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A reduction theorem for the generalised Rhodes' Type II Conjecture

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PRESENTADA PER VICENTE PÉREZ I CALABUIG
DIRIGIDA PER ADOLFO BALLESTER I BOLINCHES

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A la meua família

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Chapter 1

Introducció

L'estudi formal dels semigrups s'inicià a principis del segle XX amb els treballs de Suschkevich, Rees, Green, Lyapin, Clifford i Preston, entre altres. De fet, és durant la segona meitat del segle passat quan l'estudi dels semigrups guanyà gran rellevància a causa de l'aparició de la teoria d'autòmats que, junt amb l'aportació del treball seminal de Kleene, suposà el començament d'una estreta relació entre ambdues teories. Així, d'aquesta unió van nàixer fortes implicacions en la teoria de la computació i llenguatges informals: com la concepció de circuits, la compilació de llenguatges de programació o la cerca de cadenes de caràcters. Per altra banda, també cal destacar l'impuls que atorgaren els treballs d'Eilenberg o Schützenberger al desenvolupament de la lògica, l'àlgebra o la topologia mitjançant la teoria de semigrups.

Com en tota teoria algebraica, els principals resultats sobre els quals es construïx aquesta són aquells que intenten caracteritzar i descriure la pròpia estructura algebraica. D'aquesta manera, és clar que un major coneixement de l'estructura permet aprofundir en la teoria, derivant-se així, aplicacions a altres teories. En aquest sentit, un dels pilars en la teoria de semigrups i autòmats és el Teorema de Krohn-Rhodes, i.e. tot semigrup finit S es descompon en un producte orlat de grups, cadascun d'ells divisor de S, i en un nombre finit de semigrups aperiòdics (és a dir, semigrups amb subgrups maximals trivials). Així, segons aquest teorema, qualsevol resultat concernent a la descomposició de màquines seqüencials amb un nombre finit d'estats pot ser traslladat a un teorema sobre descomposició en producte orlat de semigrups finits; i viceversa, qualsevol descomposició en producte orlat de semigrups té la seua corresponent interpretació en termes de factoritzacions de màquines seqüencials amb un nombre finit d'estats.

El mínim nombre de grups apareixent en la descomposició de Krohn-Rhodes s'anomena la *complexitat* d'un semigrup finit. Ara bé, tot i que és ben conegut que existixen semigrups finits de complexitat arbitrària, no es coneix l'existència d'un algoritme per calcular tal complexitat d'un semigrup en general. De fet, provar l'existència de tal algoritme és un dels problemes oberts més importants en la teoria de semigrups finits, la recerca de la qual ha portat al desenvolupament d'una gran quantitat de ferramentes i idees de gran interés fora de la teoria de semigrups finits.

Pel que fa a les aportacions a la resolució del problema, en els treballs de Rhodes i Tilson [28] o Rhodes i Steinberg [27], s'han obtingut resultats rellevants proporcionant fites superiors i inferiors a la complexitat d'un semigrup finit. En particular, en el treball seminal de Rhodes i Tilson es presenta un mètode per obtenir una fita inferior de la complexitat d'un semigrup finit en termes de la longitud maximal d'una cadena de subsemigrups, la qual està formada pel que anomenen subsemigrups de Tipus I i de Tipus II. Aquesta longitud maximal resulta ser la complexitat per als casos en què el semigrup finit és invers o completament regular; no obstant això, un contraexemple per a un semigrup finit en general es construí en [26].

És en aquest punt quan entra en joc la noció de *nucli d'un semigrup finit*. A l'article de Rhodes i Tilson citat anteriorment, trobem definit el nucli d'un semigrup finit com el major subsemigrup de Tipus II dins d'aquest. De manera anàloga, també es pot veure com el conjunt d'elements relacionats amb la identitat, sota l'acció de qualsevol morfisme relacional entre el semigrup i un grup finit qualsevol. El nucli d'un semigrup finit juga un paper clau en el problema de la complexitat d'un semigrup finit, és per això que la seua computabilitat tindria importants conseqüències en la seua resolució. De fet, un dels resultats més rellevants del treball de Rhodes i Tilson és la descripció dels elements regulars del nucli d'un semigrup finit. Aquest resultat portà Rhodes a formular la seua famosa conjectura, originàriament anomenada *Type II Conjecture*: el nucli d'un semigrup finit és el subsemigrup tancat per conjugació més menut que conté els idempotents (en particular, el nucli d'un semigrup finit és computable).

Aquesta conjectura atragué l'atenció d'un gran nombre d'especialistes en teoria de semigrups al voltant de 20 anys fins que fou resolta. La primera solució, deguda a Ash en [3], usà tècniques algebraiques i mètodes de combinatòria; mentre que quasi al mateix temps, una altra solució independent fou obtinguda per Ribes i Zalesskiĭ en [29], basant-se en uns treballs anteriors de Pin [22] i Pin i Reutenauer [24] per aplicar mètodes profinits. Aquestes solucions de la conjectura de Rhodes aportaren una gran quantitat de noves idees a la teoria de semigrups. Simultàniament, experts d'altres àrees de la Matemàtica, com teoria de models (vid. [9] o [16]), s'interessaren per les noves tècniques que s'hi usaren. Els treballs de Margolis [21] i Henckell, Margolis, Pin i Rhodes [15], escrits poc després de la prova d'Ash de la Type II Conjecture, contenen informació sobre el context històric i les conseqüències d'aquesta prova.

Com tot problema amb resolució no immediata, no sols un gran nombre de conseqüències van ser obtingudes, sinó que, a més a més, altres problemes oberts en sorgiren de nou. Tenint en compte que la noció de nucli d'un semigrup finit ve associada a la varietat de tots els grups finits (per varietat de grups, s'entén una formació de grups tancada per subgrups), un problema particularment estudiat és saber què ocorre quan generalitzem la definició de nucli. Donada una varietat de grups $\mathfrak{F}$ qualsevol, direm que el $\mathfrak{F}$-*nucli* $\mathrm{K}_{\mathfrak{F}}(S)$ d'un semigrup finit S és el conjunt d'elements relacionats amb la identitat sota l'acció de qualsevol morfisme relacional entre S i un grup de la varietat $\mathfrak{F}$.

En aquest context i degut a la interdisciplinarietat de la solució de la Type II Conjecture per part de Ribes i Zalesskiĭ, on tècniques de teoria de grafs i topologia profinita entren en joc, els següents problemes es plantejaren implícitament en el treball de Margolis [21]. Per a una varietat de grups $\mathfrak{F}$ donada:

1. Decidir la computabilitat del $\mathfrak{F}$-nucli d'un semigrup finit (generalització de la Type II Conjecture de Rhodes).
2. Decidir la computabilitat de la clausura d'un subgrup finitament generat del grup lliure en la topologia pro-$\mathfrak{F}$.
3. Provar una versió general del teorema de Ribes i Zalesskiĭ en la topologia pro-$\mathfrak{F}$.

Aquests problemes han sigut resolts per a la varietat de p-grups en [30]. De fet, el problema 3 ha sigut resolt per a varietats extensibles i localment extensibles en [30] i [5], respectivament. A més, com a conseqüència d'aquests resultats, tenim que els Problemes 1 i 2 són equivalents per a aquest tipus de varietats (vid. [30], [5] i [20]). En particular, la solució al Problema 2 per a la varietat dels grups resolubles resulta particularment rellevant, ja que suposaria un gran avanç en la teoria de semigrups finits i en la teoria de complexitat computacional (vid. [10] i [31], per a més informació). Finalment, el Problema 1 ha sigut resolt per a la varietat de grups abelians en [11], qualsevol varietat de grups abelians decidible en [32] i la varietat dels grups nilpotents en [2].

Com que la generalització de la Type II Conjecture de Rhodes és per a qualsevol varietat de grups, és natural esperar que existisca algun argument que la puga provar; tanmateix, no s'ha pogut trobar cap indici de com tal prova poguera ser. Aleshores, l'estratègia a seguir en aquests casos és reduir el problema a algun tipus de semigrups especial. De fet, en els casos en què la varietat $\mathfrak{F}$ és extensible, a partir d'un teorema de Ribes i Zaleskiĭ es pot inferir que el $\mathfrak{F}$-kernel és computable si els seus elements regulars són computables.

Segons aquest plantejament, Steinberg en [33] prova un teorema de reducció per a la computabilitat dels elements regulars del $\mathfrak{F}$-nucli d'un semigrup finit. Segons aquest resultat, els elements regulars del $\mathfrak{F}$-nucli són computables si, i només si, el $\mathfrak{F}$-nucli d'un semigrup invers finit és computable. Malauradament, aquest teorema està lluny de solucionar la generalització de la conjectura de Rhodes per a varietats extensibles [4] i, per tant, es requerixen noves idees i tècniques.

El principal resultat d'aquesta tesi pretén anar un pas més enllà i aconseguir una aproximació decisiva cap a la verificació de la generalització de la conjectura de Rhodes per a varietats extensibles. Així, com a resultat principal s'afirma el següent: el $\mathfrak{F}$-nucli d'un semigrup invers finit és computable si, i només si, el $\mathfrak{F}$-nucli d'un semigrup invers finit amb tots els seus subgrups maximals en $\mathfrak{F}$ és computable. Queda recollit en el següent teorema:

Teorema A *Siga $\mathfrak{F}$ una varietat de grups. Les següents afirmacions són equivalents:*

1. *Els elements regulars del $\mathfrak{F}$-nucli d'un semigrup finit són computables.*
2. *El $\mathfrak{F}$-nucli de tot semigrup invers finit és computable.*
3. *El $\mathfrak{F}$-nucli de tot semigrup invers finit amb els subgrups maximals en $\mathfrak{F}$ és computable.*

El Teorema A és conseqüència d'un resultat més general:

Teorema B *Siga $\mathfrak{F}$ una varietat de grups. Aleshores, les següents afirmacions són equivalents:*

1. *Els elements regulars del $\mathfrak{F}$-nucli d'un semigrup finit són computables.*
2. *$K_{\mathfrak{F}}(S) \cap J$ és computable per a tot semigrup invers finit S i tota $\mathcal{J}$-classe J de S.*
3. *$\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ és computable per a tot semigrup invers finit $\bar{S}$ amb zero i amb una única $\mathcal{J}$-classe 0-minimal tal que $\bar{S}_e \in \mathfrak{F}$, per a tot idempotent $e \in \mathrm{E}(\bar{S})$.*

Com a nota del mètode de prova del resultat principal, caldria remarcar que s'ha adoptat un enfocament estructural clàssic del problema amb gran èxit. El punt de partida es troba en el treball seminal de Rhodes i Tilson [28], de manera que s'ha procurat desenvolupar aquesta tesi des d'un punt de vista "pur" de la teoria de semigrups finits. Probablement, es podria donar un altre enfocament fent ús dels anomenats grafs de Schützenberger o els grafs inevitables d'Ash, donant lloc així a proves més geomètriques de les equivalències dalt mostrades. No obstant això, s'ha preferit un també anomenat "clàssic enfocament" perquè resulta ser més transparent i autocontingut al nostre entendre.

La tesi queda organitzada com seguix. Al Capítol 3 es pretén recollir els resultats bàsics que es fan servir en semigrups i nuclis. Encara que alguns d'ells poden ser ben coneguts pels experts en teoria de semigrups, probablement hi podem trobar-ne de nous. Al Capítol 4, el principal resultat d'aquesta tesi ve presentat, així com un desenvolupament dels conceptes fonamentals que s'usen a la prova d'aquest. En particular, apareix la noció de projecció d'un element en una $\mathcal{J}$-classe 0-minimal d'un semigrup invers. Sobre aquesta noció vénen construïts el concepte de par minimal (Secció 4.1), el semigrup de projeccions d'un semigrup de Brandt (Secció 4.2) i un tipus especial de quocients (Secció 4.3). Així, es pot apreciar com els pars minimals i els quocients serviran com a nexes d'unió d'una cadena de reduccions mostrada a la Secció 4.4. La prova del principal resultat d'aquesta tesi es fonamenta en aquesta cadena de reduccions, com es pot observar a la Secció 4.5. Finalment, el Capítol 5 està dedicat a les aplicacions del Teorema principal.

Chapter 2

Introduction

The formal study of semigroups began in the early 20th century with the works of Suschkevich, Rees, Green, Lyapin, Clifford and Preston among others. In fact, during the second half of the past century, the study of finite semigroups has been of particular importance because of its close relation to theoretical computer science, which is based on the natural link between finite semigroups and finite automata via the syntactic monoid.

One of the milestones in the theory of semigroups and automata is the Krohn-Rhodes Theorem [19]. It states that every finite semigroup S divides a wreath product of finite simple groups, each of them divisor of S, and finite aperiodic semigroups, i. e. semigroups with trivial maximal subgroups. As a consequence, any result about decomposition of machines can be translated into a theorem about wreath product decompositions of finite semigroups and any wreath product decomposition of finite semigroups has a corresponding interpretation in terms of factorizations of finite state machines.

The smallest number of groups in any Kohn-Rhodes decomposition is called the *(group) complexity* of the semigroup. It is well-known that there are semigroups of arbitrary complexity. Nevertheless, there is no obvious way to compute the complexity of a finite semigroup in general. In fact, this decidability is one of the most important open problems in finite semigroup theory and the search for the solution has led to the development of many tools and ideas that are useful in finite semigroup theory and of independent interest.

However, some upper bounds and ever more precise lower bounds for complexity have been obtained. For the last ones, in 1972, Rhodes and Tilson presented in his seminal paper [28] a method to obtain a lower bound for the complexity of a semigroup, which was given by means of taking the maximal length of a chain of subsemigroups alternating what they called type I and type II subsemigroups and containing a non-aperiodic type I subsemigroup.

Indeed, this number was proved to be the complexity for inverse semigroups and complete regular semigroups, but a counterexample for a general semigroup was constructed in [26].

At this point, the notion of the kernel of a semigroup came on the scene. It was introduced by Rhodes and Tilson in the aforesaid paper as the maximal type II subsemigroup of a given semigroup (also as the set of elements related to the identity under every relational morphism between the semigroup and a group), and its computability would have important consequences in the solution of the complexity problem. In fact, one of the main results in Rhodes and Tilsons's paper is a description of the regular elements of the kernel of a semigroup. That result led to the Rhodes' conjecture, originally called the "Type II Conjecture": the kernel of a semigroup is the smallest subsemigroup containing the idempotents and closed under weak conjugation. In particular, the kernel of a semigroup is computable.

This conjecture attracted the attention of many semigroup theorists during about two decades before being solved. Its first solution, given by Ash in [3], used algebraic and combinatorial methods. Almost at the same time, an independent solution was given by Ribes and Zalesskiĭ in [29]. Their proof used profinite methods and it is based on works of Pin [22] and Pin and Reutenauer [24]. The solution of the Type II Conjecture brought many new ideas into semigroup theory, and also attracted the attention of researchers of another areas of Mathematics, such as Model Theory (see [9] or [16]). The papers by Margolis [21] and by Henckell, Margolis, Pin and Rhodes [15], written soon after Ash's proof, contain some of the history and consequences of the Type II Conjecture and extensive literature on the theme.

Like many problems with a no immediate solution, once they are solved, not only a great number of consequences spring from their solution, but also new questions related with them can be set out (see [21]). A first natural step is to extend the definition of kernel of a semigroup to an arbitrary variety of finite groups $\mathfrak{F}$: the $\mathfrak{F}$-kernel $\mathrm{K}_{\mathfrak{F}}(S)$ of a finite semigroup S is the subsemigroup of S consisting of all elements of S such that relate to the identity under every relational morphism of S with a group in $\mathfrak{F}$. The *generalised kernels* of finite semigroups are precisely the kernels associated to varieties of finite groups, i.e, subgroup-closed formations of finite groups.

In this context, due to the interdisciplinary nature of the proof of the "Type II Theorem" by Ribes and Zalesskiĭ, where profinite topology and graph theory was involved, the following problems were implicitly put forward in [21]:

1. Decide the computability of the $\mathfrak{F}$-kernel of a finite semigroup, for a given variety $\mathfrak{F}$ (*generalised Rhodes' Type II Conjecture*).

2. Decide the computability of the closure of a finitely generated subgroup

of the free group in the pro-$\mathfrak{F}$ topology, for a given variety $\mathfrak{F}$.

3. A general version in the pro-$\mathfrak{F}$ topology of the Ribes and Zalesskiĭ's theorem, for a given variety $\mathfrak{F}$.

These problems have been solved for the variety of p-groups [30]. In fact, Problem 3 has been solved for extension closed varieties and locally extensible varieties in [30] and [5], respectively. As a consequence of that, Problems 1 and 2 are equivalent for these varieties (see [30], [5] and [20]). In this context, the solution of the Problem 2 for the variety of finite soluble groups is of particular importance, since it would have interesting consequences in finite semigroup theory and computational complexity (see [10] and [31]). Moreover, Problem 1 has been solved for the abelian group variety [11], for any variety of abelian groups which has decidable membership problem and generates the abelian group variety [32] and for the variety of nilpotent groups [2].

Since generalised Rhodes' Type II Conjecture is completely general, it is natural to hope that some argument might exist that would prove it, but up to now no one seems to have an inkling of how such a proof might proceed. Failing that, one could try to reduce the problem to a question about some restricted class of finite semigroups. Indeed, in the important case where $\mathfrak{F}$ is extension closed, a theorem of Ribes and Zalesskiĭ [30] allows us to conclude that the $\mathfrak{F}$-kernel is computable if its regular elements are computable. In this context, Steinberg [33] proved a reduction theorem for the computability of the regular elements of the $\mathfrak{F}$-kernel of a finite semigroup. He showed that the membership problem for such elements is decidable if, and only if, it is decidable for inverse semigroups.

Unfortunately, Steinberg's reduction theorem is far from solving the generalised Rhodes' Type II Conjecture for extension closed varieties [4], and therefore new ideas and techniques are required.

The main result of this thesis is meant to provide a decisive step towards verifying the generalised Rhodes' Type II Conjecture for extension closed varieties.

We prove that the $\mathfrak{F}$-kernel of every finite inverse semigroup is computable if, and only if, the $\mathfrak{F}$-kernel of every finite inverse semigroup with all maximal subgroups in $\mathfrak{F}$ is computable. Thus, our principal result is contained in the following theorem.

Theorem A. *Let $\mathfrak{F}$ be a variety of groups. The following statements are pairwise equivalent:*

1. *The regular elements of the $\mathfrak{F}$-kernel of a finite semigroup are computable.*

2. *The $\mathfrak{F}$-kernel of every finite inverse semigroup is computable.*

3. *The $\mathfrak{F}$-kernel of every finite inverse semigroup whose all maximal subgroups are in $\mathfrak{F}$ is computable.*

Theorem A is a consequence of a more general result.

Theorem B. *Let $\mathfrak{F}$ be a variety of groups. Then the following statements are pairwise equivalent:*

1. *The regular elements of the $\mathfrak{F}$-kernel of every semigroup are computable.*

2. *$K_{\mathfrak{F}}(S) \cap J$ is computable for every inverse semigroup S and every $\mathcal{J}$-class J of S.*

3. *$\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ is computable for every inverse semigroup $\bar{S}$ with zero with a unique 0-minimal $\mathcal{J}$-class J such that $\bar{S}_e \in \mathfrak{F}$, for each $e \in \mathrm{E}(\bar{S})$.*

We make one remark on our method of proof. We have adopted here a classical structural approach with great success. It has the seminal paper of Rhodes and Tilson [28] as a point of departure. In fact, the emphasis throughout it is unashamedly on what might be called 'pure' semigroup theory. It probably might be possible to use an alternative approach using Schützenberger graphs or Ash inevitable graphs leading to a geometric proof of our equivalences in Theorems A and B. However, we are much in favour to use the so-called classical approach because it is more transparent and self-contained.

All semigroups and groups considered in this paper are finite.

The thesis is organised as follows. Chapter 3 of the thesis is intended to collect the basic results on semigroups and kernels. A certain amount of what is here should be considered folklore, although probably some bits are new. Chapter 4 presents our main result and develops the fundamental concepts which are used in the proof of it: we present the notion of projection onto a 0-minimal $\mathcal{J}$-class of an inverse semigroup, and on those foundations is then built a fairly natural edifice, consisting of minimal pairs (Section 4.1), the semigroup of projections of a Brandt semigroup (Section 4.2), and some sort of quotients (Section 4.3). Minimal pairs and quotients are the links in a chain of reductions that are shown in Section 4.4. Our chain of reductions is the basis of the proof of the main result that is presented in Section 4.5. Finally, Chapter 5 is referred to the applications of our main result.

Chapter 3

Preliminaries

In this section, we collect some definitions and specific notation that are needed in our main results. For further details, background and undefined notation, see [1], [8], [23], [27].

3.1 Basic results on semigroups

Recall that a semigroup is a set S that is closed under an associative binary operation. The common term for a semigroup with an identity element is a monoid. In many instances, it is more convenient to work with a monoid than with a semigroup. It is thus sometimes to speak of the semigroup with identity adjoined S^1. Like rings, but unlike a group, a semigroup may have a zero element, and may moreover be useful to adjoin a zero to a semigroup, in much the same way we can adjoin an identity. The corresponding semigroup is denoted by S^0.

The identity and zero elements are both examples of an important class of elements within a semigroup S, namely *idempotents*, that is, elements $e \in S$ such $e^2 = e$. For a subset X of S, denote by $\mathrm{E}(X)$ the subset of all idempotents of S contained in X.

If X and Y are subset of a semigroup S, we denote by $\langle X \rangle$ the subsemigroup generated by X and $XY = \{xy : x \in X, y \in Y\}$. In particular, if $X = \{x\}$ or $Y = \{y\}$, we simply write $XY = xY$ and $XY = Xy$ respectively.

Note that ideals in semigroups are defined in much the same way as in rings. A non-empty subset I of a semigroup S is said to be a *left* (*right*) *ideal* of S if $SI \subseteq I$ ($IS \subseteq S$). If I is both a left and a right ideal of S, we say that I is an *ideal* of S. In this case, the set $S/I = (S \setminus I)^0$ is a semigroup with the products not falling in $S \setminus I$ are zero. This notion of ideal leads naturally to the consideration of Green's relations and preorders [14] that are extremely important in the structural study of semigroups:

$s \leq_{\mathcal{R}} t$ if, and only if, $S^1 s \subseteq S^1 t$,	$s \,\mathcal{R}\, t$ if, and only if, $S^1 s = S^1 t$
$s \leq_{\mathcal{L}} t$ if, and only if, $sS^1 \subseteq tS^1$,	$s \,\mathcal{L}\, t$ if, and only if, $sS^1 = tS^1$
$s \leq_{\mathcal{J}} t$ if, and only if, $J_s \subseteq J_t$,	$s \,\mathcal{J}\, t$ if, and only if, $S^1 sS^1 = S^1 tS^1$
$s \leq_{\mathcal{H}} t$ if, and only if, $s \leq_{\mathcal{R}} t \wedge s \leq_{\mathcal{L}} t$,	$s\mathcal{H}t$ if, and only if, $s \,\mathcal{R}\, t \wedge s \,\mathcal{L}\, t$.

If $\mathcal{K}$ is one of the Green's relations in a semigroup S, then the equivalence classes of the relation $\mathcal{K}$ are the $\mathcal{K}$-classes. In particular, we shall use the notation K_s for the respective $\mathcal{K}$-class of an element s of S. At the same time, we can observe that each preorder associated with $\mathcal{K}$ define a partial order between the $\mathcal{K}$-classes of S: $K_s \leq_{\mathcal{K}} K_t$ if, and only if, $s \leq_{\mathcal{K}} t$. Thus, if $0 \in S$, $\{0\}$ is the minimal $\mathcal{K}$-class of S. We say that a $\mathcal{J}$-class is *0-minimal* if J is minimal between the non-zero $\mathcal{J}$-classes of S.

Moreover, if T is a subsemigroup of S we can consider the relation $\mathcal{K}$ in T, which shall be denoted by $\mathcal{K}_T$ to avoid confusion with $\mathcal{K}$ defined in S.

A subsemigroup L of a semigroup S is called a *subgroup* of S if L under the operation of S is a group. In this case, the identity e of H is an idempotent of S and L is contained in the group of the units of the subsemigroup eSe. It is well-known that this last subgroup corresponds to the $\mathcal{H}$-class of e, H_e, so that it is the *maximal subgroup* of S having e as an identity element (see [23, Proposition 1.13], for example). For every idempotent e of a semigroup, we shall denote by S_e the maximal subgroup of S at e, which is a more convenient notation than H_e, when maximal subgroups in different subsemigroups containing the same idempotent are considered.

The notion of *regularity* is also important in the theory of semigroups. It was introduced by Green in the later sections of his paper [14]: an element x of a semigroup S is called *regular* if $x \in xSx$. The semigroup S is called *regular* if every element is regular. This is a notion that had been introduced for rings by von Neumann. He developed regular rings as a tool for his study of lattices, particularly complemented modular lattices, which were then thought to provide a suitable abstract framework for certain aspects of quantum mechanics. Concerning regular semigroups, Green proved that such semigroup may be characterised as a semigroup in which each element is $\mathcal{J}$-related to at least one idempotent. In general, we say that a $\mathcal{K}$-class is *regular* if it contains an idempotent, where $\mathcal{K}$ is one of the Green's relations.

We say that two elements s, t of a semigroup are *inverse elements* if $sts = s$ and $tst = t$. A semigroup S is said to be inverse if every element has just one inverse element. Inverse semigroups are central in semigroup theory and they form one of the most studied class of semigroups. It may be that the success of the theory of inverse semigroups stems from the fact that they are very close to groups in many of their properties. Nevertheless, inverse semigroups are not generalisations for generalisation's sake, but they arose in response to certain mathematical demands: for example, to describe some invariants in differential geometry, and they are closely related to semigroups of injective partial transformations.

