On finite groups with many supersoluble subgroups

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Abstract

The solubility of a finite group with less than 6 non-supersoluble subgroups is confirmed in the paper. Moreover we prove that a finite insoluble group has exactly 6 non-supersoluble subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$. Furthermore, it is shown that a finite insoluble group has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$. This confirms a conjecture of Zarrin [*Arch. Math. (Basel)*, 99 (2012), 201–206].

Keywords: finite group; supersoluble subgroup; soluble group

Mathematics Subject Classification (2010): 20D10, 20D20

1 Introduction

Throughout this paper, G always denotes a finite group.

The results of the present article are motivated by a paper of Zarrin [13], where an extension of the classical result of Schmidt [10] about the solubility of a group with all proper subgroups nilpotent is proved. Zarrin showed that if a group G has at most 21 non-nilpotent subgroups, then G is soluble. He also proposed the following conjecture.

Conjecture 1.1. *Let G be an insoluble group. Then G has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$.*

Our first main result confirms that conjecture.

Theorem A. *Let G be an insoluble group. Then G has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$.*

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On the other hand, Huppert [7] proved that *nilpotent* in Schmidt's theorem can be replaced by *supersoluble* with the same conclusion. Therefore it seems natural to ask: What is the minimum number of non-supersoluble subgroups to guarantee solubility? Our second main result answers this question.

Theorem B. *A group with less than* 6 *non-supersoluble subgroups is soluble.*

Our last result shows that A_5 and $\mathrm{SL}_2(5)$ are the only insoluble groups with exactly 6 non-supersoluble subgroups.

Theorem C. *Let G be an insoluble group. Then G has exactly* 6 *non-supersoluble subgroups if and only if it is isomorphic to A_5 or* $\mathrm{SL}_2(5)$.

The notion that we use is standard and follows that in Doerk and Hawkes [3] or Huppert [8]. We use $\mathrm{SL}_m(q)$ and $\mathrm{PSL}_m(q)$ to denote the special linear group and the projective special linear group, respectively, of dimension m over the field with q elements, where q is a prime power.

2 Proofs

The proofs of our results depend on the following lemmas.

Lemma 2.1. *Let G be a group. The number of non-supersoluble subgroups of $G/\Phi(G)$ is less than or equal to the number of non-supersoluble subgroups of G.*

This follows from the fact that if $H/\Phi(G)$ is a non-supersoluble subgroup of $G/\Phi(G)$, then H is a non-supersoluble subgroup of G.

Recall that a minimal simple group is a simple group whose maximal subgroups are soluble. Suppose that N is a non-trivial proper normal subgroup of a group G such that $\Phi(G) = 1$ and that all maximal subgroups of G are soluble. Then there exists a maximal subgroup M of G such that $G = NM$. Since by hypothesis N and M are soluble, then G is soluble. This implies the following result.

Lemma 2.2. *Let G be a non-soluble group whose maximal subgroups are soluble. Then $G/\Phi(G)$ is a minimal simple group.*

We will use the symbol $\delta(n)$ to denote the number of natural divisors of the natural number n.

Lemma 2.3. *The number of non-supersoluble subgroups of a minimal simple group is at least* 6*. The only minimal simple group with exactly* 6 *non-supersoluble subgroups is A_5.*

Proof. By [12] (see also [8, Kapitel II, Bemerkung 7.5]), G is isomorphic to one of the following groups:

1. $\mathrm{PSL}_2(p)$, where $p > 3$ is a prime and $5 \nmid p^2 - 1$;

2. $\mathrm{PSL}_2(2^q)$, where q is a prime;

3. $\mathrm{PSL}_2(3^q)$, where q is an odd prime;

4. $\mathrm{PSL}_3(3)$;

5. a Suzuki group $\mathrm{Sz}(2^q)$, where q is an odd prime.

It will be enough to show that in all these cases the number of non-supersoluble subgroups of G is at least 6.

