Boundedness and compactness of operators related to time-frequency analysis

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I declare that this dissertation entitled *Boundedness and compactness of operators related to time-frequency analysis* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a degree of PhD in Mathematics at Universitat de València.
- Where I have consulted the published works of others, this is always clearly attributed.
- Where I have quoted from the works of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.
- I have acknowledged all main sources of help.

València, June 25th, 2018

Eva Primo Tárraga

We declare that this dissertation presented by **Eva Primo Tárraga** entitled *Boundedness and compactness of operators related to time-frequency analysis* has been done under our supervision at Universitat de València. We also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of PhD in Mathematics.

València, June 25th, 2018

Carmen Fernández Rosell **Antonio Galbis Verdú**

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According to the normative of the Universitat de València, this PhD dissertation starts with an extended abstract in one of the official languages of Valencia Community, Valencian. For the benefit of the reader, it is followed by an abstract in English. The thesis really begins at page 1.

We would like also to point out that in each chapter we have described the methodologies, objectives, results and conclusions of this work.

Resum

En aquesta tesi estudiem diferents aspectes relatius als operadors de l'anàlisi temps-freqüència. Tot operador lineal i continu $A : \mathcal{S} \to \mathcal{S}'$ es pot escriure com un operador integral

$$\langle Af, g \rangle = \langle K, g \otimes f \rangle,$$

on $K \in \mathcal{S}'$ és el nucli i $f, g \in \mathcal{S}$, o també com un operador integral de Fourier (de fet, pseudodiferencial [Grö01, Teorema 14.3.5]). Diferents condicions sobre el nucli o el símbol i la fase (en el cas dels operadors integrals de Fourier) permetràn estendre l'operador a diversos espais de funcions o distribucions.
El nostre objectiu és emprar tècniques de l'anàlisi temps-freqüència per a estudiar l'acotació i/o compacitat d'operadors integrals de Fourier, operadors integrals o multiplicadors unimodulars entre espais de modulació o de Lebesgue. També estudiem multiplicadors en espais de Hilbert separables. Tot seguit detallem el contingut de la memòria.

En el primer capítol introduïm la notació i els espais, així com les seues propietats, emprats al llarg de la memòria. En la primera secció introduïm els espais de successions ponderats.

Definició 1. *Donats I i J conjunts numerables d'índexs, una successió de nombres positius $m = (m_{i,j})_{(i,j)\in I\times J}$ i $1 \le p, q < \infty$, considerem l'espai de successions $\ell^{p,q}_m(I \times J)$ consistent en aquelles successions $x = (x_{i,j})_{(i,j)\in I\times J}$*

tals que

$$\|x\|_{\ell_m^{p,q}} := \left(\sum_{j\in J}\left(\sum_{i\in I}|x_{i,j}m_{i,j}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

Als casos $p=\infty$ o $q=\infty$, l'anterior norma es modifica de la manera usual, per exemple

$$\|x\|_{\ell_m^{\infty,q}} := \left(\sum_{j\in J}\left(\sup_{i\in I}|x_{i,j}m_{i,j}|\right)^{q}\right)^{\frac{1}{q}}.$$

També presentem alguns operadors entre ells.

Definició 2. *Donada una successió $a=(a_{i,j})_{(i,j)\in I\times J}$ de nombres complexos, denotem com D_a l'operador diagonal*

$$D_a : \mathbb{C}^{I\times J} \to \mathbb{C}^{I\times J},\ x=(x_{i,j})_{(i,j)\in I\times J} \mapsto (a_{i,j}x_{i,j})_{(i,j)\in I\times J}.$$

Definició 3. *Siga $\gamma\in\Lambda$, per a un reticle Λ en $\mathbb{R}^N$, l'operador translació $T_\gamma : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$ es defineix com*

$$T_\gamma\left(x_\lambda\right)_{\lambda\in\Lambda} = \left(x_{\lambda-\gamma}\right)_{\lambda\in\Lambda}.$$

I algunes de les propietats dels operadors diagonal i de translació sobre els espais de successions.

En la segona secció del primer capítol presentem alguns espais de funcions. Comencem amb la definició dels espais de Lebesgue ponderats amb normes mixtes.

Definició 4. *Siguen $1\le p,q<\infty$, i m una funció pes en $\mathbb{R}^{2d}$. Aleshores l'espai ponderat amb normes mixtes, $L_m^{p,q}(\mathbb{R}^{2d})$, consisteix en totes les funcions mesurables Lebesgue f tals que*

$$\|f\|_{L_m^{p,q}} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x,y)m(x,y)|^p\,dx\right)^{\frac{q}{p}}dy\right)^{\frac{1}{q}} < \infty.$$

Als casos en què $p=\infty$ o $q=\infty$, la p-norma corresponent es reemplaçada pel suprem essencial.

Després introduïm la definició de la transformada de Fourier.

Definició 5. *La transformada de Fourier d'una funció $f \in L^1(\mathbb{R}^d)$ es defineix com*

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi \mathbf{i} x \cdot \omega} dx$$

on $x \cdot \omega = x\omega$ és el producte escalar en $\mathbb{R}^d$.

Presentem també la definició de la transformada temps-curt de Fourier.

Definició 6. *La transformada temps-curt de Fourier (STFT), $V_g f$, d'una funció $f \in L^2(\mathbb{R}^d)$ respecte a una finestra $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ es defineix com*

$$V_g f(x, \xi) := \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i \xi y} f(y) \overline{g(y-x)}\, dy,$$

és a dir, la transformada de Fourier de $f\overline{T_x g}$.

Donat que l'espai de les distribucions temperades, $\mathcal{S}'(\mathbb{R}^d)$, és l'espai dual de la classe de Schwartz, $\mathcal{S}(\mathbb{R}^d)$, i que la definició de la transformada temps-curt de Fourier està basada en el producte escalar, aquesta es pot estendre $f \in \mathcal{S}'$, agafant $g \in \mathcal{S}$. En aquest cas sabem que $V_g f$ és una funció continua. D'aquesta manera podem considerar la definició dels espais de modulació ponderats.

Definició 7. *Siga una finestra no nula $g \in \mathcal{S}(\mathbb{R}^d)$, un pes m v_s-moderat, $s > 0$, i $1 \le p, q \le \infty$, l'espai de modulació $M_m^{p,q}(\mathbb{R}^d)$ consisteix en totes les distribucions temperades $f \in \mathcal{S}'(\mathbb{R}^d)$ tals que $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$, és a dir,*

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, m(x,\omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} < \infty,$$

amb els canvis usuals quan $p = \infty$ o $q = \infty$. Si $p = q$ escrivim $M_m^p(\mathbb{R}^d)$ en lloc de $M_m^{p,p}(\mathbb{R}^d)$. Aleshores $M_m^{p,q}(\mathbb{R}^d)$ és un espai de Banach, i la seua definició és independent de la finestra g escollida.

En aquesta subsecció mostrem les propietats d'aquests espais que necessitarem al llarg de la memòria. També definim els espais de Wiener.

Definició 8. *Siguen B_1 i B_2 espais de Banach, sent B_1 un espai de funcions mesurables. Fixem $g \in \mathcal{C}_0^\infty(\mathbb{R}^d) \setminus \{0\}$. L'espai d'amalgames de Wiener $W(B_1, B_2)$ amb component local B_1 i component global B_2 es defineix com l'espai de totes les funcions f que localment pertanyen a B_1 tals que $f_{B_1} \in B_2$, $f_{B_1}(x) = \|fT_x g\|_{B_1}$. $W(B_1, B_2)$ és un espai de Banach dotat amb la norma*

$$\|f\|_{W(B_1,B_2)} := \|f_{B_1}\|_{B_2} = \|\|fT_x g\|_{B_1}\|_{B_2}.$$

I algunes de les propietats dels espais de Modulació i de Wiener.

En la següent subsecció introduïm els frames de Gabor. Fixem una funció $g \in L^2(\mathbb{R}^d)$ i un reticle $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, per a $\alpha, \beta > 0$. La família de funcions $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$, on $\pi(\lambda_1, \lambda_2)g = M_{\lambda_1} T_{\lambda_2} g(t)$, es anomenada sistema de Gabor i diem que és un frame de Gabor si existeixen constants $A, B > 0$ tals que

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2, \quad \text{per a tot } f \in L^2(\mathbb{R}^d).$$

En l'última secció del primer capítol, introduïm la condició GRS per als pesos. Diem que v satisfà la condició GRS (Gelfand-Raikov-Shilov [GRS57]) quan,

$$\lim_{n\to\infty} v(nz)^{1/n} = 1 \qquad \text{per a tot } z \in \mathbb{R}^{2d}.$$

I veiem que es poden definir espais de modulació amb pesos complint aquesta condició si remplaçem la classe de Schwartz per espais de tipus Gelfand-Shilov.

Definició 9. *Siguen $s, r \geq 0$. Una funció $f \in S(\mathbb{R}^d)$ pertany a un espai tipus Gelfand-Shilov $S_r^s(\mathbb{R}^d)$ si existeixen constants $A, B > 0$ tals que*

$$|x^\alpha \partial^\beta f(x)| \lesssim A^{|\alpha|} B^{|\beta|} (\alpha!)^r (\beta!)^s, \; \textit{per a tot } \alpha, \beta \in \mathbb{N}^d.$$

Al segon capítol treballem amb multiplicadors incondicionalment convergents. Un multiplicador sobre un espai de Hilbert separable H és un operador fitat

$$M_{m,\Phi,\Psi} : H \to H,\ f \mapsto \sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n,$$

on $\Phi = (\Phi_n)_n$ i $\Psi = (\Psi_n)_n$ són successions en H i $m = (m_n)_n$ és una successió d'escalars anomenada símbol. Un multiplicador és incondicionalment convergent si la sèrie anterior convergeix incondicionalment per a cada $f \in H$. Aquests multiplicadors són una generalització dels Multiplicadors de Gabor,

$$M_m f = \sum_{\lambda \in \Lambda} m_\lambda \langle f, \pi(\lambda) g \rangle \pi(\lambda) h.$$

El quals estan inspirats en el desenvolupament en termes d'un frame de Gabor d'una funció,

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) h.$$

En aquest cas h es una finestra dual de g.

En la situació en què $\Phi = (\Phi_n)_n$ i $\Psi = (\Psi_n)_n$ són successions Bessel en H i $m \in \ell^\infty$ l'operador $M_{m,\Phi,\Psi}$ és anomenat multiplicador de Bessel. Recordem que $\Psi = (\Psi_n)_n$ s'anomenat successió de Bessel si existeix una constant $B > 0$ tal que

$$\sum_{n=1}^{\infty} |\langle f, \Psi_n \rangle|^2 \leq B \|f\|^2$$

per a cada $f \in H$. En [Bal07] es prova que tot multiplicador de Bessel és incondicionalment convergent. Balazs i Stoeva, [SB13b], donen exemples de successions que no son Bessel i símbols que no estan fitats que defineixen multiplicadors incondicionalment convergents. No obstant això, tots els exemples són obtinguts de multiplicadors Bessel després d'algun truc. De fet, Balazs i Stoeva conjecturen, en [SB13a], que qualsevol multiplicador incondicionalment convergent es pot escriure com un multiplicador de Bessel amb símbol constant.

Conjectura 10. *[SB13a, Conjecture 1] Siga*

$$M_{m,\Phi,\Psi} : H \to H$$

un multiplicador incondicionalment convergent, aleshores existeixen successions escalars $(a_n)_n$ i $(b_n)_n$ tals que

$$m_n = a_n \cdot \bar{b}_n$$

i

$$(a_n \Phi_n)_n \;\; i \;\; (b_n \Psi_n)_n$$

són successions de Bessel en H.

En [SB13a] s'obtenen diverses classes de multiplicadors per als quals la conjectura és certa. Ells proven que aquesta és la situació dels multiplicadors de la forma $M_{m,\Phi,\Phi}$ [SB13a, Proposition 4.2] i també per a multiplicadors amb la propietat de que la successió $(|m_n| \cdot ||\Phi_n|| \cdot ||\Psi_n||)_n$ és fitada inferiorment per una constant estrictament positiva, [SB13a, Proposition 1.1].

Per a començar considerem el cas en què $m_n = 1$ i $\Psi_n = g$ per a cada $n \in \mathbb{N}$. Aleshores la conjectura té una resposta positiva si, i només si, per a tota successió incondicionalment sumable $(\Phi_n)_n$ en un espai de Hilbert separable H, podem trobar $(\alpha_n)_n \in \ell^2$ tal que $(\frac{1}{\alpha_n}\Phi_n)_n$ és una successió de Bessel en H. En conseqüència, l'objectiu principal del segon capítol és analitzar l'estructura de les successions incondicionalment sumables en un espai de Hilbert separable. Primer mostrem una reformulació del nostre cas particular.

Lema 11. *Els següents enunciats són equivalents:*

(a) Cada successió incondicionalment sumable $(\Phi_n)_n$ en H es pot escriure com $\Phi_n = \alpha_n f_n$, on $(\alpha_n)_n \in \ell^2$ i $(f_n)_n$ és una successió de Bessel en H.

(b) Cada operador fitat $T : c_0 \to H$ es pot factoritzar com

$$T = A \circ D_\alpha,$$

on $D_\alpha : c_0 \to \ell^2$ és un operador diagonal i $A : \ell^2 \to H$ és un operador fitat.

Després introduïm una reformulació de la conjectura de Balazs i Stoeva, en termes similars a la reformulació en el nostre cas en particular.

Proposició 12. *Assumim que la sèrie $\sum_n \langle f, \Psi_n \rangle \Phi_n$ convergeix incondicionalment per a tota $f \in H$. Aleshores, els següents enunciats són equivalents:*

(a) Existeix $(c_n)_n$ tal que $\{\overline{c}_n \Psi_n\}_n$ i $\{\frac{1}{c_n}\Phi_n\}_n$ són successions de Bessel en H.

(b) L'operador bilineal continu

$$\begin{aligned} T : c_0 \times H \quad &\longrightarrow \quad H \\ (\alpha, f) \quad &\longmapsto \quad \sum_n \alpha_n \langle f, \Psi_n \rangle \Phi_n \end{aligned}$$

admet una factorització

$$T = B \circ D$$

on $B : \ell^2 \to H$ és un operador fitat i

$$D : c_0 \times H \to \ell^2$$

és un operador bilineal continu tal que per a cada $f \in H$,

$$D(\cdot, f) : c_0 \to \ell^2$$

és un operador diagonal.

(c) Existeixen dos operadors fitats

$$A : H \to \ell^2 \ \text{ i } \ B : \ell^2 \to H$$

tals que per a cada $\alpha \in c_0$ l'operador $T_\alpha \in L(H)$, definit per $T_\alpha(f) = \sum_n \alpha_n \langle f, \Psi_n \rangle \Phi_n$, es pot factoritzar com

$$T_\alpha = B \circ D_\alpha \circ A.$$

Emprant la reformulació del nostre cas particular arribem a un resultat que es pot veure com una millora del Teorema d'Orlicz, [DJT95, Teorema 1.11] o [Hei11, Teorema 3.16], que afirma que cada successió incondicionalment sumable en un espai de Hilbert és absolutament 2-sumable. Aquest és el resultat principal del segon capítol.

Teorema 13. *Cada successió incondicionalment sumable* $(\Phi_n)_n$ *en un espai de Hilbert separable* H *es pot expressar com* $\Phi_n = \overline{a}_n f_n$, *on* $(a_n)_n \in \ell^2$ *i* $(f_n)_n$ *és una successió de Bessel en* H.

Aquest teorema ens dóna una resposta positiva a la conjectura de Balazs i Stoeva quan $(\Psi_n)_n$ és una successió constant. A continuació considerem una situació més general.

Corol·lari 14. *Siga* $M_{m,\Phi,\Psi}$ *un multiplicador incondicionalment convergent i assumim que* 0 *no és un punt d'acumulació dèbil de la successió* $\left(\frac{\Psi_n}{\|\Psi_n\|}\right)_n$. *Aleshores, existeixen successions escalars* $(a_n)_n$ *i* $(b_n)_n$ *tals que* $m_n = a_n \cdot \overline{b}_n$ *i a més* $(a_n\Phi_n)_n$ *i* $(b_n\Psi_n)_n$ *són successions de Bessel en* H.

En aquest corol·lari la condició sobre la successió $(\Psi_n)_n$ es pot reemplaçar per una condició similar en $(\Phi_n)_n$. El resultat següent mostra que la conjectura establerta per Balazs i Stoeva en [SB13a] és certa sota la hipòtesi més forta de la convergència absoluta de la sèrie.

Teorema 15. *Siga* $M_{m,\Phi,\Psi}$ *tal que per a cada* $f \in H$, *la sèrie*

$$\sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n$$

convergeix absolutament en H. *Aleshores existeixen successions escalars* $(a_n)_n$ *i* $(b_n)_n$ *tals que* $m_n = a_n \cdot \overline{b_n}$, *i a més* $(b_n\Psi_n)_n$ *i* $(a_n\Phi_n)_n$ *són successions de Bessel en* H.

Quan la convergència absoluta es reemplaça per la convergència incondicional en l'espai $S^2(H)$ dels operadors de Hilbert-Schmidt obtenim el següent resultat.

Teorema 16. *Siga B_H la bola unitat tancada de H dotada amb la topologia dèbil i assumim que la sèrie*

$$\sum_{n=1}^{\infty} m_n \Psi_n \otimes \Phi_n$$

convergeix incondicionalment en $S^2(H)$. Aleshores, per a cada mesura de probabilitat de tipus Borel μ sobre B_H existeixen successions escalars $(a_n)_n$ i $(b_n)_n$ tals que $m_n = a_n \cdot \overline{b}_n$, $(a_n\Psi_n)_n$ és una successió de Bessel en H i $(j_\mu(b_n\Phi_n))_n$ és una successió de Bessel en $L^2(B_H, \mu)$. En particular

$$\sum_{n=1}^{\infty} |\langle f, b_n\Phi_n\rangle|^2 < \infty$$

per a μ-quasi tota $f \in B_H$.

I arribem al últim resultat del segon capítol.

Teorema 17. *Siga (X, μ) un espai de mesura finita i $H \subset L^2(X, \mu)$ un espai de Hilbert admetent un nucli de reproducció $K(x, y)$. Fixem $v(x)^{-1} := \|K(x, \cdot)\|$. Assumim que la sèrie*

$$\sum_{n=1}^{\infty} m_n \Psi_n \otimes \Phi_n$$

convergeix incondicionalment en $S^2(H)$. Aleshores existeixen successions escalars $(a_n)_n$ i $(b_n)_n$ tals que $m_n = a_n \cdot \overline{b}_n$, $(a_n\Psi_n)_n$ és una successió de Bessel en H i $(b_n\Phi_n)_n$ és una successió de Bessel en $L^2_v(X, \mu)$.

Els resultats presentats en aquest capítol estan continguts en [FGP17b] i [FGP17a].

L'objectiu del tercer capítol és estudiar la compacitat dels operadors integrals de Fourier quan actuen sobre espais de modulació ponderats. La fitació i propietats de classes de Schatten d'aquests operadors ha sigut estudiada per alguns autors sota diferents suposicions per a la fase i el símbol. Veure per exemple [RS06, CR14, Bis11, Bou97, CNR09a, CNR10b, RS06, CT07, CT09, TCG10]. No obstant això no es coneixien caracteritzacions de la compacitat. El nostre enfocament de l'estudi de la compacitat segueix el punt de vista de [CNR10b], és a dir, els nostres resultats depenen de la representació matricial del FIO respecte a un frame de Gabor.

L'operador integral de Fourier, FIO, T amb símbol $\sigma \in L^\infty(\mathbb{R}^{2d})$ i fase real Φ sobre $\mathbb{R}^{2d}$ es pot definir formalment per

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)d\eta.$$

La fórmula anterior defineix un operador continu de $\mathcal{S}(\mathbb{R}^d)$ a $\mathcal{S}'(\mathbb{R}^d)$. La fase $\Phi(x,\eta)$ és *tame*, el que significa que és diferenciable sobre $\mathbb{R}^{2d}$ i compleix les estimacions

$$|\partial_z^\alpha \Phi(z)| \le C_\alpha, \quad |\alpha| \ge 2, z \in \mathbb{R}^{2d},$$

i la condició de no degeneració

$$|\det \partial^2_{x,\eta}\Phi(x,\eta)| \ge \delta > 0, \quad (x,\eta) \in \mathbb{R}^{2d}.$$

El símbol σ sobre $\mathbb{R}^{2d}$ compleix

$$|\partial_z^\alpha \sigma(z)| \le C_\alpha, \quad \text{a.e. } z \in \mathbb{R}^{2d}, \ |\alpha| \le 2N \tag{1}$$

per a $N \in \mathbb{N}$ fixada. Ací ∂_z^α denota la derivada distribucional. Quan $\Phi(x,\eta) = x\eta$ recuperem els operadors pseudodifferencials en la forma de Kohn-Nirenberg.

Els frames permeten relacionar operadors amb matrius de la següent manera.

Definició 18. *La matriu de Gabor associada a un operador lineal i continu $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ es defineix com*

$$M(T) = \left(\langle T(\pi(\lambda)g), \pi(\mu)g\rangle\right)_{(\mu,\lambda)\in\Lambda\times\Lambda}.$$

Si T és un FIO amb símbol σ i fase Φ escrivim $M(\sigma,\Phi)$ en lloc de $M(T)$.

Teorema 19. *Siga $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ un operador lineal i continu i $\mathcal{G}(g,\Lambda)$ un frame de Gabor amb $g \in \mathcal{S}(\mathbb{R}^d)$. Aleshores:*

(1) Per a $1 \leq p,q < \infty$, T pot ser estés de manera única com un operador fitat de $M^{p,q}_{m_1}(\mathbb{R}^d)$ a $M^{p,q}_{m_2}(\mathbb{R}^d)$ si, i només si, $M(T)$ defineix un operador fitat de $\ell^{p,q}_{m_1}(\Lambda)$ a $\ell^{p,q}_{m_2}(\Lambda)$.

(2) Per a $1 \leq p,q \leq \infty$, T pot ser estés com un operador dèbil-$*$ continu de $M^{p,q}_{m_1}(\mathbb{R}^d)$ a $M^{p,q}_{m_2}(\mathbb{R}^d)$ si, i només si, $M(T)$ defineix un operador dèbil-$*$ continu de $\ell^{p,q}_{m_1}(\Lambda)$ a $\ell^{p,q}_{m_2}(\Lambda)$.

(3) Siga $1 \leq p,q \leq \infty$ i assumim que $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ és dèbil-$*$ continu. Aleshores $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ es compacte si, i només si, $M(T) : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ ho és.

El resultat clau en [CNR10b] mostra que la representació matricial d'un FIO respecte a un frame de Gabor $\mathcal{G}(g,\Lambda)$ amb $g \in \mathcal{S}(\mathbb{R}^d)$ està ben organitzada. De fet, per a una fase tame Φ i un símbol σ complint la condició (1) existeix una constant $C_N > 0$ tal que

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C_N \langle \chi(\lambda) - \mu\rangle^{-2N}, \tag{2}$$

per a cada $\lambda, \mu \in \Lambda$ on χ és la transformació canònica de la fase Φ. Com usualment, $\langle z\rangle$ és una abreviació de $(1+|z|^2)^{1/2}$. L'aplicació $(x,\xi) = \chi(y,\eta)$ és bilipschitz $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ i ve definida a partir del sistema

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta), \\ \xi = \nabla_x \Phi(x,\eta). \end{cases}$$

L'estimació (2) és una extensió del resultat previ de Gröchenig [Grö06] respecte a la quasi-diagonalització del PSDOs. Veure també [GR08]. La condició (1) sobre el símbol es pot relaxar a $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ per a alguna $s_0 > 2d$. De fet, si $\mathcal{G}(g,\Lambda)$ és un frame de Parseval, aleshores l'estimació (2) també es manté, com provaren en [CGN12],

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle \chi(\lambda) - \mu\rangle^{-s_0}, \quad \text{per a tot } \lambda, \mu \in \Lambda. \tag{3}$$

Nosaltres emprem aquesta estimació per a estudiar la compacitat dels FIOs quan actuen sobre espais de modulació ponderats. Comencem definint un espai de matrius.

Definició 20. *Siga v un pes submultiplicatiu sobre $\mathbb{R}^{2d}$ i assumim que $\psi : \Lambda \to \Lambda$ satisfà*

$$M = \sup_{\lambda \in \Lambda} card\ \psi^{-1}(\{\lambda\}) < \infty.$$

Definim $\mathcal{C}_{v,\psi}(\Lambda)$ com el conjunt de totes les matrius $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma' \in \Lambda}$ tals que

$$\|A\|_{\mathcal{C}_{v,\psi}} = \sum_{\gamma \in \Lambda} v(\gamma) \cdot \sup_{\lambda \in \Lambda} \left| a_{\psi(\lambda)+\gamma,\lambda} \right| < \infty.$$

L'estimació (3) ens permet arribar al següent resultat.

Proposició 21. *Siga T un FIO tal que la seua fase Φ siga tame i $\sigma \in M^{\infty}_{1 \otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. Aleshores, per a tota $0 \leq s < s_0 - 2d$ tenim*

$$M(\sigma, \Phi) \in \mathcal{C}_{v_s,\chi'}(\Lambda).$$

on $\chi' : \Lambda \to \Lambda$ es una versió discretitzada de la transformació canònica χ.

Les matrius que pertanyen a aquest espai tenen certes propietats.

Proposició 22. *Siga $m = (m_\lambda)_{\lambda \in \Lambda}$ una successió v-moderada positiva, $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma' \in \Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ i $1 \leq p \leq \infty$ donat. Aleshores $A : \ell^p_{m \circ \psi}(\Lambda) \to \ell^p_m(\Lambda)$ és un operador fitat, que també és dèbil-$*$ continu.*

Teorema 23. *Siguen $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma' \in \Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ i $1 \leq p \leq \infty$ donats. Aleshores, $A : \ell^p_{m \circ \psi}(\Lambda) \to \ell^p_m(\Lambda)$ és un operador compacte si, i només si*

$$a^{\gamma} := \left(a_{\psi(\lambda)+\gamma,\lambda}\right)_{\lambda \in \Lambda} \in c_0(\Lambda) \quad \textit{per a tot } \gamma \in \Lambda.$$

Aquestes propietats ens duen a caracteritzar la compacitat dels nostres operadors.

Teorema 24. *Siga T un FIO tal que la seua fase Φ es tame i $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. Aleshores, per a tota $0 \le s < s_0 - 2d$, les següents condicions són equivalents*

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *és un operador compacte.*

(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *és un operador compacte per a alguna $1 \le p < \infty$ i per a algun pes v_s-moderat m.*

(3) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *és un operador compacte per a tota $1 \le p < \infty$ i per a tot pes v_s-moderat m.*

(4) $(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle)_\lambda \in c_0(\Lambda)$ *per a tota $\mu \in \Lambda$.*

En particular deduïm que la compacitat no depèn de p o m.

Teorema 25. *Siga T un FIO tal que la seua fase Φ es tame i $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, siga $0 \le s < s_0 - 2d$. Si $\sigma \in M^0(\mathbb{R}^{2d})$, aleshores $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ és un operador compacte per $1 \le p \le \infty$ i per a cada pes v_s-moderat m.*

També veiem que l'invers del resultat anterior és cert en el cas particular de les fases quadràtiques.

Definició 26. *La aplicació $\Phi : \mathbb{R}^{2d} \to \mathbb{R}$ es diu fase quadràtica si*

$$\Phi(x,\eta) = \frac{1}{2}Ax\cdot x + Bx\cdot\eta + \frac{1}{2}C\eta\cdot\eta + \eta_0\cdot x - x_0\cdot\eta,$$

on $x_0, \eta_0 \in \mathbb{R}^d$, A, B, C són matrius reals simètriques i B no és degenerada.

Teorema 27. *Siga T un FIO amb fase quadràtica Φ i $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ i siga $0 \le s < s_0 - 2d$. Aleshores els següents enunciats són equivalents:*

(1) $\sigma \in M^0(\mathbb{R}^{2d})$.

(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *és un operador compacte per a tota $1 \le p \le \infty$ i per a cada pes v_s-moderat m.*

Els operadors que hem considerat fins ara no tenen per què ser fitats en espais de modulació amb normes mixtes, com es mostra en [CNR10b]. Per superar aquest obstacle, Cordero, Nicola i Rodino van introduir una condició extra en la fase.

$$\sup_{x',x,\eta} \left|\nabla_x \Phi(x,\eta) - \nabla_x \Phi(x',\eta)\right| < \infty. \tag{4}$$

Teorema 28. *Siga T un FIO tal que la seua fase Φ és tame i satisfà la condició (4), i $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ amb $0 \le s < s_0 - 2d$. Aleshores, $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ és un operador fitat per a totes $1 \le p,q < \infty$ i per a tot pes v_s-moderat m.*

Baix aquesta condició extra els resultats de compacitat es poden estendre a espais de modulació amb normes mixtes.

Teorema 29. *Siga T un FIO tal que la seua fase Φ és tame i satisfà la condició (4), i $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ amb $0 \le s < s_0 - 2d$. Les següents afirmacions són equivalents:*

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ es un operador compacte.

(2) $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ és un operador compacte per a alguns $1 \le p,q \le \infty$ i per a algun pes v_s-moderat m.

(3) $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ és un operador compacte per a tots $1 \le p,q \le \infty$ i per a tot pes v_s-moderat m.

Com a conseqüència recuperem i millorem alguns resultats per a PSDOs obtinguts en [FG06, FG07, FG10].

Teorema 30. *Siga $\sigma \in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ donada. Aleshores les següents afirmacions són equivalent:*

(1) $L_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ es compacte.

(2) $L_\sigma : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ es compacte per a totes p, $q \in [1,\infty]$ i tot pes v_s-moderat m.

(3) $L_\sigma : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *es compacte per algun* $p, q \in [1, \infty]$ *i algun pes* v_s*-moderat* m.

(4) $\sigma \in M^0(\mathbb{R}^{2d})$.

En l'última secció d'aquest capítol veiem que tota aquesta argumentació es pot aplicar en l'estudi dels FIOs sobre espais de modulació amb pesos GRS, imposant estimacions similars en la fase i el símbol.

Alguns dels resultats presentats en aquest capítol estan continguts en [FGP18].

L'objectiu del quart capítol és buscar condicions per a que l'operador integral,

$$Af(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \tag{5}$$

amb nucli

$$K(x,y) = \int_{\mathbb{R}^d} \Phi(u) e^{-2\pi i(\beta(|u|)u\cdot y - u\cdot x)} du, \tag{6}$$

siga fitat sobre alguns espais de Lebesgue. Aquest operador integral es pot veure com un FIO de tipus II,

$$T_{II,\varphi,\sigma} f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i(\varphi(y,u) - u\cdot x)} \sigma(y,u) f(y) dy\, du,$$

amb $\varphi(y,u) = \beta(|u|)u\cdot y$ i $\sigma(y,u) = \Phi(u)$. És interessant trobar estimacions del tipus

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\, dx < \infty, \tag{7}$$

ja que aquesta estimació implica que l'operador corresponent A és fitat sobre $L^1(\mathbb{R}^d)$. A la funció $\Phi(u)$ se li demana un bon decaïment a l'infinit, però pot no ser diferenciable a l'origen $u = 0$. Un exemple típic ve donat per funcions radials

$$\Phi(u) = \frac{|u|}{(1+|u|^2)^m} \tag{8}$$

amb un m gran i real.

La fase $\beta(r)$ és una funció real i diferenciable en $(0, +\infty)$ però pot tindre una singularitat de tipus Hölder en l'origen. Com a exemple simplificat podem considerar el cas

$$\beta(r) = a + br^\gamma, \quad 0 < r \leq 1, \tag{9}$$

per algunes $a, b \in \mathbb{R}$, $\gamma \in (0, 1)$. Quan $r \to +\infty$, assumim que $\beta(r)$ s'aproxima a una constant.

Com a cas bàsic suposem $\beta(r) = a$, $r > 0$, és una funció constant. En aquest cas,

$$K(x, y) = \mathcal{F}\Phi(ay - x)$$

i la estimació (7) s'obté si, i només si, $\Phi \in \mathcal{F}L^1(\mathbb{R}^d)$, i.e. Φ té transformada de Fourier en $L^1(\mathbb{R}^d)$.

Al cas model anterior, (9), $\beta(r)$ aproxima una constant en ambdós casos, quan $r \to 0^+$ i quan $r \to +\infty$ i és diferenciable entre ells, aleshores es pot conjecturar que una estimació a través de la transformada de Fourier es manté també en aquest cas. Però, no és el cas, fins i tot per a fases diferenciables:

Proposició 31. *En dimensió $d = 1$, per a qualsevol $1 \leq p \leq 2$, considerem la funció pes*

$$v_m(y) = (1 + |y|)^m, \quad y \in \mathbb{R},$$

amb $m \in \mathbb{R}$ tal que

$$m < \frac{1}{p} - \frac{1}{2}.$$

Siga $\beta \in \mathcal{C}^\infty((0, +\infty))$ tal que

$$\tilde{\varphi}(u) = \beta(|u|)u$$

s'estén com un difeomorfisme diferenciable no lineal $\mathbb{R} \to \mathbb{R}$ *que satisfà*

$$\tilde{\varphi}(u) = u, \quad |u| \geq 1.$$

(per tant, $\beta(|u|) = 1$*, per a* $|u| \geq 1$*). Siga* $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$*,* $\Phi(u) = 1$ *per a* $|u| \leq 1$*.*

Aleshores l'operador A *en* (5) *no es pot estendre com un operador fitat de* $L^p_{v_m}(\mathbb{R})$ *a* $L^p(\mathbb{R})$*.*

Aleshores l'estimació ponderada

$$\int_{\mathbb{R}^d} |K(x,y)|\, dx \lesssim (1+|y|)^s \tag{10}$$

no es compleix per a $s < 1/2$.

Això sembla sorprenent, però es pot considerar una manifestació del fenomen Beurling-Helson [BH53, CNR10a, LO94, Oko09, RSTT11], que, en termes generals, estableix que l'operador de canvi de variable $f \mapsto f \circ \psi$ no és fitat en $\mathcal{F}L^1(\mathbb{R}^d)$ a no ser que $\psi : \mathbb{R}^d \to \mathbb{R}^d$ siga una aplicació afí. De fet l'operador A en (5) amb nucli $K(x,y)$ en (6) es pot escriure com

$$Af = \mathcal{F}^{-1}\Phi * \mathcal{F}^{-1}(\mathcal{F}f \circ \tilde{\varphi}), \quad \text{amb } \tilde{\varphi}(u) := \beta(|u|)u.$$

Per tant, és interesant analitzar el creixement en (10). Comencem amb la continuïtat en els espais L^1 ponderats per a l'operador A en (5), en aquest cas tindrem una pèrdua de decaïment. Aquesta es pot provar a través d'una estimació de tipus Schur per al nucli K.

Teorema 32. *Considerem funcions* $\Phi \in M^1(\mathbb{R}^d)$ *i* $\beta : (0,+\infty) \to \mathbb{R}$*. A més, assumim que per algun exponent* $\gamma \in (-1,1]$*, amb* $\ell = \lfloor d/2 \rfloor + 1$*,*

$$|\partial^\alpha \beta(|u|)u| \leq C_\alpha |u|^{\gamma+1-|\alpha|}, \quad \textit{per a } 0 \neq |u| \leq 1,\ |\alpha| \leq \ell,$$

on $C_\alpha > 0$*, i*

$$|\partial^\alpha \beta(|u|)u| \leq C'_\alpha, \qquad \textit{per a } |u| \geq 1,\ 2 \leq |\alpha| \leq 2\ell,$$

amb $C'_\alpha > 0$. *Aleshores el nucli d'integració en* (6) *satisfà*

$$\int_{\mathbb{R}^d} |K(x,y)|dx \leq C(1+|y|)^{d/(\gamma+1)},$$

per a una constant $C > 0$ *independent de* y.

Aquest resultat en el model simplificat anterior seria: *Suposem* $\beta(r)$ *com en* (9) *per a* $0 < r \leq 1$, *amb* $\gamma \in (-1,1]$ *i asumim que* β *té, com a molt, creixement lineal quan* $r \to +\infty$. *Siga* Φ *com en* (8), *amb* $m > (d+1)/2$. *Aleshores* (10) *se satisfà amb* $s = d/(\gamma+1)$.

Corol·lari 33. *Considerem funcions* $\Phi \in M^1(\mathbb{R}^d)$ *i* $\beta : (0,+\infty) \to \mathbb{R}$. *A més, siga* $\ell = \lfloor d/2 \rfloor + 1$, *assumim que la funció* $\beta(|u|)$ *s'estén com una funció* $\mathcal{C}^{2\ell}$ *en* $\mathbb{R}^d$ *i satisfà*

$$|\partial^\alpha \beta(|u|)u| \leq C_\alpha, \qquad \textit{per a } u \in \mathbb{R}^d \ \textit{ i } \ 2 \leq |\alpha| \leq 2\ell.$$

Aleshores, el nucli d'integració en (4.1.2) *satisfà*

$$\int_{\mathbb{R}^d} |K(x,y)|dx \leq C(1+|y|)^{\frac{d}{2}}.$$

Aquest resultat en el model simplificat anterior seria: *Suposem que* $\tilde{\varphi}(u) := \beta(|u|)u$ *s'estén com a una funció diferenciable en* $\mathbb{R}^d$, *amb, com a molt, creixement quadràtic a l'infinit. Siga* Φ *com en* (8), *amb* $m > (d+1)/2$. *Aleshores* (4.1.6) *se satisfà per a* $s = d/2$.

Si la fase està traslladada per una constant l'estimació es manté.

Corol·lari 34. *Considerem funcions* $\Phi \in M^1(\mathbb{R}^d)$ *i* $\beta : (0,+\infty) \to \mathbb{R}$. *Assumim que per a alguna* $\gamma \in (-1,1]$ *i* $a \in \mathbb{R}$,

$$\widetilde{\beta} := \beta - a$$

satisfà, amb $\ell = \lfloor d/2 \rfloor + 1$,

$$\left|\partial^\alpha \widetilde{\beta}(|u|)u\right| \leq C_\alpha |u|^{\gamma+1-|\alpha|}, \qquad \textit{per a } \ |u| \leq 1, \ |\alpha| \leq \ell,$$

per a $C_\alpha > 0$, *i*

$$\left|\partial^\alpha \widetilde{\beta}(|u|)u\right| \leq C'_\alpha, \qquad \text{per a } \ |u| \geq 1, \ 2 \leq |\alpha| \leq 2l,$$

amb $C'_\alpha > 0$. *Aleshores el nucli d'integració en* (6) *satisfà*

$$\int_{\mathbb{R}^d} |K(x,y)|dx \leq C(1+|y|)^{d/(\gamma+1)},$$

per a una constant $C > 0$, *independent de la variable* y.

Corol·lari 35. *Assumim les hipòtesis del Corol·lari 34 i considerem la funció pes*

$$v(y) = (1+|y|)^{d/(\gamma+1)}.$$

Aleshores l'operador integral A *en* (5) *amb nucli* K *en* (6) *és fitat de* $L^1_v(\mathbb{R}^d)$ *a* $L^1(\mathbb{R}^d)$.

Una pregunta natural després d'aquests resultats seria si es poden trobar resultats similars de continuïtat en $L^2(\mathbb{R}^d)$, amb les hipòtesis anteriors. En la següent proposició veiem que no és el cas.

Proposició 36. *Siga* $d = 1$. *Existeix un operador* A *com en* (5), *amb* β *i* Φ *complint les hipòtesis del Corol·lari 34, que no és fitat en* $L^2(\mathbb{R}^d)$.

Per últim arribem a hipòtesis adients sobre les funcions Φ i β de manera que es garantisca la continuïtat en L^2 de l'operador A.

Teorema 37. *Considerem* $\Phi \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. *Siga* $\beta : (0,\infty) \to \mathbb{R}$ *complint les següents hipòtesis:*
(i) $\beta \in \mathcal{C}^1((0,\infty))$;
(ii) Existeix $\delta > 0$ *tal que* $\beta(r) \geq \delta$, *per a tot* $r > 0$;
(iii) Existeixen $B_1, B_2 > 0$, *tals que*

$$B_1 \leq \frac{d}{dr}(\beta(r)r) \leq B_2, \qquad \text{per a tot } \ r > 0.$$

Aleshores l'operador integral A *amb nucli* K *en* (6) *és fitat en* $L^2(\mathbb{R}^d)$.

Nota 38. *El resultat anterior també funciona si canviem la funció β per $-\beta$. Per tant, baix les hipòtesis del Teorema 37 amb les hipòtesis* (*ii*) *i* (*iii*) *reemplaçades per:*
(ii)' Existeix $\delta < 0$ tal que $\beta(r) \leq \delta$, per a tot $r > 0$;
(iii)' Existeixen $B_1, B_2 < 0$, tals que

$$B_1 \leq \frac{d}{dr}(\beta(r)r) \leq B_2, \qquad \textit{per a tot } r > 0.$$

Aleshores l'operador integral A amb nucli K en (6) *és fitat en $L^2(\mathbb{R}^d)$.*

Aquest resultat en el model simplificat anterior seria: *Suposem $\beta(r)$ com en* (4.1.5) *per a $0 < r \leq 1$, amb $\gamma > 0$. Siga $\Phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ amb suport en $|u| \leq 1$. Aleshores, si $a(a + (\gamma + 1)b) > 0$ l'operador A en* (5), (6) *és fitat en $L^2(\mathbb{R}^d)$.*

Els resultats presentats en aquest capítol estan continguts en [CNP18].

L'últim capítol està dedicat a l'estudi dels multiplicadors unimodulars de Fourier. Els multiplicadors unimodulars de Fourier estan formalment definits a través de la següent expressió

$$e^{i\mu(D)} f(x) := \int_{\mathbb{R}^d} e^{2\pi i x\xi} e^{i\mu(\xi)} \hat{f}(\xi)\, d\xi,$$

on μ és una funció amb valors reals.

Els multiplicadors unimodulars de Fourier representen un dels principals camps de recerca en l'anàlisi harmònica. Les connexions amb altres branques de matemàtica pura i aplicada són incontables (combinatòria, EDPs, processament de senyals, càlcul funcional, etc.).

El prototip ve donat per la fase $\mu(\xi) = |\xi|^2$. És de gran interés l'estudi de la continuïtat d'aquests operadors sobre diferents espais. Mentre que aquests operadors representen transformacions unitàries en $L^2(\mathbb{R}^d)$, la seua continuïtat en $L^p(\mathbb{R}^d)$ per a $p \neq 2$ falla, en general. Per això recentment, diversos treballs han abordat el problema de la continuïtat en altres espais de funcions. Entre aquests, els espais més convenients han resultat ser els espais de modulació $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, àmpliament utilitzats en anàlisi de temp-freqüència [Fei83, Grö01].

Ara, En [BGOR07] es va provar que el propagador de Schrödinger ($\mu(\xi) = |\xi|^2$ en (5.1.1)) és fitat en $M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$, per a tot $1 \leq p, q \leq \infty$. Aquest resultat motiva l'estudi de la continuïtat de multiplicadors unimodulars de Fourier més generals sobre espais de modulació. Breument, per a fases no fitades (suficientment derivables) les propietats que juguen un rol clau són:

Creixement i oscil·lacions de les segones derivades $\partial^\gamma \mu$, $|\gamma| = 2$.

