On the supersoluble hypercentre of a finite group

Liyun Miao* Adolfo Ballester-Bolinches† Ramón Esteban-Romero‡ Yangming Li§

Abstract

We give some sufficient conditions for a normal p-subgroup P of a finite group G to have every G-chief factor below it cyclic. The S-permutability of some p-subgroups of $\mathrm{O}^p(G)$ plays an important role. Some known results can be reproved and some others appear as corollaries of our main theorems.

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1 Introduction and statement of results

Throughout this paper all groups are finite and p denotes a fixed prime.

The motivation for this paper comes from [2, 3, 7], where some criteria for a group G to be p-supersoluble (that is, G is p-soluble with cyclic p-chief factors) in terms of the S-semipermutability of a family of subgroups of the Sylow p-subgroups were proved.

*Department of Mathematics, Shanghai University, Shanghai, 200444, China, email: mly5858syy@hotmail.com

†Department of Mathematics, Guangdong University of Education, Guangzhou, 510310, China. Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain; email: Adolfo.Ballester@uv.es

‡Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València; Camí de Vera, s/n; 46022 València, Spain; email: resteban@mat.upv.es. Current address: Departament de Matemàtiques, Universitat de València; Dr. Moliner, 50; 46100 Burjassot, València, Spain; email: Ramon.Esteban@uv.es

§Department of Mathematics, Guangdong University of Education, Guangzhou, 510310, People's Republic of China; Departament d'Àlgebra, Universitat de València, email: liyangming@gdei.edu.cn

A subgroup H of a group G is said to be *S-permutable* (see [1, Section 1.2]) in G if H permutes with all Sylow subgroups of G, and is said to be *S-semipermutable* in G ([12]) if H permutes with all Sylow q-subgroups of G for the primes q not dividing $|H|$.

Two characteristic subgroups which play an important role in this paper are the supersoluble hypercentre $\mathrm{Z}_{\mathcal{U}}(G)$ which is the the largest normal subgroup such that every G-chief factor below $\mathrm{Z}_{\mathcal{U}}(G)$ is cyclic, and the residual of a group G with respect to the formation of all abelian groups of exponent dividing $p-1$, that is, the smallest normal subgroup G^* of G for which the corresponding factor group is abelian of exponent dividing $p-1$.

For a subgroup D of a normal p-subgroup P of a group G, the following statements are pairwise equivalent:

- D is S-semipermutable in G.
- D is S-permutable in G.
- D is normalised by $\mathrm{O}^p(G)$.

If, in addition, $|D| \neq 2$ and every subgroup H of P of order $|H| = |D|$ is S-permutable in G, then P is contained in the supersoluble hypercentre of G, or equivalently, P centralised by $\mathrm{O}^p(G^*)$ ([11, Lemma 2.8]).

In particular, if a group A acts on a p-group P and all p-subgroups of P with order $|D|$ are stabilised by $\mathrm{O}^p(A)$, then every subgroup of P of order $|D|$ is S-semipermutable in the semidirect product $G = P \rtimes A$. Therefore P is centralised by $\mathrm{O}^p(A^*)$.

Berkovich and Isaacs showed in [3] that we need only to consider the noncyclic subgroups of order $|D| = p^e$ for some $e \geq 3$ to get the same result.

Theorem 1 ([3, Theorem A])**.** *Fix an integer $e \geq 3$, and let P be a p-group with $|P| > p^e$. Let A act on P, and assume that every noncyclic subgroup of P with order p^e is stabilised by $\mathrm{O}^p(A)$. Then P is centralised by $\mathrm{O}^p(A^*)$.*

As a consequence, they showed in [3, Theorem B] that if every noncyclic subgroup of order p^e is S-semipermutable in G and the Sylow subgroups of G are noncyclic of order exceeding p^e, then G is p-supersoluble.

On the other hand, Qiao, together the first and second authors of the present paper, proved in [2] the following extension of [7, Theorem B].