If X is a subset of an inverse semigroup S we denote $X^{-1} = \{x^{-1} : x \in X\}$.

Now, let us introduce some basic well-known results concerning the concepts defined up to now.

Lemma 1. *Let s, t elements of a semigroup S. Then $st \leq_{\mathcal{J}} s$ and $st \leq_{\mathcal{J}} t$.*

Lemma 2 (see [12])**.** *Let S be a semigroup (not necessarily finite). Then:*

1. *If S is regular, then $\langle \mathrm{E}(S) \rangle$ is a regular subsemigroup of S.*
2. *If T is a regular subsemigroup of S. Then, $\mathcal{K}_T = \mathcal{K}_S \cap (T \times T)$, where $\mathcal{K} = \mathcal{L}$ or $\mathcal{R}$.*

Lemma 3.

1. *A subsemigroup T of an inverse semigroup S is an inverse subsemigroup if $a \in T$ implies $a^{-1} \in T$.*
2. *If s, t are elements of an inverse semigroup S, we have that $(s^{-1})^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1}$.*
3. *S is an inverse semigroup if, and only if, S is regular and its idempotents commute. As a consequence, if S is an inverse semigroup, $\mathrm{E}(S)$ is a subsemigroup of S.*

Another keystone for the development of semigroup theory was the notion of a semigroup to be 0-simple, introduced by Rees in [25]: a semigroup S is said to be 0-*simple* if $S^2 \neq 0$ and 0 is the only proper ideal of S, or equivalently, if $SxS = S$ for all $0 \neq x \in S$. In this paper, he went on to derive a semigroup analogue of the Artin-Wedderburn Theorem for semisimple rings and algebras. His result is now known, appropriately enough, as the Rees Theorem, or, occasionally, as the Rees-Sushkevich Theorem, since it subsumes an earlier result of Sushkevich. This theorem was semigroup theory's first major structure theorem. The main ingredient of this result is the notion of *Rees matrix semigroup* which is defined in the following way: let A, B be non-empty finite sets and let G be a group. A *Rees matrix* C is a map $C : B \times A \longrightarrow G^0$. We say that the Rees matrix is *regular* if every row and every column has a non-zero entry. Then, the *Rees matrix semigroup* with sandwich matrix C is the semigroup $\mathcal{M}^0(G, A, B, C)$ with underlying set $(A \times G \times B) \cup \{0\}$ and the operation: $0 \cdot (a, g, b) = (a, g, b) \cdot 0 = 0$, and

$$(a, g, b) \cdot (a', g', b') = \begin{cases} (a, gC(b, a')g', b') & \text{if } C(b, a') \in G, \\ 0, & \text{if } C(b, a') = 0, \end{cases}$$

for all $a, a' \in A$, $b, b' \in B$, $g \in G$.

It is known that the Rees matrix semigroup with sandwich matrix C is a regular semigroup if, and only if, C is a regular matrix, and in this case $\mathcal{M}^0(G, A, B, C)$ is a 0-simple semigroup.

The Rees theorem [25] may now be stated:

Theorem 4. *Every regular Rees matrix semigroup is a* 0*-simple semigroup. Conversely, every* 0*-simple semigroup S is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$, where G is isomorphic to the maximal subgroups S_e, for all $e \neq 0$.*

In the case when a 0-simple semigroup S is also an inverse semigroup, then S is isomorphic to a Rees matrix semigroup of the form $\mathcal{M}^0(G, \Lambda, \Lambda, I_n)$, where $n = |\Lambda|$. Such semigroups are called *Brandt semigroups*, which come from the notion of a groupoid, a structure of a binary system in which products are not always defined. Groupoids where introduced by Brandt in 1927 [7] and generalise the notion of group in several equivalent ways, having applications in topology and manifold theory (see [8, Chapter 3.3], for more information).

The following lemma is elementary.

Lemma 5. *The following statements hold:*

1. *For every $(i, g, j), (i', g', j') \in \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, $0 \neq (i, g, j)(i', g', j')$ if, and only if, $j = i'$. Moreover, $(i, g, j)^{-1} = (j, g^{-1}, i)$.*

2. $\mathrm{E}(J^0) = \{0\} \cup \{(i, 1, i) : i \in \Lambda\}$ *and* $S_{(i,1,i)} = (J^0)_{(i,1,i)} = (i, G, i)$.

3. *For every $s \in S$ and $x \in J^0$, $sx, xs \in J^0$.*

Graham [13] published an influential contribution to the structural study of a 0-simple semigroup. He showed how to apply graph theory to obtain a description of the idempotent-generated subsemigroup of a 0-simple semigroup.

Theorem 6. *Let S be a 0-simple semigroup. Then there exists an isomorphism:*

$$\psi : S \longrightarrow \mathcal{M}^0(G, A, B, C)$$

from S to a Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$ such that:

- *The matrix C is the direct sum of the n matrices $C_1, \ldots, C_n$ as shown below:*

$$\begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{array}$$

- *Each matrix $C_i : B_i \times A_i \to G^0$ is regular and:*

$$\langle \mathrm{E}(S) \rangle = \bigcup_{i=1}^{n} \mathcal{M}^0(G_i, A_i, B_i, C_i)$$

  where G_i is the subgroup of G generated by all non-zero entries of C_i, for $i = 1, \ldots, n$.

Graham's result was republished ten years after by Howie (see [18]) and Houghton (see [17]). This last author added topological techniques and cohomology that have had a strong influence in the proof of the theorem presented in [27]. In the following subsection, we present a very elementary proof of that result.

We bring the section to a close with the following constructions:

Construction 1: Let J be a regular $\mathcal{J}$-class of a semigroup S. Then, J^0 is a semigroup with the operation:

$$a \cdot b = \begin{cases} ab & \text{if } ab \in J \\ 0 & \text{otherwise} \end{cases}$$

for all $a, b \in J$, and $0 \cdot a = a \cdot 0 = 0$ for all $a \in J$. It holds that J^0 is a 0-simple semigroup and then $J^0 \cong \mathcal{M}^0(G, A, B, C)$, where $G \cong (J^0)_e = S_e$, for every $e \in \mathrm{E}(J)$. In the case when S is an inverse semigroup, J^0 is a Brandt semigroup and then isomorphic to $\mathcal{M}^0(G, \Lambda, \Lambda, I_n)$, where $n = |\Lambda|$ (see [8, Chapter 3], for example).

If J is a 0-minimal $\mathcal{J}$-class of a semigroup with zero S, by Lemma 1, $J_{st} \leq_{\mathcal{J}} J_s = J_t$ for all $s, t \in J$. Hence, $J_{st} = J_s = J_t$, for every $s, t \in J$ with $st \neq 0$. Consequently, J^0 is a subsemigroup of S.

Construction 2: Let $J = J_s$ be a $\mathcal{J}$-class of a semigroup S with zero. Set $X_J := \{t \in S : s \leq_{\mathcal{J}} t\}$. By Lemma 1, $S \setminus X_J$ is an ideal of S. Therefore $S_J := S/(S \setminus X_J) = (X_J)^0$ is a semigroup with zero. Note that if S is inverse, then S_J is inverse as well.

Recall that the product in S_J is defined by:

$$s \cdot t = \begin{cases} st, & \text{if } st \in S_J, \\ 0, & \text{otherwise.} \end{cases}$$

Let $0 \neq a, b \in S_J$. Assume that $a \leq_{\mathcal{J}_S} b$. Then, there exist $x, y \in S^1$ such that $a = xby$. By Lemma 1, $x, y \in S_J^1$ and so $a \leq_{\mathcal{J}_{S_J}} b$. This shows that relation $\mathcal{J}$ coincide in both semigroups, so that J is the unique 0-minimal $\mathcal{J}$-class of S_J.

3.2 A very elementary proof of Graham's Theorem

In this subsection, S will denote a 0-simple semigroup and $T := \langle \mathrm{E}(S) \rangle$.

Our concern here is in applying basic results on regularity to the method used by Rees to prove his isomorphism theorem. This will lead us to a new proof of Graham's Theorem.

The next two lemmas proved in [25] and their corollaries are absolutely essential in our approach.

Key Lemma 1. [25, Lemmas 2.61, 2.62, 2.63] *For each pair of non-zero idempotents e and f of S, eSf is non-zero and there exist $0 \neq x \in eSf$ and $0 \neq y \in fSe$ such that $xy = e$ and $yx = f$.*

Key Lemma 2. [25, Lemma 2.7] *Let $e, f \in \mathrm{E}(S) \setminus \{0\}$. The sets eS and fS have either no non-zero elements in common or are identical. Similarly for the sets Se and Sf and, consequently, for the sets eSf and $e'Sf'$.*

Corollary 7. *Let $0 \neq ef \in T$ with $e, f \in \mathrm{E}(S)$. Then $e \, \mathcal{R}_T \, ef \, \mathcal{L}_T \, f$. In particular, $e \, \mathcal{J}_T \, f \, \mathcal{J}_T (ef)$.*

Proof. Since S is regular, T is regular by Lemma 2. Then, there exists $0 \neq a \in T$ such that $(ef)a(ef) = ef$ and $0 \neq (ef)a =: g$ is idempotent. Hence, $gS = (ef)S \subseteq eS$ and, by Key Lemma 2, $gS = (ef)S = eS$, i.e. $e \, \mathcal{R} (ef)$. Now, we can apply Lemma 2 to conclude $e \, \mathcal{R}_T \, ef$.

Analogously, we have $(ef) \, \mathcal{L}_T \, f$ and therefore $e \, \mathcal{J}_T \, f \, \mathcal{J}_T (ef)$. □

Corollary 8. *Assume that* $0 \neq e_1 \cdots e_r \in T$ *for some* $e_i \in \mathrm{E}(S)$, $1 \leq i \leq r$. *Then* $e_1 \mathcal{J}_T \ldots \mathcal{J}_T e_r \mathcal{J}_T (e_1 \cdots e_r)$.

Now, we split the proof into the following steps.

Step 1. Let $(J_T)_{e_1}, \ldots, (J_T)_{e_n}$ be the non-zero $\mathcal{J}$-classes in T. Since T is regular, we may assume that $e_1, \ldots, e_n \in \mathrm{E}(S)$. For each $k \in \{1, \ldots, n\}$, we write:

$$T^{(k)} = (J_T)^0_{e_k} \subseteq T$$

Corollary 8 ensures us that $T^{(k)}$ is a 0-simple subsemigroup of T, for each $1 \leq k \leq n$.

Step 2. Since S is regular, we have that:

$$S = \bigcup_{i,j} r_i S l_j \quad \text{where } r_i, l_j \in \mathrm{E}(S) \setminus \{0\}, \quad i = 1, \ldots, m,\ j = 1, \ldots, l$$

Moreover, by Key Lemma 2, we can choose the idempotents r_i, l_j such that $r_1 = l_1 = e_1$, and $r_i S l_j \cap r_{i'} S l_{j'} = 0$ if, and only if, either $i \neq i'$ or $j \neq j'$, for every $1 \leq i, i' \leq m$ and every $1 \leq j, j' \leq l$.

Set $A := \{1, \ldots, m\}$ and $B := \{1, \ldots, l\}$. For each $1 \leq k \leq n$, we define:

$$A_k := \{i \in A : r_i \mathcal{J}_T e_k\}, \quad B_k := \{j \in B : l_j \mathcal{J}_T e_k\}.$$

Then $\{A_k\}_{k=1}^n$, $\{B_k\}_{k=1}^n$ are partitions of A and B, respectively.

Step 3. Let $k \in \{1, \ldots, n\}$. Applying Key Lemma 1 we have:

i) There exist non-zero elements $x_{1k} \in e_1 S e_k$, $x_{k1} \in e_k S e_1$, such that $x_{1k} x_{k1} = e_1$ and $x_{k1} x_{1k} = e_k$. Hence:

$$\begin{array}{rcl} \varphi_{1k} \colon e_1 S e_1 & \to & e_k S e_k \\ s & \mapsto & x_{k1} s x_{1k} \end{array} \qquad \begin{array}{rcl} \varphi_{k1} \colon e_k S e_k & \to & e_1 S e_1 \\ s & \mapsto & x_{1k} s x_{k1} \end{array} \tag{3.1}$$

are isomorphisms.

ii) Since $T^{(k)}$ is 0-simple, for all $i \in A_k$, $j \in B_k$, there exist non-zero elements $\bar{p}_{ik} \in r_i T^{(k)} e_k$, $\bar{q}_{kj} \in e_k T^{(k)} l_j$, such that

$$0 \neq x_{1k} \bar{q}_{kj} \in e_1 S l_j, \quad 0 \neq \bar{p}_{ik} x_{k1} \in r_i S e_1$$

Then, for all $i \in A$ and $j \in B$, we define:

$$\begin{array}{ll} 0 \neq p_{i1} := \bar{p}_{ik} x_{k1} \in r_i S e_1 & \text{if } i \in A_k, \\ 0 \neq q_{1j} := x_{1k} \bar{q}_{kj} \in e_1 S l_j & \text{if } j \in B_k, \end{array}$$

Conclusion. According to Rees' Theorem, the maximal subgroups S_e, for all $0 \neq e \in \mathrm{E}(S)$, are all isomorphic. Let $G^0 := e_1Se_1 = (S_{e_1})^0$ and consider the Rees $(B \times A)$-matrix given by:

$$C(j,i) := \begin{cases} q_{1j}p_{i1} & \text{if } q_{1j}p_{i1} \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad j \in B, i \in A$$

The proof of Theorem 2.93 in [25] gives us an isomorphism

$$\psi : S \longrightarrow \mathcal{M}^0(G, A, B, C)$$

Moreover, by Corollary 7, if $j \in B_k$ and $i \in A_{k'}$ and $k \neq k'$, it follows that $l_jr_i = 0$. Therefore $C(j,i) = 0$ and then

$$C = \begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{array}$$

This proves the first statement of the theorem.

Fix $k \in \{1, \ldots, n\}$. Then, by (3.1), φ_{k1} is an isomorphism between e_kSe_k and e_1Se_1 such that $\varphi_{k1}(\bar{q}_{kj}\bar{p}_{ik}) = q_{1j}p_{i1}$, for all $i \in A_k, j \in B_k$, and φ_{k1} restricted to $T^{(k)}_{e_k}$ defines an isomorphism between $T^{(k)}_{e_k}$ and a subgroup H_k of G. Since $T^{(k)}$ is 0-simple and $\bar{p}_{ik} \in r_iT^{(k)}e_k$ and $\bar{q}_{kj} \in e_kT^{(k)}l_j$, for each $i \in A_k$ and $j \in B_k$, we can follow the proof of Theorem 2.93 in [25] to conclude that the restriction of ψ to $T^{(k)}$ defines an isomorphism between $T^{(k)}$ and $\mathcal{M}^0(H_k, A_k, B_k, \tilde{C}_k)$, where $\tilde{C}_k$ is defined by:

$$\tilde{C}_k(j,i) := \begin{cases} \varphi_{k1}(\bar{q}_{kj}\bar{p}_{ik}) & \text{if } \bar{q}_{kj}\bar{p}_{ik} \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad j \in B_k, i \in A_k$$

It is clear that $\bar{q}_{kj}\bar{p}_{ik} \neq 0$ if, and only if, $q_{1j}p_{i1} \neq 0$. Therefore $\tilde{C}_k = C_k$.

Since $T = \bigcup_{k=1}^n (T^k)$ and $\psi(T^k)$ is isomorphic to $\mathcal{M}^0(H_k, A_k, B_k, C_k)$, we have that $\psi(T)$ can be described as $\bigcup_{k=1}^n \mathcal{M}^0(H_k, A_k, B_k, C_k)$.

Moreover, $(i, g, j) \in \mathcal{M}^0(H_k, A_k, B_k, C_k)$ is a non-zero idempotent if, and only if, $C_k(j,i) \neq 0$ and $g = C_k(j,i)^{-1}$. Since every element of $T^{(k)}$ is a product of idempotents, it follows that $H_k = \langle \{0 \neq C_k(j,i) : j \in B_k, i \in A_k\} \rangle$.

The second statement of Graham's result now holds and the proof of the theorem is complete.

3.3 Kernels of semigroups

The purpose of this section is to introduce the notion of the generalised kernel of a semigroup. It is defined in terms of relational morphisms and varieties of groups.

Relational morphisms between semigroups were first introduced by J. Rhodes and B. Tilson in the mid-1970s. This very important notion, which can be seen as a generalisation of homomorphism of semigroups, is considered as one of the key tools when comparing semigroups, growing out of the basic ideas of decomposition theory. If you present a semigroup S as homomorphic image $f\colon C \to S$ of a semigroup C of a wreath product $A \wr T$, then the inverse image of f followed by the projection to T is a relational morphism from S to T. Conversely, given a relational morphism, one wants to know how to complete it to a wreath product division.

Then, a *relational morphism* $\tau\colon\ S \multimap T$ between two semigroups S and T is a map from S into $\mathcal{P}(T)$, the set of subsets of T, such that $\tau(s_1) \neq \emptyset$ and $\tau(s_1)\tau(s_2) \subseteq \tau(s_1 s_2)$, for all $s_1, s_2 \in S$. If τ is a relational morphism between monoids a third condition is required: $1 \in \tau(1)$.

A relational morphism $\tau\colon\ S \multimap T$ is said to be surjective if for every $t \in T$, there exists $s \in S$ such that $t \in \tau(s)$. Given $\tau_1\colon\ S \multimap T$ and $\tau_2\colon\ T \multimap U$ two relational morphisms, we can define:

$$(\tau_2 \circ \tau_1)(s) = \bigcup \{\tau_2(t) : t \in \tau_1(s)\}, \text{ for every } s \in S.$$

Then $\tau_2 \circ \tau_1\colon\ S \multimap U$ is again a relational morphism between S and U.

Given a relational morphism between a semigroup S and a group G, $\tau\colon\ S \multimap G$, we can consider the set $\tau^{-1}(1) := \{s \in S : 1 \in \tau(s)\}$, which is clearly a subsemigroup of S called the *kernel of* τ. In particular, for every $e \in \mathrm{E}(S)$, $e \in \tau^{-1}(1)$ because $\tau(e)$ is a subgroup of G.

Moreover, if S is an inverse semigroup and $s \in S$, then $\tau(s)\tau(s^{-1}) \subseteq \tau(ss^{-1})$. Hence $|\tau(s)| \leq |\tau(ss^{-1})|$. On the other hand, we have $\tau(ss^{-1})\tau(s) \subseteq \tau(ss^{-1}s) = \tau(s)$ and then $|\tau(ss^{-1})| \leq |\tau(s)|$. Thus, we have $1 \in \tau(s)\tau(s^{-1}) = \tau(ss^{-1})$ and therefore, there exists $x \in \tau(s)$ such that $x^{-1} \in \tau(s^{-1})$.

This leads to the following Proposition which shows the behaviour of the images of a unique 0-minimal $\mathcal{J}$-class of an inverse semigroup S under a relational morphism between S and a group G.

Proposition 9. *Let S be an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class J, such that $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda) \leq S$. Let H be a group and $\tau\colon\ S \multimap H$ be a relational morphism. Then, the following properties hold:*

1. *For every* $i \in \Lambda$, $\tau(i,1,i) =: K_i \leq H$ [1].

2. *Given* $(i,g,j) \in J$, *for every* $x \in \tau(i,g,j)$, $\tau(i,g,j) = K_i x = x K_j$.

3. *For every* $s,t \in J$, $|\tau(s)| = |\tau(t)|$ *and* $\tau(s)\tau(t) = \tau(st)$, *in case that* $st \neq 0$.

Proof. Having into account Lemma 5, the proof comes easily. 1. Clear, from the fact that we know that relational morphisms between a semigroup and a group send idempotents to subgroups.

2. Let $(i,g,j) \in J$. Since S is an inverse semigroup we know that there exists $z \in \tau(i,g,j)$ such that $z^{-1} \in \tau(j,g^{-1},i)$.

Let $x \in \tau(i,g,j)$. We have that $(i,1,i)(i,g,j) = (i,g,j)$ and therefore:

$$K_i x \subseteq \tau(i,1,i)\tau(i,g,j) \subseteq \tau(i,g,j)$$

Similarly we get that $K_i z \subseteq \tau(i,g,j)$ and then $K_i z = K_i x \subseteq \tau(i,g,j)$.

On the other hand, let $y \in \tau(i,g,j)$. Then

$$yz^{-1} \in \tau(i,g,j)\tau(j,g^{-1},i) \subseteq \tau(i,1,i) = K_i$$

i.e. $y \in K_i z = K_i x$. Hence, we can conclude that $\tau(i,g,j) = K_i x$.

Analogously, since $(i,g,j) = (i,g,j)(j,1,j)$, we can also prove that $xK_j = \tau(i,g,j)$.

3. First, we claim that for every $i,j \in \Lambda$, K_i and K_j are conjugate. In fact, by 2, $\tau(i,1,j) = K_i x = x K_j$, for every $x \in \tau(i,1,j)$. Therefore, $K_i^x = K_j$, for every $x \in \tau(i,1,j)$.

Then, again by 2, we can ensure that the image under τ of every element in J has the same cardinality. On the other hand, since τ is a relational morphism, $\tau(s)\tau(t) \subseteq \tau(st)$, for every $s,t \in J$. Moreover, if $st \neq 0$, we have that $|\tau(s)| \leq |\tau(s)\tau(t)| \leq |\tau(st)| = |\tau(s)|$ and then $|\tau(s)\tau(t)| = |\tau(st)|$, i.e. $\tau(s)\tau(t) = \tau(st)$.

□

From now on, we shall be interested in relational morphisms into groups in a variety.

Recall that a *formation* is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and subdirect products. A formation which is closed under taking subgroups is called *variety* (or sometimes *pseudovariety*).

[1] For the sake of clarity, given an element of a Rees matrix semigroup $(i,g,j) \in \mathcal{M}^0(G,A,B,C)$, we denote its image under a relational morphism $\tau(i,g,j)$ instead of $\tau((i,g,j))$

Given a non-empty formation $\mathfrak{F}$, each group G has a smallest normal subgroup whose quotient belongs to $\mathfrak{F}$; this is called the $\mathfrak{F}$-*residual* of G and it is denoted by $G^{\mathfrak{F}}$. Clearly, $G^{\mathfrak{F}}$ is a characteristic subgroup of G and it is the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{F}$ (see [6, Section 2.2] for further details).

Let $\mathfrak{F}$ be a variety of groups. We consider the intersection $\mathrm{K}_{\mathfrak{F}}(S)$ of the kernels of all relational morphisms $\tau\colon\ S \multimap G$, between S and every group $G \in \mathfrak{F}$; $\mathrm{K}_{\mathfrak{F}}(S)$ is a subsemigroup of S called the $\mathfrak{F}$-*kernel* of S. The case when $\mathfrak{F} = \mathfrak{G}$, the variety of all groups, the $\mathfrak{G}$-kernel of S is just called the *kernel of* S.

It is clear that if W is a subsemigroup of S, then $\mathrm{K}_{\mathfrak{F}}(W)$ is a subsemigroup of $\mathrm{K}_{\mathfrak{F}}(S)$.

In order to compute the $\mathfrak{F}$-kernel of a semigroup S, it is enough to consider only surjective relational morphisms since $\mathfrak{F}$ is subgroup-closed. In addition, if S is a group, and $\tau\colon\ S \multimap H$ is a surjective relational morphism between S and $H \in \mathfrak{F}$, it follows that $\tau(1)$ is a normal subgroup of H and if $x \in S$, then $y\tau(1) = \tau(x)$ for all $y \in \tau(x)$. Hence the map $\tilde{\tau}\colon S \longrightarrow H/\tau(1) \in \mathfrak{F}$, given by $\tilde{\tau}(x) = \tau(x)$ is a group homomorphism. These observations allow us to confirm that the $\mathfrak{F}$-kernel actually generalises the notion of $\mathfrak{F}$-residual.

Proposition 10 ([11, Proposition 9.6])**.** *If G is a group and $\mathfrak{F}$ is a variety, then* $\mathrm{K}_{\mathfrak{F}}(G) = G^{\mathfrak{F}}$.

The above proposition does not hold for formations in general, as the following example shows.

Example 11. Let $\mathfrak{F}$ be the formation generated by A_5, the alternating group of degree 5. Then the $\mathfrak{F}$-residual of the cyclic group G of order 2 generated by ξ is G. However, there exists a non-trivial relational morphism between G and A_5: if $a \in A_5$ with order 2, let us define $\tau\colon\ G \multimap A_5$ taking $\tau(\xi) = a$ and $\tau(1) = 1$. Hence, $\mathrm{K}_{\mathfrak{F}}(G) = \{1\}$.

In the sequel, we consider only varieties of groups when we study $\mathfrak{F}$-kernels of semigroups. Finally, we bring the section to a close by presenting the following results relating $\mathfrak{F}$-kernels and $\mathfrak{F}$-residuals of the maximal subgroups of a semigroup.

Proposition 12. *Let S be a semigroup and let $\mathfrak{F}$ be a variety of groups. Then, for every $e \in \mathrm{E}(S)$, $(S_e)^{\mathfrak{F}}$ is a subgroup of* $(\mathrm{K}_{\mathfrak{F}}(S))_e$.

Proof. Let $e \in \mathrm{E}(S)$. Since $S_e \leq S$, we have that $\mathrm{K}_{\mathfrak{F}}(S_e) \leq \mathrm{K}_{\mathfrak{F}}(S)$. Since $(S_e)^{\mathfrak{F}}$ is a subgroup of $\mathrm{K}_{\mathfrak{F}}(S_e)$ and e is the identity element of $(S_e)^{\mathfrak{F}}$, it follows that $(S_e)^{\mathfrak{F}} \leq (\mathrm{K}_{\mathfrak{F}}(S))_e$. □

Corollary 13. *Let S be an inverse semigroup and let $\mathfrak{F}$ be a variety of groups. Suppose that* $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{E}(S)$. *Then* $S_e \in \mathfrak{F}$, *for every* $e \in \mathrm{E}(S)$.