Per a posar els nostres resultats en context recordem tres resultats previs bàsics.

(a) *Sense creixement, oscil·lacions suaus* [BGOR07, Theorem 11]. *Suposem que*

$$|\partial^\gamma \mu(\xi)| \leq C, \qquad \textit{per a} \;\; \xi \in \mathbb{R}^d, \; 2 \leq |\gamma| \leq 2(\lfloor d/2 \rfloor + 1).$$

Aleshores $e^{i\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *és fitada per a tota* $1 \leq p, q \leq \infty$.

Aquest resultat generalitza el cas del propagador de Schrödinger, on les segones derivades de μ són de fet constants.

(b) *Sense creixement, oscil·lacions suaus* [CT09, Lemma 2.2]. *Suposem que*

$$\partial^\gamma \mu \in M^{\infty,1}(\mathbb{R}^d), \qquad \textit{per a} \;\; |\gamma| = 2.$$

Aleshores $e^{i\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *és fitat per a tota* $1 \leq p, q \leq \infty$.

Aquest lema proporciona un resultat parcial, però clau, des del qual es pot deduir que el símbol $\sigma(\xi) = e^{i\mu(\xi)}$ pertany a l'espai de les amalgames de Wiener, $W(\mathcal{F}L^1, L^\infty)$. Observeu que el resultat (b) millora el de (a), a causa de la inclusió $C^{d+1}(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$ ([Grö01, Theorem 14.5.3]). Notem també que $M^{\infty,1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, aleshores les segones derivades de μ no creixen a l'infinit, però poden oscil·lar, com $\cos|\xi|^\alpha$, amb $0 < \alpha \leq 1$ (cf. [BGOR07, Corollary 15]).

(c) *Creixement a l'infinit, oscil·lacions suaus* [MNR+09, Theorem 1.1]. *Siga $\alpha \geq 2$, i suposem que*

$$|\partial^\gamma \mu(\xi)| \leq C\langle\xi\rangle^{\alpha-2}, \quad \text{per a } \ 2 \leq |\gamma| \leq \lfloor d/2 \rfloor + 3.$$

Aleshores $e^{i\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ és fitat per a tota $1 \leq p, q \leq \infty$ i $\delta \geq d(\alpha-2)|1/p - 1/2|$.

Ací $M^{p,q}_\delta(\mathbb{R}^d) = M^{p,q}_{1\otimes\langle\cdot\rangle^\delta}$, on $(1 \otimes \langle\cdot\rangle^\delta)(x,\omega) = \langle\omega\rangle^\delta$, que és un espai ponderat en la freqüència.

En [BGOR07, Lemma 8] es va provar que l'operador $e^{i\mu(D)}$ és fitat en tot $M^{p,q}(\mathbb{R}^d)$ per a tota $1 \leq p, q \leq \infty$ si el seu símbol $e^{i\mu(\xi)}$ pertany a l'espai d'amalgames de Wiener $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ [Fei81a], la seua norma es defineix com

$$\|f\|_{W(\mathcal{F}L^1,L^\infty)} = \sup_{x\in\mathbb{R}^d} \|g(\cdot - x)f\|_{\mathcal{F}L^1}$$

on $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ és una finestra arbitraria. Açò suggereix buscar condicions sobre $\mu(\xi)$ en termes d'aquest espai. El nostre primer resultat en aquesta direcció és el següent.

Teorema 39. *(Sense creixement, oscil·lacions fortes). Siga $\mu \in C^2(\mathbb{R}^d)$, una funció real, que satisfà*

$$\partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d), \quad \text{per a } \ |\gamma| = 2.$$

Aleshores

$$e^{i\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

és fitat per a tota $1 \leq p, q \leq \infty$.

Observem que $M^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ aleshores aquest resultat millora el resultat anterior (b). Ací les segones derivades de μ encara estan fitades, però poden oscil·lar, com $\cos|\xi|^2$ (cf. [BGOR07, Theorem 14]). Aquest resultat està inspirat per [CT09, Lemma 2.2]. Malgrat això, el nostre resultat principal s'ocupa de la possibilitat de segones derivades no fitades, com es mostra en el següent teorema.

Teorema 40. *(Creixement a l'infinit, oscil·lacions fortes). Siga $\alpha \geq 2$. Siga $\mu \in C^2(\mathbb{R}^d)$, una funció real tal que*

$$\langle\xi\rangle^{2-\alpha}\partial^\gamma\mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d), \qquad \text{per a} \ \ |\gamma| = 2.$$

Aleshores,

$$e^{i\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

és fitat per a tota $1 \leq p, q \leq \infty$ i

$$\delta \geq d(\alpha - 2)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

La fita superior per δ coincideix amb la de (c), i també amb els exemples a [BGOR07, Teorema 16], on es van considerar fins i tot oscil·lacions més fortes, però només per a casos model.

El Teorema 39 és, per descomptat, un cas particular del Teorema 40 i s'utilitzarà com a pas en la demostració d'aquest.

Els resultats presentats en aquest capítol estan continguts en [NPT18].

Abstract

In this thesis, we study different aspects of operators related to time-frequency analysis. Every linear and continuous operator from the Schwartz class into its dual, the space of tempered distributions, can be written as an integral operator with kernel K, or also as an integral Fourier operator (in fact, pseudodifferential [Grö01, Theorem 14.3.5]). Different conditions on the kernel or the symbol and the phase (in the FIOs case) allow to extend the operator to various spaces of functions and distributions. Below we detail the contents of the memory.

At the first chapter we introduce the notation, the definitions of some spaces and the preliminary results that will be used throughout the thesis. The second chapter is devoted to the study of uncondicional multipliers. The main result, an improvement of Orlicz's theorem, shows that every unconditionally summable sequence in a Hilbert space can be factorized as the product of a square summable scalar sequence and a Bessel sequence. Some consequences on the representation of unconditionally convergent multipliers are obtained. The aim of the third chapter is to investigate compactness for Fourier integral operators when acting on weighted modulation spaces, using the matrix representation of Fourier integral operators with respect to a Gabor frame. As a consequence, we recover and improve some known results on compactness of pseudodifferential operators. At the fourth chapter we study conditions for the boundedness of Fourier integral operators with Hölder-continuous phase on Lebesgue spaces. We prove boundedness in L^1 with a precise loss of decay depending on the Hölder exponent, and we show by counterexamples that a loss occurs even in the case of smooth phases. The continuity in L^2 is studied as well by providing sufficient conditions and relevant counterexamples. At the last chapter we find some conditions for continuity of unimodular Fourier multipliers on modulation spaces. We find some results assuming that the second derivatives of the phase are bounded or, more generally, that its second derivatives belong to a particular Wiener amalgam space, in particular, its second derivatives could have strong oscillations at infinity.

Contents

Introduction

In this thesis, we study different aspects of operators related to time-frequency analysis. Every linear and continuous operator $A : \mathcal{S} \to \mathcal{S}'$ can be written as an integral operator

$$\langle Af, g\rangle = \langle K, g \otimes f\rangle,$$

where $K \in \mathcal{S}'$ is the kernel and $f, g \in \mathcal{S}$, or also as an integral Fourier operator (in fact, pseudodifferential [Grö01, Theorem 14.3.5]). Different conditions on the kernel or the symbol and the phase (in the case of Fourier integral operators) allow to extend the operator to various spaces of functions and distributions.

Our aim is to use time-frequency analysis techniques to study boundedness and/or compactness of Fourier integral operators, integral operators or unimodular multipliers on modulation or Lebesgue spaces. We study multipliers on separable Hilbert spaces too.

Below we detail the contents of the memory.

In the first chapter we introduce the notation and we recall the definitions of some spaces and preliminary results that will be used in the thesis.

The second chapter is devoted to the study of uncondicional multipliers.

A multiplier on a separable Hilbert space H is a bounded operator

$$M_{m,\Phi,\Psi} : H \to H,\ f \mapsto \sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n,$$

where $\Phi = (\Phi_n)_n$ and $\Psi = (\Psi_n)_n$ are sequences in H and $m = (m_n)_n$ is a scalar sequence called the symbol. The multiplier is said to be unconditionally convergent if the above series converges unconditionally for every $f \in H$. In the case that $\Phi = (\Phi_n)_n$ and $\Psi = (\Psi_n)_n$ are Bessel sequences in H and $m \in \ell^\infty$ the operator $M_{m,\Phi,\Psi}$ is called a Bessel multiplier. Bessel multipliers were introduced and studied by Balazs [Bal07] as a generalization of the Gabor multipliers considered by Feichtinger and Nowak in [FN03]. Balazs proved, in [Bal07], that each Bessel multiplier is unconditionally convergent. Balazs and Stoeva conjectured in [SB13a] that every unconditionally convergent multiplier can be written as a Bessel multiplier with constant symbol by shifting weights. More precisely,

Conjecture 1. *[SB13a, Conjecture 1] Let*

$$M_{m,\Phi,\Psi} : H \to H$$

be an unconditionally convergent multiplier, then there exist scalar sequences $(a_n)_n$ *and* $(b_n)_n$ *such that*

$$m_n = a_n \cdot \overline{b}_n$$

and such that

$$(a_n\Phi_n)_n \ \ and \ \ (b_n\Psi_n)_n$$

are Bessel sequences in H.

In the second chapter new situations where the conjecture of Balazs and Stoeva is still true will be presented.

The aim of the third chapter is to investigate compactness for Fourier integral operators, FIOs, when acting on weighted modulation spaces. Our results strongly depend on the matrix representation of a FIO with respect to a Gabor frame, since our approach to the study of the compactness of the FIOs follows the point of view of Cordero, Nicola and Rodino in [CNR10b].

For a function f on $\mathbb{R}^d$ the Fourier integral operator, or FIO, T with symbol $\sigma \in L^\infty(\mathbb{R}^{2d})$ and real phase Φ on $\mathbb{R}^{2d}$ can be formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta.$$

The above formula defines a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. The phase $\Phi(x,\eta)$ is tame, which means that it is smooth on $\mathbb{R}^{2d}$ and fulfills the estimates

$$|\partial_z^\alpha \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2, z \in \mathbb{R}^{2d},$$

and the nondegeneracy condition

$$|\det \partial^2_{x,\eta} \Phi(x,\eta)| \geq \delta > 0, \quad (x,\eta) \in \mathbb{R}^{2d}.$$

The symbol σ on $\mathbb{R}^{2d}$ satisfies

$$|\partial_z^\alpha \sigma(z)| \leq C_\alpha, \quad \text{a.e. } z \in \mathbb{R}^{2d}, \ |\alpha| \leq 2N$$

for a fixed $N \in \mathbb{N}$. Here ∂_z^α denotes the distributional derivative.

When $\Phi(x,\eta) = x\eta$ we recover the pseudodifferential operators (PSDOs) in the Kohn-Nirenberg form. Frames allow to represent operators in terms of matrices. The matrix representation of a FIO with respect to a Gabor frame $\mathcal{G}(g,\Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$ is well organized, this is the key result of Cordero, Nicola and Rodino in [CNR10b]. We will use a decay estimate to discuss the compactness of the FIOs when acting on weighted modulation spaces. More precisely, we prove that the FIO is compact when acting on some modulation space of the form $M^p_m(\mathbb{R}^d)$ if and only if the sequences

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle\big)_{\lambda\in\Lambda}$$

converge to zero for all $\mu \in \Lambda$, where χ' denotes a discrete version of the canonical transformation χ defined through the system

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta), \\ \xi = \nabla_x \Phi(x,\eta). \end{cases}$$

As was shown in [CNR10b], the operators we are considering may fail to be bounded on mixed modulation spaces. To overcome this obstacle, an extra condition on the phase was introduced in [CNR10b]. Under this additional condition, the compactness results are extended to weighted mixed modulation spaces. As a consequence, we recover and improve some compactness results for PSDOs obtained by Fernández and Galbis in [FG06, FG07, FG10]. In the last section we see that the obtained results can be applied on Fourier integral operators on modulation spaces with GRS-weights, under similar estimates in the phase and the symbol.

In the fourth chapter we study conditions for the boundedness of the integral operator,

$$Af(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy,$$

with kernel

$$K(x,y) = \int_{\mathbb{R}^d} \Phi(u)e^{-2\pi i(\beta(|u|)u\cdot y - u\cdot x)}du.$$

on some Lebesgue spaces, where $\beta(r)$ is real-valued and the function $\Phi(u)$ has a good decay at infinity but could be not smooth at the origin $u = 0$. This integral operator can be seen as a FIO of type II,

$$T_{II,\varphi,\sigma}f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i(\varphi(y,u) - u\cdot x)}\sigma(y,u)f(y)dy\,du,$$

It is interesting to find estimates of the type

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\,dx < \infty.$$

This estimate implies a continuity property for the corresponding operator between weighted L^1 spaces.

Therefore, the next natural question is whether under the same assumptions similar continuity estimates hold in $L^2(\mathbb{R}^d)$. This is not the case, but different sufficient conditions are given for the L^2-continuity.

In the last chapter we find some conditions for continuity of unimodular Fourier multipliers on modulation spaces. The unimodular Fourier multipliers are formally defined by

$$e^{\mathbf{i}\mu(D)} f(x) := \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} x\cdot\xi} e^{\mathbf{i}\mu(\xi)} \hat{f}(\xi)\, d\xi,$$

with real-valued μ. These operators can be seen as a PSDO,

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} x\cdot\eta} \sigma(x,\eta)\hat{f}(\eta) d\eta,$$

with symbol $\sigma(x,\eta) = e^{\mathbf{i}\mu(\eta)}$, or as a FIO

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)} \sigma(x,\eta)\hat{f}(\eta) d\eta,$$

with phase $\Phi(x,\eta) = x\cdot\eta + \frac{1}{2\pi}\mu(\eta)$ and constant symbol. Fourier multipliers represent one of the main research fields in harmonic analysis, where a number of challenging problems remains open [Ste93].

The function $\mu(\xi) = |\xi|^2$ gives us the prototype phase. In that case the operator $e^{i\mu(D)}$ is the propagator for the free Schrödinger equation. Hence it is of great interest to study the continuity of such operators on several functions spaces arising in PDEs. Such operators represent unitary transformations of $L^2(\mathbb{R}^d)$, but their continuity on $L^p(\mathbb{R}^d)$ for $p \neq 2$ may fail. Hence recently a number of works addressed the problem of the continuity in other function spaces. From these spaces, the more convenient spaces turned out to be the modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, widely used in time-frequency analysis [Fei83, Grö01]. This is so because Bényi, Gröchenig, Okoudjou and Rogers proved, in [BGOR07], that the Schrödinger propagator (hence $\mu(\xi) = |\xi|^2$ in (5.1.1)) is bounded $M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$, for every $1 \leq p, q \leq \infty$. This result motivates the study of the continuity of more general unimodular Fourier multipliers on modulation spaces, which is what we study in the last chapter.

Chapter 1

Preliminaries

In this chapter we introduce the notation, definitions, some spaces and preliminary results that will be used through all the thesis. The chapter is divided in 2 sections with the aim of giving an overview of the necessary preliminary results. Section 1.1 is devoted to the definition of sequence spaces as well as to some properties of them. In Section 1.2 we introduce some function spaces, particularly modulation spaces with polynomial weights and Wiener amalgam spaces. In Subsection 1.2.3 we describe how Gabor frames relate certain function spaces to sequence spaces. Finally, we introduce the definition of modulation spaces with GRS-weights and we apply Gabor frames to them.

1.1 Sequence spaces

Definition 1.1.1. *Given I and J countable sets of indices, a sequence of positive numbers $m = (m_{i,j})_{(i,j)\in I\times J}$ and $1 \leq p, q < \infty$, we consider the sequence space $\ell^{p,q}_m(I \times J)$ consisting of those sequences $x = (x_{i,j})_{(i,j)\in I\times J}$ such that*

$$\|x\|_{\ell^{p,q}_m} := \left(\sum_{j\in J}\left(\sum_{i\in I}|x_{i,j}m_{i,j}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

In the case that $p = \infty$ or $q = \infty$, the previous norm is modified in the usual way, for instance, if $p = \infty$,

$$\|x\|_{\ell_m^{\infty,q}} := \left(\sum_{j \in J} \left(\sup_{i \in I} |x_{i,j} m_{i,j}| \right)^q \right)^{\frac{1}{q}}.$$

If $p = q$ we have the weighted ℓ^p-spaces.

Note that it is not necessary to indicate the order of addition given that the sums are unconditionally convergent. Recall that if $p_1 \leq p_2$ and $q_1 \leq q_2$ then $\ell^{p_1,q_1}(I \times J) \subseteq \ell^{p_2,q_2}(I \times J)$.

Throughout the thesis we denote by $\ell_m^{0,q}(I \times J)$, the closed subspace of $\ell_m^{\infty,q}(I \times J)$ consisting of those sequences $x \in \ell_m^{\infty,q}(I \times J)$ such that

$$\lim_{i \in I} |x_{i,j} m_{i,j}| = 0,$$

for every $j \in J$. We denote $\ell_m^{p,0}(I \times J)$ analogously.

It turns out that $\ell_m^{0,q}(I \times J)$ (resp. $\ell_m^{p,0}(I \times J)$) coincides with the closure in $\ell_m^{\infty,q}(I \times J)$ (resp. $\ell_m^{p,\infty}(I \times J)$) of the set of those sequences with finitely many non zero coordinates, denoted $\mathbb{C}^{(I \times J)}$.

Also, $\ell_m^{0,0}(I \times J)$ coincides with the Banach space $c_{0,m}(I \times J)$ of all sequences $x = (x_{i,j})_{(i,j) \in I \times J}$ whose product with m converges to 0.

All these spaces, $\ell_m^{p,q}(I \times J)$, are Banach spaces for p, $q \in [1, \infty] \cup \{0\}$.

When p, $q \in [1, \infty) \cup \{0\}$, the dual of $\ell_m^{p,q}(I \times J)$ can be identified with $\ell_{\frac{1}{m}}^{p',q'}(I \times J)$, where p' and q' are the conjugate exponents of p and q. As usual, we agree that the conjugate exponent of 0 is 1. The duality is given by

$$\ell_m^{p,q}(I \times J) \times \ell_{\frac{1}{m}}^{p',q'}(I \times J) \to \mathbb{C}, \ (x, y) \mapsto x \cdot y = \sum_{(i,j) \in I \times J} x_{i,j} y_{i,j}.$$

Definition 1.1.2. *Given a sequence $a = (a_{i,j})_{(i,j) \in I \times J}$ of complex numbers, we denote by D_a* ***the diagonal operator***

$$D_a : \mathbb{C}^{I \times J} \to \mathbb{C}^{I \times J}, \ x = (x_{i,j})_{(i,j) \in I \times J} \mapsto (a_{i,j} x_{i,j})_{(i,j) \in I \times J}.$$

Proposition 1.1.3. *The diagonal operator D_a is bounded on $\ell_m^{p,q}(I \times J)$ if, and only if, $a \in \ell^\infty(I \times J)$ and*

$$\|D_a\| = \|a\|_\infty$$

for all p, $q \in [1, \infty] \cup \{0\}$ and every m.

Proposition 1.1.4. *The diagonal operator D_a is a compact operator on $\ell_m^{p,q}(I \times J)$ if, and only if, $a \in c_0(I \times J)$.*

The result follows from the fact that the elements of each $a \in c_0(I \times J)$ are the $\|\cdot\|_\infty$-limit of the difference between the operator D_a and its finite sections.

A set of the form $\Lambda = A\mathbb{Z}^N$, with A an invertible $N \times N$ matrix, is called a **lattice** on $\mathbb{R}^N$. We observe that $\gamma + \Lambda = \Lambda$, whenever $\gamma \in \Lambda$. If I and J are lattices in $\mathbb{R}^d$ and $\mathbb{R}^\ell$ respectively, we write $\Lambda := I \times J$, which is a lattice in $\mathbb{R}^N$ $(N = d + \ell)$.

Definition 1.1.5. *Let $\gamma \in \Lambda$, for a lattice Λ in $\mathbb{R}^N$, the* ***translation operator*** *$T_\gamma : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$ is defined by*

$$T_\gamma \left(x_\lambda\right)_{\lambda \in \Lambda} = \left(x_{\lambda - \gamma}\right)_{\lambda \in \Lambda}.$$

Let us introduce two concepts needed for the last result of the section.

A function $v : \Lambda \to (0, \infty)$ is said to be a **submultiplicative weight** if it is symmetric on each coordinate and

$$v(r + k) \le v(r)v(k), \text{ for all } r, k \in \Lambda.$$

Given a submultiplicative weight v, a sequence of positive numbers $m = (m_\gamma)_{\gamma \in \Lambda}$ is ***v*-moderate**, with constant C_m, if

$$m_{\gamma + \gamma'} \le C_m m_\gamma v(\gamma'), \text{ for all } \gamma, \gamma' \in \Lambda.$$

If m is v-moderate, then $1/m$ is also v-moderate.

Proposition 1.1.6. *Given $m = (m_\gamma)_{\gamma \in \Lambda}$, v-moderate with constant C_m, the translation operator T_γ is bounded on $\ell_m^{p,q}(\Lambda)$ for every $\gamma \in \Lambda$, and*

$$\|T_\gamma\| \le C_m v(\gamma),$$

for all p, $q \in [1, \infty] \cup \{0\}$.

1.2 Function spaces

Definition 1.2.1. *Let $1 \leq p, q < \infty$, and let m be a weight function on $\mathbb{R}^{2d}$. Then the* ***weighted mixed-norm spaces*** *$L_m^{p,q}(\mathbb{R}^{2d})$ consist of all Lebesgue measurable functions f such that*

$$\|f\|_{L_m^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,y)m(x,y)|^p \, dx \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}} < \infty.$$

In the case that $p = \infty$ or $q = \infty$, the corresponding p-norm is replaced by the essential supremum.

Let $1 \leq p, q \leq \infty$, and m be a weight function on $\mathbb{R}^{2d}$, the dual of $L_m^{p,q}(\mathbb{R}^{2d})$ can be identified with $L_{\frac{1}{m}}^{p',q'}(\mathbb{R}^{2d})$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

The space $L^2(\mathbb{R}^d)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)} dx.$$

We consider its extension to the pairs $\left(L_m^{p,q}, L_{\frac{1}{m}}^{p',q'} \right)$ with the same notation, $\langle \cdot, \cdot \rangle$.

Definition 1.2.2. *The* ***Fourier transform*** *of a function $f \in L^1(\mathbb{R}^d)$ is defined as*

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi \mathbf{i} x \cdot \omega} dx$$

where $x \cdot \omega = x\omega$ is the scalar product on $\mathbb{R}^d$.

Once introduced the Fourier transform it is time to remember that its inverse is $\mathcal{F}^{-1} = \mathcal{I}\mathcal{F}$, where $\mathcal{I}f(x) = f(-x)$.

Definition 1.2.3. *The* ***Schwartz class***, *$\mathcal{S}(\mathbb{R}^d)$, consists of all C^∞-functions f on $\mathbb{R}^d$ such that*

$$\sup_{x \in \mathbb{R}^d} |D^\alpha x^\beta f(x)| < \infty,$$

for all $\alpha, \beta \in \mathbb{Z}_+^d$.

At this point it is necessary to recall that the Fourier transform is an isomorphism on $\mathcal{S}(\mathbb{R}^d)$, in order to carry out the following argumentations.

Definition 1.2.4. *The elements in the dual space of the Schwartz class* $\mathcal{S}'(\mathbb{R}^d)$ *are called* ***tempered distributions****.*

Also, we consider the extension of the inner product on $L^2(\mathbb{R}^d)$, $\langle\cdot,\cdot\rangle$, to the pair $(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$. And by duality the Fourier transform can be extended to $\mathcal{S}'(\mathbb{R}^d)$ as follows

$$\langle \mathcal{F}(f), \mathcal{F}(\varphi)\rangle = \langle f, \varphi\rangle \text{ for } f \in \mathcal{S}(\mathbb{R}^d) \text{ and } \varphi \in \mathcal{S}'(\mathbb{R}^d).$$

Definition 1.2.5. *For* $1 \le p \le \infty$*, the* $\mathcal{F}L^p$ *spaces are defined by*

$$\mathcal{F}L^p(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \exists\, h \in L^p(\mathbb{R}^d),\, \mathcal{F}h = f\},$$

they are Banach spaces equipped with the norm

$$\|f\|_{\mathcal{F}L^p} := \|h\|_{L^p}, \quad \textit{with } \mathcal{F}h = f.$$

1.2.1 Modulation spaces with polynomial weights

We start this section introducing the necessary elements for the incoming definitions.

A function $v : \mathbb{R}^N \to (0, \infty)$ is said to be a **submultiplicative weight** if it is continuous, even on each coordinate and

$$v(r + k) \le v(r)v(k).$$

A map $m : \mathbb{R}^N \to (0, \infty)$ is said to be ***v*-moderate**, with constant C_m, when

$$m(r + k) \le C_m m(r)v(k)$$

for every $r, k \in \mathbb{R}^N$. If m is v-moderate, then $1/m$ is also v-moderate.

The **polynomial weights** are the submultiplicative weights of the form

$$v_s(r) = \langle r\rangle^s = (1 + |r|^2)^{\frac{s}{2}},\ s > 0.$$

We consider the **translation operator** and the **modulation operator** defined by

$$T_x f(t) := f(t-x) \quad \text{and} \quad M_\xi f(t) := e^{2\pi i \xi t} f(t).$$

We denote $\pi(\xi, x) f := M_\xi T_x f(t)$. Also, we have the next relations

$$\mathcal{F}(T_x f) = M_{-x}\mathcal{F} f, \quad \mathcal{F}(M_\xi f) = T_\xi \mathcal{F} f \quad \text{and} \quad M_\xi T_x = e^{2\pi i x \xi} T_x M_\xi.$$

Definition 1.2.6. *The* ***short-time Fourier transform (STFT)****, $V_g f$, of a function $f \in L^2(\mathbb{R}^d)$ with respect to the window $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ is defined by*

$$V_g f(x, \xi) := \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i \xi y} f(y) \overline{g(y-x)}\, dy,$$

i.e. the Fourier transform of $f\overline{T_x g}$.

As the previous definition relies on the scalar product, it can be extended to tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ taking the window $g \in \mathcal{S}(\mathbb{R}^d)$. When $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, $V_g f$ is continuous.

Definition 1.2.7. *Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a v_s-moderate weight m, $s > 0$, and $1 \le p, q \le \infty$, the* ***modulation space*** *$M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$, that is,*

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p \, m(x, \omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} < \infty,$$

with the usual changes when $p = \infty$ or $q = \infty$. If $p = q$ we write $M_m^p(\mathbb{R}^d)$ instead of $M_m^{p,p}(\mathbb{R}^d)$. Moreover, $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the window g (See e.g.[Grö01, Proposition 11.3.2]).

Recall that if m is v_s-moderate, $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$ then $M_v^1(\mathbb{R}^d) \subseteq M_m^{p_1,q_1}(\mathbb{R}^d) \subseteq M_m^{p_2,q_2}(\mathbb{R}^d) \subseteq M_{1/v}^\infty(\mathbb{R}^d)$.

It is well known that $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}_m(\mathbb{R}^d)$, for $1 \leq p, q < \infty$. The closure of $\mathcal{S}(\mathbb{R}^d)$ in $M^{\infty,q}_m(\mathbb{R}^d)$ is denoted $M^{0,q}_m(\mathbb{R}^d)$ and $M^{p,0}_m(\mathbb{R}^d)$ is defined similarly. In particular, the closure of $\mathcal{S}(\mathbb{R}^d)$ in $M^{\infty}(\mathbb{R}^d)$ is denoted by $M^0(\mathbb{R}^d)$ and consists of those tempered distributions whose STFT vanishes at infinity.

$M^p(\mathbb{R}^d)$ is invariant under the Fourier transform.

For p, $q \in [1, \infty) \cup \{0\}$ the dual of $M^{p,q}_m(\mathbb{R}^d)$ can be identified with $M^{p',q'}_{1/m}(\mathbb{R}^d)$, under the pairing

$$\langle f, h \rangle = \int\int_{\mathbb{R}^{2d}} V_g f(x, \omega) \overline{V_g h(x, \omega)} d(x, \omega),$$

where p' and q' are the conjugate exponents of p and q. As usual, we agree that the conjugate exponent of 0 is 1.

In particular, we recall that $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, [Grö01, Proposition 11.3.1].

The notation $A \lesssim B$ means $A \leq cB$ for a suitable constant $c > 0$ depending only on the dimension d and Lebesgue exponents $p, q, \ldots$, arising in the context, whereas $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$.

We have defined the modulation spaces, now let us see some properties. First of all, from [Grö01, Theorem 11.3.5] we infer the following result.

Proposition 1.2.8. *For $1 \leq p, q \leq \infty$, $M^{p,q}(\mathbb{R}^d)$ is invariant under time-frequency shifts, with*

$$\|T_x M_u f\|_{M^{p,q}} = \|f\|_{M^{p,q}}.$$

Proposition 1.2.9. *[KS11, Corollary 1.2] Let $k \in \mathbb{R}$. Then we have*

$W^{k,1}(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^n)$ if $k > d$. Conversely, if $W^{k,1}(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^n)$, then $k \geq d$.

Here $W^{k,1}(\mathbb{R}^d)$ is the ***Sobolev space*** *defined by*

$$W^{k,1}(\mathbb{R}^d) := \left\{ u \in L^1(\mathbb{R}^d) : D^\alpha u \in L^1(\mathbb{R}^d) \ \text{ for all } \alpha \in \mathbb{N}^d : |\alpha| \leqslant k \right\}.$$

Definition 1.2.10. *We consider the* ***dilation operator*** *defined by*

$$U_\lambda f(x) := f(\lambda x), \quad \lambda \neq 0.$$

The Fourier transform acts as $\mathcal{F}(U_\lambda f)(x) = \left(\frac{1}{\lambda}\right)^d (\mathcal{F}f)(\frac{x}{\lambda})$.

Now, to see the behaviour of the dilation operator on modulation spaces, we need to introduce some indices. For $(1/p, 1/q) \in [0,1] \times [0,1]$, we define the subsets

$$\begin{aligned} I_1 &: \max(1/p, 1/p') \leq 1/q, \qquad & I_1^* &: \min(1/p, 1/p') \geq 1/q, \\ I_2 &: \max(1/q, 1/2) \leq 1/p', \qquad & I_2^* &: \min(1/q, 1/2) \geq 1/p', \\ I_3 &: \max(1/q, 1/2) \leq 1/p, \qquad & I_3^* &: \min(1/q, 1/2) \geq 1/p, \end{aligned}$$

as shown in Figure 1.1.

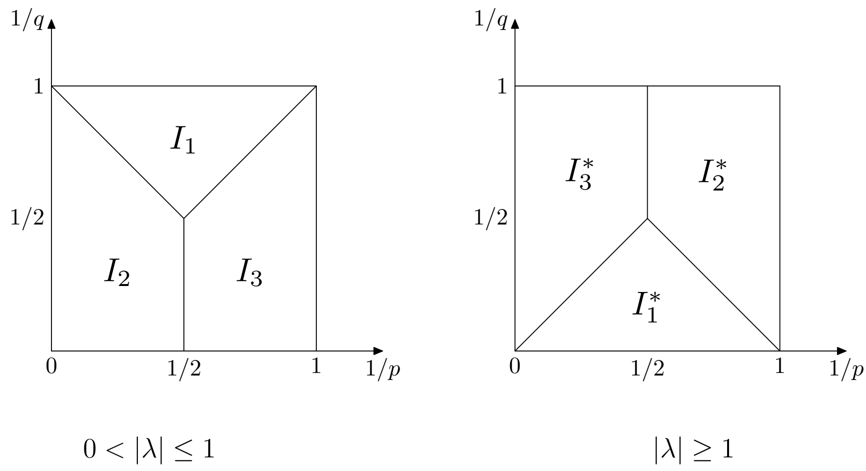

Figure 1.1: The index sets

We introduce the indices:

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p, 1/q) \in I_1^*, \\ 1/q - 1 & \text{if} \quad (1/p, 1/q) \in I_2^*, \\ -2/p + 1/q & \text{if} \quad (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p,1/q) \in I_1, \\ 1/q-1 & \text{if} \quad (1/p,1/q) \in I_2, \\ -2/p+1/q & \text{if} \quad (1/p,1/q) \in I_3. \end{cases}$$

Here is the main result about the behaviour of the dilation operator in modulation spaces.

Theorem 1.2.11. *[ST07, Theorem 3.1] Let* $1 \leq p,q \leq \infty$, *and* $\lambda \neq 0$.

(i) We have

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_1(p,q)} \|f\|_{M^{p,q}}, \textit{ for all } |\lambda| \geq 1, \textit{ for all } f \in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\alpha \in \mathbb{R}$ *such that*

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{\alpha} \|f\|_{M^{p,q}}, \textit{ for all } |\lambda| \geq 1, \textit{ for all } f \in M^{p,q}(\mathbb{R}^d),$$

then $\alpha \geq d\mu_1(p,q)$.

(ii) We have

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_2(p,q)} \|f\|_{M^{p,q}}, \textit{ for all } 0 < |\lambda| \leq 1, \textit{ for all } f \in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\beta \in \mathbb{R}$ *such that*

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{\beta} \|f\|_{M^{p,q}}, \textit{ for all } 0 < |\lambda| \leq 1, \textit{ for all } f \in M^{p,q}(\mathbb{R}^d),$$

then $\beta \leq d\mu_2(p,q)$.

1.2.2 Wiener spaces

In this section we recall the definition and some properties of Wiener amalgam spaces. We denote by $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support.

Definition 1.2.12. *Let B_1 and B_2 be Banach spaces, B_1 consisting of measurable functions. We fix $g \in \mathcal{C}_0^\infty(\mathbb{R}^d) \setminus \{0\}$. The* ***Wiener amalgam space*** *$W(B_1, B_2)$ with local component B_1 and global component B_2 is defined as the space of all functions f locally in B_1 such that $f_{B_1} \in B_2$, $f_{B_1}(x) = \|fT_x g\|_{B_1}$. $W(B_1, B_2)$ is a Banach space endowed with the norm*

$$\|f\|_{W(B_1,B_2)} := \|f_{B_1}\|_{B_2} = \|\|fT_x g\|_{B_1}\|_{B_2}.$$

Moreover, different choices of $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ generate the same space and yield equivalent norms. The Wiener spaces mainly used in this work are:

- $W(L^\infty, \ell^p)(\mathbb{R}^d)$, $1 \le p < \infty$, being Λ a lattice of $\mathbb{R}^d$, consists of those continuous functions f such that

$$\|f\|_{W(C,\ell^p)} = \left(\sum_{x\in\Lambda} \left(\sup_{y\in\mathbb{R}^d} \{|T_x f(y)|\} \right)^p \right)^{\frac{1}{p}} < \infty. \tag{1.2.1}$$

  The definition is independent of the lattice Λ.

- $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$, $1 \le p, q \le \infty$,

$$\begin{aligned} \|f\|_{W(\mathcal{F}L^p,L^q)} &= \|\|fT_x g\|_{\mathcal{F}L^p}\|_{L^q} \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathcal{F}^{-1}(fT_x g)(y)|^p dy \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}. \end{aligned} \tag{1.2.2}$$

Proposition 1.2.13. *[CN08a, Proposition 2.2] For $1 \le p, q \le \infty$, the Fourier transform maps $\mathcal{F} : W(\mathcal{F}L^p, L^q)(\mathbb{R}^d) \to W(\mathcal{F}L^q, L^p)(\mathbb{R}^d)$ continuously and it is an isomorphism when $p = q$.*

The relationship between modulation and Wiener amalgam spaces is expressed by the following result.

Proposition 1.2.14. *[CN08a, Proposition 2.4] The Fourier transform establishes an isomorphism $\mathcal{F} : M^{p,q}(\mathbb{R}^d) \to W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$, for $1 \le p, q \le \infty$. In fact, $M^p(\mathbb{R}^d) = W(\mathcal{F}L^p, L^p)(\mathbb{R}^d)$, with equivalent norms.*

The dilatation operator acts on the Wiener spaces as indicated in the following Lemma.

Lemma 1.2.15. *([CN08a, Corollary 3.2]) Let* $1 \leq p, q \leq \infty$. *Then, for every* $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$,

$$\|U_\lambda f\|_{W(\mathcal{F}L^p,L^q)} \lesssim \lambda^{d/p-d/q}\|f\|_{W(\mathcal{F}L^p,L^q)} \quad \text{for all } \ |\lambda| \geq 1$$

and

$$\|U_\lambda f\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)} \quad \text{for all } \ 0 < |\lambda| \leq 1.$$

Proposition 1.2.16. *[CN08a, Proposition 2.5] For every* $1 \leq p, q \leq \infty$ *we have*

$$\|fu\|_{W(\mathcal{F}L^p,L^q)} \lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)}\|u\|_{W(\mathcal{F}L^p,L^q)},$$

for all $f \in W(\mathcal{F}L^1, L^\infty)$ *and* $u \in W(\mathcal{F}L^p, L^q)$.*If* $p = q$, *we have*

$$\|fu\|_{M^p} \lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)}\|u\|_{M^p}.$$

Wiener amalgam spaces are invariant with respect to modulation and translation operators too [Grö01, Theorem 11.3.5].

Proposition 1.2.17. *For* $1 \leq p, q \leq \infty$, $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ *is invariant under time-frequency shifts, with*

$$\|T_x M_u f\|_{W(\mathcal{F}L^p,L^q)} = \|f\|_{W(\mathcal{F}L^p,L^q)}.$$

1.2.3 Gabor frames

In this subsection we see how Gabor frames relate modulation spaces to sequence spaces. We fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, for $\alpha, \beta > 0$. The family $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \ \lambda \in \Lambda\}$, where $\pi(\lambda_1, \lambda_2)g = M_{\lambda_1}T_{\lambda_2}g(t)$, is called a **Gabor system** and it is said to be a **Gabor frame** if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

If $A = B = 1$, then the Gabor frame is said to be a **Parseval frame**. Associated to the Gabor frame $\mathcal{G}(g, \Lambda)$ we consider the **analysis operator**

$$C_g : L^2(\mathbb{R}^d) \to \ell^2(\Lambda), \quad f \mapsto (\langle f, \pi(\lambda)g \rangle)_{\lambda\in\Lambda} ,$$

and its adjoint $D_g = C_g^*$, which is the **synthesis operator**

$$D_g : \ell^2(\Lambda) \to L^2(\mathbb{R}^d), \quad (c_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g.$$

Then $S_g = D_g \circ C_g$ is a bounded and invertible operator on $L^2(\mathbb{R}^d)$ called **frame operator**. The **canonical dual window** of g is defined as $h = S_g^{-1} g$. It turns out that $\mathcal{G}(h, \Lambda)$ is also a Gabor frame and

$$D_g \circ C_h = D_h \circ C_g = Id_{L^2(\mathbb{R}^d)}.$$

If the Gabor frame is a Parseval frame then $S_g = Id_{L^2(\mathbb{R}^d)}$ and $h = g$.

In the case that $\mathcal{G}(g, \Lambda)$ is a Gabor frame and $g \in \mathcal{S}(\mathbb{R}^d)$ then, as proved by Janssen (see [Jan95] or [Grö01, 13.5.4]), also $h = S_g^{-1}(g) \in \mathcal{S}(\mathbb{R}^d)$. Gröchenig and Leinert [GL04, 4.5] showed the existence of Parseval frames $\mathcal{G}(g, \Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$. Therefore we can reformulate some known results in the following way.

Theorem 1.2.18. *Let $g \in \mathcal{S}(\mathbb{R}^d)$. Then, for every polynomially moderate weight m and for every $1 \leq p, q \leq \infty$,*

$$C_g : M_m^{p,q}(\mathbb{R}^d) \to \ell_m^{p,q}(\Lambda) \text{ and } D_g : \ell_m^{p,q}(\Lambda) \to M_m^{p,q}(\mathbb{R}^d)$$

are bounded operators, weak-continuous, and being $h = S_g^{-1}(g)$,*

$$D_g \circ C_h = D_h \circ C_g = Id_{M_m^{p,q}(\mathbb{R}^d)}.$$

Here D_g is the transposed map of $C_g : M_{1/m}^{p',q'}(\mathbb{R}^d) \to \ell_{1/m}^{p',q'}(\Lambda)$. For $p = 1$ or $q = 1$ we take $p' = 0$ or $q' = 0$ respectively.

If $c = (c_\lambda)_{\lambda\in\Lambda}$ and $1 \leq p, q < \infty$ then $D_g(c) = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$. In the limit cases $p = \infty$ or $q = \infty$ the series in the right hand side converges to $D_g(c)$ in the weak*-topology. See for instance [FG97] or [Grö01, Chapter 12].

1.2.4 Modulation spaces with GRS-weights

Let us now see the definition of modulation spaces with GRS weights and how the Gabor frames act on them. We start introducing the so-called Gelfand-Shilov type spaces, introduced in [GS68].

Definition 1.2.19. *Let $s, r \geq 0$ be given. A function $f \in S(\mathbb{R}^d)$ is in the **Gelfand-Shilov type space** $S^s_r(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that*

$$|x^\alpha \partial^\beta f(x)| \lesssim A^{|\alpha|} B^{|\beta|} (\alpha!)^r (\beta!)^s, \text{ for all } \alpha, \beta \in \mathbb{N}^d.$$

The space $S^s_r(\mathbb{R}^d)$ is nontrivial if and only if $r + s > 1$, or $r + s = 1$ and $r, s > 0$. If $s_1 \leq s_2$ and $r_1 \leq r_2$, then $S^{s_1}_{r_1}(\mathbb{R}^d) \subseteq S^{s_2}_{r_2}(\mathbb{R}^d)$ densely. So the smallest nontrivial space with $r = s$ is provided by $S^{1/2}_{1/2}$. The action of the Fourier transform on $S^s_r(\mathbb{R}^d)$ interchanges the indices s and r, as explained in the following theorem.

Theorem 1.2.20. *For $f \in S(\mathbb{R}^d)$ we have $f \in S^s_r(\mathbb{R}^d)$ if and only if $\hat{f} \in S^r_s(\mathbb{R}^d)$.*

Therefore for $r = s$ the spaces $S^s_s(\mathbb{R}^d)$ are invariant under the action of the Fourier transform.
From now on, v denotes a non-negative funtion on $\mathbb{R}^{2d}$ satisfying the following properties:

(i) v is continuous, $v(0) = 1$, and v is even in each coordinate,

$$v(\pm z_1, \pm z_2, ..., \pm z_{2d}) = v(z_1, z_2, ..., z_{2d}),$$

(ii) v is submultiplicative,

$$v(w + z) \leq v(w)v(z), \qquad w, z \in \mathbb{R}^{2d},$$

(iii) v satisfies the **GRS-condition** (Gelfand-Raikov-Shilov [GRS57]),

$$\lim_{n\to\infty} v(nz)^{1/n} = 1, \qquad \text{for all } z \in \mathbb{R}^{2d}.$$

We call a weight satisfying properties (i)-(iii) **admissible** weight. Every weight of the form $v(z) = e^{a|z|^b}(1+|z|)^s log^r(e+|z|)$ for parameters $a, r, s \geq 0$, $0 \leq b < 1$ is admissible.