Theorem 2 ([2, Theorem 2])**.** *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $1 \leq d < |P|$. If $H \cap \mathrm{O}^p(G)$ is S-semipermutable in $\mathrm{O}^p(G)$ for all normal subgroups H of P with $|H| = d$, then either G is p-supersoluble or else $|P \cap \mathrm{O}^p(G)| > d$.*

Bearing in mind the above results, it seems to be natural to address the problem of studying the structural impact of the S-semipermutability of the subgroups of the form $H \cap \mathrm{O}^p(G)$, where H is a noncyclic subgroup of order d. The main results of this paper give the complete answer to that problem.

We begin by giving an alternative proof of Theorem 1. Then we prove the following results.

Theorem 3. *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $1 \le d < |P|$. Assume that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all noncyclic subgroups H of P with $|H| = d$. Then either $|P \cap \mathrm{O}^p(G)| > d$, or $P \cap \mathrm{O}^p(G)$ is cyclic, or else G is p-supersoluble.*

Combining Theorems 1 and 3, we have:

Corollary 4 ([3, Theorem B]). *Fix an integer $e \ge 3$, and let P be a noncyclic Sylow p-subgroup of a group G with $|P| > p^e = d$. If every noncyclic subgroup of order d is S-semipermutable in G, then G is p-supersoluble.*

Our next theorem can be regarded as an improvement of Theorem 1.

Theorem 5. *Fix an integer $e \ge 3$, and let P be a normal p-subgroup of a group G with $|P| > p^e = d$. Assume that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all noncyclic subgroups H of P with $|H| = d$. Then P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$.*

Theorem 5 does not hold for $e = 2$ as an example in [3] shows: take P a direct product of two cyclic groups of order p^2 and the semidirect product $G = P \rtimes A$, where $A = \mathrm{Aut}(P)$.

However, if we consider all subgroups of order p^2, we have:

Theorem 6. *Let P be a normal p-subgroup of a group G with $|P| > p^2$. Assume that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all subgroups H of P with $|H| = p^2$. Then P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$.*

Following [10], we say that a subgroup H of a group G is said to be *weakly S-semipermutable* in G if there exists subgroups S and T of G such that the following conditions hold:

1. T is subnormal in G and S is S-semipermutable in G.
2. $G = HT$ and $H \cap T \le S \le H$.

If H is a weakly S-semipermutable p-subgroup of G, then T is a subgroup of p-power index in G. Applying [1, Lemma 1.1.11], $U = \mathrm{O}^p(T) = \mathrm{O}^p(G)$. Therefore $H \cap U = S \cap U$ is S-semipermutable in G. Therefore, we have:

Corollary 7. *Fix an integer $e \geq 2$. Let P be a normal p-subgroup of a group G with $|P| > p^e = d$. Assume that all noncyclic subgroups of P with order d are weakly S-semipermutable in G. If either $e \geq 3$ or $e = 2$ and every subgroup of order d is weakly S-semipermutable in G, then P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$.*

Applying Corollary 7 and an standard reduction to the p-group case, we have the following partial improvement of [10, Main Theorem]:

Corollary 8. *Fix an integer $e \geq 2$. Assume that E and X are normal subgroups of a group G such that $F_p^*(E) \leq X \leq E$, where $F_p^*(E)$ is the generalised p-Fitting subgroup of E. If the Sylow p-subgroups of X are of order exceeding $d = p^e$ and every p-subgroup of X of order d is weakly S-semipermutable in G, then every p-chief factor of G below E is cyclic.*

2 Proof of Theorem 1

The main goal of this section is to present an alternative proof of Theorem 1.

Let L/M be a p-chief factor of G below P. Applying [5, B, Theorem 9.8], then L/M is of order p if and only if $G/C_G(L/M)$ is abelian of exponent dividing $p-1$, or equivalently, $G^* \leq C_G(L/M)$. Now, if P is a p-normal subgroup of G, $P \leq \mathrm{Z}_{\mathcal{U}}(G)$ if and only if G^* centralises every G-chief factor below P.

