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A kinematic method to obtain conformal factors

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Radial conformal motions are considered in conformally flat space–times and their properties are used to obtain conformal factors. The geodesic case leads directly to the conformal factor of Robertson–Walker universes. General cases admitting homogeneous expansion or orthogonal hypersurfaces of constant curvature are analyzed separately. When the two conditions above are considered together a subfamily of the Stephani perfect fluid solutions, with acceleration Fermi–Walker propagated along the flow of the fluid, follows. The corresponding conformal factors are calculated and contrasted with those associated with Robertson–Walker space–times. © *2000 American Institute of Physics.* [S0022-2488(00)03007-3]

I. INTRODUCTION

Many topics in classical and quantum gravity are set up considering conformally related metrics. The specification of a spacelike geometry in the initial-value formulation of Einstein equations, the problem of quantization in curved space–times, the Weyl unified theory of gravity and electromagnetism, the analysis of Robertson–Walker metrics in conformally flat coordinates, the avoidance of singularities in relativistic and quantum cosmology are a sample of examples where conformally equivalent geometries could play an essential role, both from kinematic and dynamic points of view.

Understanding the form of the conformal factor for particular conformally flat space–times may help us to classify them and to know how they deviate from the flat metric. An incipient attempt to classify conformally flat spaces was made by Levine;[1] he started off by considering some subalgebras of the conformal algebra in order to determine the conformal factor of a space admitting the associated group of isometries.

Another possible way to obtain conformal factors is to impose the existence of certain vector fields with particular kinematic properties, which restrict the form of the conformal factor. In particular, we can consider a timelike conformal Killing vector field, which is characterized by the kinematic properties of its unit vector **u**; it is shear-free and its acceleration is the projection (on the three-spaces orthogonal to **u**) of the gradient of a function whose derivative along **u** is a third of its expansion.[2] In fact, this characterization is conformal invariant, that is, it is the same in a conformal class of metrics. However, metrics within this class differ in the values corresponding to other kinematic properties of **u** (acceleration, expansion, etc.) and we can use this fact to determine conformal factors.

For example, it is known that Robertson–Walker space–times are conformally flat, so their metric g can be locally expressed as proportional to the Minkowski metric η. In fact, there always exist coordinate systems (t,r,θ,φ) so that $\eta=\mathrm{diag}(-1,1,r^2,r^2\sin^2\theta)$ and, depending on the curvature index k of the universe, one has $g=f(t)\eta$ if $k=0$, $g=f(u)\eta$ if $k=-1$ and $g=t^{-2}f(t/1+u)\eta$ if $k=1$ with $u=r^2-t^2$ and f as an arbitrary (positive) function of its respective argument. This result was obtained by Infeld and Schild[3] in their work on kinematic cosmology and was later considered by Tauber[4] to analyze expanding universes in these (conformally flat) coordinates.

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Could these metric expressions be deduced from the existence of a timelike congruence with particular kinematic properties?

The cosmological observer is associated with a geodesic conformal Killing field admitting orthogonal hypersurfaces. The above (t,r,θ,φ) spherical coordinates could be used to express this conformal motion, whose integral curves remain on the $\{t,r\}$-surfaces, and their components are independent of the angular coordinates (θ,φ). Then, the cosmological observer defines a geodesic conformal motion which is *radial.* Therefore, it is natural to wonder whether the existence of a radial conformal motion with certain kinematic properties can characterize the Robertson–Walker metrics as well as other generalized nonhomogeneous conformally flat cosmological models. In fact, Robertson–Walker universes are those conformally flat space–times which admit a geodesic timelike radial conformal motion, as we show in this work. Other kinematic properties over these radial conformal motions (nongeodesic with homogeneous expansion or admitting homogeneous orthogonal three-spaces) lead to a characterization of generalized conformally flat cosmologies.

In Sec. II, we obtain the general form of a radial conformal Killing vector field (RCKF) and study the first order kinematic properties for the timelike ones. The geodesic case is considered in Sec. III obtaining the aforementioned characterization of Robertson–Walker metrics. In Sec. IV, we generalize this kinematic procedure to a RCKF having homogeneous expansion; the case of vanishing expansion (that corresponds to Killing vectors) is also considered, which leads to the conformal factor of the corresponding static space–times. Finally, RCKF with orthogonal surfaces of constant curvature are considered in Sec. V and the special case of adding homogeneous expansion is analyzed confronting the emerging conformal factor to the Robertson–Walker one. Some of these results have been communicated, without proof, in the E.R.E., annual Spanish relativity meeting.[5]