From [FGT15], we have the following observation concerning submultiplicative weights satisfying the GRS-condition.

Proposition 1.2.21. *[FGT15, Proposition 1] Let v be a submultiplicative weight on $\mathbb{R}^{2d}$. Then the following conditions are equivalent:*

(1) v satisfies the GRS-condition,

(2) v satisfies $v(x) \lesssim e^{\varepsilon|x|}$, for every $\varepsilon > 0$.

We say that m is in the class of v-moderate weights $\mathcal{M}_v$, if m is a positive, even in each coordinate and continuous function on $\mathbb{R}^{2d}$ that satisfies

$$m(x+y) \leq Cv(x)m(y) \quad \text{for all } x, y \in \mathbb{R}^{2d}.$$

Now, we can define the modulation space $M_m^{p,q}$ in the following way.

Definition 1.2.22. *Let $m \in M_v$, where v is an admissible weight, and g a non-zero window function in $S_{1/2}^{1/2}(\mathbb{R}^d)$. For $1 \leq p, q \leq \infty$, the **modulation space** $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered ultra-distributions $f \in (S_{1/2}^{1/2})'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$. The norm on $M_m^{p,q}$ is*

$$\|f\|_{M_m^{p,q}(\mathbb{R}^d)} := \|V_g f\|_{L_m^{p,q}(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega) m(x,\omega)|^p dx \right)^{q/p} d\omega \right)^{1/q},$$

with the usual modifications when $p = \infty$ or $q = \infty$. We remark that the definition of $M_m^{p,q}(\mathbb{R}^d)$ is independent of the choice of the window $g \in S_{1/2}^{1/2}(\mathbb{R}^d) \setminus 0$, and different g gives rise to equivalent norms (See e.g. [Tof12, Proposition 1.11]).

For p, $q \in [1,\infty) \cup \{0\}$ the dual of $M_m^{p,q}(\mathbb{R}^d)$ can be identified with $M_{\frac{1}{m}}^{p',q'}(\mathbb{R}^d)$, under the pairing

$$\langle f, h\rangle = \int\int_{\mathbb{R}^{2d}} V_g f(x,\omega)\overline{V_g h(x,\omega)} d(x,\omega),$$

where p' and q' are the conjugate exponents of p and q. As usual, we agree that the conjugate exponent of 0 is 1.

In fact, $M_v^1(\mathbb{R}^d)$, where v is an admissible weight, is dense in $M_m^{p,q}(\mathbb{R}^d)$, for $1 \le p, q < \infty$ and $m \in \mathcal{M}_v$, [CPRT05]. Also, we have the next relationship between modulation spaces and Gelfand-Shilov type spaces from [Cor07, Corollary 3.4].

Theorem 1.2.23. *Let $m \in \mathcal{M}_v$, where v is an admissible weight, then $S_{1/2}^{1/2}(\mathbb{R}^d)$ is dense in $M_m^{p,q}(\mathbb{R}^d)$, for $1 \le p, q < \infty$.*

In particular, since M_v^1 is weak*-dense in $M_m^{p,q}(\mathbb{R}^d)$ for p or q equal to ∞, we have $S_{1/2}^{1/2}(\mathbb{R}^d)$ is weak*-dense in $M_m^{p,q}(\mathbb{R}^d)$, when p or q are equal to ∞. The closure of $S_{1/2}^{1/2}(\mathbb{R}^d)$ in $M_m^{\infty,q}(\mathbb{R}^d)$ is denoted $M_m^{0,q}(\mathbb{R}^d)$ and $M_m^{p,0}(\mathbb{R}^d)$ is defined similarly.

We recall the following theorem for a Gabor frame $\mathcal{G}(g,\Lambda)$, with a lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, from [Grö07, Theorem 6.11].

Theorem 1.2.24. *Assume that $g \in M_v^1(\mathbb{R}^d)$ for some admissible weight v and that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then the following properties hold for all $m \in \mathcal{M}_v$:*

(i) $h = S_g^{-1}(g) \in M_v^1$.

(ii) If $f \in M_m^{p,q}(\mathbb{R}^d)$, then the frame expansions

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)h = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle \pi(\lambda)g$$

converge in norm in $M_m^{p,q}(\mathbb{R}^d)$ for $1 \le p, q < \infty$ and weak when p or q are equal to ∞.*

(iii) Norm equivalence,

$$\|f\|_{M^{p,q}_m} \asymp \|\langle f, \pi(\lambda)g\rangle_{\lambda\in\Lambda}\|_{\ell^{p,q}_m}.$$

Chapter 2

Unconditionally convergent multipliers

2.1 Introduction

Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame and h a dual window. Then every function $f \in L^2(\mathbb{R}^d)$ can be expressed as

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) h.$$

Now, let $m \in \ell^\infty$ be given. The series

$$\sum_{\lambda \in \Lambda} m_\lambda \, \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) h,$$

can be interpreted as a filtered version of f. This series inspires the definition of **Gabor multipliers**

$$M_m f := \sum_{\lambda \in \Lambda} m_\lambda \, \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) h,$$

where h and g are not necessarily dual windows. Gabor multipliers are discrete versions of time-frequency localization operators introduced by

Daubechies [Dau88]. Gabor multipliers are useful tools in the analysis of pseudo-differential operators, [Grö11], and Fourier integral operators, [CGN12]. Also, they are applied in the study of multi-window spectrograms [BB00, AGR16], which can be used for spectral estimation. Due to their discrete nature, multipliers are more akin to the implementations required in acoustics [BLED10].

As a more general version of Gabor multipliers we introduce the operators called **multipliers**. A multiplier on a separable Hilbert space H is a bounded operator

$$M_{m,\Phi,\Psi} : H \to H,\ f \mapsto \sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n, \tag{2.1.1}$$

where $\Phi = (\Phi_n)_n$ and $\Psi = (\Psi_n)_n$ are sequences in H and $m = (m_n)_n$ is a scalar sequence called the symbol.

The multiplier is said to be unconditionally convergent if the above series, (2.1.1), converges unconditionally for every $f \in H$. For any (unconditionally convergent) multiplier $M_{m,\Phi,\Psi}$ its adjoint $M_{\overline{m},\Psi,\Phi}$ is also a (unconditionally convergent) multiplier.

Observe that each bounded operator T on H can be expressed as a multiplier: if $(u_n)_n$ is an orthonormal basis, we can take $\Phi_n = Tu_n$, $\Psi_n = u_n$ (alternatively $\Phi_n = u_n$, $\Psi_n = T^* u_n$) and $m_n = 1$ for each $n \in \mathbb{N}$.

In the case that $\Phi = (\Phi_n)_n$ and $\Psi = (\Psi_n)_n$ are Bessel sequences in H and $m \in \ell^\infty$ the operator $M_{m,\Phi,\Psi}$ is called a **Bessel multiplier**. Recall that $\Psi = (\Psi_n)_n$ is called a **Bessel sequence** if there is a constant $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, \Psi_n \rangle|^2 \leq B\|f\|^2,$$

for every $f \in H$. It turns out that $(\Psi_n)_n$ is a Bessel sequence if and only if there exists a bounded operator $T : \ell^2 \to H$ such that $T(e_n) = \Psi_n$, where $(e_n)_n$ denote the canonical unit vectors of ℓ^2 ([Chr03, Theorem 3.2.3]).

Bessel multipliers were introduced and studied in a systematic way by Balazs [Bal07] as a generalization of the Gabor multipliers considered in

[FN03]. In [Bal07] it is proved that each Bessel multiplier is unconditionally convergent. Balazs and Stoeva [SB13b] provide examples of non-Bessel sequences and non-bounded symbols defining unconditionally convergent multipliers. However all the examples are obtained from a Bessel multiplier after some trick. In fact, Balazs and Stoeva *conjecture* in [SB13a] that every unconditionally convergent multiplier can be written as a Bessel multiplier with constant symbol by shifting weights. More precisely,

Conjecture 2.1.1. *[SB13a, Conjecture 1] Let*

$$M_{m,\Phi,\Psi} : H \to H$$

be an unconditionally convergent multiplier, then there exist scalar sequences $(a_n)_n$ *and* $(b_n)_n$ *such that*

$$m_n = a_n \cdot \overline{b}_n$$

and

$$(a_n \Phi_n)_n \text{ and } (b_n \Psi_n)_n$$

are Bessel sequences in H.

Several classes of multipliers for which the conjecture is true are obtained in [SB13a]. For instance, they proved that this is the case for multipliers of the form $M_{m,\Phi,\Phi}$ [SB13a, Proposition 4.2] and also for multipliers with the property that the sequence $(|m_n| \cdot ||\Phi_n|| \cdot ||\Psi_n||)_n$ is bounded below by a strictly positive constant [SB13a, Proposition 1.1].

In this chapter new situations where the conjecture of Balazs and Stoeva is still true will be presented. These new situations are different in spirit to the ones considered in [SB13a]. To start we consider a particular situation: for the case that $m_n = 1$ and $\Psi_n = g$ for every $n \in \mathbb{N}$, the conjecture has a positive answer if and only if for every unconditionally summable sequence $(\Phi_n)_n$ in a separable Hilbert space H we may find $(\alpha_n)_n \in \ell^2$ such that $(\frac{1}{\alpha_n}\Phi_n)_n$ is a Bessel sequence in H. Then, the main aim of the present chapter is to analyze the structure of unconditionally summable sequences in a separable Hilbert space. Our results cannot be considered as improvements of those in [SB13a] nor can be obtained with the same techniques, they cover a completely different situation since in the cases we consider the sequence $(|m_n| \cdot ||\Phi_n|| \cdot ||\Psi_n||)_n$ converges to zero.

2.2 Auxiliary results

We need some definitions and auxiliary relationships to argue our results presented in the next section. From now on by an operator between Banach spaces we mean a bounded and linear operator.

Definition 2.2.1. *Let $1 \leq p < \infty$ and let $T : X \to Y$ be an operator between Banach spaces. We say that T is* ***p-summing*** *if there is a constant $C \geq 0$ such that*

$$\left(\sum_{i=1}^{m} \|Tx_i\|^p\right)^{\frac{1}{p}} \leq C \sup \left\{ \left(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*} \right\},$$

for every $m \in \mathbb{N}$ and every $\{x_i\}_{i=1}^m \subseteq X$. The collection of all p-summing operators from X to Y is denoted by $\Pi_p(X, Y)$.

Definition 2.2.2. *We say that an operator $T : X \to Y$ between Banach spaces is a* ***p-integral*** *operator ($1 \leq p \leq \infty$) if there are a probability measure μ on a domain Ω and (bounded linear) operators $A : L^p(\Omega, \mu) \to Y^{**}$ and $B : X \to L^\infty(\Omega, \mu)$ giving rise to the commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & Y \arrow[r, "k_Y"] & Y^{**} \\
L^\infty(\Omega, \mu) \arrow[rr, "i_p"] & & L^p(\Omega, \mu). \arrow[u, "A"']
\end{tikzcd}$$

As usual, $i_p : L^\infty(\Omega, \mu) \to L^p(\Omega, \mu)$ is the formal identity, and $k_Y : Y \to Y^{**}$ is the canonical isometric embedding. The collection of all p-integral operators from X to Y is denoted by $\mathcal{I}_p(X, Y)$.

Definition 2.2.3. *As in [PP69], let E, F be complex Banach spaces, we denote by $\mathcal{N}_p(E,F)$ the set of all operators $T : E \to F$ which can be written in the form*

$$Tu = \sum_{n=1}^{\infty} \langle u, u_n' \rangle v_n \qquad (2.2.1)$$

with

$$\left(\sum_{n=1}^{\infty} \|u_n'\|^p \right)^{\frac{1}{p}} < \infty \text{ and } \sup_{\|v'\| \le 1} \left(\sum_{n=1}^{\infty} |\langle v_n, v' \rangle|^{p'} \right)^{\frac{1}{p'}} < \infty \qquad (2.2.2)$$

when $1 \le p < \infty$ and with the additional requirement $\|u_n'\| \to 0$, $n \to \infty$, in the case $p = \infty$. The elements in $\mathcal{N}_p(E,F)$ are called ***p-nuclear operators****. It is a Banach space equipped with the norm*

$$N_p(T) = \inf \left(\sum_{n=1}^{\infty} \|u_n'\|^p \right)^{\frac{1}{p}} \sup_{\|v'\| \le 1} \left(\sum_{n=1}^{\infty} \langle v_n, v' \rangle^{p'} \right)^{\frac{1}{p'}},$$

where the infimum is taken over all the pairs $(\{u_n'\}_n, \{v_n\}_n)$ which satisfy (2.2.1) *and* (2.2.2)*.*
For $p = 1$, the space $\mathcal{N}_1(E,F)$ coincides with the space of nuclear operators of E into F, which are the ones where the series $\sum_{n=1}^{\infty} u_n'$ is absolutely convergent and $\{v_n\}_{n=1}^{\infty}$ is bounded.

We first need a particular case of [DJT95, Theorem 3.7].

Theorem 2.2.4. *Let H a Hilbert space, then every operator $T : c_0 \to H$ is 2-summing.*

Theorem 2.2.5. *[DJT95, Corollary 5.9] The 2-summing and 2-integral operators are the same.*

Theorem 2.2.6. *[Per69, Theorem 5] If E has a strongly separable dual E', then, for $1 \le p < \infty$, the set of p-nuclears operators is the same that the set of p-integral operators with equality of the corresponding norms.*

Theorem 2.2.7. *[Jar81, Theorem 19.7.4]*

(a) For every $(a_n)_n \in \ell^p$, the diagonal operator $D_a : \ell^\infty \to \ell^p$, for $1 \leq p < \infty$, is p-nuclear. Moreover, the operator $D_a : \ell^\infty \to c_0$ is also p-nuclear.

(b) An operator $S : E \to F$ is p-nuclear if, and only if, there are $(a_n)_n \in \ell^p$, $A \in \mathcal{L}(\ell^p, F)$ and $B \in \mathcal{L}(E, \ell^\infty)$ such that $S = A \circ D_a \circ B$.

Definition 2.2.8. *A **Hilbert-Schmidt operator** is a bounded operator $A : H_1 \to H_2$, with H_1 and H_2 being Hilbert spaces, such that its Hilbert-Schmidt norm is finite,*

$$\|A\|_{HS}^2 = \mathrm{Tr}(A^*A) := \sum_{i\in I} \|Ae_i\|^2$$

where $\|\cdot\|$ is the norm of H_2, $\{e_i : i \in I\}$ is an orthonormal basis of H_1, and Tr is the trace of a nonnegative self-adjoint operator. The set of Hilbert-Schmidt operators is denoted by $S^2(H_1, H_2)$.

Theorem 2.2.9. *[DJT95, Theorem 5.30] Let H_1 and H_2 be Hilbert spaces. If $1 < p < \infty$, then $\mathcal{I}_p(H_1, H_2) = \mathcal{N}_p(H_1, H_2) = S^2(H_1, H_2)$ isomorphically, and even isometrically if $p = 2$.*

2.3 Results

Now we present some results concerning unconditionally summable series, and their impact on unconditionally convergent multipliers.

We use the fact that a series $\sum_{n=1}^\infty x_n$ in a Banach space X is unconditionally convergent if and only if there exists a compact operator $T : c_0 \to X$ with the property that $T(e_n) = x_n$, where $(e_n)_n$ denote the canonical unit vectors of c_0 (see for instance [DJT95, 1.9]). We recall that, in the case that $X = H$ is a Hilbert space, every bounded operator $T : c_0 \to H$ is compact. In fact, the closed unit ball B of H is weakly compact, the transposed map $T^* : H \to \ell^1$ is a bounded operator and weak and norm convergence of sequences in ℓ^1 coincide ([DJT95, Theorem 1.7]). Therefore T^* is a compact operator and so is T.

From the previous considerations we conclude that a series $\sum_{n=1}^{\infty} x_n$ in a Hilbert space H is unconditionally convergent if and only if there exists a bounded operator $T : c_0 \to H$ with the property that $T(e_n) = x_n$. An important consequence is the fact that the unconditional convergence of $\sum_{n=1}^{\infty} x_n$ is equivalent to

$$\sum_{n=1}^{\infty} |\langle x_n, g\rangle| < \infty \quad \text{for all } g \in H.$$

This is so because if the last condition is satisfied then, by closed graph theorem, $S : H \to \ell^1$, $S(g) := (\langle x_n, g\rangle)_n$, defines a bounded operator and $T = S^* : \ell^\infty \to H$ satisfies $T(e_n) = x_n$.

For a fixed sequence $\alpha = (\alpha_n)_n$ we consider the diagonal operator $D_\alpha(x) = (\alpha_n x_n)_n$, defined in Definition 1.1.2. If $\alpha \in \ell^2$ then $D_\alpha : \ell^\infty \to \ell^2$ is a bounded operator, while $D_\alpha : \ell^2 \to \ell^2$ is a bounded operator if and only if $\alpha \in \ell^\infty$ (Proposition 1.1.3). In particular, this is the case if $\alpha \in c_0$.

Lemma 2.3.1. *The following statements are equivalent:*

(a) Every unconditionally summable sequence $(\Phi_n)_n$ in H can be written as $\Phi_n = \alpha_n f_n$, where $(\alpha_n)_n \in \ell^2$ and $(f_n)_n$ is a Bessel sequence in H.

(b) Every bounded operator $T : c_0 \to H$ can be factorized as

$$T = A \circ D_\alpha$$

where $D_\alpha : c_0 \to \ell^2$ is a diagonal operator and $A : \ell^2 \to H$ is a bounded operator.

Proof. $(a) \Rightarrow (b)$. If $T : c_0 \to H$ is bounded then $(\Phi_n)_n = (T(e_n))_n$ is unconditionally summable ([DJT95, Theorem 1.9]), hence $T(e_n) = \alpha_n f_n$, where $\alpha = (\alpha_n)_n \in \ell^2$ and $(f_n)_n$ is a Bessel sequence in H. Therefore $(f_n)_n$ defines a bounded operator $A : \ell^2 \to H$, $A(\beta) = \sum_n \beta_n f_n$, and $T = A \circ D_\alpha$.

$(b) \Rightarrow (a)$. Let $(\Phi_n)_n$ be an unconditionally summable sequence in H. Then there is a bounded operator $T : c_0 \to H$ such that $T(e_n) = \Phi_n$ and, by hypothesis, it can be factorized as $T = A \circ D_\alpha$ where $\alpha \in \ell^2$ and $A : \ell^2 \to H$ is a bounded operator. Then $(f_n)_n := (A(e_n))_n$ is a Bessel sequence in H and clearly $\Phi_n = A(\alpha_n e_n) = \alpha_n f_n$. $\square$

Now we establish a reformulation of the conjecture by Balazs and Stoeva. As a consequence, having a positive answer is equivalent to proving a bilinear version of statement (b) in Lemma 2.3.1. We will use that a sequence of scalars $\alpha = (\alpha_n)_n$ belongs to ℓ^2 if and only if $(\alpha_n \beta_n)_n \in \ell^2$ for every $(\beta_n)_n \in c_0$.

Proposition 2.3.2. *Assume that the series $\sum_n \langle f, \Psi_n \rangle \Phi_n$ converges unconditionally for all $f \in H$. Then, the following statements are equivalent:*

(a) There exists $(c_n)_n$ such that $\{\overline{c}_n \Psi_n\}_n$ and $\{\frac{1}{c_n}\Phi_n\}_n$ are Bessel sequences in H.

(b) The continuous bilinear operator

$$\begin{array}{rcll} T: & c_0 \times H & \longrightarrow & H \\ & (\alpha, f) & \longmapsto & \displaystyle\sum_n \alpha_n \langle f, \Psi_n \rangle \Phi_n \end{array}$$

admits a factorization

$$T = B \circ D$$

where

$$B : \ell^2 \to H$$

is a bounded operator and

$$D : c_0 \times H \to \ell^2$$

is a continuous bilinear operator such that for every $f \in H$,

$$D(\cdot, f) : c_0 \to \ell^2$$

is a diagonal operator.

(c) *There exist two bounded operators*

$$A : H \to \ell^2 \ \text{ and } \ B : \ell^2 \to H$$

such that for every $\alpha \in c_0$ *the operator* $T_\alpha \in L(H)$, *defined by* $T_\alpha(f) = \sum_n \alpha_n \langle f, \Psi_n \rangle \Phi_n$, *can be factorized as*

$$T_\alpha = B \circ D_\alpha \circ A.$$

Proof. Without loss of generality we assume that $\Psi_n \neq 0$ and $\Phi_n \neq 0$ for each n.

$(a) \Rightarrow (b)$ Since $\{\overline{c}_n \Psi_n\}_n$ and $\{\frac{1}{c_n}\Phi_n\}_n$ are Bessel sequences,

$$B : \ell^2 \to H, \quad B(e_n) = \frac{1}{c_n}\Phi_n$$

and

$$D : c_0 \times H \to \ell^2, \quad D(\alpha, f) = (\alpha_n \langle f, \overline{c}_n \Psi_n \rangle)_n$$

are well defined and continuous and $T = B \circ D$.

$(b) \Rightarrow (c)$ We observe that $T_\alpha = T(\alpha, \cdot)$, hence $T_\alpha = B \circ D(\alpha, \cdot)$. The fact that $D(\cdot, f) : c_0 \to \ell^2$ is a diagonal operator implies that $D(\alpha, f) = (\alpha_n \beta_n(f))_n$, with $(\beta_n(f))_n \in \ell^2$. Thus, the map

$$A : H \to \ell^2, \ f \mapsto (\beta_n(f))_n$$

is well defined. As, for every $n \in \mathbb{N}$, one has

$$\beta_n(f) = D(e_n, f)_n = \langle D(e_n, f), e_n \rangle,$$

then, A is linear and, by the closed graph theorem, continuous. Clearly,

$$T_\alpha = B \circ D_\alpha \circ A.$$

$(c) \Rightarrow (a)$ We write $A(f) \in \ell^2$ as $A(f) = (A(f)_n)_n$. We take $\alpha = e_n$ and $f = \Psi_n$. Then, $T_\alpha(\Psi_n) = \langle \Psi_n, \Psi_n \rangle \Phi_n$, and by our assumption

$$T_\alpha(\Psi_n) = B(A(\Psi_n)_n e_n) = A(\Psi_n)_n B(e_n),$$

hence

$$B(e_n) = \frac{\|\Psi_n\|^2}{A(\Psi_n)_n}\Phi_n.$$

Defining $\frac{1}{c_n} := \frac{\|\Psi_n\|^2}{A(\Psi_n)_n}$, the boundedness of B implies that $(\frac{1}{c_n}\Phi_n)_n$ is a Bessel sequence in H.

Now, with $\alpha = e_n$ and $f \in H$ we have that $T_\alpha(f) = \langle f, \Psi_n \rangle \Phi_n$, and also

$$T_\alpha(f) = B(A(f)_n e_n) = A(f)_n B(e_n) = A(f)_n \frac{1}{c_n}\Phi_n,$$

therefore,

$$A(f)_n = c_n \langle f, \Psi_n \rangle = \langle f, \overline{c}_n \Psi_n \rangle ,$$

that is,

$$A(f) = (\langle f, \overline{c}_n \Psi_n \rangle)_n.$$

Since $A : H \to \ell^2$ is bounded, we conclude that $\{\overline{c}_n \Psi_n\}_n$ is a Bessel sequence in H.

□

We recall that any bounded operator $B : c_0 \to \ell^\infty,\ B(e_j) = \left(b_j^i\right)_i$, has the property that $b^i := \left(b_j^i\right)_j \in \ell^1$ for every $i \in \mathbb{N}$ and

$$\|B\| = \sup_i \|b^i\|_{\ell^1}.$$

The next result can be viewed as an improvement of Orlicz's Theorem (see for instance [DJT95, Theorem 1.11] or [Hei11, Theorem 3.16]),which says that every unconditionally summable sequence in a Hilbert space is absolutely 2-summable. It is the main result of the chapter.

Theorem 2.3.3. *Every unconditionally summable sequence $(\Phi_n)_n$ in a separable Hilbert space H can be expressed as $\Phi_n = \overline{a}_n f_n$, where $(a_n)_n \in \ell^2$ and $(f_n)_n$ is a Bessel sequence in H.*

Proof. By Lemma 2.3.1 it is enough to show that every bounded operator $T : c_0 \to H$ can be factorized as $T = A \circ D_\alpha$, where $\alpha \in \ell^2$ and $A : \ell^2 \to H$ is a bounded operator. According to Theorem 2.2.4 and Theorem 2.2.5, T is a 2-integral operator, hence it is 2-nuclear (Theorem 2.2.6). Therefore there are bounded operators $B : c_0 \to \ell^\infty$, $S : \ell^2 \to H$ and $\lambda \in \ell^2$ such that $T = S \circ D_\lambda \circ B$, by Theorem 2.2.7. To finish it suffices to find $\alpha \in \ell^2$ and a bounded operator $\tilde{A}$ on ℓ^2 such that $D_\lambda \circ B = \tilde{A} \circ D_\alpha$, since then

$$T = A \circ D_\alpha,$$

with $A = S \circ \tilde{A}$.

As $D_\lambda \circ B = D_{t\lambda} \circ (t^{-1}B)$ for each $t > 0$, without loss of generality we can assume $\|B\| = 1$. We denote $B(e_j) = \left(b_j^i\right)_i$ and $b^i := \left(b_j^i\right)_j \in \ell^1$. We define $\alpha = (\alpha_k)_k$ such that

$$|\alpha_k|^2 := \sum_{i=1}^{\infty} |\lambda_i|^2 \cdot |b_k^i|.$$

Then

$$\sum_{k=1}^{\infty} |\alpha_k|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2 \cdot \|b^i\|_{\ell^1} \leq \|\lambda\|_{\ell^2}^2, \tag{2.3.1}$$

hence $\alpha \in \ell^2$. Next, we consider

$$f_k := \frac{1}{\alpha_k} \left(\lambda_i b_k^i\right)_i, \quad k \in \mathbb{N}.$$

Since $|b_k^i|^2 \leq |b_k^i|$, the inequality (2.3.1) implies that $f_k \in \ell^2$. To finish the proof, we have to show that there is a bounded operator $\tilde{A}$ on ℓ^2 such that $\tilde{A}(e_k) = f_k$, that is, $(f_k)_k$ is a Bessel sequence in ℓ^2. To this end, we fix $\beta = (\beta_k)_k \in \ell^2$ and $\gamma = (\gamma_k)_k \in \ell^2$. Then,

$$\sum_{k=1}^{N} |\beta_k \langle f_k, \gamma \rangle| \leq \sum_{k=1}^{N} \frac{|\beta_k|}{|\alpha_k|} \sum_{j=1}^{\infty} |\lambda_j \gamma_j| \cdot |b_k^j|,$$

for all $N \in \mathbb{N}$. As λ, $\gamma \in \ell^2$,

$$\begin{aligned}\sum_{j=1}^{\infty} |\lambda_j \gamma_j| \cdot |b_k^j| &\leq \left(\sum_{j=1}^{\infty} |\lambda_j|^2 \cdot |b_k^j| \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{\infty} |\gamma_j|^2 \cdot |b_k^j| \right)^{\frac{1}{2}} \\ &= |\alpha_k| \cdot \left(\sum_{j=1}^{\infty} |\gamma_j|^2 \cdot |b_k^j| \right)^{\frac{1}{2}}.\end{aligned}$$

Moreover

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\gamma_j|^2 \cdot |b_k^j| = \sum_{j=1}^{\infty} |\gamma_j|^2 \cdot \|b^j\|_{\ell^1} \leq \|\gamma\|_{\ell^2}^2.$$

This means that

$$\left(\left(\sum_{j=1}^{\infty} |\gamma_j|^2 \cdot |b_k^j| \right)^{\frac{1}{2}} \right)_{k \in \mathbb{N}} \in \ell^2.$$

Hence,

$$\sum_{k=1}^{\infty} |\beta_k \langle f_k, \gamma \rangle| \leq \sum_{k=1}^{\infty} |\beta_k| \cdot \left(\sum_{j=1}^{\infty} |\gamma_j|^2 \cdot |b_k^j| \right)^{\frac{1}{2}} < \infty.$$

Since this holds for every $\beta \in \ell^2$ we conclude that

$$\sum_{k=1}^{\infty} |\langle f_k, \gamma \rangle|^2 < \infty,$$

for every $\gamma \in \ell^2$. Now, the closed graph theorem gives the conclusion. □

Theorem 2.3.3 gives a positive answer to the conjecture of Balazs and Stoeva when $(\Psi_n)_n$ is a constant sequence. Next we consider a more general situation.

Corollary 2.3.4. *Let $M_{m,\Phi,\Psi}$ be an unconditionally convergent multiplier and assume that* 0 *is not a weak accumulation point of the sequence* $\left(\frac{\Psi_n}{\|\Psi_n\|}\right)_n$. *Then there exist scalar sequences $(a_n)_n$ and $(b_n)_n$ such that $m_n = a_n \cdot \overline{b_n}$ and $(a_n\Phi_n)_n$ and $(b_n\Psi_n)_n$ are Bessel sequences in H.*

Proof. In fact, our hypothesis implies the existence of finitely many elements $f_1, \ldots, f_K \in H$ with the property that

$$\sum_{k=1}^{K} \left| \left\langle f_k, \frac{\Psi_n}{\|\Psi_n\|} \right\rangle \right| \geq 1,$$

for every $n \in \mathbb{N}$. Since $M_{m,\Phi,\Psi}$ is an unconditionally convergent multiplier we have

$$\sum_{n=1}^{\infty} |m_n| \cdot |\langle f, \Psi_n\rangle| \cdot |\langle \Phi_n, g\rangle| < \infty,$$

for every $f, g \in H$. Consequently

$$\sum_{n=1}^{\infty} |m_n| \cdot \|\Psi_n\| \cdot |\langle \Phi_n, g\rangle| \leq \sum_{k=1}^{K}\sum_{n=1}^{\infty} |m_n| \cdot |\langle f_k, \Psi_n\rangle| \cdot |\langle \Phi_n, g\rangle| < \infty,$$

for every $g \in H$. It follows that the series $\sum_{n=1}^{\infty} m_n \|\Psi_n\| \Phi_n$ is unconditionally convergent and we can apply Theorem 2.3.3 to find a sequence $(b_n)_n \in \ell^2$ such that $\left(\frac{m_n}{b_n}\|\Psi_n\|\Phi_n\right)_n$ is a Bessel sequence. Since also $\left(b_n \frac{\Psi_n}{\|\Psi_n\|}\right)_n$ is a Bessel sequence, the conclusion follows. □

By Orlicz's Theorem, in the case that $(\Psi_n)_n$ is constant, the unconditional convergence of the series $\sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n$ implies that

$$\sum_{n=1}^{\infty} (|m_n| \cdot \|\Phi_n\| \cdot \|\Psi_n\|)^2 < \infty.$$

In particular, the sequence $(m_n \cdot \|\Phi_n\| \cdot \|\Psi_n\|)_n$ converges to zero and [SB13a, Proposition 1.1] cannot be applied. Obviously, in Corollary 2.3.4, the condition on the sequence $(\Psi_n)_n$ can be replaced by a similar condition on $(\Phi_n)_n$. The following result shows that the conjecture stated by Balazs and Stoeva in [SB13a] holds under the stronger hypothesis of absolute convergence of the series. The proof depends on Theorem 2.3.3 and it does not follow from the results in [SB13a].

Theorem 2.3.5. *Let $M_{m,\Phi,\Psi}$ be such that for each $f \in H$, the series*

$$\sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n$$

converges absolutely in H. Then there exist scalar sequences $(a_n)_n$ and $(b_n)_n$ such that $m_n = a_n \cdot \overline{b_n}$, and $(b_n \Psi_n)_n$ and $(a_n \Phi_n)_n$ are Bessel sequences in H.

Proof. Replacing $(m_n)_n$ and $(\Phi_n)_n$ by $(m_n||\Phi_n||)_n$ and $(\Phi_n/||\Phi_n||)_n$ we may assume that $||\Phi_n|| = 1$ for every $n \in \mathbb{N}$. The condition $(m_n \langle f, \Psi_n \rangle)_n \in \ell^1$ for every $f \in H$ implies that the sequence $(\overline{m_n}\Psi_n)_n$ is unconditionally summable in H, therefore by Theorem 2.3.3, there is $(c_n)_n \in \ell^2$ such that $\left(\frac{\overline{m_n}}{c_n}\Psi_n\right)_n$ is a Bessel sequence. As $(\overline{c_n}\Phi_n)_n$ is also a Bessel sequence, we conclude. □

As a consequence of the previous result, we prove the following result for Hilbert-Schmidt operators on H, $S^2(H)$. As usual $\mathcal{L}(H)$ denotes the space of all continuous operators on H.

Proposition 2.3.6. *If the series $\sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n$ converges absolutely for every f in H, then, the series*

$$\sum_{n=1}^{\infty} m_n \Psi_n \otimes \Phi_n$$

converges unconditionally in the Hilbert space $S^2(H)$ of Hilbert-Schmidt operators on H.

Proof. Without loss of generality we may assume that $m_n = \|\Phi_n\| = 1$. As the series

$$\sum_n |\langle f, \Psi_n \rangle|$$

converges for every $f \in H$, then the map

$$S : H \to \ell^1, \quad f \mapsto (\langle f, \Psi_n \rangle)_n$$

is bounded, and its adjoint

$$T = S^* : c_0 \to H', \quad T(e_n) = \Psi_n$$

is bounded too, then $(\Psi_n)_n = (T(e_n))_n$ is unconditionally summable,[DJT95, Theorem 1.9]. By Theorem 2.3.3, there is $(\alpha_n) \in \ell_2$ such that $\{\frac{1}{\alpha_n}\Psi_n\}$ is a Bessel sequence in H. Therefore

$$M^* f = \sum_n \frac{1}{\alpha_n} \Psi_n \alpha_n \langle f, \Phi_n \rangle$$

can be factorized as $B \circ D_\alpha \circ A$, where

$$A : H \to \ell^\infty, \, A(f) = (\langle f, \Phi_n \rangle)_n$$

and

$$B : \ell^2 \to H, \, B(x) = \sum_n x_n \frac{1}{\alpha_n} \Psi_n$$

are bounded, hence it is a 2-nuclear operator (Theorem 2.2.7). By Theorem 2.2.9, M^* is a Hilbert-Schmidt operator, consequently

$$Mf = \sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n$$

is also a Hilbert-Schmidt operator.

Using the same arguments, we see that for each $(\lambda_n) \in c_0$, the operator

$$f \mapsto \sum_n \langle f, \Psi_n \rangle \lambda_n \Phi_n,$$

is also a Hilbert-Schmidt operator. Hence the correspondence

$$U : \lambda \mapsto \sum_n \lambda_n \langle \cdot, \Psi_n \rangle \Phi_n$$

is well defined and linear from c_0 into $S^2(H)$. It is continuous when $S^2(H)$, being a subspace of $\mathcal{L}(H)$, is endowed with the strong operator topology. The closed graph theorem gives that the map above is continuous when we consider on $S^2(H)$ the Hilbert-Schmidt norm. Since

$$U(e_n) = \langle \cdot, \Psi_n \rangle \Phi_n = \Psi_n \otimes \Phi_n,$$

then

$$\sum_n \Psi_n \otimes \Phi_n$$

converges unconditionally in $S^2(H)$. □

Let B_H denote the closed unit ball of H endowed with the weak topology and μ a probability Borel measure on B_H. Then we have the canonical continuous inclusion

$$j_\mu : H \to L^2(B_H, \mu), \; j_\mu(f)\,(g) := \langle f, g \rangle \, .$$

Theorem 2.3.7. *Let B_H denote the closed unit ball of H endowed with the weak topology and assume that the series*

$$\sum_{n=1}^{\infty} m_n \Psi_n \otimes \Phi_n$$

converges unconditionally in $S^2(H)$. Then, for every probability Borel measure μ on B_H there exist scalar sequences $(a_n)_n$, $(b_n)_n$ such that $m_n = a_n \cdot \overline{b}_n$, $(a_n \Psi_n)_n$ is a Bessel sequence in H and $(j_\mu(b_n \Phi_n))_n$ is a Bessel sequence in $L^2(B_H, \mu)$. In particular

$$\sum_{n=1}^{\infty} |\langle f, b_n \Phi_n \rangle|^2 < \infty$$

for μ-almost every $f \in B_H$.

Proof. According to Theorem 2.3.3 there is $(\alpha_n)_n \in \ell^2$ such that

$$\left(\frac{m_n}{\alpha_n}\Psi_n \otimes \Phi_n\right)_n$$

is a Bessel sequence in $S^2(H)$. In particular, for some constant $C > 0$,

$$\sum_{n=1}^{\infty}\left|\frac{m_n}{\alpha_n}\langle f, \Psi_n\rangle\langle g, \Phi_n\rangle\right|^2 \leq C\|f\|^2 \cdot \|g\|^2 \tag{2.3.2}$$

for every $f, g \in B_H$. We now consider

$$a_n^2 := \left|\frac{m_n}{\alpha_n}\right|^2 \int_{B_H} |\langle g, \Phi_n\rangle|^2 \ d\mu(g).$$

After integrating in (2.3.2) we obtain that $(a_n\Psi_n)_n$ is a Bessel sequence. Moreover, for $b_n = \frac{\overline{m_n}}{a_n}$ we have

$$\sum_{n=1}^{\infty}\int_{B_H} |\langle f, b_n\Phi_n\rangle|^2 \ d\mu(f) = \sum_{n=1}^{\infty}\alpha_n^2 < \infty,$$

from where the conclusion follows. □

In the previous theorem we proved that the unconditional convergence of $\sum_{n=1}^{\infty} m_n\Psi_n \otimes \Phi_n$ in $S^2(H)$ gives a positive answer to the conjecture in a Hilbert space $\tilde{H}$ continuously containing H. Under the additional hypothesis that H is a subspace of $L^2(X, \mu)$ for some finite measure, on which evaluations are continuous, we can find a closer relation between H and $\tilde{H}$.

Definition 2.3.8. *Let X be an arbitrary set and H a Hilbert space of real-valued functions on X. The evaluation functional over the Hilbert space of functions H is a linear functional that evaluates each function at a point x,*

$$L_x : f \mapsto f(x) \quad \textit{for all } f \in H.$$

We say that H is a **reproducing kernel Hilbert space** *if, for all x in X, L_x is a bounded functional, that is, can be represented by the inner product of f with a function K_x in H,*

$$f(x) = L_x(f) = \langle f,\ K_x\rangle.$$

Since K_x is itself a function in H, it holds that for every y in X there exist a $K_y \in H$ such that

$$K_x(y) = \langle K_y,\ K_x\rangle.$$

This allows us to define the **reproducing kernel** *of H as a function $K : X \times X \to \mathbb{R}$ by*

$$K(x, y) = \langle K_y,\ K_x\rangle.$$

If $H \subset L^2(X, \mu)$ is a Hilbert space admitting a reproducing kernel $K(x, y)$, then,

$$f(x) = \int_X f(y)K(x, y)d\mu(y), \quad x \in X.$$

Theorem 2.3.9. *Let (X, μ) be a finite measure space and $H \subset L^2(X, \mu)$ a Hilbert space admitting a reproducing kernel $K(x, y)$. We put $v(x) := \|K(x, \cdot)\|^{-1}$. If the series*

$$\sum_{n=1}^{\infty} m_n \Psi_n \otimes \Phi_n$$

converges unconditionally in $S^2(H)$. Then there exist scalar sequences $(a_n)_n$ and $(b_n)_n$ such that $m_n = a_n \cdot \overline{b}_n$, $(a_n\Psi_n)_n$ is a Bessel sequence in H and $(b_n\Phi_n)_n$ is a Bessel sequence in $L^2_v(X, \mu)$.

Proof. By Theorem 2.3.3, there is $(\alpha_n)_n \in \ell^2$ such that

$$\left(\frac{m_n}{\alpha_n}\Psi_n \otimes \Phi_n\right)_n$$

is a Bessel sequence in $S^2(H)$. In particular, for some constant $C > 0$,

$$\sum_{n=1}^{\infty} \left| \frac{m_n}{\alpha_n} \langle f, \Psi_n \rangle \langle g, \Phi_n \rangle \right|^2 \leq C\|f\|^2 \cdot \|g\|^2$$

for every $f, g \in H$. For $g = v(x) \cdot K(x, \cdot)$, we obtain

$$\sum_{n=1}^{\infty} \left| \frac{m_n}{\alpha_n} \langle f, \Psi_n \rangle v(x)\Phi_n(x) \right|^2 \leq C\|f\|^2, \tag{2.3.3}$$

since $v(x) \cdot \|K(x, \cdot)\| = 1$.

We now consider

$$a_n^2 := \left| \frac{m_n}{\alpha_n} \right|^2 \int_X |v(x) \cdot \Phi_n(x)|^2 \ d\mu(x).$$

After integrating in (2.3.3) we obtain that $(a_n \Psi_n)_n$ is a Bessel sequence in H, hence in $L^2(X, \mu)$. Moreover, for $b_n = \frac{\overline{m_n}}{a_n}$ and $f \in L_v^2(X, \mu)$ we have that

$$\begin{aligned}\sum_{n=1}^{\infty} &\left| \int_X f(x) b_n \overline{\Phi_n(x)} \cdot v(x)^2 \ d\mu(x) \right|^2 \\ &\leq \sum_{n=1}^{\infty} |b_n|^2 \left(\int_X |f(x)v(x)|^2 \, d\mu(x) \right) \cdot \left(\int_X |\Phi_n(x)v(x)|^2 \, d\mu(x) \right) \\ &\leq \|f\|_{L_v^2}^2 \cdot \sum_{n=1}^{\infty} \alpha_n^2,\end{aligned}$$

which shows that $(b_n \Phi_n)_n$ is a Bessel sequence in $L_v^2(X, \mu)$. □

2.4 Conclusion

In this chapter we have proved that every unconditionally summable sequence in a separable Hilbert space can be expressed as the product of a sequence of ℓ^2 and a Bessel sequence. Then, we have improved the classical Orlicz's Theorem and we have obtained some new situations where the conjecture of Balazs and Stoeva is still true. Our results cannot be considered as improvements of those in [SB13a] nor can be obtained with the same techniques. These results are included in [FGP17b] and [FGP17a].

Chapter 3

Compactness of Fourier integral operators

3.1 Introduction

The aim of this chapter is to investigate compactness for **Fourier integral operators** (FIOs) when acting on weighted modulation spaces. The boundedness and Schatten class properties of FIOs have been studied by several authors under various assumptions on the phase and the symbol. See for instance [RS06, CR14, Bis11, Bou97, CNR09a, CNR10b, RS06, CT07, CT09, TCG10]. However no characterization seems to be known of those FIOs which are compact. Our approach to the study of the compactness of the FIOs follows the point of view of [CNR10b], which means that our results strongly depend on the matrix representation of a FIO with respect to a Gabor frame.