Consequently, Theorem 1 is equivalent to the following:

Theorem 9. *Fix an integer $e \geq 3$, and let P be a normal p-subgroup of a group G with $|P| > p^e = d$. Assume that every noncyclic subgroup of P of order d is S-permutable in G. Then P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$.*

We next lay down some results required for the proof of Theorem 9.

The following lemma proved by Su, Wang and the third author is essential.

Lemma 10 ([11, Lemma 2.8])**.** *Let P be a normal p-subgroup of G and let d be a power of p such that $1 < d < |P|$. Suppose all subgroups H of P with order d and all cyclic subgroups of P of order 4 (if P is a non-abelian 2-group and $d = 2$) are S-permutable in G. Then P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$.*

According to [6, Chapter 5, Theorems 3.11 and 3.13] and [8, Chapter IV, Satz 5.12] every nontrivial p-group P possesses a characteristic subgroup A of class at most two and of exponent p if p is odd, and of exponent at most 4 if $p = 2$ such that every nontrivial p'-automorphism of P induces a nontrivial automorphism of A. For such subgroup A, we have:

Lemma 11 ([4, Lemma 2.10]). *P is contained in* $\mathrm{Z}_{\mathcal{U}}(G)$ *if and only if A is contained in* $\mathrm{Z}_{\mathcal{U}}(G)$.

Our general hypothesis is that p is a prime and G is a group in which all members of a given family of p-subgroups are either S-permutable or S-semipermutable. Note that the p-subgroups of $G/\mathrm{O}_{p'}(G)$ are of the form $Q\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$, for Q a member of this family. Clearly the properties of G, as enunciated in the statements of our main results, are inherited by $G/\mathrm{O}_{p'}(G)$. Therefore, arguing by induction or minimal counterexample, *we may assume that* $\mathrm{O}_{p'}(G) = 1$. This fact will be used without further reference.

If X is a group, let $\mathrm{M}_c(X)$ denote the set of all cyclic maximal subgroups of X. The set of all noncyclic maximal subgroups of X will be denoted by $\mathrm{M}_{nc}(X)$.

Proof of Theorem 9. Assume that the result is false and take a counterexample (G, P) with the least $|G|+|P|$. Let A be the characteristic subgroup introduced above.

Suppose that A is a proper subgroup of P. By Lemma 11 and the choice of P, it follows that A is a noncyclic subgroup of P with order at most $d = p^e$. Let W be a subgroup of P of order pd containing A which is normalized by a Sylow p-subgroup R of G. Then A is contained in some $V \in \mathrm{M}_{nc}(W)$. Since V is S-permutable in G, it is normalised by $\mathrm{O}^p(G)$ by [1, Lemma 1.2.16]. Assume that $|\mathrm{M}_{nc}(W)| = 1$. Then V is normal in R and so V is normal in $G = R\mathrm{O}^p(G)$. Let $B \in \mathrm{M}_c(W)$. Then $W = VB$. Hence $\Phi(V)$ is a nontrivial normal subgroup of G contained in $B \cap V$. Therefore a minimal normal subgroup of G contained in $\Phi(V)$ has order p. Suppose that $V \neq W_1 \in \mathrm{M}_{nc}(W)$. Then V and W_1 are both normalized by $\mathrm{O}^p(G)$. Hence $W = VW_1$ is a normal subgroup of $G = R\mathrm{O}^p(G)$. If W were a proper subgroup of P, it would follow that $W \leq \mathrm{Z}_{\mathcal{U}}(G)$. This contradiction yields $W = P$. By Lemma 10, $|\mathrm{M}_c(W)| \geq 1$ and so P contains a minimal normal subgroup of G of order p. In both cases, we have that G has a minimal normal subgroup N of order p and there exists $B \in \mathrm{M}_c(W)$. By [6, Chapter 5, Theorem 3.11], N is contained in A. Suppose that $d > p^3$. Then $p^3 \leq d/p < |P/N|$ and every noncyclic subgroup H/N of P/N with order d/p is S-permutable in G/N. The choice of (G, P) implies that $P/N \leq Z_{\mathcal{U}}(G/N)$. Then $P \leq \mathrm{Z}_{\mathcal{U}}(G)$. This contradiction yields $d = p^3$. Therefore $V = A \in \mathrm{M}_{nc}(W)$ of order d and $W = AB$. In this case, $|A \cap B| = p^2$. Since B is cyclic, it follows that $p = 2$ and A is a normal subgroup of G of order 8 with no characteristic subgroups of order 4. It follows then A is isomorphic to the quaternion group of order 8. Assume that $|\mathrm{M}_c(W)| = 1$. Then $|\mathrm{M}_{nc}(W)| > 1$ and so $W = P$. In particular $A \cap B$ is characteristic in W and so it is normal in G. This contradiction implies that $|\mathrm{M}_c(W)| > 1$ and W contains a central cyclic subgroup E of order 4, which is contained in $\mathrm{Z}(A)$, contrary to supposition.

