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One-loop Effective Lagrangian for a Standard Model with a Heavy Charged Scalar Singlet

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Abstract

We study several problems related to the construction and the use of effective Lagrangians by considering an extension of the standard model that includes a heavy scalar singlet coupled to the leptonic doublet. Starting from the full renormalizable model, we build an effective field theory by integrating out the heavy scalar. A local effective Lagrangian (up to operators of dimension six) is obtained by expanding the one-loop effective action in inverse powers of the heavy mass. This is done by matching some Green functions calculated with both the full and the effective theories.

Using this simple example we study the renormalization of effective Lagrangians in general and discuss how they can be used to bound new physics. We also discuss the effective Lagrangian after spontaneous symmetry breaking, and the use of the standard model classical equations of motion to rewrite it in different forms. The final effective Lagrangian in the physical basis is wel suited to the study of the phenomenology of the model, which we comment on briefly. Finally, as an example of the use of our effective field theory, we consider the leptonic flavour-changing decay of the Z boson in the effective theory and compare the results obtained with the full model calculation.

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1 Introduction

Effective Lagrangians [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] have been used for a long time[1] as a systematic method to incorporate the known symmetries of a problem into the quantum field theory language. However, with the advent of Yang-Mills theories and the Higgs mechanism, which allowed the construction of physically interesting renormalizable theories, they were used only for those problems that could not be treated in any other way. Particularly important have been its applications to low-energy strong interactions in the form of the so-called Chiral Perturbation Theory [1, 2, 3, 5, 10, 11, 19] and more recently the Heavy Quark Effective Field Theory [20, 21]. Its application to weak interactions has been less intensive, since in the last decade the main effort has been in the direction of building complete renormalizable theories that could serve as alternatives or extensions of the standard model, by just enlarging the number of fermions, gauge bosons and scalars. Renormalizability was considered to be a main point and completely linked to predictability, because non-renormalizable models need an infinite number of parameters to be completely described. The striking confirmation of many of the standard model predictions in LEP experiments has started to change this point of view. Almost every one now thinks that the standard model correctly describes physics at present energies and perhaps also up to energies close to the TeV range. Of course, supersymmetric particles and other elusive particles, such as neutral heavy leptons or right-handed neutrinos, with some hidden interactions and masses much lighter than 1 TeV, are not excluded. On the other hand, the theoretical developments achieved since the beginning of the seventies have brought a new interpretation of renormalizability [19, 22, 23, 24, 25, 26]. Nowadays renormalizability is not understood as a calculational requirement or a consistency requirement. In fact one knows how to calculate, for example, with the Fermi Lagrangian, as long as one does not try to use it beyond the Fermi scale. The key idea is that one does not expect a quantum field theory, even a renormalizable one, to be valid up to arbitrarily large energy scales. Renormalizability is then seen as the physical requirement that physics at low energies cannot dramatically depend on the physics at some large scale. There could be effects of the heavy particles on the low-energy physics, but all of them must be suppressed by some power of the scale Λ at which the new physics starts. Then, for each range of energies one expects that physics is described by a Lagrangian of the form:

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \frac{1}{\Lambda}\mathcal{L}_1 + \frac{1}{\Lambda^2}\mathcal{L}_2 + \cdots \ , \tag{1.1}$$

where $\mathcal{L}_0$ is a renormalizable Lagrangian that describes the low-energy physics and $\mathcal{L}_n$ are linear combinations of non-renormalizable operators of dimension $n+4$. For processes involving energies E much smaller than Λ, the effects of the non-renormalizable operators are suppressed by $(E/\Lambda)^n$ and can systematically be computed. Of course, they depend on more arbitrary parameters, but, if the underlying theory is known and if it is renormalizable, they can, in principle, be computed in terms of the few renormalizable couplings of the underlying theory by matching the calculation of several observables or Green functions done with both theories: the full theory and the effective one. This picture is known as the decoupling case [27], as the effects of heavy particles decouple from the low-energy physics, and a well-known example is physics in the 5–80 GeV range. There, physics can be well described by a QED-QCD renormalizable gauge-invariant Lagrangian involving the five light quarks and the leptons, supplemented by four-fermion non-renormalizable interactions that

[1] Very early examples are the effective Lagrangian description of the photon-photon interactions [13, 14, 15] or Fermi's theory of weak interactions [16, 17, 18].

take into account the effects of weak interactions. These non-renormalizable couplings can be computed in terms of the standard model couplings by matching some Green functions at the Fermi scale. But once this matching is done all calculations, including QED and QCD radiative corrections, can be performed at the effective Lagrangian level. Since the full standard model is QED and QCD gauge invariant one obtains that all the effective couplings must also be QED and QCD gauge invariant.

Following the previous reasoning one can then expect that the standard model is not valid for an arbitrarily large range of energies and that it is just a low-energy approximation of a more complete theory. There are many arguments suggesting that this is the case. In particular, the so-called naturalness problem of the standard model, which will be discussed later on in the context of our effective Lagrangian, and the large amount of arbitrary input parameters needed to describe the standard model are among the most popular ones. If the standard model is just a low-energy approximation of a more complete theory, one would expect that new non-renormalizable couplings suppressed by the scale of the new physics should arise. In complete parallelism one would also expect these non-renormalizable interactions to be gauge invariant with respect to the standard model gauge group. However, a complication arises since the $SU(2) \otimes U(1)$ gauge symmetry of the standard model is spontaneously broken to $U(1)_{QED}$. Then, when building an effective theory for physics beyond the standard model, there are two possibilities:

- The gauge symmetry is linearly realized. This means that radial excitations of the scalar field are relatively light; then, for energies larger than the mass of these excitations, the symmetry is effectively restored. This is the simplest situation, in which the effective Lagrangian is constructed with exactly the same particle content as the standard model particle spectrum and non-renormalizable interactions are required to be $SU(2) \otimes U(1)$ gauge invariant.

- The gauge symmetry is non-linearly realized. That is, the physics of spontaneous symmetry breaking is non-linear. The radial excitations are heavy with respect to the Fermi scale; then, there is a wide range of energies between the Fermi scale and the scale of new physics in which the full standard gauge symmetry is present, but realized in a non-linear way. In this region of energies one can still write an effective Lagrangian, but the first term of that Lagrangian is a non-renormalizable non-linear sigma model [28, 29, 30, 31, 32, 33].

There is a big difference between these two possibilities. The second one assumes that the new physics is responsible for spontaneous symmetry breaking; as a consequence it must start not far away from the Fermi scale, since it must correct the bad behaviour of the standard model without Higgs particles. In the first one, however, the Higgs mechanism is fully implemented. Low-energy physics decouples completely from high-energy physics and the only way to get some hint on the scale of the possible new physics (apart from naturalness arguments) is just by looking at the size of the non-renormalizable interactions. Experimental bounds are then the only source of information.

A very simple and instructive illustration of the use of effective Lagrangians to bound new physics is the so-called see-saw mechanism for the generation of neutrino masses [34, 35]. It is not difficult to see that the only $SU(2) \otimes U(1)$ gauge-invariant operator of dimension five that can be built with the field content of the standard model is

$$\mathcal{L}_{see-saw} = -\frac{1}{4}\frac{1}{\Lambda}(\overline{\tilde{\ell}}\, F \vec{\tau} \ell)(\tilde{\varphi}^{\dagger} \vec{\tau} \varphi) \ , \tag{1.2}$$

where ℓ is the standard left-handed doublet of leptons, $\widetilde{\ell} = i\tau_2 \ell^c$, $\ell^c = C\overline{\ell}^T$ (C is the charge conjugation operator), φ is the Higgs doublet and $\widetilde{\varphi} = i\tau_2\varphi^*$; F is a complex symmetric matrix in flavour space ($SU(2)$ and flavour indices have been suppressed). It is clear that this Lagrangian does not conserve generational lepton numbers, but in addition it does not conserve the total lepton number, which is violated in two units. This kind of operator will be generated in any theory that does not conserve lepton number. When the Higgs develops a vacuum expectation value (VEV), it will generate a neutrino Majorana mass matrix given by

$$M_\nu = F\frac{\left\langle \varphi^{(0)} \right\rangle^2}{\Lambda} \; . \tag{1.3}$$

If we take the largest eigenvalue of F to be of order 1, the Higgs VEV $\left\langle \varphi^{(0)} \right\rangle = 174$ GeV and use the experimental bound on the τ-neutrino mass, $m_{\nu_\tau} < 31$ MeV, we find that $\Lambda > 10^6$ GeV. Should one take the cosmological bound [36] on neutrino masses, $m_{\nu_\tau} < 100$ eV, one would obtain $\Lambda > 3 \times 10^{11}$ GeV. These bounds are really impressive. But what is Λ? The Lagrangian of eq. (1.2) seems to be generated by the exchange of a scalar triplet with hypercharge 1 between the leptons and the Higgses. Then, Λ should be the mass of that triplet. However this is not the only possibility. In fact, eq. (1.2) can be identically rewritten, after a $SU(2)$ Fierz transformation, as

$$\mathcal{L}_{see-saw} = -\frac{1}{2}\frac{1}{\Lambda}(\overline{\widetilde{\ell}}\,\varphi)F(\widetilde{\varphi}^\dagger \ell) \; , \tag{1.4}$$

which suggests the exchange of a neutral heavy Majorana fermion; then Λ should be the mass of that fermion. Indeed, the original formulation of the see-saw mechanism[34, 35] was based on this possibility.

This simple example shows the power and the limitations of the effective Lagrangian approach. One can set impressive bounds on the scale of new physics, but one cannot completely disentangle its origin, at least by taking into account only the lowest-order operators.

It must also be remarked that in the effective Lagrangian approach it is essential to know what the low-energy particle spectrum is. One generally assumes that it is just the standard model particle spectrum, but there could exist light particles, completely neutral under the standard model gauge group, that interact with the standard model particles only through the exchange of new heavy particles. This is not at all an exotic situation since it is what happened with weak interactions: neutrinos are singlets under QCD and QED; however, they cannot be ignored in building an effective Lagrangian that describes weak interactions.

Keeping in mind all these limitations, one can very efficiently use the effective Lagrangian approach to new physics to set bounds on operators that violate some of the global symmetries of the standard model: lepton number conservation, baryon number conservation or flavour symmetries [37, 38, 39, 40]. These bounds are naturally implemented at tree level in the effective Lagrangian (although the effective operators could be generated through loops in the full theory). During the last years, using the precise data obtained from experiment, people have started to consider the possibility of bounding operators that do not violate any symmetry of the standard model [40, 41]. This task is much more complicated because the number of operators of a given dimension is very large [40, 42, 43] (after using the equations of motion and without taking into account flavour, Buchmüller and Wyler [40] found 80 $SU(2) \otimes U(1)$ gauge invariant dimension-six operators constructed from the standard model fields). Especially interesting is the analysis of effective operators contributing to trilinear gauge-boson couplings, since they could have important consequences at LEP2, the SSC and LHC [44, 45, 46]. Some of these operators also contribute to LEP1 observables at tree level and they are strongly bounded by the LEP very precise measurements. Others do not contribute at tree level to LEP1 observables, but only in loops [46, 47, 48, 49]. The next step is to use the effective Lagrangian at the one-loop level and try to set bounds on all operators by using radiative corrections. This analysis is even more complicated since the effective theory is non-renormalizable in the standard sense and care must be taken with divergences, especially when the full theory is not known. If the full theory is known, it is not difficult to find the right prescription to absorb all infinities in the effective theory. All couplings must be renormalized, and matching to the full theory fixes all the counterterms. If the full theory is not known, the theory still has to be renormalized and infinities absorbed in the various couplings; however, the finite parts of the counterterms remain arbitrary and cannot be determined. The way this is done has created some controversy in the literature. The most general trilinear interaction among vector bosons can be constructed by imposing only Lorentz invariance and QED gauge invariance [50, 51]. In the last years several groups have used such interactions in loops. Calculations were performed by using a momentum cut-off that was identified with the scale of new physics. Then, large effects were found since, in some cases, the diagrams were quadratically or even quartically divergent[2]. This approach is not always correct, because the effective theory also has to be renormalized, couplings must be defined at some scale and they must satisfy some renormalization group equations. Criticisms to this treatment of effective Lagrangians have already been raised by several groups [46, 47, 48, 49, 52, 53, 54, 55, 56]; however, the emphasis was put on different aspects. The authors of refs. [46, 47] stressed the importance of the gauge invariance of the effective Lagrangian under the standard model gauge group. While, for example, in refs. [52, 53] the emphasis was put on the incorrect use of cut-offs in previous calculations.

The purpose of this paper is to study some of the questions that arise when the effective Lagrangian approach to new physics is used by working out completely an example of possible new physics. We consider the simplest non-trivial extension of the standard model we could write down: a standard model supplemented by a singly charged scalar singlet coupled to leptons [57, 58]. We construct the full model and obtain a low-energy effective field theory by integrating out, at the one-loop level, the heavy scalar singlet. By construction, since the full theory is $SU(2) \otimes U(1)$ gauge invariant and we integrate out a complete scalar multiplet, we automatically obtain a $SU(2) \otimes U(1)$ gauge invariant effective theory. Therefore, we do not discuss the role of gauge invariance in the construction of the effective theory [3].

We use this example to study the renormalization of the effective Lagrangian and to study the matching conditions that relate the parameters of the effective Lagrangian to the parameters of the full Lagrangian ensuring agreement between the two theories at low energies. The possibility of using the equations of motion before or after spontaneous symmetry breaking is illustrated as well. The model is also phenomenologically interesting because the presence of the scalar gives rise to very interesting phenomena [57, 58, 65, 66, 67] such as, for example, the processes $Z \to \tau e$, $Z \to \mu e$, $\cdots$, $\mu \to e\gamma$, $\tau \to \mu\gamma$, $\cdots$, $\mu \to eee$, $\tau \to \mu ee$, $\cdots$. There are also additional contributions to the masses of the gauge bosons,

[2] A long list of references in which this method was used can be found in [46, 52].

[3] It seems, however, that if the full theory is gauge invariant with respect to some group, the effective theory should also be gauge invariant, although gauge invariance could be implemented in a non-linear way. The effective theories obtained by integrating out the standard Higgs [28, 29, 30, 31, 33] or a heavy quark [59, 60, 61, 62, 63, 64] in the standard model belong to this second type of theories.

new neutral current interactions, etc.

Of course, this analysis is quite far from being general, it is a very specific model that leads to a linearly realized gauge symmetry, but in spite of its simplicity it leads, already at the one-loop level, to many of the operators classified in refs. [40, 42, 43]. It can also help us to understand some of the tricky points discussed in the literature.

In section 2 we will introduce the notation and write down the Lagrangian of the full theory. In section 3 we obtain the tree-level and the one-loop contributions to the effective Lagrangian of diagrams with only heavy scalars in internal lines by using functional methods. When using the tree-level effective interactions at the one-loop level in the effective Lagrangian, new effective operators appear as counterterms that must be computed by matching the full theory results with the effective Lagrangian results; we do this job in section 4. Renormalization of our effective Lagrangian and the possibility of using effective Lagrangians in general at the one-loop level to bound new physics are discussed in section 5. In section 6 we discuss the use of the classical equations of motion to rewrite the effective Lagrangian in a convenient form for phenomenological analyses. We also discuss the impact of spontaneous symmetry breaking on this Lagrangian and extract the relevant interactions in terms of the physical fields. In section 7 we shortly comment on some of the most interesting phenomenological consequences of the model by using the effective Lagrangian we have obtained. Finally, in section 8 we review what we have learned about the use of effective Lagrangians with our example. We include in appendix A the calculation of the determinant of the fluctuation operator, in appendix B the calculation of the diagrams with heavy-light lines, and finally in appendix C we calculate the amplitudes for $Z \to \bar{e}_a e_b$ in both the full and the effective theory.

