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SOME CHARACTERISATIONS OF GROUPS IN WHICH NORMALITY IS A TRANSITIVE RELATION BY MEANS OF SUBGROUP EMBEDDING PROPERTIES

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Communicated by

Dedicated to the memory of Mario Curzio

Abstract. In this survey we highlight the relations between some subgroup embedding properties that characterise groups in which normality is a transitive relation in certain universes of groups with some finiteness properties.

1. Introduction

We begin by recalling the definition of the groups in which normality is a transitive relation, or, in short, T-groups.

Definition 1.1. A group G is said to be a *T-group* if $H \trianglelefteq K \trianglelefteq G$ implies $H \trianglelefteq G$.

This is equivalent to stating that all subnormal subgroups are normal. The first explicit mention we have found of T-groups in the literature corresponds to a paper of Best and Taussky [3]. Chapter 2 of [2] summarises some basic results about T-groups in finite groups. The description of T-groups in the infinite case is more complex and can be found in the celebrated paper of D. J. S. Robinson [27].

It is well known that the class of T-groups is not closed under taking subgroups. A typical example of a T-groups with subgroups that are not T-groups is the alternating group A_5 of degree 5, that is obviously a T-group since it is simple and its only subnormal subgroups are 1 and A_5, but that has a subgroup isomorphic to A_4 which is not a T-group, since the cyclic subgroups in the Klein 4-group are subnormal, but not normal in A_4. This motivates the following definition.

Definition 1.2. We say that a group G is a *$\bar{T}$-group* if every subgroup of G is a T-group.

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In the finite soluble universe, the following classical characterisation of Gaschütz characterises finite soluble T-groups. Recall that a Dedekind group is a group with all subgroups normal and that a power automorphism of a group X is an automorphism of X that stabilises all subgroups of X.

Theorem 1.3 (Gaschütz, [12])**.** *A finite soluble group G is a T-group if and only if G has an abelian Hall subgroup L of odd order such that G/L is a Dedekind group and L is acted upon by conjugation as a group of power automorphisms by G.*

This result has the virtue of showing that a finite soluble T-group is supersoluble and a $\bar{\mathrm{T}}$-group. Moreover, a finite $\bar{\mathrm{T}}$-group must be soluble, since, otherwise, if we have an insoluble $\bar{\mathrm{T}}$-group whose proper subgroups are soluble, then it is a minimal-non-supersoluble group and so it is soluble by a theorem of Doerk [8, Satz A] (see also [13, Kapitel VI, Satz 9.6]). Therefore we have the following result.

Theorem 1.4. *Let G be a finite group.*

(1) *If G is a soluble T-group, then G is a $\bar{T}$-group.*
(2) *If G is a $\bar{T}$-group, then G is soluble.*

Examples of infinite soluble T-groups that are not $\bar{\mathrm{T}}$-groups are constructed in [27] and [19].

The first class of infinite groups we will consider is the class of FC*-groups, that generalise the class of groups of FC-groups or groups in which every conjugacy class is finite.

Definition 1.5. We say that a group G is an *FC^0-group* if G is finite. By induction, we say that a group G is an *FC^{n+1}-group* if $G/\mathrm{C}_G(\langle x\rangle^G)$ is an FC^n-group for all $x \in G$. Then G is an FC*-group if G is an FC^n-group for some $n \geq 0$.

Theorem 1.6 ([11, Theorem 2.3])**.** *Let G be an FC*-group. Then the following statements are equivalent:*

(1) *G is a soluble T-group.*
(2) *G is a $\bar{T}$-group.*

The other class of groups we are interested in is the class of groups without infinite simple sections. This class contains every FC*-group and it is in fact a subclass of the class of locally graded groups.

Definition 1.7. We say that a group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image.

Theorem 1.8 ([6, Theorem 3.6])**.** (1) *Let G be a group without infinite simple sections. Then:*

(a) *G is locally graded.*
(b) *If G is a $\bar{T}$-group, then G is metabelian.*

(2) *Let G be a soluble group. Then G is a $\bar{T}$-group if and only if every ascendant subgroup of G is normal in G.*

The following question is open in the *Kourovka Notebook* (see [22, Question 14.36]).

