Intrinsic characterization of space-time symmetric tensors

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This paper essentially deals with the classification of a symmetric tensor on a four-dimensional Lorentzian space. A method is given to find the algebraic type of such a tensor. A system of concomitants of the tensor is constructed, which allows one to know the causal character of the eigenspace corresponding to a given eigenvalue, and to obtain covariantly their eigenvectors. Some algebraic as well as differential applications are considered.

I. INTRODUCTION

Second-order symmetric tensors play an important rôle in relativity. The gravitational field itself is described by a Lorentzian metric, whose energy tensor is related to physically admissible distributions of matter; Lie variations of the metric along nonconformal vectors fields, symmetric double contractions of the Riemann tensor with respect to particular directions, second-order Killing tensors, almost-product structure tensors defining totally geodesic, distortion-free, extremal, or orthogonal integrable submanifolds of the space-time, are other frequent examples of them. In general, the interpretation of a given symmetric tensor as describing a particular physical situation requires, sooner or later, the analysis of its algebraic properties.

The algebraic classification of a symmetric two-tensor, on a four-dimensional Lorentzian space is well known for a long time.[1-7] There exist four different types of such tensors, corresponding to the algebraic structure of the linear map they define. Symmetric tensors belong to type I, type II, or type III if they have only real eigenvalues and admit, respectively, four, three, or two linearly independent eigenvectors, and they belong to type IV if they have only a pair of complex conjugate eigenvalues.[8]

This classification is obtained directly by considering the possible canonical forms of a symmetric tensor in a orthonormal basis[1,2] or in a real null tetrad,[9,10] or the Jordan forms for the associated linear map.[4,5] The types and their subtypes (degenerate eigenvalues) can be denoted in several appropriate ways, being the Weiler–Segre[11] and the Plebański[3] notations the more usual ones. Criteria to distinguish the algebraic subtypes have been considered[3,9] and, recently, applied to a computer program for their determination.[12]

The invariant two-plane structure admitted by any symmetric tensor[1] on the space-time has been used as a tool for the characterization of subtypes (number and causal character of the invariant two-planes).[13] In this approach the basic object is a symmetric double two-form E, linear in the trace-free part of the symmetric tensor T (Ref. 14). The algebraic classification of E is made[15] by a method similar to the Bel's one[16] for the classification of the Weyl conformal tensor. The double two-form E allows us to introduce[17] a real differentiable map on non-null simple bivectors, analog to the Riemannian sectional curvature function, whose critical point structure is related to the invariant two-planes of T. The spacelike and timelike eigenvectors of T can be associated with another map, analog to the sectional Ricci function considered by Eisenhart.[18]

Other classification schemes have been developed considering the (Hermitian) spinor associated to N. Such spinor approaches[3,19] can be implemented using the complex null tetrad formalism[20,21] and algebraic geometry techniques.[22] The relations between these spinor treatments and their connection with standard matrix methods have been also extensively studied.[23-26]

After such a panorama, why still work on this subject? In spite of the abundant literature dedicated to it[27] some important questions still remain open. In this paper we shall mainly consider two of them, namely, that of the *causal character* of the eigenspace corresponding to every one of the eigenvalues, and that of the *explicit nature* of the eigenvectors as geometric objects depending on the given tensor.

The first of this questions has a clear meaning: to detect, prior to its explicit calculation, which is the causal orientation of the eigenspace associated to any given eigenvalue. The second of these questions needs perhaps a little explanation.

Eigenvalues and eigenvectors of a second-order mixed tensor T (linear map) are both univocally derived from it, but the nature of their dependence on T is quite different. From the known relations[28] between the coefficients of the characteristic polynomial and the simple scalar invariants[29] of T, it is clear that the eigenvalues *are concomitants*[30,31] of T: for every eigenvalue λ, there exists a scalar function l of the tensor T such that $\lambda = l(T)$.

The eigenvectors corresponding to every λ follow from their definition equations by Cramer's rule, and that whatever be the starting reference frame in which T is given. But, contrary to an extended opinion, in spite of such an apparently covariant prescription to obtain them, the eigenvectors of a tensor T are *not concomitants* of the sole tensor T (Ref. 32). What are they? Every eigenvector of T is a vector-valued double $(p+1,p)$-ary form in $i \leqslant p+1$ arbitrary directions and $j \leqslant p$ arbitrary codirections, the values of p, i, and j depending on the subtype of T and on the multiplicity of the eigenvalue. It is only the mixed tensor of order $(p+1,p+1)$ defining the $(p+1,p)$-ary form, which is a concomitant of T (Refs. 33 and 34). Denoting by s an eigenvector corresponding to the eigenvalue λ, by v the set of i arbitrary directions, and by ω the set of j arbitrary codirections, there exists a vector concomitant S of T, v, and ω such that $s = S(T,v,\omega)$.

Clearly, if T is twice covariant and the space is endowed with a metric g, the linear map associated to T is $g^{-1}T$ and vectors and covectors may by identified by g, so that the above expressions for the eigenvalues and eigenvectors of T become $\lambda = l(g,T)$ and $s = S(g,T,v)$, where now $S(g,T,v)$ is a vector-valued $(2p+1)$-ary form in $i' \leqslant i+j$ arbitrary directions.

Our present interest concerns second-order symmetric tensors on the (four-dimensional Lorentzian) space-time. For them, we shall obtain here explicitly the expressions of $S(g,T,v)$ corresponding to *all* the subtypes and *all* the eigenvalues, showing on the way, fortunately, that $p=0$ or 1. It will be then possible to construct scalar concomitants of λ, g, T, and eventually v, say $f(\lambda,g,T,v)$, whose signs indicate the causal orientation of the corresponding eigenspaces.

What is the incidence of these two questions in relativity? The expressions for $S(g,T,v)$ and $f(\lambda,g,T,v)$ constitute the *explicit general solution* to the eigenvectors problem for symmetric tensors of every subtype on the space-time. Their intrinsic and covariant form are well adapted to theoretical analysis, as well as to computational methods. But, sightly paradoxically, where these expressions play an essential role is in *differential* problems related to eigenvectors.

In many situations of interest one is led to ask for symmetric tensors T admitting one or more eigenvectors s verifying specified differential conditions, say $\mathscr{D}(s) = 0$; and one often needs to know explicitly the necessary and sufficient differential conditions on T, say $\mathscr{D}(T) = 0$, ensuring that T admits such eigenvectors s that $\mathscr{D}(s) = 0$. Our expressions for $S(g,T,v)$ give a simple systematic method to solve this problem, which has been considered up to now as a very difficult one.[35] The method is illustrated by solving the cases where $\mathscr{D}(s) = 0$ stands for the geodesic equation, $\mathscr{D}(s) \equiv i(s)\nabla s = 0$, and for the integrability conditions, $\mathscr{D}(s) \equiv s \wedge ds = 0$. It is to be noted that it gives effectively $\mathscr{D}(T)$ as a differential concomitant in the sole tensor T, because the arbitrary directions appearing in the algebraic expression of the eigenvectors may be eliminated after differentiation.

Where these "situations of interest" appear in relativity? We see them, at least, in three subjects:

(i) *Permanence* of degenerate states of a medium. As an example, let us consider an *anisotropic* perfect fluid[36] defined by a given rheology,[37] and let S be the space of states corresponding to all admissible initial data.[38] The evolution equations for the fluid do not imply, in general, that isotropic initial data remain isotropic or, in other words, *isotropic data on an instant are not permanent*. Denoting by S_m the set of anisotropic states admitting isotropic instants, one has $S = S_m \cup S_i \cup S_a$, where S_i is the set of (permanent) isotropic states and S_a the set of purely anisotropic states (having no isotropic instants). As it has been shown elsewhere for the electromagnetic case,[39] only the space $S_i \cup \bar{S}_a$, where $\bar{S}_a$ is a *proper* subset of S_a, can be characterized by differentiable conditions. The interesting[40] and larger choices of $\bar{S}_a$ involve differential conditions on the eigenvectors of the energy tensor T (Ref. 39), and one is naturally lead to formulate them, like the conservation equations, in terms of T itself.