II. RADIAL CONFORMAL MOTIONS IN CONFORMALLY FLAT SPACE–TIMES

A conformally flat space–time admits coordinates for which the metric has the local form $g=F^2\eta$, with F as a function of the coordinates (F^2 is called *conformal factor*) and η the flat metric. In spherical coordinates it results in

$$g=F^2(t,r,\theta,\varphi)[-dt\otimes dt+dr\otimes dr+r^2(d\theta\otimes d\theta+\sin^2\theta\, d\varphi\otimes d\varphi)].$$

Let us consider a field of radial directions in a conformally flat space–time,

$$\xi=\alpha(t,r,\theta,\varphi)\frac{\partial}{\partial t}+\beta(t,r,\theta,\varphi)\frac{\partial}{\partial r}.$$

The equation $\mathcal{L}_\xi g\propto g$ expresses that ξ is a conformal Killing field (or conformal motion) of g, where $\mathcal{L}_\xi$ represents the Lie derivative with respect to ξ. This condition leads to the fact that the functions α and β are independent of the angular coordinates θ and φ with the result

$$\alpha(t,r)=a(t^2+r^2)+bt+c,\quad \beta(t,r)=r(2at+b), \tag{1}$$

where a, b, and c are arbitrary constants. Then we have the following result:

Proposition 1: In a conformally flat space–time the general form of a radial conformal Killing field is

$$\xi=(a(t^2+r^2)+bt+c)\frac{\partial}{\partial t}+r(2at+b)\frac{\partial}{\partial r}, \tag{2}$$

with a, b, and c as arbitrary constants.

In a recent work[6] we have studied RCKF in Minkowski space–time. Some of their properties such as causal character (in the different domains), integral curves, vanishing shear and vorticity, are maintained invariant by conformal transformations. For instance, the function ω,

$$\omega(t,r)=\frac{a(t^2-r^2)+bt+c}{r}, \tag{3}$$

is associated with the integral curves of a general RCKF, that is, $\omega=$constant represents a hyperbolic or straight line depending on the value of the constants a, b, and c of the field. But, there also exist other kinematic properties that change from Minkowski to other conformally flat space–times.

Next, we consider timelike RCKF, their kinematic properties are those corresponding to the unit vector associated with them, $\mathbf{u}=\xi/|\xi|=(1/F\sqrt{P})\,\xi$, where

$$P=-\eta(\xi,\xi)=[a(t^2-r^2)+bt+c]^2-r^2\Delta>0 \tag{4}$$

with $\Delta\equiv b^2-4ac$. So, $\mathbf{u}$ is shear-free and vorticity-free and its expansion Θ and acceleration $\mathbf{a}$ will depend on the conformal factor of the space–time in the following way:

$$\Theta=\frac{3}{F^2\sqrt{P}}\left[\alpha\dot{F}+\beta F'+\frac{\beta}{r}F\right], \tag{5}$$

$$\mathbf{a}=\frac{A}{FP}[-\beta dt+\alpha dr]+\frac{1}{F}F_{,\theta}d\theta+\frac{1}{F}F_{,\varphi}d\varphi, \tag{6}$$

with

$$A=\alpha F'+\beta\dot{F}+2arF, \tag{7}$$

and where the dot and the prime represent the partial derivatives with respect to t and r, respectively, $F_{,\theta}=\partial F/\partial\theta$ and $F_{,\varphi}=\partial F/\partial\varphi$.

Note that the acceleration $\mathbf{a}$ of a RCKF is the orthogonal projection (to $\mathbf{u}$) of the gradient of a function because $d(\mathbf{a}-(\Theta/3)\,\mathbf{u})=0$ owing to ξ is a conformal Killing vector field. Hence, $\mathbf{u}$ is *conformally geodesic*; a conformally flat space–time exists where ξ is geodesic.[7] The space–time defined thus corresponds to the Robertson–Walker space–time, as we will prove in Sec. III.

The 1-form ξ_* associated by the metric with the field ξ,

$$\xi_*=F^2[-\alpha dt+\beta dr],$$

is integrable, that is, $\xi_*\propto ds$, with $s(t,r)$ as a potential function given by

$$s(t,r)=\begin{cases} b(t^2-r^2)+2ct & \text{if } a=0\\ \dfrac{a(t^2-r^2)-c}{2at+b} & \text{if } a\neq 0\end{cases}. \tag{8}$$

Moreover, taking in account (3) and (8), one has $g(ds,d\omega)=0$; as well as the expression (4) of P can be written as

$$P=r^2(\omega^2-\Delta)=\begin{cases} bs+c^2 & \text{if } a=0\\ (2at+b)^2(as^2+bs+c)/a & \text{if } a\neq 0\end{cases}. \tag{9}$$

From these kinematic properties of $\mathbf{u}$ and imposing different conditions over them, we can obtain different conformal factors solving the corresponding equations, as we will see in the following sections.