For a function f on $\mathbb{R}^d$ the **Fourier integral operator**, FIO, T with symbol $\sigma \in L^\infty(\mathbb{R}^{2d})$ and real phase Φ on $\mathbb{R}^{2d}$ can be formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)d\eta.$$

The above formula defines a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

The phase $\Phi(x,\eta)$ is **tame**. That it is, $\Phi(x,\eta)$ is smooth on $\mathbb{R}^{2d}$ and satisfies the estimates

$$|\partial_z^\alpha \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2, z \in \mathbb{R}^{2d}, \tag{3.1.1}$$

and the nondegeneracy condition

$$|\det\, \partial^2_{x,\eta}\, \Phi(x,\eta)| \geq \delta > 0, \quad (x,\eta) \in \mathbb{R}^{2d}. \tag{3.1.2}$$

For the symbol σ on $\mathbb{R}^{2d}$,

$$|\partial_z^\alpha \sigma(z)| \leq C_\alpha, \quad \text{a.e. } z \in \mathbb{R}^{2d},\ |\alpha| \leq 2N \tag{3.1.3}$$

holds, for a fixed $N \in \mathbb{N}$. Here ∂_z^α denotes the distributional derivative. When $\Phi(x,\eta) = x\eta$ we recover the pseudodifferential operators (PSDOs) in the Kohn-Nirenberg form.

Frames permit to represent operators in terms of matrices, the key result in [CNR10b] shows that the matrix representation of a FIO with respect to a Gabor frame $\mathcal{G}(g,\Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$ is well organized. In fact, for a tame phase function Φ and a symbol σ satisfying condition (3.1.3) there exists a constant $C_N > 0$ such that

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C_N \langle \chi(\lambda) - \mu\rangle^{-2N}, \tag{3.1.4}$$

for every $\lambda, \mu \in \Lambda$. Here $\langle z\rangle$ is an abbreviation for $(1+|z|^2)^{1/2}$, and χ is the **canonical transformation** of the phase Φ. We recall that the canonical transformation, $(x,\xi) = \chi(y,\eta)$, is a bilipschitz map $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ defined through the system

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta), \\ \xi = \nabla_x \Phi(x,\eta). \end{cases}$$

The estimate (3.1.4) is an extension of previous results of Gröchenig [Grö06] concerning almost diagonalization of PSDOs. See also [GR08]. The condition (3.1.3) on the symbol can be relaxed. In fact, if $\mathcal{G}(g,\Lambda)$ is a Parseval frame then the estimate (3.1.4) also holds under the weaker assumption that σ belongs to an appropriate modulation space (see [CGN12]).

We will use the decay estimate (3.1.4) to discuss the compactness of the FIOs when acting on weighted modulation spaces. More precisely, we prove that the FIO is compact when acting on some modulation space of the form $M^p_m(\mathbb{R}^d)$ if and only if the sequences

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle\big)_{\lambda\in\Lambda}$$

converge to zero for all $\mu \in \Lambda$, where χ' denotes a discrete version of the canonical transformation χ. This is the content of Theorem 3.3.13. In particular, it follows that compactness does not depend neither on p nor on m. To achieve our goal we need to focus our attention on a class of matrices $A = \big(a_{\gamma,\gamma'}\big)_{\gamma,\gamma'\in\Lambda}$ with the property that the decay of the coefficient $a_{\gamma,\gamma'}$ is determined by the distance of (γ,γ') to the graph of $\gamma = \chi(\gamma')$. We characterize when such a matrix defines a compact operator when acting on weighted ℓ^p spaces of sequences. For a quadratic phase Φ we completely characterize in Theorem 3.3.17 the symbols σ satisfying condition (3.1.3) for which the corresponding FIO is compact. The operators we are considering may fail to be bounded on mixed modulation spaces as was shown in [CNR10b]. To overcome this obstacle, an extra condition on the phase was introduced in [CNR10b]. Under this additional condition, the compactness results are extended to weighted mixed modulation spaces. As a consequence, we recover and improve some compactness results for PSDOs obtained in [FG06, FG07, FG10]. In the last section we see that all this argumentation can be aplied to Fourier integral operators on modulation spaces with GRS-weights, under similar conditions in the phase and the symbol.

3.2 Matrix representation of operators

Cordero, Nicola and Rodino [CNR10b] obtained a result on almost diagonalization for FIOs with respect to a Gabor frame which permitted to study boundedness of Fourier integral operators (FIOs) on weighted modulation spaces. Our aim is to use the almost diagonalization technique to study the compactness of FIOs. To this end we need to establish a clear relationship between operators acting on modulation spaces and operators acting on appropriate sequence spaces.

From now on we assume that $\mathcal{G}(g,\Lambda)$ is a Gabor frame and $g \in \mathcal{S}(\mathbb{R}^d)$. Then $h = S_g^{-1}(g) \in \mathcal{S}(\mathbb{R}^d)$ and $D_g \circ C_h = D_h \circ C_g = Id_{M_m^{p,q}(\mathbb{R}^d)}$ for all $p,q \in [1,\infty]$ and for every v-moderate weight m. The (topological) identities $\mathcal{S}'(\mathbb{R}^d) = \bigcup\{M^2_{1/v_s} : s > 0\}$ and $\mathcal{S}(\mathbb{R}^d) = \bigcap\{M^2_{v_s} : s > 0\}$ permit to conclude that

$$C_g, C_h : \mathcal{S}(\mathbb{R}^d) \to s(\Lambda)$$

and

$$C_g, C_h : \mathcal{S}'(\mathbb{R}^d) \to s'(\Lambda)$$

are topological isomorphisms into their ranges, where $s(\Lambda)$ is the space of rapidly decreasing sequences,

$$s(\Lambda) := \left\{ c \in \ell^\infty(\Lambda) : \lim_{|\lambda|\to\infty} c_\lambda \lambda^\alpha = 0, \quad \text{for all } \alpha \in \mathbb{Z}^d_+ \right\},$$

and $s'(\Lambda)$, its dual space, is endowed with the inductive topology. Moreover, every $f \in \mathcal{S}(\mathbb{R}^d)$ admits a decomposition

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle \pi(\lambda)g,$$

where the series converges in $\mathcal{S}(\mathbb{R}^d)$. Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a continuous and linear operator. For every $f \in \mathcal{S}(\mathbb{R}^d)$, Tf admits a decomposition

$$T(f) = T\left(\sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle \pi(\lambda)g\right) = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle T\left(\pi(\lambda)g\right).$$

Then,

$$\begin{aligned} C_g(T(f)) = (\langle T(f), \pi(\mu)g\rangle)_{\mu\in\Lambda} &= \left(\langle \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle\right)_{\mu\in\Lambda} \\ &= \left(\sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle \langle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle\right)_{\mu\in\Lambda} \\ &= \left(\sum_{\lambda\in\Lambda} (C_h(f))_\lambda \langle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle\right)_{\mu\in\Lambda}, \end{aligned}$$

which inspires the following definition.

Definition 3.2.1. *The **Gabor matrix** associated to a continuous and linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is defined as*

$$M(T) = (\langle T(\pi(\lambda)g), \pi(\mu)g\rangle)_{(\mu,\lambda)\in\Lambda\times\Lambda}\,.$$

If T is a FIO with symbol σ and phase Φ we write $M(\sigma, \Phi)$ instead of $M(T)$.

And from the definition we have the next expression for every $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} C_g(T(f)) &= \left(\sum_{\lambda\in\Lambda} (C_h(f))_\lambda \langle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle\right)_{\mu\in\Lambda} \\ &= \left(\sum_{\lambda\in\Lambda} M(T)_{\mu,\lambda} (C_h(f))_\lambda\right)_{\mu\in\Lambda} = M(T)(C_h(f)) \end{aligned}$$

Theorem 3.2.2. *Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a continuous and linear operator and $\mathcal{G}(g, \Lambda)$ a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$. Then*

(1) For $1 \le p, q < \infty$, T can be (uniquely) extended as a bounded operator from $M^{p,q}_{m_1}(\mathbb{R}^d)$ into $M^{p,q}_{m_2}(\mathbb{R}^d)$ if and only if $M(T)$ defines a bounded operator from $\ell^{p,q}_{m_1}(\Lambda)$ into $\ell^{p,q}_{m_2}(\Lambda)$.

(2) *For* $1 \le p, q \le \infty$, T *can be extended as a weak** *continuous operator from* $M^{p,q}_{m_1}(\mathbb{R}^d)$ *into* $M^{p,q}_{m_2}(\mathbb{R}^d)$ *if and only if* $M(T)$ *defines a weak** *continuous operator from* $\ell^{p,q}_{m_1}(\Lambda)$ *into* $\ell^{p,q}_{m_2}(\Lambda)$.

(3) *Let* $1 \le p, q \le \infty$ *and assume that* $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ *is weak** *continuous. Then* $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ *is compact if and only if* $M(T) : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ *is compact.*

Proof. Let h be the canonical dual window of g. Then we have

$$C_g \circ T = M(T) \circ C_h \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d).$$

Clearly, $M(T)$ defines a continuous operator from the range $C_h(\mathcal{S}(\mathbb{R}^d))$, which is a closed subspace of $s(\Lambda)$, into $s'(\Lambda)$. We now check that $M(T)$ defines a continuous operator from $\mathbb{C}^{(\Lambda)}$, the space of finite complex sequences, into $s'(\Lambda)$, when $\mathbb{C}^{(\Lambda)}$ is endowed with the topology inherited by $s(\Lambda)$. To this end, we fix $x \in \mathbb{C}^{(\Lambda)}$ and observe that $D_g(x) \in \mathcal{S}(\mathbb{R}^d)$, hence $M(T) \circ C_h \circ D_g(x) = C_g \circ T \circ D_g(x)$. That is,

$$(M(T)(C_h \circ D_g)(x))_\mu = \langle T\left(D_g(x)\right), \pi(\mu)g\rangle = \sum_{\lambda \in \Lambda} \langle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle \cdot x_\lambda.$$

Consequently, for every finite sequence x we have

$$M(T)(x) = M(T)(C_h \circ D_g)(x).$$

Therefore, $M(T)$ is continuous on $\mathbb{C}^{(\Lambda)}$ when this space is considered as a subspace of $s(\Lambda)$. By density, $M(T)$ defines a continuous operator from the space $s(\Lambda)$ into $s'(\Lambda)$,

$$M(T) : s(\Lambda) \xrightarrow{D_g} S(\mathbb{R}^d) \xrightarrow{C_h} C_h(\mathcal{S}(\mathbb{R}^d)) \xrightarrow{M(T)} s'(\Lambda).$$

Then we have

$$T = D_h \circ M(T) \circ C_h \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d) \tag{3.2.1}$$

and

$$M(T) = M(T) \circ C_h \circ D_g = C_g \circ T \circ D_g \quad \text{on} \quad s(\Lambda). \tag{3.2.2}$$

Let us now assume that, for $1 \leq p, q < \infty$, $M(T) : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ is bounded. From the continuity of $C_h : M^{p,q}_m(\mathbb{R}^d) \to \ell^{p,q}_m(\Lambda)$ and $D_h : \ell^{p,q}_m(\Lambda) \to M^{p,q}_m(\mathbb{R}^d)$, we deduce that $D_h \circ M(T) \circ C_h : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ is a bounded extension of T. The uniqueness follows from the facts that $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}_m(\mathbb{R}^d)$.

Conversely, let us assume that T can be extended as a bounded operator $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$, for $1 \leq p, q < \infty$. Then, from the continuity of $C_g : M^{p,q}_m(\mathbb{R}^d) \to \ell^{p,q}_m(\Lambda)$ and $D_g : \ell^{p,q}_m(\Lambda) \to M^{p,q}_m(\mathbb{R}^d)$, we deduce that $C_g \circ T \circ D_g : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ is a bounded extension of $M(T)$. The uniqueness follows from the facts that $s(\Lambda)$ is dense in $\ell^{p,q}_m(\Lambda)$. And we have (1).

To prove (2) we use the same arguments, changing the continuity of T and $M(T)$ by the weak*-continuity, and the fact that $\mathcal{S}(\mathbb{R}^d)$ is weak* dense in $M^{p,q}_m(\mathbb{R}^d)$ and $s(\Lambda)$ is weak* dense in $\ell^{p,q}_m(\Lambda)$ for $1 \leq p, q \leq \infty$.

To finish we prove (3). Recall that the compact operators are a closed ideal in the algebra of the bounded operators. As $\mathcal{S}(\mathbb{R}^d)$ is weak* dense in $M^{p,q}_m(\mathbb{R}^d)$ and $s(\Lambda)$ is weak* dense in $\ell^{p,q}_m(\Lambda)$, for $1 \leq p, q \leq \infty$, from (3.2.1), (3.2.2) and the weak*-continuity of $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$, for $1 \leq p, q \leq \infty$, we deduce that $T = D_h \circ M(T) \circ C_h : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ is compact if and only if $M(T) = C_g \circ T \circ D_g : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ is compact, $\square$

In the applications to the FIOs we will always consider $m_1 = m \circ \chi$ and $m_2 = m$. In the special case of PSDOs we will have $m_1 = m_2 = m$.

3.3 Compactness of Fourier integral operators

3.3.1 Fourier integral operators on M^p_m

Our aim is to discuss compactness properties for a FIO T whose phase is tame and with symbol $\sigma \in M^\infty_{1 \otimes v_{s_0}}(\mathbb{R}^{2d})$ for some $s_0 > 2d$. Through this section we use a fixed lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and a fixed Parseval frame $\mathcal{G}(g, \Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$. As proved in [CGN12], we have the following estimate

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle \chi(\lambda) - \mu\rangle^{-s_0}, \quad \text{for all } \lambda, \mu \in \Lambda. \tag{3.3.1}$$

And as in [CGN12], observe that any symbol satisfying condition (3.1.3) belongs to $M^{\infty}_{1\otimes v_{2N}}$.

The estimate (3.3.1) together with the results of Subsection 3.2 suggest that we should consider operators on sequence spaces defined in terms of a matrix $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda}$, where the distance of (γ, γ') to the graph of $\gamma = \chi(\gamma')$ determines the decay of the coefficient $a_{\gamma,\gamma'}$. As we cannot assure that $\chi(\lambda) \in \Lambda$, for $\lambda \in \Lambda$, we replace the canonical transformation χ by an appropriate discrete version $\chi' : \Lambda \to \Lambda$, defined as follows. We fix a symmetric relatively compact fundamental domain Q of Λ and, for every $\lambda \in \Lambda$, decompose any

$$\chi(\lambda) = r_\lambda + \chi'(\lambda),$$

where $\chi'(\lambda) \in \Lambda$ and $r_\lambda \in Q$. Since χ^{-1} is Lipschitz continuous there is $L > 0$ such that $\chi'(\lambda) = \chi'(\mu)$ implies

$$a := 2\sup_{u\in Q}\|u\| \geq \|\chi(\lambda) - \chi(\mu)\| \geq L\|\lambda - \mu\|.$$

Hence

$$\chi'^{-1}\left(\{\chi'(\lambda)\}\right) = \left\{\mu \in \Lambda : \ \chi'(\mu) = \chi'(\lambda)\right\}$$

is contained in $\overline{B\left(\lambda, \frac{a}{L}\right)} \cap \Lambda$, which is a finite set whose cardinal does not depend on λ. This suggests the condition imposed in the following definition.

Definition 3.3.1. *Let v be a submultiplicative weight on $\mathbb{R}^{2d}$ and assume that $\psi : \Lambda \to \Lambda$ satisfies*

$$M = \sup_{\lambda\in\Lambda}\left\{\operatorname{card}\left(\psi^{-1}\left(\{\lambda\}\right)\right)\right\} < \infty.$$

We define $\mathcal{C}_{v,\psi}(\Lambda)$ *as the set of all matrices* $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda}$ *such that*

$$\|A\|_{\mathcal{C}_{v,\psi}} := \sum_{\gamma\in\Lambda} v(\gamma) \cdot \sup_{\lambda\in\Lambda} \left|a_{\psi(\lambda)+\gamma,\lambda}\right| < \infty.$$

Proposition 3.3.2. *Let* T *be a FIO whose phase* Φ *is tame and* $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. *Then, for every* $0 \le s < s_0 - 2d$ *we have*

$$M(\sigma, \Phi) \in \mathcal{C}_{v_s,\chi'}.$$

Proof. We put $a_{\mu,\lambda} = \langle T\pi(\lambda)g, \pi(\mu)g\rangle$. We have to show that

$$\sum_{\gamma\in\Lambda} v_s(\gamma) \cdot \sup_{\lambda\in\Lambda} \left|a_{\chi'(\lambda)+\gamma,\lambda}\right| < \infty.$$

According to [CGN12, Theorem 3.3],

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \le C\langle\chi(\lambda) - \mu\rangle^{-s_0} = C(v_{s_0}(\chi(\lambda) - \mu))^{-1}$$

for some constant C. Since there is $r_\lambda \in Q$ such that $\chi(\lambda) = \chi'(\lambda) + r_\lambda$, we obtain

$$\begin{aligned}
|a_{\chi'(\lambda)+\gamma,\lambda}| &= |\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \gamma)g\rangle| \\
&\le C(v_{s_0}(\chi(\lambda) - \chi'(\lambda) - \gamma))^{-1} \\
&= \frac{C}{v_{s_0}(r_\lambda - \gamma)} \le \frac{Cv_{s_0}(r_\lambda)}{v_{s_0}(\gamma)} \le \frac{CR}{v_{s_0}(\gamma)},
\end{aligned}$$

where $R = \max\{v_{s_0}(r) : r \in Q\}$. Finally, using that $2d < s_0 - s$,

$$\sum_{\gamma\in\Lambda} v_s(\gamma) \cdot \sup_{\lambda\in\Lambda} \left|a_{\chi'(\lambda)+\gamma,\lambda}\right| \le CR \sum_{\gamma\in\Lambda} \frac{v_s(\gamma)}{v_{s_0}(\gamma)} < \infty.$$

□

The following almost diagonal map will play an important role when discussing compactness properties of operators defined in terms of matrices in $\mathcal{C}_{v_s,\psi}$.

Definition 3.3.3. *Let $\psi : \Lambda \to \Lambda$ be as in Definition 3.3.1 and $a \in \mathbb{C}^\Lambda$. Then*

$$D_{a,\psi} : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$$

is defined by $D_{a,\psi}(x) = y$ where

$$y_\gamma = \begin{cases} 0 & \text{if} \quad \gamma \notin \psi(\Lambda) \\ \displaystyle\sum_{\psi(\lambda)=\gamma} a_\lambda x_\lambda & \text{if} \quad \gamma \in \psi(\Lambda) \end{cases}$$

In particular, $D_{a,\psi}(e_\gamma) = a_\gamma e_{\psi(\gamma)}$. Moreover, $D_{a,\psi}\left(\mathbb{C}^{(\Lambda)}\right) \subset \mathbb{C}^{(\Lambda)}$.

The transposed map

$$D^t_{a,\psi} : \mathbb{C}^{(\Lambda)} \to \mathbb{C}^{(\Lambda)},$$

is given by

$$(D^t_{a,\psi}(x))_\lambda = \left(D^t_{a,\psi}(x), e_\lambda\right) = \left(x, a_\lambda e_{\psi(\lambda)}\right) = a_\lambda x_{\psi(\lambda)}.$$

In fact, $D^t_{a,\psi}$ can be extended as a map from $\mathbb{C}^\Lambda$ into itself. In the case that a is the constant sequence equal 1 the map $D_{a,\psi}$ is denoted by I_ψ. Then, for an arbitrary $a \in \mathbb{C}^\Lambda$ we have

$$D_{a,\psi} = I_\psi \circ D_a.$$

When ψ is the identity, $D_{a,\psi}$ is just the diagonal operator D_a.

Lemma 3.3.4. *Let*

$$M = \sup_{\lambda\in\Lambda} \left\{\text{card}\left(\psi^{-1}\left(\{\lambda\}\right)\right)\right\} < \infty.$$

Then, there is a partition $\Lambda = \bigcup_{j=1}^M \Lambda_j$ such that ψ is injective when restricted to each Λ_j.

Proof. For all $\mu \in \Lambda$, we know that there exist, at most, $\lambda_1, \lambda_2, ..., \lambda_M \in \Lambda$ such that $\mu = \psi(\lambda_j)$. We put each $\lambda_j \in \Lambda_j$, from this action we know that ψ is injective in each Λ_j. Each $\Lambda_i \subseteq \Lambda$, then $\bigcup_{j=1}^{M} \Lambda_j \subseteq \Lambda$. If $\lambda \in \Lambda$, $\psi(\lambda) \in \Lambda$, then there exist $j \in \{1, 2, .., m\}$ such that $\lambda \in \Lambda_j$, then $\Lambda \subseteq \bigcup_{j=1}^{M} \Lambda_j$ $\square$

Let $m = (m_\lambda)_{\lambda\in\Lambda}$ be a positive sequence. For any $\psi : \Lambda \to \Lambda$ as in Definition 3.3.1 we denote by $m \circ \psi$ the sequence

$$m \circ \psi = \left(m_{\psi(\lambda)}\right)_{\lambda\in\Lambda}.$$

Proposition 3.3.5. *Let $\psi : \Lambda \to \Lambda$ be as in Definition 3.3.1, $a = (a_\lambda)_{\lambda\in\Lambda}$ a sequence of complex numbers, $m = (m_\lambda)_{\lambda\in\Lambda}$ a positive sequence and $p \in [1, \infty]$. The following conditions are equivalent:*

(1) $D_{a,\psi}$ is continuous on $\ell^2(\Lambda)$.

(2) $D_{a,\psi}$ is continuous from $\ell^p_{m\circ\psi}(\Lambda)$ to $\ell^p_m(\Lambda)$.

(3) $a \in \ell^\infty(\Lambda)$.

Proof. It suffices to show the equivalence between conditions (2) and (3). Let us assume that condition (2) is satisfied. As $D_{a,\psi}(e_\lambda) = a_\lambda e_{\psi(\lambda)}$ then

$$\|D_{a,\psi}\| \geq \left\| D_{a,\psi}\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)\right\|_{\ell^p_m} = |a_\lambda|,$$

from where we get (3).

To check that (3) implies (2) let us first assume that $a \in \ell^\infty(\Lambda)$ and the restriction of ψ to the support of a, that is $\{\lambda \in \Lambda \text{ such that } a_\lambda \neq 0\}$, is injective. Then

$$\|D_{a,\psi}(x)\|_{\ell^p_m} = \|(a_\lambda x_\lambda m_{\psi(\lambda)})_\lambda\|_{\ell^p} = \|(a_\lambda x_\lambda)_\lambda\|_{\ell^p_{m\circ\psi}} \leq \|a\|_{\ell^\infty}\|x\|_{\ell^p_{m\circ\psi}}.$$

In the case that condition (3) is satisfied but ψ is not injective on the support of a we apply Lemma 3.3.4 and decompose

$$a = \sum_{j=1}^{M} a^j,$$

in such a way that the support of a^j is contained in Λ_j. Then

$$D_{a,\psi} = \sum_{j=1}^{M} D_{a^j,\psi}$$

is continuous from $\ell^p_{m\circ\psi}(\Lambda)$ to $\ell^p_m(\Lambda)$ and

$$\|D_{a,\psi}\|_{\ell^p_{m\circ\psi}\to\ell^p_m} \leq \sum_{j=1}^{M} \|a^j\|_{\ell^\infty} \leq M\|a\|_{\ell^\infty}.$$

Hence (3) implies (2) is proved. □

Remark 3.3.6. *The same argument shows that condition (3) in Proposition 3.3.5 is equivalent to being $D_{a,\psi}$ a bounded operator from $c_{0,m\circ\psi}(\lambda)$ into $c_{0,m}(\lambda)$.*

In particular, $I_\psi : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)$ is continuous. We observe that, if $p \neq q$, the map I_ψ does not need to be bounded on spaces $\ell^{p,q}_m(\Lambda)$, as can be seen in the following example.

Example 3.3.7. *Let $\psi : \mathbb{Z}\times\mathbb{Z} \mapsto \mathbb{Z}\times\mathbb{Z}$, given by $\psi(i,j) = (j,i)$. Then I_ψ is not bounded on $\ell^{2,1}(\mathbb{Z}\times\mathbb{Z})$.*

Proof. Let $(x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}} \in \ell^{2,1}(\mathbb{Z}\times\mathbb{Z})$, we calculate $I_\psi((x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}})$,

$$\begin{aligned} I_\psi((x_{i,j})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}) &= I_\psi\left(\sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} x_{i,j}e_{i,j}\right) = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} x_{i,j}I_\psi\left(e_{i,j}\right) \\ &= \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} x_{i,j}e_{\psi(i,j)} = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} x_{i,j}e_{j,i} = (x_{j,i})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}. \end{aligned}$$

We consider the sequence $(x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}}$, where

$$x_{i,j} = \frac{1}{(|i|+1)2^{|j|}},$$

we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\left|\left(\sum_{i\in\mathbb{Z}}|x_{i,j}|^2\right)^{\frac{1}{2}}\right| &= \sum_{j\in\mathbb{Z}}\left(\sum_{i\in\mathbb{Z}}\left|\frac{1}{(|i|+1)2^{|j|}}\right|^2\right)^{\frac{1}{2}}\\
&= \sum_{j\in\mathbb{Z}}\left(\sum_{i\in\mathbb{Z}}\left|\frac{1}{(|i|+1)}\right|^2\left|\frac{1}{2^{|j|}}\right|^2\right)^{\frac{1}{2}} = \sum_{j\in\mathbb{Z}}\left(\left|\frac{1}{2^{|j|}}\right|^2\sum_{i\in\mathbb{Z}}\left|\frac{1}{(|i|+1)}\right|^2\right)^{\frac{1}{2}}\\
&= \sum_{j\in\mathbb{Z}}\left|\frac{1}{2^{|j|}}\right|\left(\sum_{i\in\mathbb{Z}}\left|\frac{1}{(|i|+1)}\right|^2\right)^{\frac{1}{2}} \leq 2\sum_{j\in\mathbb{N}}\left|\frac{1}{2^{j}}\right|\left(2\sum_{i\in\mathbb{N}}\left|\frac{1}{(i+1)}\right|^2\right)^{\frac{1}{2}}\\
&\leq 4\left(2\frac{\pi^2}{6}\right)^{\frac{1}{2}} = \frac{4\pi}{\sqrt{3}} < \infty,
\end{aligned}$$

that is $(x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}} \in \ell^{2,1}(\mathbb{Z}\times\mathbb{Z})$, let us prove that $I_\psi((x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}})$ does not belong to $\ell^{2,1}(\mathbb{Z}\times\mathbb{Z})$.

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\left|\left(\sum_{i\in\mathbb{Z}}\left|\left(I_\psi((x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}})\right)_{i,j}\right|^2\right)^{\frac{1}{2}}\right| &= \sum_{j\in\mathbb{Z}}\left|\left(\sum_{i\in\mathbb{Z}}|x_{j,i}|^2\right)^{\frac{1}{2}}\right|\\
&= \sum_{j\in\mathbb{Z}}\left(\sum_{i\in\mathbb{Z}}\left|\frac{1}{(|j|+1)2^{|i|}}\right|^2\right)^{\frac{1}{2}} = \sum_{j\in\mathbb{Z}}\left(\sum_{i\in\mathbb{Z}}\left|\frac{1}{(|j|+1)}\right|^2\left|\frac{1}{2^{|i|}}\right|^2\right)^{\frac{1}{2}}\\
&= \sum_{j\in\mathbb{Z}}\left(\left|\frac{1}{(|j|+1)}\right|^2\sum_{i\in\mathbb{Z}}\left|\frac{1}{2^{|i|}}\right|^2\right)^{\frac{1}{2}} = \sum_{j\in\mathbb{Z}}\left|\frac{1}{(|j|+1)}\right|\left(\sum_{i\in\mathbb{Z}}\left|\frac{1}{2^{|i|}}\right|^2\right)^{\frac{1}{2}} = \infty.
\end{aligned}$$

Hence $I_\psi((x_{i,j})_{i,j\in\mathbb{Z}\times\mathbb{Z}}) \notin \ell^{2,1}(\mathbb{Z}\times\mathbb{Z})$. $\square$

Proposition 3.3.8. *Let* $m = (m_\lambda)_{\lambda \in \Lambda}$ *be a* v*-moderate positive sequence,* $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma' \in \Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ *and* $1 \leq p \leq \infty$ *be given. Then* $A : \ell^p_{m \circ \psi}(\Lambda) \to \ell^p_m(\Lambda)$ *is a bounded operator, which is also weak* continuous.*

Proof. It is easier to deal with the transposed map, so we first consider $b_{\gamma,\gamma'} = a_{\gamma',\gamma}$ and *claim* that $B = \left(b_{\gamma,\gamma'}\right)_{\gamma,\gamma' \in \Lambda}$ defines a bounded operator $B : \ell^q_{\frac{1}{m}}(\Lambda) \to \ell^q_{\frac{1}{m} \circ \psi}(\Lambda)$ for every $1 \leq q \leq \infty$. We should remark here that the class $\mathcal{C}_{v,\psi}(\Lambda)$ does not need to be closed under transposition. Instead we have

$$\sum_{\gamma \in \Lambda} v(\gamma) \cdot \sup_{\lambda \in \Lambda} \left| b_{\lambda,\psi(\lambda)+\gamma} \right| < \infty.$$

Using that for every $\lambda \in \Lambda$ one has $\Lambda = \psi(\lambda) + \Lambda$, and the inequality

$$1 \leq C_m \frac{m_{\psi(\lambda)}}{m_{\psi(\lambda)+\gamma}} v(\gamma), \tag{3.3.2}$$

we may write

$$\begin{aligned}
\sum_{\gamma \in \Lambda} |b_{\lambda,\gamma} x_\gamma| &= \sum_{\gamma \in \Lambda} \left| b_{\lambda,\psi(\lambda)+\gamma} x_{\psi(\lambda)+\gamma} \right| \\
&\leq \sum_{\gamma \in \Lambda} \left| b_{\lambda,\psi(\lambda)+\gamma} x_{\psi(\lambda)+\gamma} \right| C_m \frac{m_{\psi(\lambda)}}{m_{\psi(\lambda)+\gamma}} v(\gamma) \\
&\leq C_m m_{\psi(\lambda)} \left(\sum_{\gamma \in \Lambda} (v(\gamma) \sup_\lambda |b_{\lambda,\psi(\lambda)+\gamma}|)^{q'} \right)^{\frac{1}{q'}} \left(\sum_{\gamma \in \Lambda} \left(\frac{\left| x_{\psi(\lambda)+\gamma} \right|}{m_{\psi(\lambda)+\gamma}} \right)^q \right)^{\frac{1}{q}} \\
&< \infty.
\end{aligned}$$

Therefore we conclude that

$$B : \ell^q_{\frac{1}{m}}(\Lambda) \to \mathbb{C}^\Lambda$$

is a well-defined operator.

To prove that $Bx \in \ell^q_{\frac{1}{m}\circ\psi}(\Lambda)$ it is enough to check that

$$\sum_{\lambda\in\Lambda} |(Bx)_\lambda\, y_\lambda| < \infty$$

for every $y \in \ell^{q'}_{m\circ\psi}$. Here q' is the usual conjugate exponent, for $q = 1$ we consider $q' = \infty$. To this end we denote $\phi(\gamma) = v(\gamma)\sup_\lambda \left|b_{\lambda,\psi(\lambda)+\gamma}\right|$. Using again the inequality (3.3.2) we obtain

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |(Bx)_\lambda\, y_\lambda| &\leq \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Lambda} |b_{\lambda,\gamma} x_\gamma y_\lambda| = \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Lambda} \left|b_{\lambda,\psi(\lambda)+\gamma} x_{\psi(\lambda)+\gamma} y_{\lambda\in\Lambda}\right| \\
&\leq C_m \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Lambda} \left|b_{\lambda,\psi(\lambda)+\gamma}\right| v(\gamma) \frac{\left|x_{\psi(\lambda)+\gamma}\right|}{m_{\psi(\lambda)+\gamma}} |y_\lambda|\, m_{\psi(\lambda)} \\
&\leq C_m \sum_{\gamma\in\Lambda} \phi(\gamma) \sum_{\lambda\in\Lambda} \frac{\left|x_{\psi(\lambda)+\gamma}\right|}{m_{\psi(\lambda)+\gamma}} |y_\lambda|\, m_{\psi(\lambda)} \\
&\leq MC_m \|x\|_{\ell^q_{\frac{1}{m}}} \cdot \|y\|_{\ell^{q'}_{m\circ\psi}} \cdot \|A\|_{\mathcal{C}_{v,\psi}},
\end{aligned}$$

where M is the constant in Definition 3.3.1. Moreover,

$$\begin{aligned}
\|Bx\|_{\ell^q_{\frac{1}{m}\circ\psi}(\Lambda)} &= \sup_{\|y\|_{\ell^{q'}_{m\circ\psi}} \leq 1} \left\{ \sum_{\lambda\in\Lambda} |(Bx)_\lambda\, y_\lambda| \right\} \\
&\leq \sup_{\|y\|_{\ell^{q'}_{m\circ\psi}} \leq 1} \left\{ MC_m \|x\|_{\ell^q_{\frac{1}{m}}} \cdot \|y\|_{\ell^{q'}_{m\circ\psi}} \cdot \|A\|_{\mathcal{C}_{v,\psi}} \right\} \\
&\leq MC_m \|x\|_{\ell^q_{\frac{1}{m}}} \cdot \|A\|_{\mathcal{C}_{v,\psi}},
\end{aligned}$$

and $B : \ell^q_{\frac{1}{m}}(\Lambda) \to \ell^q_{\frac{1}{m}\circ\psi}(\Lambda)$ is a bounded operator for every $1 \leq q \leq \infty$. In fact, B also defines a bounded operator from $c_{0,\frac{1}{m}}$ to $c_{0,\frac{1}{m}\circ\psi}$. In fact,

$B : \ell^\infty_{\frac{1}{m}} \to \ell^\infty_{\frac{1}{m}\circ\psi}$ is continuous, $B\left(\mathbb{C}^{(\Lambda)}\right) \subset \ell^1_{\frac{1}{m}\circ\psi} \subset c_{0,\frac{1}{m}\circ\psi}$ and $c_{0,\frac{1}{m}}$ is the closure of $\mathbb{C}^{(\Lambda)}$ on $\ell^\infty_{\frac{1}{m}}$. Consequently, for every $1 \le p \le \infty$, the transposed map defines a bounded operator $A = B^t : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)$ which is also weak* continuous. □

Proposition 3.3.9. *Let $m = (m_\lambda)_{\lambda\in\Lambda}$ a v-moderate positive sequence, $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ and $1 \le p \le \infty$ be given. Then*

$$A = \sum_{\gamma\in\Lambda} (T_\gamma \circ D_{a^\gamma,\psi})$$

where $a^\gamma := (a_{\psi(\lambda)+\gamma,\lambda})_{\lambda\in\Lambda}$. The series converges absolutely in the operator norm.

Proof. Since m is v-moderate with constant C_m, by Proposition 1.1.6, we have

$$\|T_\gamma : \ell^p_m(\Lambda) \to \ell^p_m(\Lambda)\| \le C_m v(\gamma).$$

Also

$$\|D_{a^\gamma,\psi} : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)\| \le M \sup_\lambda \left|a_{\psi(\lambda)+\gamma,\lambda}\right|,$$

where M is the constant in the Definition 3.3.1. Hence

$$\sum_{\gamma\in\Lambda} \|T_\gamma \circ D_{a^\gamma,\psi}\| \le M \sum_{\gamma\in\Lambda} C_m v(\gamma) \sup_{\lambda\in\Lambda} |a_{\psi(\gamma)+\lambda,\lambda}| < \infty.$$

Consequently

$$S := \sum_{\gamma\in\Lambda} (T_\gamma \circ D_{a^\gamma,\psi})$$

defines a bounded operator from $\ell^p_{m\circ\psi}(\Lambda)$ into $\ell^p_m(\Lambda)$. With a similar argument we can decompose the transposed map in terms of operators $D^t_{a^\gamma,\psi} = I^t_\psi \circ D_{a^\gamma}$, from where we conclude that S is also weak* continuous. Moreover, A and S coincide on $\{e_\lambda : \lambda \in \Lambda\}$, from where the result follows. In fact,

$$\begin{aligned}\langle S(e_\lambda), e_\mu\rangle &= \left\langle \sum_{\gamma\in\Lambda} a_{\psi(\lambda)+\gamma,\lambda} e_{\psi(\lambda)+\gamma}, e_\mu \right\rangle = \left\langle \sum_{t\in\Lambda} a_{t,\lambda} e_t, e_\mu \right\rangle \\ &= \langle A(e_\lambda), e_\mu\rangle .\end{aligned}$$

□

The following abstract result will be useful to obtain necessary conditions for the compactness of FIOs.

Proposition 3.3.10. *Let $E = G'$ and $F = R'$ be dual Banach spaces and $T : E \to F$ be a compact operator, such that $T^t(R) \subseteq G$. If $\{x_i\}_{i\in I}$ is a sequence that converges to x in the weak* topology, $\sigma(E,G)$, then, $\{(T(x_i))\}_{i\in I}$ converges to $T(x)$.*

Proof. We first check that $\{x_i\}_{i\in I}$ is a bounded sequence in E. In fact, $\{x_i\}_{i\in I}$ is a bounded sequence in $\sigma(E,G)$. If we consider the sequence of linear operators $\{\langle x_i,\cdot\rangle\}_{i\in I}$, then for every $g\in G$, $\{\langle x_i, g\rangle\}_{i\in I}$ is a bounded sequence. By Banach-Steinhaus's Theorem, we obtain that $\{\langle x_i,\cdot\rangle\}_{i\in I}$ is uniformly bounded and we conclude that $\{x_i\}_{i\in I}$ is a bounded sequence in E. We assume that $\{T(x_i)\}_{i\in I}$ does not converge to $T(x)$ in norm. Then there are $\varepsilon > 0$ and a sequence of indices $(i_k)_{k=1}^\infty \subset I$ such that, for every k,

$$\|T(x_{i_k}) - T(x)\| > \varepsilon.$$

Since T is a compact operator, there exists a subsequence $\{T(x_{i_{k_t}})\}_t$ converging to some $y \in F$. Since $\{x_{i_{k_t}}\}_t$ $\sigma(E,G)$-converges to x we conclude that $\{T(x_{i_{k_t}})\}_t$ $\sigma(F,R)$-converges to $T(x)$. Since the norm convergence implies the $\sigma(F,R)$-convergence in F,we finally obtain that $y = T(x)$. Consequently, $\{T(x_{i_{k_t}})\}_t$ converges to $T(x)$ in norm, which is a contradiction. □

Theorem 3.3.11. *Let $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$ and $1 \le p \le \infty$ be given. Then, $A : \ell^p_{m\circ\psi}(\Lambda) \to \ell^p_m(\Lambda)$ is a compact operator if and only if*

$$a^\gamma := \left(a_{\psi(\lambda)+\gamma,\lambda}\right)_{\lambda\in\Lambda} \in c_0(\Lambda) \quad \textit{for all } \gamma\in\Lambda.$$

Proof. If $a^\gamma \in c_0(\Lambda)$ for every $\gamma \in \Lambda$, then $D_{a^\gamma,\psi} = I_\psi \circ D_{a^\gamma}$ is compact for each $\gamma \in \Lambda$. Hence, we can apply Propositions 3.3.8 and 3.3.9 to conclude that A is a compact operator.

Let us now assume that A is compact. Let

$$\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)_{\lambda\in\Lambda} \subseteq \ell^p_{m\circ\psi}(\Lambda),$$

let us see that it converges to zero in the weak* topology. Take an element of the pre-dual of $\ell^p_{m\circ\psi}(\Lambda)$, $x \in \ell^{p'}_{\frac{1}{m\circ\psi}}(\Lambda)$, being p' the conjugate exponent (for $p = 1$ we consider $p' = 0$). In particular, we know that for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that, if $|\lambda| \geq M$,

$$\left|\frac{x_\lambda}{m_{\psi(\lambda)}}\right| < \varepsilon.$$

Then,

$$\left|\left\langle \frac{e_\lambda}{m_{\psi(\lambda)}}, x\right\rangle\right| = \left|\langle e_\lambda, x\rangle \frac{1}{m_{\psi(\lambda)}}\right| = \left|x_\lambda \frac{1}{m_{\psi(\lambda)}}\right| < \varepsilon.$$

Since $\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)_{\lambda\in\Lambda}$ converges to zero in the weak* topology of $\ell^p_{m\circ\psi}(\Lambda)$, we can apply Proposition 3.3.10 to conclude that $\left(A\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)\right)_{\lambda\in\Lambda}$ converges to 0. Now, we fix $\gamma \in \Lambda$ and use that

$$\frac{m_{\psi(\lambda)+\gamma}}{m_{\psi(\lambda)}} |a_{\psi(\lambda)+\gamma,\lambda}| \leq \left\| A\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)\right\|_{\ell^p_m(\Lambda)}.$$

Since m is v-moderate we obtain

$$\left|a_{\psi(\lambda)+\gamma,\lambda}\right| \leq C_m v(\gamma) \left\| A\left(\frac{e_\lambda}{m_{\psi(\lambda)}}\right)\right\|_{\ell^p_m(\Lambda)},$$

which finishes the proof. □

We will apply Theorem 3.3.11 to the study of compactness of FIOs

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)d\eta$$

whose phase is tame and with symbol $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. As usual, χ is the canonical transformation of the symbol Φ and $\chi' : \Lambda \to \Lambda$ is its discrete version.

Theorem 3.3.12. *Let T be a FIO whose phase Φ is tame and $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. The following conditions are equivalent:*

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.

(2) $M(\sigma,\Phi) : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is compact.

(3) $(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle)_\lambda \in c_0(\Lambda)$ for every $\mu \in \Lambda$.

Proof. Since $M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$ we can apply [CNR10b, Theorem 6.1] to obtain that $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a bounded operator. From Theorem 3.2.2 we get the equivalence of conditions (1) and (2). Now it suffices to apply Proposition 3.3.2 and Theorem 3.3.11 to conclude. $\square$

We observe that, for any positive and v_s-moderate weight m,

$$\ell^p_{m\circ\chi}(\Lambda) = \ell^p_{m\circ\chi'}(\Lambda)$$

with equivalent norms and that $m \circ \chi$ is v_s-moderate whenever m is.

Theorem 3.3.13. *Let T be a FIO whose phase Φ is tame and $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$, $s_0 > 2d$. Then, for every $0 \le s < s_0 - 2d$, the following conditions are equivalent:*

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.

(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for some $1 \le p < \infty$ and for some v_s-moderate weight m.

(3) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ *is a compact operator for every* $1 \le p < \infty$ *and for every* v_s*-moderate weight* m.

Proof. From [CGN12, Corollary 5.5] and Propositions 3.3.2 and 3.3.8 we have that

$$T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d) \text{ and } M(\sigma,\Phi) : \ell^p_{m\circ\chi'}(\Lambda) \to \ell^p_m(\Lambda)$$

are bounded operators for every $1 \le p < \infty$ and for every v_s-moderate weight m. It suffices to show (2) $\Rightarrow$ (3). According to Theorems 3.2.2 and 3.3.11, condition (2) is equivalent to the fact that

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle\big)_\lambda \in c_0(\Lambda),$$

for every $\mu \in \Lambda$ and this condition does not depend on p nor on m. □

We next discuss the case $p = \infty$.