(ii) *Validity of the energy conjecture.* From Rainich,[41] one knows that Maxwell equations in the electromagnetic field variables may be *equivalently* formulated in terms of energy tensor variables. From the geometrical analysis by Misner and Wheeler[42] of the Rainich work, it has been conjectured that a necessary condition for a medium to be realistic is that it admits a Rainich theory[43] or, equivalently, that it could be described in terms of the sole *energy* tensor variables. The conjecture is verified for Maxwell electromagnetic fields[41] and for thermodynamic perfect fluids;[43] the study of its validity for other physical media involve, like for the above ones, a careful description of the differential properties of the eigenvectors of T in terms of T itself.

(iii) *Foliable* media. In Newtonian gravity, any arbitrary portion[44] of an indefinite medium may be matched with the vacuum (i.e., considered as a source of the gravitational field) or with portions of other media. This is not the case in relativity, where, *at most*, only a countable number of portions of a indefinite regular[45] continuous medium can be matched with the vacuum, and only few continuous families of portions may be matched with other media. The existence and parametric dimension of these families, that is, the "degree of foliability" of an indefinite relativistic medium, is related to the existence of eigenvectors of the energy tensor having the direction of the gradient of the associated eigenvalues. Our expressions for them allow us to write easily the corresponding equations in terms of the sole energy tensor.

The paper is organized as follows. For the sake of completeness, in Sec. II we present a brief discussion of the nature and multiplicity of the eigenvalues of a symmetric two-tensor T on a four-dimensional Lorentzian space in terms of the signs of three invariants, give the table of minimal equations for T, and describe a (optimal) method to determine the type of a given tensor. In Sec. III we obtain the explicit expressions for $S(g,T,v)$ and in Sec. IV those for $f(\lambda,g,T,v)$, the invariants whose sign gives the causal orientation of the eigenspaces. Finally, in Sec. V we present some simple applications: complete algebraic characterization of the perfect and anisotropic fluids, and differential characterization of tensors admitting a geodesic and a vorticity-free simple eigenvector.

A part of the results of this paper where communicated, without proof, to the Spanish relativistic annual meeting E.R.E. 86 (Ref. 46).

II. CHARACTERIZATION OF THE ALGEBRAIC TYPES

(a) Let T be a linear map on a four-dimensional real space, and N its trace-free part, $N = T - \frac{1}{4}(\mathrm{tr}\, T)I$. The characteristic polynomial of N is[28]

$$P(x) = x^4 - \frac{b}{2}x^2 - \frac{c}{3}x + \frac{1}{4}\left(\frac{b^2}{2} - d\right), \tag{1}$$

where

$$b \equiv \mathrm{tr}\, N^2, \quad c \equiv \mathrm{tr}\, N^3, \quad d \equiv \mathrm{tr}\, N^4. \tag{2}$$

The (real or complex) nature and multiplicity of the roots of (1) can be analyzed following standard methods. For our purposes it will be convenient to define the quantities

$$I_1 = I_3{}^3 - I_4{}^2, \tag{3}$$

$$I_2 = 2b - \sqrt{|I_3|}, \tag{4}$$

$$I_3 = 7b^2 - 12d, \tag{5}$$

$$I_4 = 3bI_3 + 4(3c^2 - b^3). \tag{6}$$

Only their signs will be needed in the present discussion. It is well known[47] that the roots of (1) are all different iff $I_1 \neq 0$, and they have the same or distinct nature according as $I_1 > 0$ or $I_1 < 0$, respectively. If $I_1 > 0$ then $I_2 \neq 0$ and the roots are real (resp., complex) if $I_2 > 0$ (resp., $I_2 < 0$); this follows directly taking into account that, for $I_3 > 0$, the polynomial

TABLE I. Nature and multiplicity of the roots of $P(x)$ in terms of the invariants I_A.

$I_1>0$	$I_2>0$		four different real roots.
	$I_2<0$		two different pairs of c.c. roots.
$I_1=0$	$I_2>0$	$I_3>0\cdots$	two equal and two different real roots.
		$I_3=0\cdots$	three equal roots.
	$I_2=0$	$I_3>0\cdots$	two pairs of two equal real roots.
		$I_3=0\cdots$	four equal roots.
	$I_2<0$	$I_4>0\cdots$	two equal real roots and one pair of c.c. ones.
		$I_4<0\cdots$	two equal pairs of c.c. roots.
$I_1<0$			two different real roots and one pair of c.c. ones.

$$p(u) = u^3 - bu^2 + (d - b^2/4)u - c^2/9 \tag{7}$$

has a relative maximum for $u = I_2/6$, and that $p(u) = 0$ is the Descartes' cubic resolvent[48] of (1). When $I_1 = 0$, the roots of (7) are $\Phi/6$, $\Phi/6$, and $b - \Phi/3$ with Φ given by

$$\Phi = 2b - \epsilon_4\sqrt{I_3}, \tag{8}$$

where ϵ_4 is the sign of I_4. Clearly,

$$\Phi^2(b - \Phi/3) = 4c^2, \tag{9}$$

and it results in the following lemma.

Lemma 1: If $I_1 = 0$ and $\Phi \neq 0$, the roots of $P(x)$ are $-c/\Phi$, $-c/\Phi$, $c/\Phi + \sqrt{\Phi/6}$, and $c/\Phi - \sqrt{\Phi/6}$. If $I_1 = \Phi = 0$ ($\Leftrightarrow b^2 = 4d$, $c = 0$), there are two pairs of equal roots given by $\pm\sqrt{b}/2$.

In the cases $I_1 = 0$, one has necessarily $I_3 \geqslant 0$ and further degenerations occur when I_2 or I_3 vanishes. As $I_2 > 0$ implies $\Phi > 0$, and $I_2 = 0$ implies $\Phi = 0$ and $b \geqslant 0$, from Lemma 1 it follows that all the roots are now real iff $I_2 \geqslant 0$. So, one has (i) one double root if $I_2 > 0$ and $I_3 > 0$, (ii) one triple root if $I_2 > 0$ and $I_3 = 0$, (iii) one pair of equal roots if $I_2 = 0$ and $I_3 > 0$, and (iv) one quadruple (zero) root if $I_2 = I_3 = 0$. On the other hand, if $I_2 < 0$ then $I_4 \neq 0$, and $\Phi = I_2$ or $\Phi = 0$ according as $I_4 > 0$ or $I_4 < 0$; in the first case there are a double real root and a pair of complex conjugate (c.c.) ones, and in the second one, two equal pairs of c.c. roots. These results are shown in Table I. Of course, this table may also be applied to a general quartic equation with real coefficients.

(b) From now on we deal with (real) second-order symmetric tensors on the four-dimensional Lorentzian space-time. As such tensors have at most a pair of complex conjugate eigenvalues,[2] from Table I the following results.

Lemma 2 (Plebański[3]): Let T be a symmetric two-tensor on a four-dimensional Lorentzian space. The nature and multiplicity of its eigenvalues are related to the signs of the invariants I_1, I_2, and I_3, according to Table II.

TABLE II. Nature and multiplicity of the eigenvalues of a symmetric tensor on the space-time.