Proof. Let S be an inverse semigroup, and $e \in \mathrm{E}(S)$. By Lemma 3, $\mathrm{E}(S)$ is a subsemigroup of S. Thus $(\mathrm{E}(S))_e = \{e\}$. By Proposition 12, $(S_e)^{\mathfrak{F}}$ is contained in $(\mathrm{K}_{\mathfrak{F}}(S))_e = (\mathrm{E}(S))_e = \{e\}$. Hence, $S_e \in \mathfrak{F}$. $\square$

Chapter 4

On the computability of the generalised kernel: a reduction theorem

The kernel of a semigroup was introduced by Rhodes aiming to treat questions related to the group-complexity of a semigroup, which is one of the most important open problems in semigroup theory. However, its definition is clearly non constructive and so the study of the computability of the kernel naturally arises. This problem, presented in the seminal paper of Rhodes and Tilson [28], had attracted the attention of many researchers during almost 20 years.

The notion of weak conjugacy is crucial here. We say that a pair of elements (s,t) of a semigroup S forms *a pair of weak conjugay* if either $sts = s$ or $tst = t$. Then, a subsemigroup $K \leq S$ is said to be closed under weak conjugacy if $sKt, tKs \subseteq K$, for every pair of conjugacy (s,t).

Rhodes and Tilson characterised the regular elements of the kernel of a semigroup S as the regular elements of the smallest subsemigroup of S closed under weak conjugation. This result led Rhodes to conjecture that the kernel of a semigroup should be the smallest subsemigroup closed under weak conjugation. The conjecture, called 'Type II Conjecture", remained open for about 20 years, was solved independently by Ash in [3] and Ribes and Zalesskiĭ in [29] after the translation of the problem into profinite topology by Pin and Reutenauer [24].

The aim of this chapter is to present a contribution to the solution of the computability of the $\mathfrak{F}$-kernel from an structural approach. If $\mathfrak{F}$ is a variety, the $\mathfrak{F}$-kernel $\mathrm{K}_{\mathfrak{F}}(S)$ of a semigroup S is computable if, and only if, $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every $\mathcal{J}$-class J of S. Hence, in the sequel, we shall be concerned about the computability of $\mathrm{K}_{\mathfrak{F}}(S) \cap J$. In this context, the following theorem of Steinberg [33] is the most optimal result so far.

Theorem 14. *Let $\mathfrak{F}$ be a variety of groups. We can compute the regular elements of* $\mathrm{K}_{\mathfrak{F}}(S)$ *for every semigroup S if, and only if,* $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap J$ *is computable for every inverse semigroup $\bar{S}$ and every $\mathcal{J}$-class J of $\bar{S}$.*

The following proposition allows us to conclude that it is enough to consider inverse semigroups with zero with a unique 0-minimal $\mathcal{J}$-class in order to compute the $\mathfrak{F}$-kernel. It was proved by Rhodes and Tilson in [28] for the variety of all groups, and it still holds for a general variety of groups $\mathfrak{F}$.

Proposition 15 ([28, Fact 2.17])**.** *Let I be an ideal of S. Then:*

$$\mathrm{K}_{\mathfrak{F}}(S/I) \setminus \{0\} = \mathrm{K}_{\mathfrak{F}}(S) \cap (S \setminus I).$$

Hence, applying Proposition 15, we get that the membership problem for $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ can be reduced to semigroups S where J is the unique 0-minimal $\mathcal{J}$-class.

Lemma 16. *Let S be a semigroup with zero and let J be a $\mathcal{J}$-class of S. Then:*

$$\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(S_J) \cap J.$$

Proof. Consider the ideal $S \setminus X_J$, where $J = J_s$ and $X_J = \{t \in S : s \leq_{\mathcal{J}} t\}$. Then $S_J = S/(S \setminus X_J)$. By Proposition 15,

$$\mathrm{K}_{\mathfrak{F}}(S_J) \setminus \{0\} = \mathrm{K}_{\mathfrak{F}}(S) \cap \big(S \setminus (S \setminus X_J)\big) = \mathrm{K}_{\mathfrak{F}}(S) \cap X_J.$$

In particular, since $J \subseteq X_J$, we have that $\mathrm{K}_{\mathfrak{F}}(S_J) \cap J = \mathrm{K}_{\mathfrak{F}}(S) \cap J$.

□

According to the construction at the end of Section 3.1, S_J is a semigroup with zero and J is the unique 0-minimal $\mathcal{J}$-class of S_J. Hence we have:

Corollary 17. *Let $\mathfrak{F}$ be a variety of groups. Then,* $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ *is computable for every inverse semigroup S and every $\mathcal{J}$-class J of S if, and only if,* $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ *is computable for every inverse semigroup $\bar{S}$ with zero having a unique* 0*-minimal $\mathcal{J}$-class $\bar{J}$.*

Finally, we can state our reduction theorem:

Theorem C. *Let $\mathfrak{F}$ be a variety of groups. Then the following statements are pairwise equivalent:*

1. *The regular elements of the $\mathfrak{F}$-kernel of every semigroup are computable.*

2. $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ *is computable for every inverse semigroup S and every $\mathcal{J}$-class J of S.*

3. $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ *is computable for every inverse semigroup $\bar{S}$ with zero with a unique* 0*-minimal $\mathcal{J}$-class J such that $\bar{S}_e \in \mathfrak{F}$, for each $e \in \mathrm{E}(\bar{S})$.*

4.1 Inverse semigroups and projections

Before proving our main result, we must deal with some structural facts concerning inverse semigroups and formalise some notation and terminology.

Throughout the section, S will be an inverse semigroup with zero and J will be a 0-minimal $\mathcal{J}$-class of S. In this case, J^0 is a Brandt subsemigroup of S which is isomorphic to a regular Rees matrix semigroup of the form $\mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, where $S_e = (J^0)_e \cong G$, for every $e \in \mathrm{E}(J)$ (see [8]). In fact, without loss of generality we can assume that $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$.

Statement 3 of Lemma 5 is telling us that S acts on J^0 by left and right multiplication. The following concepts arise naturally from these actions (see [28]).

If $s \in S$, we consider the sets:

$$\begin{aligned}&\alpha_s := \{i \in \Lambda : (i,1,i)s \neq 0\}, \quad \omega_s := \{j \in \Lambda : s(j,1,j) \neq 0\},\\ &\mathrm{E}_s := \{(i,1,i)s : i \in \alpha_s\}.\end{aligned}$$

Note that if $i \in \alpha_s$, then $0 \neq (i,1,i)s = (i', g, j)$, for some $i', j \in \Lambda$ and $g \in G$. Therefore, we get:

$$0 \neq (i,1,i)s = (i,1,i)\big((i,1,i)s\big) = (i,1,i)(i', g, j).$$

Thus $i' = i$. Moreover, $0 \neq (i,g,j)(j,1,j) = (i,1,i)s(j,1,j)$. Then $s(j,1,j) \neq 0$ and $j \in \omega_s$. Similarly, if $(i', g', j') = s(j,1,j)$, then we get $j' = j$ and:

$$(i,1,i)s(j,1,j) = \begin{cases} \big((i,1,i)s\big)(j,1,j) = (i,g,j)(j,1,j) = (i,g,j) \\ (i,1,i)\big(s(j,1,j)\big) = (i,1,i)(i',g',j) \end{cases}$$

Hence, $i' = i$ and $s(j,1,j) = (i,g,j) = (i,1,i)s$.

Consider the maps:

$$\iota_s : \alpha_s \to \omega_s, \quad \gamma_s : \alpha_s \to G$$

defined by, $\iota_s(i) = j$ and $\gamma_s(i) = g$ if, and only if, $(i,1,i)s = (i,g,j) = s(j,1,j)$. Then ι_s is a bijection and $\iota_s^{-1}(j) = i$ if, and only if, $s(j,1,j) = (i, \gamma_s(i), j) = (i,1,i)s$.

Moreover, $\mathrm{E}_s := \{(i,1,i)s : i \in \alpha_s\} = \{(i, \gamma_s(i), \iota_s(i)) : i \in \alpha_s\}$.

Note that $\alpha_0 = \omega_0 = \mathrm{E}_0 = \varnothing$.

Moreover, for every $i, j \in \Lambda$, we have:

$$|\,\mathrm{E}_s \cap (i, G, \Lambda)| = \begin{cases} 1, & \text{if } i \in \alpha_s \\ 0, & \text{otherwise} \end{cases}, \quad |\,\mathrm{E}_s \cap (\Lambda, G, j)| = \begin{cases} 1, & \text{if } j \in \omega_s \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Definition 18. The set E_s is called the *set of projections* of s onto J^0.

Let us now investigate these projections in more detail.

Proposition 19. *The following statements hold:*

1. *If $s = (i, g, j) \in J$, then $\alpha_s = \{i\}$, $\omega_s = \{j\}$, $\iota_s(i) = j$, $\gamma_s(i) = g$ and $\mathrm{E}_s = \{(i, g, j)\}$.*

2. *For every $s, t \in S$, $\alpha_{st} = \{i \in \alpha_s : \iota_s(i) \in \alpha_t\} = \iota_s^{-1}(\omega_s \cap \alpha_t)$. Moreover, $\iota_{st}(i) = \iota_t(\iota_s(i))$ and $\gamma_{st}(i) = \gamma_s(i)\gamma_t(\iota_s(i))$, for every $i \in \alpha_{st}$.*

3. *For every $s, t \in S$,*

$$\mathrm{E}_{st} = \{(i, \gamma_{st}(i), \iota_{st}(i)) : i \in \alpha_{st}\} = \mathrm{E}_s\, \mathrm{E}_t \setminus \{0\}.$$

If $(i, \gamma_{st}(i), \iota_{st}(i)) \in \mathrm{E}_{st}$, then

$$(i, \gamma_{st}(i), \iota_{st}(i)) = \big(i, \gamma_s(i), \iota_s(i)\big)\big(\iota_s(i), \gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big)$$

is the unique decomposition of $(i, \gamma_{st}(i), \iota_{st}(i))$ as a product of an element in E_s and an element of E_t.

4. *For all $s \in S$, $\alpha_{s^{-1}} = \omega_s$ and $\omega_{s^{-1}} = \alpha_s$. Moreover, $\iota_{s^{-1}} = \iota_s^{-1}$ and $\gamma_{s^{-1}}(j) = \gamma_s(i)^{-1}$, for each $j = \iota_s(i) \in \alpha_{s^{-1}}$. As a consequence, $\mathrm{E}_{s^{-1}} = (\mathrm{E}_s)^{-1}$.*

5. *If $0 \neq e \in \mathrm{E}(S)$, then $\alpha_e = \omega_e$, $\iota_e(i) = id_{\alpha_e}$ and $\gamma_e(i) = 1$, for every $i \in \alpha_e$. Hence, $\mathrm{E}_e = \{(i, 1, i) : i \in \alpha_e\}$.*

6. *For every $s, t \in S$, $\mathrm{E}_s = \mathrm{E}_t$ if, and only if, $\alpha_s = \alpha_t$, $\omega_s = \omega_t$, $\iota_s = \iota_t$ and $\gamma_s = \gamma_t$.*

Proof. 1 is clear.

2 and 3. Let $s, t \in S$ and suppose $i \in \alpha_{st}$. Then, $(i, 1, i)(st) \neq 0$ and therefore, $i \in \alpha_s$ because $(i, 1, i)s \neq 0$. Thus, we see:

$$(i, 1, i)(st) = \big((i, 1, i)s\big)t = (i, \gamma_s(i), \iota_s(i))t = (i, \gamma_s(i), \iota_s(i))(\iota_s(i), 1, \iota_s(i))t$$

i.e. $(\iota_s(i), 1, \iota_s(i))t \neq 0$ and $\iota_s(i) \in \alpha_t$. On the other hand, suppose that $\iota_s(i) \in \alpha_t$, for some $i \in \alpha_s$. Then:

$$\begin{aligned}(i, 1, i)(st) &= \big((i, 1, i)s\big)t = (i, \gamma_s(i), \iota_s(i))t = \\ &= (i, \gamma_s(i), \iota_s(i))\big((\iota_s(i), 1, \iota_s(i))t\big) = \\ &= \big(i, \gamma_s(i), \iota_s(i)\big)\big(\iota_s(i), \gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big) = \\ &= \big(i, \gamma_s(i)\gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big) \neq 0.\end{aligned}$$

Hence, we conclude $\alpha_{st} = \{i \in \alpha_s : \iota_s(i) \in \alpha_t\}$. Moreover, we have shown $\iota_{st}(i) = \iota_t(\iota_s(i))$ and $\gamma_{st}(i) = \gamma_s(i)\gamma_t(\iota_s(i))$, for every $i \in \alpha_{st}$. Consequently:

$$\mathrm{E}_{st} = \{(i, \gamma_{st}(i), \iota_{st}(i) : i \in \alpha_{st}\} = \{\big(i, \gamma_s(i)\gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big) : i \in \alpha_{st}\}.$$

Now, for every $i \in \alpha_{st}$, we can write:

$$\begin{aligned}(i, \gamma_{st}(i), \iota_{st}(i)) &= \big(i, \gamma_s(i)\gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big) = \\ &= \big(i, \gamma_s(i), \iota_s(i)\big)\big(\iota_s(i), \gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big),\end{aligned}$$

i.e. $\mathrm{E}_{st} \subseteq \mathrm{E}_s \mathrm{E}_t \setminus \{0\}$. On the other hand, let $x = (i, \gamma_s(i), \iota_s(i)) \in \mathrm{E}_s$ and $y = (i', \gamma_t(i'), \iota_t(i')) \in \mathrm{E}_t$, such that $0 \neq xy \in \mathrm{E}_s \mathrm{E}_t$. Then, $\iota_s(i) = i'$, i.e. $i \in \alpha_{st}$, $y = (\iota_s(i), \gamma_t(\iota_s(i)), \iota_t(\iota_s(i))$ and then:

$$xy = \big(i, \gamma_s(i), \iota_s(i)\big)\big(\iota_s(i), \gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big) = (i, \gamma_{st}(i), \iota_{st}(i)).$$

As a consequence, we conclude $\mathrm{E}_{st} = \mathrm{E}_s \mathrm{E}_t \setminus \{0\}$ and for every $i \in \alpha_{st}$:

$$(i, \gamma_{st}(i), \iota_{st}(i)) = \big(i, \gamma_s(i), \iota_s(i)\big)\big(\iota_s(i), \gamma_t(\iota_s(i)), \iota_t(\iota_s(i))\big)$$

is the unique decomposition of $(i, \gamma_{st}(i), \iota_{st}(i))$ as a product of an element in E_s and an element of E_t.

4. Let $s \in S$. For every $i \in \Lambda$, we have that:

$$0 \neq (i, 1, i)s \Leftrightarrow 0 \neq \big((i, 1, i)s\big)^{-1} = s^{-1}\big((i, 1, i)\big)^{-1} = s^{-1}(i, 1, i).$$

Therefore, $\alpha_{s^{-1}} = \omega_s$ and $\omega_{s^{-1}} = \alpha_s$. Moreover, for each $j \in \alpha_{s^{-1}} = \omega_s$, we have that $\iota_s^{-1}(j) = i$ if, and only if, $s(j, 1, j) = (i, \gamma_s(i), j)$. Then, for each $j \in \alpha_{s^{-1}} = \omega_s$:

$$(j, \gamma_s(i)^{-1}, i) = \big(s(j, 1, j)\big)^{-1} = (j, 1, j)s^{-1} = (j, \gamma_{s^{-1}}(j), \iota_{s^{-1}}(j)).$$

Hence, $\iota_{s^{-1}}(j) = \iota_s^{-1}(j)$ for each $j \in \alpha_{s^{-1}}$ and $\gamma_{s^{-1}}(j) = \gamma_s(i)^{-1}$, where $i = \iota_{s^{-1}}(j) \in \alpha_s$. As a consequence, $\mathrm{E}_{s^{-1}} = (\mathrm{E}_s)^{-1}$.

5. Let $0 \neq e \in \mathrm{E}(S)$ such that $e^2 = ee = e$. Then, applying Statement 2, we have that $\alpha_e = \alpha_{ee} = \iota_e^{-1}(\alpha_e \cap \omega_e)$. Hence, $\alpha_e \subseteq \omega_e$ and so $\alpha_e = \omega_e$, because $|\alpha_e| = |\omega_e|$.

Since $e^{-1} = e$, we can apply Statement 4 to conclude that $\iota_e^{-1} = \iota_e$. Hence $\iota_e = id_{\alpha_e}$. On the other hand, since $ee = e$, Statement 2 implies that $\gamma_e(i) = \gamma_{ee}(i) = \gamma_e(i)\gamma_e(\iota_e(i)) = \gamma_e(i)^2$, i.e $\gamma_e(i) = 1$, for every $i \in \alpha_e$.

6. It is obvious from the definition of $\alpha_s, \omega_s, \iota_s, \gamma_s$ and E_s. □

Let $\rho\colon S \to \mathcal{P}$ be the map defined by $\rho(s) = \mathrm{E}_s$, where $\mathcal{P}$ is the set of all projections of S onto J^0. The following example shows that ρ is not bijective in general.

Example 20. Let $S = \{0, e_1, e_2\}$ be the meet-semilattice such that $0 \leq e_1 \leq e_2$. Then $(S, \wedge)$ is an inverse semigroup, and $J_0 \lneq_{\mathcal{J}} J_{e_1} \lneq_{\mathcal{J}} J_{e_2}$. In particular, J_{e_1} is the unique 0-minimal $\mathcal{J}$-class of S. Since $0 \neq e_1 \wedge e_2 = e_2 \wedge e_1 = e_1$, we have that $\mathrm{E}_{e_2} = \{e_1\} = \mathrm{E}_{e_1}$.

The bijectivity of ρ implies the 0-minimality of J as the following proposition shows.

Proposition 21. *If ρ is bijective, then J is the unique 0-minimal $\mathcal{J}$-class of S.*

Proof. Assume, arguing by contradiction, that S has two different 0-minimal $\mathcal{J}$-classes of S, say J and J'. Let $x' \in J'$. By Lemma 1, $x'x = 0$, for every $x \in J$. Hence, $\alpha_{x'} = \omega_{x'} = \mathrm{E}_{x'} = \varnothing = \mathrm{E}_0$, contrary to assumption.

□

The above example shows the converse of the Proposition 21 does not hold.

Definition 22. We say that (S, J) is a *minimal pair* if J is the unique 0-minimal $\mathcal{J}$-class of S and ρ is a bijective map.

The set of all idempotents of a minimal pair is easily described.

Proposition 23. *Let (S, J) be a minimal pair. Then:*

$$\mathrm{E}(S) = \{s \in S : \mathrm{E}_s = \{(i, 1, i) : i \in \alpha_s\}\}.$$

Proof. Let $s \in \mathrm{E}(S)$. Applying Proposition 19, we have that $\mathrm{E}_s = \{(i, 1, i) : i \in \alpha_s\}$.

On the other hand, suppose that $s \in S$ is such that $\mathrm{E}_s = \{(i, 1, i) : i \in \alpha_s\}$. Then $\mathrm{E}_s \mathrm{E}_s \setminus\{0\} = \mathrm{E}_s$. By Proposition 19, $\mathrm{E}_{ss} = \mathrm{E}_s$ and therefore, $ss = s$ because (S, J) is a minimal pair. Thus, $s \in \mathrm{E}(S)$.

□

To underline the point that inverse semigroups with a unique 0-minimal $\mathcal{J}$-class are interesting in our context, we close with a somewhat substantial result. It shows that the relational morphisms between these semigroups and groups can be nicely characterised.

Lemma 24. *Assume that J is the unique 0-minimal $\mathcal{J}$-class of S. Let H be a group and let $\tau\colon S \multimap H$ be a relational morphism. Then, the following assertions hold:*

1. *For every $s, t \in J$ with $0 \neq st$, $\tau(s)\tau(t) = \tau(st)$.*

2. *For every* $s \in S$, $\tau(s) \subseteq \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\}$.

Conversely, let $\tau\colon J \to \mathcal{P}(H)$ *be a map satisfying the following conditions:*

1. $\tau(s)\tau(t) = \tau(st)$, *for every* $s,t \in J$ *with* $0 \neq st$,

2. $\varnothing \neq \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\}$, *for every* $0 \neq s \in S$ *and* $\tau(0) = H$.

Then the map $\tau\colon S \to \mathcal{P}(H)$ *defined by* $\tau(0) = H$ *and* $\tau(s) = \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\}$, *for every* $0 \neq s \in S$, *is a relational morphism between* S *and* H.

Proof. Let $s = (i,g,j)$ and $t = (i',g',j') \in J$ with $0 \neq st \in J$. Then, by Lemma 5, $j = i'$ and $st = (i,gg',j')$. Since τ is a relational morphism, it follows that $|\tau(s)| \leq |\tau(st)|$. On the other hand, we see that

$$(st)t^{-1} = (i,gg',j')(j',(g')^{-1},i') = (i,g,i') = (i,g,j) = s$$

and then $\tau(st)\tau(t^{-1}) \subseteq \tau(s)$. Therefore $|\tau(st)| \leq |\tau(s)|$ and $|\tau(s)| = |\tau(st)|$. Hence $\tau(s)\tau(t) = \tau(st)$.

Let $0 \neq s \in S$ and $(i,g,j) \in \mathrm{E}_s$. Then $(i,g,j) = (i,1,i)s$, for some $i \in \alpha_s$. Since $(i,1,i) \in \mathrm{E}(S)$ and $1 \in \tau(i,1,i)$, we have that

$$\tau(s) \subseteq \{1\}\tau(s) \subseteq \tau(i,1,i)\tau(s) \subseteq \tau\big((i,1,i)s\big) = \tau(i,g,j).$$

Hence, for every $0 \neq s \in s$, $\tau(s) \subseteq \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\}$.

Conversely, suppose that $\tau\colon J \to \mathcal{P}(H)$ is a map satisfying:

1. $\tau(s)\tau(t) = \tau(st)$, for every $s,t \in J$ with $0 \neq st$.

2. $\varnothing \neq \bigcap\{\tau(i,g,j)/(i,g,j) \in \mathrm{E}_s\}$, for every $0 \neq s \in S$ and $\tau(0) = H$.

We show that the map $\tau\colon S \to \mathcal{P}(H)$ defined as $\tau(0) = H$ and $\tau(s) = \bigcap\{\tau(i,g,j)/(i,g,j) \in \mathrm{E}_s\}$, for every $0 \neq s \in S$, is a relational morphism between S and H. Let $s,t \in S$ and suppose that $0 \neq st$ (if either $s = 0$, $t = 0$ or $st = 0$, then it is clear that $\tau(s)\tau(t) \subseteq \tau(st)$). Let $h \in \tau(s)$, $h' \in \tau(t)$. Then, $h \in \tau(i,g,j)$ for every $(i,g,j) \in \mathrm{E}_s$, and $h' \in \tau(i',g',j')$ for every $(i',g',j') \in \mathrm{E}_t$. By Proposition 19, $\mathrm{E}_{st} = (\mathrm{E}_s\,\mathrm{E}_t) \setminus \{0\}$. Hence if $(i'',g'',j'') \in \mathrm{E}_{st}$, there exist $(i'',g,j) \in \mathrm{E}_s$, $(j,g',j'') \in \mathrm{E}_t$ such that $(i'',g'',j'') = (i'',g,j)(j,g',j'')$. Therefore

$$hh' \in \tau(i'',g,j)\tau(j,g',j'') \subseteq \tau\big((i'',g,j)(j,g',j'')\big) = \tau(i'',g'',j'').$$

Consequently, $\tau(s)\tau(t) \subseteq \tau(st)$.

□

4.2 The semigroup of projections of a Brandt semigroup

Let B be a Brandt semigroup with zero. We shall construct an inverse semigroup with zero $\mathcal{U}_B$ with a unique 0-minimal $\mathcal{J}$-class J such that $J^0 \cong B$ and $(\mathcal{U}, J)$ is a minimal pair. The importance of the semigroup $\mathcal{U}_B$ lies in its universality: if (T, J_1) is a minimal pair and J_1^0 is isomorphic to B, then T is isomorphic to quotient of an inverse subsemigroup V of $\mathcal{U}_B$ containing J^0.

Assume that $B = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. We say that a subset E of $B \setminus \{0\}$ is a *projection subset* of B if E satisfies the following properties:

$$|\{(i,g,j) \in \mathrm{E} : g \in G, j \in \Lambda\}| = |\mathrm{E} \cap (i, G, \Lambda)| \leq 1, \quad \text{for each } i \in \Lambda. \tag{4.2}$$
$$|\{(i,g,j) \in \mathrm{E} : g \in G, i \in \Lambda\}| = |\mathrm{E} \cap (\Lambda, G, j)| \leq 1, \quad \text{for each } j \in \Lambda. \tag{4.3}$$

The set of all projections subsets of B is denoted by $\mathcal{E}_B$.

This definition is motivated by property (4.1) of the sets E_s introduced in Section 4.1. In fact, these sets are projection subsets of the Brandt semigroup J^0 considered there.

The basic properties of the projection sets are contained in the following:

Proposition 25. *1. If* $\mathrm{E} \in \mathcal{E}_B$*, then* $\mathrm{E}^{-1} \in \mathcal{E}_B$*.*

2. If $\mathrm{E}, \mathrm{E}' \in \mathcal{E}_B$*, then* $\mathrm{E}\,\mathrm{E}' \setminus \{0\} \in \mathcal{E}_B$*.*

3. For every $(i, g, j) \in B$*,* $\{(i, g, j)\} \in \mathcal{E}_B$*.*

Proof. 1. By Lemma 5, $\mathrm{E}^{-1} = \{(i, g, j)^{-1} = (j, g^{-1}, i) : (i, g, j) \in \mathrm{E}\}$. Then, we have that E^{-1} satisfies (4.2) and (4.3) if, and only if, E satisfies (4.2) and (4.3).