Theorem 3.3.14. *Let* T *be a FIO whose phase* Φ *is tame and* $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ *and let* $0 \le s < s_0 - 2d$ *and* m *a* v_s*-moderate weight. Then*

(1) T *admits a unique extension as a bounded operator*

$$T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \to M^\infty_m(\mathbb{R}^d),$$

which is also weak-continuous.*

(2) $T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \to M^\infty_m(\mathbb{R}^d)$ *is compact if and only if*

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle\big)_\lambda \in c_0(\Lambda),$$

for every $\mu \in \Lambda$.

Proof. (1) We know that $T = C_g \circ M(\sigma,\Phi) \circ C_g^*$ is bounded on $\mathcal{S}(\mathbb{R}^d)$, which is weak*-dense in $M^\infty_m(\mathbb{R}^d)$ for all m, a v_s-moderate weight. Then T admits a unique extension,

$$\tilde{T} : M^\infty_{m\circ\chi}(\mathbb{R}^d) \xrightarrow{C_g} \ell^\infty_{m\circ\chi}(\Lambda) \xrightarrow{M(\sigma,\Phi)} \ell^\infty_m(\Lambda) \xrightarrow{S^*} M^\infty_m(\mathbb{R}^d),$$

where $S = C_g : M^1_{1/m}(\mathbb{R}^d) \to \ell^1_{1/m}(\Lambda)$. In fact, all the involved maps are weak*-continuous, then $\tilde{T}$ is weak*-continuous. As the extension is unique we denote it as T.

(2) By Theorem 3.2.2, T is compact if and only if $M(\sigma, \Phi) : \ell^\infty_{m\circ\chi} \to \ell^\infty_m(\Lambda)$ is compact. Now it suffices to apply Theorem 3.3.11. □

For the proof of the next result we recall that the canonical transformation $(x, \xi) = \chi(y, \eta)$ is defined through the system

$$\begin{cases} y = \nabla_\eta \Phi(x, \eta), \\ \xi = \nabla_x \Phi(x, \eta). \end{cases}$$

Theorem 3.3.15. *Let T be a FIO whose phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ and let $0 \le s < s_0 - 2d$. If $\sigma \in M^0(\mathbb{R}^{2d})$ then $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \le p \le \infty$ and for every v_s-moderate weight m.*

Proof. It suffices to show that $M(\sigma, \Phi)_{\mu,\lambda}$ goes to zero as $|(\mu, \lambda)|$ goes to infinity. To this end we first recall the relation between the Gabor matrix of T and the STFT of σ. We denote $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \in \mathbb{R}^{2d}$. From [CNR10b, (39)] we have

$$|M(\sigma, \Phi)_{\mu,\lambda}| = |\langle T\pi(\lambda)g, \pi(\mu)g\rangle| = |V_{\Psi_{\mu_1,\lambda_2}}\sigma(z_{\lambda,\mu})|, \tag{3.3.3}$$

where

$$z_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)} = (\mu_1, \lambda_2, (\mu_2 - \nabla_x\Phi(\mu_1, \lambda_2)), (\lambda_1 - \nabla_\eta\Phi(\mu_1, \lambda_2))),$$

$$\Psi_{(\mu_1,\lambda_2)}(w) = e^{2\pi \mathbf{i}\Phi_{2,(\mu_1,\lambda_2)}(w)}\overline{g} \otimes \hat{g}$$

and $\Phi_{2,(\mu_1,\lambda_2)}$ denotes the reminder of order two of the Taylor series of Φ, that is,

$$\Phi_{2,(\mu_1,\lambda_2)}(w) = 2\sum_{|\alpha|=2}\int_0^1 (1-t)\partial^\alpha\Phi((\mu_1, \lambda_2) + tw)dt\frac{w^\alpha}{\alpha!},$$

with $(\mu_1, \lambda_2), w \in \mathbb{R}^{2d}$. By [CNR10b, 6.1] we obtain that

$$D = \{\Psi_{(\mu_1,\lambda_2)} : (\mu_1, \lambda_2) \in \mathbb{Z}^{2d}\}$$

is a relatively compact set in $S(\mathbb{R}^{2d})$. Since $\sigma \in M^0(\mathbb{R}^{2d})$,

$$S(\mathbb{R}^{2d}) \to C_0(\mathbb{R}^{2d}), \quad \Psi \mapsto V_\Psi \sigma$$

is a continuous map, hence

$$\widetilde{D} = \{V_{\Psi_{(\mu_1,\lambda_2)}} \sigma : (\mu_1, \lambda_2) \in \mathbb{Z}^{2d}\} = \{V_\Psi \sigma : \Psi \in D\}$$

is a relatively compact set in $C_0(\mathbb{R}^{2d})$. Consequently, for every $\varepsilon > 0$, there exists a finite set A with the property that for every $(\mu_1, \lambda_2) \in \mathbb{Z}^{2d}$ there is $(\widetilde{\mu_1}, \widetilde{\lambda_2}) \in A$ such that

$$\sup_{z \in \mathbb{R}^{4d}} |V_{\Psi_{(\widetilde{\mu_1},\widetilde{\lambda_2})}} \sigma(z) - V_{\Psi_{(\mu_1,\lambda_2)}} \sigma(z)| < \frac{\varepsilon}{2}.$$

We take $M_\varepsilon > 0$ so that

$$\max_{(\widetilde{\mu_1},\widetilde{\lambda_2}) \in A} \sup_{|z| > M_\varepsilon} |V_{\Psi_{(\widetilde{\mu_1},\widetilde{\lambda_2})}} \sigma(z)| < \frac{\varepsilon}{2}.$$

Now take $z \in \mathbb{R}^{4d}$ with $|z| > M_\varepsilon$. Then

$$|V_{\Psi_{(\mu_1,\lambda_2)}} \sigma(z)| \leq |V_{\Psi_{(\mu_1,\lambda_2)}} \sigma(z) - V_{\Psi_{(\widetilde{\mu_1},\widetilde{\lambda_2})}} \sigma(z)| + |V_{\Psi_{(\widetilde{\mu_1},\widetilde{\lambda_2})}} \sigma(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon,$$

for every $(\mu_1, \lambda_2) \in \mathbb{Z}^{2d}$. We conclude that $|V_{\Psi_{\mu_1,\lambda_2}} \sigma(z_{\lambda,\mu})|$ goes uniformly to zero as $|z_{\lambda,\mu}|$ goes to infinity.

Finally, we check that $M(\sigma, \Phi)_{\lambda,\mu}$ goes to zero as $|(\lambda, \mu)|$ goes to infinity. We can distinguish two cases:

- μ_1 or λ_2 goes to infinity. Then also $|z_{\lambda,\mu}|$ goes to infinity.

- Neither μ_1 nor λ_2 goes to infinity. We can assume that there exist $C > 0$ such that $|(\mu_1, \lambda_2)| \leq C$, from where it follows that $\nabla_x \Phi(\mu_1, \lambda_2)$ and $\nabla_\eta \Phi(\mu_1, \lambda_2)$ are bounded. As $|(\lambda, \mu)|$ goes to infinity then μ_2 or λ_1 goes to infinity. From the fact that $\nabla_x \Phi(\mu_1, \lambda_2)$ and $\nabla_\eta \Phi(\mu_1, \lambda_2)$ are bounded, we conclude that $|z_{\lambda,\mu}|$ goes to infinity.

From (3.3.3) we deduce that the Gabor matrix $M(\sigma, \Phi)_{\lambda,\mu}$ goes to 0 as $|(\lambda, \mu)|$ goes to infinity and the proof is complete. $\square$

We now prove that the converse is true in the particular case of quadratic phases.

Definition 3.3.16. *The map $\Phi : \mathbb{R}^{2d} \to \mathbb{R}$ is said to be a **quadratic phase** if*

$$\Phi(x, \eta) = \frac{1}{2} Ax \cdot x + Bx \cdot \eta + \frac{1}{2} C\eta \cdot \eta + \eta_0 \cdot x - x_0 \cdot \eta$$

where $x_0, \eta_0 \in \mathbb{R}^d$, A, B, C are symmetric real matrices and B is non degenerate.

Theorem 3.3.17. *Let T be a FIO with quadratic phase Φ and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ and let $0 \leq s < s_0 - 2d$. Then the following statements are equivalent:*

(1) $\sigma \in M^0(\mathbb{R}^{2d})$.

(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p \leq \infty$ and for every v_s-moderate weight m.

Proof. We need to check that (2) $\Rightarrow$ (1). We use the same notation as in the proof of Theorem 3.3.15. Since the phase Φ is quadratic then all its second partial derivatives are constant. Hence

$$\Phi_{2,(0,0)}(w) = \Phi_{2,(\mu_1,\lambda_2)}(w) \text{ and } \Psi_{(\mu_1,\lambda_2)}(w) = \Psi_{(0,0)}(w) = \Psi(w)$$

for every $(\mu_1, \lambda_2) \in \mathbb{R}^{2d}$. Consequently

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| = |V_{\Psi_{(0,0)}}\sigma(z_{\lambda,\mu})|. \tag{3.3.4}$$

We now proceed in several steps.

We first prove that

$$(\langle T\pi(\lambda)g, \pi(\mu)g\rangle)_{\lambda,\mu\in\Lambda} \in c_0(\Lambda \times \Lambda). \tag{3.3.5}$$

As $M(\sigma, \Phi) \in \mathcal{C}_{v_s,\psi}$ we have

$$\sum_{\gamma\in\Lambda} v_s(\gamma) \cdot \sup_{\lambda\in\Lambda} |M(\sigma, \Phi)_{\chi'(\lambda)+\gamma,\lambda}| < \infty.$$

In particular

$$\lim_{|\gamma|\to\infty} \sup_{\lambda\in\Lambda} |M(\sigma, \Phi)_{\chi'(\lambda)+\gamma,\lambda}| = 0.$$

So, for every $\varepsilon > 0$ there exist $\gamma_0 \in (0, \infty)$ such that for every $\lambda, \gamma \in \Lambda$ with $|\gamma| \geq \gamma_0$,

$$\varepsilon > |M(\sigma, \Phi)^t_{\lambda,\chi'(\lambda)+\gamma}| = |\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \gamma)g\rangle|. \tag{3.3.6}$$

Since T is a compact operator we can apply Theorems 3.3.12 and 3.3.13 to get

$$\left(M(\sigma, \Phi)_{\chi'(\lambda)+\gamma,\lambda}\right)_\lambda \in c_0(\Lambda)$$

for every $\gamma \in \Lambda$. That is, for every $\varepsilon > 0$ there exist $\lambda_\gamma \in (0, \infty)$ with the property that $\lambda \in \Lambda$ and $|\lambda| \geq \lambda_\gamma$ imply

$$\varepsilon > |M(\sigma, \Phi)^t_{\lambda,\chi'(\lambda)+\gamma}| = |\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \gamma)g\rangle|. \tag{3.3.7}$$

We now consider $\lambda_0 = \max_{|\gamma|\leq\gamma_0}\{\lambda_\gamma\}$. We will check that $|(\lambda, \mu)| \geq \lambda_0 + \gamma_0 + \max_{|\lambda|<\lambda_0} |\chi'(\lambda)| + 1$ implies

$$\varepsilon > |M(\sigma, \Phi)^t_{\lambda,\mu}| = |\langle T\pi(\lambda)g, \pi(\mu)g\rangle|.$$

Given (λ, μ), there is $\gamma \in \Lambda$ such that $\mu = \chi'(\lambda) + \gamma$. If $|\gamma| \geq \gamma_0$ we are done by (3.3.6). If $|\gamma| < \gamma_0$ but $|\lambda| \geq \lambda_0 \geq \lambda_\gamma$ then we are done by (3.3.7). To finish we discuss the case $|\gamma| < \gamma_0$ and $|\lambda| < \lambda_0$. We have

$$\lambda_0 + \gamma_0 + \max_{|\gamma|<\lambda_0} \left|\chi'(\gamma)\right| + 1 \leq |(\lambda, \mu)| = |\lambda| + |\chi'(\lambda)| + |\gamma|.$$

Since $|\lambda| < \lambda_0$ and $|\gamma| < \gamma_0$ we deduce

$$\max_{|\gamma|<\lambda_0} \left|\chi'(\gamma)\right| + 1 \leq \left|\chi'(\lambda)\right|,$$

which is a contradiction because $|\lambda| < \lambda_0$. Consequently

$$\varepsilon > |M(\sigma,\Phi)^t_{\lambda,\mu}| = |\langle T\pi(\lambda)g, \pi(\mu)g\rangle|$$

whenever $|(\lambda,\mu)| \geq \lambda_0 + \gamma_0 + \max_{|\lambda|<\lambda_0} |\chi'(\lambda)| + 1$ and statement (3.3.5) holds.

Secondly, we check that $G(z,w) = \langle T\pi(z)g, \pi(w)g\rangle$ goes to zero as $|(z,w)|$ goes to infinity on $\mathbb{R}^{4d}$. We have

$$\pi(u)g = \sum_{\nu\in\Lambda} \langle \pi(u)g, \pi(\nu)g\rangle \pi(\nu)g. \tag{3.3.8}$$

As $g \in S(\mathbb{R}^d) \subseteq M^1(\mathbb{R}^d)$, using the local properties of the STFT [Grö01, 12.1.11], we have $V_g g \in W(L^\infty, \ell^1)(\mathbb{R}^{2d})$. Using the norm of this space (1.2.1) we have, for every relatively compact subset $K \subset \mathbb{R}^{2d}$ there is $B > 0$ such that

$$\sum_{\nu\in\Lambda} \sup_{u\in K} |V_g g(\nu + u)| \leq B\|g\|_{M^1(\mathbb{R}^d)}.$$

In particular, we take K a symmetric and relatively compact fundamental domain of Λ and define

$$\alpha(\nu) = \sup_{u\in K} |V_g g(\nu + u)| = \sup_{u\in K} |\langle \pi(-u)g, \pi(\nu)g\rangle|.$$

Then $\alpha \in \ell^1(\Lambda)$. Given $z, w \in \mathbb{R}^{2d}$ we can decompose $z = \mu + u$ and

$w = \lambda + u'$, with $\mu, \lambda \in \Lambda$ and $u, u' \in K$. From (3.3.8) we obtain

$$\begin{aligned}
|\langle T\pi(z)g, \pi(w)g\rangle| &= |\langle T\pi(\mu+u)g, \pi(\lambda+u')g\rangle| \\
&= |\langle T\pi(\mu)\pi(u)g, \pi(\lambda)\pi(u')g\rangle| \\
&\leq \sum_{\nu,\nu'\in\Lambda} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle||\langle \pi(u)g, \pi(\nu)g\rangle||\langle \pi(u')g, \pi(\nu')g\rangle| \\
&\leq \sum_{\nu,\nu'\in\Lambda} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \\
&\leq \sum_{|\nu|,|\nu'|<M} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \\
&+ \sum_{|\nu|\geq M,\nu'\in\Lambda} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \\
&+ \sum_{|\nu|<M,|\nu'|\geq M} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \qquad (3.3.9)
\end{aligned}$$

for every $M > 0$. Let $\varepsilon > 0$ be given, take $A = \sup_{\lambda,\mu\in\Lambda} |\langle T\pi(\lambda)g, \pi(\mu)g\rangle|$, and find $M > 0$ such that

$$\sum_{|\nu|>M} \alpha(\nu) < \frac{\varepsilon}{3A\|\alpha\|_{\ell^1}}.$$

For every $\mu, \lambda \in \Lambda$ we have

$$\begin{aligned}
&\sum_{|\nu|\geq M,\nu'\in\Lambda} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \\
&\leq A \sum_{|\nu|\geq M} \alpha(\nu) \sum_{\nu'\in\Lambda} \alpha(\nu') \leq A \frac{\varepsilon}{3A\|\alpha\|_{\ell^1}} \|\alpha\|_{\ell^1} \leq \frac{\varepsilon}{3} \qquad (3.3.10)
\end{aligned}$$

and

$$\sum_{|\nu|<M,|\nu'|\geq M} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \leq \frac{\varepsilon}{3}. \qquad (3.3.11)$$

As $(\langle T\pi(\lambda)g, \pi(\mu)g\rangle)_{\lambda,\mu\in\Lambda}$ converges to 0, we can find $N \in \mathbb{N}$ such that

$$\sup_{|\lambda|+|\mu|\geq N, \lambda,\mu\in\Lambda} |\langle T\pi(\lambda)g, \pi(\mu)g\rangle| < \frac{\varepsilon}{3\|\alpha\|_{\ell^1}^2}.$$

Finally, for any $z, w \in \mathbb{R}^{2d}$ satisfying $|z|+|w| > N+2M+2\cdot\sup_{u\in C}|u|$, we obtain $|\mu|+|\lambda| > N+2M$ ($z = \mu+u$ and $w = \lambda+u'$, with $\mu, \lambda \in \Lambda$ and $u, u' \in C$) and $|\mu+\nu|+|\lambda+\nu'| > N$ whenever $|\nu|, |\nu'| < M$. Then

$$\begin{aligned}
&\sum_{|\nu|,|\nu'|<M} |\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\alpha(\nu)\alpha(\nu') \\
&\quad \leq \sup_{|\nu|,|\nu'|<M} \{|\langle T\pi(\mu+\nu)g, \pi(\lambda+\nu')g\rangle|\} \sum_{|\nu|<M} \alpha(\nu) \sum_{|\nu'|<M} \alpha(\nu') \\
&\qquad \leq \frac{\varepsilon}{3\|\alpha\|_{\ell^1}^2} \sum_{|\nu|<M} \alpha(\nu) \sum_{|\nu'|<M} \alpha(\nu') \leq \frac{\varepsilon}{3}. \qquad (3.3.12)
\end{aligned}$$

Using (3.3.9), (3.3.10), (3.3.11) and (3.3.12) we obtain

$$|\langle T\pi(z)g, \pi(w)g\rangle| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

The proof that $|\langle T\pi(z)g, \pi(w)g\rangle| \in C_0(\mathbb{R}^{4d})$ is complete.

We can now finish the proof that $\sigma \in M^0(\mathbb{R}^{2d})$. We recall that

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| = |V_\Psi\sigma(z_{\lambda,\mu})|$$

for every $\lambda, \mu \in \mathbb{R}^{2d}$ and consider $V_\Psi\sigma(a,b,c,l)$ with $(a,b,c,l) \in \mathbb{R}^{4d}$. There are unique $e, f \in \mathbb{R}^d$ such that

$$(a,b,c,l) = (a,b,e-\nabla_x\Phi(a,b), f-\nabla_\eta\Phi(a,b)) = z_{f,b,a,e}.$$

Then $|V_\Psi\sigma(a,b,c,l)| = |G(f,b,a,e)|$. If $|(a,b,c,l)|$ goes to infinity we have two possibilities:

- a or b goes to infinity. Then $|(f,b,a,e)|$ goes to infinity.

- Neither a nor b goes to infinity. We can assume that there is $A>0$ such that $|(a,b)| \leq A$, from where it follows that $\nabla_x\Phi(a,b)$ and $\nabla_\eta\Phi(a,b)$ are bounded. As $|(a,b,e-\nabla_x\Phi(a,b),f-\nabla_\eta\Phi(a,b))|$ goes to infinity and $a,b,\nabla_x\Phi(a,b),\nabla_\eta\Phi(a,b)$ are bounded, we conclude that either e or f goes to infinity. Hence $|(f,b,a,e)|$ goes to infinity.

Since $|\langle T\pi(z)g,\pi(w)g\rangle| \in C_0(\mathbb{R}^{4d})$ we can use (3.3.4) to conclude that $\sigma \in M^0(\mathbb{R}^{2d})$. $\square$

Now we consider a FIO, $T:\mathcal{S}\to\mathcal{S}'$, with symbol $\sigma\in L^\infty(\mathbb{R}^{2d})$ and real phase Φ on $\mathbb{R}^{2d}$,

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)d\eta.$$

Let us calculate the transposed map $T^*:\mathcal{S}\to\mathcal{S}'$,

$$\begin{aligned}
\langle g,T^*f\rangle = \langle Tg,f\rangle &= \int_{\mathbb{R}^d}\overline{f(x)}\left(\int_{\mathbb{R}^d} e^{2\pi\mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\,\widehat{g}(\eta)d\eta\right)dx\\
&= \int_{\mathbb{R}^{2d}} e^{2\pi\mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\,\overline{f(x)}\widehat{g}(\eta)dxd\eta\\
&= \int_{\mathbb{R}^{2d}} e^{2\pi\mathbf{i}\Phi(x,\eta)}\sigma(x,\eta)\,\overline{f(x)}\left(\int_{\mathbb{R}^d} e^{-2\pi\mathbf{i}y\eta}g(y)dy\right)dxd\eta\\
&= \int_{\mathbb{R}^{3d}} g(y)e^{2\pi\mathbf{i}[\Phi(x,\eta)-y\eta]}\sigma(x,\eta)\,\overline{f(x)}dxd\eta dy\\
&= \int_{\mathbb{R}^d} g(y)\overline{\left(\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}[\Phi(x,\eta)-y\eta]}\overline{\sigma(x,\eta)}f(x)dxd\eta\right)}dy.
\end{aligned}$$

From that we usually refer to the FIOs seen so far as FIOs of type I, $T_{I,\Phi,\sigma}$, and their adjoints are called **FIOs of type II**, $T_{II,\Phi,\overline{\sigma}}$. For a function f on $\mathbb{R}^d$ the FIO of type II, with symbol $\tau\in L^\infty(\mathbb{R}^{2d})$ and phase Φ on $\mathbb{R}^{2d}$ can be formally defined by

$$Tf(x) = T_{II,\Phi,\tau}f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}[\Phi(y,\eta)-x\eta]}\tau(y,\eta)f(y)dyd\eta.$$

Proposition 3.3.18. Let $T_{II,\Phi,\tau}$ be a FIO of type II whose phase Φ satisfy conditions (3.1.1) and (3.1.2) and symbol $\tau \in M^{\infty}_{1\otimes v_c}(\mathbb{R}^{2d})$ and let $0 \le s < c - 2d$. If $\tau \in M^0(\mathbb{R}^{2d})$ then $T_{II,\Phi,\tau} : M^p_m(\mathbb{R}^d) \to M^p_{m\circ\chi}(\mathbb{R}^d)$ is a compact operator for every $1 \le p \le \infty$ and for every v_s-moderate weight m.

Proof. If $\tau \in M^0(\mathbb{R}^{2d})$, then $\overline{\tau} \in M^0(\mathbb{R}^{2d})$. Then $T_{I,\Phi,\overline{\tau}} : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \le p \le \infty$ and for every v_s-moderate weight m by Theorem 3.3.15. Particularly $T_{I,\Phi,\overline{\tau}} : M^q_{\frac{1}{m\circ\chi}}(\mathbb{R}^d) \to M^q_{\frac{1}{m}}(\mathbb{R}^d)$ is a compact operator for every $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and for every v_s-moderate weight m . Thereby $T_{II,\Phi,\tau} : M^p_m(\mathbb{R}^d) \to M^p_{m\circ\chi}(\mathbb{R}^d)$ is a compact operator for every $1 \le p \le \infty$ and for every v_s-moderate weight m. $\square$

3.3.2 FIOs on $M^{p,q}_m$

The FIOs we are considering may fail to be bounded on mixed modulation spaces as was shown in [CNR10b]. The example was a FIO with phase $\Phi(x,\eta) = x\eta + \frac{|x|^2}{2}$, whose canonical transformation is $\chi(y,\eta) = (y, y+\eta)$. We consider the case $d = 1$ and $\Lambda = \mathbb{Z} \times \mathbb{Z}$, Let $a \in \ell^{2,1}(\mathbb{Z} \times \mathbb{Z})$,

$$\begin{aligned} I_\chi(a) &= I_\chi\left(\sum_{(y,\eta)\in\mathbb{Z}\times\mathbb{Z}} a_{y,\eta} e_{y,\eta}\right) = \sum_{(y,\eta)\in\mathbb{Z}\times\mathbb{Z}} a_{y,\eta} I_\chi\left(e_{y,\eta}\right) \\ &= \sum_{(y,\eta)\in\mathbb{Z}\times\mathbb{Z}} a_{y,\eta} e_{\chi(y,\eta)} = \sum_{(y,\eta)\in\mathbb{Z}\times\mathbb{Z}} a_{y,\eta} e_{(y,y+\eta)} \\ &= \sum_{(y,\lambda)\in\mathbb{Z}\times\mathbb{Z}} a_{y,\lambda-y} e_{(y,\lambda)} = \left(a_{y,\lambda-y}\right)_{(y,\lambda)\in\mathbb{Z}\times\mathbb{Z}}. \end{aligned}$$

Now, as in Exemple 3.3.7, we consider

$$a_{y,\eta} = \frac{1}{(|y|+1)2^{|\eta|}},$$

$a \in \ell^{2,1}(\mathbb{Z} \times \mathbb{Z})$, but we see that $I_\chi(a) \notin \ell^{2,1}(\mathbb{Z} \times \mathbb{Z})$,

$$\sum_{\lambda \in \mathbb{Z}} \left| \left(\sum_{y \in \mathbb{Z}} |(I_\chi(a))_{y,\lambda}|^2 \right)^{\frac{1}{2}} \right| \geq \sum_{\lambda \in \mathbb{Z}} \left| \left(\sum_{y = \lambda} |(I_\chi(a))_{y,\lambda}|^2 \right)^{\frac{1}{2}} \right| = \sum_{\lambda \in \mathbb{Z}} |(I_\chi(a))_{\lambda,\lambda}|$$
$$= \sum_{\lambda \in \mathbb{Z}} \left| \frac{1}{(|\lambda|+1)2^{|\lambda-\lambda|}} \right| = \sum_{\lambda \in \mathbb{Z}} \left| \frac{1}{(|\lambda|+1)} \right| = \infty.$$

That is, $I_\chi\left(\ell^{2,1}\right)$ is not contained in $\ell^{2,1}$.

To overcome this obstacle, an extra condition on the phase was introduced by Cordero, Nicola and Rodino in [CNR10b], namely

$$\sup_{x',x,\eta} \left| \nabla_x \Phi(x,\eta) - \nabla_x \Phi(x',\eta) \right| < \infty. \tag{3.3.13}$$

If $\chi = (\chi_1, \chi_2)$ is the corresponding canonical transformation, condition (3.3.13) implies that

$$\chi_2(y,\eta) = \nabla_x \Phi(\chi_1(y,\eta),\eta) = \nabla_x \Phi(0,\eta) + a(y,\eta),$$

$a(y,\eta)$ being a bounded function.

From now on, $\mathcal{G}(g,\Lambda)$ is a Parseval frame with $g \in \mathcal{S}(\mathbb{R}^d)$, $\Lambda_1 = \alpha\mathbb{Z}^d$, $\Lambda_2 = \beta\mathbb{Z}^d$ and $\Lambda = \Lambda_1 \times \Lambda_2$. If Q denotes a symmetric relatively compact fundamental domain of the lattice Λ then, there are $K \subseteq \Lambda_2$, finite, and a unique decomposition

$$\begin{aligned} \chi_1(\lambda_1,\lambda_2) &= r_1(\lambda_1,\lambda_2) + \psi_1(\lambda_1,\lambda_2), \\ \chi_2(\lambda_1,\lambda_2) &= r_2(\lambda_1,\lambda_2) + \psi_2(\lambda_2) + a(\lambda_1,\lambda_2), \end{aligned}$$

for all $(\lambda_1,\lambda_2) \in \Lambda$, where $(r_1(\lambda_1,\lambda_2), r_2(\lambda_1,\lambda_2)) \in Q$, $\psi_1(\lambda_1,\lambda_2) \in \Lambda_1$, $\psi_2(\lambda_2) \in \Lambda_2$ and $a(\lambda_1,\lambda_2) \in K$. Moreover, from conditions (3.1.1) and (3.1.2) it follows that the map

$$\mathbb{R}^d \to \mathbb{R}^d,\ \eta \mapsto \nabla_x \Phi(0,\eta),$$

is a bilipschitz global diffeomorphism, which implies that

$$\sup_{\lambda_2 \in \Lambda_2} \operatorname{card} \psi_2^{-1}(\{\lambda_2\}) < \infty.$$

This motivates the following definition.

Definition 3.3.19. *Let $\psi : \Lambda_1 \times \Lambda_2 \to \Lambda_1 \times \Lambda_2$, $\psi(i,j) = (\psi_1(i,j), \psi_2(i,j))$, be as in Definition 3.3.1, that is, ψ satisfies*

$$\sup_{\lambda \in \Lambda} \left\{ \operatorname{card} \left(\psi^{-1} \left(\{\lambda\} \right) \right) \right\} < \infty.$$

*We say that ψ is **admissible** if there exist a map $\tilde{\psi}_2 : \Lambda_2 \to \Lambda_2$ as in Definition 3.3.1 and a finite set $K \subset \Lambda_2$ such that*

$$\psi_2(i,j) = \tilde{\psi}_2(j) + a(i,j), \text{ for all } (i,j)$$

where $a(i,j) \in K$.

The discrete version $\chi' : \Lambda \to \Lambda$ of the canonical transformation associated to a phase function satisfying conditions (3.1.1),(3.1.2), (3.3.13) is admissible. From now on, given ψ admissible, to simplify the notation, we will write $\psi_2(j)$ instead of $\tilde{\psi}_2(j)$.

Given an admissible $\psi : \Lambda_1 \times \Lambda_2 \to \Lambda_1 \times \Lambda_2$, let C be the cardinal of the finite set K and $M > 0$ be such that for each $(i,j) \in \Lambda_1 \times \Lambda_2$, $\psi^{-1}(\{(i,j)\})$ has at most M elements and $\psi_2^{-1}(\{j\})$ has at most M elements for every $j \in \Lambda_2$.

Lemma 3.3.20. *Let ψ be admissible, and $M_1 = C \cdot M$. For each $j \in \Lambda_2$, we define $\psi_{1,j} : \Lambda_1 \to \Lambda_1$, as $\psi_{1,j}(i) := \psi_1(i,j)$. Then, for each $i \in \Lambda_1$ the set $\psi_{1,j}^{-1}(\{i\})$ has at most M_1 elements.*

Proof. We fix $j \in \Lambda_2$ and $i_0 \in \Lambda_1$. If $\psi_1(i,j) = \psi_1(i_0,j)$ then $\psi(i,j) = (\psi_1(i_0,j), \psi_2(j) + a(i,j))$ can take C different values. Hence, there are only $C \cdot M$ possibilities for i. $\square$

We start by analyzing the action of the basic operators $D_{a,\psi}$ on weighted sequence spaces with mixed norm $\ell_m^{p,q}$. Since $D_{a,\psi} = I_\psi \circ D_a$, we will study the continuity of I_ψ on these spaces. To this aim, we consider the transposed map $J_\psi := I_\psi^t$, with ψ admissible. We recall that for every $\lambda \in \Lambda$,

$$J_\psi(x) = (x_{\psi(\lambda)})_\lambda.$$

Proposition 3.3.21. *Let ψ be admissible, $m = (m_{i,j})_{(i,j)\in\Lambda}$ a positive sequence and $p, q \in [1,\infty] \cup \{0\}$. Then, J_ψ is continuous from $\ell_m^{p,q}(\Lambda)$ to $\ell_{m\circ\psi}^{p,q}(\Lambda)$.*

Proof. Let $x \in \ell_m^{p,q}(\Lambda)$ and put $y = x \cdot m$ and $\gamma = (i,j)$. Then

$$|y_{\psi(i,j)}| \le \sum_{k\in\Lambda_2} \left|y_{\psi_1(i,j),\psi_2(j)+k}\right|,$$

hence

$$\begin{aligned}\left(\sum_{i\in\Lambda_1} |y_{\psi(i,j)}|^p\right)^{\frac{1}{p}} &\le \sum_{k\in\Lambda_2}\left(\sum_{i\in\Lambda_1}\left|y_{\psi_1(i,j),\psi_2(j)+k}\right|^p\right)^{\frac{1}{p}} \\ &\le \sum_{k\in\Lambda_2}\left(M_1\sum_{\ell\in\Lambda_1}\left|y_{\ell,\psi_2(j)+k}\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Consequently

$$\begin{aligned}\left(\sum_{j\in\Lambda_2}\left(\sum_{i\in\Lambda_1}|y_{\psi(i,j)}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} &\le \sum_{k}\left(\sum_{j\in\Lambda_2}\left(M_1\sum_{\ell\in\Lambda_1}\left|y_{\ell,\psi_2(j)+k}\right|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &\le C\left(M\sum_{h\in\Lambda_2}\left(M_1\sum_{\ell\in\Lambda_1}|y_{\ell,h}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &= CM^{\frac{1}{q}}M_1^{\frac{1}{p}}\|y\|_{p,q},\end{aligned}$$

where we have applied triangular inequality for the norms in ℓ^p and ℓ^q, and the facts that, for each $j \in \Lambda_2$, $\psi_2(j)$ can be repeated at most M times and $\psi_1(i,j) = \psi_{1,j}(i)$ can be repeated at most M_1 times (Lemma 3.3.20).

As J_ψ maps finite supported sequences into finite supported sequences, the cases $p = 0$ or $q = 0$ follow immediately. □

For $a \in \ell^\infty(\Lambda)$ we obtain, from the decomposition $D_{a,\psi} = J_\psi^t \circ D_a$, the estimate

$$\|D_{a,\psi} : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_m(\Lambda)\| \leq C M^{\frac{1}{q}} M_1^{\frac{1}{p}} \cdot \|a\|_\infty.$$

Proposition 3.3.22. *Let $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$, with ψ admissible and $1 \leq p, q \leq \infty$ be given. Then,*

(1) A defines a bounded operator $A : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_m(\Lambda)$, which is also weak continuous.*

(2) $A = \sum_{\gamma\in\Lambda}(T_\gamma \circ D_{a^\gamma,\psi})$ where $a^\gamma := (a_{\psi(\lambda)+\gamma,\lambda})_{\lambda\in\Lambda}$. The convergence of the series is absolute.

Proof. It is easier to deal with the transposed map, so we first consider $b_{\gamma,\gamma'} = a_{\gamma',\gamma}$ and *claim* that $B = \left(b_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda}$ defines a bounded operator $B : \ell^{p,q}_{\frac{1}{m}}(\Lambda) \to \ell^{p,q}_{\frac{1}{m}\circ\psi}(\Lambda)$, for all $p, q \in [1,\infty] \cup \{0\}$. We will assume that $p, q \in [1,\infty]$. Then the case that $p = 0$ or $q = 0$ can be obtained as in the proof of Proposition 3.3.8.

As $\ell^{p,q}_{\frac{1}{m}}(\Lambda) \subset \ell^{\infty}_{\frac{1}{m}}(\Lambda)$, by Proposition 3.3.8, we obtain that

$$B : \ell^{p,q}_{\frac{1}{m}}(\Lambda) \to \mathbb{C}^\Lambda$$

is a well-defined operator. To prove that $Bx \in \ell^{p,q}_{\frac{1}{m}\circ\psi}(\Lambda)$ it is enough to check that

$$\sum_{\gamma\in\Lambda} \left|(Bx)_\gamma \, y_\gamma\right| < \infty$$

for every $y \in \ell^{p',q'}_{m\circ\psi}(\Lambda)$. We proceed as in Proposition 3.3.8. We denote $\phi(\lambda) = v(\lambda)\sup_\gamma \left|b_{\gamma,\psi(\gamma)+\lambda}\right|$. We obtain, using that translations are isometries on the spaces $\ell^{p,q}$,

$$\begin{aligned}\sum_{\gamma\in\Lambda}\left|(Bx)_\gamma\, y_\gamma\right| &\le C_m \sum_{\lambda\in\Lambda}\phi(\lambda)\sum_{\gamma\in\Lambda}\frac{x_{\psi(\gamma)+\lambda}}{m_{\psi(\gamma)+\lambda}}|y_\gamma| m_{\psi(\gamma)} \\ &\le C_m \sum_{\lambda\in\Lambda}\phi(\lambda)\cdot \|J_\psi(x)\|_{\ell^{p,q}_{\frac{1}{m}\circ\psi}}\cdot \|y\|_{\ell^{p',q'}_{m\circ\psi}}.\end{aligned}$$

(2) follows as in Proposition 3.3.8 once continuities and the estimates for the norms of the operators $D_{a^\gamma,\psi}$ are obtained. □

The characterization of compactness obtained in Theorem 3.3.11 extends to mixed spaces when ψ is admissible.

Proposition 3.3.23. *Let $A = \left(a_{\gamma,\gamma'}\right)_{\gamma,\gamma'\in\Lambda} \in \mathcal{C}_{v,\psi}(\Lambda)$, ψ admissible and $1 \le p, q \le \infty$ be given. Then, A defines a compact operator*

$$A : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_{m}(\Lambda)$$

if and only if $a^\gamma := \left(a_{\psi(\lambda)+\gamma,\lambda}\right)_{\lambda\in\Lambda} \in c_0(\Lambda)$ for all $\gamma \in \Lambda$.

The next result extends [CNR10b, Theorem 5.2] to weighted modulation spaces and also includes the cases $p = \infty$ or $q = \infty$.

Theorem 3.3.24. *Let T be a FIO whose phase Φ is tame and satisfies condition (3.3.13), and $\sigma \in M^\infty_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ with $0 \le s < s_0 - 2d$. Then, $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ is a bounded operator for every $1 \le p, q < \infty$ and for every v_s-moderate weight m.*

Proof. Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$. From [CGN12, Corollary 5.5] we have that

$$T : M^2(\mathbb{R}^d) \to M^2(\mathbb{R}^d)$$

is a bounded operator. And from Propositions 3.3.2 and 3.3.22

$$M(\sigma, \Phi) : \ell^{p,q}_{m\circ\psi}(\Lambda) \to \ell^{p,q}_{m}(\Lambda)$$

is a bounded operator for every $1 \leq p, q < \infty$ and for every v_s-moderate weight m. Then, by Theorem 3.2.2, ,

$$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_{m}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q < \infty$ and for every v_s-moderate weight m.

□

Theorem 3.3.25. *Let T be a FIO whose phase Φ is tame and satisfies condition (3.3.13), and $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ with $0 \leq s < s_0 - 2d$. and m a v_s-moderate weight. Then T admits a unique extension as a bounded operator*

$$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_{m}(\mathbb{R}^d),$$

for $1 \leq p, q \leq \infty$ and p or q equal to ∞, which is also weak-continuous.*

Proof. We know that $T = C_g \circ M(\sigma, \Phi) \circ C_g^*$ is bounded on $\mathcal{S}(\mathbb{R}^d)$, which is weak*-dense in $M^{p,q}_m(\mathbb{R}^d)$, for $1 \leq p, q \leq \infty$ and p or q equal to ∞ and for all v_s-moderate weight m. Then T admits a unique extension,

$$\tilde{T} : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \xrightarrow{C_g} \ell^{p,q}_{m\circ\chi}(\Lambda) \xrightarrow{M(\sigma,\Phi)} \ell^{p,q}_{m}(\Lambda) \xrightarrow{S^*} M^{p,q}_{m}(\mathbb{R}^d),$$

where $S = C_g : M^{p',q'}_{1/m}(\mathbb{R}^d) \to \ell^{p',q'}_{1/m}(\Lambda)$. In fact, all the involved maps are weak*-continuous, then $\tilde{T}$ is weak*-continuous. As the extension is unique we denote it by T. □

Theorem 3.3.26. *Let T be a FIO whose phase Φ is tame and satisfies condition (3.3.13), and $\sigma \in M^{\infty}_{1\otimes v_{s_0}}(\mathbb{R}^{2d})$ with $0 \leq s < s_0 - 2d$. The following conditions are equivalent:*

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.

(2) $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *is a compact operator for some* $1 \le p, q \le \infty$ *and for some* v_s*-moderate weight* m*.*

(3) $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *is a compact operator for every* $1 \le p, q \le \infty$ *and for every* v_s*-moderate weight* m*.*

Proof. Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$. From previous Theorems we have that

$$T : M^{p,q}_{m\circ\chi} \to M^{p,q}_m$$

is a (weak*-)continuous operator. We fix $1 \le p, q \le \infty$ and m. By Theorem 3.2.2,

$$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$$

is compact if, and only if,

$$M(\sigma, \Phi) : \ell^{p,q}_{m\circ\chi'}(\Lambda) \to \ell^{p,q}_m(\Lambda)$$

is a compact operator. By Propositions 3.3.2 and 3.3.23, this is exactly the case when

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \mu)g\rangle\big)_{\lambda\in\Lambda} = \big(M(\sigma, \Phi)_{\chi'(\lambda)+\mu,\lambda}\big)_{\lambda\in\Lambda} \in c_0(\Lambda),$$

for all $\mu \in \Lambda$. This condition does not depend on p, q or m. □

3.3.3 PSDOs on $M^{p,q}_m$

Now, we are going to consider compactness of pseudodifferential operators in Kohn-Nirenberg form. They are a particular case of FIOs when $\Phi(x, y) = x \cdot y$, and hence $\chi(y, \eta) = (y, \eta)$. If Λ is a regular lattice with symmetric relatively compact fundamental domain Q, the map χ' is the identity, therefore it is admissible. The class of matrices $C_{v,\chi'}$ is denoted by $\mathcal{C}_v = \mathcal{C}_v(\Lambda)$ and consists of all matrices $A = \big(a_{\gamma,\gamma'}\big)_{\gamma,\gamma'\in\Lambda}$ such that

$$\|A\|_{\mathcal{C}_v} = \sum_{\gamma\in\Lambda} v(\gamma) \cdot \sup_{\lambda\in\Lambda} |a_{\lambda,\gamma+\lambda}| < \infty.$$

According to [Grö06, Lemma 3.5], $\mathcal{C}_v$ is an algebra. Since the weight v is symmetric, it follows that

$$\sum_{\gamma\in\Lambda} v(\gamma)\cdot\sup_{\lambda\in\Lambda}|a_{\lambda,\gamma+\lambda}| = \sum_{\gamma\in\Lambda} v(\gamma)\cdot\sup_{\lambda\in\Lambda}|a_{\gamma+\lambda,\lambda}|\,.$$

This means that $A\in\mathcal{C}_v$ if and only if $A^t\in\mathcal{C}_v$. Each $A\in\mathcal{C}_v$ defines a bounded operator

$$A:\ell^{p,q}_m(\Lambda)\to\ell^{p,q}_m(\Lambda),$$

for $p,q\in[1,\infty]\cup\{0\}$ and each v-moderate sequence m. The compactness of the map is independent on p, q and m. This allows us to improve results obtained in [FG07] and [FG10].

We recall the definitions of Wigner distribution, Weyl pseudodifferential operator and pseudodifferential operator in Kohn-Nirenberg form.

Definition 3.3.27. *The* ***Wigner distribution*** *of f, $g\in L^2(\mathbb{R}^d)$ is defined as*

$$W(f,g)(x,\omega)=\int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2i\pi t\omega}dt.$$

If $f,g\in\mathcal{S}(\mathbb{R}^d)$, then $W(f,g)\in\mathcal{S}(\mathbb{R}^d)$.

Definition 3.3.28. *Given $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$ its Weyl transform or* ***Weyl pseudodifferential operator*** *is the operator $L_\sigma:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ defined by*

$$\langle L_\sigma(f),g\rangle=\langle\sigma,W(g,f)\rangle,$$

for f, $g\in\mathcal{S}(\mathbb{R}^d)$.

