Method to obtain shear-free two-fluid solutions of Einstein's equations

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We use the Einstein equations, stated as an initial-value problem (3+1 formalism), to present a method for obtaining a class of solutions which may be interpreted as the gravitational field produced by a mixture of two perfect fluids. The four-velocity of one of the components is assumed to be a shear-free, irrotational, and geodesic vector field. The solutions are given up to a set of a hyperbolic quasilinear system.

I. INTRODUCTION

In many subjects of astrophysics and cosmology the energy content is described by a single perfect fluid, when, in reality, one should distinguish two or more components. So, for example, dealing with the supernova phenomena neutrinos interact with the leptonic part of the matter. In cosmology the cold-dark-matter models distinguish a weakly interacting component and a baryonic one. In the standard cosmology, after decoupling of matter and radiation, one has a perfect-fluid energy tensor describing an isotropic radiation, and a perfect pressureless fluid describing the baryonic matter. The reason why only a vector field is considered for representing the energy content is that one expects that, because of the interaction between both components, equilibrium is soon established. However, if the two components are weakly interacting, the time taken to reach equilibrium may be significant. This situation has been outlined by Bayin[1] for the early times of a neutron star, giving some analytic solutions of Einstein's equations in the case of spherical symmetry. Letelier[2] presented a method for solving Einstein's equations in the case that each fluid component is irrotational.

The aim of this paper is to construct solutions of Einstein's equations interpretable as a mixture of perfect fluids, using the initial-value formulation of general relativity (3+1 formalism).[3,4] This method is more convenient in order to construct solutions numerically, when sufficient information about the energy-momentum tensor is available. But it is also useful in order to simplify Einstein's equations when a family of three-dimensional slices with certain geometrical properties is assumed to exist. So, for example, Stephani and Wolf[5] presented a method for finding perfect-fluid solutions with flat three-slices and an extrinsic curvature proportional to the metric tensor. General properties of solutions admitting a spacelike foliation of constant curvature are discussed by Bona and Coll.[6]

From the point of view of Einstein's equations, a two-fluid source is equivalent to considering a nonperfect fluid, with an anisotropic pressure tensor and a four-vector of "heat propagation." The latter is interpreted as the propagation of energy due to the second fluid, if the relative velocity of both components is different from zero.

The main assumption of this paper is to require a simple kinematics to one of the components: no rotation, no distortion, no acceleration, only the expansion is allowed to be different from zero. This is equivalent to requiring the existence of a slicing with extrinsic curvature proportional to the metric. So, in a sense, the intention of this work is similar to the Stephani-Wolf paper. The main difference is that we are looking for non-perfect-fluid solutions with slices not necessarily flat. This simplifies considerably the field equations but produces a problem. Which are the initial conditions in order to guarantee that one of the two components moves in the way prescribed? We do not know if we have solved the problem in general, but at least we have the method to construct a large family of solutions. We call these solutions "frozen" solutions, for the relative velocity field between both components always keeps the same direction.

II. INTIAL-VALUE FORMULATION OF EINSTEIN'S EQUATIONS

The following conventions will be used throughout the paper: units are such that $4\pi G = c = 1$, the signature of the metric is taken to be $(-+++)$, and greek and latin indices run from 0 to 3 and 1 to 3, respectively.

Let us assume that the space-time may be foliated. So, it is possible to use a coordinate system (x^0, x^1, x^2, x^3) such that on the surfaces of the foliation we have x^0=const. Let n^μ be the unitary vector orthogonal to these surfaces. The expression in coordinates is usually given in the form

$$n^\mu = \left[\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i \right] .$$

Each tensor is split into orthogonal and parallel components with respect to the covariant unitary vector n. (Henceforth all the tensorial equations will be written in covariant form.) So for the energy-momentum tensor we have

$$T = t n \otimes n + \bar{t} \otimes n + n \otimes \bar{t} + \tilde{t} ,$$

where $\otimes$ represents a tensorial product, $\overline{t}$ and $\widetilde{t}$ are a covariant vector and a covariant tensor orthogonal to n. In the case where n is a temporal vector, t will be the energy density in the proper reference system.