2. If either $\mathrm{E} = \varnothing$ or $\mathrm{E}' = \varnothing$ we are done. Otherwise, let $(i, g, j), (i, g', j') \in \mathrm{E}\,\mathrm{E}' \setminus \{0\}$. Then, by Lemma 5, we have that:

$$(i, g, j) = (i, g_1, j_1)(j_1, g_2, j), \quad \text{with } (i, g_1, j_1) \in \mathrm{E} \text{ and } (j_1, g_2, j) \in \mathrm{E}'$$
$$(i, g', j') = (i, g_1', j_1')(j_1', g_2', j), \quad \text{with } (i, g_1', j_1') \in \mathrm{E} \text{ and } (j_1', g_2', j) \in \mathrm{E}'$$

Since $\mathrm{E}, \mathrm{E}' \in \mathcal{E}_B$, we get $(i, g_1, j_1) = (i, g_1', j_1') \in \mathrm{E}$ and $(j_1, g_2, j) = (j_1', g_2', j) \in \mathrm{E}'$. Thus, $(i, g, j) = (i, g', j')$.

A similar argument establishes that if $(i, g, j), (i', g', j) \in \mathrm{E}\,\mathrm{E}' \setminus \{0\}$, then $(i, g, j) = (i', g', j)$. Consequently, $\mathrm{E}\,\mathrm{E}' \setminus \{0\}$ is a projection subset of B, as required.

The Statement 3 is clear.

□

Assume that $\mathcal{E}_B = \{\varnothing = \mathrm{E}_0, \mathrm{E}_1, \ldots, \mathrm{E}_{n_0}\}$. Let $\mathcal{U}_B := \{0 = u_0, u_1, \ldots, u_{n_0}\}$ be any set that is naturally bijective with $\mathcal{E}_B$. Write $\bar{\mathrm{E}}_{u_n} := \mathrm{E}_n$, for each $0 \leq n \leq n_0$.

Let $u_n, u_{n'} \in \mathcal{U}_B$. We define the product of u_n and $u_{n'}$ as the element $u_{n''} \in \mathcal{U}_B$ such that $\bar{\mathrm{E}}_{u_n}\bar{\mathrm{E}}_{u_{n'}} \setminus \{0\} = \bar{\mathrm{E}}_{u_{n''}}$.

With this product $\mathcal{U}_B$ becomes a semigroup in which $u_0 = 0$ is a zero element and $\bar{\mathrm{E}}_0 = \varnothing$. It will be called the *semigroup of projections of the Brandt semigroup B*.

Note that for every $u, v \in \mathcal{U}_B$, $u = v$ if, and only if, $\bar{\mathrm{E}}_u = \bar{\mathrm{E}}_v$. Moreover, if $(i, g, j) \in B \setminus \{0\}$, then $\{(i, g, j)\} \in \mathcal{E}_B$. In this case, $u := (i, g, j) \in \mathcal{U}_B$ and $\bar{\mathrm{E}}_u = \{(i, g, j)\}$. In $\mathcal{U}_B$ the product of two elements $u := (i, g, j)$ and $v = (i', g, j')$ is the same as the product $(i, g, j)(i', g, j')$ in B. Hence B can be regarded as a subsemigroup of $\mathcal{U}_B$.

In an inverse semigroup S, we have a particularly useful way of looking at the projections E_s by means of the maps $\alpha_s, \omega_s, \iota_s, \gamma_s$ (see Section 4.1). Our next aim here is to find analogues of these maps in the semigroup $\mathcal{U}_B$.

For each $u \in \mathcal{U}_B$ we define:

$$\bar{\alpha}_u := \{i \in \Lambda : |\bar{\mathrm{E}}_u \cap (i, G, \Lambda)| = 1\}, \quad \bar{\omega}_u := \{j \in \Lambda : |\bar{\mathrm{E}}_u \cap (\Lambda, G, j)| = 1\},$$
$$\bar{\iota}_u(i) \colon \bar{\alpha}_u \longrightarrow \bar{\omega}_u, \quad \bar{\gamma}_u \colon \bar{\alpha}_u \longrightarrow G,$$
$$\bar{\iota}_u(i) = j \text{ and } \bar{\gamma}_u(i) = g \Leftrightarrow \bar{\mathrm{E}}_u \cap (i, G, \Lambda) = (i, g, j) = \bar{\mathrm{E}}_u \cap (\Lambda, G, j).$$

Clearly, $\bar{\iota}_u$ is a bijection between $\bar{\alpha}_u$ and $\bar{\omega}_u$. Note that $\bar{\alpha_0}, \bar{\omega_0}, \bar{\mathrm{E}}_0 = \varnothing$, and $\bar{\mathrm{E}}_u = \{(i, \bar{\gamma}_u(i), \bar{\iota}_u(i)) : i \in \bar{\alpha}_u\}$ for all $0 \neq u \in \mathcal{U}_B$. In particular, $\bar{\mathrm{E}}_u$ is completely determined by the maps $\bar{\alpha}_u, \bar{\omega}_u, \bar{\iota}_u, \bar{\gamma}_u$, for every $u \in \mathcal{U}_B$.

We are now ready to begin the process of proving that $\mathcal{U}_B$ is an inverse semigroup with zero and a unique 0-minimal $\mathcal{J}$-class J such that $J^0 = B$. Our next result, which is the analogous version of Proposition 19, contains the key ingredients of the proof.

Proposition 26. *1. For every $u \in B \setminus \{0\}$, $\bar{\mathrm{E}}_u = \{u\}$.*

2. *For every $u, v \in \mathcal{U}_B$, $\bar{\alpha}_{uv} = (\bar{\iota}_u)^{-1}(\bar{\omega}_u \cap \bar{\alpha}_v)$, $\bar{\omega}_{uv} = \bar{\iota}_v(\bar{\omega}_u \cap \bar{\alpha}_v)$. Moreover, $\bar{\iota}_{uv}(i) = \bar{\iota}_v(\bar{\iota}_u(i))$ and $\bar{\gamma}_{uv}(i) = \bar{\gamma}_u(i)\bar{\gamma}_v(\bar{\iota}_u(i))$, for every $i \in \bar{\alpha}_{uv}$.*

3. *Let $0 \neq u \in \mathrm{E}(\mathcal{U}_B)$. Then, $\bar{\alpha}_u = \bar{\omega}_u$, $\bar{\iota}_u = id_{\bar{\alpha}_u}$, $\bar{\gamma}_u(i) = 1$, for every $i \in \bar{\alpha}_u$. In particular:*

$$\bar{\mathrm{E}}_u = \{(i, 1, i) : i \in \bar{\alpha}_u\}.$$

Conversely, if $u \in \mathcal{U}_B$ and $\bar{\mathrm{E}}_u = \{(i, 1, i) : i \in \bar{\alpha}_u\}$, then $u \in \mathrm{E}(\mathcal{U}_B)$.

4. Let $u, v \in \mathcal{U}_B$ such that $u \mathcal{J} v$. Then, $|\bar{\mathrm{E}}_u| = |\bar{\mathrm{E}}_v|$.

Proof. The Statement 1 is clear.

2. Let $u, v \in \mathcal{U}_B$ and suppose $i \in \bar{\alpha}_{uv}$. Then, there exists $g \in G$, $j \in \Lambda$, such that $(i, g, j) \in \bar{\mathrm{E}}_{uv} = \bar{\mathrm{E}}_u\bar{\mathrm{E}}_v \setminus \{0\}$. Therefore, $(i, g, j) = (i, g_1, j_1)(i_2, g_2, j)$ with $(i, g_1, j_1) \in \bar{\mathrm{E}}_u$, $(i_2, g_2, j) \in \bar{\mathrm{E}}_v$ and $j_1 = i_2$. Thus $\bar{\gamma}_u = g_1$ and $j_1 = \bar{\iota}_u(i) \in \bar{\omega}_u \cap \bar{\alpha}_v$, with $\bar{\iota}_v(j_1) = j$ and $\bar{\gamma}_v(j_1) = \bar{\gamma}_u(\bar{\iota}_u(i)) = g_2$. Then, $\bar{\iota}_{uv}(i) = j = \bar{\iota}_v(\bar{\iota}_u(i))$ and $\bar{\gamma}_{uv}(i) = g = g_1g_2 = \bar{\gamma}_u(i)\bar{\gamma}_v(\bar{\iota}_u(i))$.

On the other hand, if $i \in \bar{\omega}_u \cap \bar{\alpha}_v$, then there exists $i_1 \in \bar{\alpha}_u$ and $j \in \bar{\omega}_v$, such that $\bar{\iota}_u(i_1) = i$ and $\bar{\iota}_v(i) = j$. Thus, $(i_1, \bar{\gamma}_u(i_1), i) \in \bar{\mathrm{E}}_u$, $(i, \bar{\gamma}_v(i), j) \in \bar{\mathrm{E}}_v$, so that $(i_1, \bar{\gamma}_u(i_1)\bar{\gamma}_v(i), j) \in \bar{\mathrm{E}}_u\bar{\mathrm{E}}_v \setminus \{0\} = \bar{\mathrm{E}}_{uv}$. Therefore, $i_1 = (\bar{\iota}_u)^{-1}(i) \in \bar{\alpha}_{uv}$, $\bar{\iota}_{uv}(i_1) = j = \bar{\iota}_v(\bar{\iota}_u(i_1))$ and $\bar{\gamma}_{uv}(i_1) = \bar{\gamma}_u(i_1)\bar{\gamma}_v(i) = \bar{\gamma}_u(i_1)\bar{\gamma}_v(\bar{\iota}_u(i_1))$. Hence, for every $u, v \in \mathcal{U}_B$, we have:

$$\begin{aligned} &\bar{\alpha}_{uv} = (\bar{\iota}_u)^{-1}(\bar{\omega}_u \cap \bar{\alpha}_v), && \bar{\omega}_{uv} = \bar{\iota}_v(\bar{\omega}_u \cap \bar{\alpha}_v), \\ &\bar{\iota}_{uv}(i) = \bar{\iota}_v(\bar{\iota}_u(i)), && \bar{\gamma}_{uv}(i) = \bar{\gamma}_u(i)\bar{\gamma}_v(\bar{\iota}_u(i)), \text{ for every } i \in \bar{\alpha}_{uv}. \end{aligned}$$

3. Let $0 \neq u \in \mathrm{E}(\mathcal{U}_B)$. Then, $uu = u$ and $\bar{\alpha}_{uu} = (\bar{\iota}_u)^{-1}(\bar{\omega}_u \cap \bar{\alpha}_u) = \bar{\alpha}_u$. Thus, $\bar{\omega}_u \subseteq \bar{\alpha}_u$ and then $\bar{\alpha}_u = \bar{\omega}_u$. Moreover, $\bar{\iota}_u = \bar{\iota}_{uu} = \bar{\iota}_u \circ \bar{\iota}_u$, i.e. $\bar{\iota}_u = id_{\bar{\alpha}_u}$, so that $\bar{\gamma}_u(i) = \bar{\gamma}_{uu}(i) = \bar{\gamma}_u(i)\bar{\gamma}_u(\bar{\iota}_u(i)) = (\bar{\gamma}_u(i))^2$, for all $i \in \bar{\alpha}_u$. Hence $\bar{\gamma}_u(i) = 1$, for all $i \in \bar{\alpha}_u$. On the other hand, it is easy to check that if $u \in \mathcal{U}_B$ with $\bar{\alpha}_u = \bar{\omega}_u$, $\bar{\iota}_u = id_{\bar{\alpha}_u}$ and $\bar{\gamma}_u(i) = 1$, for all $i \in \bar{\alpha}_u$, then $\bar{\mathrm{E}}_u\bar{\mathrm{E}}_u \setminus \{0\} = \bar{\mathrm{E}}_u$. Therefore, $uu = u$ and $u \in \mathrm{E}(\mathcal{U}_B)$. Hence, we conclude:

$$\begin{aligned} \mathrm{E}(\mathcal{U}_B) &= \{u \in \mathcal{U}_B : \bar{\alpha}_u = \bar{\omega}_u, \bar{\iota}_u = id_{\bar{\alpha}_u}, \bar{\gamma}_u(i) = 1, \text{ for all } i \in \bar{\alpha}_u(i)\} = \\ &= \left\{u \in \mathcal{U}_B : \bar{\mathrm{E}}_u = \{(i, 1, i) : i \in \bar{\alpha}_u\}\right\}. \end{aligned}$$

4. Certainly $|\bar{\alpha}_u| = |\bar{\mathrm{E}}_u|$, for all $u \in \mathcal{U}_B$. Now, let $u, v \in \mathcal{U}_B$ such that $u \mathcal{J} v$. Then, there exist $w_1, w_2, w_1', w_2' \in \mathcal{U}_B$ such that $u = w_1vw_2$ and $v = w_1'uw_2'$. Applying Statement 2, we have:

$$\bar{\alpha}_u = \bar{\alpha}_{w_1vw_2} \subseteq \bar{\alpha}_{w_1v}, \quad \bar{\alpha}_v = \bar{\alpha}_{w_1'vw_2'} \subseteq \bar{\alpha}_{w_1'v}.$$

Then, $|\bar{\mathrm{E}}_u| = |\bar{\alpha}_u| \leq |\bar{\alpha}_{w_1v}| \leq |\bar{\alpha}_v| = |\bar{\mathrm{E}}_v|$. The proof that $|\bar{\mathrm{E}}_v| \leq |\bar{\mathrm{E}}_u|$ proceeds just in the same way. □

Lemma 27. *$\mathcal{U}_B$ is an inverse semigroup with zero and $J = B \setminus \{0\} = \{u \in \mathcal{U}_B : |\bar{E}_u| = 1\}$ is the unique 0-minimal $\mathcal{J}$-class of $\mathcal{U}_B$.*

Proof. We already know $\mathcal{U}_B$ is a semigroup with a zero. Now, suppose that $v \in \mathcal{U}_B$ is an inverse of an element $u \in \mathcal{U}_B$. Then $uvu = u$ and $vuv = v$. Applying Proposition 26 to $\bar{\alpha}_{uvu} = \bar{\alpha}_u$ and $\bar{\alpha}_{vuv} = \bar{\alpha}_v$, we conclude that $\bar{\alpha}_u \subseteq \bar{\alpha}_{uv} \subseteq \bar{\alpha}_u$ and $\bar{\alpha}_v \subseteq \bar{\alpha}_{vu} \subseteq \bar{\alpha}_v$. Thus, $\bar{\alpha}_{uv} = \bar{\alpha}_u$, $\bar{\alpha}_{vu} = \bar{\alpha}_v$ and then:

$$\bar{\alpha}_u = \bar{\alpha}_{uv} = (\bar{\iota}_u)^{-1}(\bar{\omega}_u \cap \bar{\alpha}_v), \quad \bar{\alpha}_v = \bar{\alpha}_{vu} = (\bar{\iota}_v)^{-1}(\bar{\omega}_v \cap \bar{\alpha}_u).$$

Hence $\bar{\omega}_u \subseteq \bar{\alpha}_v$ and $\bar{\omega}_v \subseteq \bar{\alpha}_u$. Since $|\bar{\alpha}_u| = |\bar{\omega}_u|$ and $|\bar{\alpha}_v| = |\bar{\omega}_v|$, it follows that $\bar{\alpha}_u = \bar{\omega}_v$ and $\bar{\omega}_u = \bar{\alpha}_v$.

Note that $uv \in \mathrm{E}(\mathcal{U}_B)$. Thus, by Proposition 26, we have that $id_{\bar{\alpha}_{uv}} = \bar{\iota}_{uv} = \bar{\iota}_v \circ \bar{\iota}_u$, i.e. $\bar{\iota}_v = (\bar{\iota}_u)^{-1}$. Thus, $1 = \bar{\gamma}_{uv}(i) = \bar{\gamma}_u(i)\bar{\gamma}_v(\bar{\iota}_u(i))$, i.e. $\bar{\gamma}_v(\bar{\iota}_u(i)) = \bar{\gamma}_u(i)^{-1}$, for all $i \in \bar{\alpha}_u(i)$. Therefore:

$$\begin{aligned}\bar{\mathrm{E}}_v &= \{(j, \bar{\gamma}_v(j), \bar{\iota}_v(j)) : j \in \bar{\alpha}_v = \bar{\omega}_u\} = \{(\bar{\iota}_u(i), \bar{\gamma}_v(\bar{\iota}_u(i)), i) : \bar{\iota}_u(i) \in \bar{\omega}_u\} = \\ &= \{(\bar{\iota}_u(i), \bar{\gamma}_u(i)^{-1}, i) : i \in \bar{\alpha}_u\} = \{\big(i, \bar{\gamma}_u(i), \bar{\iota}_u(i)\big)^{-1} : i \in \bar{\alpha}_u\} = (\bar{\mathrm{E}}_u)^{-1}.\end{aligned}$$

We have shown that every element of $\mathcal{U}_B$ has at most one inverse.

On the other hand, let $0 \neq u \in \mathcal{U}_B$. By Proposition 25, there exists $v \in \mathcal{U}_B$ such that $\bar{\mathrm{E}}_v = (\bar{\mathrm{E}}_u)^{-1}$. Then, $\bar{\alpha}_v = \bar{\omega}_u$, $\bar{\omega}_v = \bar{\alpha}_u$, $\bar{\iota}_v = (\bar{\iota}_u)^{-1}$ and $\bar{\gamma}_v(\bar{\iota}_u(i)) = \bar{\gamma}_u(i)^{-1}$, for every $\bar{\iota}_u(i) \in \bar{\alpha}_v$. Thus, we have:

$$\begin{aligned}&\bar{\alpha}_{uv} = (\bar{\iota}_u)^{-1}(\bar{\omega}_u \cap \bar{\alpha}_v) = (\bar{\iota}_u)^{-1}(\bar{\omega}_u \cap \bar{\omega}_u) = \bar{\alpha}_u, \quad \bar{\iota}_{uv} = \bar{\iota}_v \circ \bar{\iota}_u = id_{\bar{\alpha}_u},\\ &\bar{\omega}_{uv} = \bar{\iota}_{uv}(\bar{\alpha}_{uv}) = id_{\bar{\alpha}_u}(\bar{\alpha}_{uv}) = id_{\bar{\alpha}_u}(\bar{\alpha}_u) = \bar{\alpha}_u = \bar{\alpha}_{uv},\\ &\bar{\gamma}_{uv}(i) = \bar{\gamma}_u(i)\bar{\gamma}_v(\bar{\iota}_u(i)) = \bar{\gamma}_u(i)\bar{\gamma}_u(i)^{-1} = 1, \quad \text{for all } i \in \bar{\alpha}_{uv} = \bar{\alpha}_u.\end{aligned}$$

By Proposition 26, $uv \in \mathrm{E}(\mathcal{U}_B)$. Therefore:

$$\begin{aligned}&\bar{\alpha}_{uvu} = (\bar{\iota}_{uv})^{-1}(\bar{\omega}_{uv} \cap \bar{\alpha}_u) = id_{\bar{\alpha}_u}(\bar{\alpha}_u \cap \bar{\alpha}_u) = \bar{\alpha}_u, \quad \bar{\iota}_{uvu} = \bar{\iota}_u \circ \bar{\iota}_{uv} = \bar{\iota}_u,\\ &\bar{\omega}_{uvu} = \bar{\iota}_{uvu}(\bar{\alpha}_{uvu} = \bar{\iota}_u(\bar{\alpha}_u) = \bar{\omega}_u,\\ &\bar{\gamma}_{uvu}(i) = \bar{\gamma}_{uv}(i)\bar{\gamma}_u(\bar{\iota}_{uv}(i)) = 1\bar{\gamma}_u(i) = \bar{\gamma}_u(i), \quad \text{for every } i \in \bar{\alpha}_u.\end{aligned}$$

Thus, $\bar{\mathrm{E}}_{uvu} = \bar{\mathrm{E}}_u$ and then $uvu = u$. Similarly, $\bar{\mathrm{E}}_{vuv} = \bar{\mathrm{E}}_v$.

Hence, if $0 \neq u \in \mathcal{U}_B$, and $v \in \mathcal{U}_B$ satisfies $\bar{\mathrm{E}}_v = (\bar{\mathrm{E}}_u)^{-1}$, then v is the unique inverse of u. Consequently, $\mathcal{U}_B$ is an inverse semigroup.

We prove now that $\mathcal{U}_B$ has a unique 0-minimal $\mathcal{J}$-class J, and $J^0 = B$. By Proposition 26, we have that the sets of projections of all elements in a $\mathcal{J}$-class have the same cardinality. Moreover, $B = \{u \in \mathcal{U}_B : |\bar{\mathrm{E}}_u| = 1\} \cup \{0\}$. Write $J := \{u \in \mathcal{U}_B : |\bar{\mathrm{E}}_u| = 1\}$. Then, if $(i, g, j), (i', g', j') \in J$, it follows that $(i, g, j) = (i, 1, i')(i', g', j')(j', 1, i)$. Therefore J is a 0-minimal $\mathcal{J}$-class of $\mathcal{U}_B$.

Assume that $0 \neq J'$ is a $\mathcal{J}$-class of $\mathcal{U}_B$ such that $J' = J_s$ for some $s \in \mathcal{U}_B$. Then $|\bar{\mathrm{E}}_s| \geq 1$. Let $i \in \bar{\alpha}_s$. Then:

$$\bar{\alpha}_{(i,1,i)s} = (\bar{\iota}_{(i,1,i)})^{-1}(\bar{\omega}_{(i,1,i)} \cap \bar{\alpha}_s) = (\bar{\iota}_{(i,1,i)})^{-1}(\{i\} \cap \bar{\alpha}_s) = (\bar{\iota}_{(i,1,i)})^{-1}(\{i\}) = \{i\}$$

since $(i, 1, i) \in \mathrm{E}(J) \subseteq \mathrm{E}(\mathcal{U}_B)$. Therefore, if $v = (i, 1, i)s \in \mathcal{U}_B$, then $|\bar{\mathrm{E}}_v| = |\bar{\alpha}_v| = 1$. Hence $v \in J$ and so $J_v = J \leq_{\mathcal{J}} J_s = J'$.

□

At this point, it is reasonable to ask whether the maps $\alpha_u, \omega_u, \iota_u, \gamma_u$ defined in Section 4.1 are exactly the maps $\bar{\alpha}_u, \bar{\omega}_u, \bar{\iota}_u, \bar{\gamma}_u$. The following result provides an affirmative answer.

Proposition 28. *For every $u \in \mathcal{U}_B$, we have:*

$$\bar{\alpha}_u = \alpha_u, \quad \bar{\omega}_u = \omega_u, \quad \bar{\iota}_u = \iota_u, \quad \bar{\gamma}_u = \gamma_u, \quad \bar{\mathrm{E}}_u = \mathrm{E}_u\,.$$

Moreover, for every $u, v \in \mathcal{U}_B$, $u = v$ if, and only if, $\mathrm{E}_u = \bar{\mathrm{E}}_u = \bar{\mathrm{E}}_v = \mathrm{E}_v$. *As a consequence, $(\mathcal{U}_B, B)$ is a minimal pair.*

Proof. First, we claim that for every $u \in \mathcal{U}_B$ and for every $i \in \Lambda$, $(i,1,i)u \neq 0$ if, and only if, $i \in \bar{\alpha}_u$ and in this case, $(i,1,i)u = (i, \bar{\gamma}_u(i), \bar{\iota}_u(i))$.

By Proposition 26, we see $\varnothing \neq \bar{\alpha}_{(i,1,i)u} = (\bar{\iota}_{(i,1,i)})^{-1}(\{i\} \cap \bar{\alpha}_u)$ if, and only if, $i \in \bar{\alpha}_u$. Moreover, in this case $\bar{\alpha}_{(i,1,i)u} = \{i\}$, so that:

$$\bar{\iota}_{(i,1,i)u}(i) = \bar{\iota}_u(\bar{\iota}_{(i,1,i)}(i)) = \bar{\iota}_u(i), \quad \bar{\gamma}_u(i) = \bar{\gamma}_{(i,1,i)}(i)\bar{\gamma}_u(i) = 1\bar{\gamma}_u(i).$$

Hence $\bar{\mathrm{E}}_{(i,1,i)u} = \{(i, \bar{\gamma}_u(i), \bar{\iota}_u(i))\}$ and therefore $(i,1,i)u = (i, \bar{\gamma}_u(i), \bar{\iota}_u(i))$.

Let $u \in \mathcal{U}_B$. We may assume that $u \neq 0$. Then:

$$\alpha_u = \{i \in \Lambda : (i,1,i)u \neq 0\} = \{i \in \Lambda : i \in \bar{\alpha}_u\} = \bar{\alpha}_u,$$
$$\iota_u(i) = j,\ \gamma_u(i) = g \text{ if, and only if, } 0 \neq (i,1,i)u = (i,g,j) = (i, \bar{\gamma}_u(i), \bar{\iota}_u(i))$$

Hence, $\bar{\alpha}_u = \alpha_u$, $\bar{\iota}_u = \iota_u$, $\bar{\gamma}_u = \gamma_u$, $\bar{\omega}_u = \bar{\iota}_u(\bar{\alpha}_u) = \iota_u(\alpha_u) = \omega_u$ and

$$\bar{\mathrm{E}}_u = \{(i, \bar{\gamma}_u(i), \bar{\iota}_u(i)) : i \in \bar{\alpha}_u\} = \{(i, \gamma_u(i), \iota_u(i)) : i \in \alpha_u\} = \mathrm{E}_u\,.$$

□

Our next aim is to prove that $\mathcal{U}_B$ contains a quotient of every inverse semigroup with a unique 0-minimal $\mathcal{J}$-class J isomorphic to B.

