On the Bishop-Phelps-Bollobás type theorems

Sheldon M. Gil Dantas

Supervised by

Domingo García, Manuel Maestre and Miguel Martín

Facultat de Ciències Matemàtiques
Universitat de València

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I declare that this dissertation titled *On the Bishop-Phelps-Bollobás type theorems* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a degree of Doctor in Mathematics at Valencia University.
- Where I have consulted the published works of others, this is always clearly attributed.
- Where I have quoted from the works of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.
- I have acknowledged all main sources of help.

Valencia, 26 March 2017

Sheldon M. Gil Dantas

We declare that this dissertation presented by **Sheldon M. Gil Dantas** titled *On the Bishop-Phelps-Bollobás type theorems* has been done under our supervision at Valencia University. We also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of Doctor in Mathematics.

Valencia, 26 March 2017

Domingo García **Manuel Maestre** **Miguel Martín**

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Resum

Aquesta tesis està dedicada a estudiar la propietat de Bishop-Phelps-Bollobás en diferents contextos.

En el Capítol 1, introduïm la notació utilitzada al llarg de tota la tesis i donem un resum històric d'aquesta propietat, explicant, entre altres, els teoremes de Bishop-Phelps i Bishop-Phelps-Bollobás. El primer teorema de Bishop-Phelps [14] ens diu el següent:

Siga X un espai de Banach. Donat un número real positiu $\varepsilon > 0$ i un funcional lineal i continu $x^* \in X^*$, existeix un nou funcional $x_0^* \in X^*$ i un nou punt $x_0 \in S_X$ tals que satisfan les següents condicions:

$$|x_0^*(x_0)| = \|x_0^*\| \quad \text{i} \quad \|x_0^* - x^*\| < \varepsilon.$$

Es a dir, el teorema prova que el conjunt de tots els funcionals que alcança la norma és dens (en norma) en el dual d'un espai de Banach X.

És natural doncs preguntar-se si aquest resultat és també cert per a operadors lineals i continus. En 1963, J. Lindenstrauss [52] va donar el primer contraexemple que mostra que, en general, el resultat és fals. D'altra banda, també va presentar condicions que han de satisfer els espais de Banach per a obtindre resultats positius. Després d'això, altres autors han estudiat diverses hipòtesis que es poden exigir als espais de Banach per a saber si el conjunt dels operadors lineals i continus que alcancen la norma és dens (en norma) dins del conjunt de tots els operadors lineals i continus.

Set anys més tard, en 1970, Bollobás [15] va provar una versió més forta del teorema de Bishop-Phelps, enunciada a continuació.

Siga X un espai de Banach. Donat $\varepsilon > 0$, si $x \in B_X$ i $x^* \in B_{X^*}$ complixen

$$|1 - x^*(x)| < \frac{\varepsilon^2}{4},$$

aleshores existeixen elements $x_0 \in S_X$ i $x_0^* \in S_{X^*}$ tals que

$$|x_0^*(x_0)| = 1, \quad \|x_0 - x\| < \varepsilon \quad i \quad \|x_0^* - x^*\| < \varepsilon.$$

Amb aquest resultat Bollobás va mostrar que a més de l'aproximació per als funcionals, també és possible aproximar el punt inicial on el primer funcional x^* quasi alcança la norma, per un punt en el que el nou funcional, proper a x^*, alcança la norma.

En els últims nou anys, s'han escrit una gran quantitat d'articles en els que es presenten diferents teoremes del tipus Bishop-Phelps-Bollobás per a operadors lineals i continus. L'interès per aquests teoremes va començar amb l'article de M. Acosta, R. Aron, D. García i M. Maestre [2]. En aquest article, els autors van definir per primera vegada la propietat de Bishop-Phelps-Bollobás (BPBp). La propietat és la següent.

Diem que un parell d'espais de Banach $(X; Y)$ satisfà la BPBp si donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que per a tot $T : X \longrightarrow Y$ operador lineal de norma u i per a tot x_0 element en l'esfera unitat de X que complixen

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

existeixen un nou operador lineal $S : X \longrightarrow Y$ de norma u i un nou element x_1 en l'esfera unitat de X tals que complixen les següents condi-

cions:

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad i \quad \|S - T\| < \varepsilon.$$

És important destacar que del teorema de Bishop-Phelps-Bollobás es dedueix que el parell $(X;\mathbb{K})$ satisfà la BPBp per a tots els espais de Banach X. D'altra banda, cal assenyalar que si el parell $(X;Y)$ satisfà la BPBp, aleshores el conjunt de tots els operadors lineals i continus que alcancen la norma és dens (en norma) en el conjunt de tots els operadors lineals i continus de X en Y.

Per a finalitzar el Capítol 1, es presenten alguns dels més recents resultats relacionats amb els teoremes de Bishop-Phelps-Bollobás amb l'objectiu de posar aquesta tesis en un context actual.

A continuació, en el Capítol 2 estudiem algunes propietats semblants a la propietat de Bishop-Phelps-Bollobás. Comencem amb la propietat punt de Bishop-Phelps-Bollobás (BPBpp).

Diem que un parell d'espais de Banach $(X;Y)$ satisfà la BPBpp si donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que per a tot $T : X \longrightarrow Y$ operador lineal de norma u i per a tot x_0 element de l'esfera unitat de X tals que

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

existeix un nou operador lineal $S : X \longrightarrow Y$ de norma u que satisfà

$$\|S(x_0)\| = 1 \quad i \quad \|S - T\| < \varepsilon.$$

Destaquem que aquesta nova propietat és més forta que la BPBp. Al llarg d'aquest capítol, primer estudiem la BPBpp per a operadors lineals y després per a aplicacions bilineals, obtenint en el dos casos resultats positius d'existència de parells d'espais de Banach satisfent-les.

També és important mencionar que en la BPBpp, el punt inicial x_0, i.e., el punto en el que l'operador T quasi alcança la norma, és el mateix punt en el que l'operador proper a T també alcança la norma. D'altra banda, utilitzant la BPBpp hem provat una caracterització dels espais uniformement suaus.

Un espai de Banach X és uniformement suau si i només si el parell $(X;\mathbb{K})$ satisfà la BPBpp.

En aquest capítol també provem que si un parell $(X;Y)$ satisfà la BPBpp per a algun espai de Banach Y, l'espai de Banach X ha de ser uniformement suau. Per tant, sempre necessitem suposar que el domini X de l'operador és uniformement suau per a així poder aconseguir resultats positius sobre aquesta propietat.

A més a més, si Y té la propietat β o és un àlgebra uniforme, aleshores el parell $(X;Y)$ té la BPBpp per a tots els espais uniformement suaus X.

D'altra banda, també hem provat resultats sobre la BPBpp treballant en espais de Hilbert. Com els espais de Hilbert tenen normes transitives, mostrem que si H es un espai de Hilbert, aleshores el parell $(H;Y)$ satisfà la BPBpp per a tots els espais de Banach Y.

Acabem aquesta primera part del capítol mostrant que existeixen espais reals X uniformement suaus de dimensió 2 tals que el parell $(X;Y)$ no satisfà la propietat punt de Bishop-Phelps-Bollobás per a certs espais de Banach Y. Finalment, estenem els resultats obtinguts al considerar la propietat per a aplicacions bilineals en situacions semblants.

El contingut d'aquesta secció va ser publicat en el següent article:

S. Dantas, S. K. Kim and H. J. Lee, The Bishop-Phelps-Bollobás point property, *J. Math. Anal. Appl.* **444** (2016), no. 2, 1739–1751.

Seguint en el Capítol 2, estudiem una propietat dual a la BPBpp en dues situacions diferents. Primer, considerem que el número real positiu η que apareix en la definició (veure Definition 2.2.2) depén no solament d'$\varepsilon > 0$, però també d'un operador lineal prèviament fixat i de norma u. Anomenem aquesta propietat, propietat 1.

Diem que un parell d'espais de Banach $(X;Y)$ *satisfà la propietat* 1 *si donat* $\varepsilon > 0$ *i donat un operador lineal* T *de norma u, existeix* $\eta(\varepsilon, T) > 0$ *tal que per a tot element* x_0 *de l'esfera unitat de* X *amb*

$$\|T(x_0)\| > 1 - \eta(\varepsilon, T),$$

existeix un altre element x_1*, també en l'esfera unitat de* X*, que satisfà*

$$\|T(x_1)\| = 1 \quad i \quad \|x_1 - x_0\| < \varepsilon.$$

De la definició de la propietat 1 es dedueix que tot operador $T : X \longrightarrow Y$ alcança la norma si el parell $(X;Y)$ satisfà aquesta propietat. Per tant, i utilitzant el Teorema de James (veure Remark 2.2.3.(a)), hem de prendre com condició necessària que el domini X siga reflexiu per a així obtindre resultats positius.

Per a espais de dimensió finita hem provat el següent resultat.

El parell $(X;Y)$ *satisfà la propietat* 1 *per a tot espai de dimensió finita* X *i per a qualsevol espai de Banach* Y*.*

Un altre resultat provat s'enuncia a continuació.

Si X *es un espai de Banach reflexiu i localment uniformement convex, en particular si* X *és uniformement convex, aleshores el parell* $(X;Y)$ *satisfà la propietat* 1 *per a operadors compactes.*

Com conseqüència, el parell $(\ell_2;\ell_1)$ satisfà la propietat 1 ja que tot operador entre aquests dos espais és compacte i ℓ_2 és uniformement convex. En realitat, en aquest capítol, donem una caracterització de aquesta propietat per a tots els parells $(\ell_p;\ell_q)$.

(i) *El parell* $(\ell_p; \ell_q)$ *satisfà la propietat* 1 *per a tots* p, q *tals que* $1 \leqslant q < p < \infty$.

(ii) *El parell* $(\ell_p; \ell_q)$ *no satisfà la propietat* 1 *per a tots* p, q *tals que* $1 \leqslant p \leqslant q < \infty$.

A continuació, considerem una versió uniforme de la propietat 1 i l'anomenem propietat 2. Aquesta propietat és semblant a la propietat 1 però considerant η depenent només d'$\varepsilon > 0$.

Utilitzant el Teorema de Kim- Lee [46, Theorem 2.1] tenim la següent caracterització dels espais uniformement convexos:

Un espai de Banach X *és uniformement convex si i només si el parell* $(X; \mathbb{K})$ *té la propietat* 2.

Aquesta és una versió dual de la ja mencionada propietat punt de Bishop-Phelps-Bollobás per al parell $(X; \mathbb{K})$, que caracteritza els espais uniformement suaus. Al contrari del que hem provat amb la BPBpp, en el cas de la propietat 2 resulta més complicat aconseguir resultats positius. En efecte, entre els resultats provats destaquem els següents teoremes negatius.

Els parells $(\ell_2^2(\mathbb{R}); \ell_q^2(\mathbb{R}))$ *no satisfan la propietat* 2 *per a* $1 \leqslant q \leqslant \infty$.

I també

Donat Y *espai de Banach de dimensió* 2*, el parell* $(Y; Y)$ *no satisfà la propietat* 2.

El contingut d'aquesta secció va ser publicat en el següent article:

S. Dantas, Some kind of Bishop-Phelps-Bollobás property, *Math. Nachr.*, 2016, doi:10.1002/mana.201500487

En la Secció 2.3 treballem amb la propietat punt de Bishop-Phelps-Bollobás per a radi numèric (BPBpp-nu). Aquesta propietat va ser motivada per la recent propietat de Bishop-Phelps-Bollobás per a radi numèric (veure, per exemple, [32, 37, 47]). La seua definició és pareguda amb la BPBp però en lloc de la norma de l'operador considerem ara el radi numèric.

Diem que un espai de Banach X satisfà la propietat de Bishop-Phelps-Bollobás per a radi numèric (BPBp-nu) si donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que per a tot operador lineal i continu $T : X \longrightarrow X$ de radi numèric u i per a tot parell (x, x^*) que satisfà $\|x^*\| = \|x\| = x^*(x) = 1$ i també

$$|x^*(T(x))| > 1 - \eta(\varepsilon),$$

existeixen un nou operador lineal i continu S amb radi numèric u i un nou element (y, y^*) amb $\|y^*\| = \|y\| = y^*(y) = 1$ tals que

$$|y^*(S(y))| = 1, \quad \|y^* - x^*\| < \varepsilon, \quad \|y - x\| < \varepsilon \quad i \quad \|S - T\| < \varepsilon.$$

En el nostre cas, igual que fem amb la BPBpp, considerem el mateix punt inicial (x, x^*) per al dos operadors.

El primer resultat provat en aquesta secció ens diu que tot espai de Hilbert complex satisfà la BPBpp-nu. És per això que hem provat que alguns tipus d'operadors (auto-adjunts, anti-simètrics, unitaris i normals) definits en un espai de Hilbert complex complixen els propietats BPBpp i BPBpp-nu. Resumim aquests resultats a continuació.

Si H és un espai de Hilbert complex, aleshores

(i) *H satisfà la BPBpp-nu per a operadors auto-adjunts, anti-simètrics i unitaris;*
(ii) *$(H; H)$ satisfà la BPBpp per a operadors auto-adjunts, anti-simètrics, unitaris i normals.*

Això significa que quan es comença, per exemple, amb un operador auto-adjunt, tenim que H (respectivament el parell $(H;H)$) complix la BPBpp-nu (respectivament la BPBpp) amb S també un operador auto-adjunt.

Finalment, en l'última secció d'aquest capítol, generalitzem alguns dels resultats de [11] considerant normes absolutes. Hem provat que:

Si $|\cdot|_a$ *és una norma absoluta i el parell* $(X;Y_1 \oplus_a Y_2)$ *satisfà la BPBp, aleshores, els parells* $(X;Y_1)$ *i* $(X;Y_2)$ *també satisfan la BPBp.*

A més a més, presentem exemples de normes absolutes $|\cdot|_a$ tals que si el parell $(X_1 \oplus_a X_2;Y)$ satisfà la BPBp, aleshores els parells $(X_1;Y)$ i $(X_2;Y)$ també la satisfan.

En el Capítol 3 considerem la propietat de Bishop-Phelps-Bollobás per a operadors compactes.

Diem que un parell d'espais de Banach $(X;Y)$ *satisfà la BPBp per a operadors compactes si dotat* $\varepsilon > 0$*, existeix* $\eta(\varepsilon) > 0$ *tal que per a tot* $T : X \longrightarrow Y$ *operador lineal i compacte de norma u i per a tot* x_0 *element de l'esfera unitat de* X *tals que*

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

existeixen un nou operador lineal i compacte $S : X \longrightarrow Y$ *de norma u i un nou punt* x_1 *en l'esfera unitat de* X *que satisfan*

$$\|S(x_0)\| = 1 \quad \|x_1 - x_0\| < \varepsilon \quad i \quad \|S - T\| < \varepsilon.$$

La definició d'aquesta propietat és similar a la BPBp, però ara únicament considerem operadors compactes. Al principi d'aquest capítol donem una extensa llista d'exemples de parells d'espais de Banach $(X;Y)$ que satisfan la BPBp per a operadors compactes. Alguns d'ells ja són coneguts d'altres articles. Per exemple, el parell $(X;Y)$ té la BPBp per a operadors compactes si complixen alguna de les següents condicions:

- *X és uniformement convex i Y és arbitrari;*
- *X és arbitrari i Y és un àlgebra uniforme;*
- *$X = C_0(L)$, amb L un espai topològic de Hausdorff localment compacte, i Y és uniformement convex;*
- *X és arbitrari i Y és isomètricament isomorf a un espai L_1;*
- *$X = L_1(\mu)$, amb μ una mesura arbitrària, i Y té l'AHSP.*

D'altra banda, realitzant senzilles modificacions en les demostracions de certs resultats provats per la BPBp, obtenim més resultats positius per la versió d'aquesta propietat per a operadors compactes. La idea és agafar inicialment un operador compacte T i definir una pertorbació compacta que satisfaça les condicions de Bishop-Phelps-Bollobás (veure Definition 3.1.1). Per exemple, en la demostració de [2, Theorem 2.2], si comencem amb un operador compacte T en $\mathcal{K}(X,Y)$ de norma 1 i tal que $\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}$, llavors l'operador donat per

$$S(x) := T(x) + [(1+\eta)z_0^*(x) - T^*(y_{\alpha_0}^*)(x)]y_{\alpha_0},$$

satisfà les condicions de la BPBp i és també compacte.

Tot aquest tercer capítol està motivat per l'estudi dels operadors compactes que alcancen la norma. M. Martín va provar en [53] que existeixen espais de Banach X i Y i operadors compactes de X en Y que no poden ser aproximats per els que alcancen la norma. Va presentar també condicions que garantixen la densitat dels operadors que alcancen la norma en el conjunt dels operadors compactes.

En la primera secció del capítol, presentem algunes tècniques per a produir parells d'espais de Banach que satisfan la BPBp per a operadors compactes. Aquestes tècniques estan basades en dos resultats anteriors de J. Johnson i J. Wolfe [43] sobre operadors compactes que alcancen la norma. La idea principal de la secció és donar alguns resultats tècnics a través dels quals ens garatixen passar la propietat d'espais de successions per a espais de funcions.

Per a aconseguir estos resultats sobre els espais domini de l'operador, provem el següent lema:

Siguen X i Y dos espais de Banach. Suposem que existeix una funció $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ tal que donats $\delta \in \mathbb{R}^+$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ i $x_0 \in S_X$, podem trobar dos operadors de norma u $P \in \mathcal{L}(X;X)$ i $i \in \mathcal{L}(P(X);X)$ que compleixen les següents condicions:

(1) $\|P^* x_j^* - x_j^*\| < \delta$ *per a* $j = 1, \ldots, n$,

(2) $\|i(P(x_0)) - x_0\| < \delta$,

(3) $P \circ i = \mathrm{Id}_{P(X)}$,

(4) *el parell $(P(X);Y)$ satisfà la BPBp per a operadors compactes amb la funció η.*

Aleshores, el parell $(X;Y)$ satisfà la BPBp per a operadors compactes.

D'altra banda, el lema que necessitem per a obtindre els resultats sobre els espais imatge de l'operador és el següent:

Siguen X i Y dos espais de Banach. Suposem que

(1) *existeix una red de projeccions de norma u $\{Q_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{L}(X;Y)$ tal que per a tot $y \in Y$, la successió $\{Q_\lambda y\}$ convergeix a y en norma, i a més*

(2) *existeix una funció $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ tal que els parells $(X; Q_\lambda(Y))$ amb $\lambda \in \Lambda$ satisfan la BPBp per a operadors compactes amb la funció η.*

Aleshores, el parell $(X;Y)$ satisfà la BPBp per a operadors compactes.

Utilitzant aquests dos resultats tècnics, estem preparats per a treballar amb casos particulars d'espais de Banach en la tercera secció d'aquest capítol.

Comencem amb els espais domini de l'operador, sense perdre de vista que l'objectiu és passar la propietat d'espais de successions a espais de funcions, tal i com ja hem explicat.

Si $(c_0;Y)$ satisfà la BPBp per a operadors compactes, aleshores el parell $(C_0(L);Y)$ també satisfà la propietat per a tot L espai topològic de Hausdorff localment compacte.

Per a provar aquest resultat, necessitem prèviament una caracterització dels parells $(c_0;Y)$ que satisfan la BPBp per a operadors compactes. S'enuncia a continuació.

Siguen X i Y dos espais de Banach. Les següents afirmacions són equivalents:

(i) *el parell $(c_0(X);Y)$ té la BPBp per a operadors compactes;*

(ii) *existeix una funció $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ tal que els parells $(\ell_\infty^m(X);Y)$, amb $m \in \mathbb{N}$, tenen la BPBp per a operadors compactes amb la funció η.*

D'altra banda, quan $\mathcal{K}(X;Y) = \mathcal{L}(X;Y)$ (en particular, si algun dels espais X o Y és de dimensió finita), el resultat anterior és cert quan $(c_0(X);Y)$ o $(\ell_\infty(X);Y)$ té la BPBp. Com conseqüència, obtenim el següent resultat.

Siga Y un espai de Banach. Si el parell $(c_0;Y)$ satisfà la BPBp, aleshores el mateix parell també satisfà la BPBp per a operadors compactes.

De fet, encara que aquest teorema siga cert, no està totalment clara la relació que hi ha entre la BPBp per a operadors compactes i la BPBp. És important dir que *no és veritat* que la BPBp per a operadors compactes implique la BPBp, ja que el parell $(L_1[0,1], C[0,1])$ té la BPBp per a operadors compactes però no pot tenir la BPBp per un contraexemple donat per W. Schachermayer per a operadors que alcancen la norma entre els espais $L_1[0,1]$ i $C[0,1]$ [59]. Fins ara no hi ha resultats, ni positius ni negatius, que proven que la BPBp implique la BPBp per a operadors compactes.

Seguint en el context de les aplicacions per als espais domini, hem provat el següent resultat que ens permet passar la propietat d'espais ℓ_1 a espais L_1.

Siga μ una mesura positiva i siga X un espai de Banach tal que X^* té la propietat de Radon-Nikodým. Si Y és un espai de Banach i el parell $(\ell_1(X);Y)$ té la BPBp per a operadors compactes, aleshores el parell $(L_1(\mu,X);Y)$ també la té.

El següent lema és necessari per a provar el resultat anterior.

Siguen X i Y dos espais de Banach. Les següents afirmacions són equivalents:

(i) *per a tot $\varepsilon > 0$ existeix $0 < \xi(\varepsilon) < \varepsilon$ tal que, donats les successions $(T_k) \subset B_{\mathcal{K}(X;Y)}$ i $(x_k) \subset B_X$, i donada una sèrie convexa $\sum_{k=1}^\infty \alpha_k$ que satisfan*

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \xi(\varepsilon),$$

existeixen un subconjunt finit $A \subset \mathbb{N}$, un element $y^* \in S_{Y^*}$ i dos successions $(S_k) \subset S_{\mathcal{K}(X;Y)}$ i $(z_k) \subset S_X$ que compleixen

(a) $\sum_{k\in A} \alpha_k > 1 - \varepsilon$,

(b) $\|z_k - x_k\| < \varepsilon$ *y* $\|S_k - T_k\| < \varepsilon$ *per a tots* $k \in A$,

(c) *$y^*(S_k z_k) = 1$ per a tots $k \in A$.*

(En aquest cas, diem que el parell $(X;Y)$ satisfà l'AHSP generalitzada per a operadors compactes.);

(ii) *el parell $(\ell_1(X);Y)$ té la BPBp per a operadors compactes;*

(iii) *existeix una funció $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ tal que els parells $(\ell_1^m(X);Y)$, amb $m \in \mathbb{N}$, satisfan la BPBp per a operadors compactes amb la funció η.*

En particular, tenim la següent caracterització dels parells $(\ell_1;Y)$ que satisfan la BPBp per a operadors compactes.

Siga Y un espai de Banach. Les següents afirmacions són equivalents:

(i) *el parell $(\ell_1;Y)$ té la BPBp per a operadors compactes;*

(ii) *l'espai de Banach Y satisfà l'AHSP;*

(iii) *el parell $(\ell_1;Y)$ satisfà la BPBp;*

(iv) *el parell $(L_1(\mu);Y)$ té la BPBp per a operadors compactes per a tota mesura positiva μ;*

(v) *existeix una mesura positiva μ tal que $L_1(\mu)$ és de dimensió infinita i el parell $(L_1(\mu);Y)$ té la BPBp per a operadors compactes.*

D'altra banda, utilitzant el lema tècnic mencionat anteriorment obtenim els següents resultats per als espais imatge de l'operador compacte.

(a) *Per $1 \leqslant p < \infty$, si el parell $(X;\ell_p(Y))$ té la BPBp per a operadors compactes, aleshores el parell $(X;L_p(\mu,Y))$ també la té per a tota mesura positiva μ tal que $L_1(\mu)$ és de dimensió infinita.*

(b) *Si el parell $(X;Y)$ satisfà la BPBp per a operadors compactes, aleshores el par $(X;L_\infty(\mu,Y))$ també la satisfà per a tota mesura positiva σ-finita μ.*

(c) *Si el parell $(X;Y)$ té la BPBp per a operadors compactes, aleshores $(X;C(K,Y))$ també la té per a tot espai topològic de Hausdorff compacte K.*

Per a provar aquest resultat, necessitem del següent lema que caracteritza els parells $(X;Y)$ que satisfan la BPBp per a operadors compactes en termes de sumes directes.

Siguen X i Y dos espais de Banach i siga $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ una funció. Les següents afirmacions són equivalents:

(i) *el parell $(X;Y)$ té la BPBp per a operadors compactes amb la funció η,*

(ii) *els parells $(X;\ell_\infty^m(Y))$, amb $m \in \mathbb{N}$, satisfan la BPBp per a operadors compactes amb la funció η,*

(iii) *el parell $(X;c_0(Y))$ té la BPBp per a operadors compactes amb la funció η,*

(iv) *el parell $(X;\ell_\infty(Y))$ té la BPBp per a operadors compactes amb la funció η.*

Per a acabar el capítol i utilitzant els resultats prèviament enunciats es prova el següent teorema:

Siga K un espai topològic de Hausdorff compacte i siguen X i Y dos espais de Banach. Si μ és una mesura positiva i ν és una mesura positiva σ-finita, aleshores les següents afirmacions són certes:

(a) *Si Y té la propietat β, aleshores els parells $(X; L_\infty(\nu, Y))$ i $(X; C(K, Y))$ tenen la BPBp per a operadors compactes.*

(b) *Si Y té l'AHSP, aleshores $L_\infty(\nu, Y)$ i $C(K, Y)$ també la tenen.*

(c) *Per $1 \leqslant p < \infty$, si $\ell_p(Y)$ té l'AHSP i $L_1(\mu)$ és de dimensió infinita, aleshores $L_p(\mu, Y)$ té l'AHSP.*

El contingut d'aquest capítol va ser publicat en el següent article:

S. Dantas, D. García, M. Maestre and M. Martín, The Bishop-Phelps-Bollobás property for compact operators, *Canadian J. Math.*, 2016, http://dx.doi.org/10.4153/CJM-2016-036-6

Finalment, en el Capítol 4 estudiem diferents propietats de tipus Bishop-Phelps-Bollobás adaptades al cas d'aplicacions multilineals. Comencem enunciant la BPBp per a aplicacions multilineals.

Siguen $X_1, \ldots X_N$ i Y espais de Banach. Diem que $(X_1, \ldots, X_N; Y)$ satisfà la propietat de Bishop-Phelps-Bollobás per a aplicacions multilineals si donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que per a tota A aplicació N-lineal de $X_1 \times \ldots X_N$ en Y amb $\|A\| = 1$ i per a tot $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ que satisfan

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

existeixen una nova aplicació N-lineal $B : X_1 \times \ldots \times X_N \longrightarrow Y$ amb $\|B\| = 1$ i un nou element $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ tals que

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \quad i \quad \|B - A\| < \varepsilon.$$

Destaquem el grau de l'aplicació multilineal indicant que $(X_1, \ldots, X_N; Y)$ té la BPBp per a aplicacions N-lineals en lloc de dir que té la propietat per a aplicacions multilineals. Igual que hem fet per a BPBp per a aplicacions multilineals, també podem definir la BPBp per a aplicacions multilineals simètriques exigint que les aplicacions A i B siguen elements del conjunt de totes les aplicacions N-lineals simètriques, denotat per $\mathcal{L}_s(^N X; Y)$. En aquest cas, diem que $(^N X; Y)$ té la BPBp per aplicacions multilineals simètriques. Un cas particular apareix quan $Y = \mathbb{K}$ on denotem la BPBp per a $(X_1, \ldots, X_N; \mathbb{K})$ solament per $(X_1, \ldots, X_N)$ i diem que $(X_1, \ldots, X_N)$ té la BPBp per a formes N-lineals. Anàlogament, definim la BPBp per a polinomis homogenis.

Diem que el parell d'espais de Banach $(X; Y)$ satisfà la propietat de Bishop-Phelps-Bollobás per a polinomis homogenis si donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que per a tot P polinomi N-homogeni amb $\|P\| = 1$ i per a tot $x_0 \in S_X$ que satisfan

$$\|P(x_0)\| > 1 - \eta(\varepsilon),$$

existeixen un nou polinomi N-homogeni Q amb $\|Q\| = 1$ i un element $x_1 \in S_X$ tals que

$$\|Q(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad i \quad \|Q - P\| < \varepsilon.$$

En aquest capítol també estenem alguns resultats coneguts sobre aplicacions multilineals que alcancen la norma per la BPBp com, per exemple, la següent observació que deriva del cas lineal:

Siguen $X, X_1, \ldots, X_N$ i Y espais de Banach de dimensió finita. Aleshores les següents afirmacions són certes:

- **(i)** *$(X_1, \ldots, X_N; Y)$ té la BPBp per a aplicacions multilineals;*
- **(ii)** *$(^N X; Y)$ té la BPBp per a aplicacions multilineals simètriques;*
- **(iii)** *$(X; Y)$ té la BPBp per a polinomis N-homogenis.*

Un resultat que mereix ser destacat és un teorema d'estabilitat que permet passar la BPBp per a aplicacions multilineals de grau $N+1$ per a aplicacions multilineals de grau N.

Si $(X_1, \ldots, X_N,\, X_{N+1}; Y)$ *té la BPBp per a aplicacions* $(N+1)$*-lineals, aleshores* $(X_1, \ldots,\, X_N; Y)$ *té la BPBp per a aplicacions* N*-lineals.*

Un altre teorema que relaciona la BPBp per a aplicacions N-lineals i $(N-1)$-lineals és el següent:

Si $N \geqslant 2$ *i si* $X_1, \ldots, X_N$ *són espais de Banach amb* X_N *un espai uniformement convex, aleshores* $(X_1, \ldots, X_N)$ *té la BPBp per a aplicacions* N*-lineals si i només si* $(X_1, \ldots, X_{N-1}; X_N^*)$ *té la BPBp per a aplicacions* $(N-1)$*-lineals.*

Així mateix, també mostrem que és possible passar del cas vectorial al cas escalar en la BPBp per a aplicacions multilineals. En particular, provem la següent caracterització:

Si Y *té la propietat* β*, aleshores* $(X_1, \ldots, X_N)$ *té la BPBp per a formes* N*-lineals si i només si* $(X_1, \ldots, X_N; Y)$ *té la BPBp per a aplicacions* N*-lineals.*

D'altra banda, en aquest capítol també estudiem la BPBp per a aplicacions multilineals compactes seguint la mateixa línia del capítol anterior.

Siga Y *un espai predual de* L_1*. Suposem que* $(X_1, \ldots, X_N)$ *té la BPBp per a formes multilineals. Aleshores* $(X_1, \ldots, X_N; Y)$ *té la BPBp per a aplicacions multilineals compactes.*

Però com en [3], hem obtingut un resultat més fort:

Donat $\varepsilon \in (0,1)$*, existeix* $\eta(\varepsilon) > 0$ *tal que per a tota* A *aplicació multilineal compacta amb* $\|A\| = 1$ *i per a tot* $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$

que satisfan

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

existeixen una nova aplicació multilineal compacta B *amb* $\|B\| = 1$ *i* $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ *tals que* $\dim(B(X_1 \times \ldots \times X_N)) < \infty$,

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \;\; i \;\; \|B - A\| < \varepsilon.$$

A continuació presentem alguns exemples d'espais que compleixen la BPBp per a aplicacions multilineals compactes.

Siga Y *un espai predual de* L_1*. Si* $(X, Z; Y)$ *compleix alguna de les següents condicions, aleshores té la BPBp per a aplicacions multilineals compactes.*

(a) *Si* $X = C_0(L)$ *i* $Z = C_0(K)$ *amb* L *i* K *espais topològics de Hausdorff localment compactes.*

(b) *Si* $X = L_1(\mu)$ *i* $Z = c_0$.

(c) *Si* X *i* Z *són espais de Banach uniformement convexos.*

Seguidament, en la tercera secció d'aquest capítol caracteritzem els parells $(\ell_1(X), Y)$ que satisfan la BPBp per a formes bilineals. El resultat és el següent:

El parell $(\ell_1(X), Y)$ *té la BPBp per a formes bilineals si i només si per a tot* $\varepsilon > 0$*, existeix* $0 < \eta(\varepsilon) < \varepsilon$ *tal que, donats successions* $(T_k)_k \subset \mathcal{L}(X; Y)$ *amb* $\|T_k\| = 1$ *per a tot* k *i* $(x_k)_k \subset S_X$ *i donada una sèrie convexa* $\sum_{k=1}^{\infty} \alpha_k$ *tal que*

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k(x_k) \right\| > 1 - \eta(\varepsilon),$$

existeixen un subconjunt $A \in \mathbb{N}$*, un element* $y^* \in S_{Y^*}$ *i dos successions* $(S_k)_k \subset \mathcal{L}(X; Y)$ *amb* $\|S_k\| = 1$ *per a tot* k *i* $(z_k)_k \subset S_X$ *satisfent*

(1) $\sum_{k\in A} \alpha_k > 1 - \varepsilon$,

(2) $\|z_k - x_k\| < \varepsilon$ *i* $\|S_k - T_k\| < \varepsilon$ *per a tot* $k \in A$, *i*

(3) $y^*(S_k(x_k)) = 1$ *per a tot* $k \in A$.

També hem provat que, si H és un espai de Hilbert, aleshores el parell (X, H) té la BPBp per a formes bilineals si i només si el parell (X, H) té l'AHSP generalitzada.

Un altre camp de treball d'aquest capítol és l'estudi del radi numèric en el conjunt de les aplicacions multilineals definides en $L_1(\mu)$ amb μ una mesura arbitrària. Hem provat que per a tota aplicació $A \in \mathcal{L}(^N L_1(\mu); L_1(\mu))$, el seu radi numèric i la seua norma coincideixen.

Per a acabar el capítol, estudiem la propietat de Bishop-Phelps-Bollobás per a radi numèric en el context de les aplicacions multilineals. Mostrem que si X té dimensió finita, aleshores l'espai X satisfà aquesta propietat. D'altra banda, $L_1(\mu)$ no satisfà la propietat encara que aquest espai sí té la BPBp-nu en el cas lineal per a tota mesura μ. Un altre resultat provat és el següent: si una c_0-suma o una ℓ_1-suma satisfà la BPBp-nu per a aplicacions multilineals, aleshores tota component de la suma directa també la satisfà.

El contingut d'aquest capítol ha sigut enviat a publicació i els autors esperen una resposta definitiva de la revista:

S. Dantas, D. García, S. K. Kim, H. J. Lee and M. Maestre, The Bishop-Phelps-Bollobás theorem for multilinear mappings, *Linear Alg. Appl.*, 2017, parcialment aceptat.

Acabem la tesis presentant una llista de problemes obertes amb la intenció d'expandir nous horitzons per a continuar treballant en aquest tema. També presentem la llista dels articles publicats derivats d'aquesta tesis. A més a més, presentem tables que resumeixen de manera pràctica els parells d'espais de Banach més coneguts que satisfan la propietat de Bishop-Phelps-Bollobás per a operadors, amb l'objectiu de situar el lector en l'actual escenari del tema.

Resumen

Esta tesis está dedicada a estudiar la propiedad de Bishop-Phelps-Bollobás en diferentes contextos.

En el Capítulo 1, introducimos la notación utilizada a lo largo de toda la tesis y damos un resumen histórico de esta propiedad, explicando, entre otros, los teoremas de Bishop-Phelps y Bishop-Phelps-Bollobás. El primer teorema de Bishop-Phelps [14] nos dice lo siguiente:

Sea X un espacio de Banach. Dado un número real positivo $\varepsilon > 0$ y un funcional lineal y continuo $x^* \in X^*$, existe un nuevo funcional $x_0^* \in X^*$ y un nuevo punto $x_0 \in S_X$ tales que satisfacen las siguientes condiciones:

$$|x_0^*(x_0)| = \|x_0^*\| \quad \text{y} \quad \|x_0^* - x^*\| < \varepsilon.$$

Es decir, el teorema prueba que el conjunto de todos los funcionales que alcanzan la norma es denso (en norma) en el dual de un espacio de Banach X.

Es natural pues preguntarse si este resultado es también cierto para operadores lineales y continuos. En 1963, J. Lindenstrauss [52] dio el primer contraejemplo que muestra que, en general, el resultado es falso. Por otro lado, también presentó condiciones que deben satisfacer los espacios de Banach para obtener resultados positivos. Después de esto, otros autores han estudiado diversas hipótesis que se pueden exigir a los espacios de Banach para saber si el conjunto de los operadores lineales y

continuos que alcanzan la norma es denso (en norma) dentro del conjunto de todos los operadores lineales y continuos.

Siete años después, en 1970, Bollobás [15] probó una versión más fuerte del teorema de Bishop-Phelps enunciada a continuación:

Sea X un espacio de Banach. Dado $\varepsilon > 0$, si $x \in B_X$ y $x^* \in B_{X^*}$ cumplen

$$|1 - x^*(x)| < \frac{\varepsilon^2}{4},$$

entonces existen elementos $x_0 \in S_X$ y $x_0^* \in S_{X^*}$ tales que

$$|x_0^*(x_0)| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{y} \quad \|x_0^* - x^*\| < \varepsilon.$$

Con este resultado Bollobás mostró que además de la aproximación para los funcionales, también es posible aproximar el punto inicial donde el primer funcional x^* casi alcanza la norma, por un punto en el que el nuevo funcional, próximo a x^*, alcanza la norma.

En los últimos nueve años, se han una escrito gran cantidad de artículos en los que se presentan diferentes teoremas del tipo Bishop-Phelps-Bollobás para operadores lineales y continuos. El interés por estos teoremas comenzó con el artículo de M. Acosta, R. Aron, D. García y M. Maestre [2]. En dicho artículo, los autores definieron por primera vez la propiedad de Bishop-Phelps-Bollobás (BPBp). La propiedad es la siguiente:

Decimos que un par de espacios de Banach $(X;Y)$ satisface la BPBp si dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que para todo $T : X \longrightarrow Y$ operador lineal de norma uno y para todo x_0 elemento en la esfera unidad de X que cumplen

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

existen un nuevo operador lineal $S : X \longrightarrow Y$ de norma uno y un nuevo elemento x_1 en la esfera unidad de X tales que cumplen las siguientes *condiciones:*

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad y \quad \|S - T\| < \varepsilon.$$

Es importante destacar que del teorema de Bishop-Phelps-Bollobás se deduce que el par $(X;\mathbb{K})$ satisface la BPBp para todos los espacios de Banach X. Por otro lado, cabe señalar que si el par $(X;Y)$ satisface la BPBp, entonces el conjunto de todos los operadores lineales y continuos que alcanzan la norma es denso (en norma) en el conjunto de todos los operadores lineales y continuos de X en Y.

Para finalizar el Capítulo 1, se presentan algunos de los más recientes resultados relacionados con los teoremas de Bishop-Phelps-Bollobás con el objetivo de poner esta tesis en un contexto actual.

A continuación, en el Capítulo 2 estudiamos algunas propiedades semejantes a la propiedad de Bishop-Phelps-Bollobás. Empezamos con la propiedad punto de Bishop-Phelps-Bollobás (BPBpp).

Decimos que un par $(X;Y)$ de espacios de Banach satisface la BPBpp si dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que para cada $T : X \longrightarrow Y$ operador lineal de norma uno y para cada x_0 elemento de la esfera unidad de X tales que

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

existe un nuevo operador lineal $S : X \longrightarrow Y$ de norma uno satisfaciendo

$$\|S(x_0)\| = 1 \quad \text{y} \quad \|S - T\| < \varepsilon.$$

Destacamos que esta nueva propiedad es más fuerte que la BPBp. A lo largo de este capítulo, primero estudiamos la BPBpp para operadores lineales y luego para aplicaciones bilineales, obteniendo en ambos casos resultados positivos de existencia de pares de espacios de Banach satisfaciéndolas.

También es importante mencionar que en la BPBpp, el punto inicial x_0, i.e., el punto en el que el operador T casi alcanza la norma, es el mismo punto en el que el operador próximo a T también alcanza la norma. Por otra parte, utilizando la BPBpp hemos probado una caracterización de los espacios uniformemente suaves:

Un espacio de Banach X es uniformemente suave si y sólo si el par $(X;\mathbb{K})$ satisface la BPBpp.

En este capítulo también probamos que si un par $(X;Y)$ satisface la BPBpp para algún espacio de Banach Y, el espacio de Banach X tiene que ser uniformemente suave. Por lo tanto, siempre debemos suponer que el dominio X del operador es uniformemente suave para así poder conseguir resultados positivos sobre esta propiedad.

Además, si Y tiene la propiedad β o es un álgebra uniforme, entonces el par $(X;Y)$ tiene la BPBpp para todos los espacios uniformemente suaves X.

Por otro lado, también hemos probado resultados sobre la BPBpp trabajando en espacios de Hilbert. Al tener los espacios de Hilbert normas transitivas, mostramos que si H es un espacio de Hilbert, entonces el par $(H;Y)$ satisface la BPBpp para todos los espacios de Banach Y.

Terminamos esta primera parte del capítulo mostrando que existen espacios reales X uniformemente suaves de dimensión 2 tales que el par $(X;Y)$ no satisface la propiedad punto de Bishop-Phelps-Bollobás para ciertos espacios de Banach Y. Finalmente, extendemos los resultados obtenidos al considerar la propiedad para aplicaciones bilineales en situaciones semejantes.

El contenido de esta sección fue publicado en el siguiente artículo:

S. Dantas, S. K. Kim and H. J. Lee, The Bishop-Phelps-Bollobás point property, *J. Math. Anal. Appl.* **444** (2016), no. 2, 1739–1751.

Siguiendo en el Capítulo 2, estudiamos una propiedad dual a la BPBpp en dos situaciones distintas. Primero, consideramos que el número real positivo η que aparece en la definición (ver Definition 2.2.2) depende no sólo de $\varepsilon > 0$, sino también de un operador lineal previamente fijado y de norma uno. Llamamos a esta propiedad, propiedad 1:

Decimos que un par de espacios de Banach $(X; Y)$ satisface la propiedad 1 *si dado $\varepsilon > 0$ y dado un operador lineal T de norma uno, existe $\eta(\varepsilon, T) > 0$ tal que para todo elemento x_0 de la esfera unidad de X con*

$$\|T(x_0)\| > 1 - \eta(\varepsilon, T),$$

existe otro elemento x_1, también en la esfera unidad de X, satisfaciendo

$$\|T(x_1)\| = 1 \quad \text{y} \quad \|x_1 - x_0\| < \varepsilon.$$

De la definición de la propiedad 1 se deduce que todo operador $T : X \longrightarrow Y$ alcanza la norma si el par $(X; Y)$ satisface dicha propiedad. Por consiguiente, y utilizando el Teorema de James (ver Remark 2.2.3.(a)), hemos de tomar como condición necesaria que el dominio X sea reflexivo para así obtener resultados positivos.

Para espacios de dimensión finita hemos probado el siguiente resultado:

El par $(X; Y)$ satisface la propiedad 1 *para todo espacio de dimensión finita X y para cualquier espacio de Banach Y.*

Otro resultado probado se enuncia a continuación:

Si X es un espacio de Banach reflexivo y localmente uniformemente convexo, en particular si X es uniformemente convexo, entonces el par $(X; Y)$ satisface la propiedad 1 *para operadores compactos.*

Como consecuencia, el par $(\ell_2; \ell_1)$ satisface la propiedad 1 ya que todo operador entre estos dos espacios es compacto y ℓ_2 es uniformemente convexo. En realidad, en este capítulo, damos una caracterización de dicha propiedad para todos los pares $(\ell_p; \ell_q)$:

(i) *El par* $(\ell_p; \ell_q)$ *satisface la propiedad* 1 *para todos* p, q *tales que* $1 \leqslant q < p < \infty$.
(ii) *El par* $(\ell_p; \ell_q)$ *no satisface la propiedad* 1 *para todos* p, q *tales que* $1 \leqslant p \leqslant q < \infty$.

A continuación, consideramos una versión uniforme de la propiedad 1, a la que llamamos propiedad 2. Esta propiedad es similar a la propiedad 1 pero considerando η dependiendo sólo de $\varepsilon > 0$.

Utilizando el Teorema de Kim-Lee [46, Theorem 2.1] tenemos la siguiente caracterización de los espacios uniformemente convexos:

Un espacio de Banach X *es uniformemente convexo si y sólo si el par* $(X; \mathbb{K})$ *tiene la propiedad* 2.

Esta es una versión dual de la ya mencionada propiedad punto de Bishop-Phelps-Bollobás para el par $(X; \mathbb{K})$, que caracteriza los espacios uniformemente suaves. Al contrario de lo que hemos probado con la BPBpp, en el caso de la propiedad 2 resulta más complicado conseguir resultados positivos. En efecto, entre los resultados probados destacamos los siguientes teoremas negativos:

Los pares $(\ell_2^2(\mathbb{R}); \ell_q^2(\mathbb{R}))$ *no satisfacen la propiedad* 2 *para* $1 \leqslant q \leqslant \infty$.

y también:

Dado Y *espacio de Banach de dimensión* 2, *el par* $(Y; Y)$ *no satisface la propiedad* 2.

El contenido de esta sección fue publicado en el siguiente artículo:

S. Dantas, Some kind of Bishop-Phelps-Bollobás property, *Math. Nachr.*, 2016, doi:10.1002/mana.201500487

En la Sección 2.3 trabajamos con la propiedad punto de Bishop-Phelps-Bollobás para radio numérico (BPBpp-nu). Esta propiedad fue motivada por la reciente propiedad de Bishop-Phelps-Bollobás para radio numérico (ver, por ejemplo, [32, 37, 47]). Su definición es parecida con la BPBp pero en lugar de la norma del operador consideramos ahora el radio numérico.

Decimos que un espacio de Banach X satisface la propiedad de Bishop-Phelps-Bollobás para radio numérico (BPBp-nu) si dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que para todo operador lineal y continuo $T : X \longrightarrow X$ de radio numérico uno y para todo par (x, x^*) satisfaciendo $\|x^*\| = \|x\| = x^*(x) = 1$ y con

$$|x^*(T(x))| > 1 - \eta(\varepsilon),$$

existe un nuevo operador lineal continuo S con radio numérico uno y un nuevo elemento (y, y^*) con $\|y^*\| = \|y\| = y^*(y) = 1$ tales que

$$|y^*(S(y))| = 1, \quad \|y^* - x^*\| < \varepsilon, \quad \|y - x\| < \varepsilon \ \text{ y } \ \|S - T\| < \varepsilon.$$

En nuestro caso, igual que hacemos con la BPBpp, consideramos el mismo punto inicial (x, x^*) para ambos operadores.

El primer resultado probado en esta sección nos dice que todo espacio de Hilbert complejo satisface la BPBpp-nu. Es por ello que hemos probado que algunos tipos de operadores (auto-adjuntos, anti-simétricos, unitarios y normales) definidos en un espacio de Hilbert complejo cumplen las propiedades BPBpp y BPBpp-nu. Resumimos estos resultados a continuación:

Si H es un espacio de Hilbert complejo, entonces

(i) *H satisface la BPBpp-nu para operadores auto-adjuntos, anti-simétricos y unitarios.*
(ii) *$(H;H)$ satisface la BPBpp para operadores auto-adjuntos, anti-simétricos, unitarios y normales.*

Eso significa que cuando uno toma, por ejemplo, un operador T auto-adjunto, tenemos que H (resp. el par $(H;H)$) cumple la BPBpp-nu (resp. la BPBpp) con S también auto-adjunto.

Finalmente, en la última sección de este capítulo, generalizamos algunos de los resultados de [11] considerando normas absolutas. Hemos probado que:

Si $|\cdot|_a$ es una norma absoluta y el par $(X;Y_1 \oplus_a Y_2)$ satisface la BPBp, entonces los pares $(X;Y_1)$ y $(X;Y_2)$ también satisfacen la BPBp.

Además, presentamos ejemplos de normas absolutas $|\cdot|_a$ tales que si el par $(X_1 \oplus_a X_2;Y)$ satisface la BPBp, entonces los pares $(X_1;Y)$ y $(X_2;Y)$ también la satisfacen.

En el Capítulo 3 consideramos la propiedad de Bishop-Phelps-Bollobás para operadores compactos.

Decimos que un par $(X;Y)$ de espacios de Banach satisface la BPBp para operadores compactos si dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que para cada $T : X \longrightarrow Y$ operador lineal y compacto de norma uno y para cada x_0 elemento de la esfera unidad de X tales que

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

existen un nuevo operador lineal y compacto $S : X \longrightarrow Y$ de norma uno y un nuevo punto x_1 en la esfera unidad de X satisfaciendo

$$\|S(x_0)\| = 1 \quad \|x_1 - x_0\| < \varepsilon \quad \text{y} \quad \|S - T\| < \varepsilon.$$

La definición de esta propiedad es similar a la BPBp, pero ahora únicamente consideramos operadores compactos. Al principio de este capítulo damos una extensa lista de ejemplos de pares de espacios de Banach $(X;Y)$ que satisfacen la BPBp para operadores compactos. Algunos de ellos ya son conocidos de otros artículos. Por ejemplo, el par $(X;Y)$ tiene la BPBp para operadores compactos si cumple alguna de las siguientes condiciones:

- X *es uniformemente convexo e* Y *es arbitrario;*
- X *es arbitrario e* Y *es un álgebra uniforme;*
- $X = C_0(L)$*, siendo* L *un espacio topológico de Hausdorff localmente compacto, e* Y *es uniformemente convexo;*
- X *es arbitrario e* Y *es isométricamente isomorfo a un espacio* L_1*;*
- $X = L_1(\mu)$*, siendo* μ *es una medida arbitraria, e* Y *tiene la AHSP.*

Por otro lado, realizando sencillas modificaciones en las demostraciones de ciertos resultados probados para la BPBp, obtenemos más resultados positivos para versión de esta propiedad para operadores compactos. La idea es tomar inicialmente un operador compacto T y definir una perturbación compacta que satisfaga las condiciones de Bishop-Phelps-Bollobás (ver Definition 3.1.1). Por ejemplo, en la demostración de [2, Theorem 2.2], si partimos de un operador compacto T en $\mathcal{K}(X,Y)$ de norma 1 y tal que $\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}$, entonces el operador dado por

$$S(x) := T(x) + [(1+\eta)z_0^*(x) - T^*(y_{\alpha_0}^*)(x)]y_{\alpha_0},$$

satisface las condiciones de la BPBp y es también compacto.

Todo este tercer capítulo está motivado por el estudio de los operadores compactos que alcanzan la norma. M. Martín probó en [53] que existen espacios de Banach X e Y y operadores compactos de X en Y que no pueden ser aproximados por aquellos que alcanzan la norma. Él presentó también condiciones que garantizan la densidad de los operadores que alcanzan la norma en el conjunto de los operadores compactos.

En la primera sección del capítulo, presentamos algunas técnicas para producir pares de espacios de Banach satisfaciendo la BPBp para operadores compactos. Estas técnicas están basadas en dos resultados anteriores de J. Johnson y J. Wolfe [43] sobre operadores compactos que alcanzan la norma. La idea principal de la sección es dar algunos resultados técnicos a través de los cuales podremos pasar la propiedad para espacios de sucesiones a la propiedad para espacios de funciones.