2 The Model

The complete renormalizable model we are considering is an extension of the standard model, which contains a singly charged scalar singlet in addition to the standard model particles. This model is one of the simplest extensions one could imagine, but in spite of its simplicity, it already includes many interesting features common to any extension of the standard model containing a large mass scale compared with the Fermi scale. For completeness we list below the particle spectrum of the full model and give the corresponding transformation properties with respect to the gauge group $SU(3) \otimes SU(2) \otimes U(1)$

left-handed lepton doublets:	ℓ	$(1,\ 2, -1/2)$
	$\widetilde{\ell} = i\tau_2 \ell^c$	$(1,\ 2,\ \ 1/2)$
right-handed charged leptons:	e	$(1,\ 1, -1\ \ )$
left-handed quark doublets:	q	$(3,\ 2,\ \ 1/6)$
right-handed u-quarks:	u	$(3,\ 1,\ \ 2/3)$
right-handed d-quarks:	d	$(3,\ 1, -1/3)$
Higgs boson doublet:	φ	$(1,\ 2,\ \ 1/2)$
	$\widetilde{\varphi} = i\tau_2 \varphi^*$	$(1,\ 2, -1/2)$
gluons:	G^a_μ	$(8,\ 1,\ \ 0\ \ )$
	$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^a{}_{bc} G^b_\mu G^c_\nu$	
W bosons:	$\vec{W}_\mu$	$(1,\ 3,\ \ 0\ \ )$
	$\vec{W}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g \vec{W}_\mu \times \vec{W}_\nu$	
B bosons:	B_μ	$(1,\ 1,\ \ 0\ \ )$
	$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$	
charged scalar singlet:	h	$(1,\ 1, -1\ \ )$.

All possible $SU(2)$ and generational indices are suppressed above.

The full Lagrangian can be split into two parts:

$$\mathcal{L}_{full} = \mathcal{L}_{SM} + \mathcal{L}_h \ . \tag{2.1}$$

The first part, $\mathcal{L}_{SM}$, represents the standard model Lagrangian:

$$\begin{aligned}\mathcal{L}_{SM} = & -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a - \frac{1}{4} \vec{W}_{\mu\nu} \vec{W}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \varphi)^\dagger (D^\mu \varphi) + m^2_\varphi \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2 \\ & + i\overline{\ell}\, \not{D} \ell + i\overline{e}\, \not{D} e + i\overline{q}\, \not{D} q + i\overline{u}\, \not{D} u + i\overline{d}\, \not{D} d + (\overline{\ell} Y_e e \varphi + \overline{q} Y_d d \varphi + \overline{q} Y_u u \widetilde{\varphi} + \text{h.c.}) \ ,\end{aligned} \tag{2.2}$$

where the covariant derivative can be written in the general form

$$D_\mu = \partial_\mu - i g_s T^a_3 G^a_\mu - i g T^i_2 W^i_\mu - i g' Y B_\mu \ . \tag{2.3}$$

By T_2 and T_3 we denote the generators of the $SU(2)$ and the $SU(3)$ groups, accordingly, which are acting on the proper representations of the groups. In the case of $SU(2)$ doublets $T^i_2 = \frac{1}{2}\tau^i$, where τ^i are 2×2 Pauli matrices. For $SU(3)$-triplets $T^a_3 = \frac{1}{2}\lambda^a$, where λ^a are 3×3 Gell-Mann matrices. In the Lagrangian (2.2) the Yukawa couplings, Y_e, Y_d, Y_u, are arbitrary complex matrices in flavour space.

The Lagrangian $\mathcal{L}_h$ contains the interactions of the scalar singlet:

$$\mathcal{L}_h = (D_\mu h)^\dagger D^\mu h - m^2 |h|^2 - \alpha |h|^4 - \beta |h|^2 \varphi^\dagger \varphi + \left(\overline{\widetilde{\ell}} f \ell h^+ + \text{h.c.} \right) \tag{2.4}$$

and the covariant derivative in eq. (2.4) has the form $D_\mu = \partial_\mu + i g' B_\mu$, because the h-scalar is an $SU(2)$ singlet with hypercharge $Y = -1$.

For further applications it is convenient to rewrite the h-dependent part of the Lagrangian, using integration by parts for the covariant derivative, in the following form:

$$\mathcal{L}_h = h^+ (-D^2 - m^2 - \alpha |h|^2 - \beta \varphi^\dagger \varphi) h + \left(\overline{\widetilde{\ell}} f \ell h^+ + \text{h.c.} \right) \ . \tag{2.5}$$

The quantum numbers of the scalar singlet are such that there is no interaction between the scalar singlet h and the quarks in the Lagrangian of eq. (2.4). It is also important to note that its coupling to the leptons, f, is an antisymmetric complex matrix in flavour space. The antisymmetry of this matrix is a consequence of Fermi statistics and the fact that the current $\overline{\tilde{\ell}}_a\ell_b$ is an $SU(2)$ scalar. Then, one can write $\overline{\tilde{\ell}}_a\ell_b = -\overline{\tilde{\ell}}_b\ell_a$, from which the antisymmetry of the coupling easily follows. This property of the scalar-lepton coupling is very important since it naturally leads to a violation of the generational lepton numbers and all the interesting phenomenology related to it. However, if every lepton carries a total lepton number of 1, we can assign a total lepton number of 2 to the scalar h and, then, the Lagrangian (2.4) is invariant with respect to a global symmetry, which can be identified as conservation of the total lepton number. As a consequence, the neutrinos remain massless at all orders. Models very similar to the one defined in eq. (2.1) containing additional doublets and/or scalar singlets, in which the total lepton number is explicitly or spontaneously broken, have been used [57, 58, 66, 67, 68, 69] to generate small calculable neutrino masses.

3 Integrating out the heavy scalar

If the mass m of the scalar h is much larger than the energy available to experiment, the only effects of this particle on the low-energy observables come through virtual corrections. These effects can be taken into account by using an effective Lagrangian of the form (1.1), containing only the "light" fields of the model and where the effects of the "heavy" particle are included in the non-renormalizable terms of dimension larger than four, which are suppressed by inverse powers of the heavy mass m.

The effective Lagrangian can be defined by integrating out the heavy scalar field [7, 22, 23, 24, 25] :

$$e^{iS_{eff}} = \exp\left\{i\int d^4x\mathcal{L}_{eff}(x)\right\} \equiv \int \mathcal{D}h\mathcal{D}h^+e^{iS} = \int \mathcal{D}h\mathcal{D}h^+\exp\left\{i\int d^4x\mathcal{L}(x)\right\} \quad (3.1)$$
$$= \exp\left\{i\int d^4x\mathcal{L}_{SM}(x)\right\}\int \mathcal{D}h\mathcal{D}h^+\exp\left\{i\int d^4x\mathcal{L}_h(x)\right\} = e^{iS_{SM}}\int \mathcal{D}h\mathcal{D}h^+e^{iS_h} \ ,$$

where $\mathcal{D}h$ represents functional integration over h. The effective field theory defined by this effective Lagrangian is fully equivalent to the original theory when only Green functions with light external particles are considered. Starting from eq. (3.1) we can calculate the one-loop level effective Lagrangian by using the steepest-descent method to integrate out the heavy scalar. As we are interested in the effects of a heavy scalar ($m \geq 1$ TeV) on the physics around the M_Z scale and below, we will systematically keep only terms of order $O(1/m^2)$ through the whole calculation, neglecting all operators with higher inverse powers of the mass of the scalar singlet.

Let us denote by h_0 the solution of the classical equation of motion for the h-field, i.e.

$$\left.\frac{\delta S}{\delta h(x)}\right|_{h_0} = 0 \ . \quad (3.2)$$

Using the Lagrangian (2.5) we obtain :

$$(-D^2 - m^2 - 2\alpha\left|h_0\right|^2 - \beta\varphi^\dagger\varphi)h_0 + \overline{\tilde{\ell}}f\ell = 0 \ . \quad (3.3)$$

Then the full action can be functionally expanded around the solution $h_0(x)$

$$S = S_{SM} + S_h[h_0] + \int d^4xd^4x'\eta^\dagger(x)O(x,x')\eta(x') + \cdots \ , \quad (3.4)$$

where the fluctuation operator $O(x,x')$ is given by the second-order term in Taylor's expansion of the action S:

$$O(x,x') = \left.\frac{\delta^2 S}{\delta h(x)\delta h(x')}\right|_{h_0} \tag{3.5}$$

and $\eta(x)$ in eq. (3.4) represents the fluctuations around the classical solution, $\eta(x) = h(x) - h_0(x)$. Substituting eq. (3.4) in eq. (3.1), shifting variables from h to η in the functional integration, and using the known formula for Gaussian integration, we obtain

$$e^{iS_{eff}} = \exp\{i(S_{SM} + S_h[h_0])\} \det(O)^{-1} = \exp\{i(S_{SM} + S[h_0]) - \mathrm{Tr}\{\log(O)\}\} \ . \tag{3.6}$$

Therefore, for the one-loop effective action we have

$$S_{eff} = S_{SM} + S_h[h_0] + i\,\mathrm{Tr}\{\log(O)\} \ . \tag{3.7}$$

The first term in eq. (3.7) is just the pure standard model contribution to the effective action. Using the equation of motion, eq. (3.3), the second term, $S_h[h_0]$, can be formally written as

$$S_h[h_0] \approx -\int d^4x \overline{\tilde{\ell}}(x) f \ell(x) \frac{1}{(-D^2 - m^2 - \beta\varphi^\dagger(x)\varphi(x))} \overline{\ell}(x) f^\dagger \tilde{\ell}(x) \ , \tag{3.8}$$

where terms of order $1/m^4$ have been neglected. The Lagrangian (3.8) is highly non-local. To obtain a local version of it we could further expand the operator $1/(-D^2 - m^2 - \beta\varphi^\dagger(x)\varphi(x))$ as a power series in $1/m^2$:

$$\frac{1}{(-D^2 - m^2 - \beta\varphi^\dagger(x)\varphi(x))} = -\frac{1}{m^2} + \frac{1}{m^4}(D^2 + \beta\varphi^\dagger(x)\varphi(x)) + \cdots \ . \tag{3.9}$$

Keeping the first term we obtain the only tree-level contribution, at order $1/m^2$, of the scalar to the effective Lagrangian:

$$S_h[h_0] = \int d^4x \mathcal{L}^{(0)}(x) \tag{3.10}$$

with

$$\mathcal{L}^{(0)} = \frac{1}{m^2}(\overline{\tilde{\ell}} f \ell)(\overline{\ell} f^\dagger \tilde{\ell}) = \frac{4}{m^2} f_{ab} f^*_{a'b'} (\overline{\nu^c_{aL}} e_{bL})(\overline{e_{b'L}} \nu^c_{a'L}) \ , \tag{3.11}$$

where the summation over repeated flavour indices (a, b, a', b') is assumed. It could also be obtained by computing the tree-level diagram of fig. 1.a in the limit $q^2 \ll m^2$ (where q is the momentum of the scalar). The result in eq. (3.11) is not complete if we are going to use this Lagrangian for calculations at the loop level [70, 71]. The reason is that in order to obtain a local Lagrangian we had to use the expansion (3.9). In doing this we assumed that the derivative in eq. (3.9), or equivalently the momentum of the scalar, is negligible in comparison with the mass of the scalar. But this is not true if the h-scalar contributes inside loops, since in this case its momentum runs up to infinity. It has been shown [70, 71] that the procedure just outlined can be justified as long as one includes in the effective Lagrangian a set of local operators that compensate for the terms missed by using expansion (3.9). We are going to compute these operators in the next section.

The last piece of the effective action (3.7) is defined in terms of the fluctuation operator (3.5) and takes into account all one-loop effects with only the heavy particle in loops. In our approximation, keeping only terms $O(1/m^2)$, the fluctuation operator (3.5) can be easily calculated and leads to the following effective action

$$i\int d^4x \mathcal{L}^{(1)}(x) = \log\det(O)^{-1} = -\mathrm{Tr}\{\log(O)\} = -\mathrm{Tr}\left\{\log(-D^2 - m^2 - \beta\varphi^\dagger\varphi)\right\} \ . \tag{3.12}$$

We moved to appendix A the detailed evaluation of the determinant of a generic operator of this form as an expansion in $1/m^2$. Using the general result of appendix A, eq. (A.27), for our particular case, $U = \beta\varphi^\dagger\varphi$, $D_\mu = \partial_\mu + ig' B_\mu$, that is $A_\mu \to ig' B_\mu$ and $F_{\mu\nu} \to ig' B_{\mu\nu}$, and taking into account that $D^\mu(\varphi^\dagger\varphi) = \partial^\mu(\varphi^\dagger\varphi)$, $D^\mu B_{\mu\nu} = \partial^\mu B_{\mu\nu}$, we write this contribution to the effective Lagrangian as a sum of two parts:

$$\mathcal{L}^{(1)} = \mathcal{L}^{(1)}_{ren} + \mathcal{L}^{(1)}_{det} \; . \tag{3.13}$$

The first part, $\mathcal{L}^{(1)}_{ren}$, contains only operators with dimension not larger than four. It is given by the following expression:

$$\mathcal{L}^{(1)}_{ren} = \frac{1}{(4\pi)^2}\left(\frac{m^4}{2}\left(\frac{3}{2} + \Delta_\epsilon\right) + m^2\beta(1+\Delta_\epsilon)(\varphi^\dagger\varphi) + \frac{\beta^2}{2}\Delta_\epsilon(\varphi^\dagger\varphi)^2 - \frac{g'^2}{12}\Delta_\epsilon B_{\mu\nu}B^{\mu\nu}\right) \; . \tag{3.14}$$

All the coefficients in the above Lagrangian are ultraviolet-divergent. Here and in the rest of the paper we use dimensional regularization, where these divergences appear as simple poles, $1/\epsilon$, ($\epsilon = 2 - D/2$) in the function Δ_ϵ defined in eq. (A.29) of appendix A. As expected, all terms in eq. (3.14) are already present in the standard model Lagrangian. Therefore, they can be absorbed in a redefinition of the parameters of the standard model.

In the second part of eq. (3.13), $\mathcal{L}^{(1)}_{det}$, we included all dimension-six operators:

$$\begin{aligned}\mathcal{L}^{(1)}_{det} &= \frac{1}{m^2}\frac{1}{(4\pi)^2}\left(-\frac{\beta^3}{6}(\varphi^\dagger\varphi)^3 + \frac{\beta^2}{12}\partial_\mu(\varphi^\dagger\varphi)\partial^\mu(\varphi^\dagger\varphi)\right.\\ &\left. + \frac{g'^2\beta}{12}(\varphi^\dagger\varphi)B_{\mu\nu}B^{\mu\nu} - \frac{g'^2}{60}\partial^\mu B_{\mu\nu}\partial_\sigma B^{\sigma\nu}\right) \; .\end{aligned} \tag{3.15}$$

The coefficients of these operators are free of divergences and are suppressed by $1/m^2$. Note that the last term in eq. (A.27), which is trilinear in the field strength, is zero in the Abelian case.