A preliminary lemma, which shows that the property of being minimal pair is inherited by inverse subsemigroups of $\mathcal{U}_B$ containing B, is useful.

Lemma 29. *Let $V \subseteq \mathcal{U}_B$ be an inverse subsemigroup such that $B \subseteq V$. Then, B is the unique 0-minimal $\mathcal{J}$-class of V and (V, J) is also a minimal pair.*

Proof. Let $v, v' \in V$ such that $v \,\mathcal{J}_V\, v'$, where $\mathcal{J}_V$ is the $\mathcal{J}$-relation in V. Then $v \,\mathcal{J}_{\mathcal{U}_B}\, v'$. By Proposition 26, $|\bar{\mathrm{E}}_v| = |\bar{\mathrm{E}}_{v'}|$. Hence the sets of projections of all elements in a $\mathcal{J}_V$-class of V have the same cardinality.

On the other hand, $J := \{v \in V : |\bar{\mathrm{E}}_v| = 1\} = \{u \in U : |\bar{\mathrm{E}}_u| = 1\} = B \setminus \{0\}$ and J is a $\mathcal{J}_V$-class of V. A similar argument to those used in the proof of Lemma 27 establishes that J is the unique 0-minimal $\mathcal{J}_V$-class of V. Moreover, since $V \leq \mathcal{U}_B$, we have that $|\,\mathrm{E}_v\,| = |\,\mathrm{E}_{v'}\,|$, for every $v, v' \in V$. Hence (V, J) is a minimal pair.

□

Lemma 30. *Let S be an inverse semigroup with zero with a unique 0-minimal $\mathcal{J}$-class J. Let $\mathcal{U}_B$ the semigroup of projections of $B = J^0$. Then, there exists a homomorphism $\upsilon\colon S \longrightarrow \mathcal{U}_B$ such that*

- $\upsilon(S) =: V \subseteq \mathcal{U}_B$ *is an inverse subsemigroup of* $\mathcal{U}_B$.
- $\upsilon(J^0) = B \subseteq V$ *and* $\upsilon|_{J^0} = id_{J^0}$.
- (V, J) *is a minimal pair, where* $J = J^0 \setminus \{0\}$ *is the unique* 0*-minimal* $\mathcal{J}$*-class of* $\mathcal{U}_B$*. In particular,* $\mathrm{E}_s = \mathrm{E}_{\upsilon(s)}$*, for every* $s \in S$.

Proof. Let $\cup\colon S \longrightarrow \mathcal{U}_B$ be the map given by:

$$\cup(s) = u \Leftrightarrow \mathrm{E}_s = \mathrm{E}_u \in \mathcal{E}_B, \text{for every } s \in S.$$

We shall show that $\cup$ is a homomorphism. Let $s, s' \in S$. By Proposition 19, $\mathrm{E}_{ss'} = \mathrm{E}_s \, \mathrm{E}_{s'} \setminus \{0\}$. Suppose that $\cup(s) = u$ and $\cup(s') = u'$. Then $\mathrm{E}_{uu'} = \mathrm{E}_u \, \mathrm{E}_{u'} \setminus \{0\} = \mathrm{E}_s \, \mathrm{E}_{s'} \setminus \{0\} = \mathrm{E}_{ss'}$. Hence, $\cup(s) \cup(s') = uu' = \cup(ss')$.

Therefore $V := \cup(S)$ is a subsemigroup of $\mathcal{U}_B$. Let $u \in \cup(S)$. Then there exists $s \in S$ such that $\mathrm{E}_s = \mathrm{E}_u$. By Proposition 19, it follows that $\mathrm{E}_{s^{-1}} = (\mathrm{E}_s)^{-1} = (\mathrm{E}_u)^{-1}$. Now, the proof of Lemma 27 and Proposition 26 allow us to conclude that $\mathrm{E}_{u^{-1}} = (\mathrm{E}_u)^{-1}$. Thus, $\cup(s^{-1}) = u^{-1}$ and then, $u^{-1} \in \cup(S)$. By Lemma 3, V is an inverse subsemigroup of $\mathcal{U}_B$.

Now, recall that B can be seen as a subsemigroup of both S and $\mathcal{U}_B$. Then, for every $x \in J = B \setminus \{0\}$, we can write $(\mathrm{E}_x)_S$, if x is regarded as an element of S, and $(\mathrm{E}_x)_{\mathcal{U}_B}$, if x is regarded as an element in $\mathcal{U}_B$. If $x \in J$, then $(\mathrm{E}_x)_S = \{x\}$ by Proposition 19. On the other hand, applying Proposition 26 and Proposition 28, we have that $(\mathrm{E}_x)_{\mathcal{U}_B} = \{x\}$. In addition, $\cup(0) = 0$. Hence, we can write $\mathrm{E}_x = (\mathrm{E}_x)_S = (\mathrm{E}_x)_{\mathcal{U}_B} = \{x\}$, for every $x \in J$, so that we can conclude that $\cup(B) = B$ and $\cup|_B = id_B$.

Finally, (V, J) satisfies the hypothesis of Lemma 29. Therefore, (V, J) is a minimal pair.

□

At this point it is worth pausing to give a nice application of Lemma 30.

Assume that J is the unique 0-minimal $\mathcal{J}$-class of an inverse semigroup S. From the set of projections of S onto J^0 arises one naturally equivalence relation $\sim$ on S: $s \sim t$ if, and only if, $\mathrm{E}_s = \mathrm{E}_t$. We show that $\sim$ is in fact a congruence in S. Suppose that $\mathrm{E}_s = \mathrm{E}_t$ and $\mathrm{E}_{s'} = \mathrm{E}_{t'}$. Then $\alpha_s = \alpha_t$, $\alpha_{s'} = \alpha_{t'}$, $\iota_s = \iota_t$, $\iota_{s'} = \iota_{t'}$, $\gamma_s = \gamma_t$ and $\gamma_{s'} = \gamma_{t'}$. By Proposition 19, $\alpha_{st} = \alpha_{s't'}$, $\omega_{st} = \omega_{s't'}$, $\iota_{st} = \iota_{s't'}$ and $\gamma_{st} = \gamma_{s't'}$, so that $\mathrm{E}_{st} = \mathrm{E}_{s't'}$.

Consider the homomorphism $\cup\colon S \to \cup(S) \leq \mathcal{U}_{J^0}$ defined in Lemma 30. Then, for every $s, t \in S$, one can see that $\cup(s) = \cup(t)$ if, and only if, $\mathrm{E}_s = \mathrm{E}_t$, i.e. $s \sim t$. Hence, $S/\sim = S/\ker \cup \cong \cup(S)$. Since $(\cup(S), J)$ is a minimal pair by Lemma 30, we conclude that $S/\sim$ is an inverse semigroup with zero with a unique 0-minimal $\mathcal{J}$-class $\bar{J}$, such that $(S/\sim, \bar{J})$ is a minimal pair.

Note that if ρ_J is the natural epimorphism from S onto $S/\sim$, then $\ker \cup = \ker \rho_J$. ρ_J is called *the right Shützenberger representation of S on J* ([27, Definition 4.6.28]).

We summarise these observations in a corollary as follows.

Corollary 31. *Let S be an inverse semigroup with zero with a 0-minimal $\mathcal{J}$-class J. There exists a quotient X of S such that X is an inverse semigroup with zero having a unique 0-minimal $\mathcal{J}$-class $\bar{J}$ and $(X, \bar{J})$ is a minimal pair.*

We round this section off with a couple of results about $\mathfrak{F}$-kernels and minimal pairs. We shall have occasion to make use of them in the next sections.

Lemma 32. *Let (S, J) be a minimal pair. Consider $K = \bigcup\{(S_e)^{\mathfrak{F}} : e \in \mathrm{E}(S)\}$. Then $\mathrm{E}_s \subseteq \mathrm{K}_{\mathfrak{F}}(S) \cap J$, for every $s \in K$.*

Proof. Let $s \in K$ and let $e \in \mathrm{E}(S)$ such that $s \in (S_e)^{\mathfrak{F}}$. By Proposition 12, $s \in \mathrm{K}_{\mathfrak{F}}(S)$.

Now, let $x \in \mathrm{E}_s$. Then there exists an idempotent $e \in \mathrm{E}(J)$ such that $x = es$. Therefore, for every relational morphism $\tau\colon S \multimap F$ with $F \in \mathfrak{F}$, we have that $1 = 1 \cdot 1 \in \tau(e)\tau(s) \subseteq \tau(es) = \tau(x)$. Hence $x \in \mathrm{K}_{\mathfrak{F}}(S) \cap J$. Therefore $\mathrm{E}_s \subseteq \mathrm{K}_{\mathfrak{F}}(S) \cap J$.

□

Lemma 33. *Let (S, J) be a minimal pair. Let $K = \bigcup\{(S_e)^{\mathfrak{F}} : e \in \mathrm{E}(S)\}$. Assume that $\mathrm{E}_s \subseteq \mathrm{E}(J)$, for every $s \in K$. Then, $S_e \in \mathfrak{F}$ for every $e \in \mathrm{E}(S)$.*

Proof. Assume that $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. Let $e \in \mathrm{E}(S)$ and $s \in (S_e)^{\mathfrak{F}}$. Since $\mathrm{E}_s \subseteq \mathrm{E}(J)$, it follows that $\mathrm{E}_s = \{(i, 1, i) : i \in \alpha_s\}$. By Proposition 23, $s \in \mathrm{E}(S)$ and so $s = e$. Hence $(S_e)^{\mathfrak{F}} = \{e\}$, for every $e \in \mathrm{E}(S)$, as we wanted. □

4.3 Quotients

A subsemigroup K of an inverse semigroup S is *closed under conjugation* if sKs^{-1} $s^{-1}Ks$ are contained in K, for every $s \in S$.

If $\mathfrak{F}$ is a variety of groups, then $K_{\mathfrak{F}}(S)$ is closed under conjugation. Let $s \in S$ and $k \in K_{\mathfrak{F}}(S)$. If $\tau\colon S \multimap G \in \mathfrak{F}$ is a relational morphism, there exists $x \in \tau(s)$ such that $x^{-1} \in \tau(s^{-1})$ and therefore $1 = x \cdot 1 \cdot x^{-1} \in \tau(s)\tau(k)\tau(s^{-1}) \subseteq \tau(sks^{-1})$. Hence $1 \in \tau(sks^{-1})$. Similarly, $1 \in \tau(s^{-1}ks)$.

The intersection of any family of subsemigroups of S which are closed under conjugation is closed under conjugation as well. In particular, given a subset T of S, there exists a subsemigroup T^c of S satifsying the following properties:

1. T^c is closed under conjugation.
2. $T \subseteq T^c$.
3. R is a subsemigroup of S closed under conjugation and $T \subseteq R$, then $T^c \subseteq R$.

In particular, if $T \subseteq \mathrm{K}_{\mathfrak{F}}(S)$, then $T^c \subseteq \mathrm{K}_{\mathfrak{F}}(S)$.

In this section, (S, J) will be a minimal pair, $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, and K will be a subsemigroup of S such that K is closed under conjugation and $\mathrm{E}(J^0) \subseteq K \subseteq J^0$.

Note that J^0 is in fact closed under conjugation by the 0-minimality of J. Therefore, for every subset $T \subseteq J^0$, we have that $T^c \subseteq J^0$.

Lemma 34. *1. There exists a normal subgroup N of G such that $(i, G, i) \cap K = (i, N, i)$, for every $i \in \Lambda$. We say that $N = N_K$ is the* normal subgroup of G associated with K.

2. There exists an equivalence relation $\mathcal{R}$ on Λ defined by the rule $i, j \in \Lambda$, $i \,\mathcal{R}\, j$ if, and only if, $(i, G, j) \cap K \neq \varnothing$, for all $i, j \in \Lambda$. In this case, there exists $g \in G$, such that $(i, G, j) \cap K = (i, gN, j)$; $\mathcal{R} = \mathcal{R}_K$ will be called the equivalence relation associated with K.

Proof. Let $i \in \Lambda$ and write $e_i = (i, 1, i)$. By Lemma 5, we have $S_{e_i} = (J^0)_{e_i} = (i, G, i)$. Since $K \subseteq J^0$, then $K_{e_i} = (J^0)_{e_i} \cap K$ is a subgroup of $(J^0)_{e_i}$. In particular, there exists a subgroup N_i of G such that $K_{e_i} = (i, G, i) \cap K = (i, N_i, i)$. Let $g \in G$. Then $(i, h^g, i) = (i, g^{-1}, i)(i, h, i)(i, g, i) \in K$, because $(i, g^{-1}, i) = (i, g, i)^{-1}$ and K is closed under conjugation. Hence, $N_i \trianglelefteq G$.

On the other hand, let $i \neq i' \in \Lambda$. Then, we see that

$$(i', N_i, i') = (i', 1, i)(i, N_i, i)(i, 1, i'), \quad (i, N_{i'}, i) = (i, 1, i')(i', N_{i'}, i')(i', 1, i)$$

are both contained in K, because $(i, 1, i') = (i', 1, i)^{-1}$. Thus, $N_i = N_{i'}$. Hence, we conclude that for every $i \in \Lambda$, there exists $N \trianglelefteq G$, such that $K \cap (i, G, i) = (i, N, i)$. This proves Statement 1.

Now, we can define on Λ the following relation:

$$i\,\mathcal{R}\,j \text{ if, and only if, } (i,G,j)\cap K\neq\varnothing, \text{for every } i,j\in\Lambda.$$

Clearly, $\mathcal{R}$ is an equivalence relation, because $(i,1,i)\in K$ for every $i\in\Lambda$, and if $(i,g,j)\in K$ then $(j,g^{-1},i)=(i,g,j)^{-1}\in K$. In addition, if $(i,g,i'),(i',g',i'')\in K$ then $(i,gg',i'')\in K$.

Suppose that $i\,\mathcal{R}\,j$ and $(i,g,j)\in(i,G,j)\cap K$. Since $(j,N,j)\subseteq K$, $(i,g,j)(j,N,j)=(i,gN,j)\subseteq K\cap(i,G,j)$. Now, suppose that $(i,g',j)\in(i,G,j)\cap K$. Then, $(j,(g')^{-1},i)\in K$ and

$$(i,gN,j)(j,(g')^{-1},i)=(i,gN(g')^{-1},i)\subseteq K\cap(i,G,i)=(i,N,i).$$

Thus, $gN(g')^{-1}=N$ and then $gN=g'N$. Hence, we can conclude that $(i,G,j)\cap K=(i,gN,j)$.

□

Remark 35. Suppose that:

$$(i,G,i)\cap K=\{(i,1,i)\}, \text{ for every } i\in\Lambda,$$
$$(i,G,j)\cap K\neq\varnothing\Leftrightarrow i=j,$$

i.e. the normal subgroup associated with K is the trivial one and the equivalence relation associated with K is the identity relation. If $(i,g,j)\in K$, then $j=i$ and $g=1$. Therefore $K=\mathrm{E}(J^0)$.

Let N_K and $\mathcal{R}_K$ be the normal subgroup and the equivalence relation associated with K. Let $\tilde{G}=G/N_K$ be the quotient group and let $\tilde{\Lambda}:=\Lambda/\mathcal{R}_K=\{\tilde{i}_1,\dots,\tilde{i}_\lambda\}$ with $\tilde{i}_l=[i_l]_{\mathcal{R}_K}$, for some $i_l\in\Lambda$, for every $1\leq l\leq\lambda$, be the quotient set of Λ by $\mathcal{R}_K$.

Let $1\leq l\leq\lambda$. Then, by Lemma 34, for every $i,i'\in\tilde{i}_l$ there exists $\tilde{g}_{ii'}\in\tilde{G}$ such that $(i,G,i')\cap K=(i,\tilde{g}_{ii'},i')$. Moreover:

- For every $i,i'\in\tilde{i}_l$, $\big((i,G,i')\cap K\big)^{-1}=\big((i,\tilde{g}_{ii'},i')\big)^{-1}=(i',\tilde{g}_{ii'}^{-1},i)$. Since $|(i',\tilde{g}_{ii'}^{-1},i)|=|N_K|=|(i',G,i)\cap K|$, we have that $\tilde{g}_{ii'}^{-1}=\tilde{g}_{i'i}$.

- For every $i,i',i''\in\tilde{i}_l$, $((i,G,i')\cap K)((i',G,i'')\cap K)=(i,\tilde{g}_{ii'}\tilde{g}_{i'i''},i'')$. Since $|(i,\tilde{g}_{ii'}\tilde{g}_{i'i''},i'')|=|N_K|=|(i,G,i'')\cap K|$, we have that $\tilde{g}_{ii'}\tilde{g}_{i'i''}=\tilde{g}_{ii''}$.

- In particular, for every $i\in\tilde{i}_l$, $g_{ii}=\tilde{1}=N\in\tilde{G}$.

Then, we can consider the set $X_K := \{\tilde{g}_{ii'} \in \tilde{G} : i, i' \in \tilde{i}_l, \text{ with } 1 \leq l \leq \lambda\}$ and we call it a *set of representatives of* $\tilde{G}$ *associated with* $\mathcal{R}_K$.

Let $s \in S$ and consider the sets $\alpha_s, \omega_s \subseteq \Lambda$, the bijection ι_s between α_s and ω_s, the map γ_s from α_s to G and the set E_s of projections of s onto J^0 defined in Section 4.1. Denote by $\tilde{\alpha}_s := \alpha_s/\mathcal{R}_K$ and $\tilde{\omega}_s := \omega_s/\mathcal{R}_K$) the set of all equivalence classes of the elements of α_s and ω_s respectively.

In the sequel, we use the above notation, terminology and properties without any further reference.

Lemma 36. *1. ι_s induces a bijection $\tilde{\iota}_s$ from $\tilde{\alpha}_s$ into $\tilde{\omega}_s$.*

2. There exists a map $\tilde{\gamma}_s\colon \tilde{\alpha}_s \to \tilde{G}$, defined by $\tilde{\gamma}_s(\tilde{i}_l) = \tilde{g}_{i_l i}\widetilde{\gamma_s(i)}\tilde{g}_{\iota_s(i)i_{l'}}$, for some $i \in \alpha_s$ with $\iota_s(i) \in \tilde{i}_{l'}$ such that

$$\tilde{\mathrm{E}}_s := \{(\tilde{i}, \tilde{\gamma}_s(\tilde{i}), \tilde{\iota}_s(\tilde{i})) : \tilde{i} \in \tilde{\alpha}_s\}$$

is a projection subset of the Brandt semigroup $\tilde{J}^0 = \mathcal{M}^0(\tilde{G}, \tilde{\Lambda}, \tilde{\Lambda}, I_{\tilde{\Lambda}})$.

In particular, $\tilde{\alpha}_s = \tilde{\omega}_s = \tilde{\mathrm{E}}_s = \varnothing$ if, and only if, $s = 0$.

Proof. Given $i, i' \in \alpha_s$ such that $i \,\mathcal{R}\, i' \,\mathcal{R}\, i_l$, we have that:

$$\begin{aligned} s^{-1}((i, G, i') \cap K)s &= s^{-1}(i, 1, i)(i, \tilde{g}_{ii'}, i')(i', 1, i')s = \\ &= (\iota_s(i), \gamma_s(i)^{-1}, i)(i, \tilde{g}_{ii'}, i')(i', \gamma_s(i'), \iota_s(i')) = \\ &= (\iota_s(i), \widetilde{\gamma_s(i)}^{-1}\tilde{g}_{ii'}\widetilde{\gamma_s(i')}, \iota_s(i')) \subseteq K \cap (\iota_s(i), G, \iota_s(i')), \end{aligned}$$

i.e. $\iota_s(i) \,\mathcal{R}_K\, \iota_s(i') \,\mathcal{R}_K\, i_{l'}$, for some $1 \leq l' \leq \lambda$. Moreover:

$$|(\iota_s(i), \widetilde{\gamma_s(i)}^{-1}\tilde{g}_{ii'}\widetilde{\gamma_s(i')}, \iota_s(i'))| = |N_K| = |K \cap (\iota_s(i), G, \iota_s(i'))|$$

so that we have:

$$(\iota_s(i), G, \iota_s(i')) \cap K = (\iota_s(i), \widetilde{\gamma_s(i)}^{-1}\tilde{g}_{ii'}\widetilde{\gamma_s(i')}, \iota_s(i')).$$

Hence:

$$\tilde{g}_{\iota_s(i)\iota_s(i')} = \tilde{g}^{-1}_{i_{l'}\iota_s(i)}\tilde{g}_{i_{l'}\iota_s(i')} = \widetilde{\gamma_s(i)}^{-1}\tilde{g}_{ii'}\widetilde{\gamma_s(i')} = \widetilde{\gamma_s(i)}^{-1}\tilde{g}^{-1}_{i_l i}\tilde{g}_{i_l i'}\widetilde{\gamma_s(i')}. \tag{4.4}$$

By Proposition 19, $\iota_{s^{-1}} = \iota_s^{-1}$. Therefore if $\iota_s(i) \,\mathcal{R}\, \iota_s(i') \,\mathcal{R}\, i_{l'}$ for some $1 \leq l' \leq \lambda$, then $i \,\mathcal{R}\, i' \,\mathcal{R}\, i_l$ for some $1 \leq l \leq \lambda$.

We define $\tilde{\iota}_s\colon \tilde{\alpha}_s \to \tilde{\omega}_s$ by:

$$\tilde{\iota}_s(\tilde{i}_l) = \tilde{i}_{l'} \Leftrightarrow \text{ there exists } i \in \alpha_s \cap \tilde{i}_l \text{ and } \iota_s(i) \in \tilde{i}_{l'}.$$

According to the above discussion, $\tilde{\iota}_s$ is a bijective map. This proves Statement 1.

Define $\tilde{\gamma}_s \colon \tilde{\alpha}_s \to \tilde{G}$ by:

$$\tilde{\gamma}_s(\tilde{i}_l) = \tilde{g}_{i_l i}\widetilde{\gamma_s(i)}\tilde{g}_{\iota_s(i) i_{l'}}, \text{ where } \tilde{\iota}_s(\tilde{i}_l) = \tilde{i}_{l'} \text{ and } i \in \tilde{i}_l, \text{for every } \tilde{i}_l \in \tilde{\alpha}_s.$$

Then $\tilde{\gamma}_s$ is well-defined by (4.4). Moreover, it is a routine matter to prove that

$$\tilde{\mathrm{E}}_s := \{(\tilde{i}_l, \tilde{\gamma}_s(\tilde{i}_l), \tilde{\iota}_s(\tilde{i}_l)) : \tilde{i}_l \in \tilde{\alpha}_s\}$$

is a projection subset of the Brandt semigroup J^0.

In particular, since (S, J) is a minimal pair, recall that $\alpha_s = \omega_s = \mathrm{E}_s = \varnothing$ if, and only if, $s = 0$. As a consequence, we also have that $\tilde{\alpha}_s = \tilde{\omega}_s = \tilde{\mathrm{E}}_s = \varnothing$ if, and only if, $s = 0$.

□

The following lemma introduces an important semigroup associated with (S, J, K).

Lemma 37. *There exists a minimal pair $(S_K, \tilde{J})$ with $\tilde{J}^0 = \mathcal{M}^0(\tilde{G}, \tilde{\Lambda}, \tilde{\Lambda}, I_{\tilde{\Lambda}})$, $S_K \subseteq \mathcal{U}_{\tilde{J}^0}$ and a map $\phi_K \colon S \longrightarrow \mathcal{U}_{\tilde{J}^0}$ such that:*

1. *S_K is the inverse subsemigroup of $\mathcal{U}_{\tilde{J}^0}$ generated by $\phi_K(S)$.*
2. *For each $(i, g, j) \in J$, $\phi_K(i, g, j) = (\tilde{i}_l, \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_{l'}}, \tilde{i}_{l'})$, where $i \in \tilde{i}_l$, $j \in \tilde{i}_{l'}$, for some $1 \le l, l' \le \lambda$. As a consequence, $\phi_K(J^0) = \tilde{J}^0$ and $\phi_K(K) = \mathrm{E}(\tilde{J}^0)$.*
3. *If $u = \phi_K(s)$, for some $s \in S$, then $\mathrm{E}_u = \phi_K(\mathrm{E}_s)$.*

Proof. 1. Let $B = \mathcal{M}^0(\tilde{G}, \tilde{\Lambda}, \tilde{\Lambda}, I_{\tilde{\Lambda}})$ and consider the semigroup $\mathcal{U}_B$ of projections of B. We define a map $\phi_K \colon S \longrightarrow \mathcal{U}_B$ as follows:

$$\phi_K(s) = u \in \mathcal{U}_B \Leftrightarrow \mathrm{E}_u = \{(\tilde{i}_l, \tilde{\gamma}_s(\tilde{i}_l), \tilde{\iota}_s(\tilde{i}_l)) : \tilde{i}_l \in \tilde{\alpha}_s\} = \tilde{\mathrm{E}}_s, \text{for every } s \in S.$$

It is easy to check that $\phi_K(s)^{-1} = \phi_K(s^{-1})$. Therefore, $S_K := \langle \phi_K(S) \rangle$ is the smallest inverse subsemigroup of $\mathcal{U}_B$ containing $\phi_K(S)$. This proves Statement 1.

2. Consider a set of representatives X_K of $\tilde{G}$ associated with $\mathcal{R}_K$. Let $s = (i, g, j) \in J$ with $i \in \tilde{i}_l$ and $j \in \tilde{i}_{l'}$, for some $1 \le l, l' \le \lambda$. Then,

$$\tilde{\alpha}_s = \{\tilde{i}_l\}, \quad \tilde{\gamma}_s(\tilde{i}_l) = \tilde{g}_{i_l i}\widetilde{\gamma_s(i)}\tilde{g}_{\iota_s(i) i_{l'}} = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_{l'}}, \quad \tilde{\iota}_s(\tilde{i}_l) = \tilde{i}_{l'}.$$

Therefore, $\tilde{\mathrm{E}}_s = \{(\tilde{i}_l, \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_{l'}}, \tilde{i}_{l'})\}$ and then $\phi_K(s) = (\tilde{i}_l, \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_{l'}}, \tilde{i}_{l'})$.