Para conseguir estos resultados sobre los espacios dominio del operador, probamos el siguiente lema:

Sean X e Y dos espacios de Banach. Supongamos que existe una función $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ tal que dados $\delta \in \mathbb{R}^+$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ y $x_0 \in S_X$, podemos encontrar dos operadores de norma uno $P \in \mathcal{L}(X;X)$ e $i \in \mathcal{L}(P(X);X)$ que cumplen las siguientes condiciones:

(1) $\|P^* x_j^* - x_j^*\| < \delta$ *para* $j = 1, \ldots, n$,

(2) $\|i(P(x_0)) - x_0\| < \delta$,

(3) $P \circ i = \mathrm{Id}_{P(X)}$,

(4) *el par $(P(X);Y)$ satisface la BPBp para operadores compactos con la función η.*

Entonces, el par $(X;Y)$ satisface la BPBp para operadores compactos.

Por otro lado, el lema que necesitamos para obtener los resultados sobre los espacios imagen del operador es el siguiente:

Sean X e Y dos espacios de Banach. Supongamos que

(1) *existe una red de proyecciones de norma uno* $\{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(X;Y)$ *tal que para cada* $y \in Y$*, la sucesión* $\{Q_\lambda y\}$ *converge a* y *en norma, y además*

(2) *existe una función* $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *tal que los pares* $(X; Q_\lambda(Y))$ *con* $\lambda \in \Lambda$ *satisfacen la BPBp para operadores compactos con la función* η.

Entonces, el par $(X;Y)$ *satisface la BPBp para operadores compactos.*

Usando estos dos resultados técnicos, estamos preparados para trabajar con casos particulares de espacios de Banach en la tercera sección de este capítulo.

Comenzamos con los espacios dominio del operador, sin perder de vista que el objetivo es pasar la propiedad de espacios de sucesiones a espacios de funciones, como ya habíamos mencionado.

Si $(c_0;Y)$ *satisface la BPBp para operadores compactos, entonces el par* $(C_0(L);Y)$ *también satisface la propiedad para todo* L *espacio topológico de Hausdorff localmente compacto.*

Para probar este resultado, necesitamos previamente una caracterización de los pares $(c_0;Y)$ que satisfacen la BPBp para operadores compactos. Se enuncia a continuación.

Sean X *e* Y *dos espacios de Banach. Las siguientes afirmaciones son equivalentes:*

(i) *el par* $(c_0(X);Y)$ *tiene la BPBp para operadores compactos;*

(ii) *existe una función* $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *tal que los pares* $(\ell_\infty^m(X);Y)$*, con* $m \in \mathbb{N}$*, tienen la BPBp para operadores compactos con la función* η.

Por otro lado, cuando $\mathcal{K}(X;Y) = \mathcal{L}(X;Y)$ (en particular, si alguno de los espacios X o Y es de dimensión finita), el resultado anterior es cierto cuando $(c_0(X);Y)$ o $(\ell_\infty(X);Y)$ tiene la BPBp. Como consecuencia, obtenemos el siguiente resultado.

Sea Y un espacio de Banach. Si el par $(c_0;Y)$ satisface la BPBp, entonces el mismo par también satisface la BPBp para operadores compactos.

De hecho, aunque este teorema sea cierto, no está totalmente clara la relación que hay entre la BPBp para operadores compactos y la BPBp. Es importante mencionar que *no es verdad* que la BPBp para operadores compactos implique la BPBp, ya que el par $(L_1[0,1], C[0,1])$ tiene la BPBp para operadores compactos pero no puede tener la BPBp por un contraejemplo dado por W. Schachermayer para operadores que alcanzan la norma entre los espacios $L_1[0,1]$ y $C[0,1]$ [59]. Hasta el momento no hay resultados, ni positivos ni negativos, que prueben que la BPBp implique la BPBp para operadores compactos.

Siguiendo en el contexto de las aplicaciones para los espacios dominio, hemos probado el siguiente resultado que nos permite pasar la propiedad de espacios ℓ_1 a espacios L_1.

Sea μ una medida positiva y sea X un espacio de Banach tal que X^* tiene la propiedad de Radon-Nikodým. Si Y es un espacio de Banach y el par $(\ell_1(X);Y)$ tiene la BPBp para operadores compactos, entonces el par $(L_1(\mu,X);Y)$ también la tiene.

El siguiente lema es necesario para probar el resultado anterior.

Sean X e Y dos espacios de Banach. Las siguientes afirmaciones son equivalentes:

(i) *para todo $\varepsilon > 0$ existe $0 < \xi(\varepsilon) < \varepsilon$ tal que, dadas las sucesiones $(T_k) \subset B_{\mathcal{K}(X;Y)}$ y $(x_k) \subset B_X$, y dada una serie convexa $\sum_{k=1}^{\infty} \alpha_k$*

satisfaciendo

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \xi(\varepsilon),$$

existen un subconjunto finito $A \subset \mathbb{N}$*, un elemento* $y^* \in S_{Y^*}$ *y dos sucesiones* $(S_k) \subset S_{\mathcal{K}(X;Y)}$ *y* $(z_k) \subset S_X$ *que cumplen*

(a) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,

(b) $\|z_k - x_k\| < \varepsilon$ *y* $\|S_k - T_k\| < \varepsilon$ *para todos* $k \in A$,

(c) $y^*(S_k z_k) = 1$ *para todo* $k \in A$.

(En este caso, decimos que el par $(X;Y)$ *satisface la AHSP generalizada para operadores compactos);*

(ii) *el par* $(\ell_1(X);Y)$ *tiene la BPBp para operadores compactos;*

(iii) *existe una función* $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *tal que los pares* $(\ell_1^m(X);Y)$*, con* $m \in \mathbb{N}$*, satisfacen la BPBp para operadores compactos con la función* η*.*

En particular, tenemos la siguiente caracterización de los pares $(\ell_1;Y)$ que satisfacen la BPBp para operadores compactos.

Sea Y *un espacio de Banach. Las siguientes afirmaciones son equivalentes:*

(i) *el par* $(\ell_1;Y)$ *tiene la BPBp para operadores compactos;*

(ii) *el espacio de Banach* Y *satisface la AHSP;*

(iii) *el par* $(\ell_1;Y)$ *satisface la BPBp;*

(iv) *el par* $(L_1(\mu);Y)$ *tiene la BPBp para operadores compactos para toda medida positiva* μ*;*

(v) *existe una medida positiva μ tal que $L_1(\mu)$ es de dimensión infinita y el par $(L_1(\mu); Y)$ tiene la BPBp para operadores compactos.*

Por otro lado, utilizando el lema técnico mencionado anteriormente obtenemos los siguientes resultados para los espacios imagen del operador compacto.

(a) *Para $1 \leqslant p < \infty$, si el par $(X; \ell_p(Y))$ tiene la BPBp para operadores compactos, entonces el par $(X; L_p(\mu, Y))$ también la tiene para toda medida positiva μ tal que $L_1(\mu)$ es de dimensión infinita.*

(b) *Si el par $(X; Y)$ satisface la BPBp para operadores compactos, entonces el par $(X; L_\infty(\mu, Y))$ también la satisface para toda medida positiva σ-finita μ.*

(c) *Si el par $(X; Y)$ tiene la BPBp para operadores compactos, entonces $(X; C(K, Y))$ también la tiene para todo espacio topológico de Hausdorff compacto K.*

Para probar este resultado, necesitamos del siguiente lema que caracteriza los pares $(X; Y)$ que satisfacen la BPBp para operadores compactos en términos de sumas directas.

Sean X e Y dos espacios de Banach y sea $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ una función. Las siguientes afirmaciones son equivalentes:

(i) *el par $(X; Y)$ tiene la BPBp para operadores compactos con la función η,*

(ii) *los pares $(X; \ell_\infty^m(Y))$, con $m \in \mathbb{N}$, satisfacen la BPBp para operadores compactos con la función η,*

(iii) *el par $(X; c_0(Y))$ tiene la BPBp para operadores compactos con la función η,*

(iv) *el par* $(X; \ell_\infty(Y))$ *tiene la BPBp para operadores compactos con la función* η.

Para acabar el capítulo y utilizando los resultados previamente enunciados se prueba el siguiente teorema:

Sea K *un espacio topológico de Hausdorff compacto y sean* X *e* Y *dos espacios de Banach. Si* μ *es una medida positiva y* ν *es una medida positiva* σ*-finita, entonces las siguientes afirmaciones son ciertas:*

(a) *Si* Y *tiene la propiedad* β*, entonces los pares* $(X; L_\infty(\nu, Y))$ *y* $(X; C(K, Y))$ *tienen la BPBp para operadores compactos.*

(b) *Si* Y *tiene la AHSP, entonces* $L_\infty(\nu, Y)$ *y* $C(K, Y)$ *también la tienen.*

(c) *Para* $1 \leqslant p < \infty$*, si* $\ell_p(Y)$ *tiene la AHSP y* $L_1(\mu)$ *es de dimensión infinita, entonces* $L_p(\mu, Y)$ *tiene la AHSP.*

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Finalmente, en el Capítulo 4 estudiamos diferentes propiedades de tipo Bishop-Phelps-Bollobás adaptadas al caso de aplicaciones multilineales. Comenzamos enunciando la BPBp para aplicaciones multilineales.

Sean $X_1, \ldots X_N$ *e* Y *espacios de Banach. Decimos que* $(X_1, \ldots, X_N; Y)$ *satisface la propiedad de Bishop-Phelps-Bollobás para aplicaciones multilineales si dado* $\varepsilon > 0$*, existe* $\eta(\varepsilon) > 0$ *tal que para toda* A *aplicación* N*-lineal de* $X_1 \times \ldots X_N$ *en* Y *con* $\|A\| = 1$ *y para todo* $(x_1^0, \ldots, x_N^0) \in$

$S_{X_1} \times \ldots \times S_{X_N}$ satisfaciendo

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

existen una nueva aplicación N-lineal $B : X_1 \times \ldots \times X_N \longrightarrow Y$ con $\|B\| = 1$ y un nuevo elemento $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ tales que

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \ \text{ y } \ \|B - A\| < \varepsilon.$$

Destacamos el grado de la aplicación multilineal indicando que $(X_1, \ldots, X_N; Y)$ tiene la BPBp para aplicaciones N-lineales en lugar de decir que tiene la propiedad para aplicaciones multilineales. Al igual que hemos hecho para la BPBp para aplicaciones N-lineales, también podemos definir la BPBp para aplicaciones multilineales simétricas al exigir que las aplicaciones A y B sean elementos del conjunto de todas las aplicaciones N-lineales simétricas, denotado por $\mathcal{L}_s(^N X; Y)$. En este caso, decimos que $(^N X; Y)$ tiene la BPBp para aplicaciones multilineales simétricas. Un caso particular sucede al tomar $Y = \mathbb{K}$ en el que denotamos la BPBp para $(X_1, \ldots, X_N; \mathbb{K})$ solo por $(X_1, \ldots, X_N)$ y decimos que $(X_1, \ldots, X_N)$ tiene la BPBp para formas N-lineales.

Análogamente, definimos la BPBp para polinomios homogéneos.

Decimos que el par de espacios de Banach $(X; Y)$ satisface la propiedad de Bishop-Phelps-Bollobás para polinomios N-homogéneos si dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que para todo P polinomio N-homogéneo con $\|P\| = 1$ y para todo $x_0 \in S_X$ que satisfacen

$$\|P(x_0)\| > 1 - \eta(\varepsilon),$$

existen un nuevo polinomio N-homogéneo Q con $\|Q\| = 1$ y un elemento $x_1 \in S_X$ tales que

$$\|Q(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad y \quad \|Q - P\| < \varepsilon.$$

En este capítulo también extendemos algunos resultados conocidos sobre aplicaciones multilineales que alcanzan la norma para la BPBp como, por ejemplo, la siguiente observación que deriva del caso lineal:

Sean $X, X_1, \ldots, X_N$ e Y espacios de Banach de dimensión finita. Entonces las siguientes afirmaciones son ciertas:

(i) $(X_1, \ldots, X_N; Y)$ *tiene la BPBp para aplicaciones multilineales,*

(ii) $({}^N X; Y)$ *tiene la BPBp para aplicaciones multilineales simétricas,*

(iii) $(X; Y)$ *tiene la BPBp para polinomios N-homogéneos.*

Un resultado que merece ser destacado es un teorema de estabilidad que permite trasladar la BPBp para aplicaciones multilineales de grado $N + 1$ a aplicaciones multilineales de grado N.

Si $(X_1, \ldots, X_N,\ X_{N+1}; Y)$ tiene la BPBp para aplicaciones $(N + 1)$-lineales, entonces $(X_1, \ldots,\ X_N; Y)$ tiene la BPBp para aplicaciones N-lineales.

Otro teorema que relaciona la BPBp para aplicaciones N-lineales y $(N - 1)$-lineales es el siguiente:

Si $N \geqslant 2$ y si $X_1, \ldots, X_N$ son espacios de Banach siendo X_N un espacio uniformemente convexo, entonces $(X_1, \ldots, X_N)$ tiene la BPBp para aplicaciones N-lineales si y sólo si $(X_1, \ldots, X_{N-1}; X_N^*)$ tiene la BPBp para aplicaciones $(N - 1)$-lineales.

Asimismo, también mostramos que es posible pasar del caso vectorial al caso escalar en la BPBp para aplicaciones multilineales. En particular, probamos la siguiente caracterización:

Si Y tiene la propiedad β, entonces $(X_1, \ldots, X_N)$ tiene la BPBp para formas N-lineales si y sólo si $(X_1, \ldots, X_N; Y)$ tiene la BPBp para aplicaciones N-lineales.

Por otra parte, en este capítulo también estudiamos la BPBp para aplicaciones multilineales compactas siguiendo la misma línea del capítulo anterior.

Sea Y un espacio predual de L_1. Supongamos que $(X_1, \ldots, X_N)$ tiene la BPBp para formas multilineales. Entonces $(X_1, \ldots, X_N; Y)$ tiene la BPBp para aplicaciones multilineales compactas.

Pero al igual que prueban en [3], hemos obtenido un resultado más fuerte:

Dado $\varepsilon \in (0, 1)$, existe $\eta(\varepsilon) > 0$ tal que para todo $A \in \mathcal{K}(X_1, \ldots, X_N; Y)$ con $\|A\| = 1$ y para todo $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ que satisfacen

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

existen $B \in \mathcal{K}(X_1, \ldots, X_N; Y)$ con $\|B\| = 1$ y $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ tales que $\dim(B(X_1 \times \ldots \times X_N)) < \infty$,

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \quad y \quad \|B - A\| < \varepsilon.$$

A continuación presentamos algunos ejemplos de espacios que cumplen la BPBp para aplicaciones multilineales compactas.

Sea Y un espacio predual de L_1. Si $(X, Z; Y)$ cumple alguna de las siguientes condicones, entonces tiene la BPBp para aplicaciones multilineales compactas.

(a) *Si tomamos los espacios complejos $X = C_0(L)$ y $Z = C_0(K)$ siendo L y K espacios topológicos de Hausdorff localmente compactos.*

(b) *Si $X = L_1(\mu)$ y $Z = c_0$.*

(c) *Si X y Z son espacios de Banach uniformemente convexos.*

Seguidamente, en la tercera sección de este capítulo caracterizamos los pares $(\ell_1(X), Y)$ que satisfacen la BPBp para formas bilineales. El resultado es el siguiente:

El par $(\ell_1(X), Y)$ tiene la BPBp para formas bilineales si y sólo si para todo $\varepsilon > 0$, existe $0 < \eta(\varepsilon) < \varepsilon$ tal que, dadas sucesiones $(T_k)_k \subset \mathcal{L}(X;Y)$ con $\|T_k\| = 1$ para todo k y $(x_k)_k \subset S_X$ y dada una serie convexa $\sum_{k=1}^{\infty} \alpha_k$ tal que

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k(x_k) \right\| > 1 - \eta(\varepsilon),$$

existen un subconjunto $A \in \mathbb{N}$, un elemento $y^* \in S_{Y^*}$ y dos sucesiones $(S_k)_k \subset \mathcal{L}(X;Y)$ con $\|S_k\| = 1$ para todo k y $(z_k)_k \subset S_X$ satisfaciendo

(1) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,

(2) $\|z_k - x_k\| < \varepsilon$ *y* $\|S_k - T_k\| < \varepsilon$ *para todo* $k \in A$, *y*

(3) $y^*(S_k(x_k)) = 1$ *para todo* $k \in A$.

También hemos probado que, si H es un espacio de Hilbert, entonces el par (X, H) tiene la BPBp para formas bilineales si y sólo si el par (X, H) tiene la AHSP generalizada.

Otro campo de trabajo de este capítulo es el estudio del radio numérico en el conjunto de las aplicaciones multilineales definidas en $L_1(\mu)$, siendo μ una medida arbitraria. Hemos mostrado que para toda aplicación $A \in \mathcal{L}(^N L_1(\mu); L_1(\mu))$, su radio numérico y su norma coinciden.

Para terminar el capítulo, estudiamos la propiedad de Bishop-Phelps-Bollobás para radio numérico en el contexto de las aplicaciones multilineales. Mostramos que si X tiene dimensión finita, entonces el espacio X satisface esta propiedad. Por otro lado, $L_1(\mu)$ no satisface la propiedad aunque este espacio sí tiene la BPBp-nu en el caso lineal para toda medida μ. Otro resultado probado es el siguiente: si una c_0-suma o una ℓ_1-suma satisface la BPBp-nu para aplicaciones multilineales, entonces cada componente de la suma directa también la satisface.

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S. Dantas, D. García, S. K. Kim, H. J. Lee and M. Maestre, The Bishop-Phelps-Bollobás theorem for multilinear mappings, *Linear Alg. Appl.*, 2017, parcialmente aceptado.

Terminamos la tesis presentando una lista de problemas abiertos con la intención de expandir nuevos horizontes para continuar trabajando en este tema. También presentamos la lista de artículos publicados derivados de esta tesis. Además, presentamos tablas que resumen de manera práctica los pares de espacios de Banach más conocidos que satisfacen la propiedad de Bishop-Phelps-Bollobás para operadores, con el objetivo de situar al lector en el actual escenario del tema.

Abstract

This dissertation is devoted to the study of the Bishop-Phelps-Bollobás property in different contexts.

In Chapter 1 we introduce the notation and give a historical resume behind this property as the classics Bishop-Phelps and Bishop-Phelps-Bollobás theorems. The first theorem [14] says the following:

Let X be a Banach space. Given a positive real number $\varepsilon > 0$ and a continuous linear functional $x^* \in X^*$, there are a new functional $x_0^* \in X^*$ and a new point $x_0 \in B_X$ satisfying that

$$|x_0^*(x_0)| = \|x_0^*\| \quad \textit{and} \quad \|x_0^* - x^*\| < \varepsilon.$$

That is, that the set of all norm attaining functionals is norm dense in the dual of a Banach space X.

It is natural to ask if this theorem holds for general bounded linear operators. In 1963, J. Lindenstrauss [52] gave the first counterexample that in general this is false presenting, on the other hand, conditions on the Banach spaces to get positive results. After that, many authors studied the hypothesis that some Banach spaces must satisfy to get that the set of all bounded linear operators which attain their norm is norm dense in the set of all bounded linear operators.

Seven years later, Bollobás [15] proved a stronger version of the Bishop-Phelps theorem which we highlight as follows.

Let X be a Banach space. Given $\varepsilon > 0$, $x \in B_X$ and $x^* \in S_{X^*}$ satisfying

$$|1 - x^*(x)| < \frac{\varepsilon^2}{4},$$

there are elements $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ such that

$$|x_0^*(x_0)| = 1, \quad \|x_0 - x\| < \varepsilon \quad \textit{and} \quad \|x_0^* - x^*\| < \varepsilon.$$

With this result he showed that besides the approximation for the functionals, we can approximate the point that an initial functional almost attains its norm.

In the last nine years a lot of attention had been paid in the attempt to get Bishop-Phelps-Bollobás type theorems for bounded linear operators. This started with the seminal paper by M. Acosta, R. Aron, D. García and M. Maestre [2]. They defined the Bishop-Phelps-Bollobás property as follows.

We say that a pair $(X;Y)$ of Banach spaces has the BPBp if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever T is a norm-one linear operator from X into Y and x_0 is an element of the unit sphere of X such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are a new norm-one linear operator S and a new element x_1 on the unit sphere of X satisfying the following conditions:

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \textit{and} \quad \|S - T\| < \varepsilon.$$

Note that the Bishop-Phelps-Bollobás theorem says that the pair $(X, \mathbb{K})$ has the BPBp for all Banach spaces X. Note also that if the pair $(X;Y)$ has the BPBp, then the set of all bounded linear operators from X into

Y which attain their norm is norm dense in the set of all bounded linear operators.

We finish this chapter commenting on some important current results on this topic as the necessary reference background material to help make this dissertation entirely connected with the recent works.

In Chapter 2 we study similar properties to the BPBp. We start with the Bishop-Phelps-Bollobás *point* property (BPBpp).

We say that a pair $(X;Y)$ of Banach spaces has the BPBpp if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever T is a norm-one linear operator and x_0 is an element in the unit sphere of X such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there is a new norm-one linear operator S such that

$$\|S(x_0)\| = 1 \quad \textit{and} \quad \|S - T\| < \varepsilon.$$

Observe that this property is stronger than the BPBp. First we study it for bounded linear operators and then for bilinear mappings.

Note that here we do not change the initial point x_0, i.e., the point that T almost attain its norm is the same point that the operator which is close to T attains its norm. As a first result, we give a characterization for uniformly smooth Banach spaces via this property:

A Banach space X is uniformly smooth if and only if the pair $(X;\mathbb{K})$ has the BPBpp.

Also we prove that if a pair $(X;Y)$ has this property for some Banach space Y, the Banach space X must be uniformly smooth. For that reason, we must suppose always that the domain space X is uniformly smooth to get more positive examples.

We show that if Y has property β or it is a uniform algebra, then $(X;Y)$ has the property for all uniformly smooth Banach spaces.

On the other hand, we work on Hilbert spaces. Recalling that Hilbert spaces have transitivy norms we show that if H is a Hilbert space, then the pair $(H;Y)$ has the BPBpp for all Banach spaces Y.

We finish this first part by showing that there exist 2-dimensional uniformly smooth real Banach spaces X such that the pair $(X;Y)$ *fails* the BPBpp for some Banach space Y. After that we consider the same property for bilinear mappings on similar situations.

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Still on Chapter 2, we study the dual property of the BPBpp in two different situations. In the first situation we consider the positive real number η which appears in its definition (see Definition 2.2.2) depending not only on $\varepsilon > 0$ but also on a fixed norm-one linear operator T. We call this property as property 1.

We say that a pair $(X;Y)$ has property 1 if given $\varepsilon > 0$ and a norm one linear operator T, there is $\eta(\varepsilon, T) > 0$ such that whenever an element x_0 in the unit sphere of X satisfies

$$\|T(x_0)\| > 1 - \eta(\varepsilon, T),$$

there is another element x_1, also in the unit sphere of X, satisfying that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

Note, by the definition, that the operator $T : X \longrightarrow Y$ *must* attains its norm if the pair $(X;Y)$ has this property. So, to get positive results, we have at least suppose that the domain space X is reflexive by using James theorem (see Remark 2.2.3.(a)). For finite dimensional Banach spaces, we have the following result:

The pair $(X;Y)$ has property 1 for every Banach space Y if X is finite dimensional.

We also show that

If X is reflexive and locally uniformly convex, in particular for X uniformly convex, then the pair $(X;Y)$ has property 1 for compact operators

As a consequence the pair $(\ell_2;\ell_1)$ satisfies it since, in this particular case, every operator is compact and ℓ_2 is uniformly convex. Actually, we give the following complete characterization to this property for the pairs $(\ell_p;\ell_q)$:

(i) *The pair (ℓ_p,ℓ_q) has property 1 whenever $1 \leqslant q < p < \infty$.*
(ii) *The pair (ℓ_p,ℓ_q) fails property 1 whenever $1 \leqslant p \leqslant q < \infty$.*

After that, we consider the uniformly version of property 1 which we call as property 2. This is the same as property 1 but η depends only on $\varepsilon > 0$. By Kim-Lee theorem [46, Theorem 2.1], we have that

A Banach space X is uniformly convex if and only if the pair $(X;\mathbb{K})$ has property 2.

This is the dual version of the already mentioned BPBpp for the pair $(X;\mathbb{K})$ which characterizes uniformly smooth Banach spaces. Differently from the Bishop-Phelps-Bollobás point property, it seems to be difficult to get positive results besides the scalar case when the domain space is uniformly convex. Indeed, among other negative results, we show that

The pairs $(\ell_2^2(\mathbb{R});\ell_q^2(\mathbb{R}))$ fails property 2 for all $1 \leqslant q \leqslant \infty$.

and also

For all 2-dimensional Banach spaces Y, the pair $(Y;Y)$ cannot have property 2.

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S. Dantas, Some kind of Bishop-Phelps-Bollobás property, *Math. Nachr.*, 2016, doi:10.1002/mana.201500487

In Section 2.3 we work on the Bishop-Phelps-Bollobás point property for numerical radius (BPBpp-nu). This property was motivated by the recently study of the BPBp for numerical radius (see for example [32, 37, 47]). Its definition is similar to the BPBp but now considering numerical radius instead of the norm on the space of all bounded linear operators.

The Banach space X has the Bishop-Phelps-Bollobás property for numerical radius (BPBp-nu) if for every $\varepsilon > 0$, there exists some $\eta(\varepsilon) > 0$ such that whenever T is a operator from X into itself with numerical radius 1 and (x, x^*) satisfy $\|x^*\| = \|x\| = x^*(x) = 1$ and

$$|x^*(T(x))| > 1 - \eta(\varepsilon),$$

there are a new numerical radius one linear operator S and an element (y, y^*) with $\|y^*\| = \|y\| = y^*(y) = 1$ such that

$$|y^*(S(y))| = 1, \quad \|y^* - x^*\| < \varepsilon, \quad \|y - x\| < \varepsilon \quad \textit{and} \quad \|S - T\| < \varepsilon.$$

In our case, we are considering that the initial point (x, x^*) does not change as in the BPBpp.

As a first result, we show that the complex Hilbert spaces satisfy this property. Then we work with operators defined in complex Hilbert spaces and we give versions of the BPBpp-nu and the BPBpp for self-adjoint, anti-symmetric, unitary and normal operators. We summarize these results as follows.

If H is a complex Hilbert space, then

(i) *H has the BPBpp-nu for self-adjoint, anti-symmetric and unitary operators.*
(ii) *(H, H) has the BPBpp for self-adjoint, anti-symmetric, unitary and normal operators.*

This means that if one starts with, for example, self-adjoint operator, we have that H (resp. the pair $(H;H)$) has the BPBpp-nu (resp. the BPBpp) with S also a self-adjoint operator.

Finally, in the last section of this chapter we generalize some results from [11] by considering absolute norms. We prove that

If $|\cdot|_a$ is an absolute norm and the pair $(X; Y_1 \oplus_a Y_2)$ has the BPBp, then $(X; Y_1)$ and $(X; Y_2)$ also satisfy this property.

Moreover, we give some examples of absolute norms $|\cdot|_a$ such that if the pair $(X_1 \oplus_a X_2; Y)$ has the BPBp so do the pairs $(X_1; Y)$ and $(X_2; Y)$.

In Chapter 3 we consider the Bishop-Phelps-Bollobás property for compact operators.

We say that a pair $(X; Y)$ of Banach spaces has the BPBp for compact operators if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever T is a norm-one compact linear operator from X into Y and x_0 is an element of the unit sphere of X such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are a new norm-one compact linear operator S and a new element x_1 on the unit sphere of X satisfying the following conditions:

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \textit{and} \quad \|S - T\| < \varepsilon.$$

Its definition is similar to the BPBp but now considering just this type of operator. At the beginning of the chapter, we give an extensive list of examples of pairs $(X;Y)$ of Banach spaces which satisfy the BPBp for compact operators. Some of them are already known as, among others, the pairs $(X;Y)$ when

- X is uniformly convex and Y is arbitrary;
- X is arbitrary and Y is a uniform algebra;
- $X = C_0(L)$ and Y is uniformly convex where L is any locally compact Hausdorff topological space;
- X arbitrary and Y^* is isometrically isomorphic to a L_1-space;
- $X = L_1(\mu)$ for an arbitrary measure and Y having the AHSP.

On the other hand, others are obtained by a simple modification on the proof of some BPBp results by starting with a compact operator and defining a compact perturbation which satisfies the Bishop-Phelps-Bollobás conditions (see Definition 3.1.1). For example in the proof of [2, Theorem 2.2], if we start with a compact operator $T \in \mathcal{K}(X;Y)$ with $\|T\| = 1$ satisfying $\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}$, then the operator

$$S(x) := T(x) + [(1+\eta)z_0^*(x) - T^*(y_{\alpha_0}^*)(x)]y_{\alpha_0},$$

which satisfies the BPBp conditions, is also compact.

The hole chapter is motivated by the study of norm attaining compact operators. M. Martín [53] showed that there exist Banach spaces X and

Y, and compact operators from X into Y which *cannot* be approximated by norm attaining operators. He presented some conditions ensuring the density of norm attaining operators in the space of compact operators.

During the first section, we present some techniques to produce pairs of Banach spaces having the BPBp for compact operators. These techniques are based on two old results about norm attaining compact operators by J. Johnson and J. Wolfe [43] and the main idea here is to give some applications which carry the BPBp for compact operators from some sequence spaces to function spaces.

To get applications on the domain spaces, we prove the following technical lemma.

Let X and Y be Banach spaces. Suppose that there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that given $\delta \in \mathbb{R}^+$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $x_0 \in S_X$, we may find a norm-one operator $P \in \mathcal{L}(X;X)$ and a norm-one operator $i \in \mathcal{L}(P(X);X)$ such that

(1) $\|P^* x_j^* - x_j^*\| < \delta$ *for* $j = 1, \ldots, n$,

(2) $\|i(P(x_0)) - x_0\| < \delta$,

(3) $P \circ i = \mathrm{Id}_{P(X)}$,

(4) *the pair $(P(X); Y)$ has the BPBp for compact operators with the function η.*

Then, the pair $(X;Y)$ has the BPBp for compact operators.

On the other hand, the abstract lemma that we need to get the applications for the range spaces is the following.

Let X and Y be Banach spaces. Suppose that

(1) *there exists a net of norm-one projections $\{Q_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{L}(X;Y)$ such that $\{Q_\lambda y\} \longrightarrow y$ in norm for every $y \in Y$ and*

(2) *there exists a function* $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that the pairs* $(X; Q_\lambda(Y))$ *with* $\lambda \in \Lambda$ *have the BPBp for compact operators with the function* η.

Then the pair $(X; Y)$ *has the BPBp for compact operators.*

Using these two technical results, we will be prepare to work on concrete Banach spaces in the third section of this chapter. We start with the domain spaces trying to get results from sequences to functions spaces as the following theorem.

If $(c_0; Y)$ *has the BPBp for compact operators, then so does* $(C_0(L); Y)$ *for every locally compact Hausdorff topological space* L.

To prove this, we need a characterization for the pairs $(c_0(X), Y)$ to have the BPBp for compact operators.

Let X *and* Y *be Banach spaces. Then the following are equivalent:*

(i) *the pair* $(c_0(X); Y)$ *has the BPBp for compact operators;*

(ii) *there is a function* $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that the pairs* $(\ell_\infty^m(X); Y)$ *with* $m \in \mathbb{N}$ *have the BPBp for compact operators with the function* η.

Besides that, when $\mathcal{K}(X; Y) = \mathcal{L}(X; Y)$ (in particular, if one of the spaces X or Y is finite-dimensional), this happens when $(c_0(X); Y)$ or $(\ell_\infty(X); Y)$ has the BPBp. As a consequence we have that

Let Y *be a Banach space. If the pair* $(c_0; Y)$ *has the BPBp, then it has the BPBp for compact operators.*

In fact, although we have the above fact, it is not totally clear what is the relation between the BPBp for compact operators and the BPBp. It is worth to mentioning that *it is not true* that the BPBp for compact operators implies the BPBp for operators since the pair $(L_1[0,1], C[0,1])$ has the BPBp for compact operators but this pair cannot have the BPBp by a W. Schachermayer counterexample for norm attaning of operators from $L_1[0,1]$ into $C[0,1]$ [59]. We do not know if the BPBp implies the BPBp for compact operators.

Still on the application for the domain spaces, we get the following result which passes the BPBp for compact operators from ℓ_1 to L_1 spaces.

Let μ be a positive measure, let X be a Banach space such that X^* has the Radon-Nikodým property and let Y be a Banach space. If the pair $(\ell_1(X); Y)$ has the BPBp for compact operators, then the pair $(L_1(\mu, X); Y)$ has the BPBp for compact operators.

To prove this, we need the following useful lemma.

Then the following are equivalent:

- **(i)** *for every $\varepsilon > 0$ there exists $0 < \xi(\varepsilon) < \varepsilon$ such that given sequences $(T_k) \subset B_{\mathcal{K}(X;Y)}$ and $(x_k) \subset B_X$, and a convex series $\sum_{k=1}^{\infty} \alpha_k$ such that*

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \xi(\varepsilon),$$

  there exist a finite subset $A \subset \mathbb{N}$, $y^* \in S_{Y^*}$ and sequences $(S_k) \subset S_{\mathcal{K}(X;Y)}$, $(z_k) \subset S_X$ satisfying the following:

  - **(a)** $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,
  - **(b)** $\|z_k - x_k\| < \varepsilon$ *and* $\|S_k - T_k\| < \varepsilon$ *for all* $k \in A$,
  - **(c)** $y^*(S_k z_k) = 1$ *for every* $k \in A$.

  (in this case, we may say that the pair $(X; Y)$ has the generalized AHSP for compact operators*);*

- **(ii)** *the pair $(\ell_1(X); Y)$ has the BPBp for compact operators;*

(iii) *there is a function* $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that the pairs* $(\ell_1^m(X); Y)$ *with* $m \in \mathbb{N}$ *have the BPBp for compact operators with the function* η.

In particular, we have the following characterization for the pairs $(\ell_1; Y)$ to have the BPBp for compact operators.

The following are equivalent:

(i) *the pair* $(\ell_1; Y)$ *has the BPBp for compact operators;*

(ii) Y *has the AHSP;*

(iii) *the pair* $(\ell_1; Y)$ *has the BPBp;*

(iv) *for every positive measure* μ*, the pair* $(L_1(\mu); Y)$ *has the BPBp for compact operators;*

(v) *there is a positive measure* μ *such that* $L_1(\mu)$ *is infinite-dimensional and the pair* $(L_1(\mu); Y)$ *has the BPBp for compact operators.*

And now about the ranges spaces. We get the following applications of the abstract lemma that we had showed before.

(a) *For* $1 \leqslant p < \infty$*, if the pair* $(X; \ell_p(Y))$ *has the BPBp for compact operators, then so does* $(X; L_p(\mu, Y))$ *for every positive measure* μ *such that* $L_1(\mu)$ *is infinite-dimensional.*

(b) *If the pair* $(X; Y)$ *has the BPBp for compact operators, then so does* $(X; L_\infty(\mu, Y))$ *for every* σ*-finite positive measure* μ*.*

(c) *If the pair* $(X; Y)$ *has the BPBp for compact operators, then so does* $(X; C(K, Y))$ *for every compact Hausdorff topological space* K*.*

To prove this theorem, we need of the following useful lemma which characterizes the pair (X, Y) to have the BPBp for compact operators in terms of direct sums.

Let $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a function. The following are equivalent:

(i) *the pair $(X; Y)$ has the BPBp for compact operators with the function η,*

(ii) *the pairs $(X; \ell_\infty^m(Y))$ with $m \in \mathbb{N}$ have the BPBp for compact operators with the function η,*

(iii) *the pair $(X; c_0(Y))$ has the BPBp for compact operators with the function η,*

(iv) *the pair $(X; \ell_\infty(Y))$ has the BPBp for compact operators with the function η.*

To finish this chapter, we get the following consequences of the last theorem.

Let K be a compact Hausdorff topological space, let μ be a positive measure and let ν be a σ-finite positive measure.

(a) *If Y has property β, then $(X; L_\infty(\nu, Y))$ and $(X; C(K, Y))$ have the BPBp for compact operators.*

(b) *If Y has the AHSP, then so do $L_\infty(\nu, Y)$ and $C(K, Y)$.*

(c) *For $1 \leqslant p < \infty$, if $\ell_p(Y)$ has the AHSP and $L_1(\mu)$ is infinite-dimensional, then $L_p(\mu, Y)$ has the AHSP.*

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Finally, in Chapter 4, we study different type BPB properties adapted for multilinar mappings.

Let $X_1, \ldots X_N$ and Y be Banach spaces. We say that $(X_1, \ldots, X_N; Y)$ has the Bishop-Phelps-Bollobás property for multilinear mappings (BPBp for multilinear mappings) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever A is an N-linear mapping from $X_1 \times \ldots X_N$ into Y with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfy

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

there are a new N-linear mapping $B : X_1 \times \ldots \times X_N \longrightarrow Y$ with $\|B\| = 1$ and a new element $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ such that

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \quad \textit{and} \quad \|B - A\| < \varepsilon.$$

When it is of interest we can emphasize the degree of the multilinear mapping by saying that $(X_1, \ldots, X_N; Y)$ has the BPBp for N-linear mappings instead of the BPBp for multilinear mappings. We may also define the BPBp for symmetric multilinear mappings when we consider A and B both elements in $\mathcal{L}_s(^N X; Y)$, the set of all N-linear symmetric mappings. In this case, we say that $(^N X; Y)$ has the BPBp for symmetric multilinear mappings. When $Y = \mathbb{K}$, we denote the BPBp for $(X_1, \ldots, X_N; \mathbb{K})$ just by $(X_1, \ldots, X_N)$ and we say that $(X_1, \ldots, X_N)$ has the BPBp for N-linear forms. Analogously, we define the BPBp for homogeneous polynomials.

We say that the pair $(X; Y)$ has the Bishop-Phelps-Bollobás property for N-homogeneous polynomials if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever P is an N-homogeneous polynomial with $\|P\| = 1$ and $x_0 \in S_X$ satisfy

$$\|P(x_0)\| > 1 - \eta(\varepsilon),$$

there are a new N-homogeneous polynomial Q with $\|Q\| = 1$ and $x_1 \in S_X$ such that

$$\|Q(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|Q - P\| < \varepsilon.$$

In this chapter, we extend some known results about norm attaining multilinear mappings to the BPBp as for example the simple following observation which is came from the linear case:

Let $X, X_1, \ldots, X_N$ and Y be finite dimensional Banach spaces. Then

(i) *$(X_1, \ldots, X_N; Y)$ has the BPBp for multilinear mappings,*

(ii) *$({}^N X; Y)$ has the BPBp for symmetric multilinear mappings,*

(iii) *$(X; Y)$ has the BPBp for N-homogeneous polynomials.*

We also prove an stability result which says that we can pass from $(N+1)$-degree to a N-degree in the BPBp:

If $(X_1, \ldots, X_N,\ X_{N+1}; Y)$ has the BPBp for $(N+1)$-linear mappings, then $(X_1, \ldots,\ X_N; Y)$ has the BPBp for N-linear mappings.

We also show that

If $N \geqslant 2$ and $X_1, \ldots, X_N$ are Banach spaces where X_N is a uniformly convex space, then $(X_1, \ldots, X_N)$ has the BPBp for N-linear mappings if and only if $(X_1, \ldots, X_{N-1}; X_N^*)$ has the BPBp for $(N-1)$-linear mappings.

It is show that we may pass from the vector-valued case to the scalar-valued case in the Bishop-Phelps-Bollobás property for multilinear mappings. Moreover, we prove the following characterization:

If Y has property β, then the N-tuple $(X_1, \ldots, X_N)$ has the BPBp if and only if $(X_1, \ldots, X_N; Y)$ has the BPBp.

We also study the BPBp for compact multilinear mappings in the same vein of the previous chapter.

Let Y be a predual of an L_1-space. Suppose that the N-tuple $(X_1, \ldots, X_N)$ has the BPBp for multilinear forms. Then the pair $(X_1, \ldots, X_N; Y)$ has the BPBp for compact multilinear mappings.

But as [3], we get a little more than that:

Given $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfy

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

there are $B \in \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|B\| = 1$ and $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ such that $\dim(B(X_1 \times \ldots \times X_N)) < \infty$,

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \quad \textit{and} \quad \|B - A\| < \varepsilon.$$

We provide examples of spaces satisfying such property.

For a predual Y of an L_1-space, $(X, Z; Y)$ has the BPBp for compact bilinear mappings in the following cases.

- **(a)** *For the complex Banach spaces $X = C_0(L)$ and $Z = C_0(K)$ where L and K are locally compact topological Hausdorff spaces.*
- **(b)** *For $X = L_1(\mu)$ and $Z = c_0$.*
- **(c)** *For X and Z uniformly convex Banach spaces.*

In the third section of this chapter, we characterize the pair $(\ell_1(X), Y)$ to have BPBp for bilinear forms. We prove that

The pair $(\ell_1(X), Y)$ has the BPBp for bilinear forms if and only if for every $\varepsilon > 0$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that given sequences

$(T_k)_k \subset \mathcal{L}(X;Y)$ with $\|T_k\| = 1$ for every k, $(x_k)_k \subset S_X$ and a convex series $\sum_{k=1}^{\infty} \alpha_k$ such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k(x_k) \right\| > 1 - \eta(\varepsilon),$$

there exit a subset $A \in \mathbb{N}$, $y^* \in S_{Y^*}$ and sequences $(S_k)_k \subset \mathcal{L}(X;Y)$ with $\|S_k\| = 1$ for every k, $(z_k)_k \subset S_X$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,

(2) $\|z_k - x_k\| < \varepsilon$ *and* $\|S_k - T_k\| < \varepsilon$ *for all* $k \in A$, *and*

(3) $y^*(S_k(x_k)) = 1$ *for every* $k \in A$.

Also we show that the pair (X, H) has the BPBp for bilinear forms if and only if the pair (X, H) has the generalized AHSP where H is a Hilbert space.

Still on this chapter, we study the numerical radius on the set of all multilinear mappings defined in $L_1(\mu)$, where μ is an arbitrary measure. We prove that for every $A \in \mathcal{L}(^N L_1(\mu); L_1(\mu))$, its numerical radius and its norm coincide.

To finish the chapter, we study the Bishop-Phelps-Bollobás property for numerical radius for multilinear mappings. It is shown that if X is a finite-dimensional Banach space, then X satisfies this property. On the other hand, $L_1(\mu)$ fails it although $L_1(\mu)$ has the property in the operator case for every measure μ. We also prove that if a c_0 or a ℓ_1-sum satisfies it, then each component of the direct sum also satisfies the BPBp-nu for multilinear mappings.

The contents of this chapter was partially accepted:

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We finish this dissertation by presenting a list of open problems with the intention to expand new horizons to continue working on the subject. Also we present the list of published articles derived from this thesis. Moreover, we present tables which summary those pairs of classical Banach spaces satisfying the Bishop-Phelps-Bollobás property with the purpose to put the reader in the current scenario on this topic.

Contents

Chapter 1

Introduction

1.1 Preliminaries

We start with some definitions, conventions and notation which will be used throughout this dissertation. The set of positive integers is denoted by $\mathbb{N}$. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. We denote by $\mathbb{K}$ a field that can be either $\mathbb{R}$ or $\mathbb{C}$. We call the elements of $\mathbb{K}$ by scalars.

If X is a Banach space, then X^* denotes its topological dual space. We denote by B_X and S_X the unit ball and the unit sphere of the Banach space X, respectively, i.e.,

$$B_X = \{x \in X : \|x\| \leqslant 1\} \text{ and } S_X = \{x \in X : \|x\| = 1\}.$$

We denote by $\mathcal{L}(X;Y)$ the set of all linear continuous operators from X to Y. When $Y = \mathbb{K}$, a linear operator from X into $\mathbb{K}$ is called linear functional. We define the norm of an operator $T \in \mathcal{L}(X;Y)$ by

$$\|T\| = \sup_{x \in B_X} \|T(x)\| = \sup_{x \in S_X} \|T(x)\|.$$

We say that $T \in \mathcal{L}(X;Y)$ *attains its norm* or that it is *norm attaining* if there exists some $x_0 \in B_X$ such that $\|T\| = \|T(x_0)\|$. We denote by $\mathrm{NA}(X;Y)$ the set of all norm attaining operators.

Given $T \in \mathcal{L}(X;Y)$, we denote by $T^* \in \mathcal{L}(Y^*;X^*)$ its *adjoint operator* which is defined by

$$T^*y^*(x) = y^*(Tx) \quad (x \in X,\ y^* \in Y^*).$$

We recall that $\|T^*\| = \|T\|$ and that T^* is weak*-to-weak* continuous for every $T \in \mathcal{L}(X;Y)$.

A linear operator T is said to be a *compact operator* if $T(B_X)$ is relatively compact in Y. We denote by $\mathcal{K}(X;Y)$ as the set of all compact linear operators from X into Y. Recall that all finite rank operators in $\mathcal{L}(X;Y)$ are compact. By Schauder's Theorem, an operator in $\mathcal{L}(X;Y)$ is compact if and only if its adjoint is compact.

We also need some notation about direct sums. Let X be a Banach space, $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. By $\ell_p^m(X)$ we denote the ℓ_p-sum of m copies of X and $\ell_p(X)$ is the ℓ_p-sum of a countable infinitely many copies of X. We denote by $c_0(X)$ the c_0-sum of a countable infinitely many copies of X.

If (Ω, Σ, μ) is a positive measure space, $L_p(\mu, X)$ is the space of all strongly measurable functions $f : \Omega \longrightarrow X$ such that $\|f\|^p$ is integrable for $1 \leqslant p < \infty$ or f is essentially bounded for $p = \infty$, endowed with the natural corresponding p-norm. For $1 \leqslant p < \infty$, let p^* be the conjugate exponent, i.e. $p^* = \infty$ for $p = 1$ and p^* is determined by the equation $1/p + 1/p^* = 1$ for $1 < p < \infty$.

Let $X_1, \ldots, X_N$ and Y be Banach spaces. We denote by $\mathcal{L}(X_1, \ldots, X_N; Y)$ the set of all bounded N-linear mappings defined from $X_1 \times \ldots \times X_N$ into Y. We use the letters A, B, C or D to denote members of

$\mathcal{L}(X_1, \dots, X_N; Y)$. If $A \in \mathcal{L}(X_1, \dots, X_N; Y)$, then we define its norm by

$$\|A\| := \sup \{\|A(x_1, \dots, x_N)\| : (x_1, \dots, x_N) \in S_{X_1} \times \dots \times S_{X_N}\}.$$

Analogous to the operator case, we say that $A \in \mathcal{L}(X_1, \dots, X_N; Y)$ *attains its norm* or that it is *norm attaining* when there exists some point $(x_1^0, \dots, x_N^0) \in S_{X_1} \times \dots \times S_{X_N}$ such that $\|A(x_1^0, \dots, x_N^0)\| = \|A\|$. We denote by $\text{NA}(\mathcal{L}(X_1, \dots, X_N; Y)$ the set of all norm attaining multilinear mappings.

When $X_1 = \dots = X_N = X$, we write $\mathcal{L}(^N X; Y)$ and if moreover $Y = \mathbb{K}$ we write just $\mathcal{L}(^N X)$. When $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, \dots, X_N)$ to specify the space of all multilinear mappings from $X_1 \times \dots \times X_N$ into $\mathbb{K}$. We say that $A \in \mathcal{L}(^N X; Y)$ is *symmetric* whenever

$$A(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = A(x_1, \dots, x_N) \quad (x_1, \dots, x_N \in X)$$

for every permutation σ on $\{1, \dots, N\}$. We denote by $\mathcal{L}_s(^N X; Y)$ the set of all symmetric multilinear mappings from $X \times \dots \times X$ into Y.

A mapping $P : X \longrightarrow Y$ is said to be an *N-homogeneous polynomial* if there exists some symmetric N-linear mapping $\hat{A} \in \mathcal{L}_s(^N X; Y)$ such that

$$P(x) = \hat{A}(x, \dots, x)$$

for all $x \in X$. We denote by $\mathcal{P}(^N X; Y)$ the set of all continuous N-homogeneous polynomials from X into Y. In this space, we define the norm of P by

$$\|P\| := \sup_{x \in S_X} \|P(x)\|.$$

We say that an N-homogeneous polynomial P *attains its norm* or that it is *norm attaining* if there exists some $x_0 \in S_X$ such that $\|P(x_0)\| = \|P\|$. We denote by $\text{NA}(\mathcal{P}(^N X; Y))$ the set of all norm attaining N-homogeneous polynomials.

We say that the N-linear mapping $A \in \mathcal{L}(X_1, \ldots, X_N; Y)$ is *compact* if $A(B_{X_1} \times \ldots \times B_{X_N})$ is a precompact set in Y. We denote by $\mathcal{K}(X_1, \ldots, X_N; Y)$ the set of all N-linear compact mappings from $X_1 \times \ldots \times X_N$ into Y.

1.2 Historical background

This work was motivated by the Bishop-Phelps and the Bishop-Phelps-Bollobás theorems. In this introductory chapter we give some important historical and motivator facts about these two remarkable results and we mention some significant current research on that area.

It is well known that, in general, there exist functionals on Banach spaces which do not attain their norm. On the other hand, James proved one of the most famous theorem in the norm attaining theory which gives a characterization of reflexive Banach spaces as those Banach spaces in which all bounded linear functionals are norm attaining [39, 40].

Around the same time, Bishop and Phelps showed that every continuous linear functional can be approximated by norm attaining ones, i.e., given $x^* \in X^*$ and $\varepsilon > 0$, there are some $x_0^* \in X^*$ and $x_0 \in S_X$ such that $\|x_0^*\| = |x_0^*(x_0)|$ and $\|x_0^* - x^*\| < \varepsilon$. This theorem is known nowadays as the Bishop-Phelps theorem [14]. At the end of that article, they asked if this result could be extended to bounded linear operators, i.e., is it true that the set $\mathrm{NA}(X; Y)$ is dense in $\mathcal{L}(X; Y)$? The answer for that arises two years later with Lindenstrauss [52]. He showed that, in general, there is no version of the Bishop-Phelps theorem for operators. Nevertheless, he gave some particular cases in which such a density holds. For example, if X is any Banach space and Y is a closed subspace of ℓ_∞ containing the canonical copy of c_0, then the set $\mathrm{NA}(X; Y)$ is dense in $\mathcal{L}(X; Y)$. Also he showed that the set of all bounded linear operators such that its second adjoint operator attain the norm in X^{**} is dense in $\mathcal{L}(X;Y)$. In particular, if X is reflexive, then $\mathrm{NA}(X;Y)$ is dense in $\mathcal{L}(X;Y)$ for all Banach space Y.

After that, many positive results were obtained by using techniques which depend strongly on the particular involved Banach spaces. For example, Johnson and Wolfe proved that the set $\mathrm{NA}(C(K);C(S))$ is dense (considering just real functions) in the space $\mathcal{L}(C(K);C(S))$ for every compact and Hausdorff topological spaces K and S [43, Theorem 1]. Also, Iwanik showed the corresponding result for $L_1(\mu)$ for a finite positive Borel measure μ on the unit interval of $\mathbb{R}$ [38, Theorem 2] and Payá and Saleh proved that $\mathrm{NA}(L_1(\mu);L_\infty(\nu))$ is dense in $\mathcal{L}(L_1(\mu);L_\infty(\nu))$ for every measure μ and every localizable measure ν [58, Theorem 1]. There are much more results in this direction and we suggest the detailed survey [1] for more information about it.