$I_1>0$	Four distinct real eigenvalues		
$I_1=0$	$I_2>0$	$I_3>0$	One double real eigenvalue
		$I_3=0$	One triple real eigenvalue
	$I_2=0$	$I_3>0$	Two double real eigenvalues
		$I_3=0$	One quadruple real eigenvalue
	$I_2<0$		One double real eigenvalue and one pair of c.c. ones
$I_1<0$	Two distinct real eigenvalues and one pair of c.c. ones		

(c) From Table II, all the eigenvalues are real iff the invariants I_1 and I_2 are non-negative. This distinguishes the real types (I, II, and III) from the complex one (type IV). In order to distinguish real types between them we shall consider all the possible minimal equations. Let $m(x)$ be the minimal polynomial of N; for type I all the roots of $m(x)$ are different and for type II (resp., III) at most two (resp., three) roots of $m(x)$ are equal.

If $I_1=0$, $I_2>0$, and $I_3>0$, from (9) and Lemma 1 one has

$$m(x)=x^3-\frac{c}{\Phi}x^2+\left(\frac{c^2}{\Phi^2}-\frac{b}{2}\right)x+\frac{c}{4}\left(\frac{b}{\Phi}-1\right),$$

for type I, where, by virtue of (3), (5), and (8), Φ is given by

$$\Phi=3\{[b(4d-b^2)-4c^2]/(7b^2-12d)\}. \qquad (10)$$

One has then $m(x)=P(x)$ for type II.

If $I_1=I_2=0$ and $I_3>0$ one has $m(x)=x^2-b/4$ for type I and

$$m(x)=x^3-\frac{\epsilon}{2}\sqrt{b}\left(x^2+\frac{\epsilon}{2}\sqrt{b}x-\frac{b}{4}\right)$$

for type II, where ϵ is the sign of the eigenvalue corresponding to the null eigenvector of N.

If $I_1=I_3=0$ and $I_2>0$ ($b^3=3c^2>0$), from (8) and Lemma 1 one has

$$m(x)=x^2-(c/b)x-b/4$$

for type I, and

$$m(x)=x^3-\frac{1}{2}\left(\frac{c}{b}x^2+\frac{5}{6}bx+\frac{c}{4}\right)$$

for type II. Clearly, $m(x)=P(x)$ for type III.

If $I_1=I_2=I_3=0$ one has $m(x)=x$ for type I, $m(x)=x^2$ for type II, and $m(x)=x^3$ for type III.

All these possible minimal equations $m(N)=0$ are given in Table III and what follows.

Proposition 1: Let N be a trace-free symmetric tensor of real type. The minimal polynomial of N is the first (on the left) of the equations that it verifies in the row of Table III selected by the signs of its invariants I_1, I_2, I_3. The type of N is then that indicated in the corresponding column.

It is to be noted that Proposition 1 characterizes intrinsically the subtype of T, and that the causal character of the eigenspace associated to a double or a triple eigenvalue of a tensor of type I cannot be obtained from this proposition.

(d) Let τ be the greatest multiplicity of the roots of the minimal polynomial of T. One has $\tau=1$ for types I and IV, $\tau=2$ for type II, and $\tau=3$ for type III. Thus, the value of τ distinguishes real types among them. From

TABLE III. Determination of the algebraic type of a symmetric tensor on the space-time (Proposition 1).

$I_1>0$		$N^4-\frac{b}{2}N^2-\frac{c}{3}N+\frac{1}{4}\left(\frac{b^2}{2}-d\right)g=0$ $(\equiv P(N)=0)$		
$I_1=0$	$I_2>0,\ I_3>0$	$N^3-\frac{c}{\Phi}N^2+\left(\frac{c^2}{\Phi^2}-\frac{b}{2}\right)N+\frac{c}{4}\left(\frac{b}{\Phi}-1\right)g=0$	$P(N)=0$	
	$I_2=0,\ I_3>0$	$N^2-(b/4)g=0$	$N^3-\frac{\epsilon}{2}\sqrt{b}\left(N^2+\frac{\epsilon}{2}\sqrt{b}N-\frac{b}{4}g\right)=0$	
	$I_2>0,\ I_3=0$	$N^2-(c/b)N-(b/4)g=0$	$N^3-\frac{1}{2}\left(\frac{c}{b}N^2+\frac{5}{6}bN+\frac{c}{4}g\right)=0$	$P(N)=0$
	$I_2=I_3=0$	$N=0$	$N^2=0$	$N^3=0$
Type		I	II	III

Table III, and taking into account that $I_1 > 0$ implies $I_2 > 0$ and $I_3 > 0$, we obtain the following proposition.

Proposition 2: The greatest multiplicity τ of the roots of the minimal polynomial of a symmetric tensor of real type is given by

$$\tau = k - v, \tag{11}$$

where k is the order of its minimal equation and v is the number of the invariants I_1, I_2, I_3, which are positive.

Accordingly, the type of T is known if we know k and the signs of I_1, I_2, I_3. Clearly, iff T has type IV, one of the invariants I_1 or I_2 is negative (then $k=4$ or $k=3$ according as $I_1 < 0$ or $I_1 = 0$, respectively). If T has a real type, one needs to know the value of k to obtain its type. We shall develop an algorithm to calculate k for tensors of real type.

If A and B are two-tensors and $\mathrm{tr}\,A \neq 0$, it is convenient to consider the trace-free part of B with respect to A, $Q_A B$, defined by[49]

$$Q_A B \equiv B - (\mathrm{tr}\,B/\mathrm{tr}\,A)A.$$

Putting $Q \equiv Q_g$, one has $N = QT$ and it results $k=1$ or $k=2$ according as $N=0$ or $N \neq 0$ and $QN^2 \wedge N = 0$, respectively, $\wedge$ denoting the exterior product of tensors considered as elements (vectors) of its linear space structure.

On the other hand, real types with $b=0$ have necessarily a quadruple eigenvalue, and so, the only real case with $k>2$ and $b=0$ is type III with minimal equation $N^3 = 0$. Next, consider $k>2$ and $b \neq 0$ (necessarily $b>0$ for real types); if $b^2 = 2d$ (i.e., if N is singular) one has $k=3$ iff $N^3 \wedge N^2 \wedge N = 0$, which is equivalent to

$$Q_{N^2} N^3 \wedge N = 0. \tag{12}$$

If $b^2 \neq 2d$ then $N^3 \wedge N^2 \wedge N \neq 0$, and one has $k=3$ iff $N^3 \wedge N^2 \wedge N \wedge g = 0$, which is equivalent to

$$Q_{N^2}(N \times QN^3) \wedge Q_{N^2}(N \times QN^2) = 0, \tag{13}$$

due to the fact that N is regular and $QN^3 \wedge QN^2 \wedge N = 0$. Thus, we have the following result.

Proposition 3: The order k of the minimal equation of a symmetric tensor of real type is given by Table IV.

TABLE IV. Algorithm for obtaining the order k of the minimal equation of a symmetric tensor of real type.