To verify that $\phi_K(J^0) = \tilde{J}^0$, notice first that if $u = (\tilde{i}_l, \tilde{g}, \tilde{i}_{l'}) \in \tilde{J}$, we can take $s = (i_l, g, i_{l'}) \in J$ and then, by the above paragraph, $\phi_K(s) = u$. Thus, $\tilde{J} \subseteq \phi_K(J)$. Moreover, the above paragraph also implies that $\phi_K(J) \subseteq \tilde{J}$. By definition of φ_K, $\phi_K(s) = 0$ if, and only if, $\tilde{\mathrm{E}}_s = \varnothing$ and then, by Lemma 36 if, and only if, $s = 0$. Therefore, we conclude $\phi_K(J^0) = \tilde{J}^0$.

To prove that $\phi_K(K) = \mathrm{E}(\tilde{J}^0)$, let $s = (i, g, j) \in K$. Then, by Lemma 34, $i \,\mathcal{R}\, j \,\mathcal{R}\, i_l$ for some $1 \le l \le \lambda$ and $\tilde{g} = \tilde{g}_{ij}$. Thus, $\tilde{\alpha}_s = \{\tilde{i}_l\} = \tilde{\omega}_s$, $\tilde{\iota}_s = id_{\tilde{\alpha}_s}$ and $\tilde{\gamma}_s(\tilde{i}_l) = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_l} = \tilde{g}_{i_l i}\tilde{g}_{ij}\tilde{g}_{j i_l} = \tilde{g}_{i_l i_l} = \tilde{1}$. Then, $\phi_K(s) = (\tilde{i}_l, \tilde{1}, \tilde{i}_l) \in \mathrm{E}(B)$. Let $(\tilde{i}_l, \tilde{1}, \tilde{i}_l) \in \mathrm{E}(B)$ for some $1 \le l \le \lambda$. Then $\phi_K((i_l, 1, i_l)) = (\tilde{i}_l, \tilde{1}, \tilde{i}_l)$. Note that $\phi_K(0) = 0$ with $0 \in \mathrm{E}(J^0) \subseteq K$. Hence, we conclude that $\phi_K(K) = \mathrm{E}(\tilde{J}^0)$.

Hence, S_K is an inverse subsemigroup of $\mathcal{U}_B$ such that $B = \phi_K(J^0) \subseteq S_K \subseteq \mathcal{U}_B$. Thus, applying Lemma 29, we conclude that $(S_K, \tilde{J})$ is a minimal pair.

3. By definition of ϕ_K, we know that $\mathrm{E}_u = \tilde{\mathrm{E}}_s = \{(\tilde{i}_l, \tilde{\gamma}_s(\tilde{i}_l), \tilde{\iota}_s(\tilde{i}_l)) : \tilde{i}_l \in \tilde{\alpha}_s\}$.

Now, let $\tilde{i}_l \in \tilde{\alpha}_s$. For every $i \in \tilde{i}_l \cap \alpha_s$, we have that $\iota_s(i) \in \tilde{\iota}_s(\tilde{i}_l) = \tilde{i}_{l'}$, for some $1 \le l' \le \lambda$. Then, $\phi_K(i, \gamma_s(i), \iota_s(i)) = (\tilde{i}_l, \tilde{g}, \tilde{\iota}_s(\tilde{i}_l))$, where $\tilde{g} = \tilde{g}_{i_l i}\widetilde{\gamma_s(i)}\tilde{g}_{\iota_s(i) i_{l'}} = \tilde{\gamma}_s(\tilde{i}_l)$. Thus, $\phi_K(i, \gamma_s(i), \iota_s(i)) = (\tilde{i}_l, \tilde{\gamma}_s(\tilde{i}_l), \tilde{\iota}_s(\tilde{i}_l))$, for every $i \in \tilde{i}_l \cap \alpha_s$.

Hence, we can conclude that $\phi_K(\mathrm{E}_s) = \tilde{\mathrm{E}}_s = \mathrm{E}_u$. □

The minimal pair $(S_K, \tilde{J})$ constructed in Lemma 37 will be called *the* (K, ϕ_K)-*quotient* of (S, J), where ϕ_K is the map defined in the proof of that result.

We cannot in general assert that ϕ_K is a homomorphism. The importance of ϕ_K lies in the following results:

Corollary 38. *Let $(S_K, \tilde{J})$ be the (K, ϕ_K)-quotient of (S, J). Then, for every $s, t \in J$, we have that $\phi_K(s) = \phi_K(t)$ if, and only if, $s = xty$ with $x, y \in K$.*

Proof. Consider a set of representatives X_K of $\tilde{G}$ associated with $\mathcal{R}_K$.

Assume that $s = (i, g, j), t = (i', g', j') \in J$ and $\phi_K(s) = \phi_K(t)$. Then $\tilde{\mathrm{E}}_s = \tilde{\mathrm{E}}_t$. Hence, by Lemma 36, $\tilde{\alpha}_s = \tilde{\alpha}_t = \{\tilde{i}_l\}$ and $\tilde{\omega}_s = \tilde{\omega}_t = \{\tilde{i}_{l'}\}$, where $i, i' \in \tilde{i}_l$ and $j = \iota_s(i), j' = \iota_t(i') \in \tilde{i}_{l'}$, for some $1 \le l, l' \le \lambda$. Moreover:

$$\tilde{\gamma}_s(\tilde{i}_l) = \tilde{\gamma}_t(\tilde{i}_l) = \begin{cases} \tilde{g}_{i_l i}\widetilde{\gamma_s(i)}\tilde{g}^{-1}_{i_{l'}\iota_s(i)} = \tilde{g}_{i_l i}\tilde{g}\tilde{g}^{-1}_{i_{l'} j} \\ \tilde{g}_{i_l i'}\widetilde{\gamma_t(i')}\tilde{g}^{-1}_{i_{l'}\iota_t(i')} = \tilde{g}_{i_l i'}\tilde{g}'\tilde{g}^{-1}_{i_{l'} j'}. \end{cases}$$

Thus, $\tilde{g}_{i_l i}\tilde{g}\tilde{g}^{-1}_{i_{l'} j} = \tilde{g}_{i_l i'}\tilde{g}'\tilde{g}^{-1}_{i_{l'} j'}$ and therefore, $\tilde{g}' = \tilde{g}^{-1}_{i_l i'}\tilde{g}_{i_l i}\tilde{g}\tilde{g}^{-1}_{i_{l'} j}\tilde{g}_{i_{l'} j'}$. Then, $g' = g^{-1}_{i_l i'}g_{i_l i}g g^{-1}_{i_{l'} j}g_{i_{l'} j'}n$, for some $n \in N_K$. Hence, we can write:

$$(i', g', j') = (i', g^{-1}_{i_l i'}, i_l)(i_l, g_{i_l i}, i)(i, g, j)(j, g^{-1}_{i_{l'} j}, i_{l'})(i_{l'}, g_{i_{l'} j'}n, j'),$$

where:

$$x := (i', g_{i_l i'}^{-1}, i_l)(i_l, g_{i_l i}, i) \in (i', \tilde{g}_{i_l i'}^{-1}\tilde{g}_{i_l i}, i) = (i', \tilde{g}_{i' i}, i) \subseteq K$$
$$y := (j, g_{i_{l'} j}^{-1}, i_{l'})(i_{l'}, g_{i_{l'} j'} n, j') \in (j, \tilde{g}_{i_{l'} j}\tilde{g}_{i_{l'} j'}, j') = (j, \tilde{g}_{jj'}, j') \subseteq K.$$

Conversely, suppose that $s = (i, g, j), t = (i', g', j') \in J$ and there exist $x, y \in K$ with $s = xty$. Then, $x = (i, g_1, i') \in (i, G, i') \cap K$, $y = (j', g_2, j) \in (j', G, j) \cap K$ and $g = g_1 g' g_2$. Thus, $i \,\mathcal{R}\, i' \,\mathcal{R}\, i_l$ and $j \,\mathcal{R}\, j' \,\mathcal{R}\, i_{l'}$, for some $1 \leq l, l' \leq \lambda$. Then $\tilde{\alpha}_s = \tilde{\alpha}_t = \{\tilde{i}_l\}$ and $\tilde{\iota}_s(\tilde{i}_l) = \tilde{\iota}_t(\tilde{i}_l) = \tilde{i}_{l'}$. Moreover:

$$x \in (i, \tilde{g}_{ii'}, i') = (i, \tilde{g}_{ii_l}\tilde{g}_{i_l i'}, i') \Rightarrow x = (i, g_{ii_l} g_{i_l i'} n_1, i')$$
$$y \in (j', \tilde{g}_{j'j}, j) = (j', \tilde{g}_{j' i_{l'}}\tilde{g}_{i_{l'} j}, j) \Rightarrow y = (j', g_{j' i_{l'}} g_{i_{l'} j} n_2, j)$$

for some $n_1, n_2 \in N$. Then, $g_1 = g_{ii_l} g_{i_l i'} n_1$, $g_2 = g_{j' i_{l'}} g_{i_{l'} j} n_2$ and therefore:

$$\begin{aligned}\tilde{\gamma}_s(\tilde{i}_l) &= \tilde{g}_{i_l i}\widetilde{\gamma_s(i)}\tilde{g}_{i_{l'}\iota_s(i)}^{-1} = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{i_{l'} j}^{-1} = \tilde{g}_{i_l i}\tilde{g}_1\tilde{g}'\tilde{g}_2\tilde{g}_{i_{l'} j}^{-1} = \\ &= \tilde{g}_{i_l i}(\tilde{g}_{i_l i}^{-1}\tilde{g}_{i_l i'})\tilde{g}'(\tilde{g}_{i_{l'} j'}^{-1}\tilde{g}_{i_{l'} j})\tilde{g}_{i_{l'} j}^{-1} = \tilde{g}_{i_l i'}\tilde{g}'\tilde{g}_{i_{l'} j'}^{-1} = \\ &= \tilde{g}_{i_l i'}\widetilde{\gamma_t(i')}\tilde{g}_{i_{l'}\iota_t(i')}^{-1} = \tilde{\gamma}_t(\tilde{i}_l).\end{aligned}$$

Hence, $\tilde{\mathrm{E}}_s = \tilde{\mathrm{E}}_t = \{(\tilde{i}_l, \tilde{\gamma}_s(\tilde{i}_l), \tilde{i}_{l'})\}$, and $\phi_K(s) = \phi_K(t)$.

□

Corollary 39. *Let $(S_K, \tilde{J})$ be the (K, ϕ_K)-quotient of (S, J). The following assertions hold:*

1. *Suppose that for some $x \in J$, $\phi_K(x) = yy' \in \tilde{J}^0$ with $y, y' \in \tilde{J}^0$. Then, there exist $x_1, x_1' \in J$ such that $x = x_1 x_1'$, $\phi_K(x_1) = y$ and $\phi_K(x_1') = y'$.*

2. *Suppose that $x, y \in J$ and $0 \neq xy \in J$. Then, $\phi_K(xy) = \phi_K(x)\phi_K(y)$.*

Proof. Consider a set of representatives X_K of $\tilde{G}$ associated with $\mathcal{R}_K$.

1. Let $x = (i, g, j) \in J$ with $i \in \tilde{i}_l$, $j = \iota_x(i) \in \tilde{i}_{l'}$ and $g = \gamma_x(i)$. Then, by Lemma 36, $\tilde{\alpha}_x = \{\tilde{i}_l\}$, $\tilde{\iota}_x = \tilde{i}_{l'}$, $\tilde{\gamma}_x(\tilde{i}_l) = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_{l'}}$ and $\tilde{\mathrm{E}}_x = \{(\tilde{i}_l, \tilde{\gamma}_x(\tilde{i}_l), \tilde{i}_{l'})\}$. Thus $\phi_K(x) = (\tilde{i}_l, \tilde{\gamma}_x(\tilde{i}_l), \tilde{i}_{l'})$.

Suppose that $\phi_K(x) = yy'$ with $y, y' \in \tilde{J}^0$. Then $\mathrm{E}_{\phi_K(x)} = \mathrm{E}_y \mathrm{E}_{y'} \setminus \{0\}$ and therefore, applying Proposition 19 to the minimal pair $(S_K, \tilde{J})$, we have:

$$y = \big(\tilde{i}_l, \gamma_y(\tilde{i}_l), \iota_y(\tilde{i}_l)\big), \quad y' = \big(\iota_y(\tilde{i}_l), \gamma_{y'}(\iota_y(\tilde{i}_l)), \iota_{y'}(\iota_y(\tilde{i}_l))\big),$$

with $\gamma_x(\tilde{i}_l) = \gamma_y(\tilde{i}_l)\gamma_{y'}(\iota_y(\tilde{i}_l))$ and $\iota_x(\tilde{i}_l) = \iota_{y'}(\iota_y(\tilde{i}_l)) = \tilde{i}_{l'}$.

Assume that $\iota_y(\tilde{i}_l) = \tilde{i}_{l''} \in \tilde{\Lambda}$. Since $\phi_K(J) = \tilde{J}^0$, there exist $(i_l, g_1, i_{l''})$, $(i_{l''}, g_2, i_{l'}) \in J$ such that $\phi_K(i_l, g_1, i_{l''}) = y$ and $\phi_K(i_{l''}, g_2, i_{l'}) = y'$, where $\tilde{g}_1 = \gamma_y(\tilde{i}_l)$ and $\tilde{g}_2 = \gamma_{y'}(\iota_y(\tilde{i}_l))$. Thus:

$$\tilde{g}_1\tilde{g}_2 = \gamma_y(\tilde{i}_l)\gamma_{y'}(\iota_y(\tilde{i}_l)) = \tilde{\gamma}_x(\tilde{i}_l) = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{j i_{l'}} \Rightarrow \tilde{g} = \tilde{g}_{i_l i}^{-1}\tilde{g}_1\tilde{g}_2\tilde{g}_{j i_{l'}}^{-1}$$

and then, $g = g_{i_l i}^{-1} g_1 g_2 g_{j i_{l'}}^{-1} n$, for some $n \in N$.

Consider

$$\begin{array}{l} x_1 = (i, g_{i_l i}^{-1}, i_l)(i_l, g_1, i_{l''})(i_{l''}, 1, i_{l''}) = (i, g_{i i_l}, i_l) y (i_{l''}, 1, i_{l''}) \\ x_1' = (i_{l''}, 1, i_{l''})(i_{l''}, g_2, i_{l'})(i_{l'}, g_{j i_{l'}}^{-1} n, j) = (i_{l''}, 1, i_{l''}) y' (i_{l'}, g_{j i_{l'}}^{-1} n, j) \end{array},$$

with $(i, g_{i i_l}, i_l), (i_{l'}, g_{j i_{l'}}^{-1}, j)$, $(i_{l''}, 1, i_{l''}) \in K$. By Corollary 38, $\phi_K(x_1) = y$ and $\phi_K(x_1') = y'$.

2. Suppose that $x = (i, g, j), y = (i', g', j') \in J$ such that $0 \neq xy \in J$. Then $j = i'$ and $xy = (i, gg', j') \in J$. Assume that $i \,\mathcal{R}\, i_l$, $j = i' \,\mathcal{R}\, i_{l'}$ and $j' \,\mathcal{R}\, i_{l''}$. By Lemma 36, we have that $\tilde{\alpha}_x = \{\tilde{i}_l\}$, $\tilde{\alpha}_y = \{\tilde{i}_{l'}\}$, and:

$$\begin{array}{l} \tilde{\iota}_x(\tilde{i}_l) = \tilde{i}_{l'}, \tilde{\gamma}_x(\tilde{i}_l) = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{i_{l'} i'}^{-1}, \\ \tilde{\iota}_y(\tilde{i}_{l'}) = \tilde{i}_{l''}, \tilde{\gamma}_y(\tilde{i}_{l'}) = \tilde{g}_{i_{l'} i'}\tilde{g}'\tilde{g}_{i_{l''} j'}^{-1}, \\ \tilde{\iota}_{xy}(\tilde{i}_l) = \tilde{i}_{l''}, \tilde{\gamma}_{xy}(\tilde{i}_l) = \tilde{g}_{i_l i}\tilde{g}\tilde{g}'\tilde{g}_{i_{l''} j'}^{-1}. \end{array}$$

Then:

$$\begin{array}{l} \tilde{\iota}_{xy}(\tilde{i}_l) = \tilde{i}_{l''} = \tilde{\iota}_y(\tilde{\iota}_x(\tilde{i}_l)), \\ \tilde{\gamma}_{xy}(\tilde{i}_l) = \tilde{g}_{i_l i}\tilde{g}\tilde{g}'\tilde{g}_{i_{l''} j'}^{-1} = \tilde{g}_{i_l i}\tilde{g}\tilde{g}_{i_{l'} i'}^{-1}\tilde{g}_{i_{l'} i'}\tilde{g}'\tilde{g}_{i_{l''} j'}^{-1} = \tilde{\gamma}_x(\tilde{i}_l)\tilde{\gamma}_y(\tilde{\iota}_x(\tilde{i}_l)). \end{array}$$

Hence:

$$\begin{array}{l} \phi_K(x) = (\tilde{i}_l, \tilde{\gamma}_x(\tilde{i}_l), \tilde{\iota}_x(\tilde{i}_l)), \quad \phi_K(y) = (\tilde{i}_{l'}, \tilde{\gamma}_y(\tilde{i}_{l'}), \tilde{\iota}_y(\tilde{i}_{l'})) \\ \phi_K(xy) = (\tilde{i}_l, \tilde{\gamma}_{xy}(\tilde{i}_l), \tilde{\iota}_{xy}(\tilde{i}_l)). \end{array}$$

Consequently, $\phi_K(xy) = \phi_K(x)\phi_K(y)$.

□

Corollary 40. *Let $(S_K, \tilde{J})$ be the (K, ϕ_K)-quotient of (S, J). Then, for all $u, v \in \mathcal{U}_{\tilde{J}^0}$, we have $\phi_K^{-1}(\mathrm{E}_{uv}) \cap J \subseteq (\phi_K^{-1}(\mathrm{E}_u) \cap J)(\phi_K^{-1}(\mathrm{E}_v) \cap J)$.*

Proof. Let $x \in \phi_K^{-1}(\mathrm{E}_{uv}) \cap J$ then $\phi_K(x) \in \mathrm{E}_{uv}$. Since $\mathrm{E}_{uv} = (\mathrm{E}_u \mathrm{E}_v) \setminus \{0\}$, by Proposition 19, there exist a unique $y \in \mathrm{E}_u$ and a unique $y' \in \mathrm{E}_v$ such that $\phi_K(x) = yy'$. Applying Corollary 39, there exist $x_1, x_1' \in J$ such that $x_1 x_1' = x$ and with $\phi_K(x_1) = y$ and $\phi_K(x_1') = y'$. Thus, $x_1 \in \phi_K^{-1}(\mathrm{E}_u) \cap J$, $x_1' \in \phi_K^{-1}(\mathrm{E}_v) \cap J$ and $x \in \phi_K^{-1}(\mathrm{E}_u)\phi_K^{-1}(\mathrm{E}_v)$, as required. □

We are now ready to establish some properties of the relational morphisms between S and groups in a variety $\mathfrak{F}$ in the case where K is contained in the $\mathfrak{F}$-kernel. They turn out to be crucial in the next section.

Corollary 41. *Let $\mathfrak{F}$ be a variety. Let $(S_K, \tilde{J})$ be the (K, ϕ_K)-quotient of (S, J) and $\tau\colon S \multimap F$ a relational morphism with $F \in \mathfrak{F}$. If $K \subseteq \mathrm{K}_{\mathfrak{F}}(S)$, then the following statements hold.*

1. *If $s, t \in J$ and $\phi_K(s) = \phi_K(t)$, then $\tau(s) = \tau(t)$.*
2. *Let $0 \neq s \in S$ such that $0 \neq u = \phi_K(s) \in \mathcal{U}_{\tilde{J}^0}$. Then $\tau(s) \subseteq \tau(x)$, for all $x \in \phi_K^{-1}(\mathrm{E}_u) \cap J$.*
3. *For every $0 \neq u \in S_K$, $\bigcap\{\tau(x) : x \in \phi_K^{-1}(\mathrm{E}_u) \cap J\} \neq \varnothing$.*

Proof. 1. Let $s, t \in J$ such that $\phi_K(s) = \phi_K(t)$. Then, by Corollary 38, there exist $x, y \in K$ such that $s = xty$. Since τ is a relational morphism and $K \subseteq \mathrm{K}_{\mathfrak{F}}(S) \cap J$, it follows that $1 \in \tau(x) \cap \tau(y)$. Hence

$$\tau(s) = \tau(xty) \supseteq \tau(x)\tau(t)\tau(y) \supseteq \tau(t).$$

Similarly, $t = x'sy'$ with $x', y' \in K$. Then, $1 \in \tau(x') \cap \tau(y')$ and:

$$\tau(t) = \tau(x'sy') \supseteq \tau(x')\tau(s)\tau(y') \supseteq \tau(s).$$

Hence, $\tau(s) = \tau(t)$.

2. Assume that $0 \neq s \in S$ and $0 \neq u = \phi_K(s) \in \mathcal{U}_{\tilde{J}^0}$. Let $x \in \phi_K^{-1}(\mathrm{E}_u) \cap J$. Then, $\phi_K(x) \in \mathrm{E}_u$. Now, by Lemma 37, $\phi_K(\mathrm{E}_s) = \mathrm{E}_u$ so that there exists $y \in \mathrm{E}_s$ such that $\phi_K(y) = \phi_K(x)$. Applying Statement 1, we have that $\tau(y) = \tau(x)$. Moreover, by Lemma 24, $\tau(s) \subseteq \tau(y) = \tau(x)$.

3. Let $0 \neq u \in S_K$. Since $S_K = \langle \phi_K(S) \rangle$, it follows that $u = u_1 \cdots u_\rho$, with $u_r = \phi_K(s_r) \in \mathcal{U}_{\tilde{J}^0}$ for some $s_r \in S$, for every $1 \leq r \leq \rho$. By Corollary 40,

$$\phi_K^{-1}(\mathrm{E}_u) \cap J = \phi_K^{-1}(\mathrm{E}_{u_1 \cdots u_\rho}) \cap J \subseteq (\phi_K^{-1}(\mathrm{E}_{u_1}) \cap J) \cdots (\phi_K^{-1}(\mathrm{E}_{u_\rho}) \cap J).$$

Let $x \in \phi_K^{-1}(\mathrm{E}_u) \cap J$. Then, $x = x_1 \cdots x_\rho$, for some $x_r \in \phi_K^{-1}(\mathrm{E}_{u_r}) \cap J$, $1 \leq r \leq \rho$. According to Statement 2, we have that $\tau(s_r) \subseteq \tau(x_r)$, for every $1 \leq r \leq \rho$. Thus:

$$\tau(x) = \tau(x_1 \cdots x_\rho) \supseteq \tau(x_1) \cdots \tau(x_\rho) \supseteq \tau(s_1) \cdots \tau(s_\rho) \neq \varnothing.$$

Hence, for every $x \in \phi_K^{-1}(\mathrm{E}_u) \cap J$, $\tau(s_1) \cdots \tau(s_\rho) \subseteq \tau(x)$. Consequently:

$$\bigcap\{\tau(x) : x \in \phi_K^{-1}(\mathrm{E}_u) \cap J\} \neq \varnothing.$$

□

If $\mathcal{R}_K$ is the identity relation, that is, $i\,\mathcal{R}_K\,j$ if, and only if, $i = j$, then many complications in the foregoing account disappear. In particular, ϕ_K is a homomorphism and $S_K = \phi_K(S)$ is actually a quotient semigroup of S.

Lemma 42. *Suppose that the relation $\mathcal{R}_K$ associated with K is the identity relation. Then ϕ_K is a homomorphism and $S_K = \phi_K(S)$.*

Proof. Since $\mathcal{R}_K$ is the identity relation, then $\tilde{\Lambda} = \Lambda/\mathcal{R} = \Lambda$. By Lemma 36, $\tilde{\alpha}_s = \alpha_s$, $\tilde{\omega}_s = \omega_s$ and $\tilde{\iota}_s = \iota_s$, for every $s \in S$. Moreover, $\tilde{\gamma}_s(i) = \tilde{g}_{ii}\widetilde{\gamma_s(i)}\tilde{g}_{\iota_s(i)\iota_s(i)} = \widetilde{\gamma_s(i)}$. If $s \in S$, then $\phi_K(s) = u$ if, and only if:

$$\mathrm{E}_u = \{(i, \gamma_u(i), \iota_u(i)) : i \in \alpha_u\} = \{(i, \widetilde{\gamma_s(i)}, \iota_s(i) : i \in \alpha_s\}.$$

Therefore $\phi_K(s) = u$ if, and only if, $\alpha_s = \alpha_u$, $\omega_s = \omega_u$, $\iota_s = \iota_u$ and $\gamma_u(i) = \widetilde{\gamma_s(i)}$, for every $i \in \alpha_s = \alpha_u$.

We show now that ϕ_K is a homomorphism. Let $s, t \in S$, $\phi_K(s) = u$ and $\phi_K(t) = v$. By Proposition 19:

$$\alpha_{st} = \iota_s^{-1}(\omega_s \cap \alpha_t) = \iota_u^{-1}(\omega_u \cap \alpha_v) = \alpha_{uv}$$

and then $\omega_{st} = \iota_t(\omega_s \cap \alpha_t) = \iota_v(\omega_u \cap \alpha_v) = \omega_{uv}$. In particular, for every $i \in \alpha_{st} = \alpha_{uv}$:

$$\begin{aligned} \iota_{st}(i) &= \iota_t(\iota_s(i)) = \iota_v(\iota_u(i)) = \iota_{uv}(i), \\ \widetilde{\gamma_{st}(i)} &= \widetilde{\gamma_s(i)}\widetilde{\gamma_t(\iota_s(i))} = \gamma_u(i)\gamma_v(\iota_u(i)). \end{aligned}$$

Hence, $\phi_K(st) = uv = \phi_K(s)\phi_K(t)$. Then, by Lemma 37, $S_K = \langle\phi_K(S)\rangle = \phi_K(S)$.