III. GEODESIC CASE: CONFORMAL FACTOR OF THE ROBERTSON–WALKER UNIVERSES

First, we are going to find the conformal factor of the conformally flat space–times which admit a geodesic timelike RCKF. From (6) and (7) the condition $\mathbf{a}=0$ means that F only depends on the coordinates t and r and verifies the equation,

$$\alpha F' + \beta \dot{F} + 2arF = 0, \tag{10}$$

whose general solution is

$$F(t,r) = \begin{cases} f(s) & \text{if } a=0 \\ \dfrac{f(s)}{2at+b} & \text{if } a\neq 0 \end{cases}, \tag{11}$$

where f is an arbitrary function of its argument $s(t,r)$ given by (8).

In order to identify these conformally flat space–times, we can compare expressions (10) and (11) with Infeld–Schild's results on kinematic cosmology in Ref. 3. These authors found the conformal factor of the spherically symmetric conformally flat space–times which satisfy the postulate of spatial homogeneity. So, they obtained the local expression of the Robertson–Walker metrics in conformally flat coordinates. Our Eq. (10) is equivalent to the differential Eq. (A3) analyzed by Infeld and Schild in the Appendix of their paper.[3] The expression for the conformal factor given by (11) unifies the four cases analyzed in the mentioned Appendix, and it can be simplified taking in account the expression (9). Therefore, we can conclude the following proposition:

Proposition 2: Robertson–Walker space–times are the conformally flat space–times that admit a geodesic timelike radial conformal Killing vector field. Its metric is written as

$$g = \frac{h^2(s)}{P}\,\eta \tag{12}$$

with P given by (4), h an arbitrary function of the potential s given by (8) and η the flat metric.

Note that the norm of ξ, $|\xi| = \sqrt{-g(\xi,\xi)} = h(s)$, is constant on each homogeneous 3-space ($s=$constant) orthogonal to ξ, whose sectional curvature, $\Psi(s)$, is given by

$$\Psi(s) = -\frac{\Delta}{h^2(s)} \quad \text{with } \Delta = b^2 - 4ac. \tag{13}$$

The usual form of the Robertson–Walker metric in comoving coordinates,

$$g = -d\tau\otimes d\tau + \frac{R^2(\tau)}{(1+(k/4)\rho^2)^2}[d\rho\otimes d\rho + \rho^2(d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi)] \tag{14}$$

with $k=+1,0,-1$, is recovered when the following transformation from the conformally flat coordinates $\{t,r\}$ to the coordinates $\{\tau,\rho\}$ is carried out. The proper time τ will be obtained from:

$$\tau(t,r) = \frac{1}{2}\int \frac{h(s)}{S(s)}\,ds \tag{15}$$

considering the function $S(s)$ given by

$$S(s) = \begin{cases} bs + c^2 & \text{if } a=0 \\ as^2 + bs + c & \text{if } a\neq 0 \end{cases}. \tag{16}$$

From (13), the curvature index k is related to the sign of Δ, $k=-\mathrm{sgn}(\Delta)$, which leads to different expressions of the ρ-coordinate,

$$\rho(t,r)=\begin{cases}\dfrac{1}{\omega} & \text{if } \Delta=0\\[2ex] \dfrac{2}{\sqrt{|\Delta|}}(\omega+\sqrt{\omega^2-\Delta}) & \text{if } \Delta\neq 0\end{cases}, \tag{17}$$

where ω is given by (3). This coordinate is independent of the function $h(s)$ and hence, its form is the same for any cosmological model that we may consider. Instead, the cosmological time τ is determined when a particular, and then model-depending, specification of $h(s)$ is taken. This can be done, for instance, after analytical integration of the Einstein equations in conformally flat coordinates when a (constant) relation of proportionality between pressure and energy density is imposed.[4]

Moreover, the scale factor $R(\tau)$ of Robertson–Walker metrics is related with its only degree of freedom, that is, the conformal factor. This relationship is obtained using the coordinates transformation (15) and (17) and it results in

$$R(\tau)=\begin{cases}h(s) & \text{if } \Delta=0\\[1ex] \dfrac{h(s)}{\sqrt{|\Delta|}} & \text{if } \Delta\neq 0\end{cases}, \tag{18}$$

where s is considered as a function of τ, $s(\tau)$, from the inverse relation of (15). Clearly, the expansion of $\mathbf{u}$ is a function of s,

$$\Theta(s)=\frac{6S}{h^2}\frac{dh}{ds} \tag{19}$$

as it results from (12) and (5) or, alternatively, from (18) and (15).