Question 1.9. Are non-periodic locally graded $\bar{\mathrm{T}}$-groups soluble?

In [9], this question is reduced to the following one.

Question 1.10. Let G be a locally graded $\bar{\mathrm{T}}$-group. If G is torsion-free, can we say that it is abelian?

2. Subgroup embedding properties

In this section we present some subgroup embedding properties that have been used to characterise T-groups.

2.1. **Pseudonormal and pronormal subgroups.** The following subgroup embedding property appears in a natural way in the scope of $\bar{\mathrm{T}}$-groups. The first appearance of this property known to us is due to Peng [26].

Definition 2.1. A subgroup X of a group G is said to be *pseudonormal* [7] or *transitively normal* [18] or *to satisfy the subnormaliser condition* [24] if $\mathrm{N}_G(H) \leq \mathrm{N}_G(X)$, for each subgroup H of G such that $X \leq H \leq \mathrm{N}_G(X)$.

This is equivalent to affirming that if $H \leq L \leq G$ and H is subnormal in L, then $H \trianglelefteq L$.

Pronormality is a well-known subgroup embedding property introduced by Hall in his Cambridge lectures.

Definition 2.2. A subgroup X of a group G is said to be *pronormal* if X and X^g are conjugate in $\langle X, X^g\rangle$, for every element $g \in G$.

For instance, we have the following result for finite groups.

Theorem 2.3 ([1, Theorem A]). *Let G be a finite group, then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group.*
(2) *Every subgroup of G is pronormal.*
(3) *Every subgroup of G is pseudonormal.*

Theorem 2.3 admits an extension to FC*-groups.

Theorem 2.4. *Let G be an FC*-group, then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group.*
(2) *Every subgroup of G is pronormal.*
(3) *Every subgroup of G is pseudonormal.*

This result follows by [7, Theorem 3.1 and Corollary 3.5] and [5, Theorem 4.6] or [29, Theorem 3.3]. In general, we have the following result.

Theorem 2.5 ([7, Theorem 3.1]). *A group G is a $\bar{T}$-group if and only if all its subgroups are pseudonormal.*

With respect to pronormality, we have the next result.

Theorem 2.6 ([25]). *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *G is a $\bar{T}$-group.*
(3) *X is pronormal in G for all $X \leq G$.*

The theorem of Peng can be extended to FC*-groups.

Theorem 2.7 ([6, Theorem 3.9]). *Let G be a soluble FC*-group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *X is pronormal in G for all $X \leq G$.*

Theorem 2.8. *Let G be a group without infinite simple sections. Then G is a $\bar{T}$-group if and only if every cyclic subgroup is pronormal.*

Kovács, Neumann, and de Vries [17, Theorem 2.1] show the existence of a metabelian $\bar{\mathrm{T}}$-group that contains some non-pronormal Sylow subgroups (see also the comments after [6, Lemma 2.7] for more details). Kuzennyi and Subbotin [19, Example 2] present an example of a group with all primary subgroups pronormal, but with some non-pronormal subgroups.

2.2. **Weakly normal subgroups.** The following concept was introduced by Müller [23].

Definition 2.9 ([23]). A subgroup X of a group G is said to be *weakly normal* if $X^g \leq \mathrm{N}_G(H)$ implies $g \in \mathrm{N}_G(H)$.

Theorem 2.10 ([1, Theorem A]). *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *Every subgroup of G is weakly normal.*

Theorem 2.11 ([31, Corollary 4], [30, Theorem 2.8]). *Let G be a group. Then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group without infinite simple sections.*
(2) *G is a locally graded group whose subgroup are weakly normal.*

2.3. **$\mathcal{H}$-subgroups.** The notion of $\mathcal{H}$-subgroup is due to Bianchi, Gillio Berta Mauri, Herzog, and Verardi [4].