In 1970, Bollobás proved an improved version of the Bishop-Phelps theorem which is nowadays known as the Bishop-Phelps-Bollobás theorem. Since this result is very significant to our work, we highlight it as following.

Theorem 1.2.1 (The Bishop-Phelps-Bollobás theorem [15], [22])**.** Let X be a Banach space. Let $\varepsilon \in (0,2)$ and suppose that $x_0^* \in B_{X^*}$ and $x_0 \in B_X$ satisfy

$$\operatorname{Re} x_0^*(x_0) > 1 - \frac{\varepsilon^2}{2}. \tag{1.1}$$

Then, there are $x_1^* \in S_{X^*}$ and $x_1 \in S_X$ such that

$$|x_1^*(x_1)| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|x_1^* - x_0^*\| < \varepsilon.$$

Bollobás' result is a refinement of the Bishop-Phelps theorem since given $x^* \in B_{X^*}$ and $\varepsilon > 0$, there exists some $x_0 \in B_X$ satisfying (1.1). Applying his result, we get a new bounded linear functional x_1^* attaining its norm and it is near to x_0^*. Note that the only difference between these two theorems is that in the Bollobás' result both functionals and points where they almost attain their norm can be simultaneously approximated by norm attaining functionals and points where they attain their norm.

The Bishop-Phelps-Bollobás theorem is very useful in the study of numerical range theory. To see its application in many results, we suggest the monographs [16, 17] due by F. F. Bonsall and J. Duncan.

Since there is no version of the Bishop-Phelps theorem for bounded linear operators and the Bishop-Phelps-Bollobás theorem implies it, there is no version of the Bishop-Phelps-Bollobás theorem for this class of functions either. For that reason, in 2008, M. Acosta, R. Aron, D. García and M. Maestre introduced the Bishop-Phelps-Bollobás property which is defined as follows.

Definition 1.2.2 (The Bishop-Phelps-Bollobás property [2])**.** A pair of Banach spaces $(X;Y)$ has the *Bishop-Phelps-Bollobás property* (BPBp for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X;Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X;Y)$ with $\|S\| = 1$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon. \tag{1.2}$$

In this case, we say that the pair $(X;Y)$ has the BPBp with function $\varepsilon \longmapsto \eta(\varepsilon)$.

With this new notation, the Bishop-Phelps-Bollobás theorem just says that the pair $(X, \mathbb{K})$ has the BPBp for every Banach space X. There has been an extensive research on this topic in the same spirit of Lindenstrauss, that is, analyzing the conditions that the Banach spaces X and Y must satisfy to get a Bishop-Phelps-Bollobás type theorem for bounded linear operators from X into Y.

In [2], the authors showed that the pair $(X;Y)$ has the BPBp whenever $\dim(X) < \infty$ and $\dim(Y) < \infty$ [2, Proposition 2.4] or when Y has property β (see Definition 2.1.5) for any Banach space X [2, Theorem 2.2]. They also gave a characterization for the pair $(\ell_1;Y)$ via the geometry of the Banach space Y. More precisely, the pair $(\ell_1;Y)$ has the BPBp if and only if Y has the *approximate hyperplane series property* (AHSP, for short) (see Definition 4.2.1) [2, Theorem 4.1]. They proved that finite-dimensional Banach spaces, $L_1(\mu)$-spaces, $C(K)$-spaces and uniformly convex Banach spaces all satisfy this property [2, Proposition 3.5, 3.6, 3.7 and 3.8, respectively]. Actually, Kim, Lee and Martín defined a more general property [50, see Definition 4], the *generalized AHSP* for a pair $(X;Y)$, which characterizes when the pair $(\ell_1(X);Y)$ has the BPBp.

It is worth to mention that if the pair $(X;Y)$ has the BPBp then the set $\text{NA}(X;Y)$ is dense in $\mathcal{L}(X;Y)$. So the BPBp is stronger than the denseness of norm attaining operators. Nevertheless, the study of the BPBp is not merely a trivial extension of the corresponding study of the density of norm attaining operators. From the seminal paper [2] we know that is not always true that the density of $\text{NA}(X;Y)$ implies the BPBp for the pair $(X;Y)$ (see [2, Remark 2.5]). Actually, there are some other examples showing that but maybe one of the more remarkable is that one found in [11, Example 4.1]: there exists a sequence of two-dimensional polyhedral spaces such that, writing $\mathcal{Y}$ to denote its c_0-sum, the pair $(\ell_1^2;\mathcal{Y})$ fails the BPBp, we have the equality $\text{NA}(\ell_1^2;Y) = \mathcal{L}(\ell_1^2;Y)$ for every Banach space Y and, on the other hand, we have that the set $\text{NA}(X;\mathcal{Y})$ is dense in $\mathcal{L}(X;\mathcal{Y})$ for every Banach space X.

Many authors have studied the Bishop-Phelps-Bollobás property over the last 8 years and there is a long literature about this topic. This motivated us to create a table containing all known pairs of classic Banach spaces which satisfy this property and we invite the reader to take a look at it before going on into the next chapters. We did not include any references on it but all the remarkable results to this dissertation are referred properly over the next pages. One may find this table in the last page of this monograph.

Now let us give information about how this work is organized. Chapter 2 is devoted to the study of three similar properties to the BPBp. In the first section we start with the Bishop-Phelps-Bollobás *point* property (BPBpp for short). This is stronger than the BPBp since its definition is almost the same as Definition 1.2.2 but without changing the point x_0, i.e., if a pair $(X; Y)$ has this property, the BPBp conditions (1.2) are satisfied with a new operator S and $x_1 = x_0$. We will show positive results about it as well as cases in which it does not happen. In the second section, we study properties 1 and 2. In these cases, we are not changing the operator T in Definition 1.2.2, that is, given $\varepsilon > 0$ there exists some positive real number η such that whenever $T \in \mathcal{L}(X; Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy $\|T(x_0)\| > 1 - \eta$, we get a new vector such that T attains its norm at this vector and it is near to x_0. In property 1 we are considering this η depending not only on $\varepsilon > 0$ but also on a fixed operator T. On the other hand, property 2 studies the uniform version of property 1 in which consider η depending only on ε. As we may see, the behavior of property 2 is very different from the BPBpp although these two properties may seem to be very similar at first sight. We finish the chapter by studying the Bishop-Phelps-Bollobás point property on complex Hilbert spaces and its analogous for numerical radius.

Chapter 3 is dedicated to the study of the compact version of the Bishop-Phelps-Bollobás property. This means that we study the conditions on the domain spaces X and the range spaces Y such that given $\varepsilon > 0$, there exists a positive real number $\eta(\varepsilon) > 0$ in such way that whenever $x_0 \in S_X$ and a *compact* operator $T \in \mathcal{L}(X; Y)$ with $\|T\| = 1$ satisfy the relation $\|T(x_0)\| > 1 - \eta(\varepsilon)$, there are a new point $x_1 \in S_X$

and a new *compact* operator $S \in \mathcal{L}(X;Y)$ such that they satisfy the Bishop-Phelps-Bollobás conditions (1.2). We call this property the *BPBp for compact operators*. The strategy here is to present some technical results which allow us to carry the BPBp for compact operators from sequence to function spaces and so apply them to concrete Banach spaces.

In Chapter 4 we define and work on the BPBp, the BPBp for numerical radius and the generalized AHSP for multilinear mappings. We extend some known results about norm attaining operator to the multilinear and homogeneous polynomial cases and we characterize the pair $(\ell_1(X), Y)$ to have the BPBp for bilinear forms. We will calculate the numerical radius of a multilinear mapping defined on $L_1(\mu)$ and we present some negative results about the BPBp for numerical radius in the multilinear case.

Chapter 2

Some versions of Bishop-Phelps-Bollobás properties

As the title of this chapter suggests, we study some similar properties to the Bishop-Phelps-Bollobás property (see Definition 1.2.2). We start with a stronger property called the Bishop-Phelps-Bollobás *point* property and we will refer to it as the BPBpp to simplify the notation. We give a characterization for the uniformly smooth Banach spaces via this property, that is, we prove that a Banach space X is uniformly smooth if and only if the pair $(X; \mathbb{K})$ satisfies the BPBpp. From there on, we prove some positive results by assuming that the domain space is uniformly smooth. We also give a example of a pair of Banach spaces $(X; Y)$ such that X is uniformly smooth but the pair $(X; Y)$ fails to have the BPBpp. We finish this section by studying this property for bilinear mappings.

Section 2 is devoted to the study of two more properties which we call as properties 1 and 2. As we mention in the first chapter, in both properties we do not change the initial operator in Definition 1.2.2 of the Bishop-Phelps-Bollobás property in the sense that $S = T$ satisfies the BPBp conditions (1.2). In property 1 we consider the positive real number η depending not only on $\varepsilon > 0$ but also on a fixed operator T. On the other hand, property 2 is defined as the uniform version of property 1 where η depends only on ε. Although property 2 and the BPBpp seem to be very similar at first sight, their behavior is totally different from each other as we may see later.

In the third section we study a BPBpp version for numerical radius (BPBpp-nu, for short) and we show that the complex Hilbert spaces H satisfy it. Moreover, we study the BPBpp and BPBpp-nu for concrete operators defined in H as self-adjoint, anti-symmetric, unitary and normal operators.

In the last section we study the stability of the Bishop-Phelps-Bollobás property for some absolute sums.

2.1 The BPBpp

In this section, we present a property for a pair of Banach spaces $(X; Y)$ which insures that it is possible to approximate an operator from X into Y by operators which attain their norm at the same point where the original operator almost attains its norm. In other words, we do not change the initial point which T almost attains its norm in the BPBp (see Definition 1.2.2). We call it as the Bishop-Phelps-Bollobás *point* property. We also study it for bilinear mappings.

2.1.1 The BPBpp for operators

We start with the operators version of the Bishop-Phelps-Bollobás point property. The definition is the following.

Definition 2.1.1 (The BPBpp)**.** A pair of Banach spaces $(X; Y)$ is said to have the *Bishop-Phelps-Bollobás point property* (BPBpp, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X;Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $S \in \mathcal{L}(X;Y)$ with $\|S\| = 1$ such that

$$\|S(x)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

In this case, we say that the pair $(X;Y)$ has the BPBpp with the function $\varepsilon \longmapsto \eta(\varepsilon)$.

Note that the BPBpp is stronger than the BPBp by definition in the sense that if the pair $(X;Y)$ has the BPBpp then it has the BPBp. We also observe that it is possible to choose T in the above definition with $\|T\| \leqslant 1$ instead of $\|T\| = 1$ by changing the parameters (see Remark 3.2.1).

As our first result in this section we give a characterization for the uniformly smooth Banach spaces via the BPBpp. Before we do that, let us recall the definition of uniform smoothness. A Banach space X is said to be *uniformly smooth* if the limit

$$\lim_{t \to 0} \frac{\|z + tx\| - 1}{t} \tag{2.1}$$

exists uniformly for all $x \in B_X$ and $z \in S_X$. We also need the concept of uniformly convex Banach spaces. A Banach space X is said to be *uniformly convex* if given $\varepsilon > 0$, there exists a positive real number $\delta(\varepsilon) > 0$ such that

$$\left\| \frac{x_1 + x_2}{2} \right\| > 1 - \delta(\varepsilon) \Rightarrow \|x_1 - x_2\| < \varepsilon$$

for all $x_1, x_2 \in S_X$. If X is a uniformly convex or a uniformly smooth Banach space, then X is reflexive. Also X is uniformly smooth if and only if X^* is uniformly convex.

In what follows we use the ideas from [55, Proposition 4.10]. By [31, Theorem V.9.5, p. 447] we have that

$$\lim_{t \to 0^+} \frac{\|z + tx\| - 1}{t} = \max\{\operatorname{Re} z^*(x) : \|z^*\| = z^*(z) = 1\}. \tag{2.2}$$

Proposition 2.1.2. The Banach space X is uniformly smooth if and only if the pair $(X; \mathbb{K})$ has the BPBpp.

Proof. Suppose that X is uniformly smooth. Then X^* is uniformly convex. So, given $\varepsilon > 0$, $x_0^* \in B_{X^*}$ and $x_0 \in S_X$ such that

$$|x_0^*(x_0)| = |x_0(x_0^*)| > 1 - \eta(\varepsilon),$$

there exists $x_1^* \in S_{X^*}$ such that $|x_0(x_1^*)| = |x_1^*(x_0)| = 1$ and $\|x_1^* - x_0^*\| < \varepsilon$ by [46, Theorem 2.1] (see Theorem 2.2.1). This proves that $(X; \mathbb{K})$ has the BPBpp.

Conversely, let $\varepsilon > 0$ and consider $\eta(\varepsilon) > 0$ be the function in the definition of the BPBpp for the pair $(X; \mathbb{K})$. We prove that the limit (2.1) exists uniformly for all $x \in B_X$ and $z \in S_X$. Let $x \in B_X$, $z \in S_X$ and $0 < t < \frac{\eta(\varepsilon)}{2}$. Define

$$x_t := \frac{z + tx}{\|z + tx\|} \in S_X$$

and take $x_t^* \in S_{X^*}$ to be such that $x_t^*(x_t) = 1$. Since $x_t^*(z) = \|z + tx\| - tx_t^*(x)$, we have that

$$\operatorname{Re} x_t^*(z) > \|z\| - 2t\|x\| > 1 - \eta(\varepsilon).$$

From the assumption that the pair $(X;\mathbb{K})$ has the property, there exists $z_t^* \in S_{X^*}$ such that

$$\operatorname{Re} z_t^*(z) = 1 \quad \text{and} \quad \|z_t^* - x_t^*\| < \varepsilon.$$

Now, by the definition of the element x_t, we have that

$$\frac{\|z+tx\|-1}{t} = \frac{x_t^*(z+tx)-1}{t} \leqslant \operatorname{Re} x_t^*(x)$$

and by (2.2) we have that

$$\lim_{t\to 0^+} \frac{\|z+tx\|-1}{t} \geqslant \operatorname{Re} z_t^*(x).$$

Moreover, for $t_1 \geqslant t_2 > 0$, we have that

$$\|t_1 z + t_1 t_2 x\| \leqslant \|t_2 z + t_1 t_2 x\| + \|(t_1 - t_2) z\| = \|t_2 z + t_1 t_2 x\| + t_1 - t_2$$

and so

$$t_1(\|z + t_2 x\| - 1) \leqslant t_2(\|z + t_1 x\| - 1),$$

which implies that

$$\frac{\|z+t_2 x\|-1}{t_2} \leqslant \frac{\|z+t_1 x\|-1}{t_1}.$$

This means that the function $t \mapsto \frac{\|z+tx\|-1}{t}$ decreases when $t > 0$ goes to zero. Hence, we have

$$\begin{aligned} 0 \leqslant \frac{\|z+tx\|-1}{t} - \lim_{t\to 0^+} \frac{\|z+tx\|-1}{t} \\ \leqslant \operatorname{Re} x_t^*(x) - \operatorname{Re} z_t^*(x) \leqslant \|x_t^* - z_t^*\| < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, X is uniformly smooth. □

Remark 2.1.3. We observe that we may take x_0^* in B_{X^*} instead of S_{X^*} in the proof of Proposition 2.1.2 thanks to [46, Theorem 2.1]. So we can rewrite this result as follows: X is a uniformly smooth Banach space if and only if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that if $x_0^* \in B_{X^*}$ and $x_0 \in S_X$ are such that $|x_0^*(x_0)| > 1 - \eta(\varepsilon)$, then there exists $x_1^* \in S_{X^*}$ satisfying $|x_1^*(x_0)| = 1$ and $\|x_1^* - x_0^*\| < \varepsilon$.

As a consequence of Proposition 2.1.2, we have the following examples.

(a) If H is a Hilbert space, then the pair $(H; \mathbb{K})$ has the BPBpp.

(b) The pair $(L_p(\mu); \mathbb{K})$ has the BPBpp for every positive measure μ and every $1 < p < \infty$.

Now we start to treat the vector valued case. The next proposition is the reason that we have to assume from now on that the domain space X is uniformly smooth in order to get more examples of pairs $(X; Y)$ satisfying the BPBpp.

Proposition 2.1.4. Let X be a Banach space. Suppose that there is some Banach space Y such that the pair $(X; Y)$ has the BPBpp. Then X is uniformly smooth.

Proof. Let $\varepsilon > 0$ be given. Assume that $(X; Y)$ has the BPBpp with $\eta(\varepsilon) > 0$. We show that the pair $(X; \mathbb{K})$ has the property and we apply Proposition 2.1.2 to get that X is uniformly smooth. Let $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ be such that

$$\operatorname{Re} x_0^*(x_0) > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Define $T : X \longrightarrow Y$ by $T(x) := x_0^*(x)y_0$ for every $x \in X$ and for a fixed $y_0 \in S_Y$. Since $\|T\| = \|x_0^*\| = 1$ and

$$\|T(x_0)\| = |x_0^*(x_0)| > 1 - \eta\left(\frac{\varepsilon}{2}\right),$$

there exists $S \in \mathcal{L}(X;Y)$ with $\|S\| = 1$ such that

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \frac{\varepsilon}{2}.$$

Take $y_0^* \in S_{Y^*}$ so that $\operatorname{Re} y_0^*(S(x_0)) = |y_0^*(S(x_0))| = \|S(x_0)\| = 1$ and define $x_1^* := S^* y_0^* \in X^*$. Then we see that

$$1 \geqslant \|x_1^*\| \geqslant \operatorname{Re} x_1^*(x_0) = \operatorname{Re} S^* y_0^*(x_0) = \operatorname{Re} y_0^*(S(x_0)) = 1.$$

Hence $x_1^* \in S_{X^*}$ and it attains its norm at x_0. It remains to prove that $\|x_1^* - x_0^*\| < \varepsilon$. By using that $\|S - T\| < \frac{\varepsilon}{2}$, we get that

$$\|x_1^* - y_0^*(y_0)x_0^*\| = \|x_1^* - T^* y_0^*\| = \|S^* y_0^* - T^* y_0^*\| \leqslant \|S^* - T^*\| < \frac{\varepsilon}{2}$$

and since $\operatorname{Re} x_1^*(x_0) = 1$,

$$\operatorname{Re}(1 - y_0^*(y_0)) \leqslant \operatorname{Re}(x_1^*(x_0) - y_0^*(y_0)x_0^*(x_0)) \leqslant \|x_1^* - y_0^*(y_0)x_0^*\| < \frac{\varepsilon}{2}.$$

We can use these two inequalities to get

$$\|x_1^* - x_0^*\| \leqslant \|x_1^* - y_0^*(y_0)x_0^*\| + \|y_0^*(y_0)x_0^* - x_0^*\| < \varepsilon.$$

This proves that the pair $(X;\mathbb{K})$ has the BPBpp. Applying Proposition 2.1.2, X is uniformly smooth as desired. □

In particular, all the pairs $(X;Y)$ whenever X is not uniformly smooth, for example $X = c_0$ or $X = \ell_1$, do not have the BPBpp for any Banach space Y. For this reason, we have to assume that the domain space X is uniformly smooth if we aspire to get more examples of pairs $(X;Y)$ satisfying the property. In the next result, we prove that for such X whenever Y has the property β, the pair $(X;Y)$ satisfies the BPBpp. To do so, we use similar arguments to [2, Theorem 2.2] and [54, Theorem 4.1]. Let us remember what property β means.

Definition 2.1.5. A Banach space Y is said to have *property β with constant* $0 \leqslant \rho < 1$ if there are sets $\{y_i : i \in \Lambda\} \subset S_Y$ and $\{y_i^* : i \in \Lambda\} \subset S_{Y^*}$ such that

(i) $y_i^*(y_i) = 1$ for all $i \in \Lambda$,

(ii) $|y_i^*(y_j)| \leqslant \rho < 1$ for all $i, j \in \Lambda$ with $i \neq j$ and

(iii) $\|y\| = \sup_{i\in\Lambda} |y_i^*(y)|$ for all $y \in Y$.

The Banach spaces $c_0(\Lambda)$ and $\ell_\infty(\Lambda)$ are the most typical examples of spaces satisfying property β by taking their canonical basis and biorthogonal functionals. By [2, Theorem 2.2], when Y has property β, the pair $(X; Y)$ has the BPBp for every Banach space X. We have the analogous result for the BPBpp.

Proposition 2.1.6. Let X and Y be Banach spaces. Assume that X is uniformly smooth and that Y has property β. Then the pair $(X; Y)$ has the BPBpp.

Proof. Let $\varepsilon > 0$ be given. Proposition 2.1.2 says that there exists a positive real number $\eta(\varepsilon) > 0$ such that whenever $x_0^* \in B_{X^*}$ and $x_0 \in S_X$ satisfy $|x_0^*(x_0)| > 1 - \eta(\varepsilon)$, there is $x_1^* \in S_{X^*}$ such that $|x_1^*(x_0)| = 1$ and $\|x_1^* - x_0^*\| < \varepsilon$. Choose $\xi > 0$ such that

$$1 + \rho\left(\frac{\varepsilon}{4} + \xi\right) < \left(1 + \frac{\varepsilon}{4}\right)(1 - \xi). \tag{2.3}$$

This gives that $\xi < \frac{\varepsilon}{4}$. Let $T \in \mathcal{L}(X; Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \eta(\xi).$$

Since Y has property β, there exists some $\alpha_0 \in \Lambda$ such that $y_{\alpha_0}^*(T(x_0)) = (T^* y_{\alpha_0}^*)(x_0) > 1 - \eta(\xi)$. So there is $x_1^* \in S_{X^*}$ such that

$$|x_1^*(x_0)| = 1 \quad \text{and} \quad \|x_1^* - T^* y_{\alpha_0}^*\| < \xi.$$

Define $S : X \longrightarrow Y$ by

$$S(x) := T(x) + \left[\left(1 + \frac{\varepsilon}{4}\right) x_1^*(x) - T^* y_{\alpha_0}^*(x)\right] y_{\alpha_0} \quad (x \in X).$$

Then $S \in \mathcal{L}(X; Y)$ and $\|S - T\| < \frac{\varepsilon}{4} + \xi < \frac{\varepsilon}{2}$. Also, we have

$$S^* y^* = T^* y^* + y^*(y_{\alpha_0}) \left[\left(1 + \frac{\varepsilon}{4}\right) x_1^* - T^* y_{\alpha_0}^*\right] \quad (y^* \in Y^*) .$$

Note that $\|S\| = \sup_{\alpha \in \Lambda} \|S^* y_\alpha^*\|$. On the one hand, we have that $\|S^* y_{\alpha_0}^*\| = 1 + \frac{\varepsilon}{4}$ and on the another hand, for $\alpha \neq \alpha_0$, we have that

$$\|S^* y_\alpha^*\| \leqslant 1 + \rho\left(\frac{\varepsilon}{4} + \xi\right) < \left(1 + \frac{\varepsilon}{4}\right)(1 - \xi) < 1 + \frac{\varepsilon}{4}.$$

This shows that S^* attains its norm at $y_{\alpha_0}^* \in S_{Y^*}$. Observe that $S^* y_{\alpha_0}^* = \|S^*\| x_1^*$ and that $x_1^*(x_0) = 1$, so $\|S(x_0)\| = \|S\|$. So, if $U := \frac{S}{\|S\|}$, then $\|U(x_0)\| = 1$ and $\|U - T\| \leqslant 2\|S - T\| < \varepsilon$. Thus the pair $(X; Y)$ has the BPBpp. □

As a consequence of Proposition 2.1.6, we have the following examples.

(a) If H is a Hilbert space, then the pairs $(H; c_0)$ and $(H; \ell_\infty)$ have the BPBpp.

(b) The pairs $(L_p(\mu); c_0)$ and $(L_p(\mu); \ell_\infty)$ have the BPBpp for every positive measure μ and every $1 < p < \infty$.

(c) If X is uniformly smooth and Y is a closed subspace of ℓ_∞ containing the canonical copy of c_0, then $(X; Y)$ has the uniform BPBpp.

In fact, when the domain is a Hilbert space we get a more general result than (a) as we can see in Theorem 2.1.7. To prove this, we have to use the fact that Hilbert spaces have transitive norm, i.e., that for any fixed two norm one points x and y, there exists a linear isometry

$R : H \longrightarrow H$ such that $R(x) = y$. Moreover, if $\|y - x\|$ is small, then R can be chose so that $\|R - Id_H\|$ is also small. Let us give a proof of this result. Indeed, note first that it is enough to take $\dim(H) = 2$ by decomposing the Hilbert space H into the direct sum of a 2-dimensional space and its orthogonal complement and we define R as the identity operator outside the 2-dimensional part. Given $\varepsilon \in (0, 1)$, let x and y both in S_H be such that $\|y - x\| < \varepsilon$. Define

$$u := \frac{y - \langle y, x\rangle x}{\|y - \langle y, x\rangle x\|} \in S_H \quad \text{and} \quad v := \frac{x - \langle x, y\rangle y}{\|x - \langle x, y\rangle y\|} \in S_H.$$

Note that $\{x, u\}$ and $\{y, v\}$ are orthonormal systems in H. If we define $R : H \longrightarrow H$ by

$$R(\alpha x + \beta u) := \alpha y + \beta v \quad (\alpha x + \beta u \in H),$$

then R is a linear isometry, $R(x) = y$ and $\|R - Id_H\| < \delta(\varepsilon)$ where $\lim_{t\to 0} \delta(t) = 0$. We use this fact constantly in Section 2.3.

Theorem 2.1.7. Let H be a Hilbert space and let Y be any Banach space. Then the pair $(H; Y)$ has the BPBpp.

Proof. Let H be a Hilbert space and let $\varepsilon > 0$ be given. Since H is uniformly convex, the pair $(H; Y)$ has the BPBp for all Banach space Y (see [6, Corollary 2.3] or [46, Theorem 3.1]). Hence, there exists some function $\varepsilon \longmapsto \eta(\varepsilon)$ satisfying the BPBp for this pair. Let $T \in \mathcal{L}(H; Y)$ with $\|T\| = 1$ and $h_0 \in S_H$ be such that

$$\|T(h_0)\| > 1 - \eta(\varepsilon).$$

Then there are $\widetilde{S} \in \mathcal{L}(H; Y)$ with $\|\widetilde{S}\| = 1$ and $\widetilde{h}_0 \in S_H$ satisfying

$$\|\widetilde{S}(\widetilde{h}_0)\| = 1, \quad \|h_0 - \widetilde{h}_0\| < \varepsilon \quad \text{and} \quad \|\widetilde{S} - T\| < \varepsilon.$$

Since H is a Hilbert space, it has a transitive norm. So there is a linear isometry $R : H \longrightarrow H$ such that

$$R(h_0) = \widetilde{h}_0 \quad \text{and} \quad \|R - Id_H\| < \delta(\varepsilon)$$

where $\lim_{t\to 0} \delta(t) = 0$. Define $S := \widetilde{S} \circ R : H \longrightarrow Y$. Then $\|S\| \leqslant 1$ and

$$\|S(h_0)\| = \|\widetilde{S}(R(h_0))\| = \|\widetilde{S}(\widetilde{h}_0)\| = 1.$$

So $\|S\| = \|S(h_0)\| = 1$. Moreover,

$$\|S - T\| \leqslant \|\widetilde{S} \circ R - \widetilde{S}\| + \|\widetilde{S} - T\| \leqslant \|R - Id_H\| + \varepsilon < \delta(\varepsilon) + \varepsilon.$$

This proves that the pair $(H; Y)$ has the BPBpp as desired. □

Concerning the transitivity, it is shown in [36] that for homogeneous and non σ-finite measure μ, $L_p(\mu)$ has transitive norm when $1 \leqslant p < \infty$. However, for $p \neq 2$ it is not possible to guarantee that the isometry used there is close to the identity operator when the fixed two points are close to each other. We do not know if it is possible to extend Theorem 2.1.7 to $L_p(\mu)$-spaces. Also we do not know what happens with the pair $(X; H)$ when X is uniformly smooth and H is Hilbert.

Let K be a compact Hausdorff topological space. We denote by $C(K)$ the space of all continuous functions defined on K and $\| \, . \, \|_\infty$ denotes the supremum norm on this space. A *uniform algebra* is a $\| \, . \, \|_\infty$-closed subalgebra $A \subset C(K)$ endowed with the supremum norm that contains the constant functions and separates the points of K, i.e., for every $x, y \in K$ with $x \neq y$ there is a function $f \in A$ such that $f(x) \neq f(y)$. We said that the uniform algebra $A \in C(K)$ is *unital* if the constant function 1 belongs to A. It is known that the pair $(X; C(K))$ has the BPBp whenever X is an Asplund space [10, Corollary 2.6] and it was extended for the pair $(X; A)$ in [19, Theorem 3.6] for both unital and non-unital uniform algebra A. We use the ideas from those results to prove that the pair $(X; A)$ has the BPBpp whenever X is a uniformly smooth Banach space and A is a uniform algebra.

Theorem 2.1.8. Let X be a uniformly smooth Banach space and A be a uniform algebra. The pair $(X; A)$ has the BPBpp.

Proof. Indeed, adapt [19, Lemma 3.5] by using Proposition 2.1.2 instead of the Bishop-Phelps-Bollobás theorem. Then apply it in [19, Theorem 3.6]. Since every uniformly smooth space is reflexive and every operator from a reflexive space into A is Asplund, the result follows. □

We have the following consequence.

Corollary 2.1.9. Let X be a uniformly smooth Banach space and let K be a compact Hausdorff topological space. Then the pair $(X; C(K))$ has the BPBpp.

Next we study the stability of the property with respect to direct sums.

Proposition 2.1.10. Let X be a uniformly smooth Banach space and let $\{Y_j : j \in J\}$ be an arbitrary family of Banach spaces.

(a) The pairs $\left(X; \left(\bigoplus_{j\in J} Y_j\right)_{\ell_\infty}\right)$ and $\left(X; \left(\bigoplus_{j\in J} Y_j\right)_{c_0}\right)$ have the BPBpp if and only if, for all $j \in J$, the pair $(X; Y_j)$ satisfies it with a common function $\eta(\varepsilon) > 0$.

(b) If the pair $\left(X; \left(\bigoplus_{j\in J} Y_j\right)_{\ell_1}\right)$ has the BPBpp, then the pair $(X; Y_j)$ satisfies it as well for all $j \in J$.

Proof. For (a), use [11, Proposition 2.4] adapting it for our property. For (b), do the same by using [11, Proposition 2.7]. Alternatively, both items can be proved by adapting the more general Propositions 2.4.2 and 2.4.6 to the BPBpp. □

We do not know if the converse of Proposition 2.1.10.(b) is true even for finite sums. We finish this section by commenting that there are 2-dimensional real uniformly smooth Banach spaces X such that the pair $(X;Y)$ fails the BPBpp for some Banach space Y.

Example 2.1.11. It is proved in [46, Corollary 3.3] that a 2-dimensional real Banach space X is uniformly convex if and only if the pair $(X;Y)$ has the BPBp for all Banach spaces Y. Let X_0 be a 2-dimensional Banach space which is uniformly smooth but not strictly convex. Then, there is a Banach space Y_0 such that the pair $(X_0;Y_0)$ fails the BPBp and so it can not satisfy the BPBpp either.

2.1.2 The BPBpp for bilinear mappings

Now we study the Bishop-Phelps-Bollobás point property on another class of functions: the bilinear mappings. Before we do that, let us establish some notation, give some definitions, and recall some useful known results about the BPBp for bilinear forms.

Let X, Y and Z be Banach spaces. We denote by $\mathcal{L}(X,Y;Z)$ the set of all bilinear mappings from $X \times Y$ into Z. It is a Banach space equipped with the norm

$$\|B\| = \sup_{\substack{x \in B_X \\ y \in B_Y}} \|B(x,y)\| = \sup_{\substack{x \in S_X \\ y \in S_Y}} \|B(x,y)\|.$$

Recall that $(X,Y;Z)$ has the *Bishop-Phelps-Bollobás property for bilinear mappings* (BPBp for bilinear mappings, for short) when given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $B \in \mathcal{L}(X,Y;Z)$ with $\|B\| = 1$ and $(x_0, y_0) \in S_X \times S_Y$ are such that

$$\|B(x_0, y_0)\| > 1 - \eta(\varepsilon),$$

there are $A \in \mathcal{L}(X, Y; Z)$ with $\|A\| = 1$ and $(x_1, y_1) \in S_X \times S_Y$ such that

$$\|A(x_1, y_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon, \quad \|y_1 - y_0\| < \varepsilon \text{ and } \|A - B\| < \varepsilon.$$

In this case, we say that $(X, Y; Z)$ has the BPBp for bilinear mappings with the function $\varepsilon \longmapsto \eta(\varepsilon)$. When $Z = \mathbb{K}$, we just say that the pair (X, Y) has the BPBp for bilinear forms.

It was proved in [27, Theorem 2] that the pair (ℓ_1, ℓ_1) fails the BPBp for bilinear forms. On the other hand, if X and Y are uniformly convex Banach spaces, then $(X, Y; Z)$ has the BPBp for bilinear mappings for any Banach space Z [6, Theorem 2.2]. For the bilinear case, we introduce the following stronger property.

Definition 2.1.12. We say that $(X, Y; Z)$ has the *Bishop-Phelps-Bollobás point property for bilinear mappings* (BPBpp for bilinear mappings, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $B \in \mathcal{L}(X, Y; Z)$ with $\|B\| = 1$ and $(x_0, y_0) \in S_X \times S_Y$ satisfy

$$\|B(x_0, y_0)\| > 1 - \eta(\varepsilon),$$

there exists $A \in \mathcal{L}(X, Y; Z)$ with $\|A\| = 1$ such that

$$\|A(x_0, y_0)\| = 1 \quad \text{and} \quad \|A - B\| < \varepsilon.$$

In this case, we say that $(X, Y; Z)$ has the BPBpp for bilinear mappings with the function $\varepsilon \longmapsto \eta(\varepsilon)$.

When $Z = \mathbb{K}$, we just say that the pair (X, Y) has the BPBpp for bilinear forms. It is clear that the BPBpp for bilinear mappings implies the BPBp for bilinear mappings. Note by a routine change of parameters that we may consider $B \in \mathcal{L}(X, Y; Z)$ with $\|B\| \leqslant 1$ instead of $\|B\| = 1$ in Definition 2.1.12 (and we will use this in Theorem 2.1.16). It is worth mentioning that the pair (ℓ_1, ℓ_1) fails the BPBpp for bilinear forms since it fails the BPBp for bilinear forms.

Our first result gives a partial characterization for the pair (X, Y) to have the BPBpp for bilinear forms. It was proved in [6, Proposition 2.4] (and independently in [28, Theorem 1.1]) that if Y is a uniformly convex Banach space then the pair (X, Y) has the BPBp for bilinear forms if and only if the pair $(X; Y^*)$ has the BPBp for operators. We will do the same to our property but assuming now that Y is a Hilbert space. It is not difficult to check that if the pair (X, Y) has the BPBp for bilinear forms then the pair $(X; Y^*)$ has the BPBp for operators by using the natural identification between the Banach spaces $\mathcal{L}(X, Y)$ and $\mathcal{L}(X; Y^*)$ given by $B(x, y) := T(x)(y)$ for all $x \in X$, $y \in Y$ and $T \in \mathcal{L}(X, Y^*)$. The same happens in our case. So we have to prove the converse and we will do that in the next theorem.

Theorem 2.1.13. Let X be a uniformly smooth Banach space and let H be a Hilbert space. Then the pair (X, H) has the BPBpp for bilinear forms if and only if the pair $(X; H^*)$ has the BPBpp (for operators).

Proof. Let $\varepsilon > 0$ be given. Assume that the pair $(X; H^*)$ has the BPBpp for operators with $\eta(\varepsilon) > 0$. Consider $\delta_H(\varepsilon) > 0$ the modulus of uniform convexity of H. Let $B : X \times H \longrightarrow \mathbb{K}$ be a bilinear form with $\|B\| = 1$ and $(x_0, h_0) \in S_X \times S_H$ be such that

$$\operatorname{Re} B(x_0, h_0) > 1 - \min\{\delta_H(\varepsilon), \eta(\delta_H(\varepsilon))\}.$$

Define the bounded linear operator $T : X \longrightarrow H^*$ by $T(x)(h) := B(x, h)$ for all $x \in X$ and $h \in H$. Then $\|T\| = \|B\| = 1$ and

$$\|T(x_0)\| \geqslant \operatorname{Re} T(x_0)(h_0) = \operatorname{Re} B(x_0, h_0) > 1 - \eta(\delta_H(\varepsilon)).$$

So there exists $S \in \mathcal{L}(X; H^*)$ with $\|S\| = 1$ such that

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \delta_H(\varepsilon) < 2\varepsilon.$$

Let $h_1 \in S_H$ be such that

$$\operatorname{Re} h_1(S(x_0)) = \operatorname{Re} S(x_0)(h_1) = \|S(x_0)\| = 1$$

We prove that $\|h_0 - h_1\| < \varepsilon$. Note first that since

$$\begin{aligned} \delta_H(\varepsilon) > \|S - T\| &\geqslant \operatorname{Re} T(x_0)(h_0) - \operatorname{Re} S(x_0)(h_0) \\ &> 1 - \delta_H(\varepsilon) - \operatorname{Re} S(x_0)(h_0), \end{aligned}$$

we get that $\operatorname{Re} S(x_0)(h_0) > 1 - 2\delta_H(\varepsilon)$. Then

$$\left\| \frac{h_0 + h_1}{2} \right\| \geqslant \operatorname{Re} \left(\frac{S(x_0)(h_0) + S(x_0)(h_1)}{2} \right) > 1 - \delta_H(\varepsilon).$$

So $\|h_0 - h_1\| < \varepsilon$ as desired. Since H has a transitive norm (see the observations just before Theorem 2.1.7), we can find a linear isometry $R \in \mathcal{L}(H; H)$ such that

$$R(h_0) = h_1 \quad \text{and} \quad \|R - Id_H\| < \delta(\varepsilon)$$

where $\lim_{t \to 0} \delta(t) = 0$. We define the bilinear form $A : X \times H \longrightarrow \mathbb{K}$ by

$$A(x, h) := S(x)(R(h)) \quad ((x, h) \in X \times H)\,.$$

Then $\|A\| \leqslant 1$ and

$$|A(x_0, h_0)| = |S(x_0)(R(h_0))| = |S(x_0)(h_1)| = \operatorname{Re} S(x_0)(h_1) = 1.$$

So $\|A\| = |A(x_0, h_0)| = 1$. Moreover, for all $(x, h) \in S_X \times S_H$, we have that

$$\begin{aligned} |A(x,h) - B(x,h)| &\leqslant |S(x)(R(h)) - S(x)(h)| + |S(x)(h) - T(x)(h)| \\ &\leqslant \|R - Id_H\| + \|S - T\| \\ &< \delta(\varepsilon) + 2\varepsilon. \end{aligned}$$

Since $(x, h) \in S_X \times S_H$ is arbitrary, we get that $\|A - B\| < \delta(\varepsilon) + 2\varepsilon$. This shows that the pair (X, H) has the BPBpp for bilinear forms. □

As a consequence of Theorem 2.1.13, we have the following corollary.

Corollary 2.1.14. Let H_1 and H_2 be Hilbert spaces. Then (H_1, H_2) has the BPBpp for bilinear forms.

Proof. The pair $(H_1; H_2^*)$ has the BPBpp by Theorem 2.1.7. Hence, Theorem 2.1.13 gives the desired result. □

Remark 2.1.15. We have a direct proof for Corollary 2.1.14. By [6, Theorem 2.2], we know that (H_1, H_2) has the BPBp with some $\eta(\varepsilon) > 0$ for a given $\varepsilon > 0$ where H_1 and H_2 are Hilbert spaces. So if $B \in \mathcal{L}(H_1, H_2)$ is a bilinear form with $\|B\| = 1$ and $(h_0, h_1) \in S_H \times S_H$ satisfying

$$|B(h_0, h_1)| > 1 - \eta(\varepsilon),$$

then there are $\widetilde{A} \in \mathcal{L}(H_1, H_2)$ with $\|\widetilde{A}\| = 1$ and $(\widetilde{h}_0, \widetilde{h}_1) \in S_H \times S_H$ such that

$$|\widetilde{A}(\widetilde{h}_0, \widetilde{h}_1)| = 1, \quad \|\widetilde{h}_0 - h_0\| < \varepsilon, \quad \|\widetilde{h}_1 - h_1\| < \varepsilon \quad \text{and} \quad \|\widetilde{A} - B\| < \varepsilon.$$

Then, there are linear isometries $R_0, R_1 \in \mathcal{L}(H; H)$ such that $R_0(h_0) = \widetilde{h}_0$, $R_1(h_1) = \widetilde{h}_1$, $\|R_0 - Id_H\| < \delta_0(\varepsilon)$ and $\|R_1 - Id_H\| < \delta_1(\varepsilon)$ where

$\lim_{t\to 0}\delta_0(t) = \lim_{t\to 0}\delta_1(t) = 0$. Define $A : H_1 \times H_2 \longrightarrow \mathbb{K}$ by

$$A(x,y) := \widetilde{A}(R_0(x), R_1(y)) \quad ((x,y) \in H_1 \times H_2).$$

Then $A \in \mathcal{L}(H_1, H_2)$, $\|A\| \leqslant 1$ and

$$|A(h_0,h_1)| = |\widetilde{A}(R_0(h_0), R_1(h_1))| = |\widetilde{A}(\widetilde{h}_0, \widetilde{h}_1)| = 1.$$

So $\|A\| = |A(h_0,h_1)| = 1$. Moreover for all $(x,y) \in S_{H_1} \times S_{H_2}$,

$$\begin{aligned}
|A(x,y) - B(x,y)| &= |\widetilde{A}(R_0(x), R_1(y))| \\
&\leqslant |\widetilde{A}(R_0(x), R_1(y)) - \widetilde{A}(x, R_1(y))| + |\widetilde{A}(x, R_1(y)) - \widetilde{A}(x,y)| \\
&\qquad + |\widetilde{A}(x,y) - B(x,y)| \\
&\leqslant \|R_0 - Id_H\| + \|R_1 - Id_H\| + \|\widetilde{A} - B\| < \delta_0(\varepsilon) + \delta_1(\varepsilon) + \varepsilon.
\end{aligned}$$

This implies that (H_1, H_2) satisfies the BPBpp for bilinear forms.

In the following, we are assuming that the range space Z satisfies property β.

Theorem 2.1.16. Let X, Y and Z be Banach spaces. Suppose that the pair (X,Y) has the BPBpp for bilinear forms and that Z has property β. Then $(X,Y;Z)$ has the BPBpp for bilinear mappings.

Proof. Let $\varepsilon > 0$ be given and consider $\xi > 0$ satisfying (2.3) in Proposition 2.1.6. Suppose that Z has property β with constant $\rho \in [0,1)$ and sets $\{z_\alpha : \alpha \in \Lambda\} \subset S_Z$ and $\{z_\alpha^* : \alpha \in \Lambda\} \subset S_{Z^*}$. Let $B \in \mathcal{L}(X,Y;Z)$ with $\|B\| = 1$ and $(x_0, y_0) \in S_X \times S_Y$ be such that

$$\|B(x_0,y_0)\| > 1 - \eta(\xi),$$

where $\varepsilon \longmapsto \eta(\varepsilon)$ is the function for (X, Y) which we are assuming to have the BPBpp for bilinear forms. There exists some $\alpha_0 \in \Lambda$ such that

$$\operatorname{Re}(z_{\alpha_0}^* \circ B)(x_0, y_0) = \operatorname{Re} z_{\alpha_0}^*(B(x_0, y_0)) > 1 - \eta(\xi).$$

Then there exists $\widetilde{A} \in \mathcal{L}(X, Y)$ with $\|\widetilde{A}\| = 1$ such that

$$|\widetilde{A}(x_0, y_0)| = 1 \quad \text{and} \quad \|\widetilde{A} - (z_{\alpha_0}^* \circ B)\| < \xi.$$

Define $A : X \times Y \longrightarrow Z$ by

$$A(x, y) := B(x, y) + \left[\left(1 + \frac{\varepsilon}{4}\right) \widetilde{A}(x, y) - (z_{\alpha_0}^* \circ B)(x, y)\right] z_{\alpha_0}$$

for all $(x, y) \in X \times Y$. Notice that for all $\alpha \in \Lambda$ and $(x, y) \in X \times Y$, we have

$$\begin{aligned} z_\alpha^*(A(x, y)) = z_\alpha^*(B(x, y)) \\ + z_\alpha^*(z_{\alpha_0}) \left[\left(1 + \frac{\varepsilon}{4}\right) \widetilde{A}(x, y) - (z_{\alpha_0}^* \circ B)(x, y)\right]. \end{aligned}$$

So if $\alpha = \alpha_0$, then $z_{\alpha_0}^*(A(x, y)) = \left(1 + \frac{\varepsilon}{4}\right) \widetilde{A}(x, y)$ and this implies that $|z_{\alpha_0}^*(A(x, y))| \leqslant 1 + \frac{\varepsilon}{4}$. On the other hand, if $\alpha \neq \alpha_0$, then

$$|z_\alpha^*(A(x, y))| \leqslant 1 + \rho\left(\frac{\varepsilon}{4} + \xi\right) < \left(1 - \frac{\varepsilon}{4}\right)(1 - \xi) < 1 + \frac{\varepsilon}{4}.$$

Since $|z_{\alpha_0}^*(A(x_0, y_0))| = 1 + \frac{\varepsilon}{4}$, $\|A\| = \|A(x_0, y_0)\|$. Also, we have that $\|B - A\| < \frac{\varepsilon}{4} + \xi < \varepsilon$. So if $C := \frac{A}{\|A\|}$ then we have that $\|C(x_0, y_0)\| = 1$ and $\|C - B\| < 2\varepsilon$, proving that $(X, Y; Z)$ has the BPBpp for bilinear mappings. □

As a consequence of Theorem 2.1.16, we have the following corollary.

Corollary 2.1.17. Let H_1 and H_2 be Hilbert spaces and let Z be a Banach space with the property β. Then $(H_1, H_2; Z)$ has the BPBpp for bilinear mappings.

Proof. This is a combination of Corollary 2.1.14 (or Remark 2.1.15) and Theorem 2.1.16. □

Let us now consider compact bilinear mappings. Let X, Y and Z be Banach spaces. We say that the bilinear mapping $B : X \times Y \longrightarrow Z$ is *compact* if $B(B_X \times B_Y) \subset Z$ is precompact in Z. We denote by $\mathcal{K}(X, Y; Z)$ the set of all compact bilinear mappings from $X \times Y$ into Z. We define the *BPBpp for compact bilinear mappings* by using just compact bilinear mappings in Definition 2.1.12, i.e., we consider compact bilinear mappings A and B in that definition. It is worth mentioning that we study the BPBp for compact operators in Chapter 3 and for compact multilinear mappings in Theorem 4.1.9 and Corollary 4.1.10.

Our aim now is to prove that $(H_1, H_2; C(K))$ has the BPBpp for compact bilinear mappings whenever H_1 and H_2 are Hilbert spaces and K is a compact Hausdorff topological space. First, we prove two auxiliary results and the promised one will be a consequence of them.

For a function $\varphi : K \longrightarrow \mathcal{L}(X, Y)$, we say that φ is τ_p*-continuous* if the mapping $t \longmapsto \varphi(t)(x, y)$ is continuous on K for each $(x, y) \in X \times Y$. In the next lemma, we prove that there exists a natural (isometric) identification between the spaces $\mathcal{L}(X, Y; C(K))$ and the space of all τ_p-continuous and bounded functions from K into $\mathcal{L}(X, Y)$ endowed with the supremum norm $\|\varphi\| = \sup_{t \in K} \|\varphi(t)\|$.

Proposition 2.1.18. [31, Theorem 1, p. 490] Let X and Y be Banach spaces. Let K be a compact Hausdorff topological space. Then,

(i) there exists an isomorphic isometry between $\mathcal{L}(X, Y; C(K))$ and the set of all τ_p-continuous and bounded functions from K into $\mathcal{L}(X, Y)$ and

(ii) The subspace $\mathcal{K}(X,Y;C(K))$ of all compact bilinear mappings from $X \times Y$ into $C(K)$ corresponds to the set of all norm-continuous functions.

Proof. (i). Let $B \in \mathcal{L}(X,Y;C(K))$ and define $\varphi : K \longrightarrow \mathcal{L}(X,Y)$ by the relation

$$\varphi(t)(x,y) := B(x,y)(t) \tag{2.4}$$

for all $t \in K$ and $(x,y) \in X \times Y$. Then the function $t \longmapsto \varphi(t)(x,y)$ is continuous on K for each $(x,y) \in X \times Y$ since $B(x,y) \in C(K)$ for each $(x,y) \in X \times Y$. Conversely, if $\varphi : K \longrightarrow \mathcal{L}(X,Y)$ is a τ_p-continuous and bounded function, we define $B \in \mathcal{L}(X,Y;C(K))$ as in (2.4) and it is not difficult to see that B is a continuous bilinear mapping such that $\|B\| = \|\varphi\|$.

(ii). Let $B \in \mathcal{K}(X,Y;C(K))$. Consider $\varphi : K \longrightarrow \mathcal{L}(X,Y)$ defined by (2.4). We prove that $t \longmapsto \varphi(t)(x,y) = B(x,y)(t)$ is norm-continuous. Let $(t_\alpha)_\alpha \subset K$ be such that $t_\alpha \longrightarrow t_0 \in K$. Then

$$\begin{aligned} \|\varphi(t_\alpha) - \varphi(t)\| &= \sup_{(x,y)\in B_X \times B_Y} |\varphi(t_\alpha)(x,y) - \varphi(t_0)(x,y)| \\ &= \sup_{(x,y)\in B_X \times B_Y} |B(x,y)(t_\alpha) - B(x,y)(t_0)| \longrightarrow 0 \end{aligned}$$

since $B(B_X \times B_Y) \subset C(K)$ is equicontinuous and bounded by the Arzelà-Ascoli theorem for $C(K)$ [31, Theorem 7, p.266]. This shows that $t \longmapsto \varphi(t)(x,y)$ is norm-continuous for all $(x,y) \in X \times Y$. On the other hand, let $\varphi : K \longrightarrow \mathcal{L}(X,Y)$ be a norm-continuous function. Define again $B : X \times Y \longrightarrow C(K)$ by the relation (2.4). Given $\varepsilon > 0$ and $t_0 \in K$, there exists a neighborhood U_{t_0} of t_0 such that if $t \in U_{t_0}$, then

$\|\varphi(t) - \varphi(t_0)\| < \varepsilon$. So

$$\begin{aligned}\sup_{(x,y)\in B_X\times B_Y} |B(x,y)(t) - B(x,y)(t_0)| \\ &= \sup_{(x,y)\in B_X\times B_Y} |\varphi(t)(x,y) - \varphi(t_0)(x,y)| \\ &= \|\varphi(t) - \varphi(t_0)\| < \varepsilon.\end{aligned}$$

Hence if $t \in U_{t_0}$, then $|B(x,y)(t) - B(x,y)(t_0)| < \varepsilon$ for all $(x,y) \in B_X \times B_Y$. This shows that the set $B(B_X \times B_Y)$ is equicontinuous in $C(K)$ and since this set is already bounded, we may conclude that B is a compact bilinear mapping. □

In order to show that the pair $(H_1 \times H_2, C(K))$ has our property for compact bilinear mappings, we will prove first that it is possible to carry the property from the pair (X, Y) to $(X, Y; C(K))$ by using Proposition 2.1.18. We will prove the analogous version of this for compact operators in Chapter 3 by using different techniques (see Theorem 3.3.18.(c)).