It is interesting to note that it is possible to characterize Robertson–Walker geometries by saying that they are *the conformally flat space–times that admit an integrable and geodesic timelike conformal Killing field.* Actually, a geodesic and vorticity-free timelike conformal motion has homogeneous expansion and then, the contracted Ricci identities in a conformally flat space–time imply that this motion is associated with the 4-velocity of a perfect fluid (see, for instance, Ref. 8). Hence, the metric is necessarily the Robertson–Walker one. Therefore, in *Proposition 2*, the *radial* property of the field can be substituted by a more general one; that the field admits orthogonal surfaces. Consequently, we infer that the geodesic and vorticity-free timelike conformal motion in the Robertson–Walker space–time is necessarily *radial*, that is, it can be written in the form given by expression (2) if an appropriate spherical coordinate system is chosen.

As we have seen in Sec. II, the function s is a potential of ξ, and in particular for Robertson–Walker space–times, the 1-form ξ_* is closed, because the function

$$\bar{s}(s)=-\frac{1}{2}\int\frac{h^2(s)}{S(s)}ds$$

with $S(s)$ given by (16), allows one to write $\xi_*=d\bar{s}=-h(s)d\tau$.

A particular case results when $h(s)$ is considered as a constant function (that we can take equal to 1 without loss in generality), which corresponds to a geodesic Killing field due to the expansion vanishes according to Eq. (19). Then we arrive to the following result:

Proposition 3: The conformally flat space–times which admit a geodesic timelike radial Killing field ξ are the Minkowski space–time if $\Delta=0$, the Einstein static universe if $\Delta<0$, and the Einstein open universe if $\Delta>0$. The corresponding metric is given by $g=\eta/P$, with P given by (4) and being $\Delta=b^2-4ac$.

Note that the metric $g=(a(t^2-r^2)+bt+c)^{-2}\eta$, with $b^2=4ac$ is locally a flat metric; this corresponds to the case $\Delta=0$. According to *Propositions 2 and 3*, Robertson–Walker space–times are conformal to the static Einstein universes with conformal factor $h^2(s)$.

It is worth noting the differences between Infeld–Schild's analysis in Ref. 3 and the one presented in this section. The procedure of Infeld and Schild is based on the isometries admitted by a conformally flat space–time when spherical symmetry and spatial homogeneity are required. Instead, our approach is only based on kinematic requirements on a RCKF, without imposing any additional symmetry. In fact, in this case the spherical symmetry and the spatial homogeneity follow as a consequence of the geodesic character of the conformal field. Moreover, *Proposition 2* is a characterization of the Robertson–Walker space–times and would suggest that our procedure affords certain advantages when used to obtain conformal factors and to classify and characterize *kinematically* conformally flat space–times. This is shown in the following sections.

IV. RCKF WITH HOMOGENEOUS EXPANSION

The geodesic conformal motion of Robertson–Walker universes has homogeneous expansion, that is, the expansion only varies along the direction of the field ($\mathbf{u}\wedge d\Theta=0$). We can generalize the results of the preceding section, considering conformally flat space–times which admit a RCKF with homogeneous expansion, non-necessarily geodesic. This condition over the expansion will be interpreted in different (but equivalent) ways involving the Fermi–Walker derivative of the acceleration along the field or the Ricci tensor of the metric, Ric(g). In order to achieve this purpose, we consider a previous general property. Without any possible confusion, we will denote in the same way a field and its metrically associated 1-form.

Lemma 1: For a unitary timelike field $\mathbf{u}$, *considering its acceleration* $\mathbf{a}$, *expansion* Θ, *shear* σ, *and vorticity* Ω, *the following identity is verified:*

$$d\left(\mathbf{a}-\frac{\Theta}{3}\mathbf{u}\right)=\Big(i(\mathbf{a})(\sigma-\Omega)+\mathcal{F}_{\mathbf{u}}\mathbf{a}-\tfrac{1}{3}d\Theta\Big)\wedge\mathbf{u}-\tfrac{2}{3}\Theta\Omega+(d\mathbf{a})_{\perp}$$

with d the exterior derivative, $i(\,)$ the interior product, $\mathcal{F}_{\mathbf{u}}$ the Fermi–Walker derivative along $\mathbf{u}$, *and $\perp$ denoting the projection on the 3-space orthogonal to* $\mathbf{u}$.

Proof: First, we consider the kinematic decomposition (relative to $\mathbf{u}$) of the covariant derivative of $\mathbf{u}$ and $\mathbf{a}$, $\nabla\mathbf{u}$, and $\nabla\mathbf{a}$, respectively,

$$\nabla\mathbf{u}=-\mathbf{u}\otimes\mathbf{a}+\sigma+\Omega+\tfrac{1}{3}\Theta(g+\mathbf{u}\otimes\mathbf{u}),$$

$$\nabla\mathbf{a}=-g(\mathbf{a},\mathbf{a})\mathbf{u}\otimes\mathbf{u}+\mathbf{q}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{p}+(\nabla\mathbf{a})_{\perp},$$