Definition 3.3.29. *Given $\tau\in\mathcal{S}'(\mathbb{R}^{2d})$, the operator $K_\tau:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ defined by*

$$K_\tau:=\int_{\mathbb{R}^d}\tau(x,\omega)\hat{f}(\omega)e^{2\pi\mathbf{i}x\cdot\omega}d\omega$$

for $f,\in\mathcal{S}(\mathbb{R}^d)$, is called ***pseudodifferential operator*** *in Kohn-Nirenberg form.*

We recall that every operator from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a pseudodifferential operator L_σ with Weyl symbol σ and as a pseudodifferential operator in Kohn-Nirenberg form with symbol τ. We refer to [Grö01, Chapter 14] where the relation between σ and τ is established. In particular, for $s \geq 0$, $\sigma \in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ if and only if $\tau \in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$. For convenience, we state the results for Weyl pseudodifferential operators.

Theorem 3.3.30. *Let $\sigma \in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ be given. Then the following statements are equivalent:*

(1) $L_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is compact.*

(2) $L_\sigma : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *is compact for all* $p, q \in [1,\infty]$ *and every* v_s*-moderate weight* m.

(3) $L_\sigma : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *is compact for some* $p, q \in [1,\infty]$ *and some* v_s*-moderate weight* m.

(4) $\sigma \in M^0(\mathbb{R}^{2d})$.

Proof. Let $\mathcal{G}(g,\Lambda)$ be a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ for α, $\beta > 0$. Then, according to [Grö06, Theorem 3.2],

$$M(\sigma) := (\langle L_\sigma \pi(\lambda)g, \pi(\mu)g\rangle)_{(\mu,\lambda)\in\Lambda\times\Lambda} \in \mathcal{C}_{v_s}(\Lambda).$$

Moreover, it follows from (3.2.1) and (3.2.2) that $L_\sigma : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ is compact if and only if $M(\sigma) : \ell^{p,q}_m(\Lambda) \to \ell^{p,q}_m(\Lambda)$ is compact. Now, the equivalences among (1), (2) and (3) follow from Theorem 3.3.23. Finally, since $M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$ we can apply [FG07, Theorem 4.6] to obtain that condition (1) is equivalent to condition (4). $\square$

Alternatively we could argue as follows. According to Theorem 3.3.23, $M(\sigma) : \ell^{p,q}_m(\Lambda) \to \ell^{p,q}_m(\Lambda)$ is a compact operator if and only if

$$(\langle L_\sigma \pi(\lambda)g, \pi(\lambda+\mu)g\rangle)_{\lambda\in\Lambda} \in c_0(\Lambda) \tag{3.3.14}$$

for every $\mu \in \Lambda$. By [Grö06, 3.1],

$$|\langle L_\sigma \pi(\lambda)g, \pi(\lambda+\mu)g\rangle| = \left|V_\Phi \sigma(\lambda + \frac{\mu}{2}, j(\mu))\right|,$$

where $\Phi = W(g,g)$ and $j : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is the map $j(\xi,\omega) = (\omega, -\xi)$. This permits to prove that condition (3.3.14) is equivalent to the fact that $\sigma \in M^0(\mathbb{R}^{2d})$.

We want to finish with some comments regarding localization operators (see for instance [FG06, FG10] and the references therein). **Localization operators** are defined by means of

$$f \to L^F_{\varphi,\psi} f := \frac{1}{\langle \varphi, \psi\rangle} \int_{\mathbb{R}^{2d}} F(x,\omega) V_\varphi f(x,\omega) M_\omega T_x \psi dx d\omega.$$

We may write

$$\langle L^F_{\varphi,\psi} f, h\rangle = \langle F, V_\psi h \overline{V_\varphi f}\rangle.$$

The compact localization operators on $L^2(\mathbb{R}^d)$ were characterized in [FG06] in terms of the behaviour of the STFT of their symbols. The condition there obtained also gives compactness for the localization operators when acting on weighted modulation spaces of Hilbert type $M^2_m(\mathbb{R}^d)$ ([FG10, 5.6]). However, the reverse implication, that is, compactness on some $M^2_m(\mathbb{R}^d)$ implies compactness on L^2, could not be proved with the methods used there.

Moreover $L^F_{\varphi,\psi} = L_\sigma$ with $\sigma = F * W(\psi,\varphi)$. By [FG10, 5.2], if $F \in M^\infty(\mathbb{R}^{2d})$ and φ, $\psi \in \mathcal{S}(\mathbb{R}^d)$, then $F * W(\psi,\varphi) \in M^{\infty,1}_{1\otimes(v\circ j^{-1})}(\mathbb{R}^{2d})$ for every v, and thus, the localization operator $L^F_{\varphi,\psi}$ is continuous from $M^{p,q}_m(\mathbb{R}^d)$ into itself. Since every localization operator can be described as a PSDO in Weyl form, Theorem 3.3.30 permits to conclude that the compactness of the localization operator on a modulation class $M^p_m(\mathbb{R}^d)$ does not depend on p nor m.

Corollary 3.3.31. *Let $F \in M^\infty(\mathbb{R}^{2d})$ and $\psi, \varphi \in S(\mathbb{R}^d)$ be given. The following statements are equivalent:*

(1) $L^F_{\psi,\varphi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is compact.*

(2) $L^F_{\psi,\varphi} : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *is compact for every* $1 \le p, q < \infty$ *and each* m *v-moderated.*

(3) $L^F_{\psi,\varphi} : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ *is compact for some* $1 \le p, q < \infty$ *and some* m *v-moderated.*

Recall that, by [FG06, Proposition 3.6] [FG10, Proposition 2.3], $L^F_{\psi,\varphi}$ is compact for every pair $\psi, \varphi \in S(\mathbb{R}^d)$ if, and only if, there exist $g \in S(\mathbb{R}^d)$ such that

$$\lim_{|x|\to\infty} \sup_{|y|\le R} |V_g F(x,y)| = 0$$

for every $R > 0$.

3.4 Fourier integral operators on Modulation spaces with GRS-weights

In this section we work with admissible weights v, introduced in Subsection 1.2.4. Recall that this means that the GRS-condition holds.

3.4.1 Matrix representation

Let us see that the matrix representation that we have seen can be extended, with some particularities, to modulation spaces with GRS-weights. Let $g \in M^1_v$, then $h = S_g^{-1}(g) \in M^1_v$ by [Grö07, Theorem 6.11] and [Grö06, Theorem 2.2]. Also, we have

Theorem 3.4.1. *Let* $g \in M^1_v$*, with* v *admissible. Then, for every v-moderate weight* m *and for every* $1 \le p, q \le \infty$,

$$C_g : M^{p,q}_m(\mathbb{R}^d) \to \ell^{p,q}_m(\Lambda) \text{ and } D_g : \ell^{p,q}_m(\Lambda) \to M^{p,q}_m(\mathbb{R}^d)$$

are bounded operators, weak continuous, and*

$$D_g \circ C_h = D_h \circ C_g = Id_{M^{p,q}_m(\mathbb{R}^d)}.$$

Here D_g is the transposed map of $C_g : M^{p',q'}_{1/m}(\mathbb{R}^d) \to \ell^{p',q'}_{1/m}(\Lambda)$. For $p = 1$ or $q = 1$ we take $p' = 0$ or $q' = 0$ respectively.

Definition 3.4.2. The *Gabor matrix* associated to a continuous and linear operator $T : M^1_v(\mathbb{R}^d) \to M^\infty_{\frac{1}{v}}(\mathbb{R}^d)$ is defined as

$$M(T) = (\langle T(\pi(\lambda)g), \pi(\mu)g\rangle)_{(\mu,\lambda)\in\Lambda\times\Lambda}\,.$$

If T is a FIO with symbol σ and phase Φ we write $M(\sigma, \Phi)$ instead of $M(T)$.

Theorem 3.4.3. Let $T : M^1_v(\mathbb{R}^d) \to M^\infty_{\frac{1}{v}}(\mathbb{R}^d)$ be a continuous and linear operator and $\mathcal{G}(g, \Lambda)$ a Gabor frame with $g \in M^1_v$. Then, for all v-moderate weights m_1 and m_2, we have

(1) For $1 \le p, q < \infty$, T can be (uniquely) extended as a bounded operator from $M^{p,q}_{m_1}(\mathbb{R}^d)$ into $M^{p,q}_{m_2}(\mathbb{R}^d)$ if and only if $M(T)$ defines a bounded operator from $\ell^{p,q}_{m_1}(\Lambda)$ into $\ell^{p,q}_{m_2}(\Lambda)$.

(2) For $1 \le p, q \le \infty$, T can be extended as a weak* continuous operator from $M^{p,q}_{m_1}(\mathbb{R}^d)$ into $M^{p,q}_{m_2}(\mathbb{R}^d)$ if and only if $M(T)$ defines a weak* continuous operator from $\ell^{p,q}_{m_1}(\Lambda)$ into $\ell^{p,q}_{m_2}(\Lambda)$.

(3) Let $1 \le p, q \le \infty$ and assume that $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ is weak* continuous. Then $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ is compact if and only if $M(T) : \ell^{p,q}_{m_1}(\Lambda) \to \ell^{p,q}_{m_2}(\Lambda)$ is compact.

Proof. Let h be the canonical dual window of g. Then we have

$$C_g \circ T = M(T) \circ C_h \quad \text{on} \quad M^1_v(\mathbb{R}^d).$$

We fix $x \in \mathbb{C}^{(\Lambda)}$ and observe that $D_g(x) \in M^1_v(\mathbb{R}^d)$, hence $M(T) \circ C_h \circ D_g(x) = C_g \circ T \circ D_g(x)$. That is,

$$(M(T)(C_h \circ D_g)(x))_\mu = \langle T\left(D_g(x)\right), \pi(\mu)g\rangle = \sum_{\lambda\in\Lambda} \langle T\left(\pi(\lambda)g\right), \pi(\mu)g\rangle \cdot x_\lambda.$$

Consequently, for every finite sequence x we have

$$M(T)(x) = M(T)(C_h \circ D_g)(x).$$

First, if $T : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ defines a (weak*) continuous operator, then

$$M(T) = M(T) \circ C_h \circ D_g = C_g \circ T \circ D_g : \ell^{p,q}_{m_1}(\mathbb{R}^d) \to \ell^{p,q}_{m_2}(\mathbb{R}^d) \tag{3.4.1}$$

defines a (weak*) continuous operator.

Conversely, if $M(T) : \ell^{p,q}_{m_1}(\mathbb{R}^d) \to \ell^{p,q}_{m_2}(\mathbb{R}^d)$ defines a (weak*) continuous operator. $D_h \circ M(T) \circ C_h = \tilde{T} : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ defines a (weak*) continuous operator. We know

$$T = D_h \circ M(T) \circ C_h = \tilde{T} \quad \text{on} \quad M^1_v(\mathbb{R}^d), \tag{3.4.2}$$

which is (weak*-)dense in $M^{p,q}_{m_1}$. Then $T = \tilde{T} = D_h \circ M(T) \circ C_h : M^{p,q}_{m_1}(\mathbb{R}^d) \to M^{p,q}_{m_2}(\mathbb{R}^d)$ defines a (weak*) continuous operator.

To finish we prove (3). From the hypothesis we deduce that the identities (3.4.2) and (3.4.1) hold on $M^{p,q}_{m_1}(\mathbb{R}^d)$ and $\ell^{p,q}_{m_1}(\Lambda)$ respectively and by the fact that compact operators are an ideal in the algebra of continuous operators the conclusion follows. □

3.4.2 Fourier integral operators on M^p_m

Let T be a Fourier integral operator, with symbol σ and phase Φ on $\mathbb{R}^{2d}$, formally defined as

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta.$$

We consider, as in [CNR15a], a smooth phase $\Phi(x,\eta)$ on $\mathbb{R}^{2d}$ satisfying the estimates

$$|\partial^\alpha \Phi| \lesssim C^{|\alpha|}(\alpha!), \quad \alpha \in \mathbb{N}^{2d}, \ |\alpha| \geq 2, z \in \mathbb{R}^{2d}, \tag{3.4.3}$$

for some $C > 0$, as well as the nondegeneracy condition

$$|\det \partial^2_{x,\eta} \Phi(x,\eta)| \geq \delta > 0, \quad (x,\eta) \in \mathbb{R}^{2d}. \tag{3.4.4}$$

The symbol σ on $\mathbb{R}^{2d}$ satisfies

$$|\partial^\alpha \sigma(z)| \lesssim C^{|\alpha|}(\alpha!), \quad \alpha \in \mathbb{N}^{2d}, z \in \mathbb{R}^{2d}, \tag{3.4.5}$$

for some $C > 0$. The first two conditions of the phase allow us, as before, to consider the same canonical transformation denoted by $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$. We replace the canonical transformation χ by an appropriate discrete version $\chi' : \Lambda \to \Lambda$, defined as before, and with the same properties.

We adapt the result [CNR15a, Theorem 3.3] to our conditions on the phase and the symbol, which gives us control over the behaviour of the matrix in a new situation.

Theorem 3.4.4. *Suppose the phase Φ and symbol σ satisfy* (3.4.3)-(3.4.5) *above. Assume $g \in S^{1/2}_{1/2}(\mathbb{R}^d)$. Then there exists $\varepsilon > 0$ such that*

$$|\langle T(\pi(\lambda)g), \pi(\mu)g\rangle| \lesssim \exp(-\varepsilon|\mu - \chi(\lambda)|), \tag{3.4.6}$$

for $\lambda, \mu \in \mathbb{R}^{2d}$.

From this Theorem we deduce the next result.

Proposition 3.4.5. *Suppose the phase Φ and symbol σ satisfy* (3.4.3)-(3.4.5) *above. Assume that $g \in S^{1/2}_{1/2}(\mathbb{R}^d)$ and that v satisfies the GRS-condition. Then we have*

$$M(\sigma, \Phi) \in \mathcal{C}_{v,\chi'}.$$

Proof. We need to prove that

$$\sum_{\gamma\in\Lambda} v(\gamma) \sup_{\lambda\in\Lambda} |M(\sigma, \Phi)_{(\chi'(\lambda)+\gamma,\lambda)}| < \infty$$

As v satisfies the GRS-condition, then $v(\gamma) \lesssim e^{\varepsilon|\gamma|}$, for every $\varepsilon > 0$ and every $\gamma \in \Lambda$. Using the bound of (3.4.6),

$$\begin{aligned}\sum_{\gamma\in\Lambda} v(\gamma) \sup_{\lambda\in\Lambda} |M(\sigma,\Phi)_{(\chi'(\lambda)+\gamma,\lambda)}| &= \sum_{\gamma\in\Lambda} v(\gamma) \sup_{\lambda\in\Lambda} |\langle T(\pi(\lambda)g), \pi(\chi'(\lambda)+\gamma)g\rangle| \\ &\lesssim \sum_{\gamma\in\Lambda} v(\gamma) \sup_{\lambda\in\Lambda}\{\exp(-\varepsilon|\chi'(\lambda)+\gamma-\chi(\lambda)|)\} \\ &\lesssim \sum_{\gamma\in\Lambda} v(\gamma) \sup_{\lambda\in\Lambda}\{\exp(-\varepsilon|\gamma|+\varepsilon|r_\lambda|)\} \\ &\lesssim \sum_{\gamma\in\Lambda} \exp\left(\frac{\varepsilon}{2}|\gamma|\right) C_1 \exp(-\varepsilon|\gamma|)\} \\ &\lesssim \sum_{\gamma\in\Lambda} \exp\left(-\frac{\varepsilon}{2}|\gamma|\right) < \infty.\end{aligned}$$

□

All the results proved about the matrix and its behaviour on sequence spaces are valid here, since we have not assumed any condition on the weight v, except for the submultiplicativity.

Theorem 3.4.6. Let T be a FIO whose phase Φ and symbol σ satisfy (3.4.3)-(3.4.5). Then, $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a continuous operator for every $1 \leq p < \infty$ and for every v-moderate weight m, where v satisfies the GRS-condition.

Proof. Let $\mathcal{G}(g,\Lambda)$ be a Gabor frame with $g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \subseteq M^1_v$. From [CNR10b, Theorem 4.1] we have that

$$T : M^2(\mathbb{R}^d) \to M^2(\mathbb{R}^d)$$

is a bounded operator. And from Propositions 3.4.5 and 3.3.8

$$M(\sigma,\Phi) : \ell^p_{m\circ\chi'}(\Lambda) \to \ell^p_m(\Lambda)$$

is a bounded operator for every $1 \leq p < \infty$ and for every v-moderate weight m. We observe that, for any positive and v-moderate weight m,

$$\ell^p_{m\circ\chi}(\Lambda) = \ell^p_{m\circ\chi'}(\Lambda)$$

with equivalent norms and that $m \circ \chi$ is v-moderate whenever m is. Then, by Theorem 3.4.3,

$$T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$$

is bounded for every $1 \leq p < \infty$ and for every v-moderate weight m. $\square$

Theorem 3.4.7. Let T be a FIO whose phase Φ and symbol σ satisfy (3.4.3)-(3.4.5). Then, the following conditions are equivalent:

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.

(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for some $1 \leq p < \infty$ and for some v-moderate weight m, where v satisfies the GRS-condition.

(3) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p < \infty$ and for every v-moderate weight m, where v satisfies the GRS-condition.

(4) $(\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \mu)g\rangle)_\lambda \in c_0(\Lambda)$ for every $\mu \in \Lambda$.

Proof. Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \subseteq M^1_v$. From Theorem 3.4.6 we have that

$$T : M^p_{m\circ\chi} \to M^p_m$$

is a bounded operator. And from Propositions 3.4.5 and 3.3.11

$$M(\sigma, \Phi) : \ell^p_{m\circ\chi'}(\Lambda) \to \ell^p_m(\Lambda)$$

is a compact operator for each $1 \leq p < \infty$ and for each v-moderate weight m if, and only if,

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \mu)g\rangle\big)_{\lambda\in\Lambda} = \big(M(\sigma, \Phi)_{\chi'(\lambda)+\mu,\lambda}\big)_{\lambda\in\Lambda} \in c_0(\Lambda),$$

for all $\mu \in \Lambda$. Then, by Theorem 3.4.3,

$$T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$$

is compact for each $1 \le p < \infty$ and for each v-moderate weight m if and only if

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle\big)_{\lambda\in\Lambda} = \big(M(\sigma,\Phi)_{\chi'(\lambda)+\mu,\lambda}\big)_{\lambda\in\Lambda} \in c_0(\Lambda),$$

for all $\mu \in \Lambda$. □

We next discuss the case $p = \infty$.

Theorem 3.4.8. *Let T be a FIO whose phase Φ and symbol σ satisfy* (3.4.3)-(3.4.5) *and m a v-moderate weight, where v satisfies the GRS-condition. Then*

(1) T admits a unique extension as a bounded operator

$$T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \to M^\infty_m(\mathbb{R}^d)$$

which is also weak-continuous.*

(2) $T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \to M^\infty_m(\mathbb{R}^d)$ is compact if and only if

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda)+\mu)g\rangle\big)_\lambda \in c_0(\Lambda)$$

for every $\mu \in \Lambda$.

Proof. (1) In fact, we consider the composition

$$T : M^\infty_{m\circ\chi}(\mathbb{R}^d) \xrightarrow{C_g} \ell^\infty_{m\circ\chi}(\Lambda) \xrightarrow{M(\sigma,\Phi)} \ell^\infty_m(\Lambda) \xrightarrow{S^*} M^\infty_m(\mathbb{R}^d),$$

where $S = C_g : M^1_{1/m}(\mathbb{R}^d) \to \ell^1_{1/m}(\Lambda)$. We observe that all the involved maps are weak*-continuous. Since $S^{1/2}_{1/2}(\mathbb{R}^d)$ is weak*-dense in $M^\infty_{m\circ\chi}(\mathbb{R}^d)$ the extension is unique.

(2) By Theorem 3.4.3, T is compact if and only if $M(\sigma,\Phi) : \ell^\infty_{m\circ\chi} \to \ell^\infty_m(\Lambda)$ is compact. Now it suffices to apply Theorem 3.3.11. □

Theorem 3.4.9. Let T be a FIO whose phase Φ and symbol σ satisfy (3.4.3)-(3.4.5). If $\sigma \in M^0(\mathbb{R}^{2d})$ then $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p \leq \infty$ and for every v-moderate weight m, where v satisfies the GRS-condition.

Proof. From Theorem 3.3.15, if $\sigma \in M^0(\mathbb{R}^{2d})$ then $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator. Then, by Theorem 3.4.7, $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p \leq \infty$ and for every v-moderate weight m, where v is admissible. $\square$

We now see that the converse is true in the particular case of quadratic phases, Definition 3.3.16.

Theorem 3.4.10. Let T be a FIO whose quadratic phase Φ and symbol σ satisfy (3.4.3)-(3.4.5). Then the following statements are equivalent:

(1) $\sigma \in M^0(\mathbb{R}^{2d})$.

(2) $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p \leq \infty$ and for every v_s-moderate weight m.

Proof. From Theorem 3.3.17, $\sigma \in M^0(\mathbb{R}^{2d})$ if, and only if, $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator. Then, by Theorem 3.4.7, $T : M^p_{m\circ\chi}(\mathbb{R}^d) \to M^p_m(\mathbb{R}^d)$ is a compact operator for every $1 \leq p \leq \infty$ and for every v-moderate weight m, where v satisfies the GRS-condition. $\square$

3.4.3 Fourier Integral Operators on $M^{p,q}_m$

In the same way as before, we need to add an extra condition to consider the case $M^{p,q}_m$, namely

$$\sup_{x',x,\eta} \left| \nabla_x \Phi(x,\eta) - \nabla_x \Phi(x',\eta) \right| < \infty. \tag{3.4.7}$$

And it gives us the same properties on $\chi = (\chi_1, \chi_2)$ and $\chi' = (\chi_1', \chi_2')$. In this situation also the results proved about the matrix and its behaviour on norm-mixed sequence spaces are valid.

Theorem 3.4.11. Let T be a FIO whose phase Φ and symbol σ satisfy (3.4.3)-(3.4.5) and (3.4.7). Then, $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$ is a continuous operator for every $1 \leq p, q < \infty$ and for every v-moderate weight m, where v satisfies the GRS-condition.

Proof. Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \subseteq M^1_v$. From [CNR10b, Theorem 4.1] we have that

$$T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is a bounded operator. And from Propositions 3.4.5 and 3.3.22

$$M(\sigma, \Phi) : \ell^{p,q}_{m\circ\chi'}(\Lambda) \to \ell^{p,q}_m(\Lambda)$$

is a bounded operator for every $1 \leq p, q < \infty$ and for every v-moderate weight m. We observe that, for any positive and v-moderate weight m,

$$\ell^{p,q}_{m\circ\chi}(\Lambda) = \ell^{p,q}_{m\circ\chi'}(\Lambda)$$

with equivalent norms and that $m \circ \chi$ is v-moderate whenever m is v-moderate. Then, by Theorem 3.4.3,

$$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q < \infty$ and for every v-moderate weight m. $\square$

Theorem 3.4.12. *Let T be a FIO whose phase Φ and symbol σ satisfy* (3.4.3)-(3.4.5) *and* (3.4.7). *Then T admits a unique extension as a bounded operator*

$$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d),$$

for $1 \leq p, q \leq \infty$ and p or q equal to ∞, which is also weak-continuous.*

Proof. We know that $T = C_g \circ M(\sigma, \Phi) \circ C^*_g$ is bounded on $M^1_v(\mathbb{R}^d)$, which is weak*-dense in $M^{p,q}_m(\mathbb{R}^d)$, for $1 \leq p, q \leq \infty$ and p or q equal to ∞ and for all v_s-moderate weight m. Then T admits a unique extension,

$$\tilde{T} : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \xrightarrow{C_g} \ell^{p,q}_{m\circ\chi}(\Lambda) \xrightarrow{M(\sigma,\Phi)} \ell^{p,q}_m(\Lambda) \xrightarrow{S^*} M^{p,q}_m(\mathbb{R}^d),$$

where $S = C_g : M^{p',q'}_{1/m}(\mathbb{R}^d) \to \ell^{p',q'}_{1/m}(\Lambda)$. In fact, all the involved maps are weak*-continuous, then $\tilde{T}$ is weak*-continuous. As the extension is unique we denote it as T.

□

Theorem 3.4.13. Let T be a FIO whose phase Φ and symbol σ satisfy (3.4.3)-(3.4.5) and (3.4.7). The following conditions are equivalent:

(1) $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.

(2) $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_{m}(\mathbb{R}^d)$ is a compact operator for some $1 \leq p, q \leq \infty$ and for some v-moderate weight m, where v satisfies the GRS-condition.

(3) $T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_{m}(\mathbb{R}^d)$ is a compact operator for every $1 \leq p, q \leq \infty$ and for every v-moderate weight m, where v satisfies the GRS-condition.

Proof. Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \subseteq M^1_v$. From Theorem 3.4.11 and 3.4.12 we have that

$$T : M^{p,q}_{m\circ\chi} \to M^{p,q}_{m}$$

is a bounded operator. And from Propositions 3.4.5 and 3.3.23

$$M(\sigma, \Phi) : \ell^{p,q}_{m\circ\chi'}(\Lambda) \to \ell^{p,q}_{m}(\Lambda)$$

is a compact operator for each $1 \leq p, q < \infty$ and for each v-moderate weight m if, and only if,

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \mu)g\rangle\big)_{\lambda\in\Lambda} = \big(M(\sigma, \Phi)_{\chi'(\lambda)+\mu,\lambda}\big)_{\lambda\in\Lambda} \in c_0(\Lambda),$$

for all $\mu \in \Lambda$. Then, by Theorem 3.4.3,

$$T : M^{p,q}_{m\circ\chi}(\mathbb{R}^d) \to M^{p,q}_{m}(\mathbb{R}^d)$$

is compact for each $1 \leq p, q < \infty$ and for each v-moderate weight m if and only if

$$\big(\langle T\pi(\lambda)g, \pi(\chi'(\lambda) + \mu)g\rangle\big)_{\lambda\in\Lambda} = \big(M(\sigma, \Phi)_{\chi'(\lambda)+\mu,\lambda}\big)_{\lambda\in\Lambda} \in c_0(\Lambda),$$

for all $\mu \in \Lambda$. □

3.5 Conclusion

Summarizing, we have seen that the boundedness and compactness of Fourier integral operators on $M^{p,q}_m$ do not depend on p, q or m. We have found sufficient conditions, in some cases also necessary, for the compactness of these operators. These results improve some known results about pseudodifferential operators and localization operators. Some results of this chapter are included in [FGP18].

Chapter 4

Fourier integral operator with Hölder-continuous phase

4.1 Introduction

The aim of this chapter is to find conditions for the boundedness of the integral operator,

$$Af(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy, \tag{4.1.1}$$

with (collision) kernel

$$K(x, y) = \int_{\mathbb{R}^d} \Phi(u) e^{-2\pi \mathbf{i}(\beta(|u|) u \cdot y - u \cdot x)} du. \tag{4.1.2}$$

on some Lebesgue spaces, where the function $\Phi(u)$ has a good decay at infinity but might not be smooth at the origin $u = 0$ and $\beta(r)$ is real-valued. This integral operator can be seen as a FIO of type II,

$$T_{II,\varphi,\sigma} f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi \mathbf{i}(\varphi(y,u) - u \cdot x)} \sigma(y, u) f(y) dy\, du,$$

defined in Chapter 3, with $\varphi(y,u) = \beta(|u|)u \cdot y$ and $\sigma(y,u) = \Phi(u)$.

This operator appears in the study of the Boltzmann equation, hence it is interesting to find estimates of the type

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\, dx < \infty, \tag{4.1.3}$$

[AL17] (related references are provided by [AL10, BD98, MM06]). The estimate (4.1.3) would imply the boundedness of the corresponding operator A on $L^1(\mathbb{R}^d)$. A typical example for the function $\Phi(u)$, which has a good decay at infinity but might not be smooth at the origin $u = 0$, is given by radial functions

$$\Phi(u) = \frac{|u|}{(1+|u|^2)^m} \tag{4.1.4}$$

with large real m.

The phase $\beta(r)$ is real-valued and smooth on $(0, +\infty)$ but could have a Hölder type singularity at the origin. As an example the following over-simplified model can be considered

$$\beta(r) = a + br^\gamma, \quad 0 < r \leq 1, \tag{4.1.5}$$

for some $a, b \in \mathbb{R}$, $\gamma \in (0,1)$. As $r \to +\infty$, $\beta(r)$ is assumed to approach a constant.

As a basic case suppose $\beta(r) = a$, $r > 0$, is a constant function. Rapid granular flows are described by the Boltzmann equation and $\beta(r) = a$ corresponds to the case of inelastic interactions with constant restitution coefficient. Indeed, the loss of mechanical energy due to collisions is characterized by the restitution coefficient β which quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact velocity. Now, when $\beta(r) = a$ is constant,

$$K(x,y) = \mathcal{F}\Phi(ay - x)$$

and the estimate (4.1.3) holds if and only if $\Phi \in \mathcal{F}L^1(\mathbb{R}^d)$, i.e. Φ has Fourier transform in $L^1(\mathbb{R}^d)$. The major part of the research, at the physical as well as at the mathematical levels, has been devoted to this particular case of a constant restitution coefficient. However, as described in [BP04, AL10], a more relevant description of granular gases should involve a variable restitution coefficient $\beta(r)$.

In the model case above $\beta(r)$ approaches a constant both as $r \to 0^+$ and $r \to +\infty$ and is smooth in between, so that one could conjecture that the same estimate holds in that case. Now, this is not the case, even for smooth phases: we prove in Proposition 4.3.1 that, in dimension $d = 1$, if $\tilde{\varphi}(u) := \beta(|u|)u$ is any *nonlinear* smooth diffeomorphism $\mathbb{R} \to \mathbb{R}$ with $\tilde{\varphi}(u) = u$ (hence $\beta(|u|) = 1$) for $|u| \geq 1$, and $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\Phi \equiv 1$ on $[-1, 1]$, then the weighted estimate

$$\int_{\mathbb{R}^d} |K(x, y)|\, dx \lesssim (1 + |y|)^s \tag{4.1.6}$$

does not hold for $s < 1/2$.

This looks surprising at first glance, but it can be regarded as a manifestation of the Beurling-Helson phenomenon [BH53, CNR10a, LO94, Oko09, RSTT11], which, roughly speaking, states that the change-of-variable operator $f \mapsto f \circ \psi$ is not bounded on $\mathcal{F}L^1(\mathbb{R}^d)$ except for the case $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is an affine mapping. Indeed if we consider a function $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
Af(x) &= \int_{\mathbb{R}^d} K(x, y) f(y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(u) e^{-2\pi \mathbf{i}(\beta(|u|)u \cdot y - u \cdot x)} du f(y) dy \\
&= \int_{\mathbb{R}^d} \Phi(u) e^{2\pi \mathbf{i} u \cdot x} \left(\int_{\mathbb{R}^d} f(y) e^{-2\pi \mathbf{i} \beta(|u|) u \cdot y} dy \right) du \\
&= \int_{\mathbb{R}^d} \Phi(u) e^{2\pi \mathbf{i} u \cdot x} \mathcal{F}f(\beta(|u|)u) du = \mathcal{F}^{-1} \left[\Phi(u) \mathcal{F}f(\beta(|u|)u) \right](x),
\end{aligned}$$

and the operator A in (4.1.1) with kernel $K(x, y)$ in (4.1.2) can be written as

$$Af = \mathcal{F}^{-1}\Phi * \mathcal{F}^{-1}(\mathcal{F}f \circ \tilde{\varphi}), \quad \text{with } \tilde{\varphi}(u) := \beta(|u|)u.$$

Now, one is interested in the precise growth in (4.1.6). Let us summarize here the main results of the chapter in the special case of our oversimplified model above.

Theorem 4.3.2: *Suppose* $\beta(r)$ *as in* (4.1.5) *for* $0 < r \leq 1$, *with* $\gamma \in (-1, 1]$ *and assume* β *has at most linear growth as* $r \to +\infty$. *Let* Φ *be as in* (4.1.4), *with* $m > (d+1)/2$. *Then* (4.1.6) *holds with* $s = d/(\gamma + 1)$.

As expected, the growth in (4.1.6) is therefore the weakest one when $\gamma = 1$, being $s = d/2$ in that case. The same growth occurs for smooth phases, as the following result shows.

Theorem 4.3.3: *Suppose that* $\tilde{\varphi}(u) := \beta(|u|)u$ *extends to a smooth function on* $\mathbb{R}^d$, *with an at most quadratic growth at infinity. Let* Φ *be as in* (4.1.4), *with* $m > (d+1)/2$. *Then* (4.1.6) *holds with* $s = d/2$.

Notice that the estimate (4.1.6) implies a continuity property for the corresponding operator between weighted L^1 spaces, precisely $L^1_{v_s} \to L^1$, where $v_s(x) = (1 + |x|)^s$.

A natural question is therefore whether similar continuity estimates hold without a loss of decay at least in $L^2(\mathbb{R}^d)$, under the above assumptions. We show in Proposition 4.4.1 that, again, this is not the case. Sufficient conditions are instead given in Theorem 4.4.2 below. Here is a simplified version of Theorem 4.4.2 (and subsequent remark).

- *Suppose* $\beta(r)$ *as in* (4.1.5) *for* $0 < r \leq 1$, *with* $\gamma > 0$. *Let* $\Phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ *supported in* $|u| \leq 1$. *Then, if* $a(a + (\gamma+1)b) > 0$ *the operator* A *in* (4.1.1) *is bounded in* $L^2(\mathbb{R}^d)$.

Actually the results below are stated for β and Φ in classes of functions with minimal regularity and are inspired by the models above. It turns out that, in all the results it is sufficient to take Φ in the so-called Segal algebra $M^1(\mathbb{R}^d)$ [Fei81b, Fei89, Fei06]. Roughly speaking, a function $\Phi \in L^\infty(\mathbb{R}^d)$ belongs to $M^1(\mathbb{R}^d) = W(\mathcal{F}L^1, L^1)$ (Proposition 1.2.14) if locally has the regularity of a function in $\mathcal{F}L^1(\mathbb{R}^d)$ (in particular is continuous) and globally it decays as a function in $L^1(\mathbb{R}^d)$, but no differentiability conditions are required. We have $M^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$. To compare this space with the usual Sobolev spaces we observe that $W^{k,1}(\mathbb{R}^d) \subset M^1(\mathbb{R}^d)$ for $k \geq d+1$ (Proposition 1.2.9) but functions in $M^1(\mathbb{R}^d)$ do not need to have any derivatives. For example, the functions Φ in (4.1.4) are in $M^1(\mathbb{R}^d)$ if $m > (d+1)/2$ (see Example 4.4.4). It is important to observe that the very weak assumption $\Phi \in M^1(\mathbb{R}^d)$ prevents us to use classical tools such as stationary phase estimates; instead we use techniques and function spaces from time-frequency analysis. Recently, such function spaces and more general time-frequency analysis have been successfully applied in the study of partial differential equations with rough data by a large number of authors, see, e.g., [CN08b, RWZ16, STW11, WH07] and the references therein. We also refer to the papers [CNR10a, CR14] and the references therein for the problem of the continuity in $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and from the Hardy space to $L^1(\mathbb{R}^d)$, of general Fourier integral operators of Hörmander's type (i.e. arising from the study of hyperbolic equations). The Fourier integral operators of Hörmander's type are Fourier integral operators of type I, defined in Chapter 3, with the symbol, $\sigma(y, u)$ belonging to some Hörmander's symbol class.

In short the chapter is organized as follows.

In Section 4.2 we briefly recall the main properties and preliminary results we need in the sequel.

In Section 4.3 we study the L^1-continuity for the integral operators in (4.1.1) having phases with Hölder-type singularity at the origin. The boundedness is attained at the cost of a loss of decay. Such a loss is unavoidable, as testified by an example in dimension $d = 1$ (cf. Proposition 4.3.1).

In Section 4.4 we study the L^2-continuity properties of A in (4.1.1). Under the same assumptions of the L^1-boundedness results we provide a counterexample even in this framework (cf. Proposition 4.4.1). We then show conditions on the phase of the operators which guarantee L^2-boundedness without loss of decay.

4.2 Auxiliary results

In the sequel we list issues preparing for our later argumentation. To study the properties of our phase function, we shall rely on the following results.

Lemma 4.2.1. *([MNR$^+$09, Lemma 3.2]) Let $\varepsilon > 0$. Suppose μ is a real-*

valued function of class $\mathcal{C}^{\lfloor d/2\rfloor+1}$ *on* $\mathbb{R}^d \setminus \{0\}$ *satisfying*

$$|\partial^\alpha \mu(u)| \leq C_\alpha |u|^{\varepsilon-|\alpha|} \tag{4.2.1}$$

for $|\alpha| \leq \lfloor d/2\rfloor + 1$. *Then* $\mathcal{F}^{-1}[\eta e^{\mathbf{i}\mu}] \in L^1(\mathbb{R}^d)$ *for each* $\eta \in \mathcal{S}(\mathbb{R}^d)$ *with compact support. The norm of* $\eta e^{\mathbf{i}\mu}$ *in* $\mathcal{F}L^1(\mathbb{R}^d)$ *is indeed controlled by a constant depending only on* d, η *and the constants* C_α *in* (4.2.1).

Lemma 4.2.2. *([BGOR07, Theorem 5]) For* $d \geq 1$, *let* $l = \lfloor d/2\rfloor + 1$. *Assume that* μ *is* $2l$ *times continuously differentiable function on* $\mathbb{R}^d$ *and* $\|\partial^\alpha \mu\|_{L^\infty} \leq C_\alpha$, *for* $2 \leq |\alpha| \leq 2l$, *and some constants* C_α. *Then* $e^{\mathbf{i}\mu} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$.

The norm of $e^{\mathbf{i}\mu}$ *in* $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ *is indeed controlled by a constant depending only on* d, *and the above constants* C_α.

Lemma 4.2.3. *([CNR15b, Proposition 2.5]) Let* $h \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$ *be positively homogeneous of degree* $r > 0$, *i.e.,* $h(\lambda x) = \lambda^r h(x)$ *for* $x \neq 0$, $\lambda > 0$. *Consider* $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ *and set* $f = h\chi$. *Then, for* $\psi \in \mathcal{S}(\mathbb{R}^d)$, *there exists a constant* $C > 0$ *such that*

$$|V_\psi f(x,u)| \leq C(1+|u|)^{-r-d}, \text{ for every } x, u \in \mathbb{R}^d.$$

In order to exhibit the counterexample anticipated in the introduction we make use of a result proved in [CNR10a, Proposition 6.1] which can be stated as follows.

Proposition 4.2.4. *Let* $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ *be any nonlinear smooth diffeomorphism satisfying*

$$\tilde{\varphi}(u) = u, \quad \text{for } |u| \geq 1,$$

and let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\Phi \equiv 1$ *on* $[-1,1]^d$.

For $2 \leq p \leq \infty$, $m < d(1/2 - 1/p)$, *the so-called type I FIO* $T_{I,\varphi,\sigma}$,

$$T_{I,\varphi,\sigma} f(x) = \int e^{2\pi \mathbf{i}\varphi(x,u)} \sigma(x,u) \hat{f}(u)\, du,$$

having phase $\varphi(x,u) = \sum_{k=1}^d \tilde{\varphi}(u_k) x_k$, *and symbol* $\sigma(x,u) = \langle x\rangle^m \Phi(u)$, *does not extend to a bounded operator on* $L^p(\mathbb{R}^d)$.

Note that the phase described in this result satisfies the conditions imposed in Chapter 3. However the symbol is not bounded, hence the condition imposed in Chapter 3 are not satisfied.

4.3 Continuity in L^1 with loss of decay

We consider the integral operator A formally defined in (4.1.1) with kernel $K(x,y)$ as in (4.1.2). We assume $\Phi \in M^1(\mathbb{R}^d)$ and $\beta : (0,\infty) \to \mathbb{R}$ is a smooth function. Then, the kernel K is well-defined for every $x, y \in \mathbb{R}^d$. Indeed, since $M^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$, the integral in (4.1.2) is absolutely convergent. Inserting the kernel expression (4.1.2) in the operator A, defined in (4.1.1), and using the absolute convergence of the integrals we can apply Fubini's Theorem and infer

$$Af(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi \mathbf{i}(\beta(|u|)u\cdot y - x\cdot u)} \Phi(u) f(y)\, dy\, du. \tag{4.3.1}$$

for $f \in L^1(\mathbb{R}^d)$.

That is, the operator A can be written as a Fourier integral operator of type II. We recall that a FIO of type II with phase φ and symbol σ has the general form

$$T_{II,\varphi,\sigma} f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi \mathbf{i}(\varphi(y,u) - x\cdot u)} \sigma(y,u) f(y) dy\, du \tag{4.3.2}$$

hence

$$A = T_{II,\varphi,\sigma}, \quad \text{with } \varphi(y,u) = \beta(|u|)u\cdot y \text{ and } \sigma(y,u) = \Phi(u). \tag{4.3.3}$$

FIO's of type II are the formal adjoints of FIO's of type I, as we saw in Chapter 3.

In general we do not expect that the integral operator A in (4.1.1) with kernel K in (4.1.2) is continuous on $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, $p \ne 2$. Actually, we expect a loss of decay, as witnessed by the following example.

Proposition 4.3.1. *In dimension $d = 1$, for any $1 \leq p \leq 2$, consider the weight function*

$$v_m(y) = (1 + |y|)^m, \quad y \in \mathbb{R},$$

with $m \in \mathbb{R}$ such that

$$m < \frac{1}{p} - \frac{1}{2}.$$

Let $\beta \in \mathcal{C}^\infty((0, +\infty))$ such that

$$\tilde{\varphi}(u) = \beta(|u|)u$$

extends to a nonlinear smooth diffeomorphism $\mathbb{R} \to \mathbb{R}$ satisfying

$$\tilde{\varphi}(u) = u, \quad |u| \geq 1$$

(hence, $\beta(|u|) = 1$, for $|u| \geq 1$). Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\Phi(u) = 1$ for $|u| \leq 1$.

Then the operator A in (4.3.1) *does not extend to a bounded operator from $L^p_{v_m}(\mathbb{R})$ to $L^p(\mathbb{R})$.*

Proof. Step 1: Rephrasing the thesis. Since $v_m(y) = (1 + |y|)^m$ is a weight equivalent to $w_m(y) := \langle y \rangle^m = (1 + y^2)^{m/2}$, we can work with w_m in place of v_m. Since A can be written as a type II Fourier integral operator, the continuity of A from $L^p_{v_m}(\mathbb{R})$ to $L^p(\mathbb{R})$ is equivalent to the continuity from $L^p_{w_m}(\mathbb{R})$ to $L^p(\mathbb{R})$ of the operator $T_{II,\varphi,\sigma}$ in (4.3.2) with $\varphi(y, u) = \tilde{\varphi}(u)y$ and symbol $\sigma(y, u) = \Phi(u)$, with $\tilde{\varphi}$ and Φ as in the statement.