Theorem 2.1.19. Let X and Y be Banach spaces. Let K be a compact Hausdorff topological space. Suppose that the pair (X, Y) has the BPBpp for bilinear forms. Then $(X, Y; C(K))$ has the BPBpp for compact bilinear mappings.

Proof. Let $\varepsilon > 0$ be given. Consider $\eta(\varepsilon) > 0$ to be the BPBpp constant for the pair (X, Y). Let $B \in \mathcal{K}(X, Y; C(K))$ with $\|B\| = 1$ and $(x_0, y_0) \in S_X \times S_Y$ be such that

$$\|B(x_0, y_0)\|_\infty > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Now define $\varphi : K \longrightarrow \mathcal{L}(X, Y)$ by the relation (2.4) of Proposition 2.1.18. Since B is compact, φ is norm-continuous. Consider $t_0 \in K$ such that

$$|\varphi(t_0)(x_0, y_0)| = |B(x_0, y_0)(t_0)| > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Then there is $\widetilde{B} \in \mathcal{L}(X,Y)$ with $\|\widetilde{B}\| = 1$ such that

$$|\widetilde{B}(x_0, y_0)| = 1 \quad \text{and} \quad \|\widetilde{B} - \varphi(t_0)\| < \frac{\varepsilon}{2}.$$

Consider the retraction $r : \mathcal{L}(X,Y) \longrightarrow B_{\mathcal{L}(X,Y)}$ defined for $C \in \mathcal{L}(X,Y)$ by

$$r(C) := C \ \text{ if } \ \|C\| \leqslant 1 \ \text{ and } \ \ r(C) := \frac{1}{\|C\|}C \ \text{ if } \ \|C\| \geqslant 1.$$

Now define the norm-continuous map $\psi : K \longrightarrow \mathcal{L}(X,Y)$ by

$$\psi(t) := r(\varphi(t) + \widetilde{B} - \varphi(t_0)) \ (t \in K).$$

Then $\psi(t_0) = r(\widetilde{B}) = \widetilde{B}$. Now consider $A : X \times Y \longrightarrow C(K)$ defined by $A(x,y)(t) := \psi(t)(x,y)$ for every $t \in K$ and $(x,y) \in X \times Y$. Then $A \in \mathcal{K}(X,Y;C(K))$ with $\|A\| \leqslant 1$ and

$$1 \geqslant \|A\| \geqslant \|A(x_0,y_0)\|_\infty \geqslant |A(x_0,y_0)(t_0)| = |\widetilde{B}(x_0,y_0)| = 1.$$

Then $\|A\| = \|A(x_0,y_0)\|_\infty = 1$. It remains to prove that $\|A - B\| < \varepsilon$. To prove this, note first that if $C \in \mathcal{L}(X,Y)$ is such that $1 \leqslant \|C\| \leqslant 1 + \frac{\varepsilon}{2}$, then

$$\|r(C) - C\| = \left\| \frac{1}{\|C\|}C - C \right\| = \|C\| - 1 \leqslant \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} \|A - B\| &= \sup_{t \in K} \|\psi(t) - \varphi(t)\| \\ &= \sup_{t \in K} \|r(\varphi(t) + \widetilde{B} - \varphi(t_0)) - (\varphi(t) + \widetilde{B} - \varphi(t_0)) + \widetilde{B} - \varphi(t_0)\| \\ &\leqslant \frac{\varepsilon}{2} + \|\widetilde{B} - \varphi(t_0)\| < \varepsilon. \end{aligned}$$

This proves that $(X,Y;C(K))$ has the BPBpp for bilinear mappings. $\square$

Now we can prove the promised result.

Corollary 2.1.20. Let H_1 and H_2 be Hilbert spaces and let K be a compact Hausdorff topological space. Then the pair $(H_1, H_2; C(K))$ has the BPBpp for compact bilinear mappings.

Proof. The proof is a combination of Corollary 2.1.14 (or Remark 2.1.15) and Theorem 2.1.19. □

2.2 Properties 1 and 2

This section is devoted to the study of two properties also similar to the Bishop-Phelps-Bollobás property as the BPBpp (see Section 2.1). Indeed, properties 1 and 2 are motived by a result which gives a "dual version" of Proposition 2.1.2 proved by S. K. Kim and H. J. Lee in 2014 [46, Theorem 2.1].

Theorem 2.2.1 (Kim-Lee Theorem, [46])**.** A Banach space X is uniformly convex if and only if given $\varepsilon > 0$, there exists a positive real number $\eta(\varepsilon) > 0$ such that whenever $x_0^* \in S_{X^*}$ and $x_0 \in B_X$ satisfy

$$|x_0^*(x_0)| > 1 - \eta(\varepsilon),$$

there is $x_1 \in S_X$ such that

$$|x_0^*(x_1)| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

We observe that the theorem says that a Banach space X is uniformly convex if and only if the pair $(X; \mathbb{K})$ satisfies the Bishop-Phelps-Bollobás property without changing the initial functional x^*, that is, the functional that almost attains its norm at some point x_0 is the same functional that attains its norm at the new vector which is close to x_0. Note also that we already apply this result at the beginning of the proof of Proposition 2.1.2.

We study Theorem 2.2.1 for bounded linear operators in two cases. First, we consider the positive real number η depending on $\varepsilon > 0$ and also on a fixed operator T and we call it *property* 1. After that, we consider the uniform case of this, that is, when the number η depends only on $\varepsilon > 0$ as we are used to work when we are working with the BPBp. This property is called *property* 2.

Note that the difference between property 2 and the Bishop-Phelps-Bollobás point property is that, in the first one, we are fixing the operator and in the second one we fix the point. Despite the visual similarity between them it turns out that they are very different from each other.

We start with the formal definition of property 1 although we believe that it is already clear from the above comments.

Definition 2.2.2 (Property 1)**.** A pair of Banach spaces $(X; Y)$ has *property* 1 if given $\varepsilon > 0$ and $T \in \mathcal{L}(X; Y)$ with $\|T\| = 1$, then there exists $\eta(\varepsilon, T) > 0$ such that whenever $x_0 \in S_X$ satisfies

$$\|T(x_0)\| > 1 - \eta(\varepsilon, T),$$

there is $x_1 \in S_X$ such that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

If the above property is satisfied for every norm one compact operator, then we say that the pair $(X; Y)$ has *property* 1 *for compact operators*.

We would like to comment that D. Carando, S. Lassalle and M. Mazzitelli [18] defined a Bishop-Phelps-Bollobás type property for ideals of multilinear mappings where the positive real number η in the definition of the BPBp depends on a given $\varepsilon > 0$ and *also* on the ideal norm of the operator defined on a normed ideal of N-linear mappings. In other words, a normed ideal of N-linear mappings $\mathcal{U} = \mathcal{U}(X_1 \times \ldots \times X_N; Y)$ where $X_1, \ldots, X_N, Y$ are Banach spaces has the *weak BPBp* if for each $\Phi \in \mathcal{U}$ with $\|\Phi\| = 1$ and $\varepsilon > 0$, there exists $\eta(\varepsilon, \|\Phi\|_{\mathcal{U}}) > 0$ such that if $(x_1, \ldots, x_N) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfies $\|\Phi(x_1, \ldots, x_N)\| > 1 - \eta(\varepsilon, \|\Phi\|_{\mathcal{U}})$, then there exist $\Psi \in \mathcal{U}$ with $\|\Psi\| = 1$ and $(a_1, \ldots, a_N) \in S_{X_1} \times \ldots \times S_{X_N}$ such that $\|\Psi(a_1, \ldots, a_N)\| = 1$, $\|(a_1, \ldots, a_N) - (x_1, \ldots, x_N)\| < \varepsilon$ and $\|\Psi - \Phi\|_{\mathcal{U}} < \varepsilon$. They proved, among other things, that if $X_1, \ldots, X_N$ are uniformly convex Banach spaces then $\mathcal{U}$ has the weak BPBp for ideals of multilinear mappings for all Banach space Y. Here we will work on a different context where η depends on a fixed operator and not on the norm of the operators ideal.

Next we make simple but important observations about our property which will influence the rest of the section.

Remark 2.2.3. (a). We note that in Definition 2.2.2 the operator T must attains its norm if the pair $(X; Y)$ has property 1. So if X is not reflexive, then the pair $(X; Y)$ cannot have property 1 for any Banach space Y. Indeed, if X is not reflexive then there is a linear continuous functional $x_0^* \in S_{X^*}$ such that $|x_0^*(x)| < 1$ for all $x \in S_X$ by using James theorem. Fixing $y_0 \in S_Y$ and defining $T : X \longrightarrow Y$ by $T(x) := x_0^*(x)y_0$, we have that $\|T\| = \|x_0^*\| = 1$ and $\|T(x)\| = |x_0^*(x)| < 1$ for all $x \in S_X$. This implies that T never attains its norm and then the pair $(X; Y)$ can not satisfy the property.

(b). For every infinite-dimensional Banach space X, the pair $(X; c_0)$ fails property 1. Indeed, it is shown in [55, Lemma 2.2] that for every infinite-dimensional Banach space X, there exists a norm-one linear operator from X into c_0 which does not attain its norm.

Let us give the first result related to property 1. We assume that the domain space X is finite dimensional.

Theorem 2.2.4. Let X be a finite dimensional Banach space. Then the pair $(X; Y)$ has property 1 for all Banach spaces Y.

Proof. The proof is by contradiction. Let $T \in \mathcal{L}(X;Y)$ with $\|T\| = 1$. If the result is false for some $\varepsilon_0 > 0$, then for all $n \in \mathbb{N}$, there exists $x_n \in S_X$ such that

$$1 \geqslant \|T(x_n)\| > 1 - \frac{1}{n}$$

but $\text{dist}(x_n, NA(T)) \geqslant \varepsilon_0$ for $n \in \mathbb{N}$, where

$$NA(T) = \{z \in S_X : \|T(z)\| = 1\}$$

(which is non empty as X is finite-dimensional). Since X is finite dimensional, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \longrightarrow x_0$ for some $x_0 \in X$. This implies that $\|T(x_{n_k})\| \longrightarrow \|T(x_0)\|$ and since

$$1 \geqslant \|T(x_{n_k})\| \geqslant 1 - \frac{1}{n}$$

we get that $\|T(x_0)\| = \|x_0\| = 1$ and so $x_0 \in NA(T)$. Then

$$\varepsilon \leqslant \text{dist}(x_{n_k}, NA(T)) \leqslant \|x_{n_k} - x_0\| \overset{k\to\infty}{\longrightarrow} 0$$

which is a contradiction. So the pair $(X;Y)$ has property 1. □

Let X be a Banach space. We say that X is *locally uniformly rotund* (LUR) if for all $x, x_n \in S_X$ satisfying $\lim_n \|x_n + x\| = 2$, we have that $\lim_n \|x_n - x\| = 0$. If we assume that X is LUR and reflexive, we get that the pair $(X;Y)$ has property 1 for compact operators. In particular, we have the same result when X is uniformly convex.

Theorem 2.2.5. Let X be a reflexive Banach space which is LUR. Then the pair $(X;Y)$ has property 1 for compact operators for every Banach space Y.

Proof. As Theorem 2.2.4, the proof by contradiction. Let $T \in \mathcal{K}(X;Y)$ with $\|T\| = 1$ be a compact operator. If the result is false for some

$\varepsilon_0 > 0$, for all $n \in \mathbb{N}$, there exists $x_n \in S_X$ such that

$$1 \geqslant \|T(x_n)\| > 1 - \frac{1}{n}$$

but $\operatorname{dist}(x_n, NA(T)) \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$. Then $\|T(x_n)\| \longrightarrow 1$ as $n \to \infty$. Since X is reflexive, by Smulian's Theorem, there exists a subsequence (x_{n_k}) of (x_n) and $x_0 \in X$ such that (x_{n_k}) converges weakly to x_0. Since T is completely continuous (see, [56, Definition 3.4.33, Proposition 3.4.34 and Proposition 3.4.36]), $T(x_{n_k})$ converges in norm to $T(x_0)$ as $k \longrightarrow \infty$. Therefore $\|T(x_0)\| = \|x_0\| = 1$ and so $x_0 \in NA(T)$. Thus $NA(T) \neq \emptyset$ and

$$1 \geqslant \left\| \frac{x_n + x_0}{2} \right\| \geqslant \left\| \frac{T(x_n) + T(x_0)}{2} \right\| \overset{k\to\infty}{\longrightarrow} \|T(x_0)\| = 1.$$

This implies that $\lim_{k\to\infty} \|x_{n_k} + x_0\| = 2$ and, using that X is LUR, we get that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = 0$, which gives the following contradiction:

$$\varepsilon_0 \leqslant \operatorname{dist}(x_{n_k}, NA(T)) \leqslant \|x_{n_k} - x_0\| \overset{k\to\infty}{\longrightarrow} 0. \qquad \square$$

Corollary 2.2.6. If X uniformly convex and Y is a Banach space with Schur's property, then the pair $(X;Y)$ has property 1. In particular, $(\ell_2;\ell_1)$ has property 1.

Proof. We apply Theorem 2.2.5. To do this, we prove that every bounded linear operator $T : X \longrightarrow Y$ is compact. Indeed, since T is continuous, T is w-w continuous. Let $(x_n)_n \subset B_X$. Since X is reflexive, by the Smulian theorem, there are a subsequence of $(x_n)_n$ (which we denote again by $(x_n)_n$) and $x_0 \in X$ such that $x_n \overset{w}{\longrightarrow} x_0$. So $T(x_n) \overset{w}{\longrightarrow} T(x_0)$. Now, since Y has the Schur's property, $T(x_n) \longrightarrow T(x_0)$ in norm. So T is compact. By Theorem 2.2.5, the pair (X, Y) has property 1. $\square$

Corollary 2.2.7. If X is a reflexive Banach space which is LUR and Y is a finite dimensional Banach space, then the pair (X, Y) has property 1.

Proof. This is proved by using the fact that every bounded linear operator with finite dimensional range is compact and Theorem 2.2.5. □

What happen if the η that appears in Definition 2.2.2 of property 1 depends just on ε? As we will see, it seems to be difficult to get positive results when we add this condition in property 1. Just to help make reference, we put a name on it.

Definition 2.2.8 (Property 2)**.** We say that a pair of Banach spaces $(X;Y)$ has *property* 2 if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X;Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there is $x_1 \in S_X$ such that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

In this case, we say that the pair $(X;Y)$ has property 2 with the function $\varepsilon \longmapsto \eta(\varepsilon)$.

We observe that we already have examples of pairs of Banach spaces which satisfy property 2. The Kim-Lee Theorem says that a Banach space X is uniformly convex if and only if the pair $(X;\mathbb{K})$ has property 2. Note also that if the pair $(X;Y)$ satisfies property 2 then the pair $(X;Y)$ satisfies the BPBp. First we notice that if $(X;Y)$ has property 2 for some Banach space Y, then the Banach space X must be uniformly convex. This is the dual version of Proposition 2.1.4.

Proposition 2.2.9. Let X be a Banach space. Suppose that there is some Banach space Y such that the pair $(X;Y)$ has property 2. Then X is uniformly convex.

Proof. Let $\varepsilon \in (0,1)$ be given. Consider $\eta(\varepsilon) > 0$ to be the positive real number that satisfies property 2 for the pair $(X;Y)$. We prove that the pair $(X;\mathbb{K})$ has the same property with $\eta(\varepsilon)$ and we use the Kim-Lee Theorem to conclude the proof. Let $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ be such that

$$|x_0^*(x_0)| > 1 - \eta(\varepsilon).$$

Let $y_0 \in S_Y$ and define $T \in \mathcal{L}(X;Y)$ by $T(x) := x_0^*(x)y_0$ for all $x \in X$. So $\|T\| = \|x_0^*\| = 1$ and

$$\|T(x_0)\| = |x_0^*(x_0)| > 1 - \eta(\varepsilon).$$

Since the pair $(X;Y)$ has property 2 with $\eta(\varepsilon)$, there exists $x_1 \in S_X$ such that $\|T(x_1)\| = 1$ and $\|x_0 - x_1\| < \varepsilon$. Since $\|T(x_1)\| = |x_0^*(x_1)|$ the proof is complete. □

By this last result, since ℓ_1^2 is not uniformly convex, all the pairs $(\ell_1^2;Y)$ fail property 2 for all Banach spaces Y. What about the converse of Proposition 2.2.9? The first idea that one may have is to assume that the domain space X is a Hilbert space (since every Hilbert space is uniformly convex) and try to find some Banach space Y such that the pair (X,Y) satisfies the property. But even in this simple situation, the result fails unlike Theorem 2.1.7 for the BPBpp. In fact, the idea from now on is start to put some conditions on the Banach spaces X and Y trying get some positive result concerning property 2.

Example 2.2.10. This example works for both real and complex cases. For a given $\varepsilon > 0$, suppose that there exists $\eta(\varepsilon) > 0$ satisfying property 2 for the pair $(\ell_2^2;\ell_\infty^2)$. Let $T : \ell_2^2 \longrightarrow \ell_\infty^2$ be defined by

$$T(x,y) := \left(\left(1 - \frac{1}{2}\eta(\varepsilon)\right)x, y\right) \quad \left((x,y) \in \ell_2^2\right).$$

For every $(x, y) \in B_{\ell_2^2}$, we have

$$\|T(x,y)\|_\infty = \left\| \left(\left(1 - \frac{1}{2}\eta(\varepsilon)\right) x, y \right) \right\|_\infty \leqslant 1.$$

Since $T(e_2) = 1$, we obtain $\|T\| = 1$. Moreover,

$$\|T(e_1)\| = 1 - \frac{1}{2}\eta(\varepsilon) > 1 - \eta(\varepsilon).$$

We prove now that every $z = (a, b) \in S_{\ell_2^2}$ such that $\|T(z)\|_\infty = 1$ assumes the form $z = \lambda e_2$ for $|\lambda| = 1$. Indeed, since

$$\left|1 - \frac{1}{2}\eta(\varepsilon)\right| < 1 \quad \text{and} \quad \|T(z)\|_\infty = 1,$$

we have $|b| = 1$. Since $|a|^2 + |b|^2 = 1$, we have $a = 0$ and $b = \lambda$ with $|\lambda| = 1$. In summary, we have a norm one operator T and a norm one vector e_1 satisfying $\|T(e_1)\| > 1 - \eta(\varepsilon)$ but if T attains its norm at some point $z \in S_{\ell_2^2}$ then $z = (0, \lambda)$ with $|\lambda| = 1$. This contradicts the assumption that the pair $(\ell_2^2; \ell_\infty^2)$ has property 2 since z is far from e_1 in view of the fact that $\|e_1 - z\|_2 = \|(1, \lambda)\|_2 = \sqrt{2}$.

This shows that the pair $(\ell_2^2(\mathbb{K}); \ell_\infty^2(\mathbb{K}))$ fails property 2 for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Now what if we add on the hypothesis that both X and Y are Hilbert spaces? The answer for this question is still negative, as we may see in the next proposition.

Proposition 2.2.11. Let $1 < p \leqslant q < \infty$ (or $p < q = \infty$). Given $\beta \in (0, 1)$, there exists $T_\beta \in \mathcal{L}(\ell_p^2; \ell_q^2)$ with $\|T_\beta\| = 1$ such that

(i) $\|T_\beta(e_1)\|_q = \beta$ and

(ii) for every $z \in S_{\ell_p^2}$ such that $\|T_\beta(z)\|_q = 1$, we have $\|z - e_1\|_p = 2^{\frac{1}{p}}$.

Proof. Let $\beta \in (0, 1)$ and $1 < p \leqslant q < \infty$. Define $T_\beta : \ell_p^2 \longrightarrow \ell_q^2$ by $T_\beta(x, y) := (\beta x, y)$ for every $(x, y) \in \ell_p^2$. If $\|(x, y)\|_p = 1$, since $p \leqslant q$, we get

$$\|T_\beta(x,y)\|_q = (\beta^q|x|^q + |y|^q)^{\frac{1}{q}} \leqslant (|x|^p + |y|^p)^{\frac{1}{q}} = 1,$$

which implies that $\|T_\beta\| \leqslant 1$. Since $\|T_\beta(e_2)\|_q = \|e_2\|_q = 1$, we have $\|T_\beta\| = 1$. Now, let $z = (a,b) \in S_{\ell_p^2}$ be such that $\|T_\beta(z)\|_q = 1$. We prove that $b = \lambda e_2$ with $|\lambda| = 1$. Indeed, the equality $\|T_\beta(a,b)\|_q = 1$ implies that $\beta^q|a|^q + |b|^q = 1$ and since $|a|^p + |b|^p = 1$, we do the difference between these two equalities to get

$$(|a|^p - \beta^q|a|^q) + (|b|^p - |b|^q) = 0.$$

Since $p \leqslant q$ and $|a|, |b| \leqslant 1$, $|a|^p - \beta^q|a|^q \geqslant 0$ and $|b|^p - |b|^q \geqslant 0$. Because of the above equality, we get that $|a|^p - \beta^q|a|^q = 0 = |b|^p - |b|^q$. But $|a|^q \leqslant |a|^p$ which implies that

$$0 = |a|^p - \beta^q|a|^q \geqslant (1-\beta^q)|a|^p.$$

Thus $a = 0$ and then $b = \lambda e_2$ with $|\lambda| = 1$ as desired. So if $z \in S_{\ell_p^2}$ is such that $\|T_\beta(z)\|_q = 1$, then $\|z - e_1\|_p = 2^{\frac{1}{p}}$ which completes the proof. □

As a consequence of this last result, we get that all the pairs $(\ell_p(\mathbb{K}); \ell_q(\mathbb{K}))$ fail property 2 for $1 < p \leqslant q < \infty$ when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In particular, the pair $(\ell_2^2; \ell_2^2)$ fails it as well (and this was already known; see example right below [46, Corollary 2.4]). Moreover, since ℓ_∞^2 is isometrically isomorphic to ℓ_1^2 in the real case, the pair $(\ell_2^2; \ell_1^2)$ also fails property 2 in the real case. Next we show that the pairs $(\ell_2^2(\mathbb{R}); \ell_q^2(\mathbb{R}))$ for $1 \leqslant q < 2$ can not satisfy property 2.

Proposition 2.2.12. Let $1 \leqslant q < 2$. In the real case, given $\beta \in (0,1)$, there exists $\mathcal{L}(\ell_2^2; \ell_q^2)$ with $\|T_\beta\| = 1$ such that

(i) $\|T_\beta(e_1)\|_q = \beta$ and

(ii) for every $z \in \ell_2^2$ such that $\|T_\beta(z)\|_q = 1$ we have $\|z - e_1\|_2 = \sqrt{2}$.

Proof. Let $1 \leqslant q < 2$ and define $T : \ell_2^2 \longrightarrow \ell_q^2$ by

$$T(x, y) := \left(\frac{x - y}{2^{\frac{1}{q}}}, \frac{x + y}{2^{\frac{1}{q}}} \right)$$

for every $(x, y) \in \ell_2^2$. First note that

$$\|T(e_2)\|_q^q = \left| -\frac{1}{2^{\frac{1}{q}}} \right|^q + \left(\frac{1}{2^{\frac{1}{q}}} \right)^q = \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $\|T(e_2)\|_q = 1$. Analogously, $\|T(e_1)\|_q = 1$. Since T is a scalar multiple of the composition of a rotation in ℓ_2^2 and the identity from ℓ_2^2 into ℓ_q^2, we have that $\|T\| = 1$. Next, we show that the only points which T attains its norm are at $\pm e_1$ and $\pm e_2$. To do so, we study the norm of the operator T by using the following compact set:

$$K := \left\{ (a, b) \in \mathbb{R}^2 : a^2 + b^2 \leqslant 1, \ a, b \geqslant 0 \right\}.$$

By symmetry, the norm of T is the maximum of $\|T(z)\|$ with z in K. Let $z_0 = (a_0, b_0)$ a point of K such that T attains its norm at z_0, that is, $\|T\| = \|T(z_0)\|$. We consider K_1 as the segment that connect $(0, 0)$ with e_1, K_3 as the segment that connect $(0, 0)$ with e_2 and K_2 as the arc that connect e_1 with e_2. See Figure 2.1.

It is enough to study the values of $\|T(z)\|_q$ on the set $K_2 \setminus \{e_1, e_2\}$ since the operator T attains its norm at elements of the sphere and $\|T(e_1)\|_q = \|T(e_2)\|_q = 1$. We have

$$K_2 \setminus \{e_1, e_2\} = \left\{ (x, f(x)) : x \in \left(\frac{1}{\sqrt{2}}, 1 \right) \right\} \\ \cup \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\} \cup \left\{ (g(y), y) : y \in \left(\frac{1}{\sqrt{2}}, 1 \right) \right\},$$

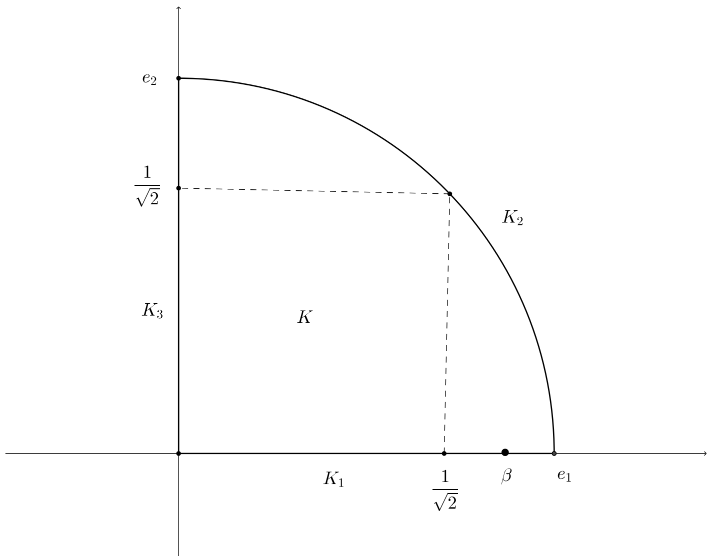

Figure 2.1

with $f : \left(\frac{1}{\sqrt{2}}, 1\right) \longrightarrow \mathbb{R}$ defined as $f(x) = (1-x^2)^{\frac{1}{2}}$ and $g : \left(\frac{1}{\sqrt{2}}, 1\right) \longrightarrow \mathbb{R}$ defined as $g(y) = (1-y^2)^{\frac{1}{2}}$. Since

$$\left\| T\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\|_q = \frac{2}{2^{\frac{1}{2}+\frac{1}{q}}} < 1$$

for every $1 \leqslant q < 2$, then $z_0 \neq \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. (On the other hand observe that if $q = 2$, then $\left\| T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \right\|_q = 1$ and if $q > 2$, then $\left\| T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \right\|_q > 1$.) Thus if $z_0 \in K_2 \setminus \{e_1, e_2\}$, then either $z_0 \in \left\{(x, f(x)) : x \in \left(\frac{1}{\sqrt{2}}, 1\right)\right\}$ and then a_0 would be a critical point of F in $\left(\frac{1}{\sqrt{2}}, 1\right)$, where

$$F(x) = \|T(x, f(x))\|_q^q = \frac{1}{2}\left[\left(x - (1-x^2)^{\frac{1}{2}}\right)^q + \left(x + (1-x^2)^{\frac{1}{2}})\right)^q\right.$$

or $z_0 \in \left\{(g(y), y) : y \in \left(\frac{1}{\sqrt{2}}, 1\right)\right\}$ and in this case b_0 would be a critical point of G in $\left(\frac{1}{\sqrt{2}}, 1\right)$, where

$$G(y) = \|T(g(y), y)\|_q^q = \frac{1}{2}\left[\left(y - (1-y^2)^{\frac{1}{2}}\right)^q + \left((1-y^2)^{\frac{1}{2}} + y\right)^q\right].$$

But, as we will see in the next lines, these can not happen because $F'(x) > 0$ and $G'(y) > 0$ for all $x, y \in \left(\frac{1}{\sqrt{2}}, 1\right)$ and then $z_0 \notin K_2 \setminus \{e_1, e_2\}$. Indeed, we consider first the case that $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$. For every $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, we get

$$\begin{aligned} F'(x) = \frac{q}{2}\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1} &\left(1 + \frac{x}{(1-x^2)^{\frac{1}{2}}}\right) \\ &+ \frac{q}{2}\left(x + (1-x^2)^{\frac{1}{2}}\right)^{q-1}\left(1 - \frac{x}{(1-x^2)^{\frac{1}{2}}}\right). \end{aligned}$$

For $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, we have that $\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1} > 0$ and since

$$\left(x + (1-x^2)^{\frac{1}{2}}\right)^{q-1} \geqslant \left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1}$$

for every x on this interval, we obtain that

$$\begin{aligned} F'(x) &\geqslant \frac{q}{2}\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1}\left(1 + \frac{x}{(1-x^2)^{\frac{1}{2}}} + 1 - \frac{x}{(1-x^2)^{\frac{1}{2}}}\right) \\ &= q\left(x - (1-x^2)^{\frac{1}{2}}\right)^{q-1}, \end{aligned}$$

for every $x \in (\frac{1}{\sqrt{2}}, 1)$ and the last expression is strictly positive. A simply change of the letter F by G and x by y imply that $G'(y) > 0$ for every $y \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

Everything we did so far was to prove that T attains its norm on K only at $z = e_1$ and $z = e_2$. Therefore, we may conclude that T attains its maximum at $\pm e_1$ and at $\pm e_2$. In other words, we proved that

$T(B_{\ell_2^2}) \cap S_{\ell_q^2} = \{\pm e_1, \pm e_2\}$. Now, for $0 < \beta < 1$, define $T_\beta : \ell_2^2 \longrightarrow \ell_q^2$ by

$$T_\beta(x,y) = \left(\frac{\beta x - y}{2^{\frac{1}{q}}}, \frac{\beta x + y}{2^{\frac{1}{q}}} \right),$$

for every $(x,y) \in \ell_2^2$. Note that $\|T_\beta(e_1)\|_q = \beta$ and $\|T_\beta(e_2)\|_q = 1$. Since $T_\beta(B_{\ell_2^2}) \subset T(B_{\ell_2^2}) \subset B_{\ell_q^2}$, then $\|T_\beta\| \leqslant 1$. Also, using that $T(B_{\ell_2^2}) \cap S_{\ell_q^2} = \{\pm e_1, \pm e_2\}$ and that $\|T_\beta(\pm e_1)\|_q < 1$, we have that $\|T_\beta(\pm e_2)\|_q = 1$. This implies that if $z \in S_{\ell_2^2}$ is such that $\|T_\beta(z)\|_q = 1$, then $z = \pm e_2$ and therefore $\|e_1 - z\|_2 = \sqrt{2}$ as we wanted. □

As a consequence of Propositions 2.2.11 and 2.2.12 we have the following corollary.

Corollary 2.2.13. The pair $(\ell_2^2(\mathbb{R}); \ell_q^2(\mathbb{R}))$ fails property 2 for every $1 \leqslant q \leqslant \infty$.

What about the case in which $1 < p \leqslant 2$ and $1 \leqslant q < 2$? We study the real case of this now. Consider $1 < p \leqslant 2$. Define $\mathrm{Id} : \ell_p^2(\mathbb{R}) \longrightarrow \ell_2^2(\mathbb{R})$ by $\mathrm{Id}(x,y) = (x,y)$ for every $(x,y) \in \ell_p^2(\mathbb{R})$. Then $\mathrm{Id}(e_1) = e_1$ and $\mathrm{Id}(e_2) = e_2$. Since $p \leqslant 2$, it is clear that $\|\mathrm{Id}\| = 1$. Given $0 < \beta < 1$, let $T_\beta : \ell_2^2(\mathbb{R}) \longrightarrow \ell_q^2(\mathbb{R})$ be as in the Proposition 2.2.12 with $1 \leqslant q < 2$. Now, define $\widetilde{T}_\beta : \ell_p^2(\mathbb{R}) \longrightarrow \ell_q^2(\mathbb{R})$ by $\widetilde{T}_\beta = T_\beta \circ \mathrm{Id}$. Then $\|\widetilde{T}_\beta\| \leqslant \|T_\beta\| \|\mathrm{Id}\| = 1$. Also, $\|\widetilde{T}_\beta(e_1)\|_q = \|(T_\beta \circ \mathrm{Id})(e_1)\|_q = \beta$ and $\|\widetilde{T}_\beta(e_2)\|_q = \|(T_\beta \circ \mathrm{Id})(e_2)\|_q = \|T_\beta(e_2)\|_q = 1$. Suppose that there exists $z \in S_{\ell_p^2(\mathbb{R})}$ such that $\|\widetilde{T}_\beta(z)\|_q = 1$. Thus $\|T_\beta(z)\|_q = 1$ and then, as we can see in the proof of the Proposition 2.2.12, z must be equals to e_2 or $-e_2$. In both cases, we have that $\|e_1 - z\|_p = 2^{\frac{1}{p}}$. We just have proved the following result.

Corollary 2.2.14. The pair $(\ell_p^2(\mathbb{R}); \ell_q^2(\mathbb{R}))$ fails property 2 for $1 < p \leqslant 2$ and $1 \leqslant q \leqslant 2$.

Next we observe that whenever we put the supremum norm in the range space, the property fails for any pair of the form $(X; \ell_\infty^2)$ which gives Example 2.2.10 in particular.

Proposition 2.2.15. The pair (X, ℓ_∞^2) fails property 2 for all Banach space X with $\dim(X) \geqslant 2$.

Proof. Suppose that there exists $\eta(\varepsilon) > 0$ that depends only on a given $\varepsilon \in (0,1)$ satisfying the property. Let $x_1^*, x_2^* \in S_{X^*}$ and $x_1, x_2 \in S_X$ be such that $x_i^*(x_j) = \delta_{ij}$ for $i, j = 1, 2$. Define $T : X \longrightarrow \ell_\infty^2$ by

$$T(x) := ((1 - \eta(\varepsilon))\, x_1^*(x), x_2^*(x)) \quad (x \in X).$$

Then $\|T(x_1)\|_\infty = 1 - \eta(\varepsilon)$ and $\|T(x_2)\|_\infty = 1$. Moreover, since $1 - \eta(\varepsilon) < 1$, we have that $\|T\| \leqslant 1$. This shows that $\|T\| = 1$. Therefore, there exists $z \in S_X$ such that

$$\|T(z)\|_\infty = 1 \quad \text{and} \quad \|z - x_1\| < \varepsilon.$$

Since $\|T(z)\|_\infty = \max\{|(1 - \eta(\varepsilon))|x_1^*(z)|, |x_2^*(z)|\}$ and $(1-\eta(\varepsilon))|x_1^*(z)| < 1$, we have that $|x_2^*(z)| = 1$. On the other hand, since $|x_2^*(z - x_1)| \leqslant \|z - x_1\| < \varepsilon$ we get a contradiction, since

$$1 = |x_2^*(z)| = |x_2^*(z - x_1) + x_2^*(x_1)| = |x_2^*(z - x_1)| < \varepsilon < 1. \qquad \square$$

We show now that if Y is a 2-dimensional Banach space, then the pair (Y, Y) does not have property 2. To do so, we use the existence of Auerbach bases for all finite dimensional Banach space (see, for example, [41, Proposition 20.21]). Let Y be an n-dimensional Banach space. Then there are elements $e_1, \ldots, e_n$ of Y and $y_1^*, \ldots, y_n^*$ of Y^* such that $\|e_i\| = \|y_i^*\| = 1$ for all i and $y_i^*(e_j) = \delta_{ij}$ for $i \neq j$. In fact, $\{e_1, \ldots, e_n\}$ is a basis of Y called an *Auerbach basis* of Y.

Proposition 2.2.16. Let Y be a 2-dimensional Banach space. Then the pair $(Y; Y)$ fails property 2.

Proof. We start by taking an Auerbach basis: let $\{e_1, e_2\}$ and $\{y_1^*, y_2^*\}$ satisfying $\|e_i\| = \|y_i^*\| = 1$ for $i = 1, 2$ and $y_i^*(e_j) = \delta_{ij}$ for $i, j =$

1, 2. Since $\{e_1, e_2\}$ is a basis for Y, every $y \in Y$ has an expression in terms of e_1, e_2, y_1^* and y_2^* given by $y = y_1^*(y)e_1 + y_2^*(y)e_2$. Given $\beta \in (0,1)$, define the continuous linear operator $T_\beta : Y \longrightarrow Y$ by $T_\beta(y) = \beta y_1^*(y)e_1 + y_2^*(y)e_2$ for all $y = y_1^*(y)e_1 + y_2^*(y)e_2 \in Y$. Then for all $y \in S_Y$, we have that

$$\begin{aligned}
\|T_\beta(y)\| &= \|\beta y_1^*(y)e_1 + y_2^*(y)e_2\| \\
&= \|\beta y_1^*(y)e_1 + \beta y_2^*(y)e_2 - \beta y_2^*(y)e_2 + y_2^*(y)e_2\| \\
&\leqslant \|\beta(y_1^*(y)e_1 + y_2^*(y)e_2)\| + \|(1-\beta)y_2^*(y)e_2\| \\
&= \beta\|y\| + (1-\beta)\|y_2^*(y)e_2\| \\
&\leqslant \beta\|y\| + (1-\beta)|y_2^*(y)| \\
&\leqslant \beta + 1 - \beta \\
&= 1.
\end{aligned}$$

Then $\|T_\beta\| \leqslant 1$. Also, note that $\|T_\beta(e_2)\| = \|e_2\| = 1$. So $\|T_\beta\| = 1$. Now let $y_0 \in S_Y$ be such that $\|T_\beta(y_0)\| = 1$. Then, using that

$$1 = \|T_\beta(y_0)\| \leqslant \beta\|y_0\| + (1-\beta)|y_2^*(y_0)| \leqslant 1,$$

we get that $|y_2^*(y_0)| = 1$ and therefore

$$\|e_1 - y_0\| \geqslant |y_2^*(e_1) - y_2^*(y_0)| = |-1| = 1.$$

Finally, if the pair $(Y;Y)$ has property 2, there exists $\eta(\varepsilon) > 0$ satisfying the property. If we put $\beta = 1 - \frac{\eta(\varepsilon)}{2}$, there exists an operator $T \in \mathcal{L}(Y;Y)$ such that $\|T\| = 1$, $\|T(e_1)\| > 1 - \eta(\varepsilon)$ and for all $y_0 \in S_Y$ which satisfies $\|T(y_0)\| = 1$ we have $\|e_1 - y_0\| \geqslant 1$. This is a contradiction and the pair $(Y;Y)$ fails property 2. □

Remark 2.2.17. If $\dim(Y) = n \geqslant 2$, the proof of Proposition 2.2.16 works as well in this situation. Indeed, for $\beta \in (0,1)$ we define $T_\beta \in$

$\mathcal{L}(Y;Y)$ by

$$T_\beta(y) = \beta y_1^*(y)e_1 + \beta y_2^*(y)e_2 + \ldots + \beta y_{n-1}^*(y)e_{n-1} + y_n^*(y)e_n$$

for all $y \in Y$, where $\{e_1, \ldots, e_n\} \subset S_Y$ and $\{y_1^*, \ldots, y_n^*\} \subset S_{Y^*}$ is given by the Auerbach basis. Then $\|T_\beta(e_i)\| = \beta$ for $i \neq n$ and $\|T_\beta(e_n)\| = 1$. To prove that $\|T_\beta\| \leqslant 1$, we add and subtract the term

$$\beta y_1^*(y)e_1 + \ldots y_{n-2}^*(y)e_{n-2} + \beta y_n^*(y)e_n$$

in $\|T_\beta(y)\|$ where $y \in S_Y$, to get $\|T_\beta(y)\| \leqslant \beta\|y\| + (1-\beta)|y_n^*(y)| \leqslant 1$. Now, if T_β attains its norm at some $y_0 \in S_Y$ then

$$\|e_i - y_0\| \geqslant |y_n^*(e_i - y_0)| = |y_n^*(y_0)| = 1$$

for all $i \neq n$.

It is clear but it is worth mentioning that if the pair $(X;Y)$ has property 2, then the pair $(X;Z)$ also has this property for all closed subspaces Z of Y. Because of that since ℓ_∞^2 is a closed subspace of $C[0,1]$ and the pair $(\ell_2^2;\ell_\infty^2)$ does not satisfy property 2 (see Example 2.2.10), the pair $(\ell_2^2;C[0,1])$ also fails this property. Another consequence of this fact is given in the next corollary.

Corollary 2.2.18. If Y is a Banach space which contains strictly convex 2-dimensional subspaces, then there exists a uniformly convex Banach space X such that the pair $(X;Y)$ fails property 2.

Proof. Indeed, let Z be a subspace of Y such that Z is stricly convex and $\dim(Z) = 2$. Then $X = Z$ is uniformly convex, since Z is finite dimensional. By Proposition 2.2.16, the pair $(X;Z)$ fails property 2 and by the above observation the pair $(X;Y)$ can not have this property. □

By Corollary 2.2.7, the pair $(\ell_2; Z)$ has property 1 if $\dim(Z) < \infty$. But in the case of property 2, we get a negative result. In fact, we show in the next proposition that the pair $(\ell_2; \ell_2^2)$ fails property 2.

Proposition 2.2.19. The pair $(\ell_2; \ell_2^2)$ fails property 2.

Proof. Suppose by contradiction that the pair $(\ell_2; \ell_2^2)$ satisfies property 2. Then given $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(\ell_2; \ell_2^2)$ with $\|T\| = 1$ and $x_0 \in S_{\ell_2}$ are such that $\|T(x_0)\|_2 > 1 - \eta(\varepsilon)$, there is $x_1 \in S_{\ell_2}$ such that $\|T(x_1)\| = 1$ and $\|x_1 - x_0\| < \varepsilon$. Since the pair $(\ell_2^2; \ell_2^2)$ fails property 2, there exists some $\varepsilon_0 > 0$, a norm one linear operator $R : \ell_2^2 \longrightarrow \ell_2^2$ and a norm one vector $(a_0, b_0) \in S_{\ell_2^2}$ with $\|R(a_0, b_0)\| > 1 - \eta(\varepsilon_0)$ such that there is no point $(c_1, c_2) \in S_{\ell_2^2}$ such that $\|R(c_1, c_2)\|_2 = 1$ and $\|(c_1, c_2) - (a_0, b_0)\|_2 < \varepsilon_0$. Let $\pi : \ell_2 \longrightarrow \ell_2^2$ be the projection on the first two coordinates, i.e., $\pi((a_n)_n) := (a_1, a_2)$ for all $(a_n)_n \in \ell_2$. Then $\|\pi\| = 1$. Define $T : \ell_2 \longrightarrow \ell_2^2$ by $T := R \circ \pi$. Then $\|T\| = \|R\| = 1$. Let $x_0 := (a_0, b_0, 0, 0, \ldots) \in S_{\ell_2}$. We have that

$$\|T(x_0)\| = \|R(a_0, b_0)\| > 1 - \eta(\varepsilon_0).$$

Then there exists $x_1 := (c_n)_n \in S_{\ell_2}$ such that $\|T(x_1)\|_2 = 1$ and $\|x_1 - x_0\|_2 < \varepsilon_0$. Since

$$1 = \|T(x_1)\|_2 = \|R(\pi(x_1))\|_2 = \|R(c_1, c_2)\|_2 \leqslant \|(c_1, c_2)\|_2 \leqslant \|x_1\|_2 = 1,$$

we get that $\|R(c_1, c_2)\|_2 = \|(c_1, c_2)\|_2 = 1$. On the other hand,

$$\|(a_0, b_0) - (c_1, c_2)\|_2 \leqslant \|x_0 - x_1\|_2 < \varepsilon_0.$$

This is a contradiction and then the pair $(\ell_2; \ell_2^2)$ fails property 2 as desired. $\square$

So, as we already mention and now the reader must be convinced of that, property 2 seems to be very strong in the sense that it could be difficult to get a positive result about it.

To finish this section, we present a complete characterization for the pairs $(\ell_p;\ell_q)$ concerning property 1. In particular, there are uniformly convex (and then reflexive) Banach spaces X such that the pair $(X;Y)$ fails property 1.

Theorem 2.2.20. The following holds.

(i) The pair $(\ell_p;\ell_q)$ has property 1 whenever $1 \leqslant q < p < \infty$.

(ii) The pair $(\ell_p;\ell_q)$ fails property 1 whenever $1 < p \leqslant q < \infty$.

Proof. (i) By Pitt's Theorem (see, for example, [9, Theorem 2.1.4]), every bounded linear operator from ℓ_p into ℓ_q with $1 \leqslant q < p < \infty$ is compact. By Theorem 2.2.5 the pair $(\ell_p;\ell_q)$ has property 1 since ℓ_p is uniformly convex for $1 < p < \infty$.

(ii) Let $\varepsilon \in (0,1)$ and $1 < p \leqslant q < \infty$. Consider ℓ_p and ℓ_q as the Banach spaces $\ell_p(\ell_p^2)$ and $\ell_q(\ell_q^2)$, respectively. For each $n \in \mathbb{N}$ define $T_n : \ell_p^2 \longrightarrow \ell_q^2$ by

$$T_n(x,y) := \left(\left(1-\frac{1}{2n}\right)x, y\right) \quad \left((x,y) \in \ell_p^2\right).$$

Let $z = ((x_n,y_n))_{n\in\mathbb{N}} \in \ell_2$ and let $T : \ell_p \longrightarrow \ell_q$ be defined by

$$T(z) := (T_n(x_n,y_n))_{n\in\mathbb{N}} = \left(\left(1-\frac{1}{2n}\right)x_n, y_n\right)_{n\in\mathbb{N}}.$$

For each $z = ((x_n, y_n))_{n\in\mathbb{N}} \in \ell_p$ we have that $\|z\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p + |y_j|^p\right)^{\frac{1}{p}}$ and then

$$\begin{aligned} \|T(z)\|_q &= \left(\sum_{j=1}^{\infty}\left(1-\frac{1}{2j}\right)^q |x_j|^q + |y_j|^q\right)^{\frac{1}{q}} \\ &\leqslant \left(\sum_{j=1}^{\infty} |x_j|^q + |y_j|^q\right)^{\frac{1}{q}} = \|z\|_q \leqslant \|z\|_p. \end{aligned}$$

So $\|T\| \leqslant 1$. We consider the vectors

$$e_{1,n} := ((0,0),\ldots,(0,0),(1,0),(0,0),\ldots) \in S_{\ell_p}$$

and

$$e_{2,n} := ((0,0),\ldots,(0,0),(0,1),(0,0),\ldots) \in S_{\ell_p}.$$

Thus we get that $\|T(e_{2,n})\|_q = \|(0,1)\|_q = 1$. So $\|T\| = 1$. Suppose that there exists $\eta(\varepsilon, T) > 0$ such that the pair $(\ell_p; \ell_q)$ has property 1. Let $n \in \mathbb{N}$ be such that $\frac{1}{2n} < \eta(\varepsilon, T)$. So since $\|T\| = \|e_{1,n}\|_p = 1$ and

$$\|T(e_{1,n})\|_q = 1 - \frac{1}{2n} > 1 - \eta(\varepsilon, T)$$

there exists $v = (u_n, w_n) \in \ell_p$ such that $\|T(v)\|_q = \|v\|_p = 1$ and $\|v - e_{1,n}\|_2 < \varepsilon$. We claim that $u_j = 0$ for all $j \in \mathbb{N}$. Indeed, suppose that there exists some $j_0 \in \mathbb{N}$ such that $u_{j_0} \neq 0$. Thus

$$\begin{aligned} \|T(v)\|_q &= \left(\sum_{j=1}^{\infty}\left(1-\frac{1}{2j}\right)^q |u_j|^q + |w_j|^q\right)^{\frac{1}{q}} \\ &= \left(\left(1-\frac{1}{2j_0}\right)^q |u_{j_0}|^q + |w_{j_0}|^q + \sum_{j\neq j_0}\left(1-\frac{1}{2j}\right)^q |u_j|^q + |w_j|^q\right)^{\frac{1}{q}} \\ &< \left(|u_{j_0}|^q + |w_{j_0}|^q + \sum_{j\neq j_0} |u_j|^q + |w_j|^q\right)^{\frac{1}{q}} = \|v\|_q \leqslant \|v\|_p = 1 \end{aligned}$$

which is a contradiction. Then $u_j = 0$ for all $j \in \mathbb{N}$ and we have

$$\|e_{1,n} - v\|_q = \left(1 + \sum_{j \neq n} |w_j|^q\right)^{\frac{1}{q}} \geqslant 1^{\frac{1}{q}} = 1 > \varepsilon.$$

This new contradiction shows that the pair $(\ell_p; \ell_q)$ fails property 1 whenever $1 < p \leqslant q < \infty$. □

2.3 The BPBpp for numerical radius on complex Hilbert spaces

Inspired by the Bishop-Phelps-Bollobás property, some authors studied the Bishop-Phelps-Bollobás property for numerical radius (see [13, 32, 37, 47]). To give this definition, we recall the concept of numerical radius. We denote by $\Pi(X)$ the set of all pairs $(x, x^*) \in S_X \times S_{X^*}$ such that $x^*(x) = 1$. Given a bounded linear operator $T : X \longrightarrow X$, we define its *numerical radius* by

$$v(T) := \sup\{|x^*(T(x))| : (x, x^*) \in \Pi(X)\}.$$

It is not difficult to see that v is a semi-norm on the Banach space $\mathcal{L}(X; X)$ of all bounded linear operators from X into X. The inequality $v(T) \leqslant \|T\|$ always holds for all $T \in \mathcal{L}(X; X)$. We refer the reader to [16, 17] for more information and background about numerical radius theory.

The Banach space X has the *Bishop-Phelps-Bollobás property for numerical radius* (BPBp-nu, for short) if for every $\varepsilon > 0$, there exists some $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X; X)$ with $v(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfy

$$|x^*(T(x))| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X;X)$ with $v(S) = 1$ and $(y, y^*) \in \Pi(X)$ such that

$$|y^*(S(y))| = 1, \ \|y^* - x^*\| < \varepsilon, \ \|y - x\| < \varepsilon \text{ and } \|S - T\| < \varepsilon.$$

The Banach spaces ℓ_1 and c_0 have the BPBp-nu ([37, Corollary 3.3 and Corollary 4.2] as well as all finite dimensional Banach spaces [47, Proposition 2] and the Banach space $L_1(\mu)$ for every measure μ [47, Theorem 4.1] (see also [32]). Besides, it is known that the Banach space $C(K)$ has this property in some cases [13, Theorem 2.2]. It is also known that the L_p-spaces satisfy the BPBp-nu when $1 < p < \infty$ [47, Examples 3.5] (see [49] for $p = 2$ in the real case).

We recall that the *numerical index* of a Banach space X is defined as follows:

$$n(X) := \inf\{v(T) : \ T \in \mathcal{L}(X;X), \ \|T\| = 1\}.$$

Equivalently, the numerical index of a Banach space X is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in \mathcal{L}(X;X)$. We note that $0 \leqslant n(X) \leqslant 1$ and $n(X) > 0$ if and only if v and $\| \, . \, \|$ are equivalent norms on $\mathcal{L}(X;X)$. The set of values of the numerical index was established in [30]:

$$\{n(X) : \ X \text{ complex Banach space}\} = [e^{-1}, 1]$$

and

$$\{n(X) : \ X \text{ real Banach space}\} = [0, 1].$$

It is known that a Hilbert space of dimension greater than one has numerical index $1/2$ in the complex case.