Definition 2.12 ([4]). A subgroup X of a group G is said to be an *$\mathcal{H}$-subgroup* or that it has the *$\mathcal{H}$-property* in G if $\mathrm{N}_G(X) \cap X^g \leq X$ for all elements g of G.

Theorem 2.13 ([4, Theorem 10]). *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*

(2) *G is a $\bar{T}$-group.*
(3) *Every subgroup of G is an $\mathcal{H}$-subgroup.*

The previous theorem also holds for groups without infinite simple sections.

Theorem 2.14 ([32, Theorem 3.2])**.** *Let G be a group without infinite simple sections. Then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group.*
(2) *Every subgroup of G has the property $\mathcal{H}$.*

2.4. **NE-subgroups.** The notion of NE-subgroup is due to Li. Here we use H^G to denote the normal closure of H in G.

Definition 2.15 ([20])**.** A subgroup H of a finite group G is called an *NE-subgroup* if it satisfies $\mathrm{N}_G(H) \cap H^G = H$.

Theorem 2.16 ([21, Theorem 3.1])**.** *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *Every subgroup of G is an NE-subgroup of G.*

Theorem 2.17 ([9])**.** *Let G be a group without infinite simple sections. Then the following statements are equivalent:*

(1) *G is a soluble $\bar{T}$-group.*
(2) *Every subgroup of G is an NE-subgroup of G.*

2.5. **φ-subgroups and cr-subgroups.** The following subgroup embedding properties were introduced by Kaplan [14].

Definition 2.18 ([14])**.** A subgroup H of a group G is said to be a *φ-subgroup* of G if, for all K, L maximal in H, if it is the case that if K, L are conjugate in G, then K, L are conjugate in H.

Definition 2.19 ([14])**.** A subgroup K of a group G is said to be a *cr-subgroup* (for "conjugation restricted") of G if there are no $A < K$, $g \in G$ such that $K = AA^g$.

Theorem 2.20 ([14, Theorem 7])**.** *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *Every subgroup of G has the property φ.*
(3) *Every subgroup of G is a cr-subgroup.*

This result admits an extension to FC*-groups.

Theorem 2.21 ([16, Theorem 5.2])**.** *Let G be a soluble FC*-group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *Every subgroup of G has the property φ.*
(3) *Every subgroup of G is a cr-subgroup.*

3. Other characterisations

The following property, introduced by Kaplan in [15], is not exactly a subgroup embedding property *per se*, but describes a class of groups in which the non-normal subgroups are embedded in the group in a particular way.

Definition 3.1 ([15])**.** A group G is said to be an *NNM-group* (for "non-normal maximal") if each non-normal subgroup of G is contained in a non-normal maximal subgroup of G.

Theorem 3.2 ([15, Theorem 1])**.** *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *All subgroups of G are NNM-groups.*

This result can be also extended to FC*-groups.

Theorem 3.3 ([11, Theorem 2.5])**.** *Let G be a FC*-group. Then the following statements are equivalent:*

(1) *G is a soluble T-group.*
(2) *All subgroups of G are NNM-groups.*

There exist examples of T-groups that are hyperfinite and FC-nilpotent, but that are not NNM-groups [11, Example 2.6].

4. Relations between subgroup embedding properties

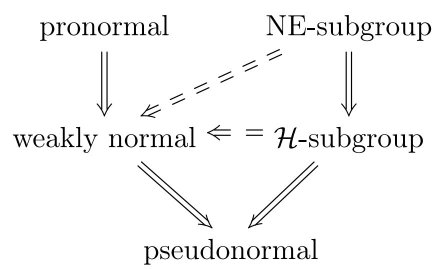
Figure 1 shows the relations between some of the subgroup embedding properties considered in this survey. The broken arrows mean implications that are only known to hold in HNN-free groups (in particular, in finite groups), but whose validity in the general case is not known. We recall that G is HNN-free if $H^g \leq H$ implies that $g \in \mathrm{N}_G(H)$ (see [28]).

