omega_{(s+1)^2-1}$ and set $\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$.

Step 3 Solve the linear system $M \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \gamma$.

Step 4 Repeat Steps 2 and 3 for all possible choices of γ_i in $\{0\} \cup \Omega_{(s+1)^2-1}$.

End

5. Numerical examples

Our algorithms can easily be implemented on a computer. We have used the MATLAB R2010b package. In this section we present some numerical examples in order to show the performance of our algorithms and demonstrate their applicability.

5.1. Case $s \geq 1$

While the computational cost of Algorithm 1 is $O(n^3)$, Algorithm 2 has computational cost of only $O(n)$. Note that the computational cost of Algorithm 3 is basically given by Step 3.

Now, we observe that the matrix M has at most rank equals 2. Then, in order to solve the system

$$\begin{bmatrix} \lambda_1 & \mu_1 \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \gamma,$$

we have to choose two linearly independent equations (only one when $\text{rank}(M) = 1$). That is, we only have to solve at most a 2×2 linear system.

Example 1. *For $s = 2$, $n = 4$, and*

$$K = \begin{bmatrix} 1.2989 & -1.2069 & 3.5632 & 3.0460 \\ 1.3793 & -2.7241 & 4.1379 & 4.8276 \\ -0.2759 & 1.3448 & -3.8276 & -3.9655 \\ 0.6437 & -2.1379 & 4.5977 & 5.2529 \end{bmatrix},$$

Algorithm 1 gives

$$A = \begin{bmatrix} -0.3820 - 0.0731i & 1.2435 - 0.0853i & -0.9103 + 0.4877i & -0.9834 + 1.1582i \\ 0.3657 + 0.5202i & 0.6035 - 0.4145i & -1.0241 - 0.3251i & -1.0180 + 0.4064i \\ -0.2845 - 0.1300i & 0.4145 - 0.7803i & 0.0894 + 0.7884i & -0.6421 + 0.9591i \\ 0.3657 + 0.5202i & -0.1036 + 0.2926i & -1.0241 - 0.3251i & -0.3109 - 0.3007i \end{bmatrix}$$

Example 2. *For the same K as in Example 1, Algorithm 2 gives*

$$B = \begin{bmatrix} -0.3901 - 0.3786i & 0.6644 + 0.7763i & 0.2438 - 1.3880i & -0.1280 - 0.6307i \\ 0.2601 + 0.4095i & 0.1463 + 0.0857i & -0.1626 - 1.3810i & -0.5039 - 0.5238i \\ -0.0650 - 0.3754i & 0.4938 - 0.5022i & 0.0406 + 0.1096i & -0.4044 + 0.1433i \\ 0.2601 + 0.4095i & -0.5608 + 0.7928i & -0.1626 - 1.3810i & 0.2032 - 1.2310i \end{bmatrix}$$

Example 3. *For the matrices A and B obtained in Examples 1 and 2, Algorithm 3 gives $c_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $c_2 = 0$.*

We now comment on the computational time in terms of n and s. For that we have used a Intel Core 2 Duo 2GHz processor. For $n = 10, 30, 50, \ldots, 500$, Figure 1 shows the computational time required for $s = 1, 5, 100, 200$. We can observe that, as expected, the computational time grows exponentially with s and n. Moreover, computation time increases more rapidly with n than with s.

5.2. Case $s = 0$

In this case, the only interesting examples correspond to those constructed in Subsection 2.2. The reason is that for every $\{K, 1\}$-potent matrix commuting with A, all possible linear combinations are $\{K, 1\}$-potent matrices.

Example 4. *If we consider the involutory matrix*

$$K = \frac{1}{9}\begin{bmatrix} 7 & -4 & 4 \\ -4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$

all possible $\{K, 1\}$-potent matrices are

$$A = \frac{1}{9}\begin{bmatrix} 4a-4c-4b+4d+e & -4a+4c-2b+2d+2e & -2a+2c-4b+4d-2e \\ -4a-2c+4b+2d+2e & 4a+2c+2b+d+4e & 2a+c+4b+2d-4e \\ -2a-4c+2b+4d-2e & 2a+4c+b+2d-4e & a+2c+2b+4d+4e \end{bmatrix}$$

for a, b, c, d, e arbitrary complex numbers.

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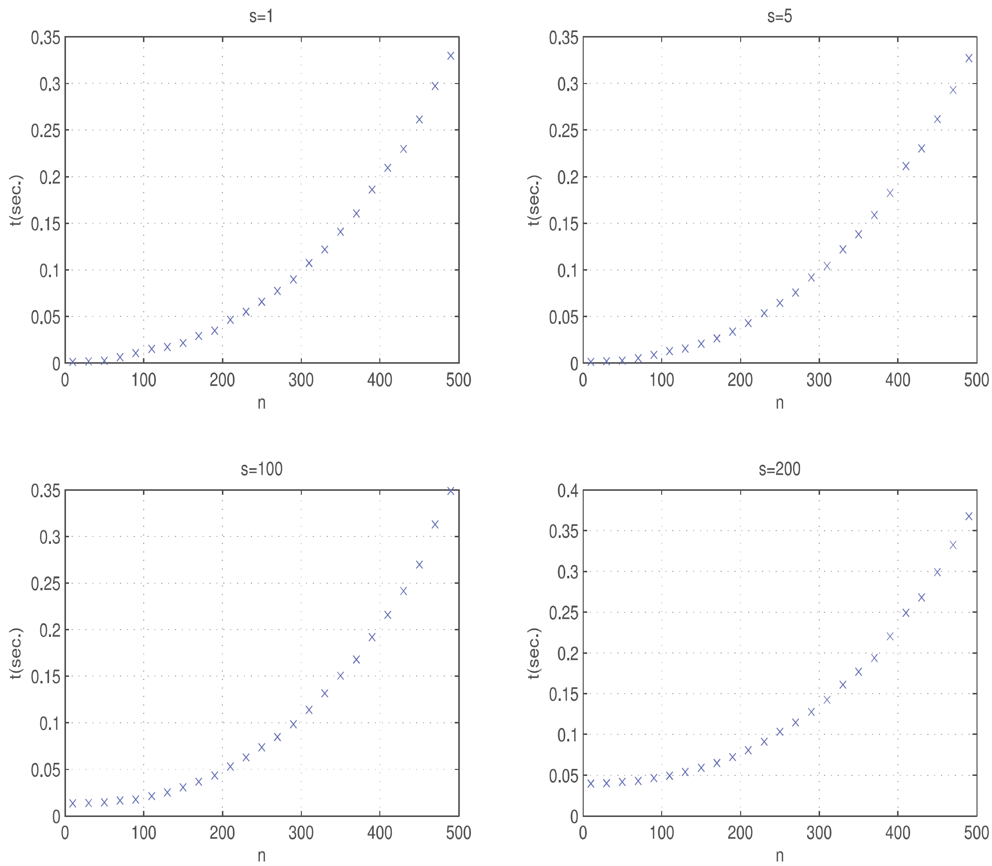

Figure 1: Time for obtaining A with n variable and $s = 1, 5, 100, 200$