In this section we study the BPBpp and the BPBpp for numerical radius on complex Hilbert spaces. The reason that we are working on Hilbert spaces is that they have transitive norm, that is, given two points

x and y in S_H, there exists a linear isometry $R \in \mathcal{L}(H;H)$ such that $R(x) = y$. Moreover, if the points x and y are close to each other, then R can be taken to be close to the identity operator on H (see observations just before Theorem 2.1.7). On the other hand, we assume that these Hilbert spaces are complex since in this case its numerical index is 1/2 and so v and $\| \,.\, \|$ are equivalent norms on $\mathcal{L}(H;H)$. Throughout this section we use these facts without any explicit mention.

Let us define the Bishop-Phelps-Bollobás point property for numerical radius.

Definition 2.3.1 (The BPBpp-nu)**.** We say that a Banach space X has the *Bishop-Phelps-Bollobás point property for numerical radius* (BPBpp-nu, for short) if given $\varepsilon > 0$, there exists some $\eta(\varepsilon) > 0$ such that whenever $(x_0, x_0^*) \in \Pi(X)$ and $T \in \mathcal{L}(X;X)$ with $v(T) = 1$ satisfy

$$|x_0^*(T(x_0))| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X;X)$ with $v(S) = 1$ such that

$$|x_0^*(S(x_0))| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

In this case, we say that X has the BPBpp-nu with the function $\varepsilon \longmapsto \eta(\varepsilon)$.

When T and S belong to a certain class of operators, we say that X has the BPBpp-nu for this class of operators. For example, if X is a Hilbert space and T and S are self-adjoint operators, we say that X has the *BPBpp-nu for self-adjoint operators*.

We note that the BPBpp-nu is the BPBpp version for numerical radius. In other words, we are starting with an operator T which almost attains its numerical radius at some point $(x_0, x_0^*) \in \Pi(X)$ and ending with another operator which attains its numerical radius also at (x_0, x_0^*) and which is close to T.

Let X be a Banach space with $n(X) = 1$ and suppose that X has the BPBpp-nu. Then X must be 1-dimensional. To show this we will prove that X is uniformly smooth by using the characterization in Proposition 2.1.2, and then we will use [45, Theorem 2.1] which says that uniformly smooth Banach spaces with numerical index 1 are 1-dimensional.

Proposition 2.3.2. Let X be a Banach space with $n(X) = 1$. If X has the BPBpp-nu, then X is a 1-dimensional Banach space.

Proof. Let $\varepsilon \in (0,1)$ be given and let $\eta(\varepsilon) > 0$ be the function for the BPBpp-nu for the Banach space X. We will show that the pair $(X;\mathbb{K})$ has the BPBpp. Let $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ be such that

$$|x_0^*(x_0)| > 1 - \eta(\varepsilon).$$

Consider $x_1^* \in S_{X^*}$ to be such that $x_1^*(x_0) = 1$ and define $T \in \mathcal{L}(X;X)$ by $T(x) := x_0^*(x)x_0$ for all $x \in X$. Then $v(T) = \|T\| = 1$, $(x_0, x_1^*) \in \Pi(X)$ and

$$|x_1^*(T(x_0))| = |x_1^*(x_0^*(x_0)x_0)| = |x_0^*(x_0)| > 1 - \eta(\varepsilon).$$

So there is $S \in \mathcal{L}(X;X)$ with $v(S) = \|S\| = 1$ such that

$$|(S^*x_1^*)(x_0)| = |x_1^*(S(x_0))| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Define $z_0^* := S^*x_1^* \in X^*$. Then $|z_0^*(x_0)| = 1$ and so $\|z_0^*\| = 1$. Also, for all $x \in S_X$,

$$\begin{aligned} |z_0^*(x) - x_0^*(x)| &= |x_1^*(S(x)) - x_0^*(x)x_1^*(x_0)| \\ &= |x_1^*(S(x)) - x_1^*(x_0^*(x)x_0)| \\ &\leqslant \|S(x) - T(x)\| \\ &< \varepsilon. \end{aligned}$$

So $\|z_0^* - x_0^*\| < \varepsilon$. This shows that the pair $(X;\mathbb{K})$ has the BPBpp and by Proposition 2.1.2, X is uniformly smooth. Finally, [45, Theorem 2.1] gives that X must be 1-dimensional. $\square$

By Proposition 2.3.2, we notice that the BPBpp-nu seems to be a strong property. Indeed, the only examples that we have satisfying it are the complex Hilbert spaces and this fact is proved in the next proposition. Before we prove that, let us remember that in a complex Hilbert space H we have that $\Pi(H) = \{(h,h) : h \in S_H\}$ and so

$$v(T) = \sup\{|\langle T(h), h\rangle| : h \in S_H\}$$

for every $T \in \mathcal{L}(H;H)$.

Proposition 2.3.3. The complex Hilbert space H has the BPBpp-nu. More precisely, given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that whenever $T \in \mathcal{L}(H;H)$ with $v(T) = 1$ and $h_0 \in S_H$ satisfy

$$|\langle T(h_0), h_0\rangle| > 1 - \eta(\varepsilon),$$

there is $S \in \mathcal{L}(H;H)$ with $v(S) = 1$ such that

$$|\langle S(h_0), h_0\rangle| = 1 \quad \text{and} \quad \|S - T\| < \beta(\varepsilon).$$

Proof. Let $\varepsilon \in (0,1)$ be given. By [49, Corollary 4.3], H has the BPBp-nu with some $\eta(\varepsilon) > 0$. Let $T \in \mathcal{L}(H;H)$ with $v(T) = 1$ and $h_0 \in S_H$ be such that

$$|\langle T(h_0), h_0\rangle| > 1 - \eta(\varepsilon).$$

Then there are $h_1 \in S_H$ and $\widetilde{S} \in \mathcal{L}(H;H)$ with $v(\widetilde{S}) = 1$ such that

$$|\langle \widetilde{S}(h_1), h_1\rangle| = 1, \quad \|h_1 - h_0\| < \varepsilon \;\text{ and }\; \|\widetilde{S} - T\| < \varepsilon.$$

Since H is a complex Hilbert space, we have that $\|\widetilde{S}\| \leqslant 2v(\widetilde{S}) = 2$. On the other hand, since H has a transitive norm and $\|h_1 - h_0\| < \varepsilon$, there exists a linear isometry $R \in \mathcal{L}(H;H)$ such that $R(h_0) = h_1$ and $\|R - Id_H\| < \delta(\varepsilon)$ with $\lim_{t\to\infty}\delta(t) = 0$. Define $S := R^* \circ \widetilde{S} \circ R \in \mathcal{L}(H)$. Then $v(S) \leqslant 1$ and

$$|\langle S(h_0), h_0\rangle| = |\langle(\widetilde{S} \circ R)(h_0), R(h_0)\rangle| = |\langle \widetilde{S}(h_1), h_1\rangle| = 1.$$

So $v(S) = 1$ and S attains its numerical radius at $(h_0, h_0) \in \Pi(H)$. Also,

$$\begin{aligned} \|S - T\| &= \|R^* \circ \widetilde{S} \circ R - T\| \\ &\leqslant \|R^* \circ \widetilde{S} \circ R - R^* \circ \widetilde{S}\| + \|R^* \circ \widetilde{S} - \widetilde{S}\| + \|\widetilde{S} - T\| \\ &\leqslant \|\widetilde{S}\|\|R - Id_H\| + \|\widetilde{S}\|\|R^* - Id_H\| + \|\widetilde{S} - T\| \\ &< 4\delta(\varepsilon) + \varepsilon =: \beta(\varepsilon). \quad \square \end{aligned}$$

From now on we assume that H is a complex Hilbert space. In the next results we will work with the BPBpp and the BPBpp-nu for self-adjoint, anti-symmetric, unitary and normal operators. We say that an operator $T \in \mathcal{L}(H;H)$ is *self-adjoint* if $T = T^*$; it is *normal* if $T^*T = TT^*$; it is *unitary* if $T^*T = TT^* = \mathrm{Id}_H$ and it is *anti-symmetric* if $T = -T^*$. We start with the self-adjoint operators.

Proposition 2.3.4. Let H be a complex Hilbert space. Then

(a) H has the BPBpp-nu for self-adjoint operators.

(b) $(H;H)$ has the BPBpp for self-adjoint operators.

Proof. (a). Let $\varepsilon \in (0,1)$ be given. By Proposition 2.3.3, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ such that H has the BPBpp-nu. Let $T \in \mathcal{L}(H;H)$ be a self-adjoint operator and $h_0 \in S_H$ be such that

$$|\langle T(h_0), h_0\rangle| > 1 - \min\{\eta(\varepsilon), \varepsilon\}.$$

Then there is $\widetilde{S} \in \mathcal{L}(H;H)$ with $v(\widetilde{S}) = 1$ such that

$$|\langle \widetilde{S}(h_0), h_0 \rangle| = 1 \quad \text{and} \quad \|\widetilde{S} - T\| < \beta(\varepsilon).$$

Since T is a self-adjoint operator, $\langle T(h_0), h_0 \rangle \in \mathbb{R}$. We may suppose that $\langle T(h_0), h_0 \rangle > 0$. We set

$$\langle \widetilde{S}(h_0), h_0 \rangle = e^{i\theta} |\langle \widetilde{S}(h_0), h_0 \rangle| = e^{i\theta} \in S_{\mathbb{C}} \text{ and } r := \langle T(h_0), h_0 \rangle \in \mathbb{R}.$$

Then we have $\langle (e^{-i\theta}\widetilde{S})(h_0), h_0 \rangle = 1$. Now since $\|\widetilde{S} - T\| < \beta(\varepsilon)$,

$$|e^{i\theta} - r| = |\langle \widetilde{S}(h_0), h_0 \rangle - \langle T(h_0), h_0 \rangle| \leqslant \|\widetilde{S} - T\| < \beta(\varepsilon).$$

So $|e^{i\theta} - 1| \leqslant |e^{i\theta} - r| + |r - 1| < \beta(\varepsilon) + \varepsilon$ and since $\|\widetilde{S}\| \leqslant 2v(\widetilde{S}) = 2$, we have that

$$\|\widetilde{S} - (e^{-i\theta}\widetilde{S})\| \leqslant |1 - e^{-i\theta}| \|\widetilde{S}\| = |1 - e^{i\theta}| \|\widetilde{S}\| < \frac{\beta(\varepsilon) + \varepsilon}{2} < \beta(\varepsilon) + \varepsilon.$$

By using this last inequality, we get that

$$\|(e^{-i\theta}\widetilde{S}) - T\| \leqslant \|(e^{-i\theta}\widetilde{S}) - \widetilde{S}\| + \|\widetilde{S} - T\| < \beta(\varepsilon) + \varepsilon + \beta(\varepsilon) = 2\beta(\varepsilon) + \varepsilon.$$

We just proved that the operator $S' := (e^{-i\theta}\widetilde{S}) \in \mathcal{L}(H;H)$ is such that

$$v(S') = \text{Re } \langle S'(h_0), h_0 \rangle = 1 \text{ and } \|S' - T\| < \gamma(\varepsilon)$$

where $\gamma(\varepsilon) = 2\beta(\varepsilon) + \varepsilon > 0$. Define $S := \frac{S' + (S')^*}{2} \in \mathcal{L}(H;H)$. Then S is self-adjoint, $v(S) \leqslant 1$,

$$|\langle S(h_0), h_0 \rangle| = \left| \frac{1}{2} \langle S'(h_0), h_0 \rangle + \frac{1}{2} \langle h_0, (S')^*(h_0) \rangle \right| = \text{Re } \langle S'(h_0), h_0 \rangle = 1.$$

and, since $T = T^*$,

$$\|S - T\| \leqslant \frac{1}{2}\|S' - T\| + \frac{1}{2}\|(S')^* - T^*\| < \gamma(\varepsilon).$$

(b). Let $\varepsilon \in (0,1)$ be given. By [33, Theorem 2.1] there is $\eta(\varepsilon) > 0$ such that H has the BPBp for self-adjoint operators. Let $T \in \mathcal{L}(H;H)$ with $\|T\| = 1$ be a self-adjoint operator and $h_0 \in S_H$ be such that

$$\|T(h_0)\| > 1 - \eta(\varepsilon),$$

There are a self-adjoint operator $\widetilde{S} \in \mathcal{L}(H)$ with $\|S\| = 1$ and a point $h_1 \in S_H$ such that

$$\|\widetilde{S}(h_1)\| = 1, \quad \|h_1 - h_0\| < \beta_1(\varepsilon) \quad \text{and} \quad \|S - T\| < \varepsilon$$

where $\lim_{t\to 0} \beta_1(t) = 0$. Since H is a Hilbert space, there is a linear isometry $R \in \mathcal{L}(H)$ such that

$$R(h_0) = h_1 \quad \text{and} \quad \|R - Id_H\| < \delta(\varepsilon)$$

with $\lim_{t\to 0} \delta(t) = 0$. Define $S := R^* \circ \widetilde{S} \circ R \in \mathcal{L}(H;H)$. Then S is self-adjoint, $\|S\| \leqslant 1$,

$$\|S(h_0)\| = \|R^*(\widetilde{S}(R(h_0))\| = \|R^*(\widetilde{S}(h_1))\| = \|\widetilde{S}(h_1)\| = 1$$

and

$$\begin{aligned}\|S - T\| &= \|R^* \circ \widetilde{S} \circ R - T\| \\ &\leqslant \|R^* \circ \widetilde{S} \circ R - R^* \circ \widetilde{S}\| + \|R^* \circ \widetilde{S} - \widetilde{S}\| + \|\widetilde{S} - T\| \\ &\leqslant \|R - Id_H\| + \|R^* - Id_H\| + \|\widetilde{S} - T\| \\ &< 2\delta(\varepsilon) + \varepsilon := \beta(\varepsilon).\end{aligned}$$

□

Corollary 2.3.5. Let H be a complex Hilbert space. Then

(a) H has the BPBpp-nu for anti-symmetric operators.

(b) (H, H) has the BPBpp for anti-symmetric operators.

Proof. Both items are consequence of Proposition 2.3.4 and the fact that an operator $T \in \mathcal{L}(H; H)$ is self-adjoint if and only if iT and $-iT$ are anti-symmetric operators. $\square$

For unitary operators, we work just with the BPBpp-nu since if $T \in \mathcal{L}(H; H)$ is unitary then T is an isometry and so $\|T(h)\| = \|h\| = 1$ for every $h \in H$. So the pair $(H; H)$ trivially has the BPBpp for unitary operators.

Proposition 2.3.6. Let H be a complex Hilbert space. Then H has the BPBpp-nu for unitary operators.

Proof. Let $\varepsilon \in (0, 1)$ be given. Let $T \in \mathcal{L}(H; H)$ be a unitary operator with $v(T) = 1$. In particular, T is normal and since H is complex we have that $\|T\| = v(T) = 1$ [35, Equation 1.9]. Now let $h_0 \in S_H$ be such that

$$\operatorname{Re} \langle T(h_0), h_0 \rangle > 1 - \frac{\varepsilon^2}{2}.$$

Then

$$\|T(h_0) - h_0\|^2 = \|T(h_0)\|^2 + \|h_0\|^2 - 2 \operatorname{Re} \langle T(h_0), h_0 \rangle < \varepsilon^2.$$

So $\|T(h_0) - h_0\| < \varepsilon$. Since $\|T(h_0)\| = \|h_0\| = 1$, there is a linear isometry $R \in \mathcal{L}(H; H)$ such that

$$R(T(h_0)) = h_0 \quad \text{and} \quad \|R - Id_H\| < \delta(\varepsilon)$$

with $\lim_{t\to 0}\delta(t) = 0$. Define $S := R \circ T \in \mathcal{L}(H;H)$. Then S is unitary, $v(S) = \|S\| \leqslant 1$,

$$|\langle S(h_0), h_0\rangle| = |\langle R(T(h_0)), h_0\rangle| = |\langle h_0, h_0\rangle| = \|h_0\|^2 = 1.$$

and

$$\|S - T\| = \|R \circ T - T\| \leqslant \|R - Id_H\|\|T\| < \delta(\varepsilon).$$

□

Next we prove the BPBpp for normal operators. The proof follows same ideas of item (b) of Proposition 2.3.4.

Proposition 2.3.7. The Hilbert space H has the BPBpp for normal operators.

Proof. Let $\varepsilon \in (0,1)$ be given. In the first part of the proof [21, Theorem 3.1] it was proved that $(H;H)$ has the BPBp for normal operators with $\varepsilon \longmapsto \eta(\varepsilon)$. Let $T \in \mathcal{L}(H;H)$ be a normal operator with $\|T\| = 1$ and $h_0 \in S_H$ satisfying $\|T(h_0)\| > 1-\eta(\varepsilon)$. Then there are a normal operator $\widetilde{S} \in \mathcal{L}(H;H)$ and $h_1 \in S_H$ such that $\|\widetilde{S}(h_1)\| = 1$, $\|h_0 - h_1\| < \sqrt{2\varepsilon} + \sqrt[4]{2\varepsilon}$ and $\|\widetilde{S} - T\| < \sqrt{2\varepsilon}$. There is a linear isometry $R \in \mathcal{L}(H;H)$ such that $R(h_0) = h_1$ and $\|R - Id_H\| < \delta(\varepsilon)$ with $\lim_{t\to 0}\delta(t) = 0$. Define $S := R^* \circ \widetilde{S} \circ R \in \mathcal{L}(H;H)$. Since $\widetilde{S}$ is normal so is S. Repeating the same arguments of item (b) of Proposition 2.3.4 we have that $\|S(h_0)\| = 1$ and $\|S - T\| < 2\delta(\varepsilon) + \sqrt{2\varepsilon}$. □

2.4 The BPBp for absolute sums

In this short section we work on the stability of the BPBp for absolute sums and it was motivated by [11] where the authors proved the following result.

Theorem 2.4.1. [11, Theorem 2.1] Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be families of Banach spaces, let X and Y be the c_0-,ℓ_1- or ℓ_∞-sum of $\{X_i\}$ and $\{Y_j\}$, respectively. If the pair (X, Y) has the BPBp, then (X_i, Y_j) has it for all $i \in I$ and $j \in J$.

Also they provided some partial converses (see [11, Proposition 2.4]). Before we give our results, let us define and recall basic facts about absolute sums. An *absolute norm* is a norm $|\ .\ |_a$ on $\mathbb{R}^2$ such that $|(1,0)|_a = |(0,1)|_a = 1$ and $|(s,t)|_a = |(|s|,|t|)|_a$ for every $s, t \in \mathbb{R}$. Given two Banach spaces Y and W and an absolute norm $|\ .\ |_a$, the *absolute sum* of Y and W with respect to $|\ .\ |_a$, denoted by $Y \oplus_a W$, is the Banach space $Y \times W$ endowed with the norm

$$\|(y,w)\|_a = |(\|y\|,\|w\|)|_a \quad (y \in Y, w \in W).$$

Examples of absolute sums are the ℓ_p-sums $\oplus_p$ for $1 \leqslant p \leqslant \infty$ associated to the ℓ_p-norm in $\mathbb{R}^2$. It is not difficult to see that $\|(y,0)\|_a = \|y\|$ for all $y \in Y$ and that

$$\|\ .\ \|_\infty \leqslant \|\ .\ \|_a \leqslant \|\ .\ \|_1 \tag{2.5}$$

for every absolute sum $\|\ .\ \|_a$. We recall also that there exists an isomorphic isometry between $(Y \oplus_a W)^*$ and $Y^* \oplus_{a^*} W^*$ where $\oplus_{a^*}$ is defined by

$$\|(y^*,w^*)\|_{a^*} := \sup_{(y,w)\in B_{Y\oplus_a W}} \|y^*\|\|y\| + \|w^*\|\|w\|$$

for every $(y^*, w^*) \in Y^* \times W^*$. The action of $(x^*, y^*) \in X^* \oplus_{a^*} Y^*$ on an element $(x, y) \in X \oplus_a Y$ is given by

$$\langle (x,y),(x^*,y^*)\rangle = x^*(x) + y^*(y).$$

Our first result says that we have stability in the BPBp when we put an absolute sum in the range space. Its proof is an extension of [11, Proposition 2.7].

Proposition 2.4.2. Let X, Y_1 and Y_2 be Banach spaces. Given an absolute norm $|\ .\ |_a$, we set $Y = Y_1 \oplus_a Y_2$. If the pair $(X; Y)$ has the BPBp, then the pairs $(X; Y_1)$ and $(X; Y_2)$ also have it.

Proof. Let $\varepsilon \in (0, 1)$ be given. Consider $\eta(\varepsilon) > 0$ to be the BPBp function for the pair $(X; Y)$. We prove that the pair $(X; Y_1)$ has the BPBp since the other case is completely analogous. Let $T_1 \in \mathcal{L}(X; Y_1)$ with $\|T_1\| = 1$ and $x_0 \in S_X$ be such that

$$\|T_1(x_0)\| > 1 - \eta(\varepsilon).$$

Define $\widetilde{T} \in \mathcal{L}(X; Y)$ by

$$\widetilde{T}(x) = (T_1(x), 0) \quad (x \in X).$$

Then $\|\widetilde{T}\| = \|T\| = 1$ and

$$\|\widetilde{T}(x_0)\|_a = \|(T_1(x_0), 0)\|_a = \|T_1(x_0)\| > 1 - \eta(\varepsilon).$$

Then there are $\widetilde{S} \in \mathcal{L}(X; Y)$ with $\|\widetilde{S}\| = 1$ and $x_1 \in S_X$ such that

$$\|\widetilde{S}(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|\widetilde{S} - T_1\| < \varepsilon.$$

Write $\widetilde{S} = (\widetilde{S}_1, \widetilde{S}_2)$, where $\widetilde{S}_j \in \mathcal{L}(X; Y_j)$ for $j = 1, 2$. By using (2.5), for all $x \in B_X$, we have

$$\begin{aligned} \|(\widetilde{S}_1(x) - T_1(x), \widetilde{S}_2(x))\|_\infty &\leqslant \|(\widetilde{S}_1(x) - T_1(x), \widetilde{S}_2(x))\|_a \\ &\leqslant \|\widetilde{S} - \widetilde{T}\|_a < \varepsilon. \end{aligned}$$

Then $\|\widetilde{S}_1 - T_1\| < \varepsilon$ and $\|\widetilde{S}_2\| < \varepsilon$. Now we consider $y^* = (y_1^*, y_2^*) \in Y_1^* \oplus_{a^*} Y_2^*$ with $\|y^*\|_{a^*} = 1$ such that

$$\operatorname{Re} y^*(\widetilde{S}(x_1)) = \operatorname{Re} y_1^*(\widetilde{S}_1(x_1)) + y_2^*(\widetilde{S}_2(x_1)) = \|\widetilde{S}(x_1)\| = 1.$$

Then

$$\begin{aligned} 1 = \operatorname{Re} y_1^*(\widetilde{S}_1(x_1)) + y_2^*(\widetilde{S}_2(x_1)) &\leqslant \|y_1^*\|\|\widetilde{S}_1(x_1)\| + \|y_2^*\|\|\widetilde{S}_2(x_1)\| \\ &\leqslant \|y^*\|_{a^*} = 1. \end{aligned}$$

So $\operatorname{Re} y_1^*(\widetilde{S}_1(x_1)) = \|y_1^*\|\|\widetilde{S}_1(x_1)\|$ and $\operatorname{Re} y_2^*(\widetilde{S}_2(x_1)) = \|y_2^*\|\|\widetilde{S}_2(x_1)\|$. Since

$$\|y_1^*\|\|\widetilde{S}_1(x_1)\| = \operatorname{Re} y_1^*(\widetilde{S}_1(x_1)) = 1 - \operatorname{Re} y_2^*(\widetilde{S}_2(x_1)) \geqslant 1 - \|\widetilde{S}_2\| > 0,$$

we have that $y_1^* \neq 0$ and $\|\widetilde{S}_1(x_1)\| \neq 0$. Define $S \in \mathcal{L}(X, Y_1)$ by

$$S_1(x) := \|y_1^*\|\widetilde{S}_1(x) + y_2^*(\widetilde{S}_2(x))\frac{\widetilde{S}_1(x_1)}{\|\widetilde{S}_1(x_1)\|} \quad (x \in X).$$

For all $x \in B_X$, $\|S_1(x)\| \leqslant \|y_1^*\|\|\widetilde{S_1}(x)\| + \|y_2^*\|\|\widetilde{S}_2(x)\| \leqslant \|y^*\|_{a^*} = 1$ and

$$\begin{aligned} \|S(x_1)\| &\geqslant \operatorname{Re} \frac{y_1^*}{\|y_1^*\|}\left(\|y_1^*\|\widetilde{S}_1(x_1) + y_2^*(\widetilde{S}_2(x_1))\frac{\widetilde{S}_1(x_1)}{\|\widetilde{S}_1(x_1)\|}\right) \\ &= \operatorname{Re} y_1^*(\widetilde{S}_1(x_1)) + \operatorname{Re} y_2^*(\widetilde{S}_2(x_1)) = 1. \end{aligned}$$

So $\|S_1\| = \|S_1(x_1)\| = 1$. Finally, since

$$1 - \|y_1^*\| \leqslant 1 - \operatorname{Re} y_1^*(\widetilde{S}_1(x_1)) = \operatorname{Re} y_2^*(\widetilde{S}_2(x_1)) \leqslant \|\widetilde{S}_2\| < \varepsilon,$$

we have

$$\|S_1(x) - T_1(x)\| \leqslant \|\|y_1^*\|\widetilde{S}_1(x) - \widetilde{S}_1(x)\| + \|\widetilde{S_1} - T_1\| + \|S_2\| < 3\varepsilon$$

for all $x \in B_X$. Thus $\|S_1 - T_1\| < 3\varepsilon$. Since we already have $\|x_1 - x_0\| < \varepsilon$, the result follows. □

So we can transfer the BPBp from a pair $(X, Y_1 \oplus_a Y_2)$ to the pairs (X, Y_1) and (X, Y_1) for every absolute sum $\oplus_a$. We do not know about the converse of this. In fact, we do not know if the pair $(X, Y_1 \oplus_1 Y_2)$ has the BPBp whenever its components (X, Y_1) and (X, Y_2) have it.

On the other hand, the proof of Proposition 2.4.2 gives the following particular important case.

Proposition 2.4.3. Let X, Y_1 and Y_2 be Banach spaces and let $1 \leqslant p \leqslant \infty$. If the pair $(X; Y_1 \oplus_p Y_2)$ has the BPBp with a function η, then so do the pair $(X; Y_j)$ with $j = 1, 2$ with the function $\eta \longmapsto \eta(\varepsilon/3)$.

Now we study the analogous problem for domain spaces. To do so, we need some definitions. For $x \in X$, we define the set $D(X, x)$ of all $x^* \in X^*$ such that $x^*(x) = \|x\|$. The set $D(X, x)$ is convex and nonempty by the Hahn-Banach theorem. We say that $x \in S_X$ is a *vertex* of B_X if $D(X, x)$ separates the points of X and we say that x is a *smooth point* of B_X if $D(X, x)$ is a singleton subset of X^*. By the remark after Theorem 4.6 of [16], we have that a vertex of B_X is an extreme point of B_X.

The next definition classifies the absolute norms in three different types through the behaviour at the vector $(1, 0)$ and, although our result is proved just for two types, we put all of them for completeness. Let $|\ .\ |_a$ be an absolute norm in $\mathbb{R}^2$.

Definition 2.4.4. We say that $|\ .\ |_a$ is of

(i) *type* 1 if the vector $(1, 0)$ is an vertex of B_X.

(ii) *type* 2 if the vector $(1, 0)$ is a smooth and extreme point of B_X.

(iii) *type* 3 if the vector $(1, 0)$ is a smooth and not extreme point of B_X.

The absolute norm $|\ .\ |_1$ is of type 1. On the other hand, $|\ .\ |_\infty$ is of type 3 and for $1 < p < \infty$, $|\ .\ |_p$ is of type 2. We have the following two characterizations.

Lemma 2.4.5. ([57, Propositions 5.3 and 5.5]) Let $|\ .\ |_a$ be an absolute norm in $\mathbb{R}^2$. Then

(a) $|\ .\ |_a$ is of type 1 if and only if there exists $K > 0$ such that $|x| + K|y| \leqslant |(x, y)|_a$.

(b) $|\ .\ |_a$ is of type 3 if and only if there exists $b_0 > 0$ such that $|(1, b_0)|_a = 1$.

Proposition 2.4.6. Let X_1, X_2 and Y be Banach spaces. Given an absolute norm $|\ .\ |_a$ of type 1 or 3, we set $X = X_1 \oplus_a X_2$. If the pair $(X; , Y)$ has the BPBp, then the pairs $(X_1; Y)$ and $(X_2; Y)$ also have it.

Proof. We show that the pair $(X_1; Y)$ has the BPBp. Let $T \in \mathcal{L}(X_1; Y)$ with $\|T\| = 1$ and $x_0 \in S_{X_1}$ be such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon).$$

Define $\widetilde{T} \in \mathcal{L}(X, Y)$ by

$$\widetilde{T}(x_1, x_2) := T(x_1) \quad ((x_1, x_2) \in X).$$

Then $\|\widetilde{T}\| = 1$ and

$$\|\widetilde{T}(x_0, 0)\| = \|T(x_0)\| > 1 - \eta(\varepsilon).$$

Since the pair $(X; Y)$ has the BPBp with η, there are $\widetilde{S} \in \mathcal{L}(X; Y)$ with $\|\widetilde{S}\| = 1$ and $(x_1', x_2') \in S_X$ such that

$$\|\widetilde{S}(x_1', x_2')\| = 1, \quad \|(x_1', x_2') - (x_0, 0)\|_a < \varepsilon \ \text{ and } \ \|\widetilde{S} - \widetilde{T}\| < \varepsilon.$$

Using (2.5), we get $\|x_1' - x_0\| < \varepsilon$ and $\|x_2'\| < \varepsilon$. Define $S \in \mathcal{L}(X_1; Y)$ by $S(x_1) := \widetilde{S}(x_1, 0)$ for all $x_1 \in X_1$. Suppose first that $|\ .\ |_a$ is an absolute norm of type 1. Then there exists $K > 0$ such that

$$\|x_1'\| + K\|x_2'\| \leqslant |(\|x_1'\|, \|x_2'\|)|_a = \|(x_1', x_2')\|_a = 1.$$

We prove that $x_2' = 0$. Note that for all $x_2 \in X_2$, we have

$$\|\widetilde{S}(0, x_2)\| = \|\widetilde{S}(0, x_2) - \widetilde{T}(0, x_2)\| \leqslant \|\widetilde{S} - \widetilde{T}\| < \varepsilon.$$

Therefore, if we assume that $x_2' \neq 0$, we get for all $\varepsilon \in (0, K)$ that

$$\begin{aligned} 1 = \|\widetilde{S}(x_1', x_2')\| = \|x_1'\| \left\| \widetilde{S}\left(\frac{x_1'}{\|x_1'\|}, 0 \right) \right\| + \|x_2'\| \left\| \widetilde{S}\left(0, \frac{x_2'}{\|x_2'\|} \right) \right\| \\ \leqslant \|x_1'\| + \varepsilon \|x_2'\| \\ < \|x_1'| + K\|x_2'\| \leqslant 1 \end{aligned}$$

which is a contradiction. Then

$$\|S(x_1')\| = \|\widetilde{S}(x_1', 0)\| = 1 = \|S\|, \quad \|S - T\| < \varepsilon \quad \text{and} \quad \|x_1' - x_0\| < \varepsilon.$$

Now assume that $|\ .\ |_a$ is an absolute norm of type 3 and let $\rho = \frac{b_0}{\varepsilon} > 0$. Consider the vector $(x_1', \rho x_2') \in X$. Note that since $\|x_2'\| < \varepsilon$,

$$\|\rho x_2'\| = \rho\|x_2'\| < \rho\varepsilon = b_0.$$

Therefore since $\|(x_1', \rho x_2')\|_a = |(\|x_1'\|, \|\rho x_2'\|)|_a$, $\|x_1'\| \leqslant 1$ and $\|\rho x_2'\| < b_0$, we have by the definition of b_0 that $\|(x_1', \rho x_2')\|_a \leqslant |(1, b_0)|_a = 1$, so that $(x_1', \rho x_2') \in B_X$. So, writting

$$(x_1', x_2') = \left(1 - \frac{\varepsilon}{b_0}\right)(x_1', 0) + \frac{\varepsilon}{b_0}(x_1', \rho x_2')$$

we get

$$1 = \|(x_1', x_2')\|_a \leqslant \left(1 - \frac{\varepsilon}{b_0}\right) \|(x_1', 0)\|_a + \frac{\varepsilon}{b_0}\|(x_1', \rho x_2')\|_a \leqslant 1$$

and

$$1 = \|\widetilde{S}(x_1', x_2')\| \leqslant \left(1 - \frac{\varepsilon}{b_0}\right) \|\widetilde{S}(x_1', 0)\| + \frac{\varepsilon}{b_0}\|\widetilde{S}(x_1, \rho x_2')\| \leqslant 1$$

which imply that $\|x_1'\| = 1$ and $\|S(x_1')\| = \|\widetilde{S}(x_1', 0)\| = 1$. Since we already have $\|S - T\| < \varepsilon$ and $\|x_1' - x_0\| < \varepsilon$, we conclude that the pair $(X_1; Y)$ has the BPBp for operators with $\eta(\varepsilon)$ for all $\varepsilon \in (0, b_0)$. $\square$

Chapter 3

The BPBp for compact operators

3.1 Introduction

In this chapter we study the Bishop-Phelps-Bollobás property for the case that T and S are compact operators in Definition 1.2.2, that is, given a compact operator T which almost attains its norm at some point, there is another compact operator S which satisfies the BPBp conditions (1.2).

Many results about density of norm attaining compact operators were given in the 1970's. For example, $\mathrm{NA}(X;Y) \cap \mathcal{K}(X;Y)$ is dense in $\mathcal{K}(X;Y)$ whenever one of the spaces X, X^*, Y or Y^* is isometrically isomorphic to an $L_1(\mu)$-space [43]. It was actually conjectured that compact operators between Banach spaces can be always approximated by norm attaining compact operators, but it has been recently shown that this is not the case: it has been shown in [53, Theorem 1] that there exist compact operators which cannot be approximated by norm attaining ones. Besides that, the author of [53] studied some conditions on the domain space or on the range space to assure the denseness of norm attaining compact operators. We refer to the survey paper [54] for a detailed account on this subject. Here we study a stronger property: the Bishop-Phelps-Bollobás version of the density of norm attaining compact operators.

Let us define formally the BPBp for compact operators. It is worth to mentioning that it already appeared (mostly without name) in some of the references cited in this dissertation (see [3] for example).

Definition 3.1.1 (BPBp for compact operators)**.** We say that a pair of Banach spaces $(X;Y)$ has the *Bishop-Phelps-Bollobás property for compact operators* (BPBp for compact operators, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{K}(X;Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{K}(X;Y)$ and $x_1 \in S_X$ such that

$$\|S\| = \|S(x_0)\| = 1, \quad \|x_0 - x_1\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

In this case, we say that the pair $(X;Y)$ has the BPBp for compact operators with the function $\varepsilon \longmapsto \eta(\varepsilon)$.

Note that we already studied something similar in Section 2.3 when we were studying the BPBpp and the BPBpp-nu for particular operators defined on complex Hilbert spaces. See also Definition 2.2.2 and Theorem 2.2.5.

An extensive list of known pairs of Banach spaces which satisfy the BPBp for compact operators is given in Examples 3.1.2. Actually, although in most of the papers that contain these results, it is not explicitly stated that the pair $(X;Y)$ satisfies the property for *compact operators*, the proofs can be easily adapted to the compact case.

Examples 3.1.2. The pair of Banach spaces $(X;Y)$ has the BPBp for compact operators when

(a) X is arbitrary and Y has property β (adapting the proof of [2, Theorem 2.2]);

(b) X is uniformly convex and Y is arbitrary (using [6, Corollary 2.3] or adapting the proof of [46, Theorem 3.1]);

(c) X is arbitrary and Y is a uniform algebra - in particular, $Y = C_0(L)$ for a locally compact Hausdorff topological space L - [19, R2 in page 380];

(d) $X = L_1(\mu)$ and $Y = L_1(\nu)$ for arbitrary measures μ and ν (adapting the proof of [26, Theorem 3.1]);

(e) $X = L_1(\mu)$ and $Y = L_\infty(\nu)$ for any measure μ and any localizable measure ν (adapting the proof of [26, Theorem 4.1]);

(g) $X = C_0(L)$ and Y is uniformly convex where L is any locally compact Hausdorff topological space [3, Theorem 3.3];

(h) X is arbitrary and Y^* is isometrically isomorphic to an $L_1(\mu)$-space [3, Theorem 4.2]; in particular, if $Y = C_0(L)$ for a locally compact Hausdorff topological space L;

(i) $X = L_1(\mu)$ for an arbitrary measure and Y having the AHSP [5, Corollary 2.4].

We invite the reader to take a look into the tables 4.1, 4.2 and 4.3 in order to compare the examples of pairs of classic Banach spaces $(X;Y)$ which satisfy the BPBp with the pairs $(X;Y)$ which satisfy the BPBp for compact operators listed above. Doing this, it is natural to ask the following questions.

(Q1) does the BPBp for compact operators imply the BPBp?

(Q2) does the BPBp imply the BPBp for compact operators?

The answer for (Q1) is **no**. Indeed, the pair $(L_1[0,1]; C[0,1])$ has the BPBp for compact operators (by any of the assertions (c), (h) or (i) of Examples 3.1.2) but $\text{NA}(L_1[0,1]; C[0,1])$ is *not* dense in $\mathcal{L}(L_1[0,1]; C[0,1])$ (by Schachermayer's counterexample [59, Theorem A]) and thus, the pair $(L_1[0,1]; C[0,1])$ does not have the BPBp. On the other hand, (Q2) seems to be an open problem.

Section 3.2 is devoted to give some technical results about the BPBp for compact operators. We will apply them in section 3.3 in order to get more examples of pairs $(X; Y)$ which satisfy it.

3.2 The tools

In this section we present abstract results about the Bishop-Phelps-Bollobás property for compact operators which makes possible to carry it from sequences spaces to functions spaces.

First we would like to mention that a routine change of parameters in Definition 3.1.1 allows us to show that we may require the conditions not only for norm one operators and vectors, but for operators and vectors with norm less than or equal to one. We prove this simple observation for the compact version of the BPBp although its proof holds for all Definitions 2.1.1, 2.1.12, 4.1.1 and 4.1.2.

Remark 3.2.1. Let X and Y be Banach spaces. The pair $(X; Y)$ has the *BPBp for compact operators* if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{K}(X; Y)$ with $\|T\| \leqslant 1$ and $x_0 \in B_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{K}(X; Y)$ and $x_1 \in S_X$ such that

$$\|S\| = \|S(x_0)\| = 1, \quad \|x_0 - x_1\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Indeed, for $\varepsilon > 0$ suppose that $(X;Y)$ has the BPBp with some function $\eta(\varepsilon) > 0$. Let $T \in \mathcal{L}(X;Y)$ with $0 < \|T\| \leqslant 1$ and $0 < \|x_0\| \leqslant 1$ satisfying

$$\|T(x_0)\| > 1 - \min\left\{\eta\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{2}\right\}.$$

Then

$$\left\|\frac{T}{\|T\|}\left(\frac{x_0}{\|x_0\|}\right)\right\| \geqslant \|T(x_0)\| > 1 - \min\left\{\eta\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{2}\right\}.$$

So there are $S \in \mathcal{K}(X;Y)$ with $\|S\| = 1$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1, \quad \left\|x_1 - \frac{x_0}{\|x_0\|}\right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\|S - \frac{T}{\|T\|}\right\| < \frac{\varepsilon}{2}.$$

Since $1 - \|T\| \leqslant 1 - \|T(x_0)\| < \frac{\varepsilon}{2}$ and $1 - \|x_0\| \leqslant 1 - \|T(x_0)\| < \frac{\varepsilon}{2}$, we have that

$$\|x_1 - x_0\| \leqslant \left\|x_1 - \frac{x_0}{\|x_0\|}\right\| + |1 - \|x_0\|| < \varepsilon$$

and

$$\|S - T\| \leqslant \left\|S - \frac{T}{\|T\|}\right\| + |1 - \|T\|| < \varepsilon.$$

We first deal with domain spaces, for which the results are based on [43, Lemma 3.1]: if a Banach space X admits a net of norm-one projections with finite rank whose adjoints converge pointwise in norm to the identity operator, then $\mathrm{NA}(X;Y) \cap \mathcal{K}(X;Y)$ is dense in $\mathcal{K}(X;Y)$ for every Banach space Y. We note that for the BPBp this result is *not* valid since the finite-dimensionality of the domain space does not guarantee the BPBp as we already mention in the first chapter of this dissertation (see [11, Example 4.1]) and we have to impose additional conditions.

The most general result that we have is the following one, from which we will deduce some particular cases.

Lemma 3.2.2 (Main technical lemma)**.** Let X and Y be Banach spaces. Suppose that there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that given $\delta \in \mathbb{R}^+$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $x_0 \in S_X$, we may find a norm-one operator $P \in \mathcal{L}(X;X)$ and a norm-one operator $i \in \mathcal{L}(P(X);X)$ such that

(1) $\|P^* x_j^* - x_j^*\| < \delta$ for $j = 1, \ldots, n$;

(2) $\|i(P(x_0)) - x_0\| < \delta$;

(3) $P \circ i = \mathrm{Id}_{P(X)}$;

(4) the pair $(P(X);Y)$ has the BPBp for compact operators with the function η.

Then, the pair $(X;Y)$ has the BPBp for compact operators.

Proof. Let $\varepsilon > 0$ be given. Define

$$\eta'(\varepsilon) = \min\left\{\frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{6}\right\}.$$

Let $T \in \mathcal{K}(X;Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \eta'(\varepsilon).$$

First we note that $T^*(B_{Y^*})$ is compact. Indeed, since B_{Y^*} is w^*-compact and T^* is weak*-to-weak* continuous, then $T^*(B_{Y^*})$ is w^*-compact, so it is norm-closed. By Schauder's theorem, T^* is a compact operator and then $\overline{T^*(B_{Y^*})}$ is a compact set. So we can conclude that $T^*(B_{Y^*})$ is compact. Because of this, we may find $x_1^*, \ldots, x_n^* \in B_{X^*}$ such that

$$\min_j \|T^* y^* - x_j^*\| < \eta'(\varepsilon) \qquad (y^* \in B_{Y^*}).$$

Let $P \in \mathcal{L}(X;X)$ and $i \in \mathcal{L}(P(X);X)$ satisfying (1)–(4) for $\delta = \eta'(\varepsilon)$. Then, for every $y^* \in B_{Y^*}$, we have

$$\begin{aligned}\|T^*y^* - P^*T^*y^*\| \leqslant \min_j \Big(\|T^*y^* - x_j^*\| + \|x_j^* - P^*x_j^*\| \\ + \|P^*x_j^* - P^*T^*y^*\| \Big) < 3\eta'(\varepsilon).\end{aligned}$$

Therefore,

$$\|T - TP\| = \|T^* - P^*T^*\| \leqslant 3\eta'(\varepsilon).$$

Next, consider $\widetilde{T} = T|_{P(X)} \in \mathcal{K}(P(X);Y)$. Then, $\|\widetilde{T}\| \leqslant 1$ and

$$\begin{aligned}\|\widetilde{T}(P(x_0))\| &\geqslant \|T(x_0)\| - \|T(x_0) - T(P(x_0))\| \\ &\geqslant \|T(x_0)\| - \|T - TP\| \\ &> 1 - \eta'(\varepsilon) - 3\eta'(\varepsilon) \\ &\geqslant 1 - \eta\left(\frac{\varepsilon}{2}\right).\end{aligned}$$

As the pair $(P(X);Y)$ has the BPBp for compact operators with the function η, there are $\widetilde{S} \in \mathcal{K}(P(X);Y)$ and $\widetilde{x}_1 \in S_{P(X)}$ such that

$$\|\widetilde{S}\| = 1 = \|\widetilde{S}(\widetilde{x}_1)\|, \quad \|P(x_0) - \widetilde{x}_1\| < \frac{\varepsilon}{2}, \quad \|\widetilde{S} - \widetilde{T}\| < \frac{\varepsilon}{2}.$$

Finally, consider $S = \widetilde{S} \circ P \in \mathcal{K}(X;Y)$ which satisfies $\|S\| \leqslant 1$ and consider $x_1 = i(\widetilde{x}_1) \in B_X$. First, we see that

$$\|Sx_1\| = \|[\widetilde{S} \circ P \circ i](\widetilde{x}_1)\| = \|\widetilde{S}(\widetilde{x}_1)\| = 1,$$

so $\|S\| = 1 = \|Sx_1\|$ (in particular, $\|x_1\| = 1$). Next,

$$\begin{aligned}\|x_1 - x_0\| &\leqslant \|i(\widetilde{x}_1) - i(P(x_0))\| + \|i(P(x_0)) - x_0\| \\ &< \|\widetilde{x}_1 - P(x_0)\| + \eta'(\varepsilon) < \frac{\varepsilon}{2} + \frac{\varepsilon}{6} < \varepsilon.\end{aligned}$$

Finally,

$$\begin{aligned}\|S - T\| &\leqslant \|S - TP\| + \|TP - T\| \\ &\leqslant \|\widetilde{S}P - TP\| + 3\eta'(\varepsilon) \\ &\leqslant \|\widetilde{S} - \widetilde{T}\| + 3\eta'(\varepsilon) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$ □

A useful particular case of the above result is the following proposition in which we assume also the pointwise convergence in X besides the pointwise convergence in X^*.

Proposition 3.2.3. Let X be a Banach space for which there exists a net $\{P_\alpha\}_{\alpha\in\Lambda}$ of rank-one projections on X such that

(1) $\{P_\alpha x\} \longrightarrow x$ for all $x \in X$ in norm and

(2) $\{P_\alpha^* x^*\} \longrightarrow x^*$ for all $x^* \in X^*$ in norm.

If for a Banach space Y there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that all the pairs $(P_\alpha(X); Y)$ with $\alpha \in \Lambda$ have the BPBp for compact operators with the function η, then the pair $(X; Y)$ has the BPBp for compact operators.

Proof. This is a consequence of Lemma 3.2.2 considering the formal inclusion as operator i. □

The requirements for the Banach space X in the above proposition are fulfilled if X has a shrinking monotone Schauder basis, that is, a monotone Schauder basis such that the biorthogonal functionals form a basis to the dual space.

Corollary 3.2.4. Let X be a Banach space with a shrinking monotone Schauder basis and let $\{P_n\}_{n\in\mathbb{N}}$ be the sequence of natural projections associated to the basis. If for a Banach space Y there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that all the pairs $(P_n(X); Y)$ with $n \in \mathbb{N}$ have the

BPBp (for compact operators) with the function η, then the pair $(X;Y)$ has the BPBp for compact operators.

Another particular case of Proposition 3.2.3 is given by the following corollary.

Corollary 3.2.5. Let X be a Banach space. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a net of norm-one projections on X such that

(1) $\alpha \preceq \beta$ implies $P_\alpha(X) \subset P_\beta(X)$ and

(2) $\{P_\alpha^* x^*\} \longrightarrow x^*$ in norm for all $x^* \in X^*$.

If for a Banach space Y there exists a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that all the pairs $(P_\alpha(X);Y)$ with $\alpha \in \Lambda$ have the BPBp for compact operators with the function η, then the pair $(X;Y)$ has the BPBp for compact operators.

Proof. We prove that $P_\alpha x \longrightarrow x$ in norm for all $x \in X$ and then we apply Proposition 3.2.3. First, we prove that $Z = \overline{\bigcup_{\alpha\in\Lambda} P_\alpha(X)}$ is the whole space X. For a contradiction, suppose that there exists some $x_0 \in X$ such that $x_0 \notin Z$. Define $\varphi : Z \oplus [x_0] \longrightarrow \mathbb{K}$ by $\varphi(z+\lambda x_0) := \lambda$ for all $z \in Z$. Then φ is a continuous linear functional on $Z \oplus [x_0]$ and $\ker\varphi = Z$. By the Hahn-Banach theorem, there exists some $x_0^* \in X^*$ such that

$$Z \subset \ker x_0^* \quad \text{and} \quad x_0^*(x_0) = 1.$$

Using the hypothesis, we get $\{P_\alpha^* x_0^*\} \longrightarrow x_0^*$ and then $(P_\alpha^* x_0^*)(x_0) \longrightarrow x_0^*(x_0) = 1$. But this cannot happen since $(P_\alpha^* x_0^*)(x_0) = x_0^*(P_\alpha(x_0)) = 0$. This contradiction gives the desired equality. Now let $x \in X$. By the first part of this proof, there exist $\alpha_0 \in \Lambda$ and $x_1 \in X$ such that

$$\|P_{\alpha_0}(x_1) - x\| < \frac{\varepsilon}{2}.$$

Since the net $(P_\alpha)_\alpha$ is ordered by inclusion, for all $\alpha \geqslant \alpha_0$ we have that $P_\alpha(P_{\alpha_0}(x_1)) = P_{\alpha_0}(x_1)$. So

$$\|P_{\alpha_0}(x_1) - P_\alpha(x)\| = \|P_\alpha(P_{\alpha_0}(x_1)) - P_\alpha(x)\| \leqslant \|P_\alpha\|\|P_{\alpha_0}(x_1) - x\| < \frac{\varepsilon}{2},$$

for $\alpha \geqslant \alpha_0$. Thus

$$\|P_\alpha(x) - x\| \leqslant \|P_\alpha(x) - P_{\alpha_0}(x_1)\| + \|P_{\alpha_0}(x_1) - x\| < \varepsilon,$$

whenever $\alpha \geqslant \alpha_0$. This proves that $\{P_\alpha x\}$ converges to x in norm for all $x \in X$ as desired. □

Our next abstract result deals with range spaces instead of domain spaces. The idea of the proof, which is an adaptation of [43, Lemma 3.4] to the BPBp, was used in [3, Theorem 4.2] to prove that every pair $(X;Y)$ has the BPBp for compact operators when Y^* is isometric to an $L_1(\mu)$-space.

Proposition 3.2.6. Let X and Y be Banach spaces. Suppose that there exists a net of norm-one projections $\{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(X;Y)$ such that $\{Q_\lambda y\} \longrightarrow y$ in norm for every $y \in Y$. If there is a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the pairs $(X; Q_\lambda(Y))$ with $\lambda \in \Lambda$ have the BPBp for compact operators with the function η, then the pair $(X;Y)$ has the BPBp for compact operators.