The following comments show that no other general implications between the subgroup embedding properties presented in Figure 1 hold, although some partial results have been obtained.

In [1, Remark 1], an example of a weakly normal subgroup that is not pronormal is presented. It is constructed from an irreducible and faithful Σ_3-module V_7 over the field of 7 elements whose restriction to the alternating group A_3 of degree 3 is a direct sum of two irreducible modules $V_7 = W_1 \oplus W_2$ of dimension 1. Let $G = \Sigma_3 \ltimes V_7$. Then $H = A_3W_1$ is weakly normal, but not pronormal in G. The fact that pronormal subgroups are weakly normal has been proved in [1, Proposition 1]. This proof is also valid in the infinite case.

In [1, Lemma 1], it is shown that weakly normal subgroups satisfy the subnormaliser condition. This proof is also valid for infinite groups. An example of Mysovskikh [24] (see also [2, Example 1.5.16])

FIGURE 1. Relations between subgroup embedding properties



shows that the converse is false. It consists of a semidirect product $G = A_4 \ltimes W$ of A_4 by an irreducible and faithful module of dimension 3 over the field of 3 elements obtained by considering the A_4-invariant subgroup $W = \langle w_4 w_1^{-1}, w_4 w_2^{-1}, w_4 w_3^{-1}\rangle$ of the base subgroup of the natural wreath product $C_3 \wr A_4$. Then $D = \langle (1,2)(3,4)\rangle W$ is pseudonormal, but not weakly normal in G.

Suppose that $H^G \cap \mathrm{N}_G(H) = H$, then $H^g \cap \mathrm{N}_G(H) \leq H^G \cap \mathrm{N}_G(H) = H$ and so all NE-subgroups are $\mathcal{H}$-subgroups. The converse is false, because in $\mathrm{SL}_2(3)$, a Sylow 3-subgroup H is an $\mathcal{H}$-subgroup that is not an NE-subgroup.

In [4, Lemma 5], it is shown that $\mathcal{H}$-subgroups are pseudonormal in finite groups. For infinite groups, the result also holds, we have to modify slightly the argument: if $H \leq K \leq \mathrm{N}_G(H)$ and $g \in \mathrm{N}_G(K)$, then $K^g = K^{g^{-1}} = K$ and so H^g, $H^{g^{-1}} \leq K \leq \mathrm{N}_G(H)$. Therefore $H^g = H^g \cap \mathrm{N}_G(H) \leq H$ and $H^{g^{-1}} = H^{g^{-1}} \cap \mathrm{N}_G(H) \leq H$, that is, $H \leq H^g$. Consequently $H^g = H$ and $g \in \mathrm{N}_G(H)$. The previous example of Mysovskikh gives a pseudonormal subgroup that is not an $\mathcal{H}$-subgroup.

The fact that $\mathcal{H}$-subgroups are weakly normal in finite groups was indicated in [1]: if $H^g \leq \mathrm{N}_G(H)$ and H is an $\mathcal{H}$-subgroup, then $H^g \leq H^g \cap \mathrm{N}_G(H) = H$. This gives that $g \in \mathrm{N}_G(H)$ if G is HNN-free. A similar argument shows that for HNN-free groups, NE-subgroups are weakly normal. However, we do not know whether these implications hold in the general case. This is left open in [32, Question 2]. Conversely, the subgroup $\langle (1,2,3,4)\rangle$ of the symmetric group Σ_4 of degree 4 is an example of a weakly normal subgroup that is not an $\mathcal{H}$-subgroup (see [1, Example 1]) and, hence, not an NE-subgroup. In [1, Theorems 4 and 5], some sufficient conditions for a weakly normal subgroup H of a supersoluble group to be an $\mathcal{H}$-subgroup are considered, namely H being a p-group or H having all its subgroups weakly normal.

The above presented group of [1, Remark 1] is also an example of an $\mathcal{H}$-subgroup that is not pronormal.

The subgroup $\langle (1,2,3,4)\rangle$ of Σ_4 of [1, Example 1] is also a pronormal subgroup, but not an $\mathcal{H}$-subgroup and, consequently, not an NE-subgroup.