$\mathbf{p}$ and $\mathbf{q}$ being orthogonal vectors to $\mathbf{u}$. The antisymmetric part of these tensors is, respectively,

$$d\mathbf{u}=\mathbf{a}\wedge\mathbf{u}+2\Omega,\quad d\mathbf{a}=(\mathbf{q}-\mathbf{p})\wedge\mathbf{u}+(d\mathbf{a})_{\perp}.$$

Then we have

$$d\left(\mathbf{a}-\frac{\Theta}{3}\mathbf{u}\right)=\left(\mathbf{q}-\mathbf{p}-\frac{1}{3}d\Theta-\frac{\Theta}{3}\mathbf{a}\right)\wedge\mathbf{u}-\frac{2}{3}\Theta\Omega+(d\mathbf{a})_{\perp},$$

and taking into account that $g(\mathbf{u},\mathbf{a})=0$ and denoting by ${}^t\nabla\mathbf{a}$ the transposed 2-tensor of $\nabla\mathbf{a}$, we can write

$$\mathbf{p}\equiv-(i(\mathbf{u})\nabla\mathbf{a})_{\perp}=-\mathcal{F}_{\mathbf{u}}\mathbf{a},$$

$$\mathbf{q} \equiv -(i(\mathbf{u})^t \nabla \mathbf{a})_\perp = i(\mathbf{a})(\sigma - \Omega) + \frac{\Theta}{3}\mathbf{a}$$

obtaining the required expression.

Now, we come back to the case of a timelike conformal Killing field, which is characterized by $\sigma=0$ and $d(\mathbf{a}-(\Theta/3)\mathbf{u})=0$ (see Ref. 2). Moreover, if this field is integrable $(\Omega=0)$ it follows that $\mathbf{u}\wedge d\mathbf{a}=0$. Hence, relatively to $\mathbf{u}$, the 2-form $d\mathbf{a}$ has a vanishing magnetic part and $(d\mathbf{a})_\perp=0$. So *Lemma 1* reduces to

$$\left(\mathcal{F}_\mathbf{u}\mathbf{a} - \tfrac{1}{3}d\Theta\right)\wedge\mathbf{u}=0,$$

and due to $g(\mathbf{u},\mathbf{a})=0$, the condition of Fermi–Walker propagated acceleration along $\mathbf{u}$ is equivalent to homogeneous expansion.

On the other hand, if we consider the contracted Ricci identities for a field with vanishing shear and vorticity (see, for instance, Ref. 8),

$$i(\mathbf{u})\mathrm{Ric}(g) = -\frac{2}{3}d\Theta + \left[\frac{\Theta^2}{3} + i(\mathbf{u})d\Theta + \delta\mathbf{a}\right]\mathbf{u},$$

where δ denotes the exterior codifferential $(\delta\mathbf{a}=-\nabla_\nu\mathbf{a}^\nu)$, it results that homogeneous expansion is equivalent to that $\mathbf{u}$ is an eigenvector of the Ricci tensor. We summarize these comments in the following proposition:

Proposition 4: For an integrable timelike conformal Killing field the following properties are equivalent:

(1) *The field has homogeneous expansion.*
(2) *Its acceleration is Fermi–Walker propagated along the field.*
(3) *The field is an eigenvector of the Ricci tensor.*

Next, we are going to find the conformal factor of the conformally flat space–times that admit a RCKF with homogeneous expansion, following a kinematic procedure as in the previous section. From Eq. (5) the condition $\mathbf{u}\wedge d\Theta=0$ leads us to the following system of differential equations for the conformal factor:

$$\left[\beta G' + \alpha \dot{G} - \frac{\beta}{r}G\right]_{,j}=0 \quad j=\theta,\varphi, \tag{20}$$

$$\alpha\beta[\ddot{G}+G''] + (\alpha^2+\beta^2)\dot{G}'=0, \tag{21}$$

where $G=1/F$, and α and β are given by (1). The solution of this system can be written as

$$G=\sqrt{P}[\mu(s)+\nu(\omega,\theta,\varphi)],$$

where μ and ν are arbitrary functions, and ω, s, and P are given, respectively, by (3), (8), and (9). So, denoting $h(s)=1/\mu(s)$, we can conclude with the following result:

Proposition 5: The metric, $\tilde{g}$, of the conformally flat space–times that admit a timelike RCKF with homogeneous expansion is given by

$$\tilde{g} = \frac{g}{(1+h\nu)^2}, \tag{22}$$

where $g=h^2(s)/P\,\eta$ is the Robertson–Walker metric given in Proposition 2 and ν is an arbitrary function independent of the potential s.

The above class of metrics admits alternative interpretations according to *Proposition 4*; as a consequence, the energy tensor of these space–times can be expressed in diagonal form.[9] Note that the conformal factor of Robertson–Walker metrics is recovered when we consider the function ν of expression (22) as a constant.