It is clear that one could also obtain the interactions in eq. (3.14) and eq. (3.15) by using ordinary Feynman rules. In fig. 2 we give the diagrams that give rise to this part of the effective Lagrangian. Functional methods provide, however, a much more elegant and compact method to calculate them.

As mentioned before, all the terms in $\mathcal{L}^{(1)}_{ren}$ can be absorbed in a redefinition of the standard model parameters. The first term in eq. (3.14) is only a renormalization of the vacuum energy. We can neglect it, or, if we prefer, it can be absorbed in the constant term of the Higgs potential of the standard model. The second term renormalizes the quadratic coupling in the Higgs potential:

$$\bar{m}^2_\varphi = m^2_\varphi - m^2\frac{\beta}{(4\pi)^2}(1+\Delta_\epsilon) \; . \tag{3.16}$$

The third one is a renormalization of the quartic coupling of the standard Higgs:

$$\bar{\lambda} = \lambda - \frac{\beta^2}{2(4\pi)^2}\Delta_\epsilon. \tag{3.17}$$

Finally, the fourth term in eq. (3.14), which is an additional contribution to the kinetic term for the B_μ-field, can be removed by wave-function renormalization:

$$\bar{B}_\mu = \left(1 + \frac{g'^2\Delta_\epsilon}{3(4\pi)^2}\right)^{1/2} B_\mu \; . \tag{3.18}$$

In order to keep the canonical form of the covariant derivative, we have to renormalize also the $U(1)_Y$ coupling g' keeping the product $g'B_\mu$ invariant:

$$\bar{g}' = \left(1 + \frac{g'^2 \Delta_\epsilon}{3(4\pi)^2}\right)^{-1/2} g' \ . \tag{3.19}$$

The above equations (3.16), (3.17) and (3.19) relate the bare parameters of the full Lagrangian m_φ, λ, and g' to the corresponding bare parameters $\bar{m}_\varphi$, $\bar{\lambda}$ and $\bar{g}'$ of the effective Lagrangian.

4 Matching the effective Lagrangian to the full Lagrangian

As commented in the previous section, it is necessary to expand the operator of eq. (3.9) in powers of $1/m^2$ to have a local effective Lagrangian, but this expansion is only valid at the classical level, since the limit of $m \to \infty$ and the functional integration over the light fields do not commute. This is just a consequence of the fact that the momentum p of the scalar inside loop diagrams runs up to infinity and then an expansion in p^2/m^2 is not appropriate. In spite of this problem it is still possible to obtain a local effective Lagrangian as an expansion in $1/m^2$ [70, 71]. It is necessary, however, to include some additional operators that are not expected from the naïve expansion in eq. (3.9). These operators appear as a result of quantum corrections, hence, they are suppressed by additional couplings and $1/(4\pi)^2$ factors.

The practical way to obtain these terms of the Lagrangian is to consider the most general linear combination of all the operators with a given dimension, allowed by symmetry, and then to extract the coefficients of this combination by matching the effective Lagrangian calculation to the full theory calculation. Most of the operators obtained should be included in any case as counterterms to cancel the divergences generated when the tree-level Lagrangian of eq. (3.11) is used inside loops. But it is important to note that the finite parts of the coefficients of those operators can only be fixed by computing the various Green functions with both the full and the effective theories and matching the results for small energies compared with the heavy mass. Some other operators, however, appear with finite coefficients. They are just a consequence of the matching procedure and cannot be obtained from the divergences that appear in the effective theory. In the diagrammatic language all these new operators are generated by Feynman diagrams, with both the heavy scalar and lepton lines in the loops. We will compute in the full theory the amplitudes corresponding to these diagrams and will compare them with the equivalent amplitudes obtained by using the tree-level effective Lagrangian, eq. (3.11), in one-loop diagrams. The difference will give us the necessary counterterms. Actually, the effective Lagrangian calculation can be avoided by splitting the scalar propagator in two parts:

$$\frac{1}{k^2 - m^2} = -\frac{1}{m^2} + \frac{1}{m^2}\frac{k^2}{(k^2 - m^2)} \ . \tag{4.1}$$

If we use this form when calculating diagrams with only one scalar propagator, the first part gives exactly what one would obtain by using the effective tree-level four-fermion interaction, eq. (3.11), while the second part gives just the counterterm we should add to the effective Lagrangian. We would also like to note that by splitting the propagator in these two pieces we have increased the degree of ultraviolet divergence in each of the two terms, with respect to the original diagram. On the other hand, the second part contains an additional factor k^2 in the numerator, which reduces the infared degree of divergence of this contribution. Therefore, any possible small momentum singularity is transmitted to the low-energy effective Lagrangian, as it should be, since infared singularities have nothing to do with the high-energy behaviour of the theory.

A technical issue about the calculation is the selection of diagrams one should compute to reconstruct the full effective Lagrangian. The effective Lagrangian must be a linear combination of gauge-invariant operators. Each of the operators gives rise to a variety of physical processes. Obviously, to obtain the coefficients of these operators it is not necessary to compute all these processes, since many of them are related by gauge invariance. Our strategy consists in computing all one-particle-irreducible diagrams with the minimal number of external particles. We keep track of all external momenta (of course only to the order p^2/m^2) without using any equation of motion for the external particles. After that, we write an effective Lagrangian that reproduces those amplitudes and when there is no ambiguity, we reconstruct gauge invariance by promoting any derivative to a covariant derivative. Sometimes, however, there is some ambiguity in the promotion of a derivative to a covariant one. Then, we compute also the diagrams with one additional external gauge boson in order to disentangle that ambiguity and we check that the gauge-invariant effective Lagrangian correctly describes all the amplitudes. Finally, as an additional check, we compute some diagrams with three external gauge bosons by using both the full and the effective Lagrangians. We do it for the special case of zero external momenta[4]. Since the full calculation is tedious, we give the details in appendix B, presenting here only the main results and an explanation of the procedure.

4.1 Self-energies of the lepton-doublet

We give in fig. 3 the diagrams contributing to the lepton wave function renormalization in the full and the effective theories. In the effective theory the wave function renormalization is zero because it is a massless tadpole-like diagram (fig. 3.b). However, in the full theory there is a non-zero contribution that must be included as an additional effective operator. From the results of appendix B we find that, for the first term in eq. (B.1), the additional operator we should include is

$$2(\Delta_\epsilon + \frac{1}{2}) i(\bar{\ell} F \not{D} \ell) \ , \tag{4.2}$$

where we have defined the matrix F as

$$F_{ab} \equiv \frac{(f^\dagger f)_{ab}}{(4\pi)^2} \ . \tag{4.3}$$

Note that even though in appendix B we have only calculated the charged scalar contribution to the self-energy of the doublet, gauge invariance requires that the partial derivative should be substituted by a covariant derivative. In this case the promotion from the derivative to the covariant derivative can be done without ambiguity. This leads to eq. (4.2). It implies that there should be an additional contribution to the coupling of the gauge bosons to the lepton doublet given by the coefficient in eq. (4.2). We will see in the next subsection that, indeed, this contribution appears.

[4] We are studying a more efficient method for the calculation of the gauge-invariant effective Lagrangian, based on a calculation with constant fields similar to the one used in appendix A.

The second term in eq. (B.1) is proportional to $\not{p}p^2$, which requires an effective Lagrangian of the form $i(\overline{\ell}_a \not{\partial}\partial^2 \ell_b)$. However, the promotion of this term to covariant derivatives is ambiguous: should we use $\not{D}D^2$, $\not{D}^3$ or $D_\mu \not{D} D^\mu$? The only way to resolve this ambiguity, as we will see below, is to perform a full calculation with one external gauge boson.

The operator in eq. (4.2) contributes to the kinetic term of the doublets. To write it again in a canonical form, we should redefine the lepton doublet as follows:

$$\ell_a \to \ell_a - \frac{(\Delta_\epsilon + \frac{1}{2})}{(4\pi)^2}(f^\dagger f)_{ac}\ell_c \ . \tag{4.4}$$

This redefinition only affects the tree-level four-fermion effective Lagrangian and the standard model Yukawa coupling. This is because the wave function renormalization appears at the one-loop level, and, as a consequence, its effect on all the other (one-loop) operators is a two-loop effect. It can be taken into account by redefining the standard model Yukawa couplings and the couplings f_{ab} that appear in eq. (3.11) as follows

$$Y_{ab} \to \ \bar{Y}_{ab} = \ Y_{ab} - \frac{(\Delta_\epsilon + \frac{1}{2})}{(4\pi)^2}(f^\dagger f Y)_{ab} \ , \tag{4.5}$$

$$f_{ab} \to \ \bar{f}_{ab} = \ f_{ab} - 2\frac{(\Delta_\epsilon + \frac{1}{2})}{(4\pi)^2}(f f^\dagger f)_{ab} \ . \tag{4.6}$$

As we see this does not change the flavour structure of the couplings, since the $\bar{f}_{ab}$ is still antisymmetric in flavour indices and does not give any new interesting process. The same thing could be said about the standard model Yukawa couplings $\bar{Y}_{ab}$.

4.2 Penguins

In fig. 4 we present the contributing diagrams to the vertex of the B_μ and $\vec{W}_\mu$ gauge bosons with leptons, with the heavy scalar running in the loop (figs. 4.a and 4.b). Note that for the $\vec{W}_\mu$ there is no diagram fig. 4.b, because the scalar is an $SU(2)$ singlet. We also give the corresponding diagrams in the effective theory (fig. 4.c). Here it is interesting to note that the amplitude corresponding to the diagrams of figs. 4.a and 4.b in the full theory is finite, when the diagrams with external wave function renormalization are included. This can be understood because the full theory is renormalizable: since all these diagrams lead to a flavour-changing vertex and as our original Lagrangian does not contain such a coupling, there is no counterterm available. In the effective theory language, this cancellation is just a consequence of gauge invariance: the divergent contributions to the vertex in the full theory can be taken into account by the gauge-invariant dimension-four operator of eq. (4.2), which can be removed by a wave function renormalization of the lepton doublet.

We give in appendix B the resulting calculation, eqs. (B.2) and (B.3), of the vertex diagrams of the B_μ and $\vec{W}_\mu$ after subtraction of the effective Lagrangian contribution (diagram 4.c). We cast this result in a form that can be easily identified with some effective operators. The first term can already be obtained, as promised, from eq. (4.2). The rest of the terms can be obtained (for both the B_μ and the $\vec{W}_\mu$ fields) from the following operators:

$$-\frac{1}{3m^2} i(\overline{\ell} F (D^2 \not{D} + \not{D} D^2)\ell) \tag{4.7}$$

$$+\frac{2}{3m^2}\left(\Delta_\epsilon + \frac{5}{3}\right)\frac{g'}{m^2}(\overline{\ell}F\gamma_\nu \ell)D_\mu B^{\mu\nu} + \frac{2}{3m^2}\left(\Delta_\epsilon + \frac{4}{3}\right)\frac{g}{m^2}(\overline{\ell}F\gamma_\nu \vec{\tau}\ell)D_\mu \vec{W}^{\mu\nu} \tag{4.8}$$

$$+\frac{g'}{4m^2}B^{\mu\nu}(\overline{\ell}Fi\sigma_{\mu\nu}\not{D}\ell + \text{h.c.}) + \frac{g}{4m^2}\vec{W}^{\mu\nu}(\overline{\ell}Fi\sigma_{\mu\nu}\vec{\tau}\not{D}\ell + \text{h.c.}) \ . \tag{4.9}$$

In these equations, D_μ is the appropriate covariant derivative for each of the fields. It is easy to see that the pure derivative part of the first term eq. (4.7), reproduces perfectly the second term in eq. (B.1) and that the ambiguity mentioned in the previous subsection has been solved completely with this calculation. To check this result further, we have computed the diagram with three W_μ gauge bosons for zero external momenta and compared the result with that obtained from eqs. (4.7)–(4.9) for constant fields.

The physical content of these operators is quite obscure. Then, we found it convenient to use the operator identity:

$$\begin{aligned} \not{D}^3 &= \frac{1}{2}(D^2 \not{D} + \not{D} D^2) \\ &+\frac{1}{8}\left(g'\sigma_{\mu\nu}B^{\mu\nu}\not{D} + g'\not{D}\sigma_{\mu\nu}B^{\mu\nu} - g\sigma_{\mu\nu}\vec{W}^{\mu\nu}\vec{\tau}\not{D} - g\not{D}\sigma_{\mu\nu}\vec{W}^{\mu\nu}\vec{\tau}\right) \end{aligned} \tag{4.10}$$

to rewrite eqs. (4.7)–(4.9) in the following form:

$$-\frac{2}{3m^2}i(\bar{\ell}F\not{D}^3\ell) \tag{4.11}$$

$$+\frac{2}{3}\left(\Delta_\epsilon+\frac{5}{3}\right)\frac{g'}{m^2}(\bar{\ell}F\gamma_\nu\ell)D_\mu B^{\mu\nu} + \frac{2}{3}\left(\Delta_\epsilon+\frac{4}{3}\right)\frac{g}{m^2}(\bar{\ell}F\gamma_\nu\vec{\tau}\ell)D_\mu\vec{W}^{\mu\nu} \tag{4.12}$$

$$+\frac{g'}{3m^2}B^{\mu\nu}(\bar{\ell}Fi\sigma_{\mu\nu}\not{D}\ell + \text{h.c.}) + \frac{g}{6m^2}\vec{W}^{\mu\nu}(\bar{\ell}Fi\sigma_{\mu\nu}\vec{\tau}\not{D}\ell + \text{h.c.})\ . \tag{4.13}$$

As we will see later, after using the equations of motion, the physical content of these operators becomes more transparent. The first term can be removed in favour of Yukawa-type couplings and does not give any interesting process. The second and the third terms will give rise to processes such as $Z \to \mu e$ and $\mu \to eee$, and finally the last terms, with the appropriate combination of W_μ^3 and B_μ, will give rise to $\mu \to e\gamma$.

4.3 Seagull diagrams

There are also some operators that involve two scalar Higgses, two lepton doublets, and one covariant derivative. They are generated by the diagrams of fig. 5 and they do not contribute to any interesting process. For completeness we give the result here. From the diagram of fig. 5.b we get the amplitude eq. (B.4), which can be obtained by using the operator

$$i\frac{\beta}{m^2}(\varphi^\dagger\varphi)(\bar{\ell}F\not{D}\ell) + \text{h.c.}\ . \tag{4.14}$$

From the diagram of fig. 5.a, and after subtraction of the effective Lagrangian contribution, fig. 5.c, we get the amplitude (B.5), which can be obtained from the operators

$$\begin{aligned} &(\Delta_\epsilon+1)\frac{i}{m^2}\left((D_\mu\varphi)^\dagger\varphi\right)(\bar{\ell}\hat{F}\gamma^\mu\ell) + (\Delta_\epsilon+1)\frac{i}{m^2}\left((D_\mu\varphi)^\dagger\vec{\tau}\varphi\right)(\bar{\ell}\hat{F}\gamma^\mu\vec{\tau}\ell) \\ &-\frac{i}{2m^2}\left(\varphi^\dagger\varphi\right)(\bar{\ell}\hat{F}\not{D}\ell_b) - \frac{i}{2m^2}\left(\varphi^\dagger\vec{\tau}\varphi\right)(\bar{\ell}\hat{F}\vec{\tau}\not{D}\ell) + \text{h.c.}\ , \end{aligned} \tag{4.15}$$

where

$$\hat{F}_{ab} = \frac{(fY_eY_e^\dagger f^\dagger)_{ba}}{(4\pi)^2}\ . \tag{4.16}$$

We used an $SU(2)$ Fierz transformation to write the operators in this form, being the original operators expressed in terms of $\tilde{\varphi}$. Clearly the effects of operators (4.15) are suppressed by at least two powers of the lepton masses.