□

4.4 Key Lemmas

In this section two lemmas that turn out to be crucial in the proof of Theorem B are proved. The first one reduces the computability of the $\mathfrak{F}$-kernel associated with a variety of groups $\mathfrak{F}$ to minimal pairs. The second one reduces the computability of the $\mathfrak{F}$-kernel of a minimal pair to the computability of the $\mathfrak{F}$-kernel of some of its quotients.

Lemma 43. *Let $\mathfrak{F}$ be a variety of groups. Then, for every inverse semigroup S with zero and a unique 0-minimal $\mathcal{J}$-class J, $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable if, and only if, for every minimal pair $(\bar{S}, \bar{J})$, $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ is computable.*

Proof. Only the sufficiency of the condition is in doubt.

Suppose that for every minimal pair $(\bar{S}, \bar{J})$, one can decide membership in $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$. Let S be an inverse semigroup S with zero with a unique 0-minimal $\mathcal{J}$-class J. By Lemma 30, there exist a minimal pair (V, J) and an epimorphism $\cup\colon S \to V$, such that $\mathrm{E}_s = \mathrm{E}_{\cup(s)}$, for every $s \in S$, and $\cup|_{J^0} = id_{J^0}$.

We prove that $\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(V) \cap J$. Let $s \in \mathrm{K}_{\mathfrak{F}}(S) \cap J$ and let $\tau\colon V \multimap F$ be a relational morphism with $F \in \mathfrak{F}$. Then, $\bar{\tau}\colon S \multimap F$, given by $\bar{\tau}(w) = \tau(\cup(w))$, is a relational morphism because is a composition of relational morphisms. Since $s \in \mathrm{K}_{\mathfrak{F}}(S) \cap J$, we have $1 \in \bar{\tau}(s)$ and then $1 \in \tau(\cup(s))$. Consequently, $s \in \mathrm{K}_{\mathfrak{F}}(V)$ and so $s \in \mathrm{K}_{\mathfrak{F}}(V) \cap J$.

On the other hand, let $s_0 \in \mathrm{K}_{\mathfrak{F}}(V) \cap J$. Let $\tau\colon S \multimap F$ be a relational morphism with $F \in \mathfrak{F}$. Let $\bar{\tau}\colon J \to \mathcal{P}(F)$ be the map given by $\bar{\tau}(s) = \tau(s)$, for every $s \in J$. Since S is an inverse semigroup with zero with a unique 0-minimal $\mathcal{J}$-class, we can apply Lemma 24 to τ to conclude that

$$\bar{\tau}(s)\bar{\tau}(t) = \tau(s)\tau(t) = \tau(st) = \bar{\tau}(st), \text{ for every } s, t \in J \text{ with } st \neq 0.$$

Moreover, for every $v \in V$, if $\cup(s) = v$ for some $s \in S$, we have

$$\begin{aligned}\varnothing \neq \tau(s) \subseteq \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\} = \\ = \bigcap\{\bar{\tau}(i,g,j) : (i,g,j) \in \mathrm{E}_v = \mathrm{E}_{\cup(s)}\}.\end{aligned}$$

Then $\bar{\tau}$ satisfies properties 1 and 2 of Lemma 24. Thus, we can define the relational morphism $\bar{\tau}\colon V \multimap F$, given by $\bar{\tau}(0) = F$ and $\bar{\tau}(v) = \bigcap\{\bar{\tau}(i,g,j) : (i,g,j) \in \mathrm{E}_v\}$. In particular, $1 \in \bar{\tau}(s_0) = \tau(s_0)$. Therefore $s_0 \in \mathrm{K}_{\mathfrak{F}}(S) \cap J$.

Consequently, we have that $\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(V) \cap J$. Since $\mathrm{K}_{\mathfrak{F}}(V) \cap J$ is computable, it follows that $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, as required.

□

Lemma 44. *Let $\mathfrak{F}$ be a variety of groups. Let (S, J) be a minimal pair and let $K \subseteq J^0$ a subsemigroup closed under conjugation such that $\mathrm{E}(J^0) \subseteq K \subseteq \mathrm{K}_{\mathfrak{F}}(S)$. Let $(S_K, \tilde{J})$ be the (K, ϕ_K)-quotient of (S, J). Then*

$$\mathrm{K}_{\mathfrak{F}}(S) \cap J = \phi_K^{-1}\big(\mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}\big) \cap J.$$

In particular, $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable if, and only if, $\mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}$ is computable.

Proof. Let $s_0 \in \mathrm{K}_{\mathfrak{F}}(S) \cap J$. We shall show that $\phi_K(s_0) \in \mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}$. Since $s_0 \in J$, then $\phi_K(s_0) \in \tilde{J}$ by Lemma 37. Let $\tau\colon S_K \multimap F$ be a relational morphism with $F \in \mathfrak{F}$.

We consider the map $\bar{\tau}\colon J \to \mathcal{P}(F)$, given by $\bar{\tau}(s) = \tau(\phi_K(s))$, for every $s \in J$. We prove that $\bar{\tau}$ satisfies the conditions of the second part of Lemma 24. On one hand, if $s_1, s_2 \in J$ and $0 \neq s_1s_2 \in J$, then, applying Corollary 39, it follows that $0 \neq \phi_K(s_1s_2) = \phi_K(s_1)\phi_K(s_2) \in \tilde{J}$. Since τ is a relational morphism, we can apply the first part of Lemma 24 to conclude that $\tau(\phi_K(s_1))\tau(\phi_K(s_2)) = \tau(\phi_K(s_1s_2))$. Then

$$\bar{\tau}(s_1)\bar{\tau}(s_2) = \tau(\phi_K(s_1))\tau(\phi_K(s_2)) = \tau(\phi_K(s_1s_2)) = \bar{\tau}(s_1s_2).$$

On the other hand, let $0 \neq s \in S$ such that $0 \neq u = \phi_K(s)$ and let $(i,g,j) \in \mathrm{E}_s$. Then, $\phi_K(i,g,j) \in \mathrm{E}_u$ and therefore $\tau(u) \subseteq \tau(\phi_K(i,g,j)) = \bar{\tau}(i,g,j)$. Thus, it holds:

$$\varnothing \neq \tau(u) \subseteq \bigcap\{\bar{\tau}(i,g,j) : (i,g,j) \in \mathrm{E}_s\}.$$

Then, by Lemma 24, the map $\bar{\tau}\colon S \multimap F$ defined by $\bar{\tau}(s) = \bigcap\{\tau(\phi_K(i,g,j)) : (i,g,j) \in \mathrm{E}_s\}$, for every $0 \neq s \in S$, and $\bar{\tau}(0) = F$ is a relational morphism.

Since $s_0 \in \mathrm{K}_{\mathfrak{F}}(S) \cap J$, $1 \in \bar{\tau}(s_0) = \tau(\phi_K(s_0))$. We conclude that $\phi_K(s_0) \in \mathrm{K}_{\mathfrak{F}}(S)$.

Now, let $s_0 \in \phi_K^{-1}\big(\mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}\big) \cap J$. Then $\phi_K(s_0) \in \mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}$. Therefore, it remains to prove that $s_0 \in \mathrm{K}_{\mathfrak{F}}(S)$. Consider an arbitrary relational morphism $\tau\colon S \multimap F$, with $F \in \mathfrak{F}$. By Corollary 41, there exists a map $\bar{\tau}\colon S_K \to \mathcal{P}(F)$ given by:

$$\bar{\tau}(u) = \bigcap\{\tau(x) : x \in \phi_K^{-1}(\mathrm{E}_u) \cap J\} \neq \varnothing, \text{ for every } 0 \neq u \in S_K, \quad \bar{\tau}(0) = F.$$

In addition, if $\phi_K(x) = \phi_K(x') = y$ then $\tau(x) = \tau(x')$, for every $x, y \in J$. Thus, if $y \in \tilde{J}^0$ then:

$$\phi_K^{-1}(\mathrm{E}_y) \cap J = \{x \in J : \phi_K(x) = y\}.$$

Therefore, $\bar{\tau}(y) = \tau(x)$, for some $x \in J$ with $\phi_K(x) = y$. In particular, $\bar{\tau}(\phi_K(s_0)) = \tau(s_0)$.

Let $u, v \in S_K$. If either $u = 0$, $v = 0$ or $uv = 0$, then it is clear that $\bar{\tau}(s)\bar{\tau}(v) \subseteq \bar{\tau}(uv)$, since $\bar{\tau}(0) = F$. Suppose that $uv \neq 0$ and let $h \in \bar{\tau}(u)$ and $h' \in \bar{\tau}(v)$. By Lemma 40, $\varphi_K^{-1}(\mathrm{E}_{uv}) \cap J \subseteq (\phi_K^{-1}(\mathrm{E}_u) \cap J)(\phi_K^{-1}(\mathrm{E}_v) \cap J)$. Hence if $x \in \phi_K^{-1}(\mathrm{E}_{uv}) \cap J$, there exist $x_1 \in \phi_K^{-1}(\mathrm{E}_u) \cap J$ and $x_2 \in \phi_K^{-1}(\mathrm{E}_v) \cap J$ such that $x = x_1x_2$, and then:

$$\tau(x) = \tau(x_1x_2) \supseteq \tau(x_1)\tau(x_2) \ni hh'.$$

Therefore, $hh' \in \bigcap\{\tau(x) : x \in \phi_K^{-1}(\mathrm{E}_{uv})\} = \bar{\tau}(uv)$. Consequently, $\bar{\tau}$ is a relational morphism.

Since $\phi_K(s_0) \in \mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}$, it follows that $1 \in \tau(s_0) = \bar{\tau}(\phi_K(s_0))$. Hence $s_0 \in \mathrm{K}_{\mathfrak{F}}(S)$, as required.

□

As a nice corollary to Lemma 44, we have the following description of the $\mathfrak{F}$-kernel of a Brandt semigroup.

Corollary 45. *Let $\mathfrak{F}$ be a variety of groups. Let $S = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$ be a Brandt semigroup. Then:*

$$\mathrm{K}_{\mathfrak{F}}(S) = \bigcup_{i\in\Lambda}(i, G^{\mathfrak{F}}, i) \cup \{0\}.$$

In particular, $\mathrm{K}_{\mathfrak{F}}(S)$ *is computable.*

Proof. Since S is a Brandt semigroup and $J = S \setminus \{0\}$, then (S, J) is a minimal pair. Let $K := \bigcup_{i\in\Lambda}(i, G^{\mathfrak{F}}, i) \cup \{0\} \subseteq J^0 = S$. Clearly, K is an inverse subsemigroup of $J^0 = S$ such that $\mathrm{E}(J^0) = \mathrm{E}(S) = \{(i, 1, i) : i \in \Lambda\} \cup \{0\} \subseteq K$. If $s = (i, g, j) \in S$, we have that

$$\begin{aligned} s^{-1}Ks &= (j, g^{-1}, i)(i, G^{\mathfrak{F}}, i)(i, g, j) \cup \{0\} = (j, (G^{\mathfrak{F}})^g, j) \cup \{0\} = \\ &= (j, G^{\mathfrak{F}}, j) \cup \{0\} \subseteq K. \end{aligned}$$

because $G^{\mathfrak{F}} \trianglelefteq G$. Thus K is closed under conjugation. Note that that $S_{(i,1,i)} = (i, G, i)$, for every $i \in \Lambda$. Hence

$$K = \{0\} \cup \left(\bigcup_{i\in\Lambda}(i, G^{\mathfrak{F}}, i)\right) = \{0\} \cup \left(\bigcup_{i\in\Lambda}(S_{(i,1,i)})^{\mathfrak{F}}\right).$$

By Proposition 12, $K \subseteq \mathrm{K}_{\mathfrak{F}}(S) \cap J^0$.

Note that $G^{\mathfrak{F}}$ and $\mathcal{R} = id$ are, respectively, the normal subgroup and the equivalence relation associated with K. Hence, if $(S_K, \tilde{J})$ is the (K, ϕ_K)-quotient of (S, J) we can apply Lemma 42, to conclude that $S_K = \phi_K(S) = \phi_K(J^0) = \tilde{J}^0$ is a subsemigroup of $\subseteq \mathcal{U}_{\tilde{J}^0}$, where $\tilde{J}^0 = \mathcal{M}^0(G/G^{\mathfrak{F}}, \Lambda, \Lambda, I_\Lambda)$. Thus, S_K is a Brandt semigroup. In addition, Lemma 42 also implies that $\phi_K(K) = \mathrm{E}(\tilde{J}^0) = \mathrm{E}(S_K)$ and $\phi_K^{-1}(\mathrm{E}(\tilde{J}^0)) = \phi_K^{-1}(\mathrm{E}(S_K)) = K$.

Now, we prove that $\mathrm{K}_{\mathfrak{F}}(S_K) = \mathrm{E}(S_K)$ by defining a relational morphism $\tau\colon\ S_K \multimap F$ with $F \in \mathfrak{F}$ and $\tau^{-1}(1) = \mathrm{E}(S_K)$.

Let p be a prime such that $C_p = \langle a\rangle \in \mathfrak{F}$ with $o(a) = p$. Then

$$F = G/G^{\mathfrak{F}} \times C_p \times \overset{|\Lambda|}{\cdots} \times C_p \in \mathfrak{F}.$$

Let $\tau\colon S_K \to \mathcal{P}(F)$ be the map given by:

$$\tau(i, gG^{\mathfrak{F}}, j) = \begin{cases} \big(gG^{\mathfrak{F}}, 1, \ldots, 1, \underset{i}{a^{-1}}, 1, \ldots, 1, \underset{j}{a}, 1, \ldots, 1\big), & \text{if } i \neq j \\ \big(gG^{\mathfrak{F}}, 1, \ldots, 1\big), & \text{if } i = j \end{cases}, \ \tau(0) = F.$$

One can easily check that τ is a relational morphism and

$$\tau^{-1}(1) = \{0\} \cup \{(i, g, j) : i = j \text{ and } g \in G^{\mathfrak{F}}\} = \{0\} \cup \{(i, G^{\mathfrak{F}}, i) : i \in \Lambda\} = \\ = \mathrm{E}(S_K).$$

By Lemma 44:

$$\mathrm{K}_{\mathfrak{F}}(S) \setminus \{0\} = \mathrm{K}_{\mathfrak{F}}(S) \cap J = \phi_K^{-1}(\mathrm{K}_{\mathfrak{F}}(S_K) \cap \tilde{J}^0) \cap J = \\ = \phi_K^{-1}(\mathrm{K}_{\mathfrak{F}}(S_K) \setminus \{0\}) \cap J = \phi_K^{-1}(\mathrm{E}(S_K) \setminus \{0\}) = K \setminus \{0\}$$

Consequently, $K_{\mathfrak{F}}(S_K) = K = \bigcup_{i \in \Lambda}(i, G^{\mathfrak{F}}, i) \cup \{0\}$.

□

4.5 Main result

The equivalence between 1 and 2 is just Theorem 14.

It is clear that 2 implies 3. Hence the circle of implications will be complete if we prove that 3 implies 2.

Suppose that $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ is computable for every inverse semigroup $\bar{S}$ with zero and a unique 0-minimal $\mathcal{J}$-class $\bar{J}$ with $\bar{S}_e \in \mathfrak{F}$, for each $e \in \mathrm{E}(\bar{S})$. We show that for every inverse semigroup S with zero $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable for every $\mathcal{J}$-class J of S.

Applying Corollary 17 and Lemma 43, it is enough to prove that $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every minimal pair (S, J).

Let (S, J) be a minimal pair. Then, we can define the following sequence $\{(S_m, J_m)\}_{m \in \mathbb{N}}$ of minimal pairs:

- $(S_1, J_1) := (S, J)$
- $(S_{m+1}, J_{m+1}) := (K_m, \phi_{K_m})$-quotient of (S_m, J_m), where $K_m = L_m^c$ and

$$L_m := \bigcup_{e \in \mathrm{E}(S_m)} \{\mathrm{E}_s : s \in (S_m)_e^{\mathfrak{F}}\} \cup \{0\}.$$

Let $m \in \mathbb{N}$. Then $e \in (S_m)_e^{\mathfrak{F}} \subseteq L_m$, for every $e \in \mathrm{E}(J_m)$. Thus, $\mathrm{E}(J_m^0) \subseteq L_m$. Moreover, by Lemma 32, $L_m \subseteq K_{\mathfrak{F}}(S_m)$. Therefore, $K_m = (L_m)^c \subseteq J_m^0$

satisfies that $\mathrm{E}(J_m^0) \subseteq K_m \subseteq \mathrm{K}_{\mathfrak{F}}(S_m) \cap J_m^0$. Hence, applying Lemma 44, we have:

$$\mathrm{K}_{\mathfrak{F}}(S_m) \cap J_m = \phi_m^{-1}(\mathrm{K}_{\mathfrak{F}}(S_{m+1}) \cap J_{m+1}) \cap J_m.$$

Suppose that $J_m^0 = \mathcal{M}^0(G_m, \Lambda_m, \Lambda_m, I_{\Lambda_m})$. Let N_m and $\mathcal{R}_m$ be the normal subgroup of G_m and the equivalence relation on Λ_m associated with K_m respectively. Then $G_{m+1} = G_m/N_m$ and $\Lambda_{m+1} = \Lambda_m/\mathcal{R}_m$. In particular, $|G_{m+1}| \leq |G_m|$ and $|\Lambda_{m+1}| \leq |\Lambda_m|$.

Since G_1 and $|\Lambda_1|$ are finite, there exists $m_0 \in \mathbb{N}$ such that $|\Lambda_{m_0+1}| = |\Lambda_{m_0}|$ and $|G_{m_0+1}| = |G_{m_0}|$. In particular, $\mathcal{R}_{m_0} = id$ and $N_{m_0} = \{1\}$. Thus, according to Remark 35, $K_{m_0} = \mathrm{E}(J_{m_0})$ and then $L_{m_0} = \mathrm{E}(J_{m_0})$.

Then, for every $e \in \mathrm{E}(S_{m_0})$ and every $s \in ((S_{m_0})_e)^{\mathfrak{F}}$, it follows that $\mathrm{E}_s \subseteq \mathrm{E}(J_{m_0})$. Applying Lemma 33, we have that $(S_{m_0})_e \in \mathfrak{F}$, for every $e \in \mathrm{E}(S_{m_0})$. Thus, by hypothesis, $\mathrm{K}_{\mathfrak{F}}(S_{m_0}) \cap J_{m_0}$ is computable.

If $m_0 = 1$ we are done. Otherwise, applying Lemma 44 $m_0 - 1$ times, we have:

$$\begin{aligned}\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(S_1) \cap J_1 &= \phi_1^{-1}(\mathrm{K}_{\mathfrak{F}}(S_2) \cap J_2) \cap J_1 = \\ &= \phi_1^{-1}\big(\phi_2^{-1}(\mathrm{K}_{\mathfrak{F}}(S_3) \cap J_3) \cap J_2\big) \cap J_1 = \ldots = \\ &= \phi_1^{-1}\big(\phi_2^{-1}\big(\ldots\big(\phi_{m_0-1}^{-1}(\mathrm{K}_{\mathfrak{F}}(S_{m_0}) \cap J_{m_0}) \cap J_{m_0-1}\big) \cap \ldots\big) \cap J_2\big) \cap J_1\end{aligned}$$

and then, $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable.

Chapter 5

Applications

The aim of this chapter is to present some applications of our main result.

5.1 Abelian kernel of an inverse semigroup

In this section, we obtain a nice description of the abelian kernel of an inverse semigroup.

The problem of computing the $\mathfrak{Ab}$-kernel of a semigroup was first solved by M. Delgado in [11]. Then, it was also solved by Steinberg in [32], where the $\mathfrak{F}$-kernel was also computed for any variety of abelian groups with decidable membership. Both solutions describe an algorithm which decides whether a given element of a semigroup S belongs to the $\mathfrak{Ab}$-kernel or not.

We use a completely different approach. In fact, we are able to describe the abelian kernel of every inverse semigroup. In addition, given an inverse semigroup S we also describe how to construct an abelian group and a relational morphism τ such that $\tau^{-1}(1) = \mathrm{K}_{\mathfrak{Ab}}(S)$.

Given an inverse semigroup S and J a $\mathcal{J}$-class of S, Lemma 16 ensures us that $\mathrm{K}_{\mathfrak{Ab}}(S) \cap J = \mathrm{K}_{\mathfrak{Ab}}(S_J) \cap J$, where S_J is an inverse semigroup with J as a unique 0-minimal $\mathcal{J}$-class. Then, by Lemma 30 and Lemma 43, there exist a minimal pair (V, J) and an epimorphism $\upsilon\colon S_J \to V$, such that $\mathrm{E}_s = \mathrm{E}_{\upsilon(s)}$ and $\mathrm{K}_{\mathfrak{Ab}}(S_J) \cap J = \mathrm{K}_{\mathfrak{Ab}}(V) \cap J$. Finally, the proof of Theorem B holds that there exists a series of minimal pairs $(S_i, J_i)_{1 \leq i \leq n+1}$ and maps φ_i, $1 \leq i \leq n$:

$$S_1 \xrightarrow{\varphi_1} S_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} S_{n+1}$$

such that $(V, J) = (S_1, J_1)$, (S_{n+1}, J_{n+1}) is a minimal pair with all maximal subgroups abelian and

$$\begin{aligned}\mathrm{K}_{\mathfrak{Ab}}(V)\cap J &= \mathrm{K}_{\mathfrak{Ab}}(S_1)\cap J_1 = \phi_1^{-1}(\mathrm{K}_{\mathfrak{Ab}}(S_2)\cap J_2)\cap J_1 = \\ &= \phi_1^{-1}\big(\phi_2^{-1}(\mathrm{K}_{\mathfrak{Ab}}(S_3)\cap J_3)\cap J_2\big)\cap J_1 = \ldots = \\ &= \phi_1^{-1}\big(\phi_2^{-1}\big(\ldots\big(\phi_n^{-1}(\mathrm{K}_{\mathfrak{Ab}}(S_{n+1})\cap J_{n+1})\cap J_n\big)\cap\ldots\big)\cap J_2\big)\cap J_1.\end{aligned}$$

Moreover, the maps φ_i are given constructively by the proof of Lemma 37. As a consequence, it is enough to give a description of $\mathrm{K}_{\mathfrak{Ab}}(\bar{S})\cap\bar{J}$, for every minimal pair $(\bar{S},\bar{J})$ with all maximal subgroups abelian.

We start with the following proposition which describes the relational morphisms between minimal pairs and abelian groups.

Proposition 46. *Let (S,J) be a minimal pair with $J^0=\mathcal{M}^0(G,\Lambda,\Lambda,I_\Lambda)$. Let $\tau\colon\ S \multimap A$, with $A\in\mathfrak{Ab}$. Then, it follows:*

1. *For every $i,j\in\Lambda$, $\tau(i,1,i)=\tau(j,1,j)=: H\trianglelefteq A$. Then, for every $(i,g,j)\in J$, $\tau(i,g,j)=Hx$, for each $x\in\tau(i,g,j)$.*
2. *For every element $s\in S$, it holds $\tau(i,g,j)=\tau(i',g',j')$, for each pair $(i,g,j),(i',g',j')\in\mathrm{E}_s$.*
3. *There exists a relational morphism $\bar{\tau}\colon\ S\multimap A/H$ having $|\bar{\tau}(s)|=1$, for every $0\neq s$, and $\bar{\tau}^{-1}(H)\cap J=\tau^{-1}(0)\cap J$.*

Proof. Without loss of generality we can suppose that $\Lambda=\{1,\ldots,\lambda\}$. Since A is an abelian group, we will use additive notation for the operation in A.

1. On one hand, we know that $\tau(1,1,1)=: H\leq A$, but since A is an abelian group, $H\trianglelefteq A$. Then, according to the Proposition 9, $\tau(i,1,i)=\tau(1,1,1)=H$, for every $i\in\Lambda$. As a consequence, for every $(i,g,j)\in J$, $\tau(i,g,j)=Hx$, for each $x\in\tau(i,g,j)$.

2. Now, let $s\in S$ and let $(i,g,j),(i',g',j')\in\mathrm{E}_s$. By Lemma 24, we know that $\tau(s)\subseteq\tau(i,g,j)\cap\tau(i',g',j')$. But then, we get that $\varnothing\neq\tau(s)\subseteq Hx\cap Hy$, where $x\in\tau(i,g,j)$ and $y\in\tau(i',g',j')$. Hence, $Hx=Hy$ and then $\tau(i,g,j)=\tau(i',g',j')$.

3. Let $0\neq s\in S$. By the previous assertions, we know that there exists $x_s\in A$ such that $\tau(i,g,j)=\tau(i',g',j')=Hx_s$, for every $(i,g,j),(i',g',j')\in\mathrm{E}_s$. Then, we can define $\bar{\tau}(s):=Hx_s\in A/H$ and $\bar{\tau}(0)=A/H$. Therefore, for every $0\neq s$, $|\bar{\tau}(s)|=1$.