As a result of replacing (22) in (5), the expansion of **u** only depends on s and it is exactly the same as the expansion of the geodesic RCKF of Robertson–Walker space–times, whose expression is (19). As a consequence, when h is a constant function, the expansion vanishes and, taking in account expression (9) and *Proposition 5*, we have the following result:

Proposition 6: The metric of the conformally flat space–times that admit a timelike radial Killing vector field can be written as

$$g=\frac{1}{r^2}f^2(\omega,\theta,\varphi)\,\eta,$$

f being an arbitrary function and $\omega(t,r)$ given by (3).

In particular, if we consider that the conformal factor is independent from the angular coordinates θ and φ, we have static spherically symmetric conformally flat space–times whose conformal factor is f^2/r^2 with f as an arbitrary function that only depends on ω. An interesting case results when the function f is a constant, that we take equal to 1, then the metric is written as $g=\eta/r^2$, which corresponds to the Bertotti–Robinson solution of the Einstein–Maxwell equations for a regular electromagnetic field.[8]

V. RCKF WITH ORTHOGONAL 3-SPACES OF CONSTANT CURVATURE

Another interesting property over RCKF that allows us to obtain other generalized conformal factors is to consider the geometry of its orthogonal surfaces. In Ref. 6 we have studied RCKF in Minkowski space–time and obtained the induced metric γ on their orthogonal surfaces,

$$\gamma=r^2(s,\omega)\left[\frac{1}{\omega^2-\Delta}d\omega\otimes d\omega+d\theta\otimes d\theta+\sin^2\theta\,d\varphi\otimes d\varphi\right] \tag{23}$$

with the coordinates s and ω given by (8) and (3), respectively. This metric γ has constant sectional curvature, that is, its double 2-form of curvature is $\mathcal{R}(\gamma)=\Phi(s)/2\,\gamma\wedge\gamma$, where $\wedge$ denotes the exterior product of double 1-forms[10] and $\Phi(s)$ has the expression,

$$\Phi(s)=\begin{cases}\dfrac{-b^2}{bs+c^2} & \text{if } a=0\\[2ex] \dfrac{-a}{as^2+bs+c} & \text{if } a\neq 0\end{cases}. \tag{24}$$

In a conformally flat space–time, the orthogonal sections to a timelike RCKF are the surfaces $s=\text{constant}$. The induced metric on these 3-spaces will be $\tilde{\gamma}=F^2\gamma$ whose curvature form can be written as

$$\mathcal{R}(\tilde{\gamma})=F^2[\mathcal{R}(\gamma)+\Sigma\wedge\gamma],$$

where

$$\Sigma=F\nabla d\frac{1}{F}-\frac{1}{2}\tilde{\gamma}(d\ln|F|,d\ln|F|)\tilde{\gamma}, \tag{25}$$

with ∇ expressing the covariant derivative with respect to the metric γ and where we have considered the expressions relating the Riemann tensors in a class of conformal metrics.[11] So, the curvature of the metric $\tilde{\gamma}$ can be written as

$$\mathcal{R}(\tilde{\gamma})=\left(\frac{\Phi(s)}{2}\gamma+\Sigma\right)\wedge\tilde{\gamma}.$$

The condition of constant sectional curvature for these 3-spaces will be that Σ is proportional to $\tilde{\gamma}$ which, taking into account the expression (25), is equivalent to the following system of equations:

$$\left[\beta\dot{G}+\alpha G'-\frac{\alpha}{r}G\right]_{,j}=0 \quad j=\theta,\varphi,$$

$$G_{,\theta\varphi}=\cot\theta\, G_{,\varphi},$$

$$G_{,\varphi\varphi}=\sin^2\theta\, G_{,\theta\theta}-\sin\theta\cos\theta\, G_{,\theta},$$

$$P(G_{,\theta\theta}+rG')=r^2(\beta^2\ddot{G}+\alpha^2G''+2\alpha\beta\dot{G}'),$$

where $G=1/F$, α and β are given by (1), the dot and the prime represent the partial derivative with respect to t and r, respectively, and subindexes denoting partial derivatives with respect to them. This system can be solved having that the solution is

$$G(t,r,\theta,\varphi)=\boldsymbol{\sigma}\cdot\mathbf{r}+H(t,r),$$

where we have denoted

$$\boldsymbol{\sigma}\cdot\mathbf{r}=\sigma_1(s)r\cos\varphi\sin\theta+\sigma_2(s)r\sin\varphi\sin\theta+\sigma_3(s)r\cos\theta,$$

with $\sigma_i(s)$ $(i=1,2,3)$ as arbitrary functions of $s(t,r)$ given by (8). The function $H(t,r)$ is

$$H(t,r)=\begin{cases} m(s)r^2+n(s) & \text{if } a=b=0\\ m(s)t+n(s) & \text{in other cases}\end{cases} \tag{26}$$

being $m(s)$ and $n(s)$ arbitrary functions of their argument s. So, we have the following result:

Proposition 7: The metric of the conformally flat space–times that admit a timelike RCKF with orthogonal surfaces of constant curvature can be written as

$$g=\frac{1}{(\boldsymbol{\sigma}\cdot\mathbf{r}+H(t,r))^2}\eta, \tag{27}$$

where the function $H(t,r)$ is given by (26).