4.4 Four-fermion interactions

In this section we shall discuss possible new four-fermion interactions that could arise at the one-loop level. These interactions can come from three different kinds of diagrams:

i) Diagrams with gauge boson corrections to the vertex of the charged scalar with the lepton doublet. Since these only contain light particles in the loop, they are fully taken into account by gauge boson corrections to the effective four-lepton interaction (3.11). In addition they do not change the flavour structure of the coupling.

ii) Box diagrams with a charged scalar and a gauge boson running in the loop (fig. 6.a).

It is easy to see that diagrams with a $\vec{W}_\mu$ exchange just cancel because of the anti-symmetry of the couplings f_{ab}. Diagrams with exchange of B_μ renormalize the tree-level four-lepton interaction without changing its flavour structure. Part of their contributions are taken into account by the effective theory diagrams of fig. 6.c. But, obviously, these diagrams do not modify the flavour structure of the tree-level effective four-fermion interaction (3.11) and, then, they cannot give rise to processes such as $\mu \to eee$ at least at order $1/m^2$.

iii) Box diagrams with two charged scalars in the loop (fig. 6.b). Diagrams of this type are finite and in the sum they give rise to the following operator:

$$-\frac{(4\pi)^2}{m^2} F_{ab} F_{cd} (\overline{\ell}_a \gamma_\mu \ell_b)(\overline{\ell}_c \gamma^\mu \ell_d) \ . \tag{4.17}$$

It can be shown that this operator does not contribute to processes with two identical fermions, such as $\mu^- \to e^+ e^- e^-$. Indeed, taking only into account, for example, the down component of the $SU(2)$ lepton doublets, the interaction among four charged leptons induced by eq. (4.17) can be written as

$$-\frac{(4\pi)^2}{m^2} F_{ab} F_{cd} (\overline{e_{aL}} \gamma_\mu e_{bL})(\overline{e_{cL}} \gamma^\mu e_{dL}) = -\frac{(4\pi)^2}{2m^2} (F_{ab} F_{cd} - F_{ad} F_{cb})(\overline{e_{aL}} \gamma_\mu e_{bL})(\overline{e_{cL}} \gamma^\mu e_{dL}) \ , \tag{4.18}$$

where, to obtain the right-hand side, we have used a Fierz transformation. We immediately see that this operator does not contribute to processes such as $\mu^- \to e^+ e^- e^-$. However it does contribute to other interesting processes such as $\tau^- \to e^+ \mu^- e^-$.

Clearly only the contributions of type (iii) are interesting and only these will be included in our effective Lagrangian.

Before we summarize the results of section 4, we note that all the results obtained in section 4 have been derived from the effective tree-level Lagrangian given in eq. (3.11). One can easily see that this Lagrangian can also be written (after a combined Fierz transformation in $SU(2)$ and Dirac space) in the more familiar form

$$\mathcal{L}^{(0)} \text{ “} = \text{”} \ \frac{1}{m^2} f_{ab} f^*_{cd} (\overline{\ell}_d \gamma^\mu \ell_b)(\overline{\ell}_c \gamma_\mu \ell_a) \ . \tag{4.19}$$

But this is true only in four dimensions: the Fierz transformations we used above cannot be done in D dimensions. Should we start with this new Lagrangian, the finite parts of the counterterms would be different when using dimensional regularization, but this formulation would be completely equivalent to the original one. We decided to keep the original form of the Lagrangian so as to keep the notation as close as possible to the full Lagrangian formulation.

Using the results of sections 3 and 4, the complete one-loop effective Lagrangian is

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \mathcal{L}^{(0)} + \mathcal{L}^{(1)}_{det} + \mathcal{L}^{(1)}_{match} \ . \tag{4.20}$$

Here $\mathcal{L}_{SM}$ is the standard model Lagrangian, eq. (2.2), $\mathcal{L}^{(0)}$ as given in eq. (3.11) is the tree-level $1/m^2$ term of the effective Lagrangian, $\mathcal{L}^{(1)}_{det}$ is given in eq. (3.15) and contains all $1/m^2$ one-loop contributions of diagrams with only singlet scalars in the loop and finally $\mathcal{L}^{(1)}_{match}$, which is the sum of the terms in eqs. (4.11)–(4.15) and (4.17), gives all the contributions coming from matching the full theory in diagrams with heavy-light particles in the loops:

$$\begin{aligned}
\mathcal{L}^{(1)}_{match} &= \frac{2}{3}(\Delta_\epsilon + \frac{5}{3})\frac{g'}{m^2}(\bar{\ell}F\gamma_\nu\ell)D_\mu B^{\mu\nu} + \frac{2}{3}(\Delta_\epsilon + \frac{4}{3})\frac{g}{m^2}(\bar{\ell}F\gamma_\nu\vec{\tau}\ell)D_\mu\vec{W}^{\mu\nu} \\
&+ \frac{1}{3}\frac{g'}{m^2}B^{\mu\nu}(\bar{\ell}Fi\sigma_{\mu\nu}\not{D}\ell + \text{h.c.}) + \frac{1}{6}\frac{g}{m^2}\vec{W}^{\mu\nu}(\bar{\ell}Fi\sigma_{\mu\nu}\vec{\tau}\not{D}\ell + \text{h.c.}) \\
&+ \left(i\frac{\beta}{m^2}(\varphi^\dagger\varphi)(\bar{\ell}F\not{D}\ell) + \text{h.c.}\right) - \frac{2}{3m^2}i(\bar{\ell}F\not{D}^3\ell) \\
&+ \Bigg((\Delta_\epsilon + 1)\frac{i}{m^2}\left((D_\mu\varphi)^\dagger\varphi\right)(\bar{\ell}\hat{F}\gamma^\mu\ell) + (\Delta_\epsilon + 1)\frac{i}{m^2}\left((D_\mu\varphi)^\dagger\vec{\tau}\varphi\right)(\bar{\ell}\hat{F}\gamma^\mu\vec{\tau}\ell) \\
&- \frac{i}{2m^2}\left(\varphi^\dagger\varphi\right)(\bar{\ell}\hat{F}\not{D}\ell) - \frac{i}{2m^2}\left(\varphi^\dagger\vec{\tau}\varphi\right)(\bar{\ell}\hat{F}\vec{\tau}\not{D}\ell) + \text{h.c.}\Bigg) \\
&- \frac{(4\pi)^2}{m^2}(\bar{\ell}F\gamma_\mu\ell)(\bar{\ell}F\gamma^\mu\ell)\ . \qquad (4.21)
\end{aligned}$$

All the couplings in this Lagrangian should be understood as bare barred effective Lagrangian couplings related to the full theory bare couplings through eqs. (3.16), (3.17), (3.19), (4.5) and (4.6). To simplify the notation we have suppressed everywhere generational indices, then F and $\hat{F}$ are the matrices defined in eqs. (4.3) and (4.16) .

5 Renormalization and operator mixing analysis of the effective Lagrangian

Equations (3.16), (3.17), (3.19), (4.5) and (4.6) relate the bare couplings and masses of the effective theory with those of the full theory. It is, however, more interesting to have the equivalent relations for renormalized couplings. If an $\overline{MS}$ scheme[5] is employed to renormalize both the full and the effective theories, we can very easily obtain the matching equations for the renormalized couplings. Let us take, for example, the gauge coupling g'. We denote the $\overline{MS}$ renormalized quantities with the same symbol as the bare quantities, but adding an additional dependence on the renormalization scale μ. All effective theory quantities will be distinguished by a bar. The standard relationship between bare and renormalized couplings in $\overline{MS}$ schemes is ($D = 4 - 2\epsilon$ and $\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} - \gamma + \log(4\pi)$):

$$\begin{aligned}
g'\mu^\epsilon &= g'(\mu) + \frac{1}{2\hat{\epsilon}}\, b_{g'}g'^3(\mu) + \cdots\ , \\
\bar{g}'\mu^\epsilon &= \bar{g}'(\mu) + \frac{1}{2\hat{\epsilon}}\, \bar{b}_{g'}\bar{g}'^3(\mu) + \cdots\ ,
\end{aligned}$$

where $b_{g'}$ and $\bar{b}_{g'}$ are the lowest-order coefficients of the β-functions for the coupling constants in the full and the effective theories, respectively. Substituting these equations in

[5]It has been customary in the recent literature to use cut-off regularization schemes when working with effective Lagrangians and then identify the cut-off with the scale of new physics. The interpretation of the ultraviolet cut-off as the scale of new physics has led to some erroneous results. The effective Lagrangian should also be renormalized, and physical results should be independent of the regularization and of the renormalization schemes. We will see later that dimensional regularization with minimal subtraction leads to the same physical results in a cleaner way; therefore, we will always use this scheme.

eq. (3.19) and equating finite terms [7, 72] we obtain the desired matching condition for renormalized couplings:

$$\bar{g}'(\mu) = g'(\mu) - \frac{g'(\mu)^3}{3(4\pi)^2}\log(\mu/m) + \cdots \quad . \tag{5.1}$$

Note that this equation can be obtained by just dropping the $1/\hat{\epsilon}$ contained in Δ_ϵ in eq. (3.19). This is not surprising since the divergent term in eq. (3.19) gives just the charged scalar contribution to the beta function of g' in the full theory. Similar arguments give for the rest of the couplings and masses the following result

$$\bar{m}^2_\varphi(\mu) = m^2_\varphi(\mu) - m^2(\mu)\frac{\beta(\mu)}{(4\pi)^2}(1 + 2\log(\mu/m)) \ , \tag{5.2}$$

$$\bar{\lambda}(\mu) = \lambda(\mu) - \frac{\beta^2(\mu)}{(4\pi)^2}\log(\mu/m) \ , \tag{5.3}$$

$$\bar{Y}_{ab}(\mu) = Y_{ab}(\mu) - \frac{(f^\dagger f Y)_{ab}(\mu)}{(4\pi)^2}(\frac{1}{2} + 2\log(\mu/m)) \tag{5.4}$$

and a matching equation for the four-lepton coupling in the effective theory expressed in terms of the full theory couplings and masses. It can be obtained by using an effective $\bar{f}_{ab}(\mu)$ defined as follows:

$$\bar{f}_{ab}(\mu) = f_{ab}(\mu) - 2\frac{(f f^\dagger f)_{ab}(\mu)}{(4\pi)^2}\left(\frac{1}{2} + 2\log(\mu/m)\right) + \cdots \quad . \tag{5.5}$$

In this equation there could also be some contributions from box diagrams, which we have not computed.

These matching conditions are, in principle, valid for an arbitrary value of the renormalization scale μ. However, it is clear that in order to avoid large logarithms they should be evaluated at some scale around the charged scalar mass and then, using the standard model renormalization group, run all the couplings to obtain their values at lower scales.

Equation (5.2) is very interesting, since it manifests in all its crudeness the so-called naturalness problem of the standard model; $\bar{m}_\varphi(\mu)$ is the mass parameter that appears in the Higgs potential part of the effective Lagrangian, and it has to be of the order of the electroweak scale. However, it is clear that if we take $m(\mu)$ very large, we should also take $m_\varphi(\mu)$ large in order to have $\bar{m}_\varphi(\mu)$ small enough. But even if we do so at some scale μ, it will be very difficult to keep $\bar{m}_\varphi(\mu)$ small at any other scale. This represents a serious fine-tuning problem, which appears when the standard model is embedded in another model containing mass scales much larger than the Fermi scale. It is important to note that by using dimensional regularization the problem appears only in the matching conditions. If a cut-off regularization scheme is used, the naturalness problem can be related to the appearance of quadratic divergences. However, this relation is not direct. Quadratic divergences can appear at the intermediate stages, but they should be absorbed in a renormalization of the mass parameters of both the full and the effective theories, since physical quantities must be cut-off-independent. After renormalization, the naturalness problem should also appear only as a fine-tuning problem in the matching conditions, as in eq. (5.2). As the main physical consequences can be obtained equally in dimensional regularization, we do not find any particular advantage by working with a cut-off regularization scheme.

If an $\overline{MS}$ scheme is used to renormalize also the higher dimension operators, we could write the effective Lagrangian in terms of a set of running couplings $c_i(\mu)$ for each of the operators

$$\mathcal{L}_{eff} = \sum_i c_i(\mu) O_i \ , \tag{5.6}$$

where the $c_i(\mu)$ are obtained from the bare coefficients in eq. (4.21) just by dropping the $1/\epsilon - \gamma + \log(4\pi)$ terms in Δ_ϵ; that is just by substituting Δ_ϵ by $2\log(\mu/m)$. Then all couplings should be understood as renormalized effective Lagrangian couplings (barred couplings) related to the full theory couplings through eqs. (5.1)–(5.5).

To simplify matters we can consider only processes that violate muon-lepton number in one unit and electron-lepton number in minus one unit (the total lepton number must be conserved). For example, we can take the two operators

$$\begin{aligned} O_1 &\equiv (\overline{\tilde{\ell}}_3 \ell_2)(\overline{\ell}_1 \tilde{\ell}_3) \\ O_2 &\equiv \frac{1}{g'} D_\mu B^{\mu\nu} (\overline{\ell}_1 \gamma_\nu \ell_2) \end{aligned}$$

The factor $1/g'$ in the penguin operator has been included for convenience. From eqs. (3.11) and (4.8), we obtain expressions for the corresponding running couplings:

$$c_1(\mu) = \frac{(f^\dagger f)_{12}(\mu)}{m^2} + \cdots \tag{5.7}$$

$$c_2(\mu) = \frac{(f^\dagger f)_{12}(\mu)}{m^2} \frac{4g'^2(\mu)}{3(4\pi)^2} \log\frac{\mu}{m} + \frac{(f^\dagger f)_{12}(\mu)}{m^2} \frac{10 g'^2(\mu)}{9(4\pi)^2} + \cdots \ , \tag{5.8}$$

where the dots represent additional one-loop contributions, which, as we will see immediately, are not important at the level we are working. Note that $(f^\dagger f)_{12} = f^*_{31} f_{32}$ because of the antisymmetry of the couplings in flavour indices. Equations (5.7)–(5.8) can be cast into the following form

$$c_1(\mu) = c_1(m)\left(1 + \gamma_{11}\log\frac{\mu}{m}\right) + c_2(m)\gamma_{12}\log\frac{\mu}{m} \tag{5.9}$$

$$c_2(\mu) = c_1(m)\gamma_{21}\log\frac{\mu}{m} + c_2(m)\left(1 + \gamma_{22}\log\frac{\mu}{m}\right) \ , \tag{5.10}$$

with

$$\gamma_{11} \approx O\left(\frac{g^2}{(4\pi)^2}\right), \quad \gamma_{12} \approx O\left(\frac{g^2}{(4\pi)^2}\right), \quad \gamma_{21} = \frac{4}{3}\frac{g'^2}{(4\pi)^2}, \quad \gamma_{22} \approx O\left(\frac{g^2}{(4\pi)^2}\right) \tag{5.11}$$

and the couplings at the scale $\mu = m$ are given by

$$c_1(m) = \frac{(f^\dagger f)_{12}(m)}{m^2}, \qquad c_2(m) = \frac{10}{9}\frac{g'^2}{(4\pi)^2}\frac{(f^\dagger f)_{12}(m)}{m^2} \ . \tag{5.12}$$

Since $c_2(m)$ appears only at the one-loop level in the full model, the last term in eq. (5.9) is a two-loop effect. This is the reason why we have not considered it in eq. (5.7). On the other hand, if μ is not very different from m we have $\gamma_{11}\log(\mu/m) \ll 1$ and $\gamma_{22}\log(\mu/m) \ll 1$; then, the diagonal elements of the anomalous dimension matrix do not need to be computed.