Einstein's equations are also split with respect to each hypersurface of the foliation, given a set of boundary equations, in which only "spatial" coordinate derivatives appear, and a set of evolution equations with respect to the coordinate x^0. The boundary conditions may be written as

$$(\mathrm{tr}K)^2-\mathrm{tr}K^2-\epsilon R=4t\ , \tag{1}$$

$$\mathrm{div}(K-\mathrm{tr}K\gamma)=-\epsilon 2\overline{t}\ . \tag{2}$$

And the evolution equations are

$$(\partial_0-L_\beta)\gamma=-2\alpha K\ , \tag{3}$$

$$(\partial_0-L_\beta)K=2\epsilon\alpha\left[\widetilde{t}-\frac{1}{m-2}\mathrm{tr}T\gamma\right]-\epsilon\alpha\,\mathrm{Ric}(\gamma)+\alpha(\mathrm{tr}KK-2K^2)+\epsilon\nabla d\alpha\ , \tag{4}$$

where $\epsilon=n^2$; γ and K are the first and second fundamental form of a slice, namely, $\gamma=g-\epsilon n\otimes n$, $K=-\nabla n$; $\mathrm{Ric}(\gamma)$ is the Ricci tensor corresponding to metric γ and $R=\mathrm{tr}\,\mathrm{Ric}(\gamma)$, where tr denotes the trace operator; ∇ is the covariant derivative with respect to the metric γ; and L_β is the Lie derivative along the shift vector β^i. In Eq. (4), m is the dimension of the manifold. In this case $m=4$, but below, in Sec. V, we will need to take $m=3$.

It is well known that, as a consequence of the Bianchi identities, the divergence of the energy-momentum tensor must be zero for any solution of the field equations. So we can write the following consequences of the field equations

$$(\partial_0-L_\beta)t=\alpha t\,\mathrm{tr}K-2i(\overline{t})d\alpha-\alpha\,\mathrm{div}\overline{t}-\epsilon\alpha\,\mathrm{tr}(K\times\widetilde{t}), \tag{5}$$

$$(\partial_0-L_\beta)\overline{t}=-\alpha\,\mathrm{div}\widetilde{t}-i(d\alpha)\widetilde{t}+\epsilon t d\alpha+\alpha\,\mathrm{tr}K\overline{t}\ , \tag{6}$$

where $i(\overline{t})$, div, $\times$, are the interior product by $\overline{t}$, divergence operator, and the contracted tensorial product, respectively.

Next we give all the quantities appearing in these equations in indices notation:

$$\begin{aligned}&\mathrm{tr}K=\gamma^{ij}K_{ij},\quad K^2=K\times K=K_i^mK_{mj}dx^i\otimes dx^j\ ,\\&\mathrm{tr}K^2=K^{im}K_{im},\quad \mathrm{div}K=\nabla^jK_{ij}dx^i,\quad \overline{t}=\overline{t}_idx^i\ .\end{aligned} \tag{7}$$

III. SHEAR-FREE AND GEODESIC SOLUTIONS

In this section we assume that the unitary vector field normal to the foliation is temporal, geodesic and shear-free. Then we take $\alpha=1$ and $\epsilon=-1$ in Eqs. (3) and (4). We choose a zero shift vector β^i. The shear-free condition allows us to write the extrinsic curvature in the form

$$K=-H\gamma\ . \tag{8}$$

Then, the boundary conditions are

$$6H^2+R=4t\ , \tag{9}$$

$$dH=\overline{t}\ . \tag{10}$$

Let us write the three-dimensional metric γ in the form

$$\gamma=\Omega^2\overline{\gamma}\ , \tag{11}$$

where $\overline{\gamma}$ is the metric on the initial hypersurface, and Ω a function defined on the space-time, taking the value 1 on the initial surface.

Equation (3) can be written in the form

$$\partial_0\Omega=\Omega H \tag{12}$$

and substituting Eq. (8) into (4) one gets

$$\partial_0H=-H^2-\tfrac{1}{3}(t+\mathrm{tr}\widetilde{t})\ , \tag{13}$$

$$\mathrm{Ric}(\gamma)-\tfrac{1}{3}R\gamma=2(\widetilde{t}-\tfrac{1}{3}\mathrm{tr}\widetilde{t}\gamma)\ . \tag{14}$$

From Eqs. (5) and (6) we get

$$\partial_0t=-\mathrm{div}\overline{t}-H(\mathrm{tr}\widetilde{t}+3t)\ , \tag{15}$$

$$\partial_0\overline{t}=-3H\overline{t}-\mathrm{div}\widetilde{t}\ . \tag{16}$$

The latter are consequences of Einstein's equations written above, but it is very useful to consider them.