Therefore $A = P$ and so P has exponent p or 4. In both cases, every subgroup of P of order d must be noncyclic. By Lemma 10, $P \leq \mathrm{Z}_{\mathcal{U}}(G)$. This final contradiction proves the theorem. □

3 Proofs of the main results

Proof of Theorem 3. The proof of this result is a modification of the proof of [2, Theorem 2], and we indicate only the necessary changes, so it had best be read with [2] open in front of the reader.

We assume the result is not true and let G be a counterexample of least order. Write $U = \mathrm{O}^p(G)$, $N = P \cap U$. Then N is a noncyclic subgroup of order at most d and G is not p-supersoluble. In particular, $N \neq 1$ and $d \geq p$.

Then $\mathrm{O}_{p'}(G) = 1$, G is p-soluble and if T is a minimal normal subgroup of G contained in U, it follows that $T \leq N$. Thus $|T| \leq d$. If N/T is cyclic, then G/T is p-supersoluble. Otherwise, G/T satisfies the hypotheses of the theorem and so G/T is p-supersoluble by minimality of G. In both cases, G/T is p-supersoluble. Since G is not p-supersoluble, it follows that $T \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $T \nleq G$. Then $G = TM$ and $P = T(P \cap M)$. Let A be a normal subgroup of P such that $|T : A| = p$. Then $A(P \cap M)$ is a normal subgroup of P and let B a normal subgroup of P of order d such that $A \leq B \leq A(P \cap M)$. Since $B = A(B \cap M)$, it follows that B is not cyclic and so $B \cap U$ is S-semipermutable in G by hypothesis. Arguing as in [2, Theorem 2], we have that $B \cap T = B \cap U \cap T$ is normalized by U. Since $T \cap M = 1$, we have that $A = A(B \cap T) = B \cap T$ is normalized by U. Then A is normal in $G = PU$, which contradicts the minimality of T as normal subgroup of G.

□

Proof of Corollary 4. Assume that the result is false and consider a counterexample G with $|G|$ as small as possible. Write $U = \mathrm{O}^p(G)$. Then $\mathrm{O}_{p'}(G) = 1$. Suppose M is a proper normal p-supersoluble subgroup with a noncyclic Sylow p-subgroup of order exceeding d. By [1, Lemma 2.1.6], M' is p-nilpotent and $\mathrm{O}_{p'}(M) = 1$. Therefore a Sylow p-subgroup of M is normal in G and contained in $\mathrm{Z}_{\mathcal{U}}(G)$ by Theorem 9. In particular, $\mathrm{O}^{p'}(G) = G$ and $G^* = G$. Write $U = \mathrm{O}^p(G)$ and $N = U \cap P$. By [8, IV, Satz 5.5] there exists an element $a \in N$ of order p or order 4 such that $a \notin \mathrm{Z}(U)$. Since P is noncyclic and $e \geq 3$, it follows that P has a noncyclic subgroup H of order d (see for instance [3, Lemma 2.3]). Then, by [9, Theorem A], H