Equations (5.9) and (5.10) are an approximate solution of the general renormalization group equation that describes mixing among the operators O_1 and O_2

$$\mu\frac{dc_i(\mu)}{d\mu} = \gamma_{ij} c_j(\mu) \ , \tag{5.13}$$

which is valid only in the case that $\gamma_{ij} \log(\mu/m) \ll 1$. If we pretend to use this solution for a wider range of μ's, the full γ_{ij} matrix should be computed and the integration of eq. (5.13) should be done by taking also into account the running of the gauge coupling constants. However, for the standard model values of the gauge coupling constants the linear approximation ($\gamma_{ij}\ \log(\mu/m) \ll\ 1$) works very well for a wide range of scales ($\gamma_{ij} \sim 10^{-3}$). Obviously, in this approximation and taking into account that only the operator in eq. (3.11) is generated at tree level it is clear that in our model it is enough to consider only mixing of the effective operators generated at the one-loop level with the tree-level operator because other mixings would represent two-loop effects. Then, at the order we are working, we have always 2×2 operator mixing.

This simple example shows us clearly what can and what cannot be obtained by using the effective Lagrangian. From the effective theory we could calculate the anomalous dimension matrix that controls the mixing among the different operators in the effective Lagrangian since the logarithmic terms are the same in the full and in the effective theories, however the boundary conditions for the renormalization group equation (5.13) can only be obtained after the matching procedure. If we do not know the full theory a set of boundary conditions can only be obtained from experiment. In this case we can still compute the anomalous dimensions in the effective theory, write down eq. (5.13), and solve it for arbitrary boundary conditions at the scale μ_0. Then the solution, within our previous approximations, is

$$
\begin{aligned}
c_1(\mu) &= c_1(\mu_0)\left(1+\gamma_{11}\log\frac{\mu}{\mu_0}\right)+c_2(\mu_0)\gamma_{12}\log\frac{\mu}{\mu_0} \\
c_2(\mu) &= c_1(\mu_0)\gamma_{21}\log\frac{\mu}{\mu_0}+c_2(\mu_0)\left(1+\gamma_{22}\log\frac{\mu}{\mu_0}\right)\ . \qquad (5.14)
\end{aligned}
$$

Now we should take the full, complete basis of operators that mix at the one-loop level, since if the boundary conditions are not known we cannot neglect *a priori* any of the mixings. To keep simplicity, however, we will discuss only two-operator mixing. The interesting point about eqs. (5.14) is that, if we know all the effective couplings at some scale μ_0, we can obtain them at some other scale (not necessarily the scale responsible for the new physics). Equations (5.14) clearly tell us that to predict the values of the couplings at any scale we need two (in the case of mixing of two operators) boundary conditions. These conditions could be obtained from experiment. For example, if we could bound the couplings of the two operators at LEP1 we could straightforwardly predict these couplings at LEP2 energies. However, if in our example we only know one of the couplings at LEP1, no matter how many loops we use to calculate the anomalous dimensions, we will not be able to bound the other operator at the same or any other scale. Only if the full theory is known can we relate the couplings of different operators as they are expressed in terms of the few parameters of the full theory. Then, bounds on one operator can be related to bounds on other operators. In our example it is clear from eq. (5.12) that

$$
c_2(m) = \frac{10}{9}\frac{g'^2}{(4\pi)^2}c_1(m) \qquad (5.15)
$$

or, for instance at the M_Z scale, by using eqs. (5.7)–(5.8) we have

$$
c_2(M_Z) = \left(\frac{4g'^2(M_Z)}{3(4\pi)^2}\log(M_Z/m)+\frac{10g'^2(M_Z)}{9(4\pi)^2}\right)c_1(M_Z)\ . \qquad (5.16)
$$

Then it is clear that a bound on $c_1(M_Z)$ implies a bound on $c_2(M_Z)$, and viceversa, and it also implies a bound on $c_1(\mu)$ and $c_2(\mu)$ at any other scale μ. But it is important to realize that this is only possible because our full theory provides us with additional relationships among couplings. Of course, one can always impose additional assumptions to obtain an estimate of the bounds on different operators.

For example, if one assumes that there are no unnatural cancellations among the couplings of different operators one can bound each of them independently from the others. In our example this would be implemented by putting, for instance, $c_1(m) \neq 0$ and $c_2(m) = 0$, then use eqs. (5.9) and (5.10) with $\mu = M_Z$ to obtain $c_1(M_Z)$ and $c_2(M_Z)$ as a function of $c_1(m)$. Clearly with this assumptions a bound on $c_2(M_Z)$ implies a bound on $c_1(m)$ and therefore also a bound on $c_1(M_Z)$. And similarly for the other coupling. The estimates so obtained are interesting and can be useful. However, we feel that they can never substitute more elaborate bounds based on a complete set of experimental data.

6 Spontaneous symmetry breaking and the use of the equations of motion

From all previous sections it is clear that our effective Lagrangian reproduces the results of the full theory at the one-loop level as long as energies smaller than the h mass are involved. However, until now we have considered the full unbroken theory, but the standard model and the extension of it we are considering now undergo spontaneous symmetry breaking (SSB) and this has to be taken into account in the analysis.

Another point that has been the origin of controversy is the possibility of using the standard model equations of motion to reduce the number of operators in the effective Lagrangian or to make its physical content more evident. It is well known that the equations of motion cannot be used naïvely in a Lagrangian if it is going to be used to calculate diagrams with particles off-the-mass-shell. In fact, the possibility of using the equations of motion to trade derivative couplings in favour of pseudo-scalar couplings has led to many wrong calculations in pion physics and in other models containing Goldstone bosons. However, the equations of motion can still be used under certain circumstances. First of all, if the Lagrangian is going to be used to calculate tree-level amplitudes with all particles on-the-mass-shell, they can be used without any problem. Moreover it can be shown [12, 73, 74] that the use of the equations of motion is equivalent to a redefinition (in general with a non-linear transformation) of the fields of the theory. If the starting theory is renormalizable, normally these non-linear transformations lead to Lagrangians that contain operators of higher dimension and which are not explicitly renormalizable. However, if the starting theory is described by an effective Lagrangian, with all kinds of non-renormalizable interactions, the effect of these transformations is equivalent to the use of the equations of motion plus a modification of the higher-order terms in the effective Lagrangian. This is not a problem since all these terms are already present in the effective Lagrangian. Then, it only amounts to a re-ordering of the effective Lagrangian [12]. There are some subtleties related to the Jacobian of the transformation and renormalization, but in general it can be shown that they do not represent a problem [73, 74]. However, one has to be careful when studying processes that are sensitive to operators of different dimensions. In our case there is no problem since the only operators that could be eliminated with the equations of motion appear at the one-loop level; then, they are going to be used uniquely as tree-level insertions. We could use the equations of motion before or after SSB, or not use them at all, but the final result should be independent from this. Intermediate steps, however, can look quite different. We found it convenient to use the equations of motion of the standard model before SSB, because they are simpler and because it makes it easier to see which new physics is generated by the effective Lagrangian. We are going to use them to substitute the derivatives of field strengths in favour of fermionic currents and new gauge boson interactions involving the standard Higgs scalar. Similarly, we will substitute covariant derivatives of the lepton doublet in favour of the right-handed leptons and the standard Higgs. Later on we will see, in a particular example, that physical observables are the same as the results one would have obtained if proceeding in a different way.

From the standard model Lagrangian, eq. (2.2), one can easily obtain the following equations of motion for the electroweak gauge bosons

$$D_\mu B^{\mu\nu} = -g'\left(J_B^\nu + \frac{i}{2}\varphi^\dagger \overleftrightarrow{D}^\nu \varphi\right) \tag{6.1}$$

$$D_\mu \vec{W}^{\mu\nu} = -g\left(\vec{J}_W^\nu + \frac{i}{2}\varphi^\dagger \overleftrightarrow{D}^\nu \vec{\tau}\varphi\right) , \tag{6.2}$$

where

$$J_B^\nu = -\frac{1}{2}\bar{\ell}\gamma^\nu\ell - \bar{e}\gamma^\nu e + \frac{1}{6}\bar{q}\gamma^\nu q + \frac{2}{3}\bar{u}\gamma^\nu u - \frac{1}{3}\bar{d}\gamma^\nu d \tag{6.3}$$

$$\vec{J}_W^\nu = \frac{1}{2}\bar{\ell}\gamma^\nu\vec{\tau}\ell + \frac{1}{2}\bar{q}\gamma^\nu\vec{\tau}q \tag{6.4}$$

are the hypercharge and $SU(2)$ fermionic currents[6]. We are also going to use the standard model equation of motion for the leptonic doublet

$$i \not{D}\ell = -Y_e e\varphi . \tag{6.5}$$

We will use eqs. (6.1)–(6.2) and (6.5) in (4.21). Every time we use the equation of motion for the leptonic doublet we obtain an additional leptonic Yukawa coupling constant Y_e. Since these couplings are small, we are going to neglect terms that contain two or more of those Yukawa couplings.

After using (6.1), (6.2) and (6.5) in $\mathcal{L}^{(1)} = \mathcal{L}^{(1)}_{det} + \mathcal{L}^{(1)}_{match}$, the resulting Lagrangian is

$$\begin{aligned}
\mathcal{L}^{(1)} = {} & \frac{1}{m^2}\frac{1}{(4\pi)^2}\left(-\frac{\beta^3}{6}(\varphi^\dagger\varphi)^3 + \frac{\beta^2}{12}\partial_\mu(\varphi^\dagger\varphi)\partial^\mu(\varphi^\dagger\varphi) + \frac{g'^2\beta}{12}(\varphi^\dagger\varphi)B_{\mu\nu}B^{\mu\nu}\right. && (6.6)\\
& \left.-\frac{g'^4}{60}\left(J_B^\mu J_{B\mu} + i(\varphi^\dagger\overleftrightarrow{D}_\mu\varphi)J_B^\mu - \frac{1}{4}(\varphi^\dagger\overleftrightarrow{D}^\mu\varphi)(\varphi^\dagger\overleftrightarrow{D}_\mu\varphi)\right)\right) && (6.7)\\
& -\frac{2}{3}\left(\Delta_\epsilon + \frac{5}{3}\right)\frac{g'^2}{m^2}(J_B^\mu + \frac{i}{2}(\varphi^\dagger\overleftrightarrow{D}^\mu\varphi))(\bar{\ell}F\gamma_\mu\ell) && (6.8)\\
& -\frac{2}{3}\left(\Delta_\epsilon + \frac{4}{3}\right)\frac{g^2}{m^2}(\vec{J}_W^\mu + \frac{i}{2}(\varphi^\dagger\overleftrightarrow{D}^\mu\vec{\tau}\varphi))(\bar{\ell}F\gamma_\mu\vec{\tau}\ell) && (6.9)\\
& -\frac{g'}{3m^2}B^{\mu\nu}(\bar{\ell}FY_e\sigma_{\mu\nu}e\varphi + \text{h.c.}) - \frac{g}{6m^2}\vec{W}^{\mu\nu}(\bar{\ell}FY_e\sigma_{\mu\nu}\vec{\tau}e\varphi + \text{h.c.}) && (6.10)\\
& +\left(\frac{\beta}{m^2}(\varphi^\dagger\varphi)(\bar{\ell}FY_e e\varphi) + \text{h.c.}\right) && (6.11)\\
& -\frac{(4\pi)^2}{m^2}(\bar{\ell}F\gamma_\mu\ell)(\bar{\ell}F\gamma^\mu\ell) . && (6.12)
\end{aligned}$$

The terms from eq. (4.21) containing $\not{D}^3$ and $\hat{F}$ have been neglected as they are suppressed by two or more leptonic Yukawa couplings. Again all couplings must be understood as

[6] Note that eqs. (6.1)–(6.4) differ from eqs. (2.13) and (2.14) in ref. [40], where some terms are missing.

bare effective Lagrangian couplings (barred couplings). We can write eqs. (6.6)–(6.12) in terms of $\overline{MS}$ renormalized couplings by just dropping the divergent contributions ($\Delta_\epsilon \to 2\log(\mu/m)$).

It is clear that after using the equations of motion one can easily see the plethora of interesting processes that a single term in eq. (4.21) can generate.

We will first study how our effective Lagrangian can modify the standard model mechanism for SSB and what its effects are on the spectrum of physical gauge bosons.

The first term in (6.6) modifies the standard model Higgs potential and produces a shift in the vacuum expectation value (VEV). Since this is a one-loop effect one should also use the full one-loop standard model effective potential to analyse this shift. This gives a global redefinition of the VEV and has only consequences in the Higgs sector of the theory: the ratio of the Higgs to the W mass and the Higgs couplings are changed, but there are no other effects. As we are not interested in Higgs physics here we are not going to compute this shift.

The Higgs dependent terms of the effective Lagrangian can have interesting effects when the Higgs develops a VEV:

$$\langle\varphi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{6.13}$$

In what follows we will substitute the Higgs field by its VEV and will neglect all the terms containing physical Higgses.

The second term in eq. (6.6), after SSB, produces a wave function renormalization of the Higgs scalar. Again, since we are not interested in Higgs interactions here we are going to neglect it.