Let us write some of these equations in a more convenient form. Using the Eqs. (10) and (13) one gets

$$\partial_0\overline{t}=-3H\overline{t}-\tfrac{1}{3}d(\eta+\mathrm{tr}\widetilde{t})\ , \tag{17}$$

where we have introduced the variable η defined by

$$\eta=t-\tfrac{3}{2}H^2\ , \tag{18}$$

which is the difference between the energy density and the "critical energy density." Taking into account Eqs. (13) and (15) we obtain an evolution equation for η:

$$\partial_0\eta=-\mathrm{div}\overline{t}-2H\eta\ . \tag{19}$$

Combining Eqs. (12) and (20) we get

$$\partial_0(\Omega^2\eta)=-\Omega^2\mathrm{div}\overline{t}\ . \tag{20}$$

And Eq. (13) may be rewritten as

$$\partial_0H=-\tfrac{3}{2}H^2=\tfrac{1}{3}(\eta+\mathrm{tr}\widetilde{t})\ . \tag{21}$$

Equations (12), (21), (20), and (17) form a quasilinear system of differential equations for the six variables $\Omega, H, \eta, \overline{t}$. One can verify directly that the solutions of the equations of evolution verify at each instant the boundary conditions (9) and (10). The trace of the tensor $\widetilde{t}$ is not determined by the field equations: then it must be an input of the problem. A solution of the system is characterized by the initial values (values on the initial sheet) $\Omega_0=1, H_0, \eta_0, \overline{t}_0$. The requirement $\overline{t}_0=dH_0$ must be satisfied on account of the boundary condition (10).

The initial metric, i.e., $\overline{\gamma}$, is constrained by the conditions (9) and (14), which can be combined in a sole equation:

$$\mathrm{Ric}(\overline{\gamma})-\tfrac{1}{2}R(\overline{\gamma})\overline{\gamma}=2[\widetilde{t}_0^T-\tfrac{1}{3}(t_0-\tfrac{3}{2}H_0^2)\overline{\gamma}]\ . \tag{22}$$

This equation is just a three-dimensional Einstein equation (the left member is the three-dimensional Einstein tensor) corresponding to the energy-momentum tensor

$$\tau=\tilde{t}_0^T-\tfrac{1}{3}\eta_0\bar{\gamma}\ . \tag{23}$$

It is worth pointing out here that $\tilde{t}_0^T$ and η_0 are constrained by the three-dimensional Bianchi identities:

$$-3\,\mathrm{div}\tilde{t}_0^T+d\eta_0=0\ . \tag{24}$$

For each solution of the evolution equations, one must compute the energy-momentum tensor, in order to interpret physically the result. Let us do this right now. The Ricci tensors of two metrics related by a conformal factor Ω are given by the well-known relation[7]

$$\begin{aligned}\mathrm{Ric}=\mathrm{Ric}(\bar{\gamma})-\bar{\nabla}d\ln\Omega+d\ln\Omega\otimes d\ln\Omega\\ -\bar{\gamma}[(d\ln\Omega)^2+\bar{\Delta}\ln\Omega]\ .\end{aligned} \tag{25}$$

And taking into account Eq. (14) we can write the orthogonal part of the energy-momentum tensor in the form

$$\begin{aligned}\tilde{t}=\tfrac{1}{3}\mathrm{tr}\tilde{t}\gamma+\tilde{t}_0^T+\tfrac{1}{2}\{-\bar{\nabla}d\ln\Omega+d\ln\Omega\otimes d\ln\Omega\\ -\tfrac{1}{3}\bar{\gamma}[(\bar{\nabla}\ln\Omega)^2-\bar{\Delta}\ln\Omega]\}\ .\end{aligned} \tag{26}$$

A summary of this section is presented in Table I.