Now, let us check that $\bar{\tau}$ is a relational morphism. Suppose that $0\neq s,t\in S$ are such that $st\neq 0$. Then, $\mathrm{E}_{st}\neq\varnothing$ and by Proposition 19, there exist $(i,g,j)\in\mathrm{E}_s$, $(j,g',i')\in\mathrm{E}_t$ such that $(i,gg',i')=(i,g,j)(j,g',i')\in\mathrm{E}_{st}$. Therefore, applying Lemma 24 to τ, we have that $\tau(i,g,j)+\tau(j,g',i)=$

$\tau(i, gg', i')$ and then $\bar{\tau}(s) + \bar{\tau}(t) = \bar{\tau}(st)$. On the other hand, it is clear that if either $s = 0$, $t = 0$ or $st = 0$, then we have that $\bar{\tau}(s)\bar{\tau}(t) \subseteq \bar{\tau}(st)$. Hence, we can conclude that $\bar{\tau}$ is a relational morphism. Moreover, it follows that for every $(i, g, j) \in J$, $\mathrm{E}_{(i,g,j)} = \{(i, g, j)\}$ and then

$$H = \bar{\tau}(i, g, j) \Leftrightarrow \tau(i, g, j) = H \Leftrightarrow 0 \in \tau(i, g, j),$$

i.e. $\bar{\tau}^{-1}(H) \cap J = \tau^{-1}(0) \cap J$. □

As a consequence, for every minimal pair (S, J), there exists a relational morphism $\tau\colon\ S \multimap A \in \mathfrak{Ab}$ such that $\tau^{-1}(1) \cap J = \mathrm{K}_{\mathfrak{Ab}}(S) \cap J$, and $|\tau(s)| = 1$ and $\tau(s) = \tau(i, g, j)$, for every $0 \neq s \in S$ and $(i, g, j) \in \mathrm{E}_s$.

Theorem 47. *Let (S, J) be a minimal pair with all maximal subgroups abelian and $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. Let $X = \{x_i : i \in \Lambda\}$ the alphabet and the infinite abelian group $F_{ab,X} \oplus G$, where $F_{ab,X}$ is the free abelian group over X. Then*

$$\mathrm{K}_{\mathfrak{Ab}}(S) \cap J = \{(i, g, j) : (-x_i + x_j, g) \in N\},$$

where $N = \langle N_s : 0 \neq s \in S\rangle \trianglelefteq F_{ab,X} \oplus G$ with

$$N_s := \{(-x_i + x_j, g) - (-x_{i'} + x_{j'}, g') : (i, g, j), (i', g', j') \in \mathrm{E}_s\},$$

for every $0 \neq s \in S$.

Proof. First, we use additive notation for the operation in G.

Without loss of generality we can suppose that $\Lambda = \{0, 1, \ldots, n\}$ for a certain $n \in \mathbb{N}$. Then, $X = \{x_i : 0 \leq i \leq n\}$ and we set $A := F_{ab,X} \oplus G$, a finite generated abelian group.

For each $0 \neq s \in S \setminus J$, we set

$$N_s := \{(-x_i + x_j, g) - (-x_{i'} + x_{j'}, g') : (i, g, j), (i', g', j') \in \mathrm{E}_s\},$$

and consider the normal subgroup $N := \langle N_s : 0 \neq s \in S\rangle \trianglelefteq A$. Then, A/N is a finitely generated abelian group and therefore, by the fundamental theorem of finitely generated abelian groups, we can write

$$\bar{A} := A/N = (\langle y_1 + N\rangle \oplus \cdots \oplus \langle y_m + N\rangle) \oplus (\langle z_1 + N\rangle \oplus \cdots \oplus \langle z_{m'} + N\rangle),$$

where $y_k \in A$ and $\langle y_k + N\rangle \cong \mathbb{Z}$, for every $1 \leq k \leq m$; $z_l \in A$ and $\langle z_l + N\rangle \cong \mathbb{Z}_{p_l^{\varepsilon_l}}$, with p_l prime and $\varepsilon_l \in \mathbb{N}$, for every $1 \leq l \leq m'$.

As a consequence, for every $(i, g, j) \in J$, we can write

$$(-x_i + x_j, g) + N = (b_1 y_1 + \ldots + b_m y_m + b'_1 z_1 + \ldots + b'_{m'} z_{m'}) + N,$$

with $b_1, \ldots, b_m, b'_1, \ldots, b'_{m'} \in \mathbb{Z}$. Write $\varepsilon_{(i,g,j)} := |b_1| + \ldots + |b_n|$ and $n_0 := \max\{\varepsilon_{(i,g,j)} : (i,g,j) \in J\}$. Consider $\bar{N} := \langle n_0 y_1, \ldots, n_0 y_m\rangle \oplus N \trianglelefteq A$. Then

$$B := A/\bar{N} = (\langle y_1 + \bar{N}\rangle \oplus \cdots \oplus \langle y_m + \bar{N}\rangle) \oplus (\langle z_1 + \bar{N}\rangle \oplus \cdots \oplus \langle z_{m'} + \bar{N}\rangle).$$

Note that $\langle y_k + \bar{N}\rangle \cong \mathbb{Z}_{n_0}$, for every $1 \leq k \leq n$, and $\langle z_l + \bar{N}\rangle \cong \mathbb{Z}_{p_l^{\varepsilon_l}}$, for every $1 \leq l \leq m$. In particular, $B \in \mathfrak{Ab}$ is a finite abelian group.

Moreover, by the choice of n_0, it follows that for every $(i,g,j) \in J$, $(-x_i + x_j, g) \in N$ if, and only if, $(-x_i + x_j, g) \in \bar{N}$. Next we prove that $\mathrm{K}_{\mathfrak{Ab}}(S) \cap J = \{(i,g,j) : (-x_i + x_j, g) \in N\}$.

Let $\tau\colon J \to \mathcal{P}(B)$, given by $\tau(i,g,j) = (-x_i + x_j, g) + \bar{N}$, for every $(i,g,j) \in J$. Note that if $0 \neq (i,g,j)(j,g',i') = (i, g+g', i')$, for some $(i,g,j), (j,g',i') \in J$, then

$$\begin{aligned}\tau(i,g+g',i') &= (-x_i + x_{i'}, g+g') + \bar{N} = (-x_i + x_j, g) + (-x_j + x_{i'}, g') + \bar{N} \\ &= \tau(i,g,j) + \tau(j,g',i').\end{aligned}$$

On the other hand, for every $s \in S$ and every $(i,g,j), (i',g',j') \in \mathrm{E}_s$:

$$\tau(i,g,j) = (-x_i + x_j, g) + \bar{N} = (-x_{i'} + x_{j'}, g') + \bar{N} = \tau(i',g',j'),$$

because $(-x_i + x_{i'}, g) - (-x_{i'} + x_{j'}, g') \in N \leq \bar{N}$. As a consequence,

$$\varnothing \neq \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\} = \tau(i_0,g_0,j_0), \quad \text{for each } (i_0,g_0,j_0) \in \mathrm{E}_s.$$

Therefore, applying Lemma 24, we have that $\tau_0\colon S \multimap B$ defined as

$$\tau_0(s) = \bigcap\{\tau(i,g,j) : (i,g,j) \in \mathrm{E}_s\} = \tau(i_0,g_0,j_0), \text{ for each } (i_0,g_0,j_0) \in \mathrm{E}_s,$$

and $\tau_0(0) = B$, is a relational morphism satisfying

$$\begin{aligned}\tau_0^{-1}(1) \cap J = \{(i,g,j) \in J : (-x_i + x_j, g) \in \bar{N}\} = \\ = \{(i,g,j) \in J : (-x_i + x_j, g) \in N\}.\end{aligned}$$

Hence, $\mathrm{K}_{\mathfrak{Ab}}(S) \cap J \subseteq \{(i,g,j) \in J : (-x_i + x_j, g) \in N\}$.

On the other hand, we know that there exists a relational morphism $\bar{\tau}\colon S \multimap \bar{B} \in \mathfrak{Ab}$ such that $\bar{\tau}^{-1}(1) \cap J = \mathrm{K}_{\mathfrak{Ab}}(S) \cap J$. According to Proposition 46, we can suppose that $|\bar{\tau}(s)| = 1$ and $\bar{\tau}(s) = \bar{\tau}(i,g,j)$, for every $0 \neq s \in S$ and $(i,g,j) \in \bar{\tau}(s)$.

Then, we can define a map $f\colon X \to \bar{B}$ given by $f(x_0) = 0$ and $f(x_i) = \bar{\tau}(0,0,i) \in \bar{B}$, for every $1 \leq i \leq n$. Therefore, by the universal property of free abelian groups, there exists a unique homomorphism $\varphi\colon F_{ab,X} \to \bar{B}$

which extends f. On the other hand, we can also consider the map $\bar{\varphi}\colon G \to \bar{B}$ given by $\bar{\varphi}(g) = \bar{\tau}(0, g, 0) \in \bar{B}$, which is a homomorphism because we know that $\bar{\tau}(0, g, 0) + \bar{\tau}(0, g', 0) = \bar{\tau}(0, g + g', 0)$, by Lemma 24.

Let $\phi\colon A \to \bar{B}$ the homomorphism defined by $\phi(w, g) = \varphi(w) + \bar{\varphi}(g)$, for every $(w, g) \in A$. Therefore, $A/\ker\phi \cong \phi(A) \leq \bar{B} \in \mathfrak{Ab}$ and let us denote $\psi\colon \phi(A) \to A/\ker\phi$ such isomorphism between both groups.

Note that for every $0 \neq s \in S$, $\bar{\tau}(s) = \bar{\tau}(i, g, j)$, for each $(i, g, j) \in \mathrm{E}_s$ and then

$$\begin{aligned}\bar{\tau}(i, g, j) = \bar{\tau}\big((i, 0, 0)(0, g, 0)(0, 0, j)\big) = \bar{\tau}(i, 0, 0) + \bar{\tau}(0, g, 0) + \bar{\tau}(0, 0, j) = \\ = -\phi(x_i) + \phi(g) + \phi(x_j) = \phi(-x_i + x_j, g) \in \phi(A),\end{aligned}$$

i.e. $\bar{\tau}(s) \in \phi(A) \leq \bar{B}$, for every $0 \neq s \in S$. Hence, we can suppose without loss of generality that $\bar{\tau}(0) = \phi(A) = \bar{B} \cong A/\ker\phi$. In addition, note that $0 = \bar{\tau}(i, g, j)$ if, and only if, $(-x_i + x_j, g) \in \ker\phi$. Therefore, $\bar{\tau}^{-1}(0) \cap J = \{(i, g, j) : (-x_i + x_j, g) \in \ker\phi\}$.

Now, let $0 \neq s \in S$ and let $(i, g, j), (i', g', j') \in \mathrm{E}_s$. Then, by definition of $\bar{\tau}$, $\bar{\tau}(i, g, j) = \bar{\tau}(i', g', j')$ and according to the previous paragraph, $\phi(-x_i + x_j, g) = \phi(-x_{i'} + x_{j'}, g)$. Thus, $(-x_i + x_j, g) - (-x_{i'} + x_{j'}, g') \in \ker\phi$. Therefore, $N_s \subseteq \ker\phi$, for every $0 \neq s \in S$, so that $N \leq \ker\phi$.

Hence, we can conclude that

$$\begin{aligned}\{(i, g, j) : (-x_i + x_j, g) \in N\} \subseteq \{(i, g, j) : (-x_i + x_j, g) \in \ker\phi\} = \\ = \bar{\tau}^{-1}(1) \cap J = \mathrm{K}_{\mathfrak{Ab}}(S) \cap J,\end{aligned}$$

and the other inclusion is proved.

□

5.2 On the computability of the $\mathfrak{F}$-kernel for extension closed varieties: a conjecture

The aim of this section is to discuss about a conjecture on the computability of the $\mathfrak{F}$-kernel for extension closed varieties.

Recall that the case when $\mathfrak{F} = \mathfrak{S}$ [1], the variety of all soluble groups, is of particular importance as we have mentioned in Chapter 3. Rhodes and Steinberg in their seminar work [27] define a semigroup to be *soluble* if all of its maximal subgroups are soluble. Hence, as a consequence of Theorem B, we can ensure that the problem of determine the computability of the soluble

[1]Usually, *soluble kernel* is used to refer to the $\mathfrak{S}$-kernel

kernel of a semigroup can be reduced to determine the computability of the soluble kernel of a soluble inverse semigroup.

On the other hand, our main theorem allows a general approach to the problem from an structural point of view. In fact, having in mind that we can see $\mathfrak{F}$-kernels as a generalisation of $\mathfrak{F}$-residuals, one can expect that in the case that $\mathfrak{F}$ is extension closed, the $\mathfrak{F}$-kernel of an inverse semigroup with all maximal subgroups in $\mathfrak{F}$ can be reduce to the idempotents.

Obviously, it is not true if $\mathfrak{F} = \mathfrak{G}_p$, the variety of all p-groups, as we can see in the following example:

Example 48. Let A be the alphabet of two letters $A = \{a, b\}$ and F_A be the free group over A. Take $H := \langle ab^2, a^2b\rangle \leq F_A$. According to [20], we can consider $S = M(H)$, the inverse semigroup of transformations of the partial inverse automaton associated with H. Then, for every prime p, we have that applying the method described in this paper to compute the pro-p closure of a finitely generated subgroup of a free group, we can see that H is p-dense in F_A and therefore, it is possible to conclude that $\mathrm{K}_{\mathfrak{G}_p}(S) \neq \mathrm{E}(S)$.

As a consequence, one needs to impose some additional condition on the extension closed variety $\mathfrak{F}$. Note that the proof of Theorem 47 strongly depends on the fact that the variety of abelian groups contain infinitely many cyclic groups of prime order. Our investigations allow us to conclude that for a general extension closed variety $\mathfrak{F}$, this condition is essential to prove that the $\mathfrak{F}$-kernel of an inverse semigroup with maximal subgroups in $\mathfrak{F}$ coincides with the idempotents subsemigroup. Note that the variety of soluble groups satisfies this condition, while the variety of p-groups does not satisfy it, for every prime p. Hence, we can state the following:

Conjecture: Let $\mathfrak{F}$ be an extension closed variety having infinitely many cyclic groups of order prime and let S be an inverse semigroup with all maximal subgroups in $\mathfrak{F}$. Then, $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{E}(S)$.

According to our main result it is enough to check the conjecture for minimal pairs (S, J) with all maximal subgroups in $\mathfrak{F}$ and J^0 aperiodic.

Theorem 49. *Let $\mathfrak{F}$ be an extension closed variety of groups. The following statements are equivalent:*

1. $\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{E}(J)$*, for every minimal pair* (S, J)*, with all maximal subgroups in* $\mathfrak{F}$

2. $\mathrm{K}_{\mathfrak{F}}(S) \cap \bar{J} = \mathrm{E}(\bar{J})$*, for every minimal pair* $(\bar{S}, \bar{J})$*, with all maximal subgroups in* $\mathfrak{F}$ *and* $\bar{J}^0$ *is aperiodic,*

Proof. Only the 2 implies 1 is in doubt.

Suppose that $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J} = \mathrm{E}(\bar{J})$ for every minimal pair $(\bar{S}, \bar{J})$, with all maximal subgroups in $\mathfrak{F}$ and $\bar{J}^0$ is aperiodic .

Let (S, J) be a minimal pair with $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$ and all maximal subgroups in $\mathfrak{F}$. We can assume without loss of generality that $\Lambda = \{1, \ldots, \lambda\}$. Then, we can consider the aperiodic Brandt semigroup $B = \mathcal{M}^0(1, \Lambda, \Lambda, I_\Lambda)$ and the semigroup $\mathcal{U}_B$ of projections.

Recall that $\mathcal{U}_B$ has a unique 0-minimal $\mathcal{J}$-class, $\bar{J} := B \setminus \{0\}$, such that $(\mathcal{U}_B, \bar{J})$ is a minimal pair. Moreover, since B is aperiodic, for every $u \in \mathcal{U}_B$, $\gamma_u(i) = 1$, for each $i \in \alpha_u$. Therefore, we can assert that for every $u, v \in \mathcal{U}_B$, $u = v$ if, and only if, $\alpha_u = \alpha_v$, $\omega_u = \omega_v$ and $\iota_u = \iota_v$. In particular, for every $u \in \mathcal{U}_B$, u^{-1} is such that $\alpha_{u^{-1}} = \omega_u$, $\omega_{u^{-1}} = \alpha_u$ and $\iota_{u^{-1}} = \iota^{-1}$.

On the other hand, we can observe that both Brandt semigroups, B and J^0, have the same set of indices. As a consequence, we can define the following map between minimal pairs, $\varphi\colon S \to \mathcal{U}_B$, given by

$$\varphi(s) = u \text{ if, and only if, } \alpha_u = \alpha_s,\ \omega_u = \omega_s \subseteq \Lambda \text{ and } \iota_u = \iota_s,$$

for every $s \in S$. In fact, φ is a homomorphism because for every $s, t \in S$, if $u = \varphi(s)$ and $v = \varphi(t)$, then according to Proposition 19

$$\alpha_{st} = \iota_s^{-1}(\omega_s \cap \alpha_t) = \iota_u^{-1}(\omega_u \cap \alpha_v) = \alpha_{uv},\ \omega_{st} = \iota_t(\iota_s(\alpha_{st})) = \iota_v(\iota_u(\alpha_{uv})) = \omega_{uv}$$

and

$$\iota_{st}(i) = \iota_t(\iota_s(i)) = \iota_v(\iota_u(i)) = \iota_{uv}(i), \quad \text{for every } i \in \alpha_s = \alpha_u.$$

Therefore, $\varphi(st) = uv = \varphi(s)\varphi(t)$. Moreover, by definition of φ, it is clear that $\varphi(s)^{-1} = \varphi(s^{-1})$ and for every $(i, 1, j) \in \bar{J}$, $\varphi(i, g, j) = (i, 1, j)$, for all $g \in G$. Thus, $\varphi(S)$ is an inverse subsemigroup of $\mathcal{U}_B$ and $B \subseteq \varphi(S)$. Hence, applying Lemma 29, we conclude that $(\varphi(S), \bar{J})$ is a minimal pair.

Then, we have that $\bar{J}^0 = B$ is aperiodic and according to [23, Theorem 5.8], we have that $\varphi(S)$ has every maximal subgroup in $\mathfrak{F}$. Therefore, by hypothesis, $\mathrm{K}_{\mathfrak{F}}(\varphi(S)) \cap \bar{J} = \mathrm{E}(\bar{J})$ and then, there exists a relational morphism $\tau_1\colon\ \varphi(S) \multimap\!\!\to A_1 \in \mathfrak{F}$ such that $\tau_1^{-1}(1) \cap \bar{J} = \mathrm{E}(\bar{J})$.

Now, we can consider $\tau_2\colon\ S \multimap\!\!\to A_1$ the relational morphism given by the composition $\tau_2(s) := \tau_1(\varphi(s))$, for every $s \in S$. Note that, by definition of φ, $\varphi^{-1}(\bar{J}) = J$. Therefore, we have that

$$\tau_2^{-1}(1) \cap J = \{s \in J : 1 \in \tau_1(\varphi(s))\} = \varphi^{-1}(\tau_1^{-1}(1) \cap \bar{J}) = \varphi^{-1}(\mathrm{E}(\bar{J})).$$

But $\mathrm{E}(\bar{J}) = \{(i, 1, i) : i \in \Lambda\}$. Hence, $\varphi^{-1}(\mathrm{E}(\bar{J})) = \{(i, g, i) : i \in \Lambda,\ g \in G\} = \tau_2^{-1}(1) \cap J$.

According to Proposition 9, we have that $\tau_2(i,1,i) = H_i \leq A_1$, for every $i \in \Lambda$. Moreover, since $\tau_2^{-1}(1) \cap J = \{(i,g,i) : i \in \Lambda,\ g \in G\}$, we can also conclude that $\tau_2(i,g,i) = H_i$, for every $g \in G$.

Then, let us fix $H := \tau_2(1,1,1) \leq A_1$. Again, by Proposition 9, we have that for every $g \in G$ and every $i \in \Lambda$, $\tau_2(1,g,i) = \tau_2(1,g,1)\tau_2(1,1,i) = H\tau_2(1,1,i)$. Moreover, it also ensures us that $\tau_2(1,1,i) = Hx$, for every $x \in \tau(1,1,i)$. Therefore, we can take $x_1 := 1 \in \tau_2(1,1,1)$ and $x_i \in \tau_2(1,1,i)$, for every $2 \leq i \leq \lambda$, so that $\tau_2(1,1,i) = Hx_i$, for every $1 \leq i \leq \lambda$. Then, for every $g \in G$, it holds $\tau_2(1,g,i) = \tau_2(1,g,1)\tau_2(1,1,i) = H(Hx_i) = Hx_i$.

Let $i \neq i' \in \Lambda$. Then, we know that there exists $x \in \tau_2(1,1,i)$ such that $x^{-1} \in \tau_2(i,1,1)$. Moreover, applying Proposition 9, we have that $Hx_i = Hx$. Then, we claim that $Hx_{i'} \neq Hx_i$. In fact, if $Hx = Hx_i = Hx_{i'}$, then there exists $h \in H$ such that $x^{-1}hx_{i'} = 1$. But then, by Proposition 9, it follows

$$1 \in x^{-1}Hx_{i'} = (x^{-1}H)(Hx_{i'}) = \tau_2(i,1,1)\tau_2(1,1,i') = \tau_2(i,1,i'),$$

which is a contradiction because $1 \in \tau_2(i,1,i')$ if, and only if, $i = i'$.

Hence, we can consider the set of right cosets of H,

$$H\backslash A_1 := \{H = Hx_1, \ldots, Hx_\lambda, Hx_{\lambda+1}, Hx_n\},$$

and the set $\Omega = \{1, \ldots, n\}$, where $|A_1 : H| = n$. We know that A_1 acts on Ω as $i \cdot x = i'$ if, and only if, $Hx_ix = Hx_{i'}$, for every $i, i' \in \Omega$. Moreover, for every $(i,g,i') \in J$, applying Proposition 9, we have that

$$\tau_2(i,g,i') = \tau_2(i,g,1)\tau_2(1,g',i') = x_i^{-1}Hx_{i'}$$

and therefore, $i \cdot x = i'$, for all $x \in \tau_2(i,g,i')$.

Now, let $s \in S$. By Lemma 24, $\tau_2(s) \subseteq \bigcap\{\tau_2(i,g,j) : (i,g,j) \in \mathrm{E}_s\}$. Recall that $\mathrm{E}_s = \{(i, \gamma_s(i), \iota_s(i)) : i \in \alpha_s\}$. Therefore, by the above paragraph, we have that $i \cdot x = \iota_s(i)$, for every $x \in \tau_2(s)$ and every $i \in \alpha_s$.

Consider $A_2 := G \wr_\Omega A_1 \in \mathfrak{F}$, with the product defined as

$$\big((g_1, \ldots, g_n), x\big)\big((g'_1, \ldots, g'_n), x'\big) = \big((g_1g'_{1\cdot x}, \ldots, g_ng'_{n\cdot x}), xx'\big),$$

for every $\big((g_1, \ldots, g_n), x\big), \big((g'_1, \ldots, g'_n), x'\big) \in A_2$.

Then, we can construct the map $\tau_3 \colon S \to \mathcal{P}(A_2)$ given by $\tau_3(0) := A_2$ and

$$\tau_3(s) := \left\{ \big((g_1, \ldots, g_n), x\big) : \begin{array}{ll} g_i = \gamma_s(i) \in G, & \text{if } i \in \alpha_s \\ g_i \in G, & \text{otherwise} \end{array},\ x \in \tau_2(s) \right\}.$$

Next, we show that τ_3 is a relational morphism. Let $s, t \in S$ such that $s \neq 0 \neq t$ and $st \neq 0$ (otherwise, it is clear that $\tau_3(s)\tau_3(t) \subseteq \tau_3(st)$ because

$\tau_3(0) = A_2$). Take $((g_1, \dots, g_n), x) \in \tau_3(s)$ and $((g'_1, \dots, g'_n), x') \in \tau_3(t)$. Then, we have seen that

$$\big((g_1, \dots, g_n), x\big)\big((g'_1, \dots, g'_n), x'\big) = \big((g_1 g'_{1\cdot x}, \dots, g_n g'_{n\cdot x}), xx'\big).$$

Let us call $g''_i := g_i g_{i\cdot x}$, for every $1 \le i \le n$. Suppose that $i \in \alpha_{st}$; then, by Proposition 19, we know that $i \in \alpha_s$, $\iota_s(i) \in \alpha_t \cap \omega_s$ and $\gamma_{st}(i) = \gamma_s(i)\gamma_t(\iota_s(i))$. On the other hand, since $x \in \tau_2(s)$, we have seen that $i \cdot x = \iota_s(i)$. Therefore

$$g''_i = g_i g_{i\cdot x} = \gamma_s(i) g_{\iota_s(i)} = \gamma_s(i)\gamma_t(\iota_s(i)) = \gamma_{st}(i).$$

Moreover, since τ_2 is a relational morphism, $xx' \in \tau_2(s)\tau_2(t) \subseteq \tau_2(st)$. Hence, we can conclude that

$$\big((g''_1, \dots, g''_n), xx'\big) = \big((g_1, \dots, g_n), x\big)\big((g'_1, \dots, g'_n), x'\big) \in \tau_3(st).$$

As a consequence, $\tau_3 \colon S \multimap A_2 \in \mathfrak{F}$ is a relational morphism. We would finish the proof if we proved that $\tau_3^{-1}(1) \cap J = \mathrm{E}(J)$.

One inclusion is clear. For the other inclusion, let $s = (i_0, g_0, j_0) \in \tau_3^{-1} \cap J$. By definition of τ_3, we have that $\gamma_s(i_0) = 1$, i.e. $g_0 = 1$, and also that $1 \in \tau_2(s)$. Since $\tau_2^{-1}(1) \cap J = \{(i, g, i) : i \in \Lambda, g \in G\}$, we can conclude that $i_0 = j_0$ and therefore, $s = (i_0, 1, i_0) \in \mathrm{E}(J)$.

□

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