Step 2: From type II FIOs to type I FIOs. By duality, the continuity of $T_{II,\varphi,\sigma}$ is equivalent to the boundedness of the adjoint $(T_{II,\varphi,\sigma})^* = T_{I,\varphi,\sigma}$, from $L^p(\mathbb{R})$ to $L^p_{w_{-m}}(\mathbb{R})$, for $2 \leq p \leq \infty$.

Step 3: Results for type I FIOs. The continuity of $T_{I,\varphi,\sigma}$ from $L^p(\mathbb{R})$ to $L^p_{w_{-m}}(\mathbb{R})$ is equivalent to the boundedness of the operator $w_{-m}T_{I,\varphi,\sigma}$ on

$L^p(\mathbb{R})$. Now, observe that

$$\begin{aligned}\langle x\rangle^{-m}T_{I,\varphi,\sigma}f(x) &= \langle x\rangle^{-m}\int e^{2\pi\mathbf{i}\varphi(x,u)}\sigma(x,u)\hat{f}(u)\,du\\ &= \int e^{2\pi\mathbf{i}\varphi(x,u)}\langle x\rangle^{-m}\sigma(x,u)\hat{f}(u)\,du\\ &= \int e^{2\pi\mathbf{i}\varphi(x,u)}\tilde{\sigma}(x,u)\hat{f}(u)\,du := T_{I,\varphi,\tilde{\sigma}}\end{aligned}$$

with $\tilde{\sigma}(x,u)=\langle x\rangle^{-m}\sigma(x,u)=\langle x\rangle^{-m}\Phi(u)$.

Now the type I FIO $T_{I,\varphi,\tilde{\sigma}}$ is not bounded on L^p, $2\leq p\leq\infty$, by Proposition 4.2.4. □

Continuity in weighted L^1 spaces, i.e. with a loss of decay, for the operator A in (4.3.1) can be proved by a Schur-type estimate for the kernel K. The following result addresses such estimates and Corollary 4.3.5 the corresponding continuity result.

Theorem 4.3.2. *Consider functions $\Phi\in M^1(\mathbb{R}^d)$ and $\beta:(0,+\infty)\to\mathbb{R}$. Moreover, assume that for some exponent $\gamma\in(-1,1]$, with $\ell=\lfloor d/2\rfloor+1$,*

$$|\partial^\alpha\beta(|u|)u|\leq C_\alpha|u|^{\gamma+1-|\alpha|},\qquad \text{for }\ 0\neq|u|\leq 1,\ |\alpha|\leq\ell, \tag{4.3.4}$$

where $C_\alpha>0$, and

$$|\partial^\alpha\beta(|u|)u|\leq C'_\alpha,\qquad \text{for }\ |u|\geq 1,\ 2\leq|\alpha|\leq 2\ell, \tag{4.3.5}$$

with $C'_\alpha>0$. Then the integral kernel in (4.1.2) satisfies

$$\int_{\mathbb{R}^d}|K(x,y)|dx\leq C(1+|y|)^{d/(\gamma+1)}, \tag{4.3.6}$$

for a suitable constant $C>0$ independent of y.

Proof. We study the cases $|y|<1$ and $|y|\geq 1$ separately. First we assume that $|y|\geq 1$. To see the estimation, we prove that

$$e^{-2\pi\mathbf{i}\beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right)\frac{u}{|y|^{1/(\gamma+1)}}\cdot y}\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^d)$$

For that, we consider a function $\chi \in C_0^\infty$, such that $\chi(u) = 1$ when $|u| \leq \frac{1}{2}$ and $\chi(u) = 0$ when $|u| \geq 1$. We want to see that:

$$e^{-2\pi \mathbf{i}\beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right)\frac{u}{|y|^{1/(\gamma+1)}}\cdot y} \cdot \chi(u) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \tag{4.3.7}$$

and

$$e^{-2\pi \mathbf{i}\beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right)\frac{u}{|y|^{1/(\gamma+1)}}\cdot y} \cdot (1-\chi(u)) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \tag{4.3.8}$$

To prove (4.3.7) we use Lemma 4.2.1. It is sufficient to verify that the phase

$$B_y(u) := 2\pi\beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right)\frac{u}{|y|^{1/(\gamma+1)}} \cdot y$$

satisfies the estimation

$$|\partial^\alpha B_y(u)| \leq C_\alpha |u|^{\gamma+1-|\alpha|}$$

for $|u| \leq 1$, since the function in (4.3.7) is zero when $|u| > 1$. Using the hypothesis (4.3.4),

$$|\partial^\alpha \beta(|u|)u| \leq C_\alpha |u|^{\gamma+1-|\alpha|},$$

for $|u| \leq 1$, we have

$$|\partial^\alpha B_y(u)| \leq \frac{2\pi C_\alpha}{|y|^{|\alpha|/(\gamma+1)}}\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right)^{\gamma+1-|\alpha|} \cdot |y| = \tilde{C}_\alpha |u|^{\gamma+1-|\alpha|} \tag{4.3.9}$$

for $|u|/(|y|^{1/(\gamma+1)}) \leq 1$, i.e. $|u| \leq |y|^{1/(\gamma+1)}$. Then, we can apply the Lemma 4.2.1 and we have (4.3.7).

To prove (4.3.8) we use Lemma 4.2.2. It is sufficient to verify that the phase satisfies the estimation

$$|\partial^\alpha B_y(u)| \leq C_\alpha$$

for $|u| \geq \frac{1}{2}$ and $2 \leq |\alpha| \leq 2\ell$. If $\frac{1}{2} \leq |u| \leq |y|^{1/(\gamma+1)}$ the last estimation follows from (4.3.9). On the other hand, if $|u| \geq |y|^{1/(\gamma+1)}$, we use the hypothesis (4.3.5),

$$|\partial^\alpha \beta(|u|)u| \leq C'_\alpha,$$

for $|u| \geq 1$ and $2 \leq |\alpha| \leq 2\ell$. Then, for $|u| \geq |y|^{1/(\gamma+1)}$,

$$|\partial^\alpha B_y(u)| \leq C'_\alpha \frac{2\pi}{|y|^{|\alpha|/(\gamma+1)}} \cdot |y| = \tilde{C'}_\alpha \frac{1}{|y|^{(|\alpha|-\gamma-1)/(\gamma+1)}} \leq \tilde{C'}_\alpha$$

for $2 \leq |\alpha| \leq 2\ell$, given that $|y| \geq 1$ and $\gamma \leq 1$. Then, we can apply Lemma 4.2.2 and we have (4.3.8).

Now from (4.3.7) and (4.3.8), we deduce

$$e^{-2\pi \mathbf{i} \beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right) \frac{u}{|y|^{1/(\gamma+1)}} \cdot y} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d),$$

where its norm in $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ depends only on d, χ, C_α and C'_α. Then, by Lemma 1.2.16, we have

$$\begin{aligned}
&\left\| e^{-2\pi \mathbf{i} \beta(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\lesssim (1+|y|^{2/(\gamma+1)})^{d/2} \left\| e^{-2\pi \mathbf{i} \beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right) \frac{u}{|y|^{1/(\gamma+1)}} \cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\lesssim (1+|y|)^{d/(\gamma+1)} \left\| e^{-2\pi \mathbf{i} \beta\left(\frac{|u|}{|y|^{1/(\gamma+1)}}\right) \frac{u}{|y|^{1/(\gamma+1)}} \cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\lesssim (1+|y|)^{d/(\gamma+1)}
\end{aligned}$$

Now, as $\Phi \in M^1(\mathbb{R}^d)$, we have that $\Phi(u) e^{-\mathbf{i}\beta(|u|)u\cdot y} \in M^1$. Also, we have that

$$K(x,y) = \mathcal{F}^{-1}\left[\Phi(u) e^{-\mathbf{i}\beta(|u|)u\cdot y}\right](x).$$

Then, by Proposition 1.2.16 for $p = 1$,

$$\begin{aligned}\int_{\mathbb{R}^d} |K(x,y)|dx &= \left\| \mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right]\right\|_{L^1} \\ &= \left\|\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right\|_{\mathcal{F}L^1} \lesssim \left\|\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right\|_{M^1} \\ &\leq \|\Phi\|_{M^1}\left\|e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\ &\leq \|\Phi\|_{M^1}C'(1+|y|)^{d/(\gamma+1)} \leq C(1+|y|)^{\frac{d}{\gamma+1}},\end{aligned}$$

for some $C > 0$.

In the case $|y| < 1$, we argue as above without the factor $1/|y|^{1/(\gamma+1)}$ and we obtain the same estimates. We start proving that

$$e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$$

For that, we consider a function $\chi \in C_0^\infty$, such that $\chi(u) = 1$ when $|u| \leq \frac{1}{2}$ and $\chi(u) = 0$ when $|u| \geq 1$. We want to see that:

$$e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y}\cdot\chi(u) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \tag{4.3.10}$$

and

$$e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y}\cdot(1-\chi(u)) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \tag{4.3.11}$$

To prove (4.3.10) we use Lemma 4.2.1. It is sufficient to verify that the phase

$$B_y(u) := 2\pi\beta\left(|u|\right)u\cdot y$$

satisfies the estimate

$$|\partial^\alpha B_y(u)| \leq C_\alpha |u|^{\gamma+1-|\alpha|}$$

for $|u| \leq 1$, since the function in (4.3.10) is zero when $|u| > 1$. Using the hypothesis (4.3.4),

$$|\partial^\alpha \beta(|u|)u| \leq C_\alpha |u|^{\gamma+1-|\alpha|},$$

for $|u| \leq 1$, we have

$$|\partial^\alpha B_y(u)| \leq 2\pi C_\alpha\left(|u|\right)^{\gamma+1-|\alpha|}\cdot|y| \leq \tilde{C}_\alpha|u|^{\gamma+1-|\alpha|}, \tag{4.3.12}$$

as $|y| < 1$, for $|u| \leq 1$. Then, we can apply the Lemma 4.2.1 and we have (4.3.10).

To prove (4.3.11) we use Lemma 4.2.2. It is sufficient to verify that the phase satisfies the estimation

$$|\partial^\alpha B_y(u)| \leq C_\alpha$$

for $|u| \geq \frac{1}{2}$ and $2 \leq |\alpha| \leq 2\ell$. If $\frac{1}{2} \leq |u| \leq 1$ the last estimation follows from (4.3.12). On the other hand, if $|u| \geq 1$, we use the hypothesis (4.3.5),

$$|\partial^\alpha \beta(|u|)u| \leq C'_\alpha,$$

for $|u| \geq 1$ and $2 \leq |\alpha| \leq 2\ell$. Then,

$$|\partial^\alpha B_y(u)| \leq C'_\alpha 2\pi |y| \leq \tilde{C'}_\alpha$$

for $2 \leq |\alpha| \leq 2\ell$, given that $|y| < 1$. Then, we can apply Lemma 4.2.2 and we have (4.3.11).

Now from (4.3.10) and (4.3.11), we deduce

$$e^{-2\pi \mathbf{i}\beta(|u|)u\cdot y} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d),$$

where its norm in $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ depends only on d, χ, C_α and C'_α. Now, as $\Phi \in M^1(\mathbb{R}^d)$, we have that $\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y} \in M^1$. Also, we have that

$$K(x,y) = \mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right](x).$$

Then, by Proposition 1.2.16 for $p = 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} |K(x,y)|dx &= \left\|\mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right]\right\|_{L^1} \\
&= \left\|\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right\|_{\mathcal{F}L^1} \lesssim \left\|\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right\|_{M^1} \\
&\leq \|\Phi\|_{M^1}\left\|e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y}\right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\leq \|\Phi\|_{M^1} C' \leq C(1+|y|)^{\frac{d}{\gamma+1}},
\end{aligned}$$

for some $C > 0$.

$\square$

In the previous result the weakest growth is reached when $\gamma = 1$, the exponent in (4.3.6) in that case being $d/2$. That growth is the same obtained even for smooth phases, as proved in the following result, and cannot be further reduced, as shown in Proposition 4.3.1.

Corollary 4.3.3. *Consider functions $\Phi \in M^1(\mathbb{R}^d)$ and $\beta : (0, +\infty) \to \mathbb{R}$. Moreover, setting $\ell = \lfloor d/2 \rfloor + 1$, assume that the function $\beta(|u|)$ extends to a $\mathcal{C}^{2\ell}$ function on $\mathbb{R}^d$ and satisfies*

$$|\partial^\alpha \beta(|u|)u| \leq C_\alpha, \quad \text{for } u \in \mathbb{R}^d \text{ and } 2 \leq |\alpha| \leq 2\ell. \tag{4.3.13}$$

Then, the integral kernel in (4.1.2) satisfies

$$\int_{\mathbb{R}^d} |K(x,y)| dx \leq C(1 + |y|)^{\frac{d}{2}}.$$

Proof. The proof uses the same arguments used in Theorem 4.3.2. We split into the cases $|y| \geq 1$ and $|y| < 1$. We study first $|y| \geq 1$ and prove that

$$e^{-2\pi \mathbf{i} \beta\left(\frac{|u|}{|y|^{1/2}}\right) \frac{u}{|y|^{1/2}} \cdot y} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d), \tag{4.3.14}$$

and its norm does not depend on y, that is, the set

$$\left\{ e^{-2\pi \mathbf{i} \beta\left(\frac{|u|}{|y|^{1/2}}\right) \frac{u}{|y|^{1/2}} \cdot y} : |y| \geq 1 \right\}$$

is a bounded set in $W(\mathcal{F}L^1, L^\infty)$. Using Lemma 4.2.2, the problem is reduced to verify that the rescaled phase

$$B_y(u) := 2\pi \beta\left(\frac{|u|}{|y|^{1/2}}\right) \frac{u}{|y|^{1/2}} \cdot y$$

satisfies the estimate

$$|\partial^\alpha B_y(u)| \leq C_\alpha, \quad \text{for all } u \in \mathbb{R}^d \text{ and } 2 \leq |\alpha| \leq 2\ell.$$

By the hypothesis (4.3.13),

$$|\partial^\alpha B_y(u)| \leq C_\alpha \frac{2\pi}{|y|^{|\alpha|/2}} \cdot |y| = C'_\alpha \frac{1}{|y|^{(|\alpha|-2)/2}} \leq C''_\alpha,$$

since $|\alpha| \geq 2$ and $|y| \geq 1$; this gives (4.3.14).

Then, by Lemma 1.2.15, we have

$$\begin{aligned}\left\| e^{-2\pi \mathbf{i}\beta(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} &\lesssim (1+|y|)^{d/2} \left\| e^{-2\pi \mathbf{i}\beta\left(\frac{|u|}{|y|^{1/2}}\right)\frac{u}{|y|^{1/2}}\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\ &\lesssim (1+|y|)^{d/2}.\end{aligned}$$

As $\Phi \in M^1(\mathbb{R}^d)$, we have that $\Phi(u)e^{-i\beta(|u|)u\cdot y} \in M^1(\mathbb{R}^d)$. Moreover, we can write

$$K(x,y) = \mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right](x).$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^d} |K(x,y)|dx &= \left\| \mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right] \right\|_{L^1} = \left\| \Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y} \right\|_{\mathcal{F}L^1} \\ &\lesssim \left\| \Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y} \right\|_{M^1} \lesssim \|\Phi\|_{M^1} \left\| e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\ &\leq C(1+|y|)^{\frac{d}{2}},\end{aligned}$$

for some $C > 0$.

The case $|y| < 1$ is obtained with the same pattern above, without the dilation factor $|y|^{-\frac{1}{2}}$. We prove that

$$e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \tag{4.3.15}$$

uniformly with respect to y. Using Lemma 4.2.2, the problem is reduced to verify that the phase

$$B_y(u) := 2\pi\beta\left(|u|\right) u \cdot y$$

satisfies the estimate

$$|\partial^\alpha B_y(u)| \leq C_\alpha, \quad \text{for all } u \in \mathbb{R}^d \text{ and } 2 \leq |\alpha| \leq 2\ell.$$

By the hypothesis (4.3.13),

$$|\partial^\alpha B_y(u)| \leq C_\alpha 2\pi |y| \leq C'_\alpha$$

since $|\alpha| \geq 2$ and $|y| < 1$; this gives (4.3.15).

As $\Phi \in M^1(\mathbb{R}^d)$, we have that $\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y} \in M^1(\mathbb{R}^d)$. Moreover, we can write

$$K(x,y) = \mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right](x).$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^d} |K(x,y)|dx &= \left\|\mathcal{F}^{-1}\left[\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right]\right\|_{L^1} = \left\|\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right\|_{\mathcal{F}L^1} \\
&\lesssim \left\|\Phi(u)e^{-\mathbf{i}\beta(|u|)u\cdot y}\right\|_{M^1} \lesssim \|\Phi\|_{M^1} \left\|e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y}\right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\leq C \leq C(1+|y|)^{\frac{d}{2}},
\end{aligned}$$

for some $C > 0$. □

Corollary 4.3.4. *Consider functions $\Phi \in M^1(\mathbb{R}^d)$ and $\beta : (0, +\infty) \to \mathbb{R}$. Assume that for some $\gamma \in (-1, 1]$ and $a \in \mathbb{R}$,*

$$\widetilde{\beta} := \beta - a \tag{4.3.16}$$

satisfies, with $\ell = \lfloor d/2 \rfloor + 1$,

$$\left|\partial^\alpha \widetilde{\beta}(|u|)u\right| \leq C_\alpha |u|^{\gamma+1-|\alpha|}, \quad \textit{for } |u| \leq 1,\ |\alpha| \leq \ell,$$

for some $C_\alpha > 0$, and

$$\left|\partial^\alpha \widetilde{\beta}(|u|)u\right| \leq C'_\alpha, \quad \textit{for } |u| \geq 1,\ 2 \leq |\alpha| \leq 2\ell,$$

with $C'_\alpha > 0$. Then the integral kernel in (4.1.2) satisfies

$$\int_{\mathbb{R}^d} |K(x,y)|dx \leq C(1+|y|)^{d/(\gamma+1)}, \tag{4.3.17}$$

for a suitable constant $C > 0$, independent of the variable y.

Proof. By the proof of Theorem 4.3.2 we know that

$$e^{-2\pi \mathbf{i}\widetilde{\beta}(|u|)u\cdot y} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$$

with

$$\left\| e^{-2\pi \mathbf{i}\widetilde{\beta}(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim (1+|y|)^{d/(\gamma+1)}.$$

By (4.3.16),

$$e^{-2\pi \mathbf{i}\beta(|u|)u\cdot y} = e^{-2\pi \mathbf{i} a u\cdot y} \cdot e^{-2\pi \mathbf{i}\widetilde{\beta}(|u|)u\cdot y} = M_{-ay} e^{-2\pi \mathbf{i}\widetilde{\beta}(|u|)u\cdot y}.$$

Using the invariance property of $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ with respect to time-frequency shifts in (1.2.17):

$$\begin{aligned}
\left\| e^{-2\pi \mathbf{i}\beta(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} &= \left\| M_{-ay} e^{-2\pi \mathbf{i}\widetilde{\beta}(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\asymp \left\| e^{-2\pi \mathbf{i}\widetilde{\beta}(|u|)u\cdot y} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\leq C(1+|y|)^{d/(\gamma+1)}.
\end{aligned}$$

This concludes the proof. □

We finish this section by using the previous results for the integral kernel $K(x, y)$ to obtain the L^1-boundedness for the corresponding operator A. The cost is a loss of decay, as explained below.

Corollary 4.3.5. *Assume the hypotheses of Corollary 4.3.4 and consider the weight function*

$$v(y) = (1+|y|)^{d/(\gamma+1)}.$$

Then the integral operator A in (4.3.1) *with kernel K in* (4.1.2) *is bounded from $L^1_v(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Proof. By Corollary 4.3.4, we know that the kernel $K(x, y)$ satisfies the estimate in (4.3.17). Let $f \in L^1_v(\mathbb{R}^d)$; using Fubini's Theorem and the estimate in (4.3.17),

$$\begin{aligned}\|Af(x)\|_{L^1} &= \int_{\mathbb{R}^d} |Af(x)|dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K(x,y)f(y)dy \right| dx \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)||f(y)|dydx = \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |K(x,y)|dx \right) dy \\ &\leq \int_{\mathbb{R}^d} |f(y)|C(1+|y|)^{d/(\gamma+1)}dy = C\|f\|_{L^1_v},\end{aligned}$$

as desired. □

Note that in this proposition $d/(\gamma+1) \geq d/2 \geq 1/2$, while in the Proposition 4.3.1 the index $m < 1/2$.

4.4 Continuity in L^2

A natural question is whether the assumptions of Corollary 4.3.4, that give continuity of the operator A on $L^1(\mathbb{R}^d)$ with a loss of decay, guarantee at least continuity of A on $L^2(\mathbb{R}^d)$ without any loss. The answer is negative even in dimension $d = 1$, as shown by the following result.

Proposition 4.4.1. *Let $d = 1$. There exists an operator A as in* (4.3.1), *with β and Φ satisfying the assumptions of Corollary 4.3.4, that is not bounded on $L^2(\mathbb{R})$.*

Proof. Let $h(r)$ be a function such that $\Phi(u) = h(|u|) \in \mathcal{C}_0^\infty(\mathbb{R})$, and $h(0) \neq 0$. For $\gamma \in (0,1)$, set $\beta(u) = u^\gamma$. Finally, take $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(u) = 1$ when $u \in \text{supp}(\Phi)$, and consider the function $\tilde{\beta}(u) = \chi(u)\beta(u)$. Let A be the operator with integral kernel

$$K(x,y) = \int_{\mathbb{R}} h(|u|)e^{-2\pi \mathbf{i}(\tilde{\beta}(|u|)u\cdot y - u\cdot x)}du = \int_{\mathbb{R}} h(|u|)e^{-2\pi \mathbf{i}(\beta(|u|)u\cdot y - u\cdot x)}du.$$

We now show that A is not bounded on $L^2(\mathbb{R})$.

For $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} Af(x) = \int_{\mathbb{R}} K(x,y)f(y)dy &= \int_{\mathbb{R}}\int_{\mathbb{R}} h(|u|)e^{-2\pi\mathbf{i}(\beta(|u|)u\cdot y - u\cdot x)}f(y)dudy \\ &= \int_{\mathbb{R}} h(|u|)e^{2\pi\mathbf{i}u\cdot x}\left(\int_{\mathbb{R}} f(y)e^{-2\pi\mathbf{i}\beta(|u|)u\cdot y}dy\right)du \\ &= \int_{\mathbb{R}} h(|u|)e^{2\pi\mathbf{i}u\cdot x}\mathcal{F}f(\beta(|u|)u)du = \mathcal{F}^{-1}\left[h(|u|)\mathcal{F}f(\beta(|u|)u)\right](x). \end{aligned}$$

Then, by Parseval's Theorem,

$$\|Af\|_2^2 = \|h(|u|)\mathcal{F}f(\beta(|u|)u)\|_2^2 = \int_{\mathbb{R}} |h(|u|)|^2|\mathcal{F}f(\beta(|u|)u)|^2dx.$$

We perform the change of variable

$$\tilde{u} = \beta(|u|)u = |u|^\gamma u = \begin{cases} u^{\gamma+1}, & u \geq 0, \\ -|u|^{\gamma+1}, & u < 0, \end{cases}$$

so that

$$u = \begin{cases} \tilde{u}^{\frac{1}{\gamma+1}}, & \tilde{u} \geq 0, \\ -(-\tilde{u})^{\frac{1}{\gamma+1}}, & \tilde{u} < 0, \end{cases}$$

and $du = \frac{1}{1+\gamma}|\tilde{u}|^{\frac{1}{1+\gamma}-1}d\tilde{u}$. In this way, we obtain

$$\begin{aligned} \|Af\|_2^2 &= \int_{\mathbb{R}} |h(|u|)|^2|\mathcal{F}f(\beta(|u|)u)|^2du \\ &= \frac{1}{1+\gamma}\int_{\mathbb{R}} |\tilde{u}|^{\frac{1}{1+\gamma}-1}|h(|\tilde{u}|^{\frac{1}{1+\gamma}})|^2|\mathcal{F}f(\tilde{u})|^2d\tilde{u}. \end{aligned}$$

Now, the last expression is controlled by $C\|f\|_{L^2}^2$, for a suitable constant $C > 0$ and for every $f \in \mathcal{S}(\mathbb{R})$, if and only if

$$|\tilde{u}|^{-\frac{\gamma}{1+\gamma}}|h(|\tilde{u}|^{\frac{1}{1+\gamma}})|^2 \in L^\infty(\mathbb{R}).$$

But, this fails since $-\gamma/(1+\gamma) < 0$ and $|h(|u|)| \geq \delta > 0$ in a neighborhood of 0, notice that $h(|\tilde{u}|^{\frac{1}{1+\gamma}})$ has compact support. $\square$

We now look for suitable assumptions on the functions Φ and β which guarantee L^2-continuity of the operator A. A successful choice is shown below.

Theorem 4.4.2. *Consider $\Phi \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Let $\beta : (0,\infty) \to \mathbb{R}$ satisfy the following assumptions:*
(i) $\beta \in \mathcal{C}^1((0,\infty))$.
(ii) There exists $\delta > 0$ such that $\beta(r) \geq \delta$, for all $r > 0$.
(iii) There exist $B_1, B_2 > 0$, such that

$$B_1 \leq \frac{d}{dr}(\beta(r)r) \leq B_2, \quad \text{for all } r > 0.$$

Then the integral operator A with kernel K in (4.1.2) *is bounded on $L^2(\mathbb{R}^d)$.*

Proof. We first observe that, since $\Phi \in L^1(\mathbb{R}^d)$, the integral defining the kernel $K(x,y)$ is absolutely convergent and K is well-defined. Let $f \in \mathcal{S}(\mathbb{R}^d)$, using Fubini's Theorem, we can write

$$\begin{aligned} Af(x) &= \int_{\mathbb{R}^d} K(x,y)f(y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi(u)e^{-2\pi \mathbf{i}(\beta(|u|)u\cdot y - u\cdot x)} f(y)dudy \\ &= \int_{\mathbb{R}^d} \Phi(u)e^{2\pi \mathbf{i} u\cdot x}\left(\int_{\mathbb{R}^d} f(y)e^{-2\pi \mathbf{i}\beta(|u|)u\cdot y}dy\right)du \\ &= \int_{\mathbb{R}^d} \Phi(u)e^{2\pi \mathbf{i} u\cdot x}\mathcal{F}f(\beta(|u|)u)du = \mathcal{F}^{-1}\left[\Phi(u)\mathcal{F}f(\beta(|u|)u)\right](x). \end{aligned}$$

Then, by Parseval's Theorem,

$$\|Af\|_2^2 = \|\Phi(u)\mathcal{F}f(\beta(|u|)u)\|_2^2 = \int_{\mathbb{R}^d} |\Phi(u)|^2|\mathcal{F}f(\beta(|u|)u)|^2du.$$

Changing to polar coordinates $u = r\theta$, with $r > 0$ and $\theta \in \mathbb{S}^{d-1}$, we have $du = r^{d-1}drd\theta$ and

$$\begin{aligned} \|Af\|_2^2 &= \int_{\mathbb{R}^d} |\Phi(u)|^2|\mathcal{F}f(\beta(|u|)u)|^2du \\ &= \int_0^\infty \int_{S^{d-1}} |\Phi(r\theta)|^2|\mathcal{F}f(\beta(r)r\theta)|^2 r^{d-1}d\theta dr. \end{aligned}$$

Observe that the function $\varphi(r) := \beta(r)r$ is strictly increasing by assumption (iii). Performing the change of variable $\tilde{r} = \varphi(r)$, as $r \in (0,\infty)$, there exists an $a \geq 0$ such that $\varphi((0,+\infty)) = (a,\infty) \subseteq (0,\infty)$. Moreover, $\varphi(r)$ has an inverse $\varphi^{-1}(\tilde{r}) = r$ such that

$$B_2^{-1} \leq \frac{d}{d\tilde{r}}(\varphi^{-1}(\tilde{r})) \leq B_1^{-1} \qquad \text{for all } \tilde{r} > 0.$$

Further, by assumption (ii),

$$\frac{1}{\delta} \geq \frac{1}{\beta(r)} = \frac{r}{\beta(r)r} = \frac{\varphi^{-1}(\tilde{r})}{\tilde{r}}.$$

Then we can write,

$$\begin{aligned}
\|Af\|_2^2 &= \int_0^\infty \int_{S^{d-1}} |\Phi(r\theta)|^2 |\mathcal{F}f(\beta(r)r\theta)|^2 r^{d-1} d\theta dr \\
&\leq \int_0^\infty \int_{S^{d-1}} |\Phi(\varphi^{-1}(\tilde{r})\theta)|^2 |\mathcal{F}f(\tilde{r}\theta)|^2 (\varphi^{-1}(\tilde{r}))^{d-1} \frac{d}{d\tilde{r}}(\varphi^{-1}(\tilde{r})) d\theta d\tilde{r} \\
&\leq \sup_{\tilde{r},\theta} \left\{ |\Phi(\varphi^{-1}(\tilde{r})\theta)|^2 \left(\frac{\varphi^{-1}(\tilde{r})}{\tilde{r}} \right)^{d-1} \frac{d}{d\tilde{r}}(\varphi^{-1}(\tilde{r})) \right\} \times \\
&\quad \times \int_0^\infty \int_{S^{d-1}} |\mathcal{F}f(\tilde{r}\theta)|^2 (\tilde{r})^{d-1} d\theta d\tilde{r} \leq \|\Phi\|_\infty^2 \left(\frac{1}{\delta}\right)^{d-1} B_1^{-1} \|f\|_2^2.
\end{aligned}$$

This gives $\|Af\|_2 \leq C\|f\|_2$, for every $f \in \mathcal{S}(\mathbb{R}^d)$. By a density argument we obtain the claim for every $f \in L^2(\mathbb{R}^d)$. □

Remark 4.4.3. *The previous proof still works if we change the function β with $-\beta$. Hence, under the assumptions of Theorem 4.4.2 with assumptions (ii) and (iii) replaced by:*
(ii)' There exists $\delta < 0$ such that $\beta(r) \leq \delta$, for all $r > 0$.
(iii)' There exist $B_1, B_2 < 0$, such that

$$B_1 \leq \frac{d}{dr}(\beta(r)r) \leq B_2, \qquad \textit{for all } r > 0;$$

the integral operator A with kernel K in (4.1.2) *is bounded on $L^2(\mathbb{R}^d)$.*

We now exhibit a class of examples of functions $\Phi \in M^1(\mathbb{R}^d)$, hence fulfilling the assumptions of Theorem 4.3.2 and related corollaries, as well as those of Theorem 4.4.2, which are of special interest in the study of Boltzmann equation, cf. [AL10].

Example 4.4.4. *Consider the function*

$$\Phi(u) = \frac{|u|}{(1+|u|^2)^m}, \quad \textit{for } m > \frac{d+1}{2}.$$

Then $\Phi \in M^1(\mathbb{R}^d)$ *(Observe that* $\Phi(u) = h(|u|)$*, in this situation).*

Proof. We consider a function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, such that $\chi(u) = 1$ when $|u| \leq 1/2$ and $\chi(u) = 0$ when $|u| \geq 1$. We write

$$\Phi(u) = \Phi(u)\chi(u) + \Phi(u)(1-\chi(u))$$

and show that

$$\Phi(u)\chi(u) \in M^1(\mathbb{R}^d) \tag{4.4.1}$$

and

$$\Phi(u)(1-\chi(u)) \in M^1(\mathbb{R}^d). \tag{4.4.2}$$

To prove (4.4.1), we choose another cut-off function $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\widetilde{\chi}(u) = 1 \quad \text{for } u \in \operatorname{supp}\chi;$$

then $\widetilde{\chi}\cdot\chi = \chi$. Consider now the function $h(u)=|u|$, which is in $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$ and positively homogeneous of degree 1 and set $f = h\chi$. Lemma 4.2.3 gives, for $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$|V_\psi f(x,\xi)| \leq C(1+|\xi|)^{-(d+1)}$$

hence, by (1.2.2),

$$\begin{aligned}
\||u|\chi(u)\|_{W(\mathcal{F}L^1,L^\infty)} &= \|f\|_{W(\mathcal{F}L^1,L^\infty)} = \|\|fT_x\overline{\psi}\|_{\mathcal{F}L^1}\|_{L^\infty} \\
&= \left\|\left\|\mathcal{F}(f\overline{T_x\psi})\right\|_{L^1}\right\|_{L^\infty} = \|\|V_\psi f(x,\cdot)\|_{L^1}\|_{L^\infty} \\
&= \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|V_\psi f(x,\xi)|d\xi < \infty,
\end{aligned}$$

that is $|u|\chi \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$. Since

$$\frac{\widetilde{\chi}(u)}{(1+|u|^2)^m} \in \mathcal{S}(\mathbb{R}^d) \subseteq M^1(\mathbb{R}^d).$$

we can write

$$\Phi(u)\chi(u) = |u|\chi(u) \cdot \frac{\widetilde{\chi}(u)}{(1+|u|^2)^m} \in M^1(\mathbb{R}^d),$$

$$V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i \xi y} f(y)\overline{g(y-x)}\, dy,$$

i.e. the Fourier transform of $f\overline{T_x g}$.

Finally, to show (4.4.2), we observe that $\Phi(u)(1-\chi(u)) = 0$ for $|u| \leq 1/2$, hence the singularity at the origin is removed and $\Phi(u)(1-\chi(u)) \in W^{k,1}(\mathbb{R}^d)$ for all $k \in \mathbb{N}$, provided that $2m-1 > d$. We then choose $k > d$ and apply the inclusion relations between the Potential Sobolev space $W^{k,1}(\mathbb{R}^d)$ and the Feichtinger's algebra $M^1(\mathbb{R}^d)$ in Lemma 1.2.9, which gives (4.4.2). $\square$

4.5 Conclusion

Summing up, we have found results about boundedness of a particular type of Fourier integral operators with Hölder continuous phase. We have seen that the sufficient conditions for boundedness on L^1 do not work for L^2. So we looked for some other sufficient conditions for boundedness on L^2. These results are included in [CNP18].

Chapter 5

Unimodular Fourier multipliers

5.1 Introduction

The aim of this chapter is to look for conditions for continuity of unimodular Fourier multipliers on modulation spaces. The **unimodular Fourier multipliers** are formally defined by

$$e^{\mathbf{i}\mu(D)}f(x) := \int_{\mathbb{R}^d} e^{2\pi\mathbf{i}x\cdot\xi} e^{\mathbf{i}\mu(\xi)} \hat{f}(\xi)\, d\xi, \tag{5.1.1}$$

with real-valued μ. These operators can be seen as a PSDO,

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi\mathbf{i}x\cdot\eta} \sigma(x,\eta)\hat{f}(\eta)d\eta,$$

with symbol $\sigma(x,\eta) = e^{\mathbf{i}\mu(\eta)}$, or as a FIO

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi\mathbf{i}\Phi(x,\eta)} \sigma(x,\eta)\hat{f}(\eta)d\eta,$$

with phase $\Phi(x,\eta) = x\cdot\eta + \frac{1}{2\pi}\mu(\eta)$ and constant symbol. Fourier multipliers represent one of the main research fields in harmonic analysis, where a number of challenging problems remains open [Ste93]. The connections with other branches of pure and applied mathematics are uncountable (combinatorics, PDEs, signal processing, functional calculus, etc.).

The prototype is given by $\mu(\xi) = |\xi|^2$, which satisfies the hypothesis in Chapter 3. In that case the operator $e^{\mathbf{i}\mu(D)}$ is the propagator for the free Schrödinger equation, and similarly for other constant coefficient equations. Hence it is of great interest to study the continuity of such operators on several functions spaces arising in PDEs. Whereas such operators represent unitary transformations of $L^2(\mathbb{R}^d)$, their continuity on $L^p(\mathbb{R}^d)$ for $p \neq 2$ in general fails. Recently a number of works addressed the problem of the continuity in other function spaces. Among those, the more convenient spaces, at least in the case of the Schrödinger model, turned out to be the modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, widely used in time-frequency analysis [Fei83, Grö01]. The basic reason is that the Schrödinger propagator is sparse with respect to Gabor frames [CNR09b].

It is known (see e.g. [Tof04, Proposition 1.5] and [BGOR07]) that the Schrödinger propagator ($\mu(\xi) = |\xi|^2$ in (5.1.1)) is bounded $M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$, for every $1 \leq p, q \leq \infty$. This result motivated the study of the continuity of more general unimodular Fourier multipliers on modulation spaces. The recent bibliography in this connection is quite large; see e.g. [BGOR07, BO09, CFS12, CFSZ13, CT09, CN09, CS14, DDS13, GWZ17, KKI14, MNR+09, Son14, ZCFG16, ZCFG15, ZCG14]. In short, it turns out that, for unbounded (smooth enough) phases, the properties which play a key role are:

Growth and oscillations of the second derivatives $\partial^\gamma \mu$, $|\gamma| = 2$.

To put our results in context, let us just recall three basic facts.

(a) *No growth, mild oscillations* [BGOR07, Theorem 11]. *Suppose that*

$$|\partial^\gamma \mu(\xi)| \leq C, \quad \textit{for } \xi \in \mathbb{R}^d,\ 2 \leq |\gamma| \leq 2(\lfloor d/2 \rfloor + 1).$$

Then $e^{\mathbf{i}\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ is bounded for every $1 \leq p, q \leq \infty$.

This result generalizes the case of the Schrödinger propagator, where the second derivatives of μ are in fact constants.

(b) *No growth, mild oscillations* [CT09, Lemma 2.2]. *Suppose that*

$$\partial^\gamma \mu \in M^{\infty,1}(\mathbb{R}^d), \quad \textit{for } |\gamma| = 2.$$

Then $e^{\mathbf{i}\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is bounded for every* $1 \le p, q \le \infty$.

Actually, [CT09, Lemma 2.2] provides a partial but key result in this connection, from which it is easy to deduce that the symbol $\sigma(\xi) = e^{\mathbf{i}\mu(\xi)}$ is then in the Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)$, which is sufficient to conclude (see also [Bou97, CT07, TCG10]). The result in (b) is also a particular case of [CNR15c, Theorem 2.3].

Observe that the result in (b) improves that in (a), because of the embedding $C^{d+1}(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$ ([Grö01, Theorem 14.5.3]). We also notice that $M^{\infty,1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, so that here the second derivatives of μ do not grow at infinity, but they could oscillate, say, as $\cos|\xi|^\alpha$, with $0 < \alpha \le 1$ (cf. [BGOR07, Corollary 15]).

(c) *Growth at infinity, mild oscillations* [MNR$^+$09, Theorem 1.1]. *Let* $\alpha \ge 2$, *and suppose that*

$$|\partial^\gamma \mu(\xi)| \le C\langle \xi \rangle^{\alpha-2}, \quad \textit{for } 2 \le |\gamma| \le \lfloor d/2 \rfloor + 3.$$

Then $e^{\mathbf{i}\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is bounded for every* $1 \le p, q \le \infty$ *and* $\delta \ge d(\alpha - 2)|1/p - 1/2|$.

Here $M^{p,q}_\delta(\mathbb{R}^d) = M^{p,q}_{1\otimes\langle\cdot\rangle^\delta}$, where $(1 \otimes \langle \cdot \rangle^\delta)(x, \omega) = \langle \omega \rangle^\delta$, that is a modulation space weighted in frequency, so that we have in fact a loss of derivatives, which is proved to be sharp.

Now, it was proved in [BGOR07, Lemma 8] that, more generally, the operator $e^{\mathbf{i}\mu(D)}$ is bounded on all $M^{p,q}(\mathbb{R}^d)$ for every $1 \le p, q \le \infty$ if its symbol $e^{\mathbf{i}\mu(\xi)}$ belongs to the Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ [Fei81a], whose norm is defined as

$$\|f\|_{W(\mathcal{F}L^1,L^\infty)} = \sup_{x\in\mathbb{R}^d} \|g(\cdot - x)f\|_{\mathcal{F}L^1}$$

where $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is an arbitrary window. This suggests to look at conditions on $\mu(\xi)$ in terms of this space, rather than modulation spaces. Here is our first result in this direction.

Theorem 5.2.4. *(No growth, strong oscillations). Let $\mu \in C^2(\mathbb{R}^d)$, real-valued, satisfying*

$$\partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \quad \textit{for } |\gamma| = 2.$$

Then

$$e^{\mathbf{i}\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q \leq \infty$.

Observe that $M^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ so that this result improves that in (b) above. Here the second derivatives of μ are still bounded, but they are allowed to oscillate, say, as $\cos |\xi|^2$ (cf. [BGOR07, Theorem 14]). This result is strongly inspired by [CT09, Lemma 2.2] and in fact the proof is similar. However, our main result deals with the case of possibly unbounded second derivatives, as stated in the following theorem.

Theorem 5.3.7. *(Growth at infinity, strong oscillations). Let $\alpha \geq 2$. Let $\mu \in C^2(\mathbb{R}^d)$, real-valued and such that*

$$\langle \xi \rangle^{2-\alpha} \partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \quad \textit{for } |\gamma| = 2.$$

Then

$$e^{\mathbf{i}\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q \leq \infty$ and

$$\delta \geq d(\alpha - 2) \left| \frac{1}{p} - \frac{1}{2} \right|.$$

The above threshold for δ agrees with that in (c), and also with the examples in [BGOR07, Theorem 16], where even stronger oscillations were considered, but only for model cases.

Theorem 5.2.4 is of course a particular case of Theorem 5.3.7 and will be used as a step in the proof of the latter.

In short the chapter is organized as follows. Section 5.2 is devoted to the proof of Theorem 5.2.4, whereas in Section 5.3 we prove Theorem 5.3.7.

5.2 No growth, strong oscillations

This section is devoted to the proof of Theorem 5.2.4. We begin with a preliminary result which is strongly inspired by [CT09, Lemmas 2.1 and 2.2], where a similar investigation is carried on in the framework of modulation spaces (as opposite to the Wiener amalgam spaces considered here).

Lemma 5.2.1. *Let $f \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ and $\chi \in C_0^\infty(B)$, where B is an open ball with center at the origin. Let*

$$g_{x_0}(x) = \chi(x - x_0) \int_0^1 (1-t) f(t(x - x_0) + x_0)\, dt,$$

for some $x_0 \in \mathbb{R}^d$.