IV. TWO-PERFECT-FLUID SOLUTIONS

Let T_1 and T_2 be the energy-momentum tensors of two perfect fluids with velocities, densities, and pressures n,ρ_1,p_1 and u,ρ_2,p_2, respectively. Let V be the relative velocity: that is,

$$u=\frac{1}{\sqrt{1-V^2}}(n+V)\ . \tag{27}$$

By splitting the energy-momentum tensor of the mixture of both components, i.e., $T=T_1+T_2$ in parts parallel and orthogonal to the velocity n, of one of the fluids one gets

$$T=tn\otimes n+\bar{t}\otimes n+n\otimes\bar{t}+\tilde{t}\ , \tag{28}$$

$$\tilde{t}=a\gamma+b\bar{t}\otimes\bar{t}\ , \tag{29}$$

where

$$\begin{aligned}t&=\rho_1-p_2+\frac{\rho_2+p_2}{1-V^2},\quad b=\frac{1-V^2}{\rho_2+p_2}\ ,\\ a&=p_1+p_2,\quad \bar{t}=\frac{\rho_2+p_2}{1-V^2}V\ .\end{aligned} \tag{30}$$

Conversely, an energy-momentum tensor with an orthogonal part relative to n as given by Eq. (28) and such that $b|\bar{t}|<1$, may be interpreted as the mixture of two perfect fluids, n being the velocity of one of them. The densities, pressures, and relative velocities are given by the equations

$$\begin{aligned}&\rho_1-p_2=t-\frac{1}{b},\quad \rho_2+p_2=\frac{1}{b}-b\bar{t}^2\ ,\\ &p_1+p_2=a,\quad V=b\bar{t}\ .\end{aligned} \tag{31}$$

The question arises here, what initial conditions must be taken in order to guarantee that the orthogonal component of the energy-momentum tensor keeps the form characteristic of two perfect fluids?

We have not treated this point in all its generality, but we have found a sufficient condition that allows us to construct specific examples. Let us state a preliminary result.

Lemma. All solutions of Einstein's equations, satisfying the requirements of the previous section (see Table I), plus the condition $\bar{t}=\Gamma\bar{t}_0$, satisfy

$$\begin{aligned}&\Omega=\Omega(x^0,H_0),\quad H=H(x^0,H_0)\ ,\\ &\Gamma=\frac{\partial^2\ln\Omega}{\partial x^0\partial H_0}=\frac{\partial H}{\partial H_0},\quad \eta+\mathrm{tr}\tilde{t}=3F(x^0,H_0)\ .\end{aligned} \tag{32}$$

To prove this, we apply the differential operator d to Eq. (12) and take into account Eq. (10), obtaining $\partial_0(d\ln\Omega)=\Gamma\bar{t}_0$. As $d\ln\Omega|_{x^0=0}=0$, one gets $d\ln\Omega \propto dH_0$. Therefore, Ω depends only on H_0 and x^0. Tak-

TABLE I. Shear-free geodesic non-perfect-fluid solutions.

Initial metric	
Initial conditions	$\tilde{t}_0^T,\eta_0$ such that $-3\,\mathrm{div}\tilde{t}_0^T+d\eta_0=0$
Three-dimensional field equations	$\mathrm{Ric}(\bar{\gamma})-\frac{1}{2}R(\bar{\gamma})\bar{\gamma}=2(\tilde{t}_0^T-\frac{1}{3}\eta_0\bar{\gamma})$
Evolution	
Initial conditions	$\Omega_0=1,\eta_0,H_0,\ \ \bar{t}_0=dH_0$
Arbitrary function	$\mathrm{tr}\tilde{t}$
Evolution equations	$\partial_0\Omega=\Omega H$
	$\partial_0 H=-\frac{3}{2}H^2-\frac{1}{3}(\eta+\mathrm{tr}\tilde{t})$
	$\partial_0(\Omega^2\eta)=-\Omega^2\mathrm{div}\bar{t}$
	$\partial_0\bar{t}=-3H\bar{t}-\frac{1}{3}d(\eta+\mathrm{tr}\tilde{t})$
Energy-momentum tensor	
	$T=tn\otimes n+\bar{t}\otimes n+n\otimes\bar{t}+\tilde{t}$
	$\tilde{t}=\frac{1}{3}\mathrm{tr}\tilde{t}\gamma+\tilde{t}_0^T+\frac{1}{2}\{-\bar{\nabla}d\ln\Omega+\bar{\nabla}\ln\Omega\otimes\bar{\nabla}\ln\Omega$
	$-\frac{1}{3}\bar{\gamma}[(\bar{\nabla}\ln\Omega)^2-\bar{\Delta}\ln\Omega]\}$