N	$(=0) \Rightarrow$		$k=1$
$(\neq 0)$			
$\downarrow$			
$QN^2 \wedge N$	$(=0) \Rightarrow$		$k=2$
$(\neq 0)$			
$\downarrow$			
$\mathrm{tr}\,N^2$	$(=0) \Rightarrow$		$k=3$
$(\neq 0)$			
$\downarrow$			
$\mathrm{tr}^2 N^2 - 2\,\mathrm{tr}\,N^4$	$(=0) \rightarrow$	$Q_{N^2}N^3 \wedge N$ $(=0) \Rightarrow$	$k=3$
$(\neq 0)$		$(\neq 0) \Rightarrow$	$k=4$
$\downarrow$			
$Q_{N^2}(N \times QN^3)$	$(=0) \Rightarrow$		$k=3$
$\wedge Q_{N^2}(N \times QN^2)$			
$(\neq 0) \Rightarrow$			$k=4$

III. COVARIANT DETERMINATION OF EIGENSPACES

(a) From the minimal polynomial $m(x)$ and for each eigenvalue λ of T, let us consider the polynomial

$$p_\lambda(x) = m(x)/(x-\lambda), \tag{14}$$

and let T_λ be the (nonzero) tensor, defined by

$$T_\lambda \equiv p_\lambda(T), \tag{15}$$

in terms of which the minimal equation of T can be written as

$$m(T) = (T - \lambda g)T_\lambda = 0. \tag{16}$$

Thus, T_λ is an *eigentensor* of T that commutes with it:

$$TT_\lambda = T_\lambda T = \lambda T_\lambda, \tag{17}$$

and from (16) it results directly.

Proposition 4: The image of T_λ as a linear map, is contained into the eigenspace E_λ of eigenvalue λ: $\mathrm{Im}\,T_\lambda \subset E_\lambda$.

It is then natural to ask for the cases where $E_\lambda = \mathrm{Im}\,T_\lambda$. Let us begin considering some remarkable properties of the T_λ's. From (15)–(17) one has

$$T_\lambda{}^2 = p_\lambda(T)T_\lambda = p_\lambda(\lambda)T_\lambda \tag{18}$$

and

$$T_\lambda T_\mu = 0 \tag{19}$$

for any pair of distinct eigenvalues λ and μ.

On the other hand, (14) and (15) can be written as

$$p_\lambda(x) = \sum_{p=0}^{k-1} \alpha_p x^p, \quad T_\lambda = \sum_{p=0}^{k-1} \alpha_p T^p,$$

where, in general, the coefficients α_p's are complex, and hence

$$\mathrm{tr}\,T_\lambda = \sum_{p=0}^{k-1} \alpha_p\,\mathrm{tr}\,T^p = \sum_{p=0}^{k-1} \alpha_p \sum_\mu m_\mu \mu^p$$

$$= \sum_\mu m_\mu p_\lambda(\mu)$$

being m_μ, the multiplicity of the eigenvalue μ of T. From (14), $p_\lambda(\mu) = 0$ for every $\mu \neq \lambda$ and then

$$\operatorname{tr} T_\lambda = m_\lambda p_\lambda(\lambda) \tag{20}$$

and (18) leads to

$$T_\lambda^2 = (\operatorname{tr} T_\lambda / m_\lambda) T_\lambda. \tag{21}$$

Let n_λ be the multiplicity of λ as root of the minimal polynomial ($1 < n_\lambda < m_\lambda$); Eq. (14) may be written as

$$p_\lambda(x) = (x-\lambda)^{n_\lambda - 1} \prod_{\mu \neq \lambda} (x-\mu)^{n_\mu}, \tag{22}$$

and the following results.

Proposition 5: The eigentensor T_λ has nonzero trace if, and only if, λ is a single root of the minimal polynomial.

On the other hand, if $n_\lambda = 1$, from (15), (20), and (22) one has

$$T_\lambda(s_\lambda) = \prod_{\mu \neq \lambda} (\lambda - \mu)^{n_\mu} s_\lambda = p_\lambda(\lambda) s_\lambda = \frac{\operatorname{tr} T_\lambda}{m_\lambda} s_\lambda \neq 0$$

for every eigenvector s_λ corresponding to λ, that is $E_\lambda \subset \operatorname{Im} T_\lambda$; taking into account Proposition 4, we have the following.

Proposition 6: For every eigenvalue λ with $n_\lambda = 1$ one has $E_\lambda = \operatorname{Im} T_\lambda$.

Suppose now that T is of type II or III; from Proposition 6, each eigentensor T_λ with $n_\lambda = 1$ generates the corresponding E_λ, but if $n_\lambda > 1$, from Proposition 5 and Eq. (21) it results

$$g(T_\lambda u, T_\lambda v) = T_\lambda^2(u,v) = 0$$

for arbitrary vectors u and v. Taking into account Proposition 6 we have

Proposition 7: (i) For every tensor T of type I or IV and for every eigenvalue λ, the action of T_λ on arbitrary directions generates all the directions of the eigenspace E_λ. (ii) For every T of type II or III and for the eigenvalue λ such that $n_\lambda > 1$, the action of T_λ on arbitrary directions gives the null direction of the eigenspace E_λ.

(b) We will now consider separately every one of the four algebraic types, and obtain explicitly the eigentensors T_λ of T. They will be expressed in terms of $N \equiv QT$, simpler to compute. For tensors of type II and III with $m_\lambda > n_\lambda > 1$ the image of the corresponding T_λ only provides the null eigendirection of T, so that calculation of the associated eigenspace will be needed to construct additional eigentensors: those generating the ternary forms ($p=1$) indicated in the Introduction.

Type I. If $I_1 > 0$ then $m(x) = P(x)$ and the eigentensor associated to the eigenvalue ν of N is $N_\nu = p_\nu(N)$ with

$$p_\nu(x) = \frac{m(x)}{x-\nu} = x^3 + \nu x^2 + \left(\nu^2 - \frac{b}{2}\right) x + \nu\left(\nu^2 - \frac{b}{2}\right) - \frac{c}{3}.$$

If $I_1 = 0, I_2 > 0$, and $I_3 > 0$, the eigentensor associated to the the double eigenvalue ν_d is $N_d = (N - \nu_+ g) \times (N - \nu_- g)$ and those associated to the single eigenvalues $\nu_\pm$ are $N_\pm = (N - \nu_d g)(N - \nu_\mp g)$.

If $I_1 = I_2 = 0$ and $I_3 > 0$, the eigentensors associated to the pair of double eigenvalues ($\pm\sqrt{b}/2$) are now $N_\pm = N \pm \sqrt{b}/2\, g$.

If $I_1 = I_3 = 0$ and $I_2 > 0$, denoting by ν_s the single eigenvalue and by ν_t the triple one, the associated eigentensors are given by $N_s = N - \nu_t g$ and $N_t = N - \nu_s g$, respectively.

If $I_1 = I_2 = I_3 = 0$, the eigentensor is g.

Thus, in account of Lemma 1 and Eqs. (8)–(10), and denoting by $i(v)$ the interior product by the vector v (Ref. 50), we have the following theorem.

Theorem 1: The eigenvectors s of a symmetric tensor T of type I are *linear* forms in *one* arbitrary direction v, $s = i(v) T_\lambda$, where the T_λ's are the second-order eigentensors given in Table V.

Note that $T_\lambda = N_\nu$, N being the trace-free part of T, $N = T - \operatorname{tr} T/4$, and $\nu = \lambda - \operatorname{tr} T/4$. In explicit computations it is easier to evaluate N_ν in terms of N: This is the reason for the expressions given in Table V.

Type II. Now $I_1 = 0$, and there are only three linearly independent eigenvectors: one null (from now on denoted by l) and two spacelike. Let T_l be the eigentensor giving l.

If $I_2 > 0$ and $I_3 > 0$, then $T_l = (N - \nu_d g)(N - \nu_+ g)(N - \nu_- g)$, and the eigentensors giving the spacelike eigenvectors are $T_\pm = (N - \nu_d g)^2 (N - \nu_\mp g)$.