Finally, in [9], we prove that the $\bar{\mathrm{T}}$-groups with no infinite subgroups coincide with the locally graded groups whose subgroups are NE-subgroups. Based on this result, we see that the constructions of

Kovács, Neumann, and de Vries and Kuzennyi and Subbotin that we have mentioned after Theorem 2.8 give examples of NE-subgroups that are not pronormal.

On the other hand, the properties of being φ-subgroups and cr-subgroups seem to be essentially different from the other properties we have considered before. We know by [14, Theorem 4.1] that φ-subgroups that are normal are cr-subgroups and that soluble cr-subgroups are φ-subgroup, with some counterexamples given when the hypothesis of normality or solubility of the corresponding subgroup is removed. It is clear that all subgroups of prime order are both φ-subgroups and cr-subgroups, but $K = \langle(1,2)(3,4)\rangle$ is not pseudonormal in the alternating group A_4. Moreover, the fact that $V = \langle(1,2)(3,4),(1,3)(2,4)\rangle$ has two subgroups $\langle(1,2)(3,4)\rangle$, $\langle(1,3)(2,4)\rangle$ that are not conjugate in G, shows that normal subgroups are not necessarily φ-subgroups nor cr-subgroups.

5. Systems of subgroups satisfying embedding properties and $\bar{\mathrm{T}}$-groups

We can summarise the previous characterisations of $\bar{\mathrm{T}}$-groups by means of subgroup embedding properties, as well as other characterisations presented in [7, 9, 32], in the following results.

Theorem 5.1. *Let G be a periodic, locally graded group. The following statements are pairwise equivalent.*

(1) *G is a $\bar{T}$-group.*
(2) *G is locally finite and all cyclic subgroups of G are pronormal.*
(3) *All subgroups of G are $\mathcal{H}$-subgroups.*
(4) *G is locally finite and all cyclic subgroups of G are $\mathcal{H}$-subgroups (see* [32, Theorem 3.1]*).*
(5) *All subgroups of G are weakly normal.*
(6) *G is locally finite and all cyclic subgroups of G are weakly normal (see* [7, Lemma 3.2]*).*
(7) *All subgroups of G are NE-subgroups.*
(8) *G is locally finite and all cyclic subgroups of G are NE-subgroups (see* [9]*).*
(9) *All subgroups of G are pseudonormal.*
(10) *G is locally finite and all cyclic subgroups of G are pseudonormal (see* [7, Lemma 3.2]*).*

Moreover, if one of the above conditions hold, then G is metabelian.

Theorem 5.2. *Let G be a non-periodic group without infinite simple sections. The following statements are pairwise equivalent.*

(1) *All subgroups of G are pronormal.*
(2) *All cyclic subgroups of G are pronormal.*
(3) *All subgroups of G are NE-subgroups.*
(4) *All subgroups of G are weakly normal.*
(5) *All subgroups of G are $\mathcal{H}$-subgroups.*
(6) *G is abelian.*

In the comment after [32, Theorem 3.1] it is shown that the dihedral infinite group D_∞ is an example of a group with all cyclic subgroups $\mathcal{H}$-subgroups, but with a non-$\mathcal{H}$-subgroup. In [9] we also show that this group has all cyclic subgroups NE-subgroups, but it contains a subgroup which is not an NE-subgroup.

We do not know whether a non-periodic soluble group with all cyclic subgroups weakly normal is abelian.

We conclude by summarising the characterisations for $\bar{\mathrm{T}}$-groups that hold in the universe of all FC*-groups.

Theorem 5.3. *Let G be a soluble FC*-group. The following statements are pairwise equivalent.*

(1) *G is a T-group.*
(2) *G is a $\bar{T}$-group.*
(3) *All subgroups of G are NNM-groups.*
(4) *All subgroups of G are cr-subgroups.*
(5) *All subgroups of G are φ-subgroups.*

A scheme of the characterisations contained in this section can be found in [10].

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