ing into account this result, and considering again Eq. (12) we get $H=H(x^0,H_0)$. Substituting $\bar{t}=\Gamma\bar{t}_0$ into Eq. (21) one gets that $\eta+\mathrm{tr}\tilde{t}$ is also a function of x^0 and H_0. Finally, from the relation $d\ln\Omega \propto dH_0$, we get

$$\partial_0(d\ln\Omega)=\frac{\partial^2\ln\Omega}{\partial x^0\partial H_0}dH_0\ ,$$

and considering $\partial_0(d\ln\Omega)=\Gamma\bar{t}_0$ one gets $\Gamma=(\partial^2\ln\Omega)/(\partial x^0\partial H_0)=\partial H/\partial H_0$. Using this lemma, we can state a sufficient condition to guarantee the two-perfect-fluid condition.

Theorem. A solution of Einstein's equations, verifying the condition $\bar{t}\propto\bar{t}_0$ and the requirements of the previous section (Table I), may be interpreted as the sum of two perfect fluids, if the initial conditions $\{\bar{t}_0,\tilde{t}_0^T,\bar{\gamma}\}$ satisfy

(i) $\tilde{t}_0^T=b_0(\bar{t}_0\otimes\bar{t}_0-\frac{1}{3}\bar{t}_0^2\bar{\gamma})$,

(ii) $\bar{\nabla}\bar{t}_0=\lambda_0\bar{t}_0\otimes\bar{t}_0+\mu_0\bar{\gamma}$.

To prove this let us consider the expression (26) for the energy-momentum tensor. Taking into account the results of the lemma we have

$$d\ln\Omega=\frac{\partial\ln\Omega}{\partial H_0}\bar{t}_0\ ,$$
$$\bar{\nabla}d\ln\Omega=\frac{\partial\ln\Omega}{\partial H_0}\bar{\nabla}\bar{t}_0+\frac{\partial^2\ln\Omega}{\partial H_0^2}\bar{t}_0\otimes\bar{t}_0 \tag{33}$$

and taking into account the condition (ii), one gets

$$\bar{\nabla}d\ln\Omega=(\lambda_0\beta+\beta')\bar{t}_0\otimes\bar{t}_0+\mu_0\beta\bar{\gamma} \tag{34}$$

with

$$\beta=\frac{\partial\ln\Omega}{\partial H_0}\quad\text{and}\quad\beta'=\frac{\partial\beta}{\partial H_0}\ . \tag{35}$$

By substituting Eqs. (33) and (34) into the expression (26) we get the characteristic energy-momentum tensor of a two-component fluid [Eq. (29)], with

$$a\Omega^2=\tfrac{1}{3}\Omega^2\mathrm{tr}\tilde{t}-\tfrac{1}{3}\bar{t}_0^2b\Gamma^2\ , \tag{36}$$

$$b\Gamma^2=\tfrac{1}{2}(2b_0+\beta^2-\beta'-\lambda_0\beta)\ . \tag{37}$$

The following remark will be very useful in order to solve the three-dimensional Einstein equations. Let us write $\bar{t}_0=\kappa v$, with v a unitary spatial vector, $v^2=+1$. It is easy to prove, as a direct consequence of the requirement (ii) above, that v is geodesic, irrotational, and shear-free, and its length κ depends only on H_0.

This means that the initial sheet $x^0=0$ admits also a foliation with a unitary normal v which is also shear-free and geodesic. Therefore we can try to solve the field equations by the same procedure we used at the beginning of this paper, that is we shall split the three-dimensional Einstein equations with respect to the foliation defined by the vector field v ($2+1$ formalism).

A resume of the results of the section is presented in Table II.

V. THE INITIAL METRIC FOR TWO-PERFECT FLUID SOLUTIONS

To determine the initial metric $\bar{\gamma}$, one must solve the three-dimensional Einstein equations (22). We will develop the idea stated at the end of the previous section, that is to solve the three-dimensional Einstein equations by applying the $2+1$ formalism. Taking into account the condition (i) above, one can write the source tensor τ in the form

$$\tau=b_0\bar{t}_0\otimes\bar{t}_0-\tfrac{1}{3}(b_0\kappa^2+\eta_0)\bar{\gamma}\ . \tag{38}$$

The basic variables are, as is well known, the metric induced on the sheets,

$$\hat{\sigma}=\bar{\gamma}-v\otimes v \tag{39}$$

and its extrinsic curvature, $\hat{K}=-\frac{1}{2}L_v\hat{\sigma}=-\hat{\nabla}v$, which,

TABLE II. Shear-free and geodesic two-perfect-fluid solutions.