The third term clearly gives a wave function renormalization of the B_μ gauge boson. Then, one has to diagonalize simultaneously the kinetic term and the mass terms for the gauge bosons. After SSB we get the following kinetic term for the B_μ gauge boson

$$-\frac{1}{4}(1-\delta_m)B_{\mu\nu}B^{\mu\nu} \ , \tag{6.14}$$

where

$$\delta_m \equiv \frac{g'^2\beta}{3(4\pi)^2}\frac{v^2}{m^2} \ . \tag{6.15}$$

The W gauge boson kinetic term and the mass terms remain unchanged. Then we can recover the canonical kinetic term by redefining the B_μ field as follows

$$B_\mu \to (1-\delta_m)^{-1/2}B_\mu \ ; \tag{6.16}$$

this will change the mass matrix of the gauge bosons. However as the B_μ field comes always with g' we can recover the standard model form by redefining g' as well

$$\hat{g}' \equiv (1-\delta_m)^{-1/2}g' \ . \tag{6.17}$$

It is not difficult to see that after SSB the last term in eq. (6.7) gives

$$\frac{1}{4}v^2\frac{\hat{g}'^4}{60(4\pi)^2}\frac{v^2}{m^2}(gW^{3\mu}-\hat{g}'B^\mu)(gW^3_\mu-\hat{g}'B_\mu) \ , \tag{6.18}$$

where we have already included the redefinition of g' from eq. (6.17). This term is an additional contribution to the neutral gauge bosons mass term. Adding it to the standard model contribution and diagonalizing the full gauge boson mass matrix, we find that the

W^3_μ and B_μ gauge bosons can be expressed in terms of the physical photon A_μ,and the Z-boson Z_μ as follows:

$$\begin{aligned} W^3_\mu &= \hat{s}_W A_\mu + \hat{c}_W Z_\mu \\ B_\mu &= \hat{c}_W A_\mu - \hat{s}_W Z_\mu \ , \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} \hat{s}_W &= \frac{\hat{g}'}{\sqrt{g^2+\hat{g}'^2}} \approx s_W\left(1+\frac{1}{2}c_W^2\delta_m\right) \\ \hat{c}_W &= \frac{g}{\sqrt{g^2+\hat{g}'^2}} \approx c_W\left(1-\frac{1}{2}s_W^2\delta_m\right) \end{aligned} \tag{6.20}$$

and

$$s_W = \frac{g'}{\sqrt{g^2+g'^2}} \tag{6.21}$$

is the tree-level weak mixing angle in the pure standard model. Similarly the relation between the electric charge and the gauge coupling is modified in comparison with the standard model one:

$$e = \hat{g}'\hat{c}_W = g\hat{s}_W \approx g s_W\left(1-\frac{1}{2}c_W^2\delta_m\right) \ . \tag{6.22}$$

With the same notation the physical masses are

$$\begin{aligned} m_W^2 &= \frac{1}{2}g^2v^2 \\ m_Z^2 &= \frac{m_W^2}{\hat{c}_W^2}(1+\delta_Z) \approx \frac{m_W^2}{c_W^2}(1+s_W^2\delta_m+\delta_Z) \ , \end{aligned} \tag{6.23}$$

where

$$\delta_Z = \frac{\hat{g}'^4}{60(4\pi)^2}\frac{v^2}{m^2} \tag{6.24}$$

comes from eq. (6.18).

From these equations it is clear that only the δ_Z correction, which produces a relative shift form the standard relation between the masses of the Z and the W could in principle be observed; however, it is very small. All the other shifts related to δ_m can be absorbed in the definition of the coupling $\hat{g}'$ or, equivalently, in the weak mixing angle $\hat{s}_W$ and then are not observable.

Now we can go back to eqs. (6.6)–(6.12) and write them in terms of the physical gauge bosons. The result we have found (neglecting all Higgs interactions) is:

$$\begin{aligned} \mathcal{L}^{(1)} &= -\frac{g^2}{2m_W^2}\delta_Z(c_W^2 J_A^\mu - J_Z^\mu)(c_W^2 J_{A\mu} - J_{Z\mu}) + \frac{g}{\hat{c}_W}\delta_Z Z_\mu (c_W^2 J_A^\mu - J_Z^\mu) && (6.25) \\ &+\frac{2}{3}\frac{g}{m^2\hat{c}_W}\left(-(1-2\hat{s}_W^2)\left(\Delta_\epsilon+\frac{4}{3}\right)+\hat{s}_W^2\frac{1}{3}\right)\left(M_Z^2 Z^\mu + \frac{g}{\hat{c}_W}J_Z^\mu\right)(\overline{\nu_L}F\gamma_\mu\nu_L) && (6.26) \\ &+\frac{2}{3}\frac{g}{m^2\hat{c}_W}\left(\left(\Delta_\epsilon+\frac{4}{3}\right)+\hat{s}_W^2\frac{1}{3}\right)\left(M_Z^2 Z^\mu + \frac{g}{\hat{c}_W}J_Z^\mu\right)(\overline{e_L}F\gamma_\mu e_L) && (6.27) \\ &-\frac{2}{3}\left(\Delta_\epsilon+\frac{4}{3}\right)\frac{g}{m^2}\left((\sqrt{2}M_W^2 W_\mu^+ + J_\mu^\dagger)(\overline{\nu_L}F\gamma^\mu e_L) + \text{h.c.}\right) && (6.28) \\ &-\frac{2}{9}\frac{e^2}{m^2}J_A^\mu(\overline{e}F\gamma_\mu e) \ -\frac{2}{3}\frac{e^2}{m^2}\left(\Delta_\epsilon+\frac{5}{3}\right)J_A^\mu(\overline{\nu_L}F\gamma_\mu\nu_L) && (6.29) \end{aligned}$$

$$-\frac{1}{6}\frac{e}{m^2}A^{\mu\nu}\left((\overline{e_L}FM_e\sigma_{\mu\nu}e_R)+\text{h.c.}\right) \tag{6.30}$$

$$+\frac{1}{6}\frac{g}{m^2\hat{c}_W}(1+\hat{s}_W^2)Z^{\mu\nu}\left((\overline{e_L}FM_e\sigma_{\mu\nu}e_R)+\text{h.c.}\right) \tag{6.31}$$

$$-\frac{1}{3\sqrt{2}}\frac{g}{m^2}\left(W^+_{\mu\nu}(\overline{\nu_L}FM_e\sigma_{\mu\nu}e_R)+\text{h.c.}\right) \tag{6.32}$$

$$-\frac{(4\pi)^2}{m^2}\left((\overline{e_L}F\gamma^\mu e_L)(\overline{e_L}F\gamma_\mu e_L)+(\overline{\nu_L}F\gamma^\mu\nu_L)(\overline{\nu_L}F\gamma_\mu\nu_L)\right.$$
$$\left.+\,2(\overline{e_L}F\gamma^\mu e_L)(\overline{\nu_L}F\gamma_\mu\nu_L)\right) \tag{6.33}$$

Here $M_e = Y_e v$ is the charged lepton mass matrix and $A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$ are the field strengths of the photon and the Z gauge boson, respectively. ν_L and e_L are the components of the left-handed lepton doublet and e_R are the right-handed parts of the charged leptons. We have also defined the following neutral currents:

$$J_A^\mu = \sum_f Q_f \overline{f}\gamma^\mu f$$
$$J_Z^\mu = \sum_f \overline{f}(v_f - a_f\gamma_5)\gamma^\mu f \tag{6.34}$$

where

$$v_f = \frac{1}{2}T_f^3 - s_W^2 Q_f, \qquad a_f = -\frac{1}{2}T_f^3 \tag{6.35}$$

and $J_\mu^\dagger$ is the usual charged current. Couplings with more than one gauge boson have not been included, and the term (6.11) has been removed because after SSB it can be absorbed in a renormalization of the charged lepton mass matrix.

As we have seen, by using the equations of motion we have obtained a Lagrangian that displays its physical content in a very transparent way. However, as stressed before, the use of the equations of motion is not necessary. We could have started with the unbroken Lagrangian, eqs. (4.21) and (3.15), and then let the Higgs field acquire a VEV without using the equations of motion at all. After diagonalization of the gauge boson mass matrices we would have obtained a quite different effective Lagrangian from that in eqs. (6.25)–(6.33). However, both Lagrangians give the same physics (at least at the level of precision we are working here). For example, the penguin operators in $\mathcal{L}^{(1)}$ (that is the first two operators in the Lagrangian of eq. (4.21)) when written in terms of the physical fields, give the following interactions between Z-bosons and leptons:

$$-\frac{g}{m^2\hat{c}_W}\frac{2}{3}\left(-(1-2s_W^2)\left(\Delta_\epsilon+\frac{4}{3}\right)+\frac{1}{3}s_W^2\right)\partial_\mu Z^{\mu\nu}(\overline{\nu_L}F\gamma_\nu\nu_L)$$
$$-\frac{g}{m^2\hat{c}_W}\frac{2}{3}\left(\left(\Delta_\epsilon+\frac{4}{3}\right)+\frac{1}{3}s_W^2\right)\partial_\mu Z^{\mu\nu}(\overline{e_L}F\gamma_\nu e_L)\ . \tag{6.36}$$

Clearly, from this interaction and for on-the-mass-shell particles, we get the same amplitude for the Z decay into a pair of different leptons as that obtained from eqs. (6.26) and (6.27). From eq. (6.36) we see that Z-exchange processes at low-energy, $q^2 \ll M_Z^2$, are suppressed by q^2/M_Z^2. We get exactly the same answer from the compensation of the contributions from the two terms (the Z^μ and the J_Z^μ current terms) in eq. (6.26) .

7 Phenomenological consequences

The terms in $\mathcal{L}^{(1)}$ give many interesting effects, apart from the shifts in the gauge-boson masses, which have already been commented on. For example, eq. (6.25) gives extra contributions to flavour-conserving neutral-current processes. Equation (6.29) gives rise to four-fermion processes with non-conservation of generational lepton numbers; for example they give rise to $\mu^- \to e^- e^- e^+$ and similar processes. To compute these one has to take into account also the contributions coming from one-loop diagrams, with one insertion of the tree-level four-lepton interaction. The couplings in eq. (6.27) allow the decay of the Z into different charged leptons. Equation (6.30) leads to $\mu \to e\gamma$ and similar processes. The amplitude for this process can easily be obtained from eq. (6.30). Taking into account that, without loss of generality, M_e can be taken diagonal with diagonal elements m_a it is given by

$$T(e_b \to e_a \gamma) = -i\frac{e}{3} F_{ab} \bar{u}(p_a) \sigma_{\mu\nu} q^\nu (m_b R + m_a L) u(p_b) \epsilon^\mu(q) \; ; \tag{7.1}$$

L and R are, respectively, the left-handed and right-handed chirality operators. The amplitude eq. (7.1) is in complete agreement with the results obtained in refs. [65, 66, 67].

We are not going to study in detail the phenomenology of this model here. An analysis of this based on our effective Lagrangian, will be presented elsewhere. However, to give a flavour of the applicability of our effective theory, and to show how one can work with it, we will comment on the calculation of the decay $Z \to e_a^+ \, e_b^-$ with both the full and the effective theories.

In the effective theory we have two amplitudes, one due to the loop diagram of fig. 4.c and the other coming from eq. (6.27). As expected, even though each of them contains a divergent contribution Δ_ϵ, they give a finite answer when summed. The total amplitude is given in appendix C, eq. (C.2).

We also have performed a complete calculation of the amplitude of this process in the full theory. After taking into account the diagrams with self-energy insertions to the external fermion legs, the final answer is finite. The amplitude is given in eq. (C.4) in terms of the functions $f_1(w)$ and $f_2(w)$, as explained in appendix C. In the limit of $m \gg M_Z$ (that is, $w \to 0$), the results obtained in the full and in the effective theories are identical, as they should be. In fact all matching conditions are designed with just this in view.

From the full model amplitude of eq. (C.4), we obtain the following branching ratio of the flavour-violating decay width ($a \neq b$) relative to the standard model flavour-conserving one,

$$BR(Z \to \overline{e}_a e_b) = \frac{\Gamma(Z \to \overline{e}_a e_b)}{\Gamma(Z \to \overline{e}_a e_a)} = |F_{ab}|^2 \, \frac{8 \, |f_1(w) + \hat{s}_W^2 f_2(w)|^2}{1 + (1 - 4\hat{s}_W^2)^2} \; . \tag{7.2}$$

The branching ratio in the effective Lagrangian is given by the same expression, but with $f_1(w)$ and $f_2(w)$ given in eq. (C.9). In fig. 7, we present this branching ratio in both the full and the effective theories as a function of m. Charged scalar Yukawa couplings, $(f^\dagger f)_{ab}$, are taken to be equal to 1. From the figure it is clear that for masses of the scalar h larger than the mass of the Z boson the effective theory gives a good approximation.

8 Discussion and conclusions

In this paper we have studied some questions related to the construction and the use of effective Lagrangians, by considering an extension of the standard model that includes a heavy charged scalar coupled to the leptonic doublet.

Starting from the full renormalizable model, we have built a low-energy effective field theory by integrating out the heavy scalar. This was done at the one-loop level and keeping only operators of dimension six or less.

Functional methods were used to obtain, in a compact and gauge-invariant form, all operators generated by tree-level and one-loop diagrams containing only heavy scalars in the internal lines. The result includes only diagrams which are one-particle-irreducible with respect to the light particles. In appendix A we give a detailed explanation of this calculation. This is not the complete answer for the effective Lagrangian, since there are many one-loop diagrams in the full theory, with heavy and light particles in the loop that are not completely taken into account by one-loop diagrams involving tree-level effective couplings. To obtain these additional contributions we have calculated several Green functions in both the full and the effective theories and required that the results of both calculations match for small momenta compared with the heavy scalar mass. The result can be expressed as a linear combination of gauge-invariant operators, which must be included in the effective Lagrangian.

Adding both contributions, from the operators corresponding to the diagrams with only heavy lines and from those with heavy and light internal particles, we have obtained the complete bare one-loop effective Lagrangian, including operators up to dimension six. It contains several operators with infinite coefficients (dimensional continuation was employed to regularize all divergent integrals, then UV divergences appear as poles in $1/(D-4)$). By using an $\overline{MS}$ scheme to renormalize both the full and the effective theories, we obtained the matching conditions for the running couplings, which express the renormalized couplings of the effective theory in terms of the renormalized couplings of the full model. To avoid large logarithms, these matching conditions should be evaluated at a scale around the heavy scalar mass and then the renormalization group used to bring the couplings to any other lower scale.

We used this simple example to discuss the behaviour of the couplings of a generic effective Lagrangian. In general all the operators in the effective Lagrangian with the same quantum numbers mix under the renormalization group. The effective couplings obey a standard renormalization group first-order differential equation controlled by the anomalous-dimension matrix. It is clear that the only information that can be extracted from the effective Lagrangian, without knowing the full theory, is this anomalous-dimension matrix. However, to obtain the couplings it is necessary to solve the renormalization group equations, and this requires the knowledge of n boundary conditions (if n-operators are involved in the mixing). One-loop calculations with the effective Lagrangian can only be used to relate the couplings of operators at different scales, but give no information at all on the actual value of those couplings. Only experiment, or a more complete theory, can give new information on the values of the effective Lagrangian couplings. This trivial observation is of importance when using the effective Lagrangian approach to classify the sort of new physics one could find in future experiments.

After renormalization we obtain an effective Lagrangian that can be split in three pieces. The first one, $\mathcal{L}_{SM}$, is just the standard model Lagrangian (expressed in terms of effective couplings); the second one is the four-lepton interaction obtained from the full theory at tree level, $\mathcal{L}^{(0)}$; the third piece, $\mathcal{L}^{(1)}$ contains all dimension-six operators generated at the one-loop level in the full model.

This effective Lagrangian was rewritten by using the standard model classical equations of motion in order to display its physical content more transparently. Some caution is needed when using the equations of motion, especially for operators that are going to be inserted in loop diagrams. In our case, however, the equations of motion are used only in operators that are generated at one loop in the full theory and that are supposed to be used only at tree level in the effective Lagrangian. Then, the use of equations of motion is completely legitimate. We used them for the unbroken theory, but it is worth while to stress that the equations of motion might also be used after SSB (or might not be used at all) and this would not change any physical result.