Proof. Let $\varepsilon > 0$ be given. Define

$$\eta'(\varepsilon) = \frac{1}{2}\min\left\{\eta\left(\frac{\varepsilon}{2}\right), \varepsilon\right\}.$$

Let $T \in \mathcal{K}(X;Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ such that

$$\|T(x_0)\| > 1 - \eta'(\varepsilon).$$

As $T(B_X)$ is relatively compact, we may find $y_1, \dots, y_m \in Y$ such that

$$\min_j \|T(x) - y_j\| < \frac{\eta'(\varepsilon)}{3} \quad (x \in B_X).$$

By hypothesis, there is $\lambda \in \Lambda$ such that

$$\|Q_\lambda(y_j) - y_j\| < \frac{\eta'(\varepsilon)}{3} \qquad (j = 1, \dots, m).$$

Now, for every $x \in B_X$, we have

$$\begin{aligned}\|Tx - Q_\lambda Tx\| &\leqslant \min_j \{\|Tx - y_j\| + \|y_j - Q_\lambda(y_j)\| + \|Q_\lambda(y_j) - Q_\lambda Tx\|\} \\ &< \min_j 2\|Tx - y_j\| + \frac{\eta'(\varepsilon)}{3} < \eta'(\varepsilon).\end{aligned}$$

Therefore,

$$\|T - Q_\lambda T\| \leqslant \eta'(\varepsilon).$$

The operator $\widetilde{T} = Q_\lambda \circ T \in \mathcal{K}(X; Q_\lambda(Y))$ is such that $\|\widetilde{T}\| \leqslant 1$ and satisfies

$$\|\widetilde{T}(x_0)\| \geqslant \|T(x_0)\| - \|Q_\lambda T - T\| > 1 - 2\eta'(\varepsilon) \geqslant 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Then, there exists $\widetilde{S} \in \mathcal{K}(X; Q_\lambda(Y))$ with $\|\widetilde{S}\| = 1$ and $x_1 \in S_X$ such that

$$\|\widetilde{S}(x_1)\| = 1, \quad \|\widetilde{S} - \widetilde{T}\| < \frac{\varepsilon}{2} \ \text{ and } \ \|x_0 - x_1\| < \frac{\varepsilon}{2} < \varepsilon.$$

If we write $S \in \mathcal{K}(X; Y)$ to denote the operator $\widetilde{S}$ viewed as an operator with range in Y, we have that $\|S\| = \|S(x_1)\| = 1$ and

$$\|S - T\| \leqslant \|\widetilde{S} - \widetilde{T}\| + \|\widetilde{T} - T\| < \frac{\varepsilon}{2} + \eta'(\varepsilon) \leqslant \varepsilon. \qquad \square$$

In the next section we also need some technical results concerning direct sums. Note that both items (a) and (b) are particular cases of Propositions 2.4.2 and 2.4.6 adapted for compact operators.

Lemma 3.2.7. Let X, X_1, X_2, Y, Y_1 and Y_2 be Banach spaces.

(a) If $(X_1 \oplus_1 X_2; Y)$ or $(X_1 \oplus_\infty X_2; Y)$ has the BPBp for compact operators with a function η, then so do the pairs $(X_j; Y)$ with $j = 1, 2$ with the same function η.

(b) If $(X; Y_1 \oplus_1 Y_2)$ or $(X; Y_1 \oplus_\infty Y_2)$ has the BPBp for compact operators with a function η, then so do the pairs $(X; Y_j)$ for $j = 1, 2$ with the same function η.

3.3 Applications

In this section we give some applications of the abstract results that we proved in Section 3.2. The idea here is to use Lemma 3.2.2, its consequences and Proposition 3.2.6 to transfer the Bishop-Phelps-Bollobás property for compact operators from sequence spaces, such as c_0 or ℓ_p, to function spaces, such as $C_0(L)$ or $L_p(\mu)$. By doing that, we will be able to give more examples of pairs which satisfy the BPBp for compact operators beyond Examples 3.1.2.

The first application is the following sufficient condition for the pair $(C_0(L); Y)$ to have the BPBp for compact operators.

Theorem 3.3.1. Let L be a locally compact Hausdorff topological space and let Y be a Banach space. If the pair $(c_0; Y)$ has the BPBp for compact operators, then $(C_0(L); Y)$ has the BPBp for compact operators.

To prove this theorem we use Lemma 3.2.2. Nevertheless, we need two preliminary results in order to show that the hypothesis of that lemma are satisfied. The first one is the following lemma, for which we only need the case $X = \mathbb{K}$, but we state it in the general form for completeness.

Lemma 3.3.2. Let X and Y be Banach spaces. Then the following are equivalent:

(i) the pair $(c_0(X); Y)$ has the BPBp for compact operators;

(ii) there is a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the pairs $(\ell_\infty^m(X); Y)$ with $m \in \mathbb{N}$ have the BPBp for compact operators with the function η.

Moreover, when $\mathcal{K}(X; Y) = \mathcal{L}(X; Y)$ (in particular, if one of the spaces X or Y is finite-dimensional), this happens when $(c_0(X); Y)$ or $(\ell_\infty(X); Y)$ has the BPBp.

Proof. First note that each $\ell_\infty^m(X)$ is an ℓ_∞-summand in $c_0(X)$. So (i) implies (ii) because of Lemma 3.2.7.(a). Now if we suppose that the pairs $(\ell_\infty^m(X); Y)$ have the BPBp for compact operators with a function η, we can construct the sequence of projections satisfying conditions (1) and (2) in Proposition 3.2.3. So (ii) implies (i).

When $\mathcal{K}(X, Y) = \mathcal{L}(X; Y)$, if $(c_0(X); Y)$ or $(\ell_\infty(X); Y)$ has the BPBp, then [11, Proposition 2.6] gives that each $(\ell_\infty^m(X); Y)$ has the BPBp with the same function η. Since every operator from $\ell_\infty^m(X)$ into Y is compact, (ii) holds. Now using that $\mathcal{K}(X; Y)$ is equals to $\mathcal{L}(X; Y)$ is not difficult to see that (ii) implies (i) also in this case. □

In particular, we have the following consequence.

Corollary 3.3.3. Let Y be a Banach space. If the pair $(c_0; Y)$ has the BPBp, then it has the BPBp for compact operators.

The next preliminary result is based on [3, Proposition 3.2] and gives the possibility to apply Lemma 3.2.2 when the domain space is a $C_0(L)$-space. We recall that $C_0(L)^*$ can be identified with the space of regular Borel measures on L by the Riesz representation theorem.

Lemma 3.3.4 (Extension of [3, Proposition 3.2])**.** Let L be a locally compact Hausdorff topological space. Given $\delta > 0$, $\mu_1, \ldots, \mu_n \in B_{C_0(L)^*}$ and $f_0 \in B_{C_0(L)}$, there exist a norm-one projection $P \in \mathcal{L}(C_0(L), C_0(L))$ and a norm-one operator $i \in \mathcal{L}(P(C_0(L)), C_0(L))$ such that:

(1) $\|P^*\mu_j - \mu_j\| < \delta$ for $j = 1, \ldots, n$;

(2) $\|i(P(f_0)) - f_0\| < \delta$;

(3) $P \circ i = \mathrm{Id}_{P(C_0(L))}$;

(4) $P(C_0(L))$ is isometrically isomorphic to ℓ_∞^m for some $m \in \mathbb{N}$.

Proof. Almost everything is given by [3, Proposition 3.2] and its proof. We have to define the operator i and, to do so, we need to give some details which already appear in that proposition.

First, we note that we may suppose that $\|f_0\| = 1$. Indeed, if $f_0 = 0$, then (2) is always true and, as P is a projection, (3) is true by taking i to be the inclusion of $P(C_0(L))$ into $C_0(L)$. In the case that $f_0 \neq 0$, we use $f_0/\|f_0\|$ and the result for f_0 will follows.

Let $\mu_0 \in S_{C_0(L)^*}$ be such that $\mu_0(f_0) = \|f_0\| = 1$. By the Riesz representation theorem, we may view $\mu_0, \mu_1, \ldots, \mu_n$ as Borel measures on L. Consider the finite positive regular measure $\mu = \sum_{j=0}^{n} |\mu_j|$. We use the Radon-Nikodým theorem, the density of simple functions on $L_1(\mu)$, the regularity of μ, Urysohn's lemma, and the continuity of f_0 as in the proof of [3, Proposition 3.2] to get the following: there are a finite collection $K_1, \ldots, K_m$ of pairwise disjoint compact subsets of L with $\mu(K_k) > 0$ for $k = 1, \ldots, m$ and a collection of continuous functions with pairwise disjoint compact support $\varphi_1, \ldots, \varphi_m$ with values in $[0, 1]$ with $\varphi_k = 1$ on K_k for every $k = 1, \ldots, m$, in such a way that, if we define

$$P(f) = \sum_{k=1}^{m} \frac{1}{\mu(K_k)} \left(\int_{K_k} f \, d\mu \right) \varphi_k \qquad \big(f \in C_0(L)\big),$$

one has

(a) $P \in \mathcal{L}(C_0(L), C_0(L))$ is a norm-one projection;

(b) $\|P^*\mu_j - \mu_j\| < \delta/2$ for every $j = 0, 1, \ldots, n$;

(c) $P(C_0(L))$ is the linear span of $\{\varphi_1, \ldots, \varphi_m\}$ and so, it is isometrically isomorphic to ℓ_∞^m;

(d) $\sup\limits_{t,s \in K_k} |f_0(t) - f_0(s)| < \delta/2$ for $k = 1, \ldots, m$;

(e) $\sup\left\{ \Big|[Pf_0](t) - f_0(t)\Big| \,:\, t \in \bigcup_{k=1}^m K_k \right\} < \delta/2.$

Note that (b) and (c) are (1) and (4) of our lemma. Next, we use (b) with $j = 0$ to get that

$$\|P(f_0)\| \geqslant |\mu_0(P(f_0))| \geqslant |\mu_0(f_0)| - \|P^*\mu_0 - \mu_0\| > 1 - \frac{\delta}{2},$$

and consider $\Upsilon_0 \in P(C_0(L))^*$ such that

$$\|\Upsilon_0\| = 1 \quad \text{and} \quad \Upsilon_0(P(f_0)) = \|P(f_0)\| > 1 - \frac{\delta}{2}.$$

On the other hand, we may use (d) to get a compactly supported continuous function $\Psi : L \longrightarrow [0, 1]$ such that $\Psi = 1$ on $\bigcup_{k=1}^m K_k$ and such that

$$\sup_{t \in \operatorname{supp}(\Psi)} \Big|[Pf_0](t) - f_0(t)\Big| < \frac{\delta}{2}.$$

We are now ready to define the operator $i \in \mathcal{L}(P(X); C_0(L))$ as follows:

$$\left[i \left(\sum_{k=1}^m \alpha_k \varphi_k \right) \right](t) = \Psi(t) \left(\sum_{k=1}^m \alpha_k \varphi_k(t) \right) + \Big(1 - \Psi(t)\Big) \Upsilon_0 \left(\sum_{k=1}^m \alpha_k \varphi_k \right) f_0(t)$$

for every $t \in L$ and every $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. Then, i is linear, $\|i\| \leqslant 1$ and $P \circ i = \mathrm{Id}_{P(X)}$. This gives (3). Finally, since

$$\begin{aligned} i(Pf_0) - f_0 &= \Psi(Pf_0) + (1-\Psi)\Upsilon_0(Pf_0)f_0 - f_0 \\ &= \Psi(Pf_0) + (1-\Psi)\|Pf_0\|f_0 - f_0 + \Psi f_0 - \Psi f_0 \\ &= \Psi(Pf_0 - f_0) - (1-\Psi)(1-\|Pf_0\|)f_0 \end{aligned}$$

we have that

$$\begin{aligned} \|i(P(f_0)) - f_0\| &\leqslant \|\Psi(Pf_0 - f_0)\| + \|(1-\Psi)(1-\|P(f_0)\|)f_0\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

by using that $\|1-\Psi\|_\infty \leqslant 1$, $|1-\|P(f_0)\|| < \frac{\delta}{2}$ and $\|f_0\| = 1$. This gives (2) and it finishes the proof. □

Now we combine these results to get Theorem 3.3.1.

Proof of Theorem 3.3.1. By Lemma 3.3.4 and Lemma 3.3.2, we are in the hypotheses of Lemma 3.2.2, so the result follows. □

A family of Banach spaces Y for which the pair $(c_0; Y)$ satisfies the Bishop-Phelps-Bollobás property has been recently discovered in [7]. This family strictly contains the uniformly convex Banach spaces and the Banach spaces with property β. By using Theorem 3.3.1, one gets that the pair $(C_0(L), Y)$ has the BPBp for compact operators for all elements Y in that family. To explain this application better we need a definition. Let Y be a Banach space, $E \subset S_Y$ and $F : E \longrightarrow S_{Y^*}$. We say that the family E is *uniformly strongly exposed* by F if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$y \in B_Y, \quad e \in E, \quad \operatorname{Re} F(e)(y) > 1 - \delta \;\Rightarrow\; \|y - e\| < \varepsilon.$$

The promised application is the following.

Corollary 3.3.5. Let Y be a Banach space. Suppose that there exist a set I, $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$, a subset $E \subset S_Y$, a mapping $F : E \longrightarrow S_{Y^*}$ and $0 \leqslant \rho < 1$ satisfying that

(1) $y_i^*(y_i) = 1$, $\forall i \in I$,

(2) $|y_i^*(y_j)| \leqslant \rho$, $\forall i, j \in I$, $i \neq j$,

(3) E is uniformly strongly exposed by F,

(4) $|F(e)(y_i)| \leqslant \rho$, $\forall e \in E$, $i \in I$,

(5) $\|y\| = \max\{\sup\{|y_i^*(y)| : i \in I\}, \sup\{|F(e)(y)| : e \in E\}\}$ for all $y \in Y$.

Then, for every locally compact Hausdorff topological space L, the pair $(C_0(L); Y)$ has the BPBp for compact operators.

Proof. Applying [7, Theorem 2.4] we get that the pair $(c_0; Y)$ has the BPBp. By Corollary 3.3.3, $(c_0; Y)$ has the BPBp for compact operators. Now we apply Theorem 3.3.1 to get the property for the pair $(C_0(L); Y)$ as desired. □

We observe that this result covers the already known cases when Y is uniformly convex ($I = \emptyset$) and when Y has property β ($E = \emptyset$) (see items (a) and (b) of Examples 3.1.2). It was proved in the cited paper [7] that there are examples of Banach spaces Y satisfying the requirements of Corollary 3.3.5 which are neither uniformly convex nor satisfy property β, even in dimension two.

Now we work with ℓ_1-preduals spaces. By using Corollary 3.2.5, we will prove the following.

Theorem 3.3.6. Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1 and let Y be a Banach space. If the pair $(c_0; Y)$ has the BPBp for compact operators, then $(X; Y)$ has the BPBp for compact operators.

To do so, we use the ideas of [34, Theorem 1.1, Theorem 3.2 and Corollary 4.1]. More specifically, in order to apply Corollary 3.2.5, we construct a net of w^*-continuous finite-rank contractive projections in X^* which converges pointwise to the identity operator in this space and such that their adjoint operators restricted to $X \subset X^{**}$ is ordered by inclusion.

Let $(e_n^*)_n$ be a Schauder basis of X^* isometrically equivalent to the canonical basis of ℓ_1. For every $n \in \mathbb{N}$, we put $Y_n := [e_1^*, \ldots, e_n^*]$, the linear span of $\{e_1^*, \ldots, e_n^*\}$. It was contructed in the proof of Corollary 4.1 of [34] a sequence $Q_n : X^* \longrightarrow X^*$ of w^*-continuous contractive projections such that $Q_n^*(X^*) = Y_n$ for each $n \in \mathbb{N}$. We prove that this sequence satisfies the hypothesis of Corollary 3.2.5, that is, for any $x^* \in X^*$, $\{Q_n x^*\} \longrightarrow x^*$ in norm and Q_n^* restricted to X is ordered by inclusion and it is isometric to ℓ_∞^n for each $n \in \mathbb{N}$. We start with a general lemma which has its own interest.

Lemma 3.3.7. Let E be a Banach space and consider M, N to be closed subspaces of E^* such that

$$E^* = M \oplus N,$$

where $\dim M < \infty$. Consider $Q : E^* \longrightarrow M$ to be the projection on the first coordinate and $\widetilde{Q} := i \circ Q : E^* \longrightarrow E^*$ where $i : M \hookrightarrow E^*$ is the inclusion of M in E^*. If Q is w^*-continuous then

$$(\widetilde{Q})^*(E^{**}) = (\widetilde{Q})^*(E) = Q^*(M^*).$$

Proof. Suppose that Q is w^*-continuous. First we prove that $\varphi \circ Q$ is w^*-continuous for all $\varphi \in M^*$ and $\psi \circ \widetilde{Q}$ is w^*-continuous for all $\psi \in E^{**}$. Indeed, let $(x_\alpha^*) \subset E^*$ be a net such that $x_\alpha^* \xrightarrow{w^*} x_0^*$ for some $x_0^* \in E^*$. Since Q is w^*-continuous, $Q(x_\alpha^*) \xrightarrow{w^*} Q(x_0^*)$. We are assuming that M as finite dimension, so $Q(x_\alpha^*) \longrightarrow Q(x_0^*)$ in norm. Thus for all $\varphi \in M^*$,

$\varphi(Q(x^*_\alpha)) \longrightarrow \varphi(Q(x^*_0))$ which shows that $\varphi \circ Q$ is w^*-continuous for all $\varphi \in M^*$. On the other hand, $\widetilde{Q}(x^*_\alpha) = i \circ Q(x^*_\alpha) \longrightarrow i \circ Q(x^*_0) = \widetilde{Q}(x^*_0)$ in norm and then $\psi \circ \widetilde{Q}(x^*_\alpha) \longrightarrow \psi \circ \widetilde{Q}(x^*_0)$ for all $\psi \in E^{**}$. By these observations,

$$Q^* \circ \varphi = \varphi \circ Q \in E \quad \text{and} \quad (\widetilde{Q})^* \circ \psi = \psi \circ \widetilde{Q} \in E$$

for all $\varphi \in M^*$ and $\psi \in E^{**}$. Let $\psi \in E^{**} = M^* \oplus N^*$. So there are unique $\varphi_1 \in M^*$ and $\varphi_2 \in N^*$ such that for all $e^* \in E^*$,

$$\phi(e^*) = \varphi_1(Q(e^*)) + \varphi_2((Id_{E^*} - Q)(e^*)).$$

Then

$$\begin{aligned}(\widetilde{Q})^*(\phi) = \phi \circ \widetilde{Q} &= \phi \circ i \circ Q \\ &= \varphi_1(Q(i \circ Q)) + \varphi_2((Id_{E^*} - Q)(i \circ Q)) \\ &= \varphi_1 \circ Q + \varphi_2(0) \\ &= Q^* \circ \varphi_1.\end{aligned}$$

This proves the lemma. □

Lemma 3.3.8. Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1. Let $(e^*_n)_n$ be the Schauder basis of X^* isometrically equivalent to the usual ℓ_1-basis. Denote by Y_n the linear span of $\{e^*_1, \ldots, e^*_n\}$. There exists a sequence $(P_n)_n$ in X of contractive projections such that

(a) $P^*_n : X^* \longrightarrow Y_n$ is w^*-continuous,

(b) $P_n(X) \subset P_{n+1}(X)$ for all $n \in \mathbb{N}$,

(c) $P_n(X)$ is isometric to ℓ^n_∞ for all $n \in \mathbb{N}$,

(d) $\{P^*_n x^*\} \longrightarrow x^*$ in norm for all $x^* \in X^*$,

(e) $\{P_n x\} \longrightarrow x$ in norm for all $x \in X$.

Proof. By Corollary 4.1 of [34], there exists a sequence of w^*-continuous contractive projections $Q_n : X^* \longrightarrow X^*$ with $Q_n(X^*) = Y_n$ and $Q_n Q_{n+1} = Q_n$ for all $n \in \mathbb{N}$. Given $x^* \in X^*$, we prove that $\{Q_n x^*\} \longrightarrow x^*$ in norm. Indeed, since $x^* = \sum_{j=1}^{\infty} a_j e_j^*$ with $a_j \in \mathbb{K}$ for all $j \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left\| x^* - \sum_{j=1}^{n} a_j e_j^* \right\| < \frac{\varepsilon}{2},$$

for all $n \geqslant n_0$. On the other hand, since $\sum_{j=1}^{n} a_j e_j^* \in Y_n$, it follows that $Q_n \left(\sum_{j=1}^{n} a_j e_j^* \right) = \sum_{j=1}^{n} a_j e_j^*$. Thus, for all $n \geqslant n_0$, we get that

$$\left\| Q_n(x^*) - \sum_{j=1}^{n} a_j e_j^* \right\| = \left\| Q_n(x^*) - Q_n \left(\sum_{j=1}^{n} a_j e_j^* \right) \right\| \leqslant \left\| x^* - \sum_{j=1}^{n} a_j e_j^* \right\| < \frac{\varepsilon}{2}.$$

So, we conclude that

$$\| Q_n(x^*) - x^* \| \leqslant \left\| Q_n(x^*) - \sum_{j=1}^{n} a_j e_j^* \right\| + \left\| \sum_{j=1}^{n} a_j e_j^* - x^* \right\| < \varepsilon,$$

for all $n \geqslant n_0$. Now we prove that if $n \geqslant m$ then $Q_n^*(X) \subset Q_m^*(X)$. Indeed, by Lemma 3.3.7, $Q_n^*(Y_n^*) = Q_n^*(X)$ for all $n \in \mathbb{N}$. Since $Q_n Q_{n+1} = Q_n$ for all $n \in \mathbb{N}$, it follows that $Q_{n+1}^* Q_n^* = Q_n^*$ for all $n \in \mathbb{N}$. So we get

$$Q_n^*(X) = Q_{n+1}^*(Q_n^*(X)) \subset Q_{n+1}^*(X),$$

for all $n \in \mathbb{N}$. Consequently, $Q_n^* x \longrightarrow x$ in norm for all $x \in X$ (see the proof of Corollary 3.2.5). Finally, we note that $Q_n^*(X) = Q_n^*(Y_n^*)$ is isometric to ℓ_∞^n for all $n \in \mathbb{N}$. To finish the proof, we put $P_n := Q_n^*$ restricted to X (and then $P_n^* = Q_n$) for all $n \in \mathbb{N}$ to get properties (a)–(e). □

Proof of Theorem 3.3.6. Suppose that X^* is isometrically isomorphic to ℓ_1 and that the pair $(c_0; Y)$ has the BPBp for compact operators. By Lemma 3.3.2, the pair $(\ell_\infty^n; Y)$ has the BPBp for compact operators with the same function η for every $n \in \mathbb{N}$. Consider the sequence $(P_n)_n$ of Lemma 3.3.8. Since $P_n(X)$ is isometric to ℓ_∞^n, then the pair $(P_n(X); Y)$ has the BPBp for compact operators with the same function η. Now, applying Corollary 3.2.5 we get that the pair $(X; Y)$ has the BPBp for compact operators. □

Using the same arguments of Corollary 3.3.5 we get the following examples of pairs of Banach spaces $(X; Y)$ satisfying the Bishop-Phelps-Bollobás property for compact operators.

Corollary 3.3.9. Let Y be a Banach space. Suppose that there exist a set I, $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$, a subset $E \subset S_Y$, a mapping $F : E \longrightarrow S_{Y^*}$ and $0 \leqslant \rho < 1$ satisfying that

(1) $y_i^*(y_i) = 1, \forall i \in I$;

(2) $|y_i^*(y_j)| \leqslant \rho, \forall i, j \in I, i \neq j$;

(3) E is uniformly strongly exposed by F;

(4) $|F(e)(y_i)| \leqslant \rho, \forall e \in E, i \in I$;

(5) $\|y\| = \max\Big\{\sup\{|y_i^*(y)| : i \in I\}, \sup\{|F(e)(y)| : e \in E\}\Big\}$ for any $y \in Y$.

Then, for every Banach space X such that X^* is isometrically isomorphic to ℓ_1, the pair (X, Y) has the BPBp for compact operators.

Observe again that this result covers the cases when Y is uniformly convex $(I = \emptyset)$ and when Y has property β $(E = \emptyset)$. If Y has property β, the result was already known (see Examples 3.1.2.(a)), but it was unknown for uniformly convex spaces.

Corollary 3.3.10. Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1 and let Y be a uniformly convex Banach space. Then $(X;Y)$ has the BPBp for compact operators.

Remark 3.3.11. We do not know whether Theorem 3.3.6 can be extended to general L_1-predual spaces.

Next we will give a result for $L_1(\mu, X)$-spaces (see Section 1.1 for notations). This is an extension of Examples 3.1.2.(i).

Theorem 3.3.12. Let μ be a positive measure, let X be a Banach space such that X^* has the Radon-Nikodým property and let Y be a Banach space. If the pair $(\ell_1(X);Y)$ has the BPBp for compact operators, then the pair $(L_1(\mu, X);Y)$ has the BPBp for compact operators.

We first need the following lemma which is similar to Lemma 3.3.2 but now for ℓ_1-spaces. This lemma gives a version for compact operators of [50, Theorem 6] which characterizes the pairs $(\ell_1(X);Y)$ to have the BPBp. Observe that, in this case, no assumption on X is needed.

Lemma 3.3.13. Let X and Y be Banach spaces. Then the following are equivalent:

(i) for every $\varepsilon > 0$ there exists $0 < \xi(\varepsilon) < \varepsilon$ such that given sequences $(T_k) \subset B_{\mathcal{K}(X;Y)}$ and $(x_k) \subset B_X$, and a convex series $\sum_{k=1}^{\infty} \alpha_k$ such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \xi(\varepsilon),$$

there exist a finite subset $A \subset \mathbb{N}$, $y^* \in S_{Y^*}$ and sequences $(S_k) \subset S_{\mathcal{K}(X;Y)}$, $(z_k) \subset S_X$ satisfying the following:

(a) $\sum_{k\in A} \alpha_k > 1 - \varepsilon$,

(b) $\|z_k - x_k\| < \varepsilon$ and $\|S_k - T_k\| < \varepsilon$ for all $k \in A$,

(c) $y^*(S_k z_k) = 1$ for every $k \in A$.

(in this case, we may say that the pair $(X;Y)$ has the *generalized AHSP for compact operators*);

(ii) the pair $(\ell_1(X);Y)$ has the BPBp for compact operators;

(iii) there is a function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the pairs $(\ell_1^m(X);Y)$ with $m \in \mathbb{N}$ have the BPBp for compact operators with the function η.

Moreover, if $\mathcal{K}(X;Y) = \mathcal{L}(X;Y)$ (in particular, if one of the spaces X or Y is finite-dimensional), then the above is equivalent to

(iv) the pair $(\ell_1(X);Y)$ has the BPBp.

Proof. (i) $\Rightarrow$ (ii) is an adaptation of the proof of [50, Theorem 6] to compact operators. Indeed, if we suppose that the set A is finite and all the operators T_k's and S_k's are compact, then the operator $S : \ell_1(X) \longrightarrow Y$ defined there is also compact. (ii) implies (iii) follows from Lemma 3.2.7.(a) since each $\ell_1^m(X)$ is an ℓ_1-summand in $\ell_1(X)$. Finally, for (iii) implies (i), we can again adapt the proof of [50, Theorem 6] to the case of compact operators, using that in our item (i) we may reduce to finite sums instead of series (using the analogous for compact operators of [50, Remark 5.a]).

If $\mathcal{K}(X,Y) = \mathcal{L}(X,Y)$, item (iii) is equivalent to the fact that all the pairs $(\ell_1^m(X),Y)$ with $m \in \mathbb{N}$ have the BPBp with the same function η. This is equivalent to (iv) by [50, Theorem 6]. $\square$

In particular, we have the following characterization for the pairs $(\ell_1;Y)$ to have the BPBp for compact operators.

Corollary 3.3.14. Let Y be a Banach space. The following are equivalent:

(i) the pair $(\ell_1;Y)$ has the BPBp for compact operators;

(ii) Y has the AHSP;

(iii) the pair $(\ell_1; Y)$ has the BPBp;

(iv) for every positive measure μ, the pair $(L_1(\mu); Y)$ has the BPBp for compact operators;

(v) there is a positive measure μ such that $L_1(\mu)$ is infinite-dimensional and the pair $(L_1(\mu); Y)$ has the BPBp for compact operators.

Proof. We already know that (ii) and (iii) are equivalent by [2, Theorem 4.1]. Applying Lemma 3.3.13 we get the equivalence between (i) and (iii). To prove that (ii) implies (iv) go to Examples 3.1.2.(i). It is clear that (iv) implies (v). Finally, (v) implies (ii) is proved in [5, Corollary 2.4]. □

To prove Theorem 3.3.12 we need the following modification of [29, Lemma III.2.1, p. 67] to put it in the conditions of Proposition 3.2.3.

Lemma 3.3.15. Let (Ω, Σ, μ) be a measure space such that $L_1(\mu)$ is infinite-dimensional, let X be a Banach space and let $\varepsilon > 0$.

(a) For $1 \leqslant p < \infty$, given $f_1, \ldots, f_n$ in $L_p(\mu, X)$ there exists a norm-one projection $P : L_p(\mu, X) \longrightarrow L_p(\mu, X)$ such that $P(L_p(\mu, X))$ is isometrically isomorphic to $\ell_p(X)$ and

$$\|P(f_j) - f_j\| < \varepsilon,$$

for every $j = 1, \ldots, n$.

(b) If μ is a finite measure then, for $1 \leqslant p < \infty$, $f_1, \ldots, f_n$ in $L_p(\mu, X)$, $g_1, \ldots, g_m$ in $L_{p^*}(\mu, X^*)$ and $\varepsilon > 0$, there exists a norm-one projection $P : L_p(\mu, X) \longrightarrow L_p(\mu, X)$ such that $P(L_p(\mu, X))$ is isometrically isomorphic to $\ell_p(X)$ and such that

$$\|f_j - Pf_j\|_p < \varepsilon \quad \text{and} \quad \|g_k - P^* g_k\|_{p^*} < \varepsilon,$$

for all $j = 1, \dots, n$ and $k = 1, \dots, m$.

(c) If μ is a finite measure then, given $g_1, \dots, g_m$ in $L_\infty(\mu, X)$ and $\varepsilon > 0$ there exists a norm-one projection $P : L_\infty(\mu, X) \longrightarrow L_\infty(\mu, X)$ such that $P(L_\infty(\mu, X))$ is isometrically isomorphic to $\ell_\infty(X)$ and such that

$$\|g_k - Pg_k\|_\infty < \varepsilon,$$

for all $k = 1, \dots, m$.

Proof. Since $L_1(\mu)$ is infinite-dimensional, there exists a sequence $\mathcal{A} = (A_i)_{i=1}^\infty$ of pairwise disjoint measurable sets such that $0 < \mu(A_i) < \infty$ for every $i \in \mathbb{N}$.

(a). Fix $1 \leqslant p < \infty$. Let $f_1, \dots, f_n$ in $L_p(\mu, X)$. As, by definition, simple functions are dense in $L_p(\mu, X)$, for each $j \in \{1, \dots, n\}$ we can find a finite family $\mathcal{A}_j$ of pairwise disjoint measurable sets of positive and finite measure, and $x_A \in X$ for every $A \in \mathcal{A}_j$ such that the vector-valued simple function $\phi_j = \sum_{A \in \mathcal{A}_j} x_A \chi_A$ satisfies

$$\|f_j - \phi_j\|_p < \varepsilon$$

for every $j = 1, \dots, n$. Define

$$\Omega_1 = \left[\bigcup_{i=1}^{\infty} A_i\right] \cup \left[\bigcup_{j=1}^{n} \bigcup_{A \in \mathcal{A}_j} A\right]$$

and consider $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ to be an infinite countable partition of Ω_1 such that all their elements are measurable with $0 < \mu(B_i) < \infty$ for every $i \in \mathbb{N}$ and such that $\mathcal{B}$ is a refinement of the above families. In the case $p = 1$, we have that

$$\sum_{i=1}^{\infty} \left\| \frac{1}{\mu(B_i)} \left(\int_{B_i} f \, d\mu \right) \chi_{B_i} \right\|_1 = \sum_{i=1}^{\infty} \left\| \int_{B_i} f \, d\mu \right\| \leqslant \|f\|_1$$

for every $f \in L_1(\mu, X)$. For $1 < p < \infty$, let us observe that for any $r, s \in \mathbb{N}$ with $1 \leqslant r < s$, the fact that $\mathcal{B}$ is a partition leads to

$$\left\|\sum_{i=r}^{s} \frac{1}{\mu(B_i)}\Big(\int_{B_i} f d\mu\Big)\chi_{B_i}(t)\right\|^p = \sum_{i=r}^{s} \frac{1}{\mu(B_i)^p} \left\|\int_{B_i} f d\mu\right\|^p \chi_{B_i}(t)$$

for every t in Ω. Hence

$$\begin{aligned}\left\|\sum_{i=r}^{s} \frac{1}{\mu(B_i)}\Big(\int_{B_i} f d\mu\Big)\chi_{B_i}\right\|_p^p &= \int_\Omega \Big(\sum_{i=r}^{s} \frac{1}{\mu(B_i)^p} \left\|\int_{B_i} f d\mu\right\|^p \chi_{B_i}\Big) d\mu \\ &= \sum_{i=r}^{s} \frac{1}{\mu(B_i)^p} \left\|\int_{B_i} f d\mu\right\|^p \int_\Omega \chi_{B_i} d\mu.\end{aligned}$$

But

$$\begin{aligned}\sum_{i=r}^{s} \frac{1}{\mu(B_i)^p} \left\|\int_{B_i} f d\mu\right\|^p \int_\Omega \chi_{B_i} d\mu &= \sum_{i=r}^{s} \mu(B_i)^{1-p} \left\|\int_{B_i} f d\mu\right\|^p \\ &\leqslant \sum_{i=r}^{s} \mu(B_i)^{1-p} \int_\Omega \|f\|^p d\mu \Big(\int_\Omega \chi_{B_i} d\mu\Big)^{\frac{p}{p^*}} \\ &= \int_\Omega \|f\|^p d\mu = \|f\|_p^p\end{aligned}$$

for every $1 \leqslant r < s$. Thus, for $1 \leqslant p < \infty$, we can define $P : L_p(\mu, X) \longrightarrow L_p(\mu, X)$ by

$$P(f) = \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)}\Big(\int_{B_i} f d\mu\Big)\chi_{B_i} \qquad \big(f \in L_p(\mu, X)\big).$$

Since $P(\chi_{B_i}) = \chi_{B_i}$ for every i, we have that P is a norm-one projection such that $P(L_p(\mu, X))$ is isometrically isomorphic to $\ell_p(X)$. Moreover, for each $j \in \{1, \ldots, n\}$, there exists a sequence $(x_{ij})_{i\in\mathbb{N}}$ in X such that

$$\phi_j = \sum_{i=1}^{\infty} x_{ij}\chi_{B_i},$$

where the equality holds both pointwise and with respect to the p-norm. This implies that

$$\|P(f_j) - f_j\|_p = \|\phi_j - f_j\|_p < \varepsilon,$$

for every $j = 1, \dots, n$.

(b). If we now assume that μ is a finite measure, the sequence $\mathcal{A} = (A_i)_{i=1}^{\infty}$ of pairwise disjoint sets of positive measure can be assumed to be a partition of Ω. As above, given $f_1, \dots, f_n$ in $L_p(\mu, X)$, we can find $\phi_1, \dots, \phi_n$ simple functions $\phi_j = \sum_{A \in \mathcal{A}_j} x_A \chi_A$, such that

$$\|f_j - \phi_j\|_p < \varepsilon,$$

for every $j = 1, \dots, n$, where now each $\mathcal{A}_j$ is a partition of Ω of measurable sets of positive measure. Let us take $g_1, \dots, g_m$ in $L_{p^*}(\mu, X^*)$. We distinguish two subcases again. If $p > 1$, then, by definition of $L_{p^*}(\mu, X^*)$, for each $j \in \{1, \dots, m\}$, we can find a finite family $\mathcal{C}_j$ of pairwise disjoint measurable sets of positive and finite measure, and $x_C^* \in X^*$ such that the vector-valued simple function $\eta_k = \sum_{C \in \mathcal{C}_k} x_C^* \chi_C$ satisfies that

$$\|g_k - \eta_k\|_{q^*} < \varepsilon$$

for every $k = 1, \dots, m$. Observe that, since μ is finite, by adding a suitable null characteristic function, we may and do assume that each $\mathcal{C}_k$ is actually a measurable partition of Ω. If $p = 1$, then $p^* = \infty$. In that case, by [29, p. 97], there exists a measurable, bounded and countably valued mapping $\eta_k : \Omega \longrightarrow X^*$ such that

$$\|g_k - \eta_k\|_\infty < \frac{\varepsilon}{2}$$

for every $k = 1, \dots, m$. Thus, again there exists a countable partition $\mathcal{C}_k$ of Ω and vectors $x_C^* \in X^*$ such that $\eta_k(t) = \sum_{C \in \mathcal{C}_k} x_C^* \chi_C(t)$, for every

$t \in \Omega$. In both cases, we can find $\mathcal{B} = (B_i)_{i=1}^{\infty}$ a partition of Ω of sets of positive measure that is a refinement of all the partitions $\mathcal{A}$, $\mathcal{A}_j$, $\mathcal{C}_k$ for every j and k. As in (a), if we define $P : L_p(\mu, X) \longrightarrow L_p(\mu, X)$ by

$$P(f) = \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)} \Big(\int_{B_i} f d\mu \Big) \chi_{B_i},$$

we have that P is a norm-one projection such that $P(L_p(\mu, X))$ is isometrically isomorphic to $\ell_p(X)$ and $\|P(f_j) - f_j\|_p = \|\phi_j - f_j\|_p < \varepsilon$ for every $j = 1, \ldots, n$. Furthermore, $L_{p^*}(\mu, X^*)$ is isometrically isomorphic to a subspace of $L_p(\mu, X)^*$ (see e.g. [29, p. 97]), and

$$\begin{aligned}
\Big[P^*(x^* \chi_{B_r})\Big](f) &= \int_{\Omega} x^* \Big(P(f)(t)\Big) \chi_{B_r}(t)\, d\mu(t) \\
&= \int_{\Omega} \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)} x^* \left(\int_{B_i} f d\mu\right) \chi_{B_i}(t) \chi_{B_r}(t)\, d\mu(t) \\
&= \frac{1}{\mu(B_r)} x^* \left(\int_{B_r} f d\mu\right) \int_{\Omega} \chi_{B_r}(t)\, d\mu(t) = \Big[x^* \chi_{B_r}\Big](f)
\end{aligned}$$

for every f in $L_p(\mu, X)$, every x^* in X^* and every r. We know that there exists $x^*_{ik} \in X^*$ such that $\eta_k(t) = \sum_{i=1}^{\infty} x^*_{ik} \chi_{B_i}(t)$, pointwise in Ω and convergent with the $\|\cdot\|_{p^*}$-norm for $1 < p < \infty$. Hence, for $1 < p < \infty$, we obtain that $P^*(\eta_k) = \eta_k$ for every k. For $p = 1$ the equality holds too. But we need to do some extra work to prove it. We have

$$\begin{aligned}
\left[P^* \Big(\sum_{r=1}^{l} x^*_{rk} \chi_{B_r}\Big)\right](f) &= \sum_{r=1}^{l} x^*_{rk} \left(\int_{B_r} f(t)\, d\mu(t)\right) \\
&= \int_{\Omega} \sum_{r=1}^{l} x^*_r (f(t)) \chi_{B_r}(t)\, d\mu(t).
\end{aligned}$$

But

$$\sum_{r=1}^{l} |x^*_{rk}(f(t)) \chi_{B_r}(t)| \leqslant \sum_{r=1}^{l} \|x^*_{rk}\| \|f(t)\| \chi_{B_r}(t) \leqslant M \|f(t)\|,$$

for every t in Ω and every $l \in \mathbb{N}$, where $M := \max\{\|g_1\|_\infty, \dots, \|g_m\|_\infty\} + \varepsilon$. Thus the series $\sum_{r=1}^{\infty} x_{rk}^*(f(t))\chi_{B_r}(t)$ is absolutely convergent for every f and every t, and we obtain that

$$\eta_k(t)((f)(t)) = \sum_{r=1}^{\infty} x_{rk}^*(f(t))\chi_{B_r}(t).$$

Moreover, by the Lebesgue Dominated Theorem

$$\begin{aligned}\big[\eta_k\big](f) = \int_\Omega \eta_k(f(t))\, d\mu(t) &= \int_\Omega \sum_{r=1}^{\infty} x_{rk}^*(f(t))\chi_{B_r}(t)\, d\mu(t) \\ &= \sum_{r=1}^{\infty} \int_\Omega x_{rk}^*(f(t))\chi_{B_r}(t)\, d\mu(t),\end{aligned}$$

for all f in $L_1(\mu, X)$. On the other hand, since the series defining P converges in the $\|.\|_1$-norm,

$$\begin{aligned}\big[P^*(\eta_k)\big](f) &= \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)}[\eta_k]\left(\left(\int_{B_i} f(u)\, d\mu(u)\right)\chi_{B_i}\right) \\ &= \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)} \int_\Omega \sum_{r=1}^{\infty} x_{rk}^* \left(\int_{B_i} f(u)\, d\mu(u)\right)\chi_{B_r}(t)\chi_{B_i}(t)\, d\mu(t) \\ &= \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)} \int_\Omega x_{ik}^* \left(\int_{B_i} f(u)\, d\mu(u)\right)\chi_{B_i}(t)\, d\mu(t) \\ &= \sum_{i=1}^{\infty} x_{ik}^* \left(\int_{B_i} f(u) d\mu(u)\right) = \sum_{i=1}^{\infty} \int_\Omega x_i^*(f(u))\chi_{B_i}(u)\, d\mu(u).\end{aligned}$$

Thus, for $1 \leqslant p < \infty$,

$$\begin{aligned}\|P^*(g_k) - g_k\|_{p^*} &\leqslant \|P^*(g_k) - P^*(\eta_k)\|_{p^*} + \|\eta_k - g_k\|_{p^*} \\ &\leqslant 2\|\eta_k - g_k\|_{p^*} < \varepsilon,\end{aligned}$$

for every $k = 1, \dots, m$.

(c). To prove this, we follow the lines of (b). Given $g_1, \ldots, g_m$ in $L_\infty(\mu, X)$ and $\varepsilon > 0$, for each $k \in \{1, \ldots, m\}$ there exists a measurable, bounded and countably valued mapping $\eta_k : \Omega \longrightarrow X$ such that

$$\|g_k - \eta_k\|_\infty < \varepsilon,$$

and hence, we have a countable partition $\mathcal{C}_k$ of Ω and points $x_C \in X$ for every $C \in \mathcal{C}_k$ such that $\eta_k(t) = \sum_{C \in \mathcal{C}_k} x_C \chi_C(t)$, for every $t \in \Omega$. Again, we take an infinite countable partition $\mathcal{B} = \{B_i : i =\in \mathbb{N}\}$ of Ω of sets of positive measure that is a refinement of $\mathcal{A}$ and $\mathcal{C}_k$ for all k. Finally, if we define $P : L_\infty(\mu, X) \longrightarrow L_\infty(\mu, X)$ by

$$P(f) = \sum_{i=1}^{\infty} \frac{1}{\mu(B_i)} \left(\int_{B_i} f d\mu \right) \chi_{B_i},$$

we have that P is a norm-one projection such that $P(L_\infty(\mu, X))$ is isometrically isomorphic to $\ell_\infty(X)$ and

$$\|P(g_k) - g_k\|_\infty = \|\eta_k - g_k\|_\infty < \varepsilon,$$

for every $k = 1, \ldots, m$. □

Proof of Theorem 3.3.12. If $L_1(\mu)$ is finite-dimensional, the result is a consequence of Lemma 3.3.13. Let us suppose that $L_1(\mu)$ is infinite-dimensional in the rest of the proof.

Let us start with the case when μ is finite. As X^* has the Radon-Nikodým property, we have that $L_\infty(\mu, X^*) = L_1(\mu, X)^*$ (see e.g. [29, Theorem IV.1.1 in p. 98]), so Lemma 3.3.15.(b) gives a net $\{P_\lambda\}_{\lambda \in \Lambda}$ of norm-one projections on $L_1(\mu, X)$ such that $\{P_\lambda f\} \longrightarrow f$ in norm for every $f \in L_1(\mu, X)$, $\{P_\lambda^* g\} \longrightarrow g$ in norm for every $g \in L_\infty(\mu, X^*) = L_1(\mu, X)^*$ and $P_\lambda(L_1(\mu, X))$ is isometrically isomorphic to $\ell_1(X)$. Now, we may apply Proposition 3.2.3 to conclude the result.

If μ is σ-finite, we may use [20, Proposition 1.6.1] to reduce to the previous case: there is a finite measure ν such that $L_1(\mu, X)$ is isometrically isomorphic to $L_1(\nu, X)$. Let us also observe that we actually get that there exists a common function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, depending only on X and Y, such that all the spaces $(L_1(\mu, X), Y)$ have the BPBp for compact operators with the function η when μ is σ-finite.

Finally, for the general case, we may adapt an argument from the proof of [26, Proposition 2.1]. Let $0 < \varepsilon < 1$ and $T \in \mathcal{K}(L_1(\mu, X); Y)$ with $\|T\| = 1$ and $f_0 \in S_{L_1(\mu,X)}$ satisfying

$$\|Tf_0\| > 1 - \eta(\varepsilon),$$

where η is the universal function for all σ-finite measures given in the previous case. Pick a sequence $\{f_n\}_{n\in\mathbb{N}}$ in the unit sphere of $L_1(\mu, X)$ such that $\lim_{n\to\infty} \|Tf_n\| = 1$. Then, there is a measurable set A such that the measure $\mu|_A$ is σ-finite and the support of all the f_n, $n \geqslant 0$, are contained in A. Then, consider T_1 to be the restriction of T to $L_1(\mu|_A, X)$, which satisfies $\|T_1\| = 1$ and $\|T_1 f_0\| > 1 - \eta(\varepsilon)$. By the assumption on η, there exist a compact norm-one operator $S_1 : L_1(\mu|_A, X) \longrightarrow Y$ and a norm-one vector $g \in L_1(\mu|_A, X)$ such that

$$\|S_1 g\| = 1, \quad \|T_1 - S_1\| < \varepsilon \quad \text{and} \quad \|f_0 - g\| < \varepsilon.$$

Let $P : L_1(\mu, X) \longrightarrow L_1(\mu|_A, X)$ denote the restriction operator. Then

$$S = (S_1 \circ P) + T \circ (\mathrm{Id} - P)$$

is a compact norm-one operator from $L_1(\mu, X)$ into Y, g can be viewed as a norm-one element in $L_1(\mu, X)$ (just extending by 0), $\|Sg\| = 1$, $\|S - T\| < \varepsilon$ and $\|f_0 - g\| < \varepsilon$. $\square$

When X is just the base field, we recover the result for pairs of the form $(L_1(\mu); Y)$ from [5, Corollary 2.4] (see Examples 3.1.2.(i)). More applications of Theorem 3.3.12 for the vector-valued case can be given using the results of [50].

Corollary 3.3.16. Let μ be a positive measure and let X, Y be Banach spaces. The pair $(L_1(\mu, X); Y)$ has the BPBp for compact operators in the following cases:

(a) if X and Y are finite-dimensional;

(b) if X^* has the Radon-Nikodým property, Y is a Hilbert space and the pair $(X; Y)$ has the BPBp for compact operators;

(c) in particular, if Y is a Hilbert space and $X = c_0$ or $X = L_p(\nu)$ for any positive measure ν and $1 < p < \infty$.

Proof. (a). When X and Y are finite-dimensional, it is shown in [50, Proposition 7] that the pair $(\ell_1(X); Y)$ has the BPBp. Since X and Y are finite-dimensional, Lemma 3.3.13 gives that the pair $(\ell_1(X); Y)$ has the BPBp for compact operators. Now, Theorem 3.3.12 applies since X^* trivially has the RNP.

(b). If we only consider finite convex sums instead of convex series, we may repeat the proof of [50, Proposition 9] but using only compact operators to get item (i) of Lemma 3.3.13. Then, we have that $(\ell_1(X); Y)$ has the BPBp for compact operators. If X^* has the Radon-Nikodým property, Theorem 3.3.12 finishes the proof.

(c). It follows from (b) and Examples 3.1.2. □

The proof of Theorem 3.3.12 can be adapted to pairs of the form $(L_p(\mu, X); Y)$ for $1 < p < \infty$, but only when the measure μ satisfies that $L_1(\mu)$ is infinite-dimensional.

Proposition 3.3.17. Let $1 < p < \infty$, let μ be a positive measure such that $L_1(\mu)$ is infinite-dimensional, let X be a Banach space such that X^* has the Radon-Nikodým property, and let Y be a Banach space. If the pair $(\ell_p(X); Y)$ has the BPBp for compact operators, then so does the pair $(L_p(\mu, X); Y)$.

Let us observe that, in this case, the scalar-valued version of the result has no interest, as the spaces $L_p(\mu)$ are uniformly convex for $1 < p < \infty$ and we may use Examples 3.1.2.(b).

Now we work with the results in the range spaces. In those cases, we are using Proposition 3.2.6.

Theorem 3.3.18. Let X and Y be Banach spaces.

(a) For $1 \leqslant p < \infty$, if the pair $(X; \ell_p(Y))$ has the BPBp for compact operators, then so does $(X; L_p(\mu, Y))$ for every positive measure μ such that $L_1(\mu)$ is infinite-dimensional.

(b) If the pair $(X; Y)$ has the BPBp for compact operators, then so does $(X; L_\infty(\mu, Y))$ for every σ-finite positive measure μ.

(c) If the pair $(X; Y)$ has the BPBp for compact operators, then so does $(X; C(K, Y))$ for every compact Hausdorff topological space K.

To prove this theorem we need the following result which reminds Lemma 3.3.2 but now for range spaces.

Lemma 3.3.19. Let X, Y be Banach spaces and let $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a function. The following are equivalent:

(i) the pair $(X; Y)$ has the BPBp for compact operators with the function η,

(ii) the pairs $(X; \ell_\infty^m(Y))$ with $m \in \mathbb{N}$ have the BPBp for compact operators with the function η,

(iii) the pair $(X; c_0(Y))$ has the BPBp for compact operators with the function η,

(iv) the pair $(X; \ell_\infty(Y))$ has the BPBp for compact operators with the function η.

Proof. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) can be proved by adapting the proof of [11, Proposition 2.4] to compact operators. Finally, the fact that any of the assertions (ii), (iii) or (iv) implies (i) is a consequence of Lemma 3.2.7.(b). □

We are now ready to present the proof of the theorem.

Proof of Theorem 3.3.18. (a). Fix $1 \leqslant p < \infty$. Suppose that the pair $(X; \ell_p(Y))$ has the BPBp for compact operators. If $L_1(\mu)$ is infinite-dimensional, Lemma 3.3.15.(a) provides a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_p(\mu, Y)$ such that $\{Q_\lambda f\} \longrightarrow f$ in norm for every $f \in L_p(\mu, Y)$ and $Q_\lambda(L_p(\mu, Y))$ is isometrically isomorphic to $\ell_p(Y)$. By Proposition 3.2.6, we get that the pair $(X; L_p(\mu, Y))$ has the BPBp for compact operators.

(b). Suppose that the pair $(X; Y)$ has the BPBp for compact operators. If $L_\infty(\mu)$ is finite-dimensional, the result is a consequence of Lemma 3.3.19. Otherwise, if $L_\infty(\mu)$ is infinite-dimensional, we may suppose that the measure μ is finite by using [20, Proposition 1.6.1]. Lemma 3.3.15.(c) provides a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $L_\infty(\mu, X)$ such that $\{Q_\lambda f\} \longrightarrow f$ in norm for every $f \in L_\infty(\mu, Y)$ and $Q_\lambda(L_\infty(\mu, X))$ is isometrically isomorphic to $\ell_\infty(Y)$. Now, Lemma 3.3.19 gives that all the pairs $(X, Q_\lambda(L_\infty(\mu, X)))$ have the BPBp for compact operators with the same function, and so the result follows again from Proposition 3.2.6.