Initial metric	
Initial conditions	$\tilde{t}_0^T=b_0(\bar{t}_0\otimes\bar{t}_0-\frac{1}{3}\kappa^2\bar{\gamma})$, $\bar{t}_0=dH_0$
	η_0 such that $-3\,\mathrm{div}\tilde{t}_0^T+d\eta_0=0$
Three-dimensional field equations	$\mathrm{Ric}(\bar{\gamma})-\frac{1}{2}R\bar{\gamma}=2(\tilde{t}_0^T-\frac{1}{3}\eta_0\bar{\gamma})$
Evolution	
Initial conditions	$\Omega_0=1,\eta_0,H_0,b_0,\ \bar{t}_0=dH_0$
Arbitrary function	$\mathrm{tr}\tilde{t}$
Evolution equations	$\partial_0\Omega=\Omega H$
	$\partial_0 H=-\frac{3}{2}H^2-\frac{1}{3}(\eta+\mathrm{tr}\tilde{t})$
	$\partial_0(\Omega^2\eta)=-\Omega^2\mathrm{div}(\Gamma\bar{t}_0)$
	$\Gamma=H'$
	$\bar{t}=\Gamma\bar{t}_0$
Energy-momentum tensor	
$T=tn\otimes n+\bar{t}\otimes n+n\otimes\bar{t}+\tilde{t},\ \ \tilde{t}=a\gamma+b\bar{t}\otimes\bar{t}$	
$\Omega^2a=\frac{1}{3}\Omega^2\mathrm{tr}\tilde{t}-\frac{1}{3}\kappa^2b\Gamma^2$	
$b\Gamma^2=\frac{1}{2}(2b_0+\beta^2-\beta'-\lambda_0\beta)$	

taking into account the fact that it is geodesic and shear-free, will be of the form

$$\hat{K} = -h\hat{\sigma} \; . \tag{40}$$

Let us introduce coordinates x,y,z, adapted to the foliation, with lapse function equal 1 and the shift vector equal to zero. In these coordinates we can write

$$v = dz, \quad \bar{\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_{AB} \end{bmatrix} , \tag{41}$$

where $A,B = 1,2$; $x^1 = x$, $x^2 = y$. The tensor τ can be written

$$\tau = \tau v \otimes v + \bar{\tau} \tilde{\otimes} v + \tilde{\tau}, \quad \tau = \tfrac{2}{3} b_0 \kappa^2 - \tfrac{1}{3}\eta_0 ,$$
$$\bar{\tau} = 0, \quad \tilde{\tau} = -\tfrac{1}{3}(b_0\kappa^2 + \eta_0)\hat{\sigma} \; . \tag{42}$$

The three-dimensional Einstein equations can be split using Eqs. (1)–(6) with $m = 3$, and $\epsilon = +1$, in a set of boundary conditions

$$\mathrm{tr}\,\mathrm{Ric}(\hat{\sigma}) = 2h^2 - 4\tau , \tag{43}$$

$$\partial_A h = 0 \tag{44}$$

and a set of evolution equations

$$\partial_z \hat{\sigma} = 2h\hat{\sigma} , \tag{45}$$

$$\partial_z \hat{K} = 2(\tilde{\tau} - \mathrm{tr}\tau\hat{\sigma}) - \mathrm{Ric}(\hat{\sigma}) \; . \tag{46}$$

The Bianchi identities give us

$$\frac{\partial\tau}{\partial z} = -2\tau h + h\xi(z) , \tag{47}$$

$$d(\mathrm{tr}\tilde{\tau}) \propto dz , \tag{48}$$

where we have written $\xi(z) = \mathrm{tr}\tilde{\tau}$. All of the two-dimensional metric is conformally flat, and its Ricci tensor is proportional to the metric. So, we introduce the conformal factor ω by

$$\hat{\sigma} = \omega^2 \delta , \tag{49}$$

where δ is the flat metric. Then Eq. (43) is equivalent to

$$\mathrm{Ric}(\hat{\sigma}) = (h^2 - 2\tau)\hat{\sigma} \tag{50}$$

or what is the same

$$\Delta_\delta \ln\omega = (2\tau - h^2)\omega^2 , \tag{51}$$

where Δ_δ is the Laplacian operator corresponding to the metric δ. Substituting (50) into Eq. (46) and taking into account that $\mathrm{tr}\tau = \tau + \mathrm{tr}\tilde{\tau}$, we write the latter in the form