The Riemann curvature of the homogeneous synchronization associated with this conformal motion is $\mathcal{R}(\tilde{\gamma})=\tilde{\Phi}(s)/2\tilde{\gamma}\wedge\tilde{\gamma}$, where the sectional curvature $\tilde{\Phi}$ is written as

$$\tilde{\Phi}=\frac{-P}{r^4}H_{,\omega}^{2}-\frac{2\alpha}{r^3}HH_{,\omega}-\frac{\beta^2}{Pr^2}H^2-\sigma^2,$$

with $\sigma^2=\sigma_1^2+\sigma_2^2+\sigma_3^2$. Note that actually the sectional curvature is a constant on each surface $s=$constant because its partial derivatives with respect to ω, θ, and φ are zero. In fact, we can write $\tilde{\Phi}$ as a function of s in each case in the following way:

$$\Phi(s)=\begin{cases} 4mn-\sigma^2 & \text{if } a=b=0 \\ \dfrac{b}{bs+c^2}[sm^2-bn^2+2cmn]-\sigma^2 & \text{if } a=0,\ b\neq 0 \\ \dfrac{a}{as^2+bs+c}\left[\dfrac{bs+c}{a}m^2-n^2-2smn\right]-\sigma^2 & \text{if } a\neq 0. \end{cases} \tag{28}$$

The study of the energy tensor T for these space–times leads to the nullity of the anisotropic pressure tensor (trace-free part of the orthogonal projection of T relatively to $\mathbf{u}$). Effectively, since $\mathbf{u}$ is shear-free and vorticity-free, it defines an *umbilical synchronization* (foliation of 3-spaces whose extrinsic curvature is proportional to their induced metric), which is of constant curvature if, and only if, the Ricci tensor of $\tilde{\gamma}$ has vanishing trace-free part (that is, $\mathrm{Ric}(\tilde{\gamma})\propto\tilde{\gamma}$). Then, the Gauss–Codazzi relations for these conformally flat space-times imply that the orthogonal projection (to $\mathbf{u}$) of $\mathrm{Ric}(g)$ is also proportional to $\tilde{\gamma}$, that is, the anisotropic pressure tensor vanishes. Furthermore, from *Lemma 1* of Sec. IV, the Fermi derivative of the acceleration of the RCKF along its direction is proportional to the orthogonal projection of the gradient of the expansion,

$$\mathcal{F}_{\mathbf{u}}\mathbf{a}=-\tfrac{1}{3}(d\Theta)_{\perp}. \tag{29}$$

In order to analyze how the metrics given in *Proposition 7* generalize the Robertson–Walker cosmologies, we write expression (26) as

$$H(t,r)=\begin{cases} \tilde{f}(s)+\lambda(s)r^2 & \text{if } a=b=0 \\ \tilde{f}(s)+\lambda(s)t & \text{if } a=0,\ b\neq 0 \\ (2at+b)\tilde{f}(s)+\lambda(s) & \text{if } a\neq 0 \end{cases}$$

with $\tilde{f}(s)$ and $\lambda(s)$ as arbitrary functions. So, the metric (27) can be expressed in the following way;

$$g=\left[\frac{F_0}{1+(\boldsymbol{\sigma}\cdot\mathbf{r}+\lambda(s)\,\delta(t,r))F_0}\right]^2\eta, \tag{30}$$

where F_0^2 is the conformal factor of the Robertson–Walker metrics given by (11) and the function $\delta(t,r)$ is

$$\delta(t,r)=\begin{cases} r^2 & \text{if } a=b=0 \\ t & \text{if } a=0,\ b\neq 0. \\ 1 & \text{if } a\neq 0 \end{cases}$$