We also discussed the consequences of SSB on our effective Lagrangian by substituting the VEV of the doublet and neglecting all Higgs interactions. Apart from a negligible modification of the relation between the masses of the vector bosons, the most interesting consequence of the model is due to the different operators contributing to processes with violation of the generational lepton numbers. We made a short review of some processes generated by our one-loop effective Lagrangian, $e_a^- \to e_b^- \gamma$, $e_a^- \to e_b^- e_c^+ e_c^+$, etc (where a, b, c are different flavour indices). A more detailed phenomenological analysis will be presented elsewhere.

Finally, to see how one can use our effective Lagrangian, we have calculated the decay width of the Z gauge boson to a pair of different leptons. It contains contributions from one-loop diagrams with one insertion of the tree-level four-fermion operator and direct contributions from operators generated at the one-loop level. We have done also the calculation in the full theory and compared the results. For a mass of the charged scalar m larger than the Z mass, the effective theory calculation gives a good approximation.

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APPENDICES

A The determinant of the fluctuation operator

In this appendix we calculate the effective Lagrangian contribution coming from the determinant of a fluctuation operator of the form $O = (-D^2 - m^2 - U(x))$. It is given by

$$i\int d^4x\mathcal{L}^{(1)}(x) = \log\det(O)^{-1} = -\operatorname{Tr}\{\log(O)\} = -\operatorname{Tr}\left\{\log(-D^2 - m^2 - U(x))\right\} \ , \quad \text{(A.1)}$$

where D_μ is the covariant derivative for a generic gauge group, for example an $SU(N)$, and $U(x)$ is a general matrix valued function of x.

Powerful methods to calculate traces of these kinds of operators have been developed in the last years [31, 75, 76, 77, 78, 79, 80, 81]. We will employ some techniques developed in all of those papers, but we will follow more closely those in ref. [81].

We will use the following notation

$$D_\mu = \partial_\mu + A_\mu(x), \qquad A_\mu(x) = -igT^a A^a_\mu(x) \ , \quad \text{(A.2)}$$

where T^a are the generators of the gauge group. They satisfy the normalization condition $\operatorname{tr}\left\{T^aT^b\right\} = \frac{1}{2}\delta^{ab}$. Then, the covariant derivative acts on $U(x)$ as follows:

$$D_\mu U(x) = \partial_\mu U(x) + [A_\mu(x), U(x)] \ , \quad \text{(A.3)}$$

The field strength tensor is defined as

$$F_{\mu\nu} = [D_\mu, D_\nu] \ . \quad \text{(A.4)}$$

We will use the metric $(+,-,-,-)$ and the following conventions for the momentum operator and plane wave states:

$$\hat{p}_\mu = i\partial_\mu, \qquad \langle x|p\rangle = e^{-ipx} \ . \quad \text{(A.5)}$$

The normalization of the states is (in D dimensions)

$$\int_x |x\rangle\langle x| = 1, \qquad \int_p |p\rangle\langle p| = 1 \quad \text{(A.6)}$$

and $\int_x = \int d^Dx$ and $\int_p = \int d^Dp/(2\pi)^4$.

Let O be an operator; then we understand the trace of that operator to be

$$\operatorname{Tr}\{O\} = \int_x \operatorname{tr}\{\langle x| O |x\rangle\} = \int_p \operatorname{tr}\{\langle p| O |p\rangle\} \ , \quad \text{(A.7)}$$

where "Tr" means trace over all degrees of freedom, space-time and internal, while "tr" is the trace over only the internal degrees of freedom. Then

$$\operatorname{Tr}\{O\} = \int_x\int_p \operatorname{tr}\left\{e^{ipx}\vec{O}_x e^{-ipx}\right\} \ . \quad \text{(A.8)}$$

Here O_x is the operator O in the representation of positions $\langle x| O |\phi\rangle = \vec{O}_x\langle x|\phi\rangle = \vec{O}_x\phi(x)$. In our case we have

$$O = \log(\Pi^2 - m^2 - U), \qquad \text{where} \qquad \Pi_\mu = iD_\mu = \hat{p}_\mu + iA_\mu \ . \quad \text{(A.9)}$$

By using eq. (A.8) and the operator identity $e^{ipx} f(\Pi) e^{-ipx} = f(\Pi + p)$, we find

$$i\,\mathrm{Tr}\left\{\log(\Pi^2 - m^2 - U)\right\} = i \int_x \int_p \mathrm{tr}\left\{\log(p^2 - m^2 + 2p\Pi + \Pi^2 - U)\right\} \mathbf{1} \;, \qquad (A.10)$$

where the factor $\mathbf{1}$ at the end indicates that the operators act on the identity. Now we can expand the logarithm in the following form

$$i \int_x \int_p \mathrm{tr}\left\{\log(p^2 - m^2) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{(2p\Pi + \Pi^2 - U)^n}{(p^2 - m^2)^n}\right\} \mathbf{1} \;. \qquad (A.11)$$

The first term is the usual Coleman-Weinberg term. It is a constant term, which only contributes to the energy density, and we will drop it. Comparing eq. (A.11) with the expression for the effective Lagrangian, eq. (A.1), we find

$$\mathcal{L}^{(1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{tr}\left\{ i \int_p \frac{(2p\Pi + \Pi^2 - U)^n}{(p^2 - m^2)^n} \right\} \mathbf{1} \;. \qquad (A.12)$$

It is clear that eq. (A.12) gives an expansion in powers of $1/m^2$

$$\mathcal{L}^{(1)} = \sum_{n=1}^{\infty} \frac{c_n}{m^{2n-4}} O_n \;, \qquad (A.13)$$

where O_n are traces of gauge-invariant operators of dimension $2n$ built from A_μ and U (and their covariant derivatives). In general

$$O_n = \sum_i \gamma_i^{(n)} \tilde{O}_n^{(i)} \;. \qquad (A.14)$$

Here $\tilde{O}_n^{(i)}$ is a linearly independent set of traces of operators of dimension $2n$ (operators that can be related using partial integration of D_μ, cyclic permutations inside the trace, or Bianchi identities are not considered to be linearly independent). We use the following basis for these traces of gauge-invariant operators:

$$\begin{aligned}
\tilde{O}_1 &= (\mathrm{tr}\{U\}) \\
\tilde{O}_2 &= (\mathrm{tr}\left\{U^2\right\},\ \mathrm{tr}\{F_{\mu\nu}F^{\mu\nu}\}) \\
\tilde{O}_3 &= (\mathrm{tr}\left\{U^3\right\},\ \mathrm{tr}\left\{(D_\mu U)^2\right\},\ \mathrm{tr}\{F_{\mu\nu} U F^{\mu\nu}\},\ \mathrm{tr}\{D_\mu F^{\mu\nu} D^\sigma F_{\sigma\nu}\},\ \mathrm{tr}\{F_{\mu\nu} F^{\nu\sigma} F_\sigma{}^\mu\}) \;.
\end{aligned}$$

The normalization of the O_n is such that

$$c_n = \left(\frac{m^2}{4\pi\mu^2}\right)^{D/2-2} \frac{1}{(4\pi)^2} \Gamma(n - D/2) \;. \qquad (A.15)$$

All the integrals will be dimensionally regularized and μ is the dimensional-regularization scale. As we will see, with this normalization the coefficients $\gamma_i^{(n)}$ will be just numbers and independent from D. Then, our task is to evaluate the coefficients $\gamma_i^{(n)}$. They could be evaluated directly by expanding eq. (A.12), performing the momentum integrals and using integration by parts to rewrite it in a canonical form. However, it is better to use another method [81, 82]. The trick is as follows: as we know that eqs. (A.12), (A.13) and (A.14) are valid for any A_μ and any U, with the $\gamma_i^{(n)}$ independent from A_μ and U, we can compute $\gamma_i^{(n)}$ using a particular configuration of A_μ and U. Then, the resulting $\gamma_i^{(n)}$ will be valid for any A_μ and U. As we want to compute (A.12) and we have $2p\Pi + \Pi^2 - U = 2ipA + 2p\hat{p} + \hat{p}^2 + i(\hat{p}A + A\hat{p}) - A^2 - U$, it is obvious that if we choose $\partial_\mu A_\nu = 0$ and $U = -A^2$ we will have $(2p\Pi + \Pi^2 - U)^n \mathbf{1} = (2ipA)^n$. Thus, to calculate eq. (A.12) we will choose the following (constant) configuration for the field A_μ:

$$N_\mu \equiv A_\mu \qquad \text{such as} \qquad \partial_\nu A_\mu = 0 \qquad \text{and} \qquad U = -A^2 \ . \tag{A.16}$$

To avoid confusion we will denote the field A_μ as N_μ in this special configuration. In this configuration we obviously have

$$U = -N^2, \qquad F_{\mu\nu} = [N_\mu, N_\nu], \qquad D_\mu G = [N_\mu, G] \ , \tag{A.17}$$

where G is any matrix valued function of A_μ and U in the special configuration (hence, constant with respect to space-time variables). The important point is that eq. (A.17) can be inverted, thus allowing us to pass from the special configuration to the general configuration.

Since in the special configuration (A.17) we have

$$(2p\Pi + \Pi^2 - U)^n \ \mathbf{1} = (2ipN)^n$$

eq. (A.12) reads

$$\begin{aligned} \mathcal{L}^{(1)} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{tr} \left\{ i \int_p \frac{(2ipN)^n}{(p^2 - m^2)^n} \right\} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n}{2n} i \int_p \frac{p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2n}}}{(p^2 - m^2)^{2n}} \operatorname{tr} \left\{ N^{\mu_1} N^{\mu_2} \cdots N^{\mu_{2n}} \right\} \ , \end{aligned} \tag{A.18}$$

where we have taken into account that integrals with an odd number of p's cancel under symmetric integration and we have redefined $n \to 2n$. Now we can use that

$$i \int_p \frac{p^{\mu_1} p^{\mu_2} \cdots p^{\mu_{2n}}}{(p^2 - m^2)^{2n}} = (-1)^{n+1} \left(\frac{m^2}{4\pi\mu^2} \right)^{D/2-2} \frac{m^4}{(4\pi)^2} \frac{\Gamma(n - D/2)}{m^{2n} 2^n \Gamma(2n)} S_n^{\mu_1 \mu_2 \cdots \mu_{2n}} \ . \tag{A.19}$$

Here $S_n^{\mu_1 \mu_2 \cdots \mu_{2n}}$ is the completely symmetric tensor with $2n$ indices built only with the metric tensor $g_{\mu\nu}$. For instance $S_1^{\mu_1 \mu_2} = g^{\mu_1 \mu_2}$ and $S_2^{\mu_1 \mu_2 \mu_3 \mu_4} = g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} + g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}$. In general it contains $(2n-1)!! = (2n-1)(2n-3)\cdots 1$ terms. Inserting eq. (A.19) in eq. (A.18), we finally obtain

$$\mathcal{L}^{(1)} = \sum_{n=1}^{\infty} \left(\frac{m^2}{4\pi\mu^2} \right)^{D/2-2} \frac{m^4}{(4\pi)^2} \frac{\Gamma(n - D/2)}{m^{2n}} \frac{2^n}{(2n)!} S_n^{\mu_1 \mu_2 \cdots \mu_{2n}} \operatorname{tr} \left\{ N^{\mu_1} N^{\mu_2} \cdots N^{\mu_{2n}} \right\} \ . \tag{A.20}$$

Comparing this result with eq. (A.13) and eq. (A.15) we find that in the special configuration (A.17) we have

$$O_n = \frac{2^n}{(2n)!} \operatorname{tr} \left\{ S_n(N) \right\} \ , \tag{A.21}$$

where $S_n(N) \equiv S_n^{\mu_1 \mu_2 \cdots \mu_{2n}} N_{\mu_1} N_{\mu_2} \cdots N_{\mu_{2n}}$ represents the sum of all possible products of $2n$ N_μ's contracted pairwise. For instance, $S_1(N) = N^2$ and $S_2(N) = (N^2)^2 + N_\mu N_\nu N^\mu N^\nu + N_\mu N^2 N^\mu$. Traces of different operators can be related by using the cyclic property of the trace. We find for example $\operatorname{tr}\{S_2(N)\} = 2 \operatorname{tr}\{(N^2)^2\} + \operatorname{tr}\{N_\mu N_\nu N^\mu N^\nu\}$. Then, in the special configuration we have

$$O_n = \sum_i \delta_i^{(n)} \hat{O}_n^{(i)} \ , \tag{A.22}$$

where $\hat{O}_n^{(i)}$ are traces of strings of N_μ's contracted pairwise and the coefficients $\delta_i^{(n)}$ include the normalization factor $2^n/(2n)!$.

If we define the set of linearly independent traces built from N_μ as follows:

$$\begin{aligned}
\hat{O}_1 &= (\mathrm{tr}\left\{N^2\right\}) \\
\hat{O}_2 &= (\mathrm{tr}\left\{(N_\mu N_\nu)^2\right\}, \mathrm{tr}\left\{(N^2)^2\right\}) \\
\hat{O}_3 &= (\mathrm{tr}\left\{(N_\mu N_\nu N_\sigma)^2\right\}, \mathrm{tr}\left\{(N^2)^3\right\}, \mathrm{tr}\left\{(N^2 N_\mu)^2\right\}, \mathrm{tr}\left\{(N_\mu N_\nu N^\mu)^2\right\}, \mathrm{tr}\left\{N^2 (N_\mu N_\nu)^2\right\}) ,
\end{aligned} \tag{A.23}$$

we find

$$\begin{aligned}
O_1 &= \hat{O}_1 \\
O_2 &= \frac{1}{6}\hat{O}_2^{(1)} + \frac{1}{3}\hat{O}_2^{(2)} \\
O_3 &= \frac{1}{90}\hat{O}_3^{(1)} + \frac{1}{45}\hat{O}_3^{(2)} + \frac{1}{30}\hat{O}_3^{(3)} + \frac{1}{30}\hat{O}_3^{(4)} + \frac{1}{15}\hat{O}_3^{(5)} ,
\end{aligned}$$

but we know from eq. (A.14) that in the general case O_n can be expressed as a linear combination of traces of gauge-invariant operators $\tilde{O}_n^{(i)}$. Then, in the special configuration we can write O_n in terms of the two sets of operators

$$O_n = \sum_i \delta_i^{(n)} \hat{O}_n^{(i)} = \sum_i \gamma_i^{(n)} \tilde{O}_n^{(i)} , \tag{A.24}$$

but in the special configuration we can relate the operators $\tilde{O}_n^{(i)}$ with the operators $\hat{O}_n^{(i)}$ by using eq. (A.17). In general we will have

$$\tilde{O}_n^{(i)} = \sum_j (P_n)_{ij} \hat{O}_n^{(j)} \qquad \text{and} \qquad \delta_i^{(n)} = \sum_j \gamma_j^{(n)} (P_n)_{ji} . \tag{A.25}$$

The important point of these equations is that the above linear transformation P_n can can be inverted. This will allow us to compute the γ's in terms of the already known δ's. In matrix form we obtain

$$\gamma^{(n)} = (P_n^T)^{-1} \delta^{(n)} , \tag{A.26}$$

where P_n^T is the transposed matrix of P_n. We skip the details of a rather tedious calculation and quote only the final results valid for an arbitrary configuration

$$\begin{aligned}
O_1 &= -\mathrm{tr}\left\{U\right\} \\
O_2 &= \frac{1}{2}\mathrm{tr}\left\{U^2\right\} + \frac{1}{12}\mathrm{tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} \\
O_3 &= -\frac{1}{6}\mathrm{tr}\left\{U^3\right\} + \frac{1}{12}\mathrm{tr}\left\{(D_\mu U)^2\right\} - \frac{1}{12}\mathrm{tr}\left\{F_{\mu\nu} U F^{\mu\nu}\right\} \\
&\quad + \frac{1}{60}\mathrm{tr}\left\{D_\mu F^{\mu\nu} D^\sigma F_{\sigma\nu}\right\} - \frac{1}{90}\mathrm{tr}\left\{F_{\mu\nu}F^{\nu\sigma}F_\sigma{}^\mu\right\} .
\end{aligned} \tag{A.27}$$

These results agree with the results obtained in [81] by using proper time methods (after passing to Minkowski space).