$$\partial_z h = -h^2 + \mathrm{tr}\tilde{\tau} \; . \tag{52}$$

From the boundary condition (44), we know that h only depends on z; therefore we can write $\omega^2 = r^2(z)\phi^2(x^1, x^2)$, with $r(z)$ taken to be equal 1 on the initial surface $z = 0$. Equation (51) can now be written

$$\Delta_\delta \ln\phi = (2\tau_0 - h_0^2)\phi^2 \tag{53}$$

and the evolution equations simplified to

$$\frac{dr}{dz} = rh , \tag{54}$$

$$\frac{dh}{dz} = -h^2 + \xi(z) \; . \tag{55}$$

This system admits a constant of motion

$$\partial_z[(h^2 - 2\tau)\omega^2] = 0 \tag{56}$$

from which we can get

$$\tau = (2\tau_0 - h_0^2)\frac{r_0^2}{2r^2} + \tfrac{1}{2}h^2 \; . \tag{57}$$

For some purposes it will be useful to introduce the radial coordinate $r = r(z)$; then we can write

$$\gamma_{ij} dx^i dx^j = e^{\lambda(r)} dr^2 + r^2 \phi^2(x^A)\delta_{AB} dx^A dx^B ,$$
$$e^{\lambda(r)} = \left[\frac{dz}{dr} \right]^2 . \tag{58}$$

From Eqs. (54) and (55), it is easy to get the equation for $\lambda(r)$,

$$\frac{de^{-\lambda}}{dr} = 2r\xi(r) , \tag{59}$$

and using (58) and (54) one determines h as a function of r:

$$h^2 = r^{-2} e^{-\lambda} \; . \tag{60}$$

Finally, from Eq. (42) we get

$$b_0\kappa^2 = (2\tau_0 - h_0^2)\frac{r_0^2}{2r^2} + \tfrac{1}{2}h^2 - \tfrac{1}{2}\xi(r) , \tag{61}$$

$$\eta_0 = -(2\tau_0 - h_0^2)\frac{r_0^2}{2r^2} - \tfrac{1}{2}h^2 - \xi(r) \tag{62}$$

that, with Eqs. (57) and (38) determines the source τ of the three-dimensional Einstein equations.

VI. THE EVOLUTION EQUATIONS FOR TWO-FLUID SOLUTIONS

Here we study the evolution of the three-dimensional metric with the time x^0. For that, we need to solve the quasilinear system given by Eqs. (12), (21), and (20):

$$\partial_0 \Omega = \Omega H ,$$
$$\partial_0 H = -\tfrac{3}{2}H^2 - \tfrac{1}{3}(\eta + \mathrm{tr}\tilde{t}) ,$$
$$\partial_0(\Omega^2\eta) = -\Omega^2 \mathrm{div}(\Gamma\bar{t}_0) \; .$$

Substituting in the last equation

$$\Omega^2 \mathrm{div}(\Gamma\bar{t}_0) = -\kappa^2\Gamma' + \Gamma(\lambda_0\kappa^2 + 3\mu_0)$$

we get

$$\Omega^2\eta = \eta_0 + \lambda_0\kappa^2 + 3\mu_0 - (\lambda_0\kappa^2 + 3\mu_0)\beta - \kappa^2\beta' \; . \tag{63}$$

The space-time function $\mathrm{tr}\tilde{t}$ in the evolution equation is related to the sum of the pressures, which we denote by a, by Eqs. (36) and (37), i.e.,

$$\Omega^2 \mathrm{tr}\tilde{t} = 3\Omega^2 a + \tfrac{1}{2}\kappa^2(2b_0 + \beta^2 - \beta' - \lambda_0\beta) . \quad (64)$$