If $I_2 = 0$ and $I_3 > 0$, then $T_l = N^2 - b/4g$, and the eigentensor giving the spacelike two-eigenplane is $T_\epsilon = (N + \epsilon\sqrt{b}/2\, g)(N - \epsilon\sqrt{b}/2\, g)^2$, ϵ being the sign of the eigenvalue corresponding to l. Note that if the algebraic type of N has been obtained using Proposition 1 then ϵ is known from the minimal equation of N. Otherwise, if the type has been obtained using Proposition 2 (with $k=3$ given by Table IV) then ϵ must be calculated from $N(l) = \epsilon\sqrt{b}/2l$, taking into account that $\{l\}$ is the image of T_l.

If $I_2 > 0$ and $I_3 = 0$, then $T_l = (N - \nu_s g)(N - \nu_t g)$, and the eigentensor giving the spacelike eigendirection $\{e\}$ is $T_e = (N - \nu_t g)^2$. The eigenspace associated to the triple eigenvalue ν_t is the null two-plane orthogonal to l and e, and it is given by $*(l \wedge e)$, $*$ being the Hodge operator and $\wedge$ denoting the exterior product. The eigenvectors s_t are then given by $i(v)*(l \wedge e)$, where

TABLE V. Second-order eigentensors of a tensor T of type I (Theorem 1). Φ is given by Eq. (10).

Invariants		type I	Eigenvalues $\lambda = \nu + \mathrm{tr}\, T/4$
$I_1 > 0$		$T_\lambda = N^3 + \nu N^2 + \left(\nu^2 - \frac{b}{2}\right) N + \left[\nu\left(\nu^2 - \frac{b}{2}\right) - \frac{c}{3}\right] g$	λ = any eigenvalue
$I_1 = 0$	$I_2 > 0,\ I_3 > 0$	$T_d = N^2 - 2\frac{c}{\Phi} N + \left(\frac{3c^2}{\Phi^2} - \frac{b}{2}\right) g$	λ_d = double eigenvalue
		$T_\pm = N^2 \pm \sqrt{\frac{\Phi}{6}}\, N - \frac{c}{\Phi}\left(\frac{c}{\Phi} \mp \sqrt{\frac{\Phi}{6}}\right) g$	$\lambda_\pm$ = simple eigenvalues
	$I_2 = 0,\ I_3 > 0$	$T_\pm = N \pm \frac{1}{2}\sqrt{b}\, g$	$\lambda_\pm$ = doubles eigenvalues
	$I_2 > 0,\ I_3 = 0$	$T_t = N - \frac{3}{2}(c/b) g,\quad T_s = N + \frac{1}{2}(c/b) g$	λ_t = triple eigenvalue λ_s = simple eigenvalue
	$I_2 = I_3 = 0$	$T_\lambda = g$	λ = quadruple eigenvalue

$l = i(v)T_l$ and $e = i(v)T_e$, for arbitrary v.

If $I_2 = I_3 = 0$, then $N_l = N$ and the eigenspace associated is now the null three-plane given by $*l$.

Thus, in account of Lemma 1 and Eqs. (8)–(10), we have the following.

Theorem 2: (i) The null eigenvector l of a symmetric tensor T of type II, and the spacelike eigenvectors not contained in the null eigenspace, are *linear* forms in *one* arbitrary direction v, $s = i(v)T_\lambda$, where the T_λ's are the second-order eigentensors given in Table VI. (ii) The spacelike eigenvectors s contained in the null eigenspace, which appears in the cases $I_3 = 0$, are *cubic* forms in *one* arbitrary direction v when $I_2 > 0$, and in *two* arbitrary directions v, u when $I_2 = 0$; they are given by

$$s = i(v)*\{i(v)N \wedge i(v)N^2\}$$

and

$$s = i(v)i(u)*i(v)N,$$

respectively.

TABLE VI. Second-order eigentensors of a tensor T of type II (Theorem 2). Φ is given by Eq. (10) and ϵ is the sign of the eigenvalue of N associated to the null eigenvector l given by the eigentensor T_l. The spacelike eigenvectors not contained in the null eigenspace are given by T_e.

Invariants		Type II
$I_1 = 0$	$I_2 > 0$	$T_l = N^3 - \frac{c}{\Phi} N^2 + \left(\frac{c^2}{\Phi^2} - \frac{b}{2}\right) N + \frac{c}{4}\left(\frac{b}{\Phi} - 1\right) g$
	$I_3 > 0$	$T_{e\pm} = N^3 + \left(\frac{c}{\Phi} \pm \sqrt{\Phi/6}\right) N^2 - \frac{c}{\Phi}\left(\frac{c}{\Phi} \mp 2\sqrt{\Phi/6}\right) N - \frac{c^2}{\Phi^2}\left(\frac{c}{\Phi} \mp \sqrt{\Phi/6}\right) g$
	$I_2 = 0$ $I_3 > 0$	$T_l = N^2 - (b/4) g$ $T_e = N^2 - \epsilon\sqrt{b}\, N + (b/4) g$
	$I_2 > 0$ $I_3 = 0$	$T_l = N^2 - (c/b) N - (b/4) g$ $T_e = N^2 + (c/b) N + (b/12) g$
	$I_2 = 0$ $I_3 = 0$	$T_l = N$

Type III. There are now only two linearly independent eigenvectors: one null, l, and one spacelike, e. Let T_l be the eigentensor giving l. In this case it is $I_1 = I_3 = 0$, and there are two cases, according to $I_2 > 0$ or $I_2 = 0$. If $I_2 > 0$, then $T_l = (N - \nu_s\, g)(N - \nu_t\, g)^2$, and the eigentensor giving the spacelike eigenvector e is $T_e = (N - \nu_t\, g)^3$. If $I_2 = 0$, then $T_l = N^2$; in this case we can write $N = l \otimes p + p \otimes l$ where p is a spacelike vector orthogonal to the null two-eigenplane II_l associated to the quadruple eigenvalue. One has then $i(v)N = g(v,l)p$

TABLE VII. Second-order eigentensors of a tensor T of type III (Theorem 3). The null eigenvector l is given by the eigentensor T_l. The spacelike eigenvectors not contained in the null eigenspace are given by T_e.

Invariants		Type III
$I_1 = 0$	$I_2 > 0$	$T_l = N^3 - \frac{1}{2}\left(\frac{c}{b} N^2 + \frac{5}{6} bN + \frac{c}{4} g\right)$
	$I_3 = 0$	$T_e = N^3 + \frac{1}{4}\left(6\frac{c}{b} N^2 + bN + \frac{c}{6} g\right)$
	$I_2 = 0$ $I_3 = 0$	$T_l = N^2$

$+g(v,p)l$: Π_l is the two-plane othogonal to $i(v)N$ and $i(v)N^2$ for any timelike vector v. Thus, we have the following theorem.

Theorem 3: (i) The null eigenvector l of a symmetric tensor T of type III, as well as the spacelike one e when $I_2 > 0$, are *linear* forms in *one* arbitrary direction v, $s = i(v)T_\lambda$, where the T_λ's are the second-order eigentensors given in Table VII. (ii) The spacelike eigenvectors s of the null eigenplane, corresponding to $I_2 = 0$, are *cubic* forms in *one* arbitrary direction v; they are given by

$$s=i(v)*\{i(v)N\wedge i(v)N^2\}.$$

Type IV. This type corresponds to a "partially complex type I," so that it is easy to show the following.

Theorem 4: The (spacelike) eigenvectors of a symmetric tensor T of type IV are *linear* forms in *one* arbitrary direction v, $s = i(v)T_\lambda$, where the T_λ's are the second-order eigentensors,

$$T_\nu=N^3+\nu N^2+\left(\nu^2-\frac{b}{2}\right)N+\left[\nu\left(\nu^2-\frac{b}{2}\right)-\frac{c}{3}\right]g$$

for the single real eigenvalue ν, and

$$T_d=N^2-\frac{2c}{I_2}N+\left(3\left(\frac{c}{I_2}\right)^2-\frac{b}{2}\right)g$$

for the spacelike eigenplane when $I_1 = 0, I_2 < 0$.