(c). Following step-by-step the proof of [44, Theorem 4], by using peak partitions of unity and extending the scalar-valued case to the vector-valued case, we may find a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of norm-one projections on $C(K,Y)$ such that $\{Q_\lambda f\} \longrightarrow f$ in norm for every $f \in C(K,Y)$ and $Q_\lambda(C(K,Y))$ is isometrically isomorphic to $\ell_\infty^m(Y)$. Now, the result follows one more time from Lemma 3.3.19 and Proposition 3.2.6. □

In particular, Theorem 3.3.18.(c) gives that the pair $(X, C(K))$ has the BPBp for compact operators for every Banach space X (see Theorem 2.1.19 and Theorem 4.1.9 for similar cases). More consequences of Theorem 3.3.18 are the following.

Corollary 3.3.20. Let X, Y be Banach spaces, let K be a compact Hausdorff topological space, let μ be a positive measure and let ν be a σ-finite positive measure.

(a) If Y has property β, then $(X; L_\infty(\nu, Y))$ and $(X; C(K,Y))$ have the BPBp for compact operators.

(b) If Y has the AHSP, then so do $L_\infty(\nu, Y)$ and $C(K,Y)$.

(c) For $1 \leqslant p < \infty$, if $\ell_p(Y)$ has the AHSP and $L_1(\mu)$ is infinite-dimensional, then $L_p(\mu, Y)$ has the AHSP.

Proof. (a). If Y has property β, then Example 3.1.2.(a) says that the pair $(X; Y)$ has the BPBp for compact operators. By items (b) and (c) of Theorem 3.3.18 we have the result.

(b) and (c). Here we use Corollary 3.3.14 which says that a Banach space Z has the AHSP if and only if the pair $(\ell_1; Z)$ has the BPBp for compact operators. With this in mind, both statements are direct consequences of Theorem 3.3.18. □

Chapter 4

The BPBp for multilinear mappings

In this chapter, we study the Bishop-Phelps-Bollobás property, the Bishop-Phelps-Bollobás property for numerical radius and the generalized AHSP for multilinear mappings. See Section 1.1 for multilinear and homogeneous polynomials notations.

4.1 Extending some known results

We dedicate this section to extend some known results about norm attaining and Bishop-Phelps-Bollobás property for operators to the multilinear and polynomial cases. We start by defining the BPBp for these type of functions.

Definition 4.1.1. (BPBp for multilinear mappings) Let $X_1, \ldots X_N$ and Y be Banach spaces. We say that $(X_1, \ldots, X_N; Y)$ has the *Bishop-Phelps-Bollobás property for multilinear mappings* (BPBp for multilinear mappings, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{L}(X_1, \ldots, X_N; Y)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in$

$S_{X_1} \times \ldots \times S_{X_N}$ satisfy

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

there are $B \in \mathcal{L}(X_1, \ldots, X_N; Y)$ with $\|B\| = 1$ and $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ such that

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \ \text{ and } \ \|B - A\| < \varepsilon. \quad (4.1)$$

In this case, we say that $(X_1, \ldots, X_N; Y)$ has the BPBp for multilinear mappings with function $\varepsilon \longmapsto \eta(\varepsilon)$.

When it is of interest we can emphasize the degree of the multilinear mapping by saying that $(X_1, \ldots, X_N; Y)$ has the *BPBp for N-linear mappings* instead of the BPBp for multilinear mappings. We may also define the *BPBp for symmetric multilinear mappings* when in Definition 4.1.1 we consider A and B both elements in $\mathcal{L}_s(^N X; Y)$. In this case, we say that $(^N X; Y)$ has the BPBp for symmetric multilinear mappings. When $Y = \mathbb{K}$, we denote the BPBp for $(X_1, \ldots, X_N; \mathbb{K})$ just by $(X_1, \ldots, X_N)$ and we say that $(X_1, \ldots, X_N)$ has the *BPBp for N-linear forms*. Analogously, we define the BPBp for homogeneous polynomials as follows.

Definition 4.1.2. (BPBp for homogeneous polynomials) Let X and Y be Banach spaces. We say that the pair $(X; Y)$ has the *Bishop-Phelps-Bollobás property for N-homogeneous polynomials* if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $P \in \mathcal{P}(^N X; Y)$ with $\|P\| = 1$ and $x_0 \in S_X$ satisfy

$$\|P(x_0)\| > 1 - \eta(\varepsilon),$$

there are $Q \in \mathcal{P}(^N X; Y)$ with $\|Q\| = 1$ and $x_1 \in S_X$ such that

$$\|Q(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|Q - P\| < \varepsilon.$$

It is worth to mentioning that using a routinely change of parameters in Definitions 4.1.1 and 4.1.2, we may consider the given elements in the unit ball of their respectively spaces instead of norm-one elements (see Remark 3.2.1). For example, in the multilinear mapping case, we can say that $(X_1, \ldots, X_N; Y)$ has the BPBp for multilinear mappings if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{L}(X_1, \ldots, X_N; Y)$ with $\|A\| \leqslant 1$ and $(x_1^0, \ldots, x_N^0) \in B_{X_1} \times \ldots \times B_{X_N}$ satisfy

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

there are $B \in \mathcal{L}(X_1, \ldots, X_N; Y)$ with $\|B\| = 1$ and $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfying conditions (4.1).

Let us comment on some known results about the BPBp for those classes of functions. It was shown in [48, Theorem 2 and Corollary 3] that $(C_0(K), C_0(L))$ and (c_0, c_0) have the BPBp for bilinear forms in the complex case for all locally compact Hausdorff topological spaces K and L. On the other hand, $(L_1[0,1], L_1[0,1])$ fails the BPBp for bilinear forms [23, Theorem 3]. The pair (H, H) has the BPBp for symmetric bilinear forms on a Hilbert space H [33, Theorem 3.2 and Theorem 3.4]. In [6, Theorem 2.2] it was shown that if $X_1, \ldots, X_N$ are uniformly convex Banach spaces, then $(X_1, \ldots X_N; Y)$ has the BPBp for multilinear mappings for any Banach space Y. Also, if X is a uniformly convex Banach space then $(X; Y)$ has the BPBp for N-homogeneous polynomials for any Banach space Y [4, Theorem 3.1].

As we said before, we are trying to extend some known BPBp and norm attaing operators results to the multilinear case. Nevertheless, we already can discard some possible extensions.

- We cannot expect a BPBp version for multilinear mappings of [12, Theorem 3] which says that if X satisfies property α then the set $\operatorname{NA} \mathcal{L}(^N X)$ is dense in $\mathcal{L}(^N X)$, since a typical example of a

Banach space with this property is ℓ_1 and (ℓ_1, ℓ_1) fails the BPBp for bilinear forms [27, Theorem 2].

- There is no BPBp version for multilinear mappings of [2, Theorem 2.2] when we assume that the range space Y has property β since $\mathbb{K}$ satisfies property β and (ℓ_1, ℓ_1) fails the BPBp for bilinear forms [27, Theorem 2].
- The same arguments of [27, Theorem 2] gives a proof that (ℓ_1, ℓ_1) *does not have* the BPBp for symmetric bilinear forms although the set $\operatorname{NA} \mathcal{L}_s(^N\ell_1; Y)$ is dense in $\mathcal{L}_s(^N\ell_1; Y)$ for every $N \in \mathbb{N}$ and every Banach space Y [25, Theorem 2.4(b)].

We observe also that although (ℓ_1, ℓ_1) fails the BPBp for bilinear forms, the set $\operatorname{NA} \mathcal{L}(^N\ell_1; Y)$ is dense in $\mathcal{L}(^N\ell_1; Y)$ for every $N \in \mathbb{N}$ and every Banach space Y [25, Theorem 2.4(a)]. So, analogously to the operator case, the BPBp for multilinear mapping is not just a trivial generalization of the norm attaining multilinear mappings denseness on $\mathcal{L}(X_1, \ldots, X_N; Y)$.

We start our results now. It is known that for all finite-dimensional Banach spaces X and Y, the pair $(X; Y)$ has the BPBp [2, Proposition 2.4]. Our first result gives the analogous version for multilinear mappings and homogeneous polynomials and its proof is just an easy modification of [2, Proposition 2.4] to these cases.

Proposition 4.1.3. Let $X, X_1, \ldots, X_N$ and Y be finite dimensional Banach spaces. Then

(i) $(X_1, \ldots, X_N; Y)$ has the BPBp for multilinear mappings,

(ii) $(^N X; Y)$ has the BPBp for symmetric multilinear mappings and

(iii) $(X; Y)$ has the BPBp for N-homogeneous polynomials.

The next result concerns stability of the BPBp for multilinear mappings. In [8, Proposition 3.1] and [42, Proposition 2.1] it was proved that if X is a Banach space and $N \in \mathbb{N}$ is a natural number, then the set $\operatorname{NA}\mathcal{L}(^N X;Y)$ is dense in $\mathcal{L}(^N X;Y)$ whenever the set $\operatorname{NA}\mathcal{L}(^{N+1}X;Y)$ is dense in $\mathcal{L}(^{N+1}X;Y)$. In the next proposition we show that the analogous result for the Bishop-Phelps-Bollobás property holds as well.

Proposition 4.1.4. Let $X_1, \ldots X_N, X_{N+1}$ and Y be Banach spaces. If $(X_1, \ldots, X_N,\, X_{N+1}; Y)$ has the BPBp for $(N+1)$-linear mappings, then $(X_1, \ldots,\, X_N; Y)$ has the BPBp for N-linear mappings.

Proof. Let $\varepsilon \in (0,1)$ be given. Let $\eta(\varepsilon) > 0$ be the BPBp constant for $(X_1, \ldots,\, X_N, X_{N+1}; Y)$. Let $A \in \mathcal{L}(X_1, \ldots, X_N, X_{N+1}; Y)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ be such that

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Let $x_{N+1}^0 \in S_{X_{N+1}}$ and take $x_{N+1}^* \in S_{X_{N+1}^*}$ to be such that $x_{N+1}^*(x_{N+1}^0) = 1$. Define $\widetilde{A} : X_1 \times \ldots \times X_N \times X_{N+1} \longrightarrow Y$ by

$$\widetilde{A}(x_1, \ldots, x_n, x_{N+1}) := x_{N+1}^*(x_{N+1})A(x_1, \ldots, x_N)$$

for every $(x_1, \ldots, x_N, x_{N+1}) \in X_1 \times \ldots \times X_N \times X_{N+1}$. Then $\|\widetilde{A}\| \leqslant 1$ and

$$\|\widetilde{A}(x_1^0, \ldots, x_N^0, x_{N+1}^0)\| > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

Then there are $\widetilde{B} \in \mathcal{L}(^{N+1}X_1, \ldots, X_N, X_{N+1}; Y)$ with $\|\widetilde{B}\| = 1$ and $(z_1^0, \ldots, z_N^0, z_{N+1}^0) \in S_{X_1} \times \ldots \times S_{X_N} \times S_{X_{N+1}}$ such that

$$\|\widetilde{B}(z_1^0, \ldots, z_N^0, z_{N+1}^0)\| = 1, \ \max_{1 \leqslant j \leqslant N+1} \|z_j^0 - x_j^0\| < \frac{\varepsilon}{2} \text{ and } \|\widetilde{A} - \widetilde{B}\| < \frac{\varepsilon}{2}.$$

In particular, $\|z_j^0 - x_j^0\| < \varepsilon$ for all $j = 1, \ldots, N$. Since $\|\widetilde{B} - \widetilde{A}\| < \frac{\varepsilon}{2}$, it follows that $x_{N+1}^*(z_{N+1}^0) \neq 0$ since

$$|x_{N+1}^*(z_{N+1}^0)| \geqslant |x_{N+1}^*(x_{N+1}^0)| - \|x_{N+1}^*\| \|z_{N+1}^0 - x_{N+1}^0\| > 1 - \frac{\varepsilon}{2} > 0.$$

Now we define $C : X_1 \times \ldots \times X_N \longrightarrow Y$ by

$$C(x_1, \ldots, x_N) := \frac{1}{x^*_{N+1}(z^0_{N+1})} \widetilde{B}(x_1, \ldots, x_N, z^0_{N+1})$$

for all $(x_1, \ldots, x_N) \in X_1 \times \ldots \times X_N$. Is is not difficult to prove that $\|C\| = \|C(z^0_1, \ldots, z^0_N)\|$ and that $\|C - A\| < \varepsilon$. To finish the proof, we put $B := \frac{C}{\|C\|}$. Therefore,

$$\|B\| = \left\|B\left(z^0_1, \ldots, z^0_N\right)\right\| = 1, \ \|B-A\| < 2\varepsilon \text{ and } \max_{1\leqslant j\leqslant N} \|z^0_j - x^0_j\| < \varepsilon. \ \square$$

We observe that the converse of Proposition 4.1.4 is no longer true by using again [27, Theorem 1]. Now we observe that, since there is a natural (isometric) identification between the Banach spaces $\mathcal{L}(^N X_1, \ldots, X_N; Y)$ and $\mathcal{L}(^k X_1, \ldots, X_k;\, \mathcal{L}(^{N-k} X_{k+1}, \ldots, X_N; Y))$, we have the following result.

Proposition 4.1.5. Suppose that $X_1, \ldots, X_N$ and Y are Banach spaces. If the pair $(X_1, \ldots, X_N;\, Y)$ has the BPBp for N-linear mappings, then the pair $(X_1, \ldots, X_k; \mathcal{L}(^{N-k} X_{k+1}, \ldots, X_N; Y))$ has the BPBp for k-linear mappings.

We observe that the converse of Proposition 4.1.5 is not true in general. If it were true, then it would hold for $N = 2$, $k = 1$ and $Y = \mathbb{K}$. But this would imply that if the pair $(X; Y^*)$ has the BPBp for operators then (X, Y) has the BPBp for bilinear forms which is false in general. For this, we take again $X = Y = \ell_1$ and use [2, Theorem 4.1] which gives that the pair (ℓ_1, ℓ_∞) has the BPBp for operators and [27, Theorem 1] which shows that (ℓ_1, ℓ_1) fails the BPBp for bilinear forms. However, we have the following result which is the multilinear version of [6, Proposition 2.4] or [28, Theorem 1.1].

Proposition 4.1.6. Suppose that $X_1, \ldots,\, X_N$ are Banach spaces and let X_N be uniformly convex. Then $(X_1, \ldots, X_N)$ has the BPBp for

N-linear forms if and only if $(X_1, \ldots, X_{N-1};\ X_N^*)$ has the BPBp for $(N-1)$-linear mappings.

Proof. If $(X_1, \ldots, X_N)$ has the BPBp for N-linear mappings, then $(X_1, \ldots, X_{N-1}; X_N^*)$ has the BPBp for $(N-1)$-linear mappings by using Proposition 4.1.5 with $k = N-1$ and $Y = \mathbb{K}$. Now let $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ be given. Assume that $(X_1, \ldots, X_{N-1}; X_N^*)$ has the BPBp for $(N-1)$-linear mappings with function $\eta(\varepsilon) > 0$ and consider $\delta_{X_N}(\varepsilon) > 0$ the modulus of convexity of the space X_N. We take $\xi > 0$ satisfying that

$$\xi < N\xi < \min\left\{\frac{\delta_{X_N}(\varepsilon)}{2}, \varepsilon\right\} \quad \text{and} \quad \eta'(\xi) := \min\left\{\frac{\delta_{X_N}(\xi)}{2}, \eta(\xi)\right\}.$$

Let $A \in \mathcal{L}(X_1, \ldots, X_N)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ be such that

$$\operatorname{Re} A(x_1^0, \ldots, x_N^0) > 1 - \eta'(\xi).$$

Define $\widetilde{A} : X_1 \times \ldots \times X_{N-1} \longrightarrow X_N^*$ by

$$\widetilde{A}(x_1, \ldots, x_{N-1})(x_N) := A(x_1, \ldots, x_{N-1}, x_N)$$

for all $(x_1, \ldots, x_{N-1}) \in X_1 \times \ldots \times X_{N-1}$ and $x_N \in X_N$. Then $\|\widetilde{A}\| = \|A\| = 1$ and

$$\|\widetilde{A}(x_1^0, \ldots, x_{N-1}^0)\| \geqslant \operatorname{Re} A(x_1^0, \ldots, x_{N-1}^0, x_N^0) > 1 - \eta'(\xi).$$

So, there are $\widetilde{B} \in \mathcal{L}(X_1, \ldots, X_{N-1}; X_N^*)$ with $\|\widetilde{B}\| = 1$ and $(z_1^0, \ldots, z_{N-1}^0) \in S_{X_1} \times \ldots \times S_{X_{N-1}}$ such that

$$\|\widetilde{B}(z_1^0, \ldots, z_{N-1}^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N-1} \|z_j^0 - x_j^0\| < \xi < \varepsilon \quad \text{and} \quad \|\widetilde{B} - \widetilde{A}\| < \xi.$$

Since X_N is reflexive, there is $z_N^0 \in S_{X_N}$ such that

$$\operatorname{Re} \widetilde{B}(z_1^0, \ldots, z_{N-1}^0)(z_N^0) = \|\widetilde{B}(z_1^0, \ldots, z_{N-1}^0)\| = 1.$$

We will prove that $\|z_N^0 - x_N^0\| < \varepsilon$. Indeed, since

$$\operatorname{Re} \widetilde{A}(z_1^0, \ldots, z_{N-1}^0)(x_N^0) - \operatorname{Re} \widetilde{B}(z_1^0, \ldots z_{N-1}^0)(x_N^0) < \xi$$

then

$$\begin{aligned} \operatorname{Re} \widetilde{B}(z_1^0, \ldots, z_{N-1}^0)(x_N^0) &> \operatorname{Re} \widetilde{A}(z_1^0, \ldots, z_{N_1}^0)(x_N^0) - \xi \\ &\geqslant \operatorname{Re} A(x_1^0, x_2^0, \ldots, x_{N-1}^0, x_N^0) - \sum_{j=1}^{N-1} \|x_j^0 - z_j^0\| - \xi > 1 - \delta_{X_N}(\varepsilon). \end{aligned}$$

This implies that

$$\left\| \frac{x_N^0 + z_N^0}{2} \right\| > 1 - \frac{\delta_{X_N}(\varepsilon)}{2}$$

and then $\|z_N^0 - x_N^0\| < \varepsilon$. Now defining $B : X_1 \times \ldots \times X_N \longrightarrow \mathbb{K}$ by

$$B(x_1, \ldots, x_N) := \widetilde{B}(x_1, \ldots, x_{N-1})(x_N)$$

for all $(x_1, \ldots, x_{N-1}, x_N) \in X_1 \times \ldots \times X_{N-1} \times X_N$, we have that

$$\left\| B(z_1^0, \ldots, z_N^0) \right\| = \operatorname{Re} \widetilde{B}(z_1^0, \ldots, z_{N-1}^0)(z_N^0) = 1$$

as well as

$$\max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \quad \text{and} \quad \|B - A\| \leqslant \|\widetilde{B} - \widetilde{A}\| < \xi < \varepsilon.$$

This shows that $(X_1, \ldots, X_N)$ has the BPBp for N-linear forms. $\square$

In the next result we prove that we may pass from the vector-valued to the scalar-valued case in the BPBp for multilinear mappings.

Proposition 4.1.7. Suppose that $X_1, \ldots, X_N$ and Y are Banach spaces. If $(X_1, \ldots, X_N; Y)$ has the BPBp for multilinear mappings, then $(X_1, \ldots, X_N)$ has the BPBp for multilinear forms.

Proof. Let $\varepsilon \in (0,1)$ be given and consider $\eta(\varepsilon) > 0$ be the BPBp constant for $(X_1, \ldots, X_N; Y)$. Let $A \in \mathcal{L}(X_1, \ldots, X_N)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ be such that

$$\operatorname{Re} A(x_1^0, \ldots, x_N^0) > 1 - \eta\left(\frac{\varepsilon^2}{16}\right).$$

Define $\widetilde{A} : X_1 \times \ldots \times X_N \longrightarrow Y$ by

$$\widetilde{A}(x_1, \ldots, x_N) := A(x_1, \ldots, x_N)y_0$$

for some $y_0 \in S_Y$ and for every $(x_1, \ldots, x_N) \in X_1 \times \ldots \times X_N$. Then $\|\widetilde{A}\| = \|A\| = 1$ and

$$\|\widetilde{A}(x_1^0, \ldots, x_N^0)\| > 1 - \eta\left(\frac{\varepsilon^2}{16}\right).$$

Then there are $\widetilde{B} \in \mathcal{L}(X_1, \ldots, X_N; Y)$ with $\|B\| = 1$ and $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ such that

$$\|\widetilde{B}(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \frac{\varepsilon^2}{16} < \varepsilon \text{ and } \|\widetilde{B} - \widetilde{A}\| < \frac{\varepsilon^2}{16}.$$

Let $y_0^* \in S_{Y^*}$ be such that

$$\operatorname{Re} y_0^*(\widetilde{B}(z_1^0, \ldots, z_N^0)) = \|\widetilde{B}(z_1^0, \ldots, z_N^0)\| = 1.$$

Define $B := y_0^* \circ \widetilde{B} \in \mathcal{L}(X_1, \ldots, X_N)$. Then $\|B\| \leqslant 1$ and $|B\,(z_1^0, \ldots, z_N^0)| = 1$. So $\|B\| = 1$ and B attains its norm at $(z_1^0, \ldots, z_N^0)$ which is close to $(x_1^0, \ldots, x_N^0)$. It remains to prove that B is close to A. Indeed, first we note that

$$\begin{aligned}\operatorname{Re}(1 - y_0^*(y_0)) &\leqslant \operatorname{Re}(1 - y_0^*(y_0)A(z_1^0, \dots, z_N^0)) \\ &= \operatorname{Re}(y_0^*(\widetilde{B}(z_1^0, \dots, z_N^0)) - A(z_1^0, \dots, z_N^0)y_0) \\ &\leqslant \|\widetilde{B} - \widetilde{A}\| < \frac{\varepsilon^2}{16}.\end{aligned}$$

So, we have a complex number $a := y_0^*(y_0)$ such that $|a| \leqslant 1$ and $\operatorname{Re}(1-a) = 1 - \operatorname{Re} a < \frac{\varepsilon^2}{16}$. This implies that $(\operatorname{Re} a)^2 + (\operatorname{Im} a)^2 \leqslant 1$ and then

$$(\operatorname{Im} a)^2 \leqslant 1 - (\operatorname{Re} a)^2 = (1 + \operatorname{Re} a)(1 - \operatorname{Re} a) < 2 \cdot \frac{\varepsilon^2}{16} = \frac{\varepsilon^2}{8}.$$

By this, we get that

$$|1 - a| = \sqrt{(1 - \operatorname{Re})^2 + (\operatorname{Im} a)^2} < \sqrt{\left(\frac{\varepsilon^2}{16}\right)^2 + \frac{\varepsilon^2}{8}} < \frac{\varepsilon}{2}.$$

So $|1 - y_0^*(y_0)| < \frac{\varepsilon}{2}$ and then

$$\begin{aligned}\|B - A\| &\leqslant \|y_0^* \circ \widetilde{B} - y_0^* \circ \widetilde{A}\| + \|y_0^* \circ \widetilde{A} - A\| \\ &\leqslant \|\widetilde{B} - \widetilde{A}\| + |1 - y_0^*(y_0)| < \frac{\varepsilon^2}{16} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$ □

It is clear that the converse of the previous proposition is false because of Lindenstrauss' counterexample for the Bishop-Phelps theorem for operators. Nevertheless, we have the following consequence by using [4, Proposition 3.3].

Corollary 4.1.8. Let $X_1, \dots, X_N$ and Y be Banach spaces. Assume that Y has property β. The N-tuple $(X_1, \dots, X_N)$ has the BPBp for multilinear forms if and only if $(X_1, \dots, X_N; Y)$ has the BPBp for multilinear mappings.

Proof. By [4, Proposition 3.3], if $(X_1, \dots, X_N)$ has the BPBp for multilinear forms and Y has property β, then $(X_1, \dots, X_N; Y)$ has the BPBp for multilinear mappings. Now we apply Proposition 4.1.7 to finish the proof. □

It was proved in Section 3.3 that if the pair $(X; Y)$ has the BPBp for compact operators, then the pair $(X; C(K, Y))$ has the same property for every compact Hausdorff topological space K (see Theorem 3.3.18.(c)). In particular, $(X; C(K))$ has the BPBp for compact operators. Analogous to the operator case, we say that $(X_1, \dots, X_N; Y)$ has the *BPBp for compact multilinear mappings* when in Definition 4.1.1 we consider A and B as compact multilinear mappings. We already have some examples of this property:

- adapting [4, Proposition 3.3], if N-tuple $(X_1, \dots, X_N)$ has the BPBp for multilinear forms, then $(X_1, \dots, X_N; Y)$ has the BPBp for compact multilinear mappings whenever Y has property β;
- when we assume that $X_1, \dots, X_N$ are uniformly convex Banach spaces, we can adapt [6, Theorem 2.2] to get that $(X_1, \dots, X_N; Y)$ has the BPBp for compact multilinear mappings for all Banach space Y.

In [3, Theorem 4.2] it was proved that the pair $(X; Y)$ has the BPBp for compact operators for any Banach space X and for a predual of an L_1-space Y. We will prove the analogous result for multilinear mappings. To do so, we will use the fact that preduals of L_1-spaces have a strong form of the metric approximation property.

Theorem 4.1.9. Let $X_1, \dots, X_N$ be Banach spaces and let Y be a predual of an L_1-space. Suppose that the N-tuple $(X_1, \dots, X_N)$ has the BPBp for multilinear forms. Then the pair $(X_1, \dots, X_N; Y)$ has the BPBp for compact multilinear mappings. More precisely, given $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ satisfy

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \eta(\varepsilon),$$

there are $B \in \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|B\| = 1$ and $(z_1^0, \ldots, z_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ such that $\dim(B(X_1 \times \ldots \times X_N)) < \infty$,

$$\|B(z_1^0, \ldots, z_N^0)\| = 1, \quad \max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon \text{ and } \|B - A\| < \varepsilon.$$

Proof. Suppose that $(X_1, \ldots, X_N)$ has the BPBp for multilinear forms. Let $\varepsilon \in (0, 1)$ and $m \in \mathbb{N}$. By using an adaptation of the compact case of [4, Proposition 3.3], we have that $(X_1, \ldots, X_N; \ell_\infty^m)$ has the BPBp for compact multilinear mappings with some $\eta(\varepsilon) > 0$. Let $A \in \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|A\| = 1$ and $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \ldots \times S_{X_N}$ be such that

$$\|A(x_1^0, \ldots, x_N^0)\| > 1 - \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right).$$

By the observation just before the start of this theorem, Y has the metric approximation property and therefore there exists a finite rank operator $F : Y \longrightarrow Y$ with $\|F\| \leqslant 1$ such that, for all $y \in A(B_{X_1} \times \ldots \times B_{X_N}) \subset \overline{A(B_{X_1} \times \ldots \times B_{X_N})}$, we have that

$$\|F(y) - y\| < \min\left\{\frac{\varepsilon}{8}, \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right)\right\}.$$

Then $\|FA\| \neq 0$ and we may define $A' := \frac{1}{\|FA\|}FA$. Thus $A' \in \mathcal{K}(X_1, \ldots, X_N; Y)$ with $\|A'\| = 1$. Moreover,

$$\|A' - A\| = \left\|\frac{FA}{\|FA\|} - A\right\| \leqslant |1 - \|FA\|| + \|FA - A\| \leqslant 2\|FA - A\| < \frac{\varepsilon}{4}$$

and

$$\begin{aligned}\|A'(x_1^0,\ldots,x_N^0)\| &\geqslant \|FA(x_1^0,\ldots,x_N^0)\| \\ &\geqslant \|A(x_1^0,\ldots,x_N^0)\| - \|FA - A\| \\ &> 1 - \frac{1}{2}\eta\left(\frac{\varepsilon}{2}\right).\end{aligned}$$

Since $\dim(A'(X_1 \times \ldots \times X_N)) < \infty$, there is $k \in \mathbb{N}$ such that for every $(x_1,\ldots,x_N) \in X_1 \times \ldots \times X_N$,

$$A'(x_1,\ldots,x_N) = \sum_{i=1}^{k} A_i(x_1,\ldots,x_N)y_i$$

for some $A_i \in \mathcal{K}(X_1,\ldots,X_N) \setminus \{0\}$ and $y_i \in B_Y$ for $i = 1,\ldots,k$. We set $M := \max\{\|x_i^0\| : j = 1,\ldots,N\}$ and we choose α such that

$$0 < \alpha < \min\left\{\frac{1}{4kM}\eta\left(\frac{\varepsilon}{2}\right), \frac{1}{8kM}\varepsilon\right\}.$$

By [51, Theorem 3.1], there are a natural number $m \in \mathbb{N}$ and a subspace E of Y which is linearly isometric to ℓ_∞^m such that $d(y,E) < \alpha$ for every $y \in B_Y \cap A'(X_1 \times \ldots \times X_N)$. In particular, for $i = 1,\ldots,k$, there is $e_i \in E$ such that $\|e_i - y_i\| < \alpha$. Define $C \in \mathcal{K}(X_1,\ldots,X_N;E)$ to be such that

$$C(x_1,\ldots,x_N) := \sum_{i=1}^{k} A_i(x_1,\ldots,x_N)e_i$$

for every $(x_1,\ldots,x_N) \in X_1 \times \ldots \times X_N$. Then $\|C - A'\| < kM\alpha$. This implies that $0 < 1 - kM\alpha < \|C\| < 1 + kM\alpha$. Moreover,

$$\begin{aligned}\|C(x_1^0,\ldots,x_N^0)\| &> \|A'(x_1^0,\ldots,x_N^0)\| - \|C - A'\| \\ &> 1 - \frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right) - kM\alpha.\end{aligned}$$

and then

$$\begin{aligned}\left\|\left(\frac{C}{\|C\|}\right)\left(x_1^0,\dots,x_N^0\right)\right\| &> \frac{1-\frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right)-kM\alpha}{1+kM\alpha}\\ &> \frac{1-\frac{1}{2}\eta\left(\frac{\varepsilon}{2}\right)}{1+\frac{1}{4}\eta\left(\frac{\varepsilon}{2}\right)} > 1-\eta\left(\frac{\varepsilon}{2}\right).\end{aligned}$$

Since E is isometric to ℓ_∞^m, we have that $(X_1,\dots,X_N;E)$ has the BPBp for compact multilinear mappings with the function η and so there are $B\in\mathcal{L}(X_1,\dots,X_N;E)\subset\mathcal{K}(X_1,\dots,X_N;Y)$ with $\|B\|=1$ and $(z_1^0,\dots,z_N^0)\in S_{X_1}\times\dots\times S_{X_N}$ such that

$$\|B(z_1^0,\dots,z_N^0)\|=1,\quad \max_{1\leqslant j\leqslant N}\|z_j^0-x_j^0\|<\frac{\varepsilon}{2}<\varepsilon \text{ and } \left\|B-\frac{C}{\|C\|}\right\|<\frac{\varepsilon}{2}.$$

Since $\dim(B(X_1\times\dots\times X_N))<\infty$, it remains to prove that $\|B-A\|<\varepsilon$. This is true since

$$\begin{aligned}\|B-A\| &\leqslant \left\|B-\frac{C}{\|C\|}\right\|+\left\|\frac{C}{\|C\|}-C\right\|+\|C-A'\|+\|A'-A\|\\ &< \frac{\varepsilon}{2}+2kM\alpha+\frac{\varepsilon}{4}<\varepsilon.\end{aligned}$$

This shows that $(X_1,\dots,X_N;Y)$ has the BPBp for compact multilinear mappings. □

It is known that the pair $(C_0(L),C_0(K))$ has the BPBp for bilinear forms in the complex case [50, Theorem 2] and $(L_1(\mu),c_0)$ has the BPBp for bilinear forms [4, Corollary 2.7(2)]. Also, [6, Theorem 2.2] shows that (X_1,X_2) has the same property whenever X_1 and X_2 are uniformly convex spaces. Hence, we deduce the following corollary.

Corollary 4.1.10. For a predual Y of an L_1-space, $(X,Z;Y)$ has the BPBp for compact bilinear mappings in the following cases.

(a) For the complex Banach spaces $X=C_0(L)$ and $Z=C_0(K)$ where L and K are locally compact topological Hausdorff spaces.

(b) For $X = L_1(\mu)$ and $Z = c_0$.

(c) For X and Z uniformly convex Banach spaces.

Adapting Theorem 4.1.9 we can prove the analogous result for compact symmetric multilinear mappings and for compact N-homogeneous polynomials.

4.2 The generalized AHSP for bilinear forms

In this section we study the generalized approximate hyperplane series property for bilinear forms which was motivated by the AHSP and the generalized AHSP (see [2] and [50]). The AHSP appears for the first time in [2] where the authors were interested to characterize the pairs $(\ell_1; Y)$ to have the BPBp for operators. We recall its definition.

Definition 4.2.1 (AHSP)**.** A Banach space X is said to have the *approximate hyperplane series property* (AHSP, for short) if for every $\varepsilon > 0$, there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every sequence $(x_k)_k \subset S_X$ and every convex series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta$$

there are subsets $A \subset \mathbb{N}$ and $\{z_k : k \in A\}$, and $x^* \in S_{X^*}$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \eta$,

(2) $\|z_k - x_k\| < \varepsilon$ for all $k \in A$,

(3) $x^*(z_k) = 1$ for all $k \in A$.

They showed that the pair $(\ell_1; Y)$ has the BPBp for operators if and only if the Banach space Y has the AHSP [2, Theorem 4.1]. On the other hand, Kim, Lee and Martín defined the generalized AHSP (see [50, Definition 4]) and they also use this property to characterize the pair $(\ell_1(X); Y)$ to have the BPBp for operators.

Definition 4.2.2 (generalized AHSP)**.** A pair of Banach spaces $(X; Y)$ is said to have the *generalized AHSP* if for every $\varepsilon > 0$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that given sequences $(T_k)_k \subset \mathcal{L}(X; Y)$ with $\|T_k\| = 1$ for every k, $(x_k)_k \subset S_X$ and a convex series $\sum_{k=1}^{\infty} \alpha_k$ such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k(x_k) \right\| > 1 - \eta(\varepsilon),$$

there exit a subset $A \in \mathbb{N}$, $y^* \in S_{Y^*}$ and sequences $(S_k)_k \subset \mathcal{L}(X; Y)$ with $\|S_k\| = 1$ for every k, $(z_k)_k \subset S_X$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,

(2) $\|z_k - x_k\| < \varepsilon$ and $\|S_k - T_k\| < \varepsilon$ for all $k \in A$, and

(3) $y^*(S_k(x_k)) = 1$ for every $k \in A$.

Based on Definitions 4.2.1 and 4.2.2, we study the analogous property for bilinear forms. Note that, although there are no bilinear forms in this definition, we put its name as *generalized AHSP for bilinear forms* since it implies the BPBp for bilinear forms for the pair $(\ell_1(X), Y)$.

Definition 4.2.3 (generalized AHSP for bilinear forms)**.** Let X and Y be Banach spaces. We say that the pair (X, Y) has the *generalized approximate hyperplane series property for bilinear forms* (generalized AHSP for bilinear forms, for short) if for every $\varepsilon > 0$, there is $0 < \eta(\varepsilon) < \varepsilon$ such that for given sequences $(T_k)_k \subset S_{\mathcal{L}(X;Y^*)}$ and $(x_k)_k \subset S_X$, an element $y_0 \in S_Y$ and a convex series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\operatorname{Re} \sum_{k=1}^{\infty} \alpha_k T_k(x_k)(y_0) > 1 - \eta(\varepsilon),$$

there are $u_0 \in S_Y$, a subset $A \subset \mathbb{N}$ and sequences $(S_k)_k \subset \mathcal{L}(X; Y^*)$ with $\|S_k\| = 1$ for every k and $(z_k)_k \subset S_X$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,

(2) $\|z_k - x_k\| < \varepsilon$ and $\|S_k - T_k\| < \varepsilon$ for all $k \in A$,

(3) $\|u_0 - y_0\| < \varepsilon$ and

(4) $S_k(z_k)(u_0) = 1$ for all $k \in A$.

As the first result in this section, we observe that the pair (X, Y) has the generalized AHSP for bilinear forms whenever X and Y are finite dimensional Banach spaces. To do so, we will use the following lemma which is proved in [2].

Lemma 4.2.4. [2, Lemma 3.3] Let $\{c_n\}$ be a sequence of scalars with $|c_n| \leqslant 1$ for every $n \in \mathbb{N}$ and let $\sum_{n=1}^{\infty} \alpha_n$ be a convex series such that $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$ for some $\eta > 0$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$ satisfies the estimate

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1 - r}.$$

Proposition 4.2.5. For all finite dimensional Banach spaces X and Y, the pair (X, Y) has the generalized AHSP for bilinear forms.

Proof. For arbitrary $\varepsilon > 0$, we first prove that there exists a positive real number $\eta(\varepsilon) > 0$ satisfying the following. For each $y_0 \in S_Y$, there is $u_0 \in S_Y$ with $\|u_0 - y_0\| < \varepsilon$ such that whenever $(x, T) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ satisfies

$$\operatorname{Re} T(x)(y_0) > 1 - \eta(\varepsilon),$$

there exists $(z, S) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ with

$$S(z)(u_0) = 1 \quad \text{and} \quad \|S - T\| < \varepsilon \quad \text{and} \quad \|z - x\| < \varepsilon.$$

Indeed, otherwise, we can choose some $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$, there is $y_n \in S_Y$ in such way that for each $u \in S_Y$ with $\|u - y_n\| < \varepsilon_0$, there exists $(x_n^u, T_n^u) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ with

$$\operatorname{Re} T_n^u(x_n^u)(y_n) > 1 - \frac{1}{n}$$

such that if $(z, S) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ satisfies $S(z)(u) = 1$ then

$$\max\{\|S - T_n^u\|, \|z - x_n^u\|\} \geqslant \varepsilon_0.$$

Since Y is finite dimensional, we assume that y_n converges to $y_\infty \in S_Y$ and $\|y_n - y_\infty\| < \varepsilon$ for each n. Using compactness again, we may assume that $(x_n^{y_\infty}, T_n^{y_\infty})$ converges to $(x_\infty, T_\infty) \in S_X \times S_{\mathcal{L}(X;Y^*)}$. Then $T_\infty(x_\infty)(y_\infty) = 1$ and this gives a contradiction since we would have that

$$\max\{\|T_n^{y_\infty} - T_\infty\|, \|x_n^{y_\infty} - x_\infty\|\} \geqslant \varepsilon_0.$$

Now we are ready to prove the result. Let $\varepsilon \in (0, 1)$ and consider $0 < \eta(\varepsilon) < \varepsilon$ as above. Let sequences $(T_k)_k \subset S_{\mathcal{L}(X;Y^*)}$ and $(x_k)_k \subset S_X$, a convex series $\sum_{k=1}^{\infty} \alpha_k$ and an element $y_0 \in S_Y$ be such that

$$\operatorname{Re} \sum_{k=1}^{\infty} \alpha_k T_k(x_k)(y_0) > 1 - (\eta(\varepsilon))^2.$$

Let $A := \{k \in \mathbb{N} : \operatorname{Re} T_k(x_k)(y_0) > 1 - \eta(\varepsilon)\}$. By Lemma 4.2.4, we get that

$$\sum_{k \in A} \alpha_k > 1 - \varepsilon.$$

Also, there are $u_0 \in S_Y$ and for each $k \in A$, $(z_k, S_k) \in S_X \times S_{\mathcal{L}(X;Y^*)}$ satisfying that

$$S_k(z_k)(u_0) = 1, \quad \|u_0 - y_0\| < \varepsilon, \quad \|z_k - x_k\| < \varepsilon \quad \text{and} \quad \|S_k - T_k\| < \varepsilon.$$

for all $k \in A$. This shows that the pair (X, Y) has the generalized AHSP for bilinear forms. □

As we mentioned before it is easy to see that if the pair (X, Y) has the generalized AHSP for bilinear forms, then it has the BPBp for bilinear forms. In the following proposition we prove the converse when the range space is a Hilbert space.

Proposition 4.2.6. Let X be a Banach space and let H be a Hilbert space. The pair (X, H) has the BPBp for bilinear forms if and only if the pair (X, H) has the generalized AHSP.

Proof. Assume that the pair (X, H) has the BPBp for bilinear forms with function $\eta(\cdot)$. Note that since H is a Hilbert space, there exists a function $\xi(\cdot) > 0$ satisfying that $\lim_{t \to 0} \xi(t) = 0$ and that for every $\varepsilon > 0$ and points $h_1, h_2 \in S_H$ with $\|h_1 - h_2\| < \varepsilon$, there exits a linear isometry $R : H \longrightarrow H$ such that

$$R(h_1) = h_2 \quad \text{and} \quad \|R - Id_H\| < \xi(\varepsilon).$$

Fix $\varepsilon > 0$ and choose $\varepsilon' > 0$ so that

$$\sqrt{2(\eta(\varepsilon') + 3\varepsilon)} + \varepsilon' + \xi(\varepsilon') < \varepsilon.$$

Consider sequences $(T_k)_k \subset S_{\mathcal{L}(X;H^*)}$ and $(x_k)_k \subset S_X$, an element $h_0 \in S_H$ and a convex series $\sum_{n=1}^{\infty} \alpha_n$ such that

$$\operatorname{Re} \sum_{n=1}^{\infty} \alpha_k T_k(x_k)(h_0) > 1 - (\eta(\varepsilon'))^2 .$$

By Lemma 4.2.4 we get that

$$\sum_{k\in A} \alpha_k > 1-\eta(\varepsilon') > 1-\varepsilon'$$

where $A := \{k \in \mathbb{N} : \operatorname{Re} T_k(x_k)(y_0) > 1-\eta(\varepsilon')\}$. For each $k \in A$, we define a bilinear form B_k on $X \times H$ by $B_k(x,h) := T_k(x)(h)$ for all $(x,h) \in X \times H$. Then $\|B_k\| = \|T_k\| = 1$ for all $k \in A$ and

$$\operatorname{Re} B_k(x_k, h_0) = \operatorname{Re} T_k(x_k)(h_0) > 1-\eta(\varepsilon'),$$

for every $k \in A$. From the assumption, there are a bilinear form C_k and $(z_k, u_k) \in S_X \times S_H$ such that $|C_k(z_k,u_k)| = 1 = \|C_k\|$, $\|z_k - x_k\| < \varepsilon'$, $\|u_k - h_0\| < \varepsilon$ and $\|C_k - B_k\| < \varepsilon'$ for all $k \in A$.

Now choose a scalar c_k with $|c_k| = 1$, such that $c_k C_k(z_k,u_k) = 1$. Then,

$$\begin{aligned}|1-c_k| &\leqslant \sqrt{2(1-\operatorname{Re} c_k)} = \sqrt{2(1-\operatorname{Re} C_k(z_k,u_k))} \\ &\leqslant \sqrt{2(|1-\operatorname{Re} B_k(x_k,h_0)| + |\operatorname{Re} B_k(x_k,h_0) - \operatorname{Re} B_k(z_k,u_k)| + |\operatorname{Re} B_k(z_k,u_k) - \operatorname{Re} C_k(z_k,u_k)|)} < \sqrt{2(\eta(\varepsilon')+3\varepsilon')}\end{aligned}$$

For each $k \in A$, there exits a linear isometry $R_k : H \longrightarrow H$ such that

$$R_k(h_0) = u_k \quad \text{and} \quad \|R_k - Id_H\| < \xi(\varepsilon').$$

Define $S_k \in \mathcal{L}(X; H^*)$ by $S_k(x)(h) := c_k C_k(x, R_k(h))$ for every $x \in X$ and $h \in H$. Then

$$\|S_k - T_k\| < \sqrt{2(\eta(\varepsilon')+3\varepsilon')} + \varepsilon' + \xi(\varepsilon') < \varepsilon \text{ and } \|S_k\| = S_k(x_k)(h_0) = 1$$

for every $k \in A$. □

We now show that the generalized AHSP for bilinear forms characterizes the pair $(\ell_1(X), Y)$ to have the BPBp for bilinear forms. This is the analogous version of [50, Theorem 6].

Theorem 4.2.7. Let X and Y be Banach spaces. The pair (X, Y) has the generalized AHSP for bilinear forms if and only if the pair $(\ell_1(X), Y)$ has the BPBp for bilinear forms.

Proof. The proof will be given for complex Banach spaces since the real case is not only similar but also simpler.

Let $\varepsilon \in (0, 1)$ be given. Suppose that the pair (X, Y) has the generalized AHSP for bilinear forms with $0 < \eta(\varepsilon) < \varepsilon$. Let B be a bilinear form defined on $\ell_1(X) \times Y$ with $\|B\| = 1$ and $(x_0, y_0) \in S_{\ell_1(X)} \times S_Y$ satisfying

$$|B(x_0, y_0)| > 1 - \eta\left(\frac{\varepsilon}{3}\right).$$

Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ be such that $\alpha B(x_0, y_0) = B(x_0, \alpha y_0) = |B(x_0, y_0)|$. Define $T : \ell_1(X) \longrightarrow Y^*$ by

$$T(x)(y) := B(x, y) \quad (x \in \ell_1(X) \text{ and } y \in Y).$$

Then $\|T\| = \|B\| = 1$. We denote by T_k the restriction of T to the k-th coordinate X of $\ell_1(X)$. Then, we see that $T(x) = \sum_{k \in \mathbb{N}} T_k(x_k)$ for every $x = (x_k)_k \in \ell_1(X)$. Since $x_0 \in S_{\ell_1(X)}$, we may write $x_0 = (\alpha_k x_k^0)_k$ with $\sum_{k=1}^{\infty} \alpha_k = 1$, $\alpha_k \geqslant 0$ and $x_k^0 \in S_X$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} \alpha_k T_k(x_k^0)(\alpha y_0) &= \alpha \sum_{k=1}^{\infty} T_k(\alpha_k x_k^0)(y_0) \\ &= \alpha T(x_0)(y_0) \\ &= \alpha B(x_0, y_0) = |B(x_0, y_0)| > 1 - \eta\left(\frac{\varepsilon}{3}\right). \end{aligned}$$

Then there are $u_0 \in S_Y$, a set $A \subset \mathbb{N}$ and sequences $(S_k)_k \subset S_{\mathcal{L}(X, Y^*)}$ and $(z_k)_k \subset S_X$ such that

(1) $\sum_{k\in A}\alpha_k > 1-\frac{\varepsilon}{3}$,

(2) $\|z_k - x_k^0\| < \frac{\varepsilon}{3}$ and $\|S_k - T_k\| < \frac{\varepsilon}{3}$ for all $k\in A$,

(3) $\|u_0 - \alpha y_0\| < \frac{\varepsilon}{3} < \varepsilon$,

(4) $S_k(z_k)(u_0) = 1$ for all $k\in A$.

Define $S:\ell_1(X)\longrightarrow Y^*$ by

$$S(x) := \sum_{k\in A} S_k(x_k) + \sum_{k\in\mathbb{N}\setminus A} T_k(x_k) \quad (x=(x_k)_k\in\ell_1(X))\,.$$

Then $\|S\|=1$. Define now a bilinear form C on $\ell_1(X)\times Y$ by $C(x,y) := S(x)(y)$ for all $(x,y)\in\ell_1(X)\times Y$. So, we have that $\|C\| = \|S\| = 1$ and $\|C-B\|<\varepsilon$. Let

$$\beta_k = \frac{\alpha_k}{\sum_{k\in A}\alpha_k} \text{ for } k\in A \ \text{ and } \ \beta_k = 0 \text{ otherwise.}$$

We define $z_0 := (\beta_k z_k)_k \in \ell_1(X)$ where $z_k = x_k^0$ for all $k\in\mathbb{N}\setminus A$. Then $\|z_0\|_1 = \sum_{k\in A}\beta_k = 1$ and

$$\|z_0 - x_0\|_1 = \sum_{k\in A}\left\|\beta_k z_k - \alpha_k x_k^0\right\| + \sum_{k\in\mathbb{N}\setminus A}\|\alpha_k x_k^0\|$$

$$= \sum_{k\in A}\left\|\frac{\alpha_k}{\sum_{j\in A}\alpha_j} z_k - \alpha_k z_k\right\| + \sum_{k\in A}\|\alpha_k z_k - \alpha_k x_k^0\| + \sum_{k\in\mathbb{N}\setminus A}\alpha_k$$

$$\overset{(1),(2)}{<} 1 - \sum_{k\in A}\alpha_k + \frac{\varepsilon}{3}\sum_{k\in A}\alpha_k + \frac{\varepsilon}{3} \overset{(1)}{<} \varepsilon.$$

By using (3), we get that $\|\alpha^{-1}u_0 - y_0\|_1 = \|u_0 - \alpha y_0\|_1 < \varepsilon$. Finally,

$$\begin{aligned} 1 \geqslant |C(z_0, \alpha^{-1}u_0)| &= |S(z_0)(u_0)| = \left|\sum_{k\in A} S_k(\beta_k z_k)(u_0)\right| \\ &= \left|\sum_{k\in A} \beta_k S_k(z_k)(u_0)\right| \\ &= \sum_{k\in A} \beta_k = 1. \end{aligned}$$

So $\|C\| = |C(z_0, \alpha^{-1}u_0)| = 1$ and this proves that the pair $(\ell_1(X), Y)$ has the BPBp for bilinear forms.