The effective Lagrangian obtained from the determinant of the fluctuation operator is given by eq. (A.13) with the O_n given above and with the c_n given in eq. (A.15). For $D = 4 - 2\epsilon$ and in the limit $\epsilon \to 0$ we obtain

$$\begin{aligned}
c_1 &= -(\Delta_\epsilon + 1)\frac{1}{(4\pi)^2} \\
c_2 &= \Delta_\epsilon \frac{1}{(4\pi)^2} \\
c_3 &= (n-3)!\frac{1}{(4\pi)^2} \qquad n \geq 3 ,
\end{aligned} \tag{A.28}$$

where

$$\Delta_\epsilon = \frac{1}{\epsilon} - \gamma + \log(4\pi) + 2\log(\mu/m) \equiv \frac{1}{\hat{\epsilon}} + 2\log(\mu/m) \tag{A.29}$$

contains the divergent part when $D \to 4$ and only appears in the first two coefficients.

B Calculation of processes with heavy-light particles in the loops

Here we collect the relevant amplitudes needed to obtain the effective operators generated by one-loop diagrams in the full theory with heavy-light particles in the loops. We have always used dimensional regularization with anticommuting γ_5.

B.1 Lepton doublet self-energies

From the self-energy diagrams in the full theory (fig. 3) we find

$$T_{(3)} = \frac{(f^\dagger f)_{ab}}{(4\pi)^2}\left(2\left(\Delta_\epsilon + \frac{1}{2}\right) + \frac{2}{3}\frac{p^2}{m^2}\right)\bar{u}(p)\,\not{p}Lu(p)\ , \tag{B.1}$$

where $L = \frac{1}{2}(1-\gamma_5)$ is the left-handed chirality projector. In the effective theory, the doublet self-energy diagram with the four-fermion coupling (fig. 3.b) is zero in dimensional regularization because it is a massless tadpole-like diagram.

B.2 Penguins

The diagrams of figs. 4.a and 4.b give the following amplitudes for the coupling of the B_μ to the lepton doublet

$$\begin{aligned} T^B_{4a} &= F_{ab}\frac{g'}{m^2}\bar{u}(p_2)\left\{m^2\left(\Delta_\epsilon + \frac{1}{2}\right)\gamma_\mu - \frac{2}{3}\left(\Delta_\epsilon + \frac{4}{3}\right)(q^2\gamma_\mu - \not{q}q_\mu)\right. \\ &\quad + \frac{1}{6}\left((\not{p}_1 + \not{p}_2)(p_1+p_2)_\mu + (p_1^2+p_2^2)\gamma_\mu\right) \\ &\quad \left. + \frac{i}{2}(\not{p}_2\sigma_{\mu\nu}q^\nu + \sigma_{\mu\nu}q^\nu\,\not{p}_1)\right\}u(p_1)\epsilon^\mu(q) \\ T^B_{4b} &= F_{ab}\frac{g'}{m^2}\bar{u}(p_2)\left\{-m^2 2\left(\Delta_\epsilon + \frac{1}{2}\right)\gamma_\mu - \frac{2}{9}(q^2\gamma_\mu - \not{q}q_\mu)\right. \\ &\quad \left. - \frac{1}{3}\left((\not{p}_1 + \not{p}_2)(p_1+p_2)_\mu + (p_1^2+p_2^2)\gamma_\mu\right)\right\}u(p_1)\epsilon^\mu(q)\ . \end{aligned}$$

Here $q = p_2 - p_1$, and we have kept all external momenta off-the-mass-shell. In the diagram 4.a we split the charged scalar propagator as described in section 4. The result given here corresponds to the difference between the full theory diagram (fig. 4.a) and the vertex obtained in the effective theory with the four-fermion interaction (fig. 4.c).

The full amplitude for the B_μ vertex is

$$\begin{aligned} T^B &= F_{ab}\frac{g'}{m^2}\bar{u}(p_2)\left\{-m^2\left(\Delta_\epsilon + \frac{1}{2}\right)\gamma_\mu - \frac{2}{3}\left(\Delta_\epsilon + \frac{5}{3}\right)(q^2\gamma_\mu - \not{q}q_\mu)\right. \\ &\quad - \frac{1}{6}((\not{p}_1 + \not{p}_2)(p_1+p_2)_\mu + (p_1^2+p_2^2)\gamma_\mu) \\ &\quad \left. + \frac{i}{2}(\not{p}_2\sigma_{\mu\nu}q^\nu + \sigma_{\mu\nu}q^\nu\,\not{p}_1)\right\}u(p_1)\epsilon^\mu(q)\ . \end{aligned} \tag{B.2}$$

For the vertex of the $\vec{W}_\mu$ only the diagram 4.a exists since it does not couple to the $SU(2)$ singlet scalar. The result is

$$
\begin{aligned}
T^W \;=\; & F_{ab}\,\frac{g}{m^2}\,\bar{u}(p_2)\left\{m^2\left(\Delta_\epsilon+\frac{1}{2}\right)\gamma_\mu-\frac{2}{3}\left(\Delta_\epsilon+\frac{4}{3}\right)(q^2\gamma_\mu-\not{q}q_\mu)\right.\\
&+\frac{1}{6}\left((\not{p}_1+\not{p}_2)(p_1+p_2)_\mu+(p_1^2+p_2^2)\gamma_\mu\right)\\
&\left.+\frac{i}{2}(\not{p}_2\sigma_{\mu\nu}q^\nu+\sigma_{\mu\nu}q^\nu\,\not{p}_1)\right\}\tau_i\,u(p_1)\epsilon_i^\mu(q)\ . \qquad \text{(B.3)}
\end{aligned}
$$

From these amplitudes together with the self-energy amplitudes, one can easily reconstruct the gauge-invariant operators that generate them as done in section 4.

B.3 Seagull diagrams

From the diagram of fig. 5.b we obtain the amplitude

$$\frac{\beta}{m^2}F_{ab}\bar{u}(p_2)(\not{p}_1+\not{p}_2)u(p_1)\ , \qquad \text{(B.4)}$$

while for the diagram of fig. 5.a we get

$$\frac{1}{m^2}\hat{F}_{ab}\bar{u}(p_2)\left(2(\Delta_\epsilon+1)(\not{q}_1+\not{q}_2)-(\not{p}_1+\not{p}_2)\right)u(p_1)\ . \qquad \text{(B.5)}$$

Here q_1 and q_2 are the momenta of an incoming $\widetilde{\varphi}$ and an outgoing $\widetilde{\varphi}$ respectively, and $\hat{F}$ is defined in eq. (4.16). As always, we have used the splitting of the charged scalar propagator $1/(k^2-m^2)=-1/m^2+k^2/(m^2(k^2-m^2))$ and have only computed the second term. The first term gives rise to the effective Lagrangian contribution given by the diagram of fig. 5.c.

From eq. (B.5) we obtain the following effective operators

$$(\Delta_\epsilon+1)2\frac{i}{m^2}(\bar{\ell}\gamma^\mu(D_\mu\widetilde{\varphi})\hat{F}\widetilde{\varphi}^\dagger\ell)-\frac{i}{m^2}(\bar{\ell}\widetilde{\varphi}\hat{F}\widetilde{\varphi}^\dagger\,\not{D}\ell)+\text{h.c.}\ , \qquad \text{(B.6)}$$

which can be written in the form of eq. (4.15) after using the following $SU(2)$ Fierz transformations:

$$
\begin{aligned}
((D_\mu\widetilde{\varphi})\widetilde{\varphi}^\dagger) &= \frac{1}{2}((D_\mu\varphi)^\dagger\varphi)\mathbf{1}+\frac{1}{2}((D_\mu\varphi)^\dagger\vec{\tau}\varphi)\vec{\tau}\\
(\widetilde{\varphi}\widetilde{\varphi}^\dagger) &= \frac{1}{2}(\varphi^\dagger\varphi)\mathbf{1}+\frac{1}{2}(\varphi^\dagger\vec{\tau}\varphi)\vec{\tau}\ .
\end{aligned}
$$

Here $\mathbf{1}$ is the 2×2 identity matrix.

B.4 Four-fermion interactions

The amplitude corresponding to the box with the two charged scalars in the loop (fig. 6.c and crossed) has the following form

$$-\frac{(4\pi)^2}{m^2}F_{ab}F_{cd}(\bar{u}(p_1)\gamma_\mu u(p_2))(\bar{u}(p_3)\gamma_\mu u(p_4))\ . \qquad \text{(B.7)}$$

C Lepton-flavour-changing Z decay

In this appendix we give some results of the calculation of the decay width for $Z \to \bar{e}_a e_b$ done with both the full and the effective Lagrangians.

We show in fig. 4.c the diagram responsible for this decay in the effective theory (the wavy line is, in this case, a Z gauge boson, the external leptons are charged leptons or neutrinos, and the lines in the loop represent neutrinos or charged leptons, respectively). The corresponding amplitudes are (for massless external leptons and all particles on-the-mass-shell):

$$T(Z \to \overline{e}_a e_b)_{(4.c)} = \frac{g}{\hat{c}_W} F_{ab} \frac{2}{3} \frac{M_Z^2}{m^2} \left(\log(M_Z^2/m^2) - i\pi - \Delta_\epsilon - \frac{5}{3} \right) \overline{u}(p_2)\, \not{\epsilon}(q) L v(p_1) \; . \quad \text{(C.1)}$$

To this amplitude we should add the contribution coming from the operators generated through matching, eq. (6.27). The total result is

$$T(Z \to \overline{e}_a e_b)_{eff} = \frac{g}{\hat{c}_W} F_{ab} \frac{2}{3} \frac{M_Z^2}{m^2} \left(\log(M_Z^2/m^2) - i\pi - \frac{1}{3} + \hat{s}_W^2 \frac{1}{3} \right) \overline{u}(p_2)\, \not{\epsilon}(q) L v(p_1) \; . \quad \text{(C.2)}$$

For the Z decay to neutrinos, a similar calculation gives the following result:

$$T(Z \to \overline{\nu}_a \nu_b)_{eff} =$$
$$\frac{g}{\hat{c}_W} F_{ab} \frac{2}{3} \frac{M_Z^2}{m^2} \left(-(1 - 2\hat{s}_W^2)(\log(M_Z^2/m^2) - i\pi - \frac{1}{3}) + \hat{s}_W^2 \frac{1}{3} \right) \overline{u}(p_2)\, \not{\epsilon}(q) L v(p_1) \; . \quad \text{(C.3)}$$

In both cases the divergent term coming from the loop integration has been cancelled by the one-loop effective operator contribution.

Previous results should be compared with the full Lagrangian amplitudes,

$$T(Z \to \overline{e}_a e_b)_{full} = \frac{g}{c_W} F_{ab} \left(-f_1(w) - s_W^2 f_2(w) \right) \bar{u}(p_2)\, \not{\epsilon}(q) L v(p_1) \quad \text{(C.4)}$$

$$T(Z \to \overline{\nu}_a \nu_b)_{full} = \frac{g}{c_W} F_{ab} \left((1 - 2s_W^2) f_1(w) - s_W^2 f_2(w) \right) \bar{u}(p_2)\, \not{\epsilon}(q) L v(p_1) \; , \quad \text{(C.5)}$$

where, using the variable

$$w = -\frac{M_Z^2}{m^2} - i\eta \qquad (\eta \to 0^+) \; , \quad \text{(C.6)}$$

we have

$$f_1(w) = \frac{1}{2} + \frac{2}{w} - \frac{2+w}{w} \log(w) - \frac{2}{w^2} \log(1-w)\log(w) - \frac{2}{w^2} \mathrm{Li}_2(w) \quad \text{(C.7)}$$

and

$$f_2(w) = -5 - \frac{4}{w} + \frac{8}{w^2} \left(\mathrm{Li}_2\left(\frac{2}{1 - \sqrt{1 + \frac{4}{w}}} \right) + \mathrm{Li}_2\left(\frac{2}{1 + \sqrt{1 + \frac{4}{w}}} \right) \right)$$
$$+2\left(1 + \frac{2}{w}\right) \sqrt{1 + \frac{4}{w}} \log\left(\frac{1 + \sqrt{1 + \frac{4}{w}}}{-1 + \sqrt{1 + \frac{4}{w}}} \right) \; . \quad \text{(C.8)}$$

Here $Li_2(w)$ is the dilogarithmic function. The functions $f_1(w)$ and $f_2(w)$ have the following asymptotic values for $w \to 0$:

$$f_1(w) \to \frac{2}{3} w \left(\log(w) - \frac{1}{3} \right) , \qquad f_2(w) \to \frac{2}{9} w \; . \quad \text{(C.9)}$$

These results are in complete agreement with the effective Lagrangian calculation. In fact the logarithmic contribution coming from $f_1(w)$ can easily be obtained from the calculation of diagram 4.c. However, the non-logarithmic part of that amplitude is arbitrary, in fact divergent. The only way to fix it is by matching the full theory.

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Figure captions

Figure 1: Diagram a) is the tree-level diagram contributing to the effective Lagrangian. The solid lines represent lepton doublets and the thick dashed line represents the heavy scalar. Diagram b) represents the four-lepton interaction in the effective theory. The symbol $\otimes$ means a scalar current insertion.

Figure 2: Diagrammatic representation of the contributions to the one-loop effective Lagrangian given by the determinant of the fluctuation operator. Gauge bosons are represented by wavy lines and the standard Higgs is represented by thin dashed lines.

Figure 3: Matching conditions: Self-energy diagrams. a) in the full theory and b) in the effective one.

Figure 4: Matching conditions: Penguin diagrams a) and b) in the full theory and c) in the effective one.

Figure 5: Matching conditions: Seagull diagrams a) and b) in the full theory and c) in the effective one.

Figure 6: Matching conditions: Box diagrams a) and b) in the full theory and c) in the effective one.

Figure 7: The family lepton-number violating branching ratio $BR(Z \to \tau^- e^+) = \Gamma(Z \to \tau^- e^+)/\Gamma(Z \to e^- e^+)$ computed in both the full (solid line) and the effective (dashed line) theories as a function of the charged scalar mass. The couplings $F_{\tau e}$ are taken to be equal to 1.

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