IV. CAUSAL CHARACTER OF THE EIGENSPACES

(a) The above covariant method to obtain the eigenvectors allows us to construct a simple criterion to find the causal character of the eigenspaces. For tensors of type II and III, such a character follows directly from the preceding results: the eigenspace E_λ corresponding to the eigenvalue λ with $n_\lambda > 1$ is null, and the eigenspaces E_μ with $\mu\neq\lambda$ are necessarily spacelike. Thus, taking into account Proposition 5 we have the following theorem.

Theorem 5: If T is a tensor of type II or III the eigenspace E_λ is null or spacelike according as $\mathrm{tr}\, T_\lambda = 0$ or $\mathrm{tr}\, T_\lambda\neq 0$, respectively.

Tensors of type I require a more careful treatment. Consider first the case of a single eigenvalue λ: the corresponding eigendirection $\{v_\lambda\}$ is given by $T_\lambda x = \alpha_x v_\lambda$, for arbitrary x, where α_x is a real factor depending on x. Taking into account Eq. (21) and Proposition 5 one has

$$T_\lambda(x,x)=[g(v_\lambda,v_\lambda)/\mathrm{tr}\, T_\lambda]\alpha_x^2, \tag{23}$$

that is, the sign of $T_\lambda(x,x)$ is independent of x and gives the causal character of v_λ. In fact, remembering that the *Riemannian trace*, $\mathrm{tr}_u A$, of the two-tensor A (with respect to the unit timelike vector u) is defined by

$$\mathrm{tr}_u A\equiv 2A(u,u)+\epsilon_\sigma\,\mathrm{tr}\, A, \tag{24}$$

where ϵ_σ is the sign of the signature of g, in an orthonormal basis $\{e_a\}_{a=0}^3$ it follows that

$$\mathrm{tr}_{e_0} A=\sum_{a=0}^{3} A(e_a,e_a), \tag{25}$$

thus we get the following results.

Proposition 8: For the single eigenvalues λ of a tensor of type I, the sign of $\mathrm{tr}_u\, T_\lambda$ is independent of u.

Theorem 6: The eigenvector associated to a single eigenvalue λ of a tensor of type I is *spacelike* (resp., *timelike*) if, and only if, $\epsilon_\sigma\,\mathrm{tr}\, T_\lambda\,\mathrm{tr}_u\, T_\lambda$ is *positive* (resp., *negative*).

Consider the case of a double eigenvalue of a tensor of type I. Let $\mathscr{G}\equiv\frac{1}{2}\, g\wedge g$ be the Cartan metric induced by g on the bivector space and denote by F any simple bivector associated to the two-eigenplane E_λ; then E_λ is spacelike (resp., timelike) iff $\mathscr{G}(F,F)$ is positive (resp., negative). Consider now the following double two-form $\mathscr{T}_\lambda$ (Ref. 14), constructed from the eigentensor T_λ:

$$\mathscr{T}_\lambda\equiv T_\lambda\wedge T_\lambda. \tag{26}$$

From Eq. (21) it follows[51] that

$$\mathscr{T}_\lambda\times\mathscr{T}_\lambda=\tfrac{1}{2}(\mathrm{tr}\, T_\lambda)^2\mathscr{T}_\lambda. \tag{27}$$

For arbitrary simple bivectors X (and considerig $\mathscr{T}_\lambda$ as a linear map on the bivector space) $F_X\equiv\mathscr{T}_\lambda(X)$ is, when nonzero, simple and related to E_λ. Taking into account that

$$\mathscr{G}(F_X,F_X)=\tfrac{1}{2}(\mathrm{tr}\, T_\lambda)^2\mathscr{T}_\lambda(X,X),$$

the following proposition results.

Proposition 9: For a double eigenvalue λ of a tensor of type I, the sign of $\mathscr{T}_\lambda(X,X)$, when nonzero, is independent of X.

When λ is a triple eigenvalue of a tensor of type I, the causal character of the three-eigenplane E_λ is obtained by complementarity from Proposition 8, and one has finally the following theorem.

Theorem 7: For a tensor of type I, (i) The two-eigenplane associated to a double eigenvalue λ is *spacelike* (resp., *timelike*) if, and only if, $\mathscr{T}_\lambda(X,X)$ is *positive* (resp., *negative*) where $\mathscr{T}_\lambda\equiv T_\lambda\wedge T_\lambda$, and T_λ is the eigentensor associated to λ. (ii) The three-eigenplane associated to a triple eigenvalue is *spacelike* (resp., *timelike*) if, and only if, $\epsilon_\sigma\,\mathrm{tr}\, T_\mu\,\mathrm{tr}_u\, T_\mu$ is *negative* (resp., *positive*), where μ is the single eigenvalue of T.

V. APPLICATIONS

(a) The above results allows us to directly classify algebraically a *given* tensor. These results are also useful in the reciprocal problem: To obtain the complete algebraic characterization of *classes* of tensors admitting a given general form.

As an example, we shall characterize the tensors whose Segre[11] characteristic is $\{1,(111)\}$, that is, those of the form

$$T=(\rho+p)u\otimes u+\epsilon_\sigma pg,$$

where u is a unit timelike vector, $u^2=-\epsilon_\sigma$. They represent (Pascalian, heat flow-free) *perfect fluids* with unit velocity u, proper energy density ρ, and pressure p.

From Table III, tensors of type I admitting a strict triple eigenvalue are characterized by the minimal equation $N^2=c/bN+b/4g$ with $c\neq0$ and $b>0$, and the following lemma follows.

Lemma 3: (Taub's lemma[52]). A symmetric tensor T is of type I and admits a strict triple eigenvalue iff, it satisfies the relations

$$QT^2=\chi QT,\quad 4\,\mathrm{tr}\,T^2>(\mathrm{tr}\,T)^2,$$

with $2\chi\neq\mathrm{tr}\,T$, Q being the trace-removing operator.

In order to ensure that the single eigenvector u is timelike, we have to consider the eigentensor T_s associated to the simple eigenvalue λ_s of T (see Table V). Because $\mathrm{tr}\,T_s=\lambda_s-\lambda_t=2c/b$, from Theorem 6 one has that u is timelike iff $\epsilon\epsilon_\sigma\,\mathrm{tr}_x\,T_s<0$, where ϵ is the sign of c and x is an arbitrary unit timelike vector. Taking into account that $b^3=3c^2$ and $\chi=\lambda_s+\lambda_t=\mathrm{tr}\,T/2+c/b$, we have the following result.

Theorem 8: In a space-time with signature $2\epsilon_\sigma$, a symmetric tensor T defines algebraically a perfect fluid if, and only if, it satisfies

$$(i)\ \ QT^2=\chi QT,\quad 2\chi\neq\mathrm{tr}\,T,\quad 4\,\mathrm{tr}\,T^2>(\mathrm{tr}\,T)^2,$$

$$(ii)\ \ (2\chi-\mathrm{tr}\,T)(2\epsilon_\sigma T(x,x)+\chi)<0,$$

for any unit timelike vector x.

From Lemma 1, the eigenvalues are given by $\lambda_s=(3c/b+\mathrm{tr}\,T/2)/2$, and $\lambda_t=(-c/b+\mathrm{tr}\,T/2)/2$, and taking into account the expression for T_s the following results.