is contained in the soluble radical A of G. Suppose that $a \notin A$. Then a subgroup X of order d of $A\langle a\rangle$ containing $\langle a\rangle$ is noncyclic and so $X \leq A$, contrary to the choice of a. Therefore $a \in A$. Let $B = P \cap A$. Then $a \in B$ and $|B| \geq d$. Assume that $G = A$. By Theorem 3, either N has order greater than d or N is cyclic. If either $U < G$ and $|N| > d$ or N is cyclic, then G is p-supersoluble, contrary to assumption. Hence $G = U = \mathrm{O}^{p'}(G)$ which is also a contradiction. Hence $AP < G$. Then, by the choice of G, AP is p-supersoluble and so B is a normal subgroup of G. Assume that $|B| > d$. Then B is contained in $\mathrm{Z}_{\mathcal{U}}(G)$ and $a \in \mathrm{Z}(U)$, contrary to assumption. Hence $|B| = d$ and B is the unique noncyclic subgroup of G of order d. By [3, Lemma 2.3], B is abelian and has a cyclic maximal subgroup. Then $\Phi(B)$ and $\Omega_1(B)$ are two characteristic subgroups of B contained in $\mathrm{Z}_{\mathcal{U}}(G)$ such that $B = \Phi(B)\Omega_1(B)$. Consequently, $B \leq \mathrm{Z}_{\mathcal{U}}(G)$, final contradiction.

□

Proof of Theorem 5. We suppose that the theorem is false and derive a contradiction. Let (G, P) be a counterexample with $|G| + |P|$ minimal. Let $P_0 \in Syl_p(G)$ and write $U = \mathrm{O}^p(G)$ and $N = P \cap U$.

Suppose that $N = P$. Then, by Theorem 9, P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$. Therefore, we may assume that N is a proper subgroup of P. Moreover, it is clear that every G-chief factor laying between N and P is central in G. Hence N is contained in $\mathrm{Z}_{\mathcal{U}}(G)$ if and only if P is contained in $\mathrm{Z}_{\mathcal{U}}(G)$. By the choice of the pair (G, P), $|N| \leq d$ and $1 \neq N$ is not cyclic.

Let $\mathcal{H}$ denote the set of all noncyclic subgroups of P with order d. Since P is not cyclic and $e \geq 3$, we have that $\mathcal{H}$ is not empty. By hypothesis, $H \cap U$ is S-semipermutable in G for each $H \in \mathcal{H}$.

The proof will follow as a consequence of the following two steps.

(1) *If T is a minimal normal subgroup of G contained in N, then $P/T \leq \mathrm{Z}_{\mathcal{U}}(G/T)$.*

Let T be a minimal normal subgroup of G contained in N. Then $|T| \leq d$. If N/T is cyclic, it follows that $N/T \leq \mathrm{Z}_{\mathcal{U}}(G/T)$. Hence $P/T \leq \mathrm{Z}_{\mathcal{U}}(G/T)$. Therefore we may assume that $d/|T| \geq p^2$ and N/T is noncyclic.

Suppose that $d/|T| = p^2$. Then $|N/T| = p^2$. If N/T were not contained in $\mathrm{Z}_{\mathcal{U}}(G/T)$, we would have that N/T would be a minimal normal subgroup of G/T. Let S/N be a minimal normal subgroup of G/N contained in P/N. Then $|S/N| = p$ and so $|S/T| = p^3$. If S/T had a cyclic maximal subgroup, N/T would contain a normal subgroup of G/T of order p. This contradiction shows that every maximal subgroup of S/T is noncyclic. Let $N/T \neq M/T$ a maximal subgroup of S/T and let A/T be a subgroup of order p in $N/T \cap Z(P_0/T)$ and let B/T a subgroup of order p of M/T such that B/T is not contained in N/T and $H/T = (A/T)(B/T)$ is of order p^2. Then $H \in \mathcal{H}$ and

$(H \cap U)/T = (H/T) \cap (U/T)$ is S-semipermutable in G/T. Since $N/T \cap B/T = 1$, we have that $A/T = (H \cap U)/T$ is S-semipermutable in G/T. As $A/T \leq N/T \leq O_p(G/T)$, we have that A/T is S-permutable in G/T. Then A/T is normalized by $O^p(G/T)$ and we get that $A/T \trianglelefteq G/T$ since $A/T \leq Z(P_0/T)$, contradicting the minimal condition of N/T as normal subgroup of G/T. Consequently, $P/T \leq \mathrm{Z}_{\mathcal{U}}(G/T)$.