From the relations $dH_0 = \kappa\, dz$, and taking into account Eq. (54) we get

$$\partial_{H_0} = \frac{rh}{\kappa}\partial_r .$$

By substituting this expression into the evolution equations we get

$$\partial_0\Omega = \Omega H ,$$

$$\partial_0 H = c_3 \frac{rh}{\kappa\Omega^2}\partial_r\beta - \tfrac{3}{2}H^2 - \Omega^{-2}(c_1 + c_2\beta + \tfrac{1}{3}c_3\beta^2) - a , \quad (65)$$

$$\partial_0\beta = \frac{rh}{\kappa}\partial_r H ,$$

where c_1, c_2, c_3 are given by

$$c_1 = \tfrac{1}{3}[\eta_0 + (\lambda_0 + b_0)\kappa^2 + 3\mu_0] ,$$

$$c_2 = -(\mu_0 + \tfrac{1}{2}\lambda_0\kappa^2) , \quad (66)$$

$$c_3 = \tfrac{1}{2}\kappa^2 .$$

It is easy to derive the following expression for λ_0 and μ_0:

$$\lambda_0 = \frac{h}{\kappa^2}\left[r\frac{d\kappa}{dr} - \kappa \right] , \quad (67)$$

$$\mu_0 = \kappa(r)h(r) .$$

In consequence, c_1, c_2, c_3, depend only on the coordinate r.

The evolution equations (65) can be put in matrix form

$$\partial_0 u = A\partial_r u + F ,$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & N & 0 \end{bmatrix}, \quad M = \frac{c_3 rh}{\kappa\Omega^2} ,$$

$$N = \frac{rh}{\kappa}, \quad u = (\Omega, H, \beta) ,$$

This is a strictly hyperbolic system, because the matrix A has real and distinct eigenvalues $\lambda = 0$, $\lambda_i = \pm\sqrt{c_3}/\Omega$.

VII. SUMMARY AND CONCLUSIONS

We have obtained a family of solutions of Einstein's equations, up to a strictly hyperbolic system of differential equations. The metric has been written in the form

$$ds^2 = -(dx^0)^2 + \Omega(x^0, r)[e^{\lambda(r)}dr^2 + r^2\phi^2(x^A)\delta_{AB}dx^A dx^B] .$$

A solution is determined by giving (1) two real numbers h_0, r_0; (2) a function $\tau_0(x^1, x^2)$ defined on the initial surface $r = r_0$, $x^0 = 0$; (3) two functions of r: $\kappa(r)$, $\xi(r)$; and (4) a space-time function depending on x^0 and r: $a(x^0, r)$.

The evolution factor Ω is determined by solving the strictly hyperbolic quasilinear system (65). By using Eq. (31), the family of solutions may be interpreted as a two-component fluid.

The arbitrary function τ_0 decides the geometric properties of the two-surfaces $x^0 = \text{const}$, $r = \text{const}$. So, taking τ_0 constant we get spheres, hyperboloids or two planes depending on the sign of $2\tau_0 - h_0^2$. The function ϕ must be a solution of the elliptical equation (53).

The arbitrary function $a(x^0, r)$ may be specified by an election of the equation of the state of each component, or by the law of interaction of the two components. Finally, the functions κ and ξ may be chosen according to the total density and the sum of the density and pressure of one of the components.

A summary of the results is presented in Table III.

TABLE III. Results of this paper.

Four-dimensional metric
$ds^2 = -(dx^0)^2 + \Omega(x^0, r)[e^{\lambda(r)}dr^2 + r^2\phi(x^A)\delta_{AB}dx^A dx^B]$
Degrees of freedom
$h_0, r_0, \tau_0(x^A), \xi(r), \kappa(r), a(x^0, r)$
Spatial metric
$\gamma_{ij}dx^i dx^j = e^{\lambda(r)}dr^2 + r^2\phi^2(x^A)\delta_{AB}dx^A dx^B$
$e^{-\lambda} = -2\int_{r_0}^{r} s\xi(s)ds + r_0^2 h_0^2$
$\Delta_\delta \ln\phi = (2\tau_0 - h_0^2)\phi^2$
Evolution equations
$\partial_0\Omega = \Omega H$
$\partial_0 H = c_3 \frac{rh}{\kappa\Omega^2}\partial_r\beta - \frac{3}{2}H^2 - \Omega^{-2}(c_1 + c_2\beta + \frac{1}{3}c_3\beta^2) - a$
$\partial_0\beta = \frac{rh}{\kappa}\partial_r H$
$c_1 = \frac{1}{3}\left[-\frac{3}{2}\xi(r) + r\frac{d\kappa}{dr} + 2\kappa h \right]$
$c_2 = -\frac{h}{2}\left[r\frac{d\kappa}{dr} + \kappa \right]$
$c_3 = \frac{1}{2}\kappa^2$

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