Now we assume that the pair $(\ell_1(X), Y)$ has the BPBp for bilinear forms with some function $\eta(\varepsilon) > 0$ and we assume that $\eta(\varepsilon) < \varepsilon$ for a given $\varepsilon > 0$. Let $\xi(\varepsilon) > 0$ be such that

$$\xi(\varepsilon) + \frac{2\xi(\varepsilon)}{\varepsilon} < \varepsilon \quad \text{and} \quad \xi(\varepsilon) + \sqrt{2(\eta(\xi(\varepsilon)) + 3\xi(\varepsilon))} < \varepsilon. \tag{4.2}$$

Consider sequences $(T_k)_k \in \mathcal{L}(X; Y^*)$ with $\|T_k\| = 1$ for all $k \in \mathbb{N}$ and $(x_k^0)_k \subset S_X$, a convex series $\sum_{k=1}^{\infty} \alpha_k$ and an element $y_0 \in S_Y$ to be such that

$$\operatorname{Re} \sum_{k=1}^{\infty} \alpha_k T_k(x_k^0)(y_0) > 1 - \eta(\xi(\varepsilon)).$$

Define B a bilinear form on $\ell_1(X) \times Y$ by

$$B(x, y) := \sum_{k=1}^{\infty} T_k(x_k)(y) \quad (((x_k)_k, y) = (x, y) \in \ell_1(X) \times Y).$$

Then $\|B\| = \|T_k\| = 1$. Putting $x_0 := (\alpha_k x_k^0) \in S_{\ell_1(X)}$, we have

$$\operatorname{Re} B(x_0, y_0) = \operatorname{Re} \sum_{k=1}^{\infty} T_k\left(\alpha_k x_k^0\right)(y_0) > 1 - \eta(\xi(\varepsilon)).$$

Since $(\ell_1(X), Y)$ has the BPBp for bilinear forms, there are a bilinear form C on $\ell_1(X) \times Y$ and $(z_0, u_0) \in S_{\ell_1(X)} \times S_Y$ such that $|C(z_0, u_0)| = 1 =$

$\|C\|$, $\|z_0 - x_0\| < \xi(\varepsilon) < \varepsilon$, $\|u_0 - y_0\| < \xi(\varepsilon) < \varepsilon$ and $\|C - B\| < \xi(\varepsilon) < \varepsilon$. Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ be such that $\alpha C(z_0, u_0) = C(z_0, \alpha u_0) = 1$. Note that we have

$$\begin{aligned} \operatorname{Re} C(z_0, u_0) &\geqslant \operatorname{Re} B(x_0, y_0) - \|C - B\| - \|z_0 - x_0\| - \|u_0 - y_0\| \\ &> 1 - \eta(\xi(\varepsilon)) - 3\xi(\varepsilon). \end{aligned}$$

Since $|\alpha| = 1$, we see that

$$|1-\alpha| = \sqrt{(1 - \operatorname{Re}\alpha)^2 + \operatorname{Im}\alpha^2} = \sqrt{2(1 - \operatorname{Re}\alpha)} = \sqrt{2(1 - \operatorname{Re} C(z_0, u_0))}.$$

From the last inequality, we conclude that

$$\|y_0 - \alpha u_0\| \leqslant \|y_0 - u_0\| + |1 - \alpha| < \xi(\varepsilon) + \sqrt{2(\eta(\xi(\varepsilon)) + 3\xi(\varepsilon))} \overset{(4.2)}{<} \varepsilon.$$

To find the desired set A, we write $z_0 = (z_k^0)_k \in S_{\ell_1(X)}$ and consider the set $\widetilde{A} := \{k \in \mathbb{N} : \|z_k^0\| = 0\}$. First we note that

$$\xi(\varepsilon) > \|x_0 - z_0\|_1 = \sum_{k\in\mathbb{N}} \|\alpha_k x_k^0 - z_k^0\| \geqslant \sum_{k\in\widetilde{A}} \alpha_k \tag{4.3}$$

and also that

$$\begin{aligned} \|x_0 - z_0\|_1 &\geqslant \sum_{k\in\mathbb{N}\setminus\widetilde{A}} \left\|\alpha_k x_k^0 - z_k^0\right\| = \sum_{k\in\mathbb{N}\setminus\widetilde{A}} \alpha_k \left\| x_k^0 - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|}\right\| \\ &\geqslant \sum_{k\in\mathbb{N}\setminus\widetilde{A}} \left(\alpha_k \left\| x_k^0 - \frac{z_k^0}{\|z_k^0\|}\right\| - \left\| \frac{z_k^0}{\|z_k^0\|} - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|}\right\|\right). \end{aligned}$$

Observe that for $k \in \mathbb{N} \setminus \widetilde{A}$

$$\begin{aligned} \left\| \frac{z_k^0}{\|z_k^0\|} - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|}\right\| = \left| 1 - \frac{\|z_k^0\|}{\alpha_k}\right| &= \left| \|x_k^0\| - \frac{\|z_k^0\|}{\alpha_k} \cdot \left\|\frac{z_k^0}{\|z_k^0\|}\right\|\right| \\ &\leqslant \left\| x_k^0 - \frac{\|z_k^0\|}{\alpha_k} \cdot \frac{z_k^0}{\|z_k^0\|}\right\|. \end{aligned}$$

So

$$\begin{aligned}\sum_{k\in\mathbb{N}\setminus\widetilde{A}}\alpha_k\left\|x_k^0-\frac{z_k^0}{\|z_k^0\|}\right\| &\leqslant \|x_0-z_0\|_1+\sum_{k\in\mathbb{N}\setminus\widetilde{A}}\alpha_k\left\|x_k^0-\frac{\|z_k^0\|}{\alpha_k}\cdot\frac{z_k^0}{\|z_k^0\|}\right\|\\ &=\|x_0-z_0\|_1+\sum_{k\in\mathbb{N}\setminus\widetilde{A}}\left\|\alpha_k x_k^0-z_k^0\right\|\\ &\leqslant 2\left\|x_0-z_0\right\|_1<2\xi(\varepsilon).\end{aligned}$$

Define $A:=\left\{k\in\mathbb{N}:\left\|x_k^0-\frac{z_k^0}{\|z_k^0\|}\right\|<\varepsilon,\ \|z_k^0\|\neq 0\right\}$. We see that

$$\begin{aligned}2\xi(\varepsilon)>\sum_{k\in\mathbb{N}\setminus\widetilde{A}}\alpha_k\left\|x_k^0-\frac{z_k^0}{\|z_k^0\|}\right\| &\geqslant \sum_{k\in(\mathbb{N}\setminus\widetilde{A})\setminus A}\alpha_k\left\|x_k^0-\frac{z_k^0}{\|z_k^0\|}\right\|\\ &\geqslant \varepsilon\sum_{k\in(\mathbb{N}\setminus\widetilde{A})\setminus A}\alpha_k.\end{aligned}\tag{4.4}$$

Since

$$1=\sum_{k\in\mathbb{N}}\alpha_k=\sum_{k\in A}\alpha_k+\sum_{k\in(\mathbb{N}\setminus\widetilde{A})\setminus A}\alpha_k+\sum_{k\in\widetilde{A}}\alpha_k$$

we get that

$$\begin{aligned}\sum_{k\in A}\alpha_k=1-\sum_{k\in\widetilde{A}}\alpha_k-\sum_{k\in(\mathbb{N}\setminus\widetilde{A})\setminus A}\alpha_k &\overset{(4.3)}{>} 1-\xi(\varepsilon)-\sum_{k\in(\mathbb{N}\setminus\widetilde{A})\setminus A}\alpha_k\\ &\overset{(4.4)}{>} 1-\xi(\varepsilon)-\frac{2\xi(\varepsilon)}{\varepsilon}\\ &\overset{(4.2)}{>} 1-\varepsilon.\end{aligned}$$

Now we define $S:\ell_1(X)\longrightarrow Y^*$ by $S(x)(y):=C(x,y)$ for all $x\in\ell_1(X)$ and $y\in Y$ and let S_k be the restriction of S to the k-th coordinate X of $\ell_1(X)$. Then, we have that $\|S_k-T_k\|<\|C-B\|<\varepsilon$ and

$$\sum_{k=1}^{\infty}S_k(z_k^0)(\alpha u_0)=C(z_0,\alpha u_0)=\alpha C(z_0,u_0)=|C(z_0,u_0)|=1.$$

Since $1 = \|z_0\|_1 = \sum_{k=1}^{\infty} \|z_k^0\|$, we get that $S_k(z_k^0)(\alpha u_0) = \|z_k^0\|$ for all $k \in \mathbb{N}$. Therefore, for every $k \in A$, we see that $S_k\left(\frac{z_k^0}{\|z_k^0\|}\right)(\alpha u_0) = 1$ which implies $\|S_k\| = 1$. The proof ends if we choose the set A, the sequences $(S_k)_k \subset \mathcal{L}(X; Y^*)$ which satisfies $\|S_k\| = 1$ for all $k \in \mathbb{N}$ and $(z_k)_k := \left(\frac{z_k^0}{\|z_k^0\|}\right)_k \subset S_X$ and the element $\alpha u_0 \in S_Y$. □

4.3 The numerical radius on $\mathcal{L}(^N L_1(\mu); L_1(\mu))$

Consider the set $\Pi_N(X)$ of all elements $(x_1, \ldots, x_N, x^*)$ in $S_X^N \times S_{X^*}$ such that $x^*(x_1) = \ldots = x^*(x_N) = 1$. When $N = 1$, we denote $\Pi_1(X)$ just by $\Pi(X)$. The *numerical radius of an N-linear mapping* $A \in \mathcal{L}(^N X; X)$ is the number

$$v(A) := \sup \{|x^*(A(x_1, \ldots, x_N))| : (x_1, \ldots, x_N, x^*) \in \Pi_N(X)\}.$$

As in the operator case, we have that v is a semi-norm on $\mathcal{L}(^N X; X)$ such that $v(A) \leqslant \|A\|$ for all $A \in \mathcal{L}(^N X; X)$. On the other hand, the equality is not true in general. Nevertheless, in [25, Theorem 3.1(i) and Theorem 3.2] it was proved that $v(A) = \|A\|$ for every $A \in \mathcal{L}(^N c_0; c_0)$ or $A \in \mathcal{L}(^N \ell_1; \ell_1)$. Also in [24, Theorem 3.2] it was proved that $v(L) = \|L\|$ for every $L \in \mathcal{L}(^N A_D; A_D)$ where A_D is the disc algebra. In this section we prove the same result but now on $L_1(\mu)$-spaces. In what follows, $\langle x, x^* \rangle$ means the action $x^*(x)$ for $x \in X$ and $x^* \in X^*$.

Theorem 4.3.1. Let μ be an arbitrary measure. For each positive integer N and each $A \in \mathcal{L}(^N L_1(\mu); L_1(\mu))$, we have $v(A) = \|A\|$.

Proof. Since $v(A) \leqslant \|A\|$, we need to prove the another inequality. Without loss of generality, suppose that $\|A\| = 1$ and we prove that $v(A) > 1 - \varepsilon$ for any fixed $\varepsilon > 0$. Choose $f_1, \ldots, f_N \in S_{L_1(\mu)}$ such that

$$\|A(f_1, \ldots, f_N)\|_1 > 1 - \frac{\varepsilon}{2}. \tag{4.5}$$

Consider

$$\Omega' = \left(\bigcup_{i=1}^{N}\{t \in \Omega \ : \ |f_i(t)| > 0\}\right) \cup \{t \in \Omega \ : \ |A(f_1, \ldots, f_N)(t)| > 0\}.$$

Since given $f \in L_1(\mu)$ and $r > 0$ the set $\{t \in \Omega \ : \ |f(t)| > r\}$ is a measurable set of finite measure, there exists a sequence $(\Omega_k)_k$ of pairwise disjoint measurable sets having finite measure such that $\Omega' = \bigcup_{k=1}^{\infty} \Omega_k$. For a finite partition $\pi \subset \Sigma$ of Ω_k let us denote by $E_\pi : L_1(\mu) \longrightarrow L_1(\mu)$ the contractive projection given by

$$E_\pi(f) := \sum_{F\in\pi} \left(\frac{1}{\mu(F)} \int_F f d\mu\right) \chi_F \quad (f \in L_1(\mu)).$$

Fixed k, since Ω_k has finite measure, we can apply [29, Lemma III.2.1, pg. 67] to the finite measurable subset Ω_k, $f_i\chi_{\Omega_k}$ and $A(f_1, \ldots, f_N)\chi_{\Omega_k}$ to find a finite partition π_k of Ω_k such that

$$\|E_{\pi_k} f_j - f_j\chi_{\Omega_k}\|_1 < \frac{\varepsilon}{2^{k+1}(N+1)}$$

and that

$$\|E_{\pi_k} A(f_1, \ldots, f_N) - A(f_1, \ldots, f_N)\chi_{\Omega_k}\|_1 < \frac{\varepsilon}{2^{k+1}(N+1)},$$

for every k. Now we take

$$\pi = \bigcup_{k=1}^{\infty} \pi_k = \{F_i \ : \ i \in \mathbb{N}\} \subset \Sigma.$$

We have that π is a countable family of measurable sets with finite measure that is a partition of Ω'. We define $E_\pi : L_1(\mu) \longrightarrow L_1(\mu)$ by

$$E_\pi(f) = \sum_{k=1}^{\infty} E_{\pi_k}(f) \quad (f \in L_1(\mu))$$

and then we have that E_π is a contractive projection such that

$$\|E_\pi f_j - f_j\|_1 \leqslant \sum_{k=1}^{\infty} \|E_{\pi_k} f_j - f_j \chi_{\Omega_k}\|_1 < \frac{\varepsilon}{2(N+1)}, \tag{4.6}$$

for $j = 1, \ldots, N$. Analogously,

$$\|E_\pi A(f_1, \ldots, f_N) - A(f_1, \ldots, f_N)\|_1 < \frac{\varepsilon}{2(N+1)}. \tag{4.7}$$

We claim that we have the following inequality

$$\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N)\|_1 > 1 - \varepsilon. \tag{4.8}$$

Indeed, note first that since A is N-linear mapping, we have

$$\begin{aligned} A(E_\pi f_1, \ldots E_\pi f_N) - A(f_1, \ldots, f_N) = A(E_\pi f_1 - f_1, \ldots, E_\pi f_N) + \\ A(f_1, E_\pi f_2 - f_2, \ldots, E_\pi f_N) + \ldots \\ + A(f_1, \ldots, f_{n-1}, E_\pi f_n - f_n), \end{aligned}$$

which shows that

$$\begin{aligned} \|A(E_\pi f_1, \ldots E_\pi f_N) - A(f_1, \ldots, f_N)\|_1 &\leqslant \sum_{j=1}^{N} \|E_\pi f_j - f_j\|_1 \\ &< N \cdot \frac{\varepsilon}{2(N+1)}. \end{aligned}$$

Since $\|E_\pi\| \leqslant 1$,

$$\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N) - E_\pi A(f_1, \ldots, f_N)\|_1 < N \cdot \frac{\varepsilon}{2(N+1)}.$$

On the other hand, by using (4.7) and (4.5), we get that

$$\begin{aligned} \|E_\pi A(f_1, \ldots, f_N)\|_1 &> \|A(f_1, \ldots, f_N)\|_1 - \frac{\varepsilon}{2(N+1)} \\ &> 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2(N+1)}. \end{aligned}$$

So

$$\begin{aligned}\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N)\|_1 \geqslant &\|E_\pi A(f_1, \ldots, f_N)\|_1 - \\ &\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N) - E_\pi A(f_1, \ldots, f_N)\|_1 \\ >& 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2(N+1)} - N \cdot \frac{\varepsilon}{2(N+1)} = 1 - \varepsilon.\end{aligned}$$

Recall that $\pi = \{F_i \ : \ i \in \mathbb{N}\} \subset \Sigma$. For each $j = 1, \ldots, N$, we put

$$E_\pi f_j = \sum_i a_i^j \cdot \frac{1}{\mu(F_i)} \chi_{F_i} \quad \text{with} \quad a_i^j = \int_{F_i} f_j d\mu.$$

By using (4.6),

$$\begin{aligned}1 - \frac{\varepsilon}{2(N+1)} < \|E_\pi f_j\|_1 = \int_\Omega |E_\pi f_j| d\mu &= \int_\Omega \left| \sum_i a_i^j \frac{1}{\mu(F_i)} \chi_{F_i} \right| d\mu \\ &= \int_\Omega \sum_i |a_i^j| \frac{1}{\mu(F_i)} \chi_{F_i} d\mu \\ &= \sum_i |a_i^j| \leqslant 1.\end{aligned}$$

Hence, from (4.8) we have that

$$\begin{aligned}1 - \varepsilon <&\|E_\pi A(E_\pi f_1, \ldots, E_\pi f_N)\|_1 \\ =& \left\| E_\pi A \left(\sum_i a_i^1 \frac{1}{\mu(F_i)} \chi_{F_i}, \ldots, \sum_i a_i^N \frac{1}{\mu(F_i)} \chi_{F_i} \right) \right\|_1 \\ =& \left\| \sum_{l_1, \ldots, l_N \in \mathbb{N}} a_{l_1}^1 \cdots a_{l_N}^N E_\pi A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1 \\ \leqslant& \sum_{l_1, \ldots, l_N \in \mathbb{N}} |a_{l_1}^1| \cdots |a_{l_N}^N| \left\| E_\pi A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1 .\end{aligned}$$

and so we conclude that there exist $l_1, \ldots, l_N \in \mathbb{N}$ such that

$$\left\| E_\pi A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1 > 1 - \varepsilon. \tag{4.9}$$

Now we write

$$E_\pi A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) = \sum_i a_i \frac{1}{\mu(F_i)} \chi_{F_i},$$

where

$$a_i = \int_{F_i} A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) d\mu.$$

Define on $L_\infty(\mu)$ the element

$$g := \sum_i c_i \chi_{F_i}, \text{ where } |c_i| = 1, \; c_i a_i = |a_i| \text{ for all } i \in \mathbb{N}.$$

Then $\|g\|_\infty = 1$, we note that for all $j = 1, \ldots, N$, we have that

$$\left\langle \frac{\overline{c_{l_j}}}{\mu(F_{l_j})} \chi_{F_{l_j}}, g \right\rangle = \int_\Omega \frac{\overline{c_{l_j}}}{\mu(F_{l_j})} \chi_{F_{l_j}} \cdot g \, d\mu = \frac{1}{\mu(F_{l_j})} \int_\Omega [\chi_{F_{l_j}}]^2 d\mu = 1$$

and also since

$$\begin{aligned} &\left| \left\langle E_\pi A \left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{\overline{c_{l_N}}}{\mu(F_{l_N})} \chi_{F_{l_N}} \right), g \right\rangle \right| = \\ &\quad \left| \left\langle E_\pi A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right), g \right\rangle \right| = \\ &\quad \left| \int_\Omega \left(\sum_i c_i \chi_{F_i} \right) \cdot \left(\sum_i a_i \frac{1}{\mu(F_i)} \chi_{F_i} \right) d\mu \right| = \\ &\quad \sum_i |a_i| = \left\| E_\pi A \left(\frac{1}{\mu(F_{l_1})} \chi_{F_{l_1}}, \ldots, \frac{1}{\mu(F_{l_N})} \chi_{F_{l_N}} \right) \right\|_1, \end{aligned}$$

we have that

$$\left|\left\langle E_\pi A\left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})}\chi_{F_{l_1}},\ldots,\frac{\overline{c_{l_N}}}{\mu(F_{l_N})}\chi_{F_{l_N}}\right),g\right\rangle\right| \overset{(4.9)}{>} 1-\varepsilon.$$

Finally, we consider the adjoint operator $E_\pi^*: L_\infty(\mu) \longrightarrow L_\infty(\mu)$. Then $\|E_\pi^*(g)\|_\infty \leqslant 1$. Also for all $j = 1,\ldots,N$,

$$\left\langle \frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}}, E_\pi^*(g)\right\rangle = \left\langle E_\pi\left(\frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}}\right),g\right\rangle = \left\langle \frac{\overline{c_{l_j}}}{\mu(F_{l_j})}\chi_{F_{l_j}},g\right\rangle = 1$$

and

$$\left|\left\langle A\left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})}\chi_{F_{l_1}},\ldots,\frac{\overline{c_{l_N}}}{\mu(F_{l_N})}\chi_{F_{l_N}}\right),E_\pi^*(g)\right\rangle\right| = \left|\left\langle E_\pi A\left(\frac{\overline{c_{l_1}}}{\mu(F_{l_1})}\chi_{F_{l_1}},\ldots,\frac{\overline{c_{l_N}}}{\mu(F_{l_N})}\chi_{F_{l_N}}\right),g\right\rangle\right| > 1-\varepsilon.$$

Then $v(A) > 1-\varepsilon$. This completes the proof. □

4.4 The BPBp-nu for multilinear mappings

In this section we study the Bishop-Phelps-Bollobás property for numerical radius in the multilinear case (for the operator case, see its definition at the beginning of Section 2.3). We define this property as follows.

Definition 4.4.1. We say that a Banach space X has the *Bishop-Phelps-Bollobás property for numerical radius for multilinear mappings* (BPBp-nu for multilinear mappings, for short) if for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{L}(^N X; X)$ with $v(A) = 1$ and $(x_1^0,\ldots,x_N^0,x_0^*) \in \Pi_N(X)$ satisfy

$$|x_0^*(A(x_1^0,\ldots,x_N^0))| > 1-\eta(\varepsilon),$$

there are $B \in \mathcal{L}(^N X; X)$ with $v(B) = 1$ and $(z_1^0, \ldots, z_N^0, z_0^*) \in \Pi_N(X)$ such that

(1) $|z_0^*(B(z_1^0, \ldots, z_N^0))| = 1$,

(2) $\max_{1 \leqslant j \leqslant N} \|z_j^0 - x_j^0\| < \varepsilon$,

(3) $\|z_0^* - x_0^*\| < \varepsilon$,

(4) $\|B - A\| < \varepsilon$.

If $N = 1$, then we go back to the operator case. In [47, Proposition 2] it was proved that X has the BPBp-nu when $\dim(X) < \infty$. We have the analogous version for multilinear mappings. Its proof is an easy adaptation of [47, Proposition 2] for this class of functions.

Proposition 4.4.2. Let X be a finite-dimensional Banach space. Then X has the BPBp-nu for multilinear mappings.

We prove that the infinite dimensional Banach space $L_1(\mu)$ fails the BPBp-nu for bilinear mappings although $L_1(\nu)$ has it in the operator case for every measure ν (see [32, Theorem 9] or [47, Theorem 4.1]). It is worth to mentioning that if μ_1 and μ_2 are measures such that $L_1(\mu_1)$ and $L_1(\mu_2)$ are infinite dimensional Banach spaces, then the pair $(L_1(\mu_1), L_1(\mu_2))$ *fails* the Bishop-Phelps-Bollobás property for bilinear forms [4, Corollary 2.7.(4)].

Theorem 4.4.3. The infinite dimensional Banach space $L_1(\mu)$ does not satisfy the BPBp-nu for bilinear mappings.

Proof. Since $L_1(\mu)$ is infinite dimensional, we may consider measurable subsets $(E_n), (F_n) \subset \Sigma$ such that

(i) $0 < \mu(E_n) < \infty$ and $0 < \mu(F_n) < \infty$ for all $n \in \mathbb{N}$ with

(ii) $E_i \cap E_j = \emptyset = F_i \cap F_j$ when $i \neq j$.

We put $E_0 := \Omega \setminus (\bigcup_{n=1}^{\infty} E_n) \in \Sigma$ and $F_0 := \Omega \setminus (\bigcup_{n=1}^{\infty} F_n) \in \Sigma$. Define the bilinear mapping $A : L_1(\mu) \times L_1(\mu) \longrightarrow L_1(\mu)$ by

$$A(f,g) := \sum_{i=1}^{\infty} \left(\int_{E_i} f d\mu \right) \sum_{\substack{j=1, \\ j\neq i}}^{\infty} \left(\int_{F_j} g d\mu \right) \frac{\chi_{E_1}}{\mu(E_1)} \quad (f,g \in L_1(\mu)).$$

For

$$f_0^n := \sum_{r=1}^{2n^2} \frac{1}{2n^2} \cdot \frac{\chi_{E_r}}{\mu(E_r)} \quad \text{and} \quad g_0^n := \sum_{s=1}^{2n^2} \frac{1}{2n^2} \cdot \frac{\chi_{F_s}}{\mu(F_s)}.$$

Then $f_0^n, g_0^n \in S_{L_1(\mu)}$ and

$$A(f_0^n, g_0^n) = \left(1 - \frac{1}{2n^2}\right) \frac{\chi_{E_1}}{\mu(E_1)}.$$

This shows that $\|A\| = 1$ and so we have $v(A) = \|A\| = 1$ by Theorem 4.3.1. For a contradiction, suppose that $L_1(\mu)$ has the BPBp-nu for bilinear mappings with some $\eta(\varepsilon) > 0$. Choose $n_0 \in \mathbb{N}$ to be such that $\frac{1}{2n_0^2} < \eta(1/2)$ and we consider $\chi_\Omega \in S_{L_\infty(\mu)}$. Then, $\langle f_0^{n_0}, \chi_\Omega \rangle = \langle g_0^{n_0}, \chi_\Omega \rangle = 1$ and

$$\langle A(f_0^{n_0}, g_0^{n_0}), \chi_\Omega \rangle = 1 - \frac{1}{2n_0^2} > 1 - \eta\left(\frac{1}{2}\right).$$

Then there are $B \in \mathcal{L}(^2L_1(\mu); L_1(\mu))$ with $v(B) = 1$, $\overline{f}, \overline{g} \in L_1(\mu)$ and $h \in L_\infty(\mu)$ such that

(a) $|\langle B(\overline{f}, \overline{g}), h \rangle| = 1 = \langle \overline{f}, h \rangle = \langle \overline{g}, h \rangle = \|\overline{f}\|_1 = \|\overline{g}\|_1 = \|h\|_\infty$,

(b) $\|\overline{f} - f_0^{n_0}\|_1 < \frac{1}{2}$,

(c) $\|\overline{g} - g_0^{n_0}\|_1 < \frac{1}{2}$,

(d) $\|h - \chi_\Omega\|_\infty < \frac{1}{2}$,

(e) $\|B - A\| < \frac{1}{2}$.

We prove that (b) or (c) is false. For all $n = 0, 1, 2, \ldots$, set

$$a_n := \int_{E_n} |\overline{f}|\chi_{E_n} d\mu \quad \text{and} \quad b_n := \int_{F_n} |\overline{g}|\chi_{F_n} d\mu.$$

We have that

$$1 = |\langle B(\overline{f}, \overline{g}), h\rangle| \leqslant \|h\|_\infty \|B(\overline{f}, \overline{g})\|_1 \leqslant \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m = \|\overline{f}\|_1 \|\overline{g}\|_1 = 1.$$

This shows that

$$\|B(\overline{f} \cdot \chi_{E_n}, \overline{g} \cdot \chi_{F_m})\| = a_n \cdot b_m$$

for all $n, m = 0, 1, 2, \ldots$. Considering $N := \{n \in \mathbb{N} : a_n > 0\}$ and $M := \{m \in \mathbb{N} : b_m > 0\}$, we have that

$$\left\| B\left(\frac{1}{a_n}\overline{f} \cdot \chi_{E_n}, \frac{1}{b_m}\overline{g} \cdot \chi_{F_m}\right)\right\|_1 = 1 \text{ for all } (n, m) \in N \times M. \tag{4.10}$$

We observe now that $0 \notin N$. Indeed, if $0 \in N$, then by using (4.10), we have that

$$\left\| B\left(\frac{1}{a_0}\overline{f} \cdot \chi_{E_0}, \frac{1}{b_m}\overline{g} \cdot \chi_{F_m}\right)\right\|_1 = 1$$

for all $m \in M$. On the other hand,

$$\begin{aligned} &A\left(\frac{1}{a_0}\overline{f}\chi_{E_0}, \frac{1}{b_m}\overline{g} \cdot \chi_{F_m}\right) \\ &\qquad = \sum_{i=1}^{\infty} \left(\int_{E_i} \frac{1}{a_0}\overline{f} \cdot \chi_{E_0} d\mu\right) \sum_{\substack{j=1,\\ j\neq i}}^{\infty} \left(\int_{F_j} \frac{1}{b_m}\overline{g} \cdot \chi_{F_m}\right) \frac{\chi_{E_1}}{\mu(E_1)} = 0 \end{aligned}$$

because of (iii). This implies that

$$\frac{1}{2} > \|A - B\| \geqslant \left\| B\left(\frac{1}{a_0}\overline{f} \cdot \chi_{E_0}, \frac{1}{b_m}\overline{g} \cdot \chi_{F_m}\right)\right\|_1 = 1$$

which is a contradiction. By using exactly the same arguments, we may prove that $0 \notin M$. Now we note that $N \cap M = \emptyset$. Indeed, if there exists some $n \neq 0$ such that $n \in N \cap M$, then by using again (4.10), we have the same contradiction as before: $\frac{1}{2} > \|A - B\| \geqslant 1$.

Now we suppose that (b) is true and we get that (c) does not hold. We shall show that the numbers of elements in the set $\{1, \ldots, 2n^2\} \cap N$ is $> n^2$. Otherwise, there exists a set

$$S := \{j \in \mathbb{N} : 1 \leqslant j \leqslant 2n^2\}$$

with $|S| = n^2 + n_0$ for some $n_0 \in \mathbb{N}$ such that $a_j = \int_{E_j} |\overline{f}|\chi_{E_j} d\mu = 0$ for all $j \in S$. This implies that if $j \in S$, $\overline{f}$ is 0 almost everywhere in E_j. Then

$$\begin{aligned}
\|\overline{f} - f_0\|_1 = \int_{E_0} |\overline{f} - f_0|\chi_{E_0} d\mu + \sum_{j=1}^{2n^2} \int_{E_j} |\overline{f} - f_0|\chi_{E_j} d\mu \\
+ \sum_{k=2n^2+1}^{\infty} \int_{E_k} |\overline{f} - f_0|\chi_{E_k} d\mu \\
= \int_{E_0} |\overline{f} - f_0|\chi_{E_0} d\mu + \sum_{j\in S} \int_{E_j} |f_0|\chi_{E_j} d\mu + \\
\sum_{j\notin S} \int_{E_j} |\overline{f} - f_0|\chi_{E_j} d\mu + \sum_{k=2n^2+1}^{\infty} \int_{E_k} |\overline{f} - f_0|\chi_{E_k} d\mu \\
\geqslant \sum_{j\in S} \int_{E_j} |f_0|\chi_{E_j} d\mu.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{j\in S} \int_{E_j} f_0 \cdot \chi_{E_j} d\mu = \frac{1}{2n^2} \sum_{j\in S} \frac{1}{\mu(E_j)} \int_{E_j} \chi_{E_j} d\mu \\
= \frac{1}{2n^2} \cdot (n^2 + n_0) = \frac{1}{2} + \frac{n_0}{2n^2} \geqslant \frac{1}{2}.
\end{aligned}$$

This implies that $\|\overline{f}-f_0\|_1 \geqslant \frac{1}{2}$ which is a contradiction. Since $N \cap M = \emptyset$, the numbers of elements in the set $\{1, \ldots, 2n^2\} \cap M$ is $< n^2$ which implies that $\|\overline{g} - g_0\|_1 \geqslant \frac{1}{2}$ by using the same arguments as before. This contradicts (c) and then $L_1(\mu)$ fails the property. □

In [47, Proposition 4] it was proved that if X is a Banach space which is uniformly convex and uniformly smooth, we get a weak version of the BPBp-nu for operators in the sense that the new operator which is close to the fixed one does not have numerical radius equals to one but it just attains its numerical radius at some point of $\Pi(X)$. The same kind of result holds for the multilinear, symmetric and polynomial cases.

Proposition 4.4.4. Let X be a uniformly convex and uniformly smooth Banach space. Given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A_0 \in \mathcal{L}^N(X, X)$ with $v(A_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy

$$|x_0^*(A_0(x_0, \ldots, x_0))| > 1 - \eta(\varepsilon),$$

there are $B_0 \in \mathcal{L}^N(X, X)$ and $(z_0, z_0^*) \in \Pi(X)$ such that

(1) $v(B_0) = |z_0^*(B_0(z_0, \ldots, z_0))|$,

(2) $\|z_0 - x_0\| < \varepsilon$,

(3) $\|z_0^* - x_0^*\| < \varepsilon$,

(4) $\|B_0 - A_0\| < \varepsilon$.

Proof. The proof is just a slight modification of [47, Proposition 4] for the multilinear case since if $(x_1, \ldots, x_N, x^*) \in \Pi_N(X)$, then $x_1 = \ldots = x_N =: x_0$ because X is uniformly convex. □

In the next proposition, we deal with direct sums again. We prove the multilinear version of [47, Lemma 19]. First we need some notation.

Given a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of Banach spaces, we denote by $[\bigoplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ and $[\bigoplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}$ the c_0 and ℓ_1-sum of $\{X_\lambda\}$. Letting $X = [\bigoplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ or $[\bigoplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}$, we consider $X^* = [\bigoplus_{\lambda\in\Lambda} X_\lambda^*]_{\ell_1}$ or $[\bigoplus_{\lambda\in\Lambda} X_\lambda^*]_{\ell_\infty}$ to be their dual spaces, respectively. We denote by $P_\lambda : X \longrightarrow X_\lambda$ the norm-one linear projection from X onto X_λ and by $Q_\lambda : X^* \longrightarrow X_\lambda^*$ the norm-one linear projection from X^* onto X_λ^*. We denote an element $x \in \bigoplus_{\lambda\in\Lambda} X_\lambda$ by $x = (x_\lambda)_{\lambda\in\Lambda}$. We also use the inclusion $\overline{P_\lambda} : X_\lambda \longrightarrow [\bigoplus_{\lambda\in\Lambda} X_\lambda]$. In X^* we use Q_λ and $\overline{Q_\lambda}$.

Proposition 4.4.5. Let $\{X_k : k \in \mathbb{N}\}$ be a family of Banach spaces. Let $X = [\bigoplus_{k=1}^{\infty} X_k]_{c_0}$ or $X = [\bigoplus_{k=1}^{\infty} X_k]_{\ell_1}$. If X has the BPBp-nu for multilinear mappings then so does X_j for all $j \in \mathbb{N}$.

Proof. Let $\varepsilon \in (0,1)$ and consider $\eta(\varepsilon) > 0$ the BPBp-nu constant for the space X. Fix $j \in \mathbb{N}$. Let $A_j \in \mathcal{L}(^N X_j; X_j)$ with $v(A_j) = 1$ and $(x_1^j, \dots, x_N^j; x_j^*) \in \Pi_N(X_j)$ be such that

$$|x_j^*(A_j(x_1^j, \dots, x_N^j))| > 1 - \eta(\varepsilon).$$

We will prove that there are $B_j \in \mathcal{L}(^N X_j; X_j)$ with $v(B_j) = 1$ and $(z_1^j, \dots, z_N^j, z_j^*) \in \Pi_N(X_j)$ such that $|z_j^*(B_j(z_1^j, \dots, z_N^j))| = 1$, $\|z_j^* - x_j^*\| < \varepsilon$, $\max_{1\leqslant l\leqslant N} \|z_l^j - x_l^j\| < \varepsilon$ and $\|B_j - A_j\| < \varepsilon$. Define $A \in \mathcal{L}(^N X; X)$ by

$$A(z_1, \dots, z_N) := (\overline{P_j} \circ A_j)\,(P_j(z_1), \dots, P_j(z_N))$$

for all $z_1, \dots, z_N \in X$. Then $v(A) = v(A_j) = 1$. Consider for each $l = 1, \dots, N$, the point $x_l := \overline{P_j}(x_l^j) \in S_X$ and $x^* := \overline{Q_j}(x_j^*) \in S_{X^*}$. Then $x^*(x_l) = 1$ for all $l = 1, \dots, N$ and $|x^*(A(x_1, \dots, x_N))| = |x_j^*(A_j(x_1^j, \dots, x_N^j))| > 1 - \eta(\varepsilon)$. Since X has the BPBp-nu for multilinear mappings with $\eta(\varepsilon) > 0$, there are $B \in \mathcal{L}(^N X; X)$ with $v(B) = 1$ and $(z_1, \dots, z_N, z^*) \in \Pi_N(X)$ such that

(i) $|z^*(B(z_1, \dots, z_N))| = 1$,

(ii) $\|z^* - x^*\| < \varepsilon$,

(iii) $\max_{1 \leqslant l \leqslant N} \|z_l - x_l\| < \varepsilon$,

(iv) $\|B - A\| < \varepsilon$.

Now we write $B = (D_1, D_2, \ldots)$ with $D_k \in \mathcal{L}(^N X; X_k)$ for each $k \in \mathbb{N}$. We define the N-linear mapping $B_j \in \mathcal{L}(^N X_j; X_j)$ by

$$B_j(y_1^j, \ldots, y_N^j) := D_j(\overline{P_j}(y_1^j), \ldots, \overline{P_j}(y_N^j))$$

for all $y_1^j, \ldots, y_N^j \in X_j$. Then $v(B_j) \leqslant v(B) = 1$ and $\|B_j - A_j\| \leqslant \|B - A\| < \varepsilon$ by using (iv). Now by using (ii) and (iii), we get that

$$\|Q_j(z^*) - x_j^*\| \leqslant \|z^* - x^*\| < \varepsilon$$

and

$$\max_{1 \leqslant l \leqslant N} \|P_j(z_l) - x_l^j\| \leqslant \max_{1 \leqslant l \leqslant N} \|z_l - x_l\| < \varepsilon.$$

To finish the proof, we have to prove that $(P_j(z_1), \ldots, P_j(z_N), Q_j(z^*)) \in \Pi_N(X_j)$ and

$$|\langle B_j(P_j(z_1), \ldots, P_j(z_N)), Q_j(z^*)\rangle| = 1. \tag{4.11}$$

To do so we consider first $X = [\bigoplus_{k=1}^{\infty} X_k]_{c_0}$. Since $(z_1, \ldots, z_N, z^*) \in \Pi_N(X)$, for all $l = 1, \ldots, N$, we have

$$1 = z^*(z_l) = \sum_{n \in \mathbb{N}} Q_n(z^*) P_n(z_l) \leqslant \sum_{n \in \mathbb{N}} \|Q_n(z^*)\| \|P_n(z_l)\|.$$

But when $n \neq l$ and $l = 1, \ldots, N$, we have

$$\|P_n(z_l)\| = \|P_n(z_l) - P_n(x_l)\| \leqslant \|z_l - x_l\|_\infty < \varepsilon$$

and then

$$\begin{aligned}1 \leqslant \sum_{n\in\mathbb{N}} \|Q_n(z^*)\|\|P_n(z_l)\| &< \|Q_j(z^*)\|\|P_j(z_l)\| + \varepsilon \sum_{\substack{n\in\mathbb{N},\\ n\neq j}} \|Q_n(z^*)\| \\ &\leqslant \|Q_j(z^*)\| + \varepsilon \sum_{\substack{n\in\mathbb{N}\\ n\neq j}} \|Q_n(z^*)\| \\ &< \|z^*\|_1 = 1.\end{aligned}$$

This implies that $\|Q_j(z^*)\| = 1$ and $Q_n(z^*) = 0$ for all $n \neq j$. Then $Q_j(z^*)(P_j(z_l)) = z^*(z_l) = 1$ for all $l = 1, \ldots, N$. In particular, $P_j(z_l) \in S_{X_j}$ for all $l = 1, \ldots, N$. Now we prove (4.11). First note that we can write $(z_1, \ldots, z_N)$ as

$$\Bigg((1-\varepsilon)\overline{P_j}P_j(z_1) + \varepsilon\left(\overline{P_j}P_j(z_1) + \frac{1}{\varepsilon}\left(z_1 - \overline{P_j}P_j(z_1)\right)\right), \ldots, \\ (1-\varepsilon)\overline{P_j}P_j(z_N) \\ + \varepsilon\left(\overline{P_j}P_j(z_N) + \frac{1}{\varepsilon}\left(z_N - \overline{P_j}P_j(z_N)\right)\right)\Bigg)$$

Because of that, we have that

$$1 = |z^*B(z_1, \ldots, z_N)| \leqslant \\ (1-\varepsilon)^N |z^*B(\overline{P_j}P_j(z_1), \ldots, \overline{P_j}P_j(z_N))| + \\ \sum_{\substack{\gamma_l\in\{1-\varepsilon,\varepsilon\}\\ l\in\{1,\ldots,N\}\\ \gamma_1\cdots\gamma_N\neq(1-\varepsilon)^N}} \gamma_1\cdots\gamma_N |z^*B(Z, \ldots, Z)|.$$

where $Z \in \left\{\overline{P_j}P_j(z_l), \overline{P_j}P_j(z_l) + \frac{1}{\varepsilon}\left(z_l - \overline{P_j}P_j(z_l)\right) : l = 1, \ldots, N\right\}$. Now, since for every $l = 1, \ldots, N$,

$$Q_j(z^*)(P_j(z_l)) = 1 \quad \text{and} \quad Q_n(z^*) = 0 \ \ \forall n \neq j,$$

we have that

$$|z^*B(\overline{P_j}P_j(z_1),\dots,\overline{P_j}P_j(z_N))| \leqslant v(B)=1$$

and $|z^*B(Z,\dots,Z)| \leqslant v(B)=1$, whenever

$$Z \in \left\{\overline{P_j}P_j(z_l), \overline{P_j}P_j(z_l)+\frac{1}{\varepsilon}\left(z_l-\overline{P_j}P_j(z_l)\right) : l=1,\dots,N\right\}.$$

Then, we get that

$$\begin{aligned}1 &\leqslant (1-\varepsilon)^N|z^*B(\overline{P_j}P_j(z_1),\dots,\overline{P_j}P_j(z_N)| \\ &\qquad + \sum_{\substack{\gamma_l\in\{1-\varepsilon,\varepsilon\}\\ l\in\{1,\dots,N\}\\ \gamma_1\cdots\gamma_N\neq(1-\varepsilon)^N}} \gamma_1\cdots\gamma_N|z^*B(Z,\dots,Z)| \\ &\leqslant (1-\varepsilon)^N + \sum_{\substack{\gamma_j\in\{1-\varepsilon,\varepsilon\}\\ l\in\{1,\dots,N\}\\ \gamma_1\cdots\gamma_N\neq(1-\varepsilon)^N}} \gamma_1\cdots\gamma_N = (\varepsilon+(1-\varepsilon))^N = 1.\end{aligned}$$

Therefore $|z^*B(\overline{P_j}P_j(z_1),\dots,\overline{P_j}P_j(z_N))| = 1$ and then

$$\begin{aligned}&|\langle B_j(P_j(z_1),\dots,P_j(z_N)),Q_j(z^*)\rangle| \\ &\qquad = |\langle D_j(\overline{P_j}P_j(z_1),\dots,\overline{P_j}P_j(z_N),Q_j(z^*)\rangle| \\ &\qquad\qquad = |z^*B(\overline{P_j}P_j(z_1),\dots,\overline{P_j}P_j(z_N))| = 1\end{aligned}$$

This proves (4.11) and $v(B_j)=1$, and it completes the proof in the c_0-sum case. Similarly, we can prove it for ℓ_1-sums. □

Open problems

During the preparation of this dissertation, there were some questions that we could not answer.

In Theorem 2.1.7 we prove that the pair $(H; Y)$ has the Bishop-Phelps-Bollobás point property for all Hilbert spaces H and any Banach space Y. To do so, we use the fact that Hilbert spaces have transitive norm, i.e., given two norm-one points x and y, there exists a linear isometry R such that $R(x) = y$. Moreover, if the points x and y are close from each other, we may choose the isometry R satisfying $\|R - \mathrm{Id}_H\| < \delta(\varepsilon)$ for some function $\delta(\varepsilon) > 0$ with $\lim_{t\to 0} \delta(t) = 0$. We do not know if we can use similar ideas for L_p-spaces.

1. Does the pair $(L_p(\mu), Y)$ satisfy the BPBpp for operators for all Banach spaces Y and $1 < p < \infty$?

It could be interesting to give a characterization for the pair $(X; Y)$ to have property 1 (see Definition 2.2.2).

2. It is possible to give a characterization for the pair $(X; Y)$ to have property 1?

About property 2 (see Definition 2.2.8), we have just one positive result which is the Kim-Lee theorem [46, Theorem 2.1]. So, it is natural to ask if

3. Is there a Banach space $Y \neq \mathbb{K}$ such that the pair $(X;Y)$ satisfies property 2?

In Section 2.3 we studied the BPBpp and the BPBpp for numerical radius on Hilbert spaces for self-adjoint, anti-symmetric, normal and unitary operators. It is true that

4. H, a complex Hilbert space, satisfies the Bishop-Phelps-Bollobás point property for numerical radius for normal operators?

Also about this topic, we can consider other types of functions as hermitian and symmetric bilinear forms. For a complex Hilbert space, we recall that a hermitian form is a bilinear form B on $H \times H$ such that $B(x,y) = \overline{B(y,x)}$ for every $(x,y) \in H \times H$.

5. Let H be a complex Hilbert space. It is true that the pair (H,H) has the BPBpp for hermitian forms?

6. Let H be a real or complex Hilbert space. It is true that the pair (H,H) has the BPBpp for symmetric forms?

In Chapter 3 we have seen that there are some cases in which the pair $(X;Y)$ has both BPBp and BPBp for compact operators. We know that there are pairs of Banach spaces which have the BPBp for compact operators but fail the BPBp, but we do not know if the BPBp implies the BPBp for compact operators.

7. If the pair $(X;Y)$ has the BPBp, does it have the BPBp for compact operators?

About multilinear mappings (see Chapter 4) we have the following questions.

8. Is there some infinite dimensional Banach space (non Hilbert) X such that (X, X) has the BPBp for symmetric bilinear forms?

9. For every Banach space X, is it true that the set of all N homogeneous continuous polynomials whose canonical extension to X^{**} attain their norm is dense in $\mathcal{P}(^N X)$ for $N > 2$?

Finally, we finish it with an old problem:

10. Is the Bishop-Phelps theorem true for linear operators from any Banach space X into euclidian space $\mathbb{R}^2$?

Derived works

Some results of this dissertation were submitted and published in specific journals during the PhD procedure. We list them specifying the chapter/section which they correspond.

Chapter 2, Section 2.1

- *Title*: The Bishop-Phelps-Bollobás point property
  Authors: S. Dantas, S. K. Kim and H. J. Lee
  Journal: Journal of Mathematical Analysis and Applications
  Year of publication: 2016
  http://dx.doi.org/10.1016/j.jmaa.2016.07.009

Chapter 2, Section 2.2

- *Title*: Some kind of Bishop-Phelps-Bollobás property
  Author: S. Dantas
  Journal: Mathematische Nachrichten
  Year of publication: to appear
  http://onlinelibrary.wiley.com/doi/10.1002/mana.201500487/full

Chapter 3

- *Title*: The Bishop-Phelps-Bollobás property for compact operators
  Authors: S. Dantas, D. García, M. Maestre and M. Martín
  Journal: Canadian Journal of Mathematics
  Year of publication: to appear
  http://dx.doi.org/10.4153/CJM-2016-036-6

Chapter 4

- *Title*: On the Bishop-Phelps-Bollobás theorem for multilinear mappings
  Authors: S. Dantas, D. García, S. K. Kim, H. J. Lee and M. Maestre
  Journal: Linear Algebra and its Applications
  (Partially accepted)

Classical Banach spaces with the BPBp

In this final section, we present three tables which give a summary of those pairs of classical Banach spaces pairs **satisfying** the Bishop-Phelps-Bollobás property **for operators**. We would like to emphasize that these tables clarified our minds to think about new problems on this topic at the beginning of the thesis.

We used some symbols to indicate the results in different cases. Each of them means that the pair of Banach spaces $(X;Y)$ has the BPBp in that case. The symbols are:

`FD`	for finite-dimensional Banach spaces,
$\checkmark$	for both real and complex cases,
$\checkmark_{\mathbb{R}}$	just for the real case,
$\checkmark_{\mathbb{C}}$	just for the complex case,
$\checkmark_{\bullet}$	for localizable measures,
$\checkmark_{\circ}$	for locally compact metrizable spaces,
$\checkmark_{\sigma}$	for a σ-finite measures.

In what follows, $p, q \in (1, \infty)$. In general, μ and ν are arbitrary measures, K and S are compact Hausdorff topological spaces and L_1, L_2 are locally compact Hausdorff topological spaces, except when we use the symbols above to indicate specific cases.

		RANGE SPACES									
		FD	ℓ_1^n	ℓ_p^n	ℓ_q^n	ℓ_∞^n	c_0	ℓ_1	ℓ_p	ℓ_q	ℓ_∞
DOMAIN SPACES	FD	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$				$\checkmark$
	ℓ_1^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$				$\checkmark$
	ℓ_p^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_q^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_∞^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$				$\checkmark$
	c_0						$\checkmark$				$\checkmark$
	ℓ_1	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_p	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_q	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_∞						$\checkmark$				$\checkmark$
	$L_1(\mu)$						$\checkmark$	$\checkmark_\sigma$			$\checkmark$
	$L_1(\nu)$						$\checkmark$	$\checkmark_\sigma$			$\checkmark$
	$L_p(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_p(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_q(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_q(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_\infty(\mu)$						$\checkmark$				$\checkmark$
	$L_\infty(\nu)$						$\checkmark$				$\checkmark$
	$C(K)$			$\checkmark_\mathbb{R}$	$\checkmark_\mathbb{R}$		$\checkmark$		$\checkmark_\mathbb{R}$	$\checkmark_\mathbb{R}$	$\checkmark$
	$C(S)$			$\checkmark_\mathbb{R}$	$\checkmark_\mathbb{R}$		$\checkmark$		$\checkmark_\mathbb{R}$	$\checkmark_\mathbb{R}$	$\checkmark$
	$C_0(L_1)$						$\checkmark$				$\checkmark$
	$C_0(L_2)$						$\checkmark$				$\checkmark$

Table 4.1

		RANGE SPACES					
		$L_1(\mu)$	$L_1(\nu)$	$L_p(\mu)$	$L_p(\nu)$	$L_q(\mu)$	$L_q(\nu)$
DOMAIN SPACES	FD						
	ℓ_1^n						
	ℓ_p^n	✓	✓	✓	✓	✓	✓
	ℓ_q^n	✓	✓	✓	✓	✓	✓
	ℓ_∞^n						
	c_0			✓	✓	✓	✓
	ℓ_1	✓	✓	✓	✓	✓	✓
	ℓ_p	✓	✓	✓	✓	✓	✓
	ℓ_q	✓	✓	✓	✓	✓	✓
	ℓ_∞						
	$L_1(\mu)$	✓	✓	✓	✓	✓	✓
	$L_1(\nu)$	✓	✓	✓	✓	✓	✓
	$L_p(\nu)$	✓	✓	✓	✓	✓	✓
	$L_p(\mu)$	✓	✓	✓	✓	✓	✓
	$L_q(\mu)$	✓	✓	✓	✓	✓	✓
	$L_q(\nu)$	✓	✓	✓	✓	✓	✓
	$L_\infty(\mu)$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓	✓	✓	✓
	$L_\infty(\nu)$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓	✓	✓	✓
	$C(K)$			✓$_{\mathbb{R}}$	✓$_{\mathbb{R}}$	✓$_{\mathbb{R}}$	✓$_{\mathbb{R}}$
	$C(S)$			✓$_{\mathbb{R}}$	✓$_{\mathbb{R}}$	✓$_{\mathbb{R}}$	✓$_{\mathbb{R}}$
	$C_0(L_1)$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$
	$C_0(L_2)$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$	✓$_{\mathbb{C}}$

Table 4.2

		RANGE SPACES					
		$L_\infty(\mu)$	$L_\infty(\nu)$	$C(K)$	$C(S)$	$C_0(L_1)$	$C_0(L_2)$
DOMAIN SPACES	FD	$\checkmark_{\sigma}$	$\checkmark_{\sigma}$				
	ℓ_1^n						
	ℓ_p^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_q^n	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_∞^n						
	c_0	$\checkmark_{\sigma}$	$\checkmark_{\sigma}$			$\checkmark$	$\checkmark$
	ℓ_1			$\checkmark$	$\checkmark$		
	ℓ_p	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_q	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	ℓ_∞						
	$L_1(\mu)$	$\checkmark_{\bullet}$	$\checkmark_{\bullet}$				
	$L_1(\nu)$	$\checkmark_{\bullet}$	$\checkmark_{\bullet}$				
	$L_p(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_p(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_q(\mu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_q(\nu)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$L_\infty(\mu)$						
	$L_\infty(\nu)$						
	$C(K)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$		
	$C(S)$			$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$		
	$C_0(L_1)$					$\checkmark_{\circ}$	$\checkmark_{\circ}$
	$C_0(L_2)$					$\checkmark_{\circ}$	$\checkmark_{\circ}$

Table 4.3

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