The special case of homogeneous expansion corresponds to a family of perfect fluid solutions whose acceleration is Fermi–Walker propagated along the fluid flow, according to (29); and, from (20) and (21), the metric has the expression (30) with $\boldsymbol{\sigma}$ and λ as (four) arbitrary constants. When the expansion is nonzero, these metrics belong to the conformally flat class of perfect fluids called Stephani universes.[8,12] In particular, when $\boldsymbol{\sigma}=0$, the metric (30) is expressed as a one-parameter deformation of the Robertson–Walker metric $g_0=F_0^2\,\eta$, that is

$$g_\lambda=\frac{g_0}{[1+\lambda\,\delta(t,r)F_0]^2}. \tag{31}$$

The acceleration of the fluid results from (6),

TABLE I. Coefficients A, B, C, and D that appear in the expressions (32) and (33) of the energy density and the pressure, respectively; we denote $\mu(s)=1/h(s)$ and the symbols (*) and (**) represent the following functions: (*) $(2/\sqrt{S})\{b(5bt+3c)\mu+2S[3(2bt+c)\mu'+2t(S\mu')']\}$; (**) $(4a/\sqrt{aS})\{[3as+b+[a(bt+2c)/(2at+b)]]\mu+S[3\mu'+[2/(2at+b)](S\mu')']\}$.

	$a=b=0$	$a=0$ and $b\neq 0$	$a\neq 0$
$\delta(t,r)$	r^2	t	1
A	0	3	0
$B(s)$	$12c\mu(s)$	$12c(\sqrt{S}\mu)'$	$-12(\sqrt{aS}\mu)'$
$C(r)$	$4r^2$	-3	0
$D(t,r)$	$8c(c^2r^2\mu''-\mu)$	(*)	(**)

$$\mathbf{a}_\lambda=\frac{\lambda F_0}{P(1+\lambda\,\delta F_0)}(\alpha\delta'+\beta\,\dot{\delta}-2ar\delta)(-\beta dt+\alpha dr)$$

which is different from zero, but for $\lambda=0$, as it can be seen for each value of the function $\delta(t,r)$. And, from Eq. (5), the homogeneous expansion is

$$\Theta_\lambda=\Theta_0-\frac{3\lambda}{\sqrt{P}}\left(\alpha\,\dot{\delta}-\frac{\beta}{r}\delta\right)$$

being Θ_0 the expansion (19) of the field on the Robertson–Walker space–time with metric $g_0=[h^2(s)/P]\,\eta$. The energy density ρ_λ and the pressure p_λ of the fluid are

$$\rho_\lambda=\rho_0+A\lambda^2+B(s)\,\lambda, \tag{32}$$

$$p_\lambda=p_0+C(r)\lambda^2+D(t,r)\lambda, \tag{33}$$

where ρ_0 and p_0 are, respectively, the energy density and the pressure of the Robertson–Walker universe,

$$\rho_0=\frac{3}{h^2}\left[-\Delta+\left(\frac{2Sh'}{h}\right)^2\right],$$

$$p_0=-\frac{\rho_0}{3}-\frac{8S}{h^2}\left(\frac{Sh'}{h}\right)',$$

with S as in (16); and with the coefficients A, $B(s)$, $C(r)$, and $D(t,r)$ listed in Table I. As it is known, the energy density (32) is homogeneous; and the form of the functions $C(r)$ and $D(t,r)$ give the inhomogeneities of the pressure (33).

VI. DISCUSSION AND COMMENTS

The main subject of this paper has been to consider conformal flatness over space–time metrics and connect it with several types of radial conformal motions. The form of the conformal factor is obtained considering different kinematical properties for these motions. In this sense, the conformal factor of Robertson–Walker geometries has been interpreted from a kinematical point of view; it follows from the existence of a geodesic radial conformal motion. As a generalization, *Proposition 5* provides the general metric form when the conformal field has homogeneous expansion, and *Proposition 7* gives the metric when the field has orthogonal 3-spaces of constant curvature. Essentially, both of these propositions would develop the original Infeld–Schild program[3] of using conformally flat coordinates in relativistic cosmology.

The results we have obtained could be helpful when dealing with physical and geometrical interpretations of generalized conformally flat cosmologies. In this way, some general comments about the energy tensor of these space–times have been done. For instance, *Proposition 4* allows us alternative interpretations of the homogeneous expansion case.

However, an accurate study of the energetic contents needs to be done in order to arrive at some realistic cosmological models. We have not considered any usual energy condition that restricts the algebraic type of the matter tensor. Such a study would involve an additional algebraic treatment on the results presented in this paper. For example, for a perfect fluid family of metrics like (31), the inequalities $-\rho_\lambda \leqslant p_\lambda \leqslant \rho_\lambda$ will impose new restrictions over the coefficients given in Table I, as it can be easily analyzed.

An interpretation of the energy content of a conformally flat metric as two perfect fluid components with noncollinear velocities could be also interesting in cosmology. This would require that the corresponding algebraic conditions[13] have to be satisfied. In fact, the energy tensor of a conformally flat space–time can always be decomposed, relatively to a radial conformal Killing observer, in a mixture of two components, being one of them a perfect fluid whose 4-velocity is collinear with this conformal field. Most of these comments are being developed at the present.

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