Proposition 10: The energy density ρ, the pressure p, and the direction of the unit velocity u of a perfect fluid energy tensor T are given by

$$\rho=\tfrac{1}{2}\epsilon_\sigma(\mathrm{tr}\,T-3\chi),\quad p=\tfrac{1}{2}\epsilon_\sigma(\mathrm{tr}\,T-\chi)$$

and

$$u\propto i(x)T-\epsilon_\sigma px,$$

where

$$\chi=\tfrac{1}{2}\left(\mathrm{tr}\,T+\epsilon\sqrt{\tfrac{1}{3}(4\,\mathrm{tr}\,T^2-(\mathrm{tr}\,T)^2)}\right),$$

ϵ being the sign of $\mathrm{tr}\,T^3-6\,\mathrm{tr}\,T\,\mathrm{tr}\,T^2+8\,\mathrm{tr}\,T^3$.

One may also characterize the class of tensors admitting a timelike three-eigenplane, that is, those of Segre characteristic $\{(1,11)1\}$. In this case the expression of condition (ii) in Theorem 8 must be positive for an arbitrary unit timelike vector. Neutrino energy tensors N of type I satisfying $N(u,u)\neq0$, for any observer u, and whose (geodesic) principal null congruence has shear σ and twist ω restricted by $|\sigma|^2=4\omega^2\neq0$, belong to this class.[53]

For macroscopic matter one assumes generally the *Plebański energy conditions,*[54] which state that, for any observer, the energy density is non-negative and the Poynting vector is nonspacelike. For perfect fluids these conditions are equivalent to the inequalities $-\rho<p\leqslant\rho$, that is, $0\geqslant2\epsilon_\sigma\chi<\epsilon_\sigma\,\mathrm{tr}\,T$, and thus, condition (ii) in Theorem 8 reduces to $2T(x,x)+\epsilon_\sigma\chi>0$.

These results have been considered elsewhere[43] for the construction of the Rainich theory of the thermodynamic perfect fluid.

(b) As another example, we shall now characterize intrinsically the energy tensors describing an anisotropic perfect fluid[36] with two equal pressures. Their Segre characteristic, $\{1,1(11)\}$, has to be distinguished from the $\{(1,1)11\}$, corresponding to energy tensors of certain classes of fermionic fields.[55] From Table III, these tensors satisfy a minimal equation of the form $Q(N^3-\kappa N^2)=(b/2-\kappa^2)N$, and the characteristic equation implies $(b(4d-b^2)-4c^2)\kappa=cI_3/3$. Taking into account Eqs. (3)–(6), and Theorems 6 and 7, we have the following results.

Lemma 4: A symmetric tensor T is of type I and admits a strict double eigenvalue iff its trace-free part $N\equiv QT$ satisfies

$$Q\{N^3-\kappa N^2+(\kappa^2-b/2)N\}=0$$

and

$$b>0,\quad 7b^2>12d>3(b^2+4c^2/b),$$

where $b\equiv\mathrm{tr}\,N^2$, $c\equiv\mathrm{tr}\,N^3$, $d\equiv\mathrm{tr}\,N^4$, $\kappa\equiv c/\Phi$, Φ being given by Eq. (10).

Theorem 9: A symmetric tensor T of type I with a strict double eigenvalue is algebraically the energy tensor of an anisotropic perfect fluid with two equal pressures iff one of the two following equivalent conditions occurs:

$$(a)\quad tr\,T_+\,\mathrm{tr}\,T_-\,\mathrm{tr}_x\,T_+\,\mathrm{tr}_x\,T_-<0,$$

$$(b)\quad (T_d\wedge T_d)(X,X)>0,$$

where x is any unit timelike vector, X any arbitrary simple bivector, and T_+, T_-, and T_d are the concomitants of T given in Table V.

The characterization of Segre class $\{(1,1)11\}$ is similar to that of the above theorem, but the inequalities (a) and (b) are inverted.

Note that the previous statement is independent of the sign of the space-time metric signature ϵ_σ. However, ϵ_σ appears in the characterization of anisotropic perfect fluids when they satisfy the Plebański conditions. In this case these conditions are $\epsilon_\sigma(\lambda_0 + \lambda_i) < 0$ and $\epsilon_\sigma(\lambda_0 - \lambda_i) < 0$ ($i=1,2$), λ_0 and λ_1 being the single eigenvalues associated to the timelike and the spacelike directions, respectively, and λ_2 is now the double one. Hence, from Lemma 1 we have $\lambda_0 = c/\Phi - \epsilon_\sigma\sqrt{\Phi/6} + \text{tr}\, T/4$, and the condition (a) in Theorem 9 may be simplified. The following theorem results.

Theorem 10: In a space-time with metric g of signature $2\epsilon_\sigma$, a symmetric two-tensor defines algebraically an anisotropic perfect fluid with two equal pressures subject to the Plebański energy conditions iff its trace-free part $N=\mathbf{Q}T$ satisfies the conditions of Lemma 4 and

$$\epsilon_\sigma(4\kappa+\text{tr}\, T)<0, \quad 2\epsilon_\sigma\, \kappa<\sqrt{\Phi/6}>\epsilon_\sigma \text{tr}\, T/2,$$

$$\epsilon_\sigma \text{tr}\, T_0\, \text{tr}_x\, T_0<0,$$

where x is any unit timelike vector and T_0 is defined by

$$T_0 \equiv N^2-\epsilon_\sigma\sqrt{\Phi/6}N-\kappa(\kappa+\epsilon_\sigma\sqrt{\Phi/6})g.$$

Denoting by ρ the proper energy density, by p_1 the anisotropic pressure, by u the unit velocity, and by e the unit vector along the anisotropy of T, and taking into account Theorem 1, the following proposition results.

Proposition 11: The physical variables of an anisotropic perfect fluid tensor T with two equal pressures and satisfying the Plebański conditions are given by

$$\rho=\sqrt{\Phi/6}-\epsilon_\sigma(\alpha+\kappa),$$

$$p_1=\sqrt{\Phi/6}+\epsilon_\sigma(\alpha+\kappa),$$

$$p_2=\epsilon_\sigma(\alpha-\kappa),$$

$$u \propto i(v)\{N^2-\epsilon_\sigma\sqrt{\Phi/6}N-\kappa(\kappa+\epsilon_\sigma\sqrt{\Phi/6})g\},$$

$$e \propto i(v)\{N^2+\epsilon_\sigma\sqrt{\Phi/6}N-\kappa(\kappa-\epsilon_\sigma\sqrt{\Phi/6})g\},$$

where v is an arbitrary direction, and $\alpha \equiv \text{tr}\, T/4$.

When Plebański conditions are assumed, the sum of two perfect fluid energy tensors is a tensor of Segre characteristic $\{1,1(11)\}$ (Ref. 56). Its intrinsic characterization has been also obtained[57] using results of this paper.