Suppose that $p^3 \leq d/|T| < |P/T|$. Let H/T be a noncyclic subgroup of P/T of order $d/|T|$, then $H \in \mathcal{H}$. Hence $H/T \cap O^p(G/T) = H/T \cap U/T = (H \cap U)/T$ is S-semipermutable in G/T. Therefore the pair $(G/T, P/T)$ satisfies the hypothesis of the theorem. The choice of (G, P) implies that $P/T \leq \mathrm{Z}_{\mathcal{U}}(G/T)$.

(2) $N \cap \Phi(P) \neq 1$.

Assume that $N \cap \Phi(P) = 1$. Let T be a minimal normal subgroup of G contained in N. Applying [5, A, Theorem 9.2], there exists a subgroup M of P such $P = TM$ and $T \cap M = 1$. Let T_1 be a normal subgroup of P_0, which is also a maximal subgroup of T and let B a subgroup of M such that $H = T_1B$ is of order d. Then $H \in \mathcal{H}$ and $H \cap U$ is S-semipermutable in G Hence $H \cap N = H \cap U \cap N$ is normalised by every Sylow q-subgroup of G for all primes $q \neq p$. Thus $H \cap N$ is normal in U and so is $T_1 = H \cap T$. Therefore T_1 is normal in $G = UP_0$. This contradiction shows that $N \cap \Phi(P) \neq 1$.

By Steps (1) and (2), it follows that $P/\Phi(P) \leq \mathrm{Z}_{\mathcal{U}}(G/\Phi(P))$. We can then apply [5, IV, Theorem 6.7] to conclude that $P \leq \mathrm{Z}_{\mathcal{U}}(G)$, the final contradiction. □

Proof of Theorem 6. The first part of the proof runs parallel to that of Theorem 5, so we may be brief. Starting with a minimal countexample (G, P), we assume first that $N = P \cap U$, where $U = \mathrm{O}^p(G)$, is a proper subgroup of P. Since N is not contained in $\mathrm{Z}_{\mathcal{U}}(G)$, it follows that N is a noncyclic normal subgroup of G with order at most p^2. Therefore N is a minimal normal subgroup of G. Note that P/N is contained in $\mathrm{Z}_{\mathcal{U}}(G/N)$. Therefore there exists a normal subgroup S of G contained in P such that $|S : N| = p$. Then $|S| = p^3$ and S is noncyclic. If S had a cyclic maximal subgroup, then $\Phi(S)$ would be a normal subgroup of G of order p contained in N, which would contradict our assumption. Hence there exists a noncyclic maximal subgroup M of S such that $N \neq M$. Let A be a maximal subgroup of N such that A is a normal subgroup of a Sylow p-subgroup P_0 of G. Then $|A| = p$. Assume that A is contained in M. Since M is has order p^2 and $A = N \cap M$, it follows that A is S-semipermutable in G. Assume that $A \cap M = 1$. Then $S = AM$. Let B be a subgroup of M of order p which is not contained in N. Then $H = AB$ has order p^2 and so $A = H \cap N$ is S-semipermutable in G. In both cases, it follows that A is normalised by $\mathrm{O}^p(G)$ and so A is a normal subgroup of $G = P_0 \, \mathrm{O}^p(G)$. This contradiction yields $P = N$. In this case, every subgroup of P of order p^2 is S-semipermutable in G. Applying [3, Corollary B], $P \leq \mathrm{Z}_{\mathcal{U}}(G)$, which is the final contradiction. $\square$

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