Then $g_{x_0} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$, and for some constant C independent of x_0 and f we have

$$\|g_{x_0}\|_{W(\mathcal{F}L^1,L^\infty)} \leq C \|f\|_{W(\mathcal{F}L^1,L^\infty)}.$$

Proof. Using Lemma 1.2.15, Proposition 1.2.16 and Proposition 1.2.17 we have

$$\begin{aligned}
\|g_{x_0}\|_{W(\mathcal{F}L^1,L^\infty)} &= \left\| \chi(x - x_0) \int_0^1 (1-t) f(t(x - x_0) + x_0) dt \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \|\chi(x - x_0)\|_{W(\mathcal{F}L^1,L^\infty)} \left\| \int_0^1 (1-t) f(tx + (1-t)x_0) dt \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\leq \|\chi(x - x_0)\|_{W(\mathcal{F}L^1,L^\infty)} \int_0^1 (1-t) \left\| f(tx + (1-t)x_0) \right\|_{W(\mathcal{F}L^1,L^\infty)} dt \\
&= \|\chi\|_{W(\mathcal{F}L^1,L^\infty)} \int_0^1 (1-t) \left\| f(tx) \right\|_{W(\mathcal{F}L^1,L^\infty)} dt \\
&\lesssim \|\chi\|_{W(\mathcal{F}L^1,L^\infty)} \int_0^1 (1-t) \|f\|_{W(\mathcal{F}L^1,L^\infty)} dt \\
&\lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)} .
\end{aligned}$$

□

Lemma 5.2.2. *Assume that $B \subset \mathbb{R}^n$ is an open ball, $\mu \in C^2(\mathbb{R}^d)$ is real-valued and satisfies $\partial^\gamma \mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ for all multi-indices γ with $|\gamma| = 2$ and that $f \in M^1(\mathbb{R}^d) \cap \mathcal{E}'(B)$. Then $fe^{\mathbf{i}\mu} \in M^1(\mathbb{R}^d)$ and for some constants C, C' which only depend on d and the radius of the ball B we have*

$$\|fe^{i\mu}\|_{M^1} \le C\|f\|_{M^1} \exp\Big(C' \sum_{|\gamma|=2} \|\partial^\gamma \mu\|_{W(\mathcal{F}L^1, L^\infty)}\Big).$$

Proof. We may assume that B is the unit ball centered at the origin. By Taylor expansion it follows that $\mu = \psi_1 + \psi_2$, where

$$\psi_1(x) = \mu(0) + \langle \nabla\mu(0), x\rangle, \quad \psi_2(x) = \sum_{|\gamma|=2} \frac{2}{\gamma!} \int_0^1 (1-t)\partial^\gamma \mu(tx) dt\, x^\gamma.$$

Since modulations do not affect the modulation space norms we have

$$\|fe^{\mathbf{i}\psi_1}\|_{M^1} = \|f\|_{M^1}.$$

Furthermore, if $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfies $\chi(x) = 1$ on B, then it follows from the previous lemma that, for some constant $C_1 > 0$,

$$\|\chi\psi_2\|_{W(\mathcal{F}L^1, L^\infty)} \le C_1 \sum_{|\gamma|=2} \|\partial^\gamma \mu\|_{W(\mathcal{F}L^1, L^\infty)}.$$

Hence, by Proposition 1.2.16, for some $C_2 \ge 1$ we have

$$\begin{aligned}
\|e^{\mathbf{i}\chi\psi_2}\|_{W(\mathcal{F}L^1, L^\infty)} &= \Big\|\sum_{n=0}^\infty \frac{(\mathbf{i}\chi\psi_2)^n}{n!}\Big\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\le \sum_{n=0}^\infty \frac{C_2^{n-1}}{n!} \|\chi\psi_2\|^n_{W(\mathcal{F}L^1, L^\infty)} \\
&\le \exp\Big(C_2\|\chi\psi_2\|_{W(\mathcal{F}L^1, L^\infty)}\Big) \\
&\le \exp\Big(C_1C_2 \sum_{|\gamma|=2} \|\partial^\gamma \mu\|_{W(\mathcal{F}L^1, L^\infty)}\Big).
\end{aligned}$$

Using Proposition 1.2.16 again, this gives

$$\begin{aligned}\|fe^{\mathbf{i}\mu}\|_{M^1} &= \|fe^{\mathbf{i}\psi_1}e^{\mathbf{i}\chi\psi_2}\|_{M^1} \lesssim \|fe^{\mathbf{i}\psi_1}\|_{M^1}\|e^{\mathbf{i}\chi\psi_2}\|_{W(\mathcal{F}L^1,L^\infty)} \\ &\leq C\|f\|_{M^1}\exp\Big(C'\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\Big).\end{aligned}$$

□

We recall a known result (see e.g. [BGOR07, Lemma 8]), needed in the proof of Theorem 5.2.4.

Lemma 5.2.3. *Let $\sigma \in W(\mathcal{F}L^1, L^\infty)$. Then,*

$$\sigma(D) : M^{p,q} \to M^{p,q}$$

is bounded, for every $1 \leq p, q \leq \infty$.

Proof. We can write $\sigma(D) = \mathcal{F}^{-1}\circ A_\sigma \circ \mathcal{F}$, where $A_\sigma f(\xi) = \sigma(\xi)f(\xi)$. Using Proposition 1.2.16 we have

$$\begin{aligned}\|A_\sigma f\|_{W(\mathcal{F}L^p,L^q)} &= \|\sigma f\|_{W(\mathcal{F}L^p,L^q)} \\ &\lesssim \|\sigma\|_{W(\mathcal{F}L^1,L^\infty)}\|f\|_{W(\mathcal{F}L^p,L^q)},\end{aligned}$$

so that $A_\sigma : W(\mathcal{F}L^p, L^q) \to W(\mathcal{F}L^p, L^q)$ is bounded, for every $1 \leq p, q \leq \infty$. Hence, since the Fourier transform establishes an isomorphism $\mathcal{F} : M^{p,q} \to W(\mathcal{F}L^p, L^q)$, we see that $\sigma(D) : M^{p,q} \to M^{p,q}$ is bounded too. □

Let us now prove the Theorem 5.2.4.

Theorem 5.2.4. *(No growth, strong oscillations). Let $\mu \in C^2(\mathbb{R}^d)$, real-valued, satisfying*

$$\partial^\gamma\mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \quad \textit{for } |\gamma| = 2.$$

Then

$$e^{\mathbf{i}\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q \leq \infty$.

Proof. Let us first show that $e^{\mathbf{i}\mu(x)} \in W(\mathcal{F}L^1, L^\infty)$. We know that there exists $\chi \in C_0^\infty(\mathbb{R}^d)$ (cf. [Fei81a, Fei83]) such that

$$\|e^{\mathbf{i}\mu(x)}\|_{W(\mathcal{F}L^1,L^\infty)} = \sup_{k\in\mathbb{Z}^d}\{\|\chi(x-k)e^{\mathbf{i}\mu(x)}\|_{\mathcal{F}L^1}\} \asymp \sup_{k\in\mathbb{Z}^d}\{\|\chi(x-k)e^{\mathbf{i}\mu(x)}\|_{M^1}\},$$

where the last equivalence follows from the fact that for functions supported in a ball the $\mathcal{F}L^1$ and M^1 norms are equivalent, with constants depending only on the radius of the ball.

Hence, using Lemma 5.2.2 we can continue our estimate as

$$\begin{aligned} &\leq \sup_{k\in\mathbb{Z}^d}\left\{C'\|\chi(x-k)\|_{M^1}\exp\left(C\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\right)\right\} \\ &= C\|\chi\|_{M^1}\exp\left(C'\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\right). \end{aligned}$$

Hence $e^{\mathbf{i}\mu(x)} \in W(\mathcal{F}L^1, L^\infty)$ and by Lemma 5.2.3 we deduce that

$$e^{i\mu(D)} : M^{p,q} \to M^{p,q}$$

is bounded, for every $1 \leq p, q \leq \infty$. □

5.3 Growth at infinity, strong oscillations

In this section we prove Theorem 5.3.7. To this end we begin with the following auxiliary results.

Lemma 5.3.1. *Let $\alpha \geq 2$. Let $\mu(\xi)$ be a real-valued C^2 function, satisfying*

$$\langle\xi\rangle^{2-\alpha}\partial^\gamma\mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \text{ for } |\gamma| = 2.$$

Then,

(i) $\langle\xi\rangle^{-\alpha}\mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$,

(ii) $\langle\xi\rangle^{1-\alpha}\partial^\gamma\mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ *for* $|\gamma| = 1$.

Proof. To prove (i), consider a Taylor expansion

$$\mu(\xi) = \mu(0) + \langle \nabla\mu(0), \xi\rangle + \sum_{|\gamma|=2} \frac{2}{\gamma!} \int_0^1 (1-t)\partial^\gamma \mu(t\xi)dt\, \xi^\gamma.$$

Hence

$$\langle\xi\rangle^{-\alpha}\mu(\xi) = \mu(0)\langle\xi\rangle^{-\alpha} + \langle \nabla\mu(0), \xi\rangle\langle\xi\rangle^{-\alpha} + \sum_{|\gamma|=2} \frac{2}{\gamma!} \int_0^1 (1-t)\partial^\gamma \mu(t\xi)dt\, \xi^\gamma\langle\xi\rangle^{-\alpha}, \quad (5.3.1)$$

where

$$\mu(0)\langle\xi\rangle^{-\alpha} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d), \quad \langle \nabla\mu(0), \xi\rangle\langle\xi\rangle^{-\alpha} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d),$$

because $\alpha \geq 2$. Here we used the fact that the functions $\langle\xi\rangle^{-\alpha}$ and $\xi_j\langle\xi\rangle^{-\alpha}$ are bounded together with their derivatives of every order, so that they belong to $M^{\infty,1}(\mathbb{R}^d)$ ([Grö01, Theorem 14.5.3]) and hence to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ as well.

Let us show that the last amount in (5.3.1) belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ too. We have

$$\begin{aligned}
&\left\| \int_0^1 (1-t)\partial^\gamma \mu(t\xi)dt\xi^\gamma\langle\xi\rangle^{-\alpha} \right\|_{W(\mathcal{F}L^1, L^\infty)} \\
&\leq \int_0^1 \left\| (1-t)\partial^\gamma \mu(t\xi)\xi^\gamma\langle\xi\rangle^{-\alpha} \right\|_{W(\mathcal{F}L^1, L^\infty)} dt \\
&= \int_0^1 \left\| (1-t)\partial^\gamma \mu(t\xi)\langle t\xi\rangle^{2-\alpha}\langle t\xi\rangle^{-2+\alpha}\, \xi^\gamma\langle\xi\rangle^{-\alpha} \right\|_{W(\mathcal{F}L^1, L^\infty)} dt \\
&\lesssim \int_0^1 \left\| (1-t)\langle t\rangle^{-2+\alpha}\partial^\gamma \mu(t\xi)\langle t\xi\rangle^{2-\alpha}\, \xi^\gamma\langle\xi\rangle^{-\alpha}\langle\xi\rangle^{-2+\alpha} \right\|_{W(\mathcal{F}L^1, L^\infty)} dt.
\end{aligned}$$

Using Proposition 1.2.16 and Corollary 1.2.15 we can continue the above estimate as

$$\begin{aligned}
&\lesssim \int_0^1 (1-t)\langle t\rangle^{-2+\alpha} dt \left\|\partial^\gamma \mu(t\xi)\langle t\xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \left\|\xi^\gamma\langle\xi\rangle^{-2}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \int_0^1 (1-t)\langle t\rangle^{-2+\alpha} dt \left\|\partial^\gamma \mu(\xi)\langle \xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \left\|\xi^\gamma\langle\xi\rangle^{-2}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \left\|\partial^\gamma \mu(\xi)\langle \xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \left\|\xi^\gamma\langle\xi\rangle^{-2}\right\|_{W(\mathcal{F}L^1,L^\infty)}.
\end{aligned}$$

This concludes the proof of (i) because, arguing as above, we have $\xi^\gamma\langle\xi\rangle^{-2} \in M^{\infty,1} \subset W(\mathcal{F}L^1, L^\infty)$, whereas $\partial^\gamma\mu(\xi)\langle\xi\rangle^{2-\alpha} \in W(\mathcal{F}L^1, L^\infty)$ by assumption.

To prove (ii), consider the Taylor expansion of $\partial^\gamma\mu$, for $|\gamma| = 1$

$$\partial^\gamma\mu(\xi) = \partial^\gamma\mu(0) + \sum_{|\beta|=1} \int_0^1 \partial^{\gamma+\beta}\mu(t\xi)dt\,\xi^\beta,$$

so that

$$\langle\xi\rangle^{1-\alpha}\partial^\gamma\mu(\xi) = \partial^\gamma\mu(0)\langle\xi\rangle^{1-\alpha} + \sum_{|\beta|=1} \int_0^1 \partial^{\gamma+\beta}\mu(t\xi)dt\,\xi^\beta\langle\xi\rangle^{1-\alpha}.$$

Now $\partial^\gamma\mu(0)\langle\xi\rangle^{1-\alpha} \in W(\mathcal{F}L^1, L^\infty)$, because $\alpha \geq 2$, and arguing as above

$$\begin{aligned}
&\left\|\int_0^1 \partial^{\gamma+\beta}\mu(t\xi)dt\,\xi^\beta\langle\xi\rangle^{1-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&= \left\|\int_0^1 \partial^{\gamma+\beta}\mu(t\xi)\langle t\xi\rangle^{2-\alpha}\langle t\xi\rangle^{-2+\alpha}dt\,\xi^\beta\langle\xi\rangle^{1-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \left\|\partial^{\gamma+\beta}\mu(\xi)\langle\xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \left\|\xi^\beta\langle\xi\rangle^{-1}\right\|_{W(\mathcal{F}L^1,L^\infty)},
\end{aligned}$$

where $\xi^\beta\langle\xi\rangle^{-1} \in M^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ because $|\beta| = 1$, and moreover $\partial^{\,\gamma+\beta}\mu(\xi)\,\langle\xi\rangle^{2-\alpha} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ by assumption, because $|\gamma+\beta| = 2$. $\square$

Here is the basic complex interpolation result (see e.g. [Fei81a, Fei83, FG89] and [WH07, Theorem 2.3] for a direct proof).

Proposition 5.3.2. *Let* $0 < \theta < 1$, $p_j, q_j \in [1, \infty]$ *and* $\delta_j \in \mathbb{R}$, $j = 1, 2$. *Set*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \delta = (1-\theta)\delta_1 + \theta\delta_2.$$

Then

$$(M^{p_1,q_1}_{\delta_1}(\mathbb{R}^d), M^{p_2,q_2}_{\delta_2}(\mathbb{R}^d))_{[\theta]} = M^{p,q}_{\delta}(\mathbb{R}^d).$$

We observe that, by complex interpolation of weighted modulation spaces it suffices to prove the conclusion of Theorem 5.3.7 when (p, q) is one of the four vertices of the interpolation square, $(1, 1)$, $(1, \infty)$, $(\infty, 1)$, (∞, ∞), with $\delta = d(\alpha - 2)/2$, as well as for the points $(2, 1)$, $(2, \infty)$ with $\delta = 0$. To this end, we reduce matters to the case of unweighted modulation spaces by means of the following lemma.

Lemma 5.3.3. *A multiplier* $\sigma(D)$ *is bounded from* $M^{p,q}_{\delta}(\mathbb{R}^d)$ *to* $M^{p,q}(\mathbb{R}^d)$ *if and only if the multiplier* $\sigma(D)\langle D\rangle^{-\delta}$ *is bounded on* $M^{p,q}(\mathbb{R}^d)$.

Proof. We know e.g. from [Tof04, Theorem 2.2, Corollary 2.3] that $\langle D\rangle^t$ defines an isomorphism $M^{p,q}_s(\mathbb{R}^d) \to M^{p,q}_{s-t}(\mathbb{R}^d)$ for every $s, t \in \mathbb{R}$, so that the conclusion is immediate. □

Therefore we may work with the operator

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} x\cdot\xi} e^{\mathbf{i}\mu(\xi)} \langle\xi\rangle^{-\delta} \hat{f}(\xi) d\xi.$$

We have to prove that T is bounded on $M^{1,1}$, $M^{1,\infty}$, $M^{\infty,1}$, $M^{\infty,\infty}$ for $\delta = \frac{d(\alpha-2)}{2}$, and on $M^{2,1}$ and $M^{2,\infty}$ for $\delta = 0$.

5.3.1 Boundedness on $M^{1,1}$ and $M^{\infty,1}$ for $\delta = \frac{d(\alpha-2)}{2}$

To prove this boundedness we need the following lemma (cf. [TCG10, Proposition 1.4] and [CNR09a, ST07]).

Lemma 5.3.4. *Let χ be a smooth function supported on $B_0^{-1} \le |\xi| \le B_0$ for some $B_0 > 0$. Then, for $1 \le p \le \infty$,*

$$\sum_{j=1}^{\infty} \|\chi(2^{-j}D)f\|_{M^{p,1}} \le C\|f\|_{M^{p,1}}.$$

Proof. We will use the following characterization of the $M^{p,q}$ norm, [Tri83]: let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\varphi(\xi) \ge 0$, $\sum_{m\in\mathbb{Z}^d} \varphi(\xi - m) = 1$, for all $\xi \in \mathbb{R}^d$. Then

$$\|f\|_{M^{p,q}} \asymp \Big(\sum_{m\in\mathbb{Z}^d} \|\varphi(D-m)f\|_{L^p}^q \Big)^{1/q}.$$

Hence it turns out

$$\begin{aligned}
\sum_{j=1}^{\infty} \|\chi(2^{-j}D)f\|_{M^{p,1}} &\asymp \sum_{j=1}^{\infty} \sum_{m\in\mathbb{Z}^d} \|\varphi(D-m)\chi(2^{-j}D)f\|_{L^p} \\
&= \sum_{m\in\mathbb{Z}^d} \sum_{j=1}^{\infty} \|\varphi(D-m)\chi(2^{-j}D)f\|_{L^p} \\
&= \sum_{m\in\mathbb{Z}^d} \sum_{j=1}^{\infty} \|\chi(2^{-j}D)\varphi(D-m)f\|_{L^p}.
\end{aligned}$$

Now, the number of indices $j \ge 1$ for which $\operatorname{supp} \chi(2^{-j}\cdot) \cap \operatorname{supp} \varphi(\cdot - m) \ne 0$ is finite for every m, and even uniformly bounded with respect to m. Hence the last expression is

$$\lesssim \sum_{m\in\mathbb{Z}^d} \sup_{j\ge 1} \|\chi(2^{-j}D)\varphi(D-m)f\|_{L^p}.$$

Since the operators $\chi(2^{-j}D)$ are uniformly bounded on L^p we can continue the estimate as

$$\lesssim \sum_{m\in\mathbb{Z}^d} \|\varphi(D-m)f\|_{L^p} \asymp \|f\|_{M^{p,1}}.$$

$\square$

Consider now a Littlewood-Paley decomposition of the frequency domain. Namely, fix a smooth function ψ_0 such that $\psi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\psi_0(\xi) = 0$ for $|\xi| \geq 2$. Set $\psi(\xi) = \psi_0(\xi) - \psi_0(2\xi)$. Then $\psi_j(\xi) := \psi(2^{-j}\xi)$ for $j \geq 1$ is supported where $2^{j-1} \leq |\xi| \leq 2^{j+1}$. We can write

$$T = T^{(0)} + \sum_{j=1}^{\infty} T^{(j)} \tag{5.3.2}$$

where $T^{(j)}$ is the Fourier multiplier with symbol $\sigma_j(\xi) := e^{\mathbf{i}\mu(\xi)}\psi_j(\xi)\langle\xi\rangle^{-\delta}$, $j \geq 0$.

Now, $T^{(0)}$ is bounded on $M^{p,q}$ for every $1 \leq p, q \leq \infty$ as a consequence of Lemma 5.2.3, because $\sigma_0 \in M^1 \subset W(FL^1, L^\infty)$ by Lemma 5.2.2.

Consider now the above sum over $j \geq 1$. Let

$$\lambda_j = 2^{-\frac{\alpha-2}{2}j},$$

and consider the operators $\tilde{T}^{(j)}$ defined by

$$\begin{aligned}
\tilde{T}^{(j)} f(x) &= (U_{\lambda_j^{-1}} T^{(j)} U_{\lambda_j}) f(x) = (T^{(j)} U_{\lambda_j} f)(\lambda_j^{-1} x) \\
&= \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} \lambda_j^{-1} x\xi} e^{\mathbf{i}\mu(\xi)} \psi_j(\xi) \langle\xi\rangle^{-\delta} (U_{\lambda_j} f)\widehat{\ }(\xi) d\xi \\
&= \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} \lambda_j^{-1} x\xi} e^{\mathbf{i}\mu(\xi)} \psi_j(\xi) \langle\xi\rangle^{-\delta} \hat{f}(\lambda_j^{-1}\xi) \lambda_j^{-d} d\xi \\
&= \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} x\xi} e^{\mathbf{i}\mu(\lambda_j\xi)} \psi_j(\lambda_j\xi) \langle\lambda_j\xi\rangle^{-\delta} \hat{f}(\xi) d\xi.
\end{aligned}$$

Then we also have the next relationship,

$$T^{(j)} = U_{\lambda_j} \tilde{T}^{(j)} U_{\lambda_j^{-1}}. \tag{5.3.3}$$

Let $\chi_j(\xi) := \chi(2^{-j}\xi)$ with $\chi \in C_0^\infty(\mathbb{R}^d)$ supported where $\frac{1}{4} \leq |\xi| \leq 4$ and $\chi(\xi) = 1$ on the support of ψ, so that $\chi_j(\xi) = 1$ on the support of ψ_j. We can therefore write

$$\tilde{T}^{(j)} f(x) = \int_{\mathbb{R}^d} e^{2\pi \mathbf{i} x\xi} e^{\mathbf{i}\chi_j(\lambda_j\xi)\mu(\lambda_j\xi)} \psi_j(\lambda_j\xi) \langle\lambda_j\xi\rangle^{-\delta} \hat{f}(\xi) d\xi,$$

hence

$$\tilde{T}^{(j)} = A_j B_j, \tag{5.3.4}$$

where

$$A_j = e^{\mathbf{i}(\chi_j\mu)(\lambda_j D)}, \qquad B_j = \psi_j(\lambda_j D)\langle\lambda_j D\rangle^{-\delta}.$$

Taking into account that $\psi_j(\lambda_j\xi)$ is supported on $2^{j-1} \le |\lambda_j\xi| \le 2^{j+1}$, for $\alpha \in \mathbb{Z}^d_+$ we have $C_\alpha \ge 0$ such that,

$$\begin{cases} |\partial^\alpha(\psi_j(\lambda_j\xi))| \le C_\alpha, & \text{for } 2^{j-1} \le |\lambda_j\xi| \le 2^{j+1}, \\ |\partial^\alpha(\psi_j(\lambda_j\xi))| = 0, & \text{in other case.} \end{cases}$$

And on the support of $\psi_j(\lambda_j\xi)$ we have $\lambda_j|\xi| \asymp 2^j$. As $\delta = d(\alpha-2)/2$, we have the following estimate for $\gamma \in \mathbb{Z}^d_+$:

$$\begin{aligned} |\partial^\gamma(\psi_j(\lambda_j\xi)\langle\lambda_j\xi\rangle^{-\delta})| &= \left| \sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} \partial^\alpha(\psi_j(\lambda_j\xi))\partial^\beta(\langle\lambda_j\xi\rangle^{-\delta}) \right| \\ &\le \sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} |\partial^\alpha(\psi_j(\lambda_j\xi))| \left(\langle\lambda_j\xi\rangle^{-\delta-\beta}\right) \\ &\le \sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} C_\alpha(\langle\lambda_j\xi\rangle^{-\delta}) \lesssim (\langle\lambda_j\xi\rangle^{-\delta}) \\ &\lesssim 2^{-\frac{d(\alpha-2)}{2}j}, \qquad \text{for all } \gamma \in \mathbb{Z}^d_+. \end{aligned}$$

Then, by the classical boundedness results of pseudodifferential operators on modulation spaces (see e.g. [Grö01, Theorems 14.5.2, 14.5.2]) we have

$$\|B_j\|_{M^{p,q}\to M^{p,q}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j}, \tag{5.3.5}$$

for every $1 \le p, q \le \infty$.

Let us now prove that

$$\|A_j\|_{M^{p,q}\to M^{p,q}} \lesssim 1, \tag{5.3.6}$$

for all $j \ge 1$ and for every $1 \le p, q \le \infty$.

Using Theorem 5.2.4 it is sufficient to check that

$$\|\partial^\gamma[\chi_j(\lambda_j\xi)\mu(\lambda_j\xi)]\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim 1,$$

for $|\gamma| = 2$ and all $j \geq 1$ (actually we are using the fact that the operator norm of the multiplier in Theorem 5.2.4 is bounded when $\partial^\gamma\mu$, $|\gamma| = 2$, belong to a bounded subset of $W(\mathcal{F}L^1, L^\infty)$).

For $|\gamma| = 2$, we have

$$\partial^\gamma[\chi_j(\lambda_j\xi)\mu(\lambda_j\xi)] = \lambda_j^2\partial^\gamma[\chi_j\mu](\lambda_j\xi),$$

and by Leibniz' formula it is enough to prove that

$$\lambda_j^2\|(\partial^\gamma\chi_j)\mu\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim 1 \qquad |\gamma| = 2 \tag{5.3.7}$$

$$\lambda_j^2\|\partial^\gamma\chi_j\partial^\beta\mu\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim 1 \qquad |\gamma| = |\beta| = 1 \tag{5.3.8}$$

$$\lambda_j^2\|\chi_j\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim 1 \qquad |\gamma| = 2. \tag{5.3.9}$$

First, let us prove (5.3.7). Using Lemma 5.3.1 (i), Proposition 1.2.16 and the embeddings $C^{d+1}(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ ([Grö01, Theorem 14.5.3]) we can estimate

$$\begin{aligned}\lambda_j^2\|(\partial^\gamma\chi_j)\mu\|_{W(\mathcal{F}L^1,L^\infty)} &\lesssim \lambda_j^2\|\langle\xi\rangle^{-\alpha}\mu\|_{W(\mathcal{F}L^1,L^\infty)}\|\langle\xi\rangle^\alpha\partial^\gamma\chi_j\|_{W(\mathcal{F}L^1,L^\infty)}\\ &\lesssim \lambda_j^2\sum_{\beta\leq d+1}\|\partial^\beta[\langle\xi\rangle^\alpha\partial^\gamma\chi_j]\|_{L^\infty}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\partial^\beta[\langle\xi\rangle^\alpha\partial^\gamma\chi_j(\xi)]| &= \left|\sum_{\nu\leq\beta}\binom{\beta}{\nu}\partial^\nu\langle\xi\rangle^\alpha\partial^{\gamma+\beta-\nu}\chi_j(\xi)\right|\\ &\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}\langle\xi\rangle^{\alpha-|\nu|}2^{-j|\gamma+\beta-\nu|}\left|(\partial^{\gamma+\beta-\nu}\chi)(2^{-j}\xi)\right|\\ &\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}2^{(\alpha-|\nu|)j}2^{-2j} \lesssim 2^{j(\alpha-2)},\end{aligned}$$

because on the support of χ_j, $|\xi| \asymp 2^j$ and $|\gamma+\beta-\nu| \geq 2$. Thus

$$\lambda_j^2 \|(\partial^\gamma \chi_j)\mu\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim \lambda_j^2 2^{j(\alpha-2)} = 1.$$

Now, let us prove (5.3.8). Using Lemma 5.3.1 (ii) and arguing as above we write

$$\begin{aligned}
\lambda_j^2 \|\partial^\gamma \chi_j \partial^\beta \mu\|_{W(\mathcal{F}L^1,L^\infty)} &\lesssim \lambda_j^2 \|\langle\xi\rangle^{1-\alpha} \partial^\beta \mu\|_{W(\mathcal{F}L^1,L^\infty)} \|\langle\xi\rangle^{\alpha-1} \partial^\gamma \chi_j\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \lambda_j^2 \sum_{\beta \leq d+1} \|\partial^\beta [\langle\xi\rangle^{\alpha-1} \partial^\gamma \chi_j]\|_{L^\infty}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\partial^\beta [\langle\xi\rangle^{\alpha-1} \partial^\gamma \chi_j(\xi)]| &\lesssim \sum_{\nu \leq \beta} \binom{\beta}{\nu} \langle\xi\rangle^{\alpha-1-|\nu|} 2^{-j|\gamma+\beta-\nu|} \left|(\partial^{\gamma+\beta-\nu}\chi)(2^{-j}\xi)\right| \\
&\lesssim \sum_{\nu \leq \beta} \binom{\beta}{\nu} 2^{(\alpha-1-|\nu|)j} 2^{-j} \lesssim 2^{j(\alpha-2)},
\end{aligned}$$

because now $|\gamma+\beta-\nu| \geq 1$. Thus

$$\lambda_j^2 \|\partial^\gamma \chi_j \partial^\beta \mu\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim \lambda_j^2 2^{j(\alpha-2)} = 1.$$

Finally, let us prove (5.3.9), using the hypothesis

$$\langle\xi\rangle^{2-\alpha} \partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$$

for $|\gamma| = 2$, we have

$$\begin{aligned}
\lambda_j^2 \|\chi_j \partial^\gamma \mu\|_{W(\mathcal{F}L^1,L^\infty)} &\lesssim \lambda_j^2 \|\langle\xi\rangle^{2-\alpha} \partial^\gamma \mu\|_{W(\mathcal{F}L^1,L^\infty)} \|\langle\xi\rangle^{\alpha-2} \chi_j\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \lambda_j^2 \sum_{\beta \leq d+1} \|\partial^\beta [\langle\xi\rangle^{\alpha-2} \chi_j]\|_{L^\infty}.
\end{aligned}$$

Moreover, arguing as above

$$\begin{aligned}
|\partial^\beta [\langle\xi\rangle^{\alpha-2} \chi_j(\xi)]| &\lesssim \sum_{\nu \leq \beta} \binom{\beta}{\nu} \langle\xi\rangle^{\alpha-2-|\nu|} 2^{-j|\beta-\nu|} \left|(\partial^{\beta-\nu}\chi)(2^{-j}\xi)\right| \\
&\lesssim \sum_{\nu \leq \beta} \binom{\beta}{\nu} 2^{(\alpha-2-|\nu|)j} 2^0 \lesssim 2^{j(\alpha-2)}.
\end{aligned}$$

Thus
$$\lambda_j^2\|\chi_j\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim\lambda_j^2 2^{j(\alpha-2)}=1.$$
Hence, the estimate (5.3.6) is proved. And by (5.3.4), (5.3.5) and (5.3.6) we have
$$\|\tilde{T}^{(j)}f\|_{M^{p,q}}=\|A_jB_jf\|_{M^{p,q}}\lesssim 2^{-\frac{d(\alpha-2)}{2}j}\|f\|_{M^{p,q}}, \tag{5.3.10}$$
for every $1\leq p,q\leq\infty$.

We recall the decomposition (5.3.3)
$$T^{(j)}=U_{\lambda_j}\tilde{T}^{(j)}U_{\lambda_j^{-1}}.$$

Now, combine the estimate from $\tilde{T}^{(j)}$, (5.3.10), with those for the dilation operator, given in Theorem 1.2.11. For $p=1,\infty$ and $q=1$ they read
$$\|U_{\lambda_j}f\|_{M^{1,1}}\lesssim 2^{\frac{d(\alpha-2)}{2}j}\|f\|_{M^{1,1}},$$
$$\|U_{\lambda_j}f\|_{M^{\infty,1}}\lesssim\|f\|_{M^{\infty,1}},$$
and
$$\|U_{\lambda_j^{-1}}f\|_{M^{1,1}}\lesssim\|f\|_{M^{1,1}},$$
$$\|U_{\lambda_j^{-1}}f\|_{M^{\infty,1}}\lesssim 2^{\frac{d(\alpha-2)}{2}j}\|f\|_{M^{\infty,1}}.$$
Therefore we obtain, for $p=1,\infty$,
$$\|T^{(j)}f\|_{M^{p,1}}\lesssim 2^{-\frac{d(\alpha-2)}{2}j}2^{\frac{d(\alpha-2)}{2}j}\|f\|_{M^{p,1}}=\|f\|_{M^{p,1}}.$$

Finally, to sum these last estimates over $j\geq 1$ we take advantage of the fact we are working with functions which are localized in shells of the frequency domain. Precisely, let χ as before, namely a smooth function satisfying $\chi(\xi)=1$ for $1/2\leq|\xi|\leq 2$ and $\chi(\xi)=0$ for $|\xi|\leq 1/4$ and $|\xi|\geq 4$ (so that $\chi\psi=\psi$). With $\chi_j(\xi)=\chi(2^{-j}\xi)$ and $p=1,\infty$ we have
$$\|T^{(j)}f\|_{M^{p,1}}=\|T^{(j)}(\chi(2^{-j}D)f)\|_{M^{p,1}}\lesssim\|\chi(2^{-j}D)f\|_{M^{p,1}},$$

so that Lemma 5.3.4 gives us

$$\left\| \sum_{j\geq 1} T^{(j)} f \right\|_{M^{p,1}} \leq \sum_{j\geq 1} \|T^{(j)} f\|_{M^{p,1}} \lesssim \|f\|_{M^{p,1}}.$$

5.3.2 Boundedness on $M^{1,\infty}$ and $M^{\infty,\infty}$ for $\delta = \frac{d(\alpha-2)}{2}$

To prove this we first establish the following lemma (cf. [CNR09a, ST07]).

Lemma 5.3.5. *For $k \geq 0$, let $f_k \in \mathcal{S}(\mathbb{R}^d)$ satisfy* $\operatorname{supp} \hat{f}_0 \subset B_2(0)$ *and*

$$\operatorname{supp} \hat{f}_k \subset \{\xi \in \mathbb{R}^d : \; 2^{k-1} \leq |\xi| \leq 2^{k+1}\}, \quad k \geq 1.$$

Then, if the sequence f_k is bounded in $M^{p,\infty}(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$, the series $\sum_{k=0}^{\infty} f_k$ converges in $M^{p,\infty}(\mathbb{R}^d)$ and

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{M^{p,\infty}} \lesssim \sup_{k\geq 0} \|f_k\|_{M^{p,\infty}}.$$

Proof. Let $x \in \mathbb{R}^d$, we define $K_x \subseteq \mathbb{N}$ such that $k \in K_x$ if, and only if, $f_k(x) \neq 0$. For the properties of each f_k, K_x has, at most, 3 elements for each $x \in \mathbb{R}^d$. Then, since the sequence f_k is bounded,

$$\sum_{k=0}^{\infty} f_k(x) = \sum_{k \in K_x} f_k(x) < \infty.$$

And the convergence of the series $\sum_{k=0}^{\infty} f_k$ in $M^{p,\infty}(\mathbb{R}^d)$ is proved. We now prove the desired estimate.

Choose a window function g with $\operatorname{supp} \hat{g} \subset B_{1/2}(0)$. We can write

$$V_g(f_k)(x,\xi) = (\hat{f}_k * M_{-x}\hat{\overline{g}})(\xi).$$

Hence, $\operatorname{supp} V_g(f_0) \subset B_{5/2}(0) \subset B_{2^2}(0)$, and

$$\begin{aligned} \operatorname{supp} V_g(f_k) &\subset \{(x,\xi) \in \mathbb{R}^{2d} : \; 2^{k-1} - 2^{-1} \leq |\xi| \leq 2^{k+1} + 2^{-1}\} \\ &\subset \{(x,\xi) \in \mathbb{R}^{2d} : \; 2^{k-2} \leq |\xi| \leq 2^{k+2}\}, \end{aligned}$$

for $k \geq 1$. Hence, for each ξ, there are at most four nonzero terms in the sum $\sum_{k=0}^{\infty} \|V_g(f_k)(\cdot,\xi)\|_{L^p}$. Using this fact we obtain

$$\begin{aligned}\left\|\sum_{k=0}^{\infty} f_k\right\|_{M^{p,\infty}} &\asymp \left\|\sum_{k=0}^{\infty} V_g(f_k)\right\|_{L^{p,\infty}} \leq \sup_{\xi\in\mathbb{R}^d} \sum_{k=0}^{\infty} \|V_g(f_k)(\cdot,\xi)\|_{L^p} \\ &\leq 4 \sup_{k\geq 0} \sup_{\xi\in\mathbb{R}^d} \|V_g(f_k)(\cdot,\xi)\|_{L^p} = 4 \sup_{k\geq 0} \|V_g(f_k)\|_{L^{p,\infty}} \\ &\asymp \sup_{k\geq 0} \|f_k\|_{M^{p,\infty}}.\end{aligned}$$

□

We now consider the same decomposition as above, namely (5.3.2), and the operators $\tilde{T}^{(j)}$ in (5.3.3), $j \geq 1$. From (5.3.10) for $q = \infty$ we have the following estimate:

$$\|\tilde{T}^{(j)} f\|_{M^{p,\infty}} \leq 2^{-\frac{d(\alpha-2)}{2}j} \|f\|_{M^{p,\infty}}.$$

We then combine this estimate with those for the dilation operator which here read

$$\|U_{\lambda_j} f\|_{M^{1,\infty}} \lesssim 2^{d(\alpha-2)j} \|f\|_{M^{1,\infty}},$$

$$\|U_{\lambda_j} f\|_{M^{\infty,\infty}} \lesssim 2^{\frac{d(\alpha-2)}{2}j} \|f\|_{M^{\infty,\infty}},$$

and

$$\|U_{\lambda_j^{-1}} f\|_{M^{1,\infty}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j} \|f\|_{M^{1,\infty}},$$

$$\|U_{\lambda_j^{-1}} f\|_{M^{\infty,\infty}} \lesssim \|f\|_{M^{\infty,\infty}}.$$

Therefore we obtain, for $p = 1, \infty$,

$$\|T^{(j)} f\|_{M^{p,\infty}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j} 2^{\frac{d(\alpha-2)}{2}j} \|f\|_{M^{p,\infty}} = \|f\|_{M^{p,\infty}}.$$

We finally conclude by applying Lemma 5.3.5: for $p = 1, \infty$,

$$\left\|\sum_{j=1}^{\infty} T^{(j)} f\right\|_{M^{p,\infty}} \lesssim \sup_{j\geq 1} \|T^{(j)} f\|_{M^{p,\infty}} \lesssim \|f\|_{M^{p,\infty}}.$$

5.3.3 Boundedness on $M^{2,1}$ and $M^{2,\infty}$ for $\delta = 0$

Indeed, we will prove boundedness on $M^{2,q}$ for every $1 \leq q \leq \infty$ and $\delta = 0$. This is a special case of the following result.

Proposition 5.3.6. *Any Fourier multiplier T with symbol $\sigma \in L^\infty$ is bounded on $M^{2,q}$ for every $1 \leq q \leq \infty$.*

Proof. The desired result follows at once from the estimate

$$\|\sigma(D)f\|_{L^2} \leq \|\sigma\|_{L^\infty}\|f\|_{L^2},$$

for $f \in L^2$, and the fact that the Fourier multipliers which are bounded on L^2 are the same that the Fourier multipliers which are bounded on $M^{2,q}$, [FN06, Theorem 17 (3)].

We provide a direct proof for the benefit of the reader. Namely

$$\begin{aligned}
\|Tf\|_{M^{2,q}} &= \left\| \left\| M_x\hat{\overline{g}} * \left(\sigma\hat{f}\right)\right\|_{L^2_x}\right\|_{L^q} \\
&= \left\| \left\| \int e^{2\pi i x(\xi-y)}\hat{\overline{g}}(\xi-y)\sigma(y)\hat{f}(y)dy\right\|_{L^2_x}\right\|_{L^q_\xi} \\
&= \left\| \left\| \sigma\hat{f}T_\xi\hat{\overline{g}}\right\|_{L^2}\right\|_{L^q_\xi},
\end{aligned}$$

where we used Parseval's formula. In particular, this computation with $\sigma \equiv 1$ gives $\|f\|_{M^{2,q}} = \|\|\hat{f}T_\xi\hat{g}\|_{L^2}\|_{L^q_\xi}$, so we deduce at once the desired estimate

$$\|Tf\|_{M^{2,q}} \lesssim \|\sigma\|_{L^\infty}\|f\|_{M^{2,q}}.$$

$\square$

5.3.4 Boundedness from $M^{p,q}_\delta(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$

Finally, by complex interpolation, Proposition 5.3.2, and Lemma 5.3.3 we conclude the main result of the chapter.

Theorem 5.3.7. *Let $\alpha \geq 2$. Let $\mu \in C^2(\mathbb{R}^d)$, real-valued and such that*

$$\langle \xi \rangle^{2-\alpha} \partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \quad \textit{for } |\gamma| = 2.$$

Then

$$e^{\mathbf{i}\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q \leq \infty$ and

$$\delta \geq d(\alpha - 2) \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Proof. From the Subsections 5.3.1, 5.3.2 and 5.3.3, and Lemma 5.3.3, we have the boundedness of

$$e^{\mathbf{i}\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

when the set (p, q, δ) is equal to $(1, 1, d(\alpha - 2)/2)$, $(1, \infty, d(\alpha - 2)/2)$, $(\infty, 1, d(\alpha - 2)/2)$, $(\infty, \infty, d(\alpha - 2)/2)$ or $(2, q, 0)$, with $1 \leq q \leq \infty$. From Proposition 5.3.2, complex interpolation, we deduce the boundedness when the set (p, q, δ) is equal to $(1, q, d(\alpha - 2)/2)$, $(\infty, q, d(\alpha - 2)/2)$ or $(2, q, 0)$, with $1 \leq q \leq \infty$.

Now we set a pair (p, q), with $1 \leq q \leq \infty$ and $1 < p < \infty$, the extreme cases are proved. If $1 < p \leq 2$, there exist $\theta \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1 - \theta}{1} + \frac{\theta}{2}.$$

Moreover,

$$\frac{1}{p} - \frac{1}{2} = 1 - \frac{\theta}{2} - \frac{1}{2} = \frac{1}{2} - \frac{\theta}{2} = \frac{1 - \theta}{2}$$

Then by Proposition 5.3.2 (complex interpolation) we have the boundedness for (p, q, δ), where

$$\delta = \frac{(1 - \theta) d(\alpha - 2)}{2} + \theta 0 = d(\alpha - 2) \frac{(1 - \theta)}{2} = d(\alpha - 2) \left(\frac{1}{p} - \frac{1}{2} \right).$$

If $2 \le p < \infty$, there exist $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{\infty}.$$

Moreover,

$$\frac{1}{2} - \frac{1}{p} = -\frac{1}{2} + \frac{\theta}{2} + \frac{1}{2} = \frac{\theta}{2}$$

Then by Proposition 5.3.2, complex interpolation, we have the boundedness for (p, q, δ), where

$$\delta = (1-\theta)0 + \frac{\theta d(\alpha-2)}{2} = d(\alpha-2)\frac{\theta}{2} = d(\alpha-2)\left(\frac{1}{2} - \frac{1}{p}\right).$$

Then, we have that

$$e^{\mathbf{i}\mu(D)} : M^{p,q}_{\delta}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \le p, q \le \infty$ and

$$\delta = d(\alpha-2)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Let $\delta_0 = d(\alpha-2)\left|\frac{1}{p} - \frac{1}{2}\right|$ and $\delta \ge \delta_0$. Let $f \in M^{p,q}_{\delta} \subseteq M^{p,q}_{\delta_0}$, and we have $\|f\|_{M^{p,q}_{\delta_0}} \lesssim \|f\|_{M^{p,q}_{\delta}}$. Then

$$\left\|e^{\mathbf{i}\mu(D)} f\right\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}_{\delta_0}} \lesssim \|f\|_{M^{p,q}_{\delta}}.$$

We can conclude that

$$e^{\mathbf{i}\mu(D)} : M^{p,q}_{\delta}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \le p, q \le \infty$ and

$$\delta \ge d(\alpha-2)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

$\square$

5.4 Conclusion

In conclusion, we have found results about boundedness of unimodular Fourier multipliers on modulation spaces, when the partial derivatives of its phase, or some expression relative to the partial derivatives of its phase, belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$. These results are included in [NPT18].

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