(c) As we mentioned in the Introduction, in some physical situations one has to known, *in terms of the sole tensor* T, the differential conditions verified by some of its eigenvectors. Let us begin by considering what (necessary and sufficient) conditions a tensor T must verify in order that it admits a geodesic single eigenvector s, $i(s)\nabla s\wedge s=0$. According to Proposition 7, single eigenvectors are all given by linear forms in an arbitrary direction x, $s=i(x)T_\lambda$, where T_λ is the corresponding concomitant of T (the eigentensor associated to eigenvalue of s). Thus, in any local frame $\{e_a\}$, the geodesic equation may be written as

$$x^p(T_\lambda)^s_p\nabla_s(x^q(T_\lambda)_{q[a})(T_\lambda)_{b]r}x^r=0,$$

where the bracket on the indices indicate antisymmetrization. T_λ being of rank 1, $T_\lambda\otimes T_\lambda$ is totally symmetric, and the expansion of the above expression gives

$$x^px^qx^r(T_\lambda)^s_p\nabla_s(T_\lambda)_{q[a}(T_\lambda)_{b]r}=0. \tag{28}$$

But the tensor $\mathscr{Z}_\lambda$ whose components are

$$(\mathscr{Z}_\lambda)_{ab,cde}\equiv(T_\lambda)^s_c\nabla_s(T_\lambda)_{d[a}(T_\lambda)_{b]e}$$

is symmetric under permutations of the three indices (cde), as it follows by covariant derivation of $T_\lambda\otimes T_\lambda$, so that Eq. (28) vanishes iff $\mathscr{Z}_\lambda=0$. Now on account of the rank of T_λ and integrating by parts, $\mathscr{Z}_\lambda$ may be written as

$$(\mathscr{Z}_\lambda)_{ab,cde}=-\nabla_s(T_\lambda)^s_{[a}(T_\lambda)_{b]c}(T_\lambda)_{de},$$

and we thus have the following.

Theorem 11: A single eigenvector s corresponding to the eigenvalue λ of a symmetric tensor T is geodesic if, and only if, the eigentensor T_λ verifies

$$\delta T_\lambda\wedge T_\lambda=0,$$

where δ is the divergence operator and $\wedge$ is the exterior product on vector-valued one-forms.

Suppose now s is vorticity-free, $v\wedge dv=0$. In a local frame we have

$$\oint_{abc} x^p(T_\lambda)_{pa}[\nabla_b(x^q(T_\lambda)_{qc})-\nabla_c(x^q(T_\lambda)_{qb})]=0,$$

where $\oint$ denotes cyclic sum. In this expression two terms appear:

$$\text{I:}\quad x^px^q\oint_{abc}(T_\lambda)_{pa}[\nabla_b(T_\lambda)_{qc}-\nabla_c(T_\lambda)_{qb}],$$

$$\text{II:}\quad x^p\nabla_mx^q\oint_{abc}(T_\lambda)_{pa}[\delta^m_b(T_\lambda)_{qc}-\delta^m_c(T_\lambda)_{qb}].$$

Because of the rank of T_λ, II vanishes and the tensor $\mathscr{W}_\lambda$ having components

$$(\mathscr{W}_\lambda)_{abc,de} \equiv \oint_{abc} (T_\lambda)_{da}[\nabla_b(T_\lambda)_{ec} - \nabla_c(T_\lambda)_{eb}]$$

is symmetric in its two last indices. Consequently, s is integrable iff $\mathscr{W}_\lambda$ vanishes, and the following theorem results.

Theorem 12: A single eigenvector s associated to the eigenvalue λ of a symmetric tensor T is vorticity-free if, and only if, the eigentensor T_λ verifies

$$T_\lambda \wedge \mathbf{d}T_\lambda = 0,$$

where $\wedge$ and $\mathbf{d}$ are, respectively, the exterior product and the exterior differential acting on vector-valued forms.

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[27] See the references quoted in Refs. 6 and 12.
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[29] That is, the n invariants given by the traces of the first n powers of T in dimension n.
[30] For the notion of *concomitant* see, for instance, J. A. Schouten in Ref. 32, p. 15, or G. B. Gurewich in Ref. 28, p. 140.
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[32] A hint, but not a proof, of that is the fact that, for an even-order tensor, the elementary proceses of addition, isomerism, multiplication, and contraction (see Ref. 32, p. 15), do not allow us to obtain an odd-order tensor.
[33] That is to say, in an arbitrary frame, the eigenvector s^a is obtained from a $(p+1,p+1)$-tensor $S^{\alpha\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_{p+1}}$ by saturation of the μ's and ν's indices by i arbitrary vectors $v^\mu_{(a)}$, $1 \leqslant a \leqslant i$, and j arbitrary covectors $\omega^{(b)}_\nu$, $1 \leqslant b \leqslant j$. It is only the tensor $S^{\alpha\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_{p+1}}$ that is a concomitant of T.
[34] The detailed proof of this result will be given elsewhere. To our knowledge only two particular cases of them have been considered in the literature: the so-called *Frobenius covariants* [G. Frobenius, Crelles J. **86**, 44 (1879)]; J. Wellstein, Arch. Math. Phys. **3**, 229 (1903); H. Schwerdtfeger, *Les Fonctions de Matrices* (Hermann, Paris, 1938), Vol. 649, pp. 17–24) and the Gantmacher's *reduced* adjoint matrix (quoted in Ref. 28, pp. 90–92).
[35] Apart from very particular cases related to almost product structures, where the relation P^2 = identity defining them trivializes a great part of the difficulties, the only "general" result known up to now is the Haantjes theorem [J. Haantjes, Proc. Konink. Nederl. Akad. Van Wetens. Amsterdam, A **58**(2), 158 (1955)] for symmetric tensors of type I, which gives the differential system that T must verify in order that *all* the eigenvectors be integrable. What about the cases in which *not all* them are integrable, or the cases in which T is not of type I, or those where the differential system considered is not the integrability one? All these questions remained open until now.
[36] That is, a fluid with zero heat conductivity and generically unequal (anisotropic) pressures.
[37] Complete set of constitutive equations.
[38] In the sense of test solutions on a background metric as well as in the sense of metric solutions to the Einstein equations. The first space is, in general, greater than the second one, but our present arguments apply equally to both spaces.
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[40] That is, those $\bar{S}_a$ containing the usual physical anisotropic solutions of interest.
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[44] Here, the word "portion" means "matter contained in a compact surface" in the Newtonian context, and "energy tensor distribution in a timelike generalized cylinder" in the relativistic context.
[45] "Regular" means here that its energy tensor is of rank four *almost* everywhere.

[46] C. Bona, B. Coll, and J. A. Morales, E. R. E. 86, Pub. Universitat de València, 1986.
[47] See, for instance, W. S. Burnside and A. W. Panton, *Theory of Equations* (Dublin U.P., Dublin, 1886), 2nd ed., p. 142.
[48] G. Chrystal, *Algebra, an Elementary Textbook* (Chelsea, New York, 1964), Pt. I, 7nd ed., p. 553.
[49] It is J. Olivert who suggested to us the use of this operator.
[50] In local coordinates $(i(v)A)_{b\cdots k} = v^a A_{ab\cdots k}$.
[51] If P, Q, R, and S are double one-forms, one has $(P\wedge Q)\times(R\wedge S) = (P\times R)\wedge(Q\times S) + (P\times S)\wedge(Q\times R)$, where $\times$ denotes the cross-product of tensors, that is, the contraction of the tensorial product. For double two-forms $\times$ is the usual matricial product on bivector space (contraction by pairs of indices).
[52] A. H. Taub in *Relativity Theory and Astrophysics*, Lectures in Applied Mathematics (Am. Math Soc., Providence, RI, 1967), Vol. 8, p. 170. The Taub's conditions for a (1,1) tensor to be the energy tensor for a perfect fluid would be also *sufficient* in absence of a *given* metric.
[53] See G. S. Hall and D. A. Negm, Int. J. Theor. Phys. **25**, 405 (1986); B. Kuchowicz, Gen. Rel. Grav. **5**, 201, (1974).
[54] J. Plebański seems to have been the first to give these conditions (see Ref. 3, p. 1011), in *The Large Scale Structure of Space-Time* (Cambridge U.P., Cambridge, 1973); S. W. Hawking and G. F. R. Ellis call them the weak and dominant energy conditions, seeming to be unaware of Plebański's work.
[55] For examples of physical situations where such a distintion is essential see Refs. 3, 6, and 53.
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