The subgroups of $\mathrm{PSL}_2(p^f)$ have been studied in [2] (see also [8, Kapitel II, Satz 8.27]). These subgroups fall into the following classes:

1. elementary abelian p-groups;

2. cyclic p-groups of order z, where z divides $(p^f \pm 1)/k$ and $k = \gcd(p^f - 1, 2)$;

3. dihedral groups of order $2z$ where z is as in 2 above;

4. alternating groups A_4 for $p > 2$ or $p = 2$ and $f \equiv 0 \pmod 2$;

5. symmetric groups Σ_4 for $p^{2f} - 1 \equiv 0 \pmod{16}$;

6. alternating groups A_5 for $p = 5$ or $p^{2f} - 1 \equiv 0 \pmod 5$;

7. semidirect products of elementary abelian groups of order p^m with cyclic groups of order t; here $t \mid p^m - 1$ and $t \mid p^f - 1$;

8. groups $\mathrm{PSL}_2(p^m)$ for $m \mid f$ and $\mathrm{PGL}_2(p^m)$ for $2m \mid f$.

Recall that, by [8, Kapitel II, Hilfssatz 6.2],

$$|\mathrm{PSL}_2(p^f)| = p^f(p^f - 1)(p^f + 1)/\gcd(2, p^f - 1).$$

Assume that $G \cong \mathrm{PSL}_2(p)$ with $p > 3$ a prime and $5 \nmid p^2 - 1$. Since $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$, we can assume that $p > 5$. Therefore the only non-supersoluble proper subgroups of G are of the form A_4 for $p > 2$ and, when $p^2 - 1 \equiv 0 \pmod{16}$, Σ_4. If $p^2 - 1 \equiv 0 \pmod{16}$, there are two conjugacy classes of subgroups isomorphic to A_4 with normaliser isomorphic to Σ_4. In this case, the number of non-supersoluble proper subgroups isomorphic to A_4 or Σ_4 of G is $4p(p-1)(p+1)/(2 \cdot 24) = p(p-1)(p+1)/12$. Therefore the number of non-supersoluble subgroups is $p(p-1)(p+1)/12 + 1 \geq 7 \cdot 6 \cdot 8/12 + 1 = 29$. Note that, in the previous argument, we add 1 because we are counting the non-supersoluble subgroups of G, not only the proper non-supersoluble subgroups of G. Otherwise, there is a unique conjugacy class of self-normalising subgroups isomorphic to A_4. The number of such subgroups is the index of its normaliser, namely $p(p-1)(p-2)/24$. Hence the number of non-supersoluble subgroups is $p(p-1)(p+1)/24 + 1 \geq 11 \cdot 10 \cdot 12/24 + 1 = 56$.

Assume now that $G \cong \mathrm{PSL}_2(2^q)$, with q a prime number. If $q = 2$, then $G \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ has 5 subgroups isomorphic to A_4 and so it has 6 non-supersoluble subgroups. Therefore we can suppose that $q \geq 3$. In this case, the only possibility for a proper non-supersoluble subgroup of G has the following structure: It must be a semidirect product of an elementary abelian group of order 2^m with a cyclic group of order t with $t \mid 2^q - 1$ and $t \mid 2^m - 1$. Since q is a prime, $m = q$. The normalisers of all these subgroups are semidirect products of an elementary abelian group of order 2^q with a cyclic subgroup of order $2^q - 1$. Each of these normalisers in G has order $2^q(2^q - 1)$ and index 2^q+1 in G. It follows that the number of all non-supersoluble proper subgroups of G is $(2^q+1)\big(\delta(2^q-1)-1\big)$. Hence the number of non-supersoluble subgroups of G is greater than or equal to $2^q + 1 + 1 = 2^q + 2 \geq 2^3 + 2 = 10$.

Assume now that $G \cong \mathrm{PSL}_2(3^q)$ with q an odd prime. There is a unique conjugacy class of self-normalising subgroups isomorphic to A_4 and, since $3^{2q} - 1 \equiv 8 \pmod{16}$, there are no symmetric subgroups. The number of subgroups isomorphic to A_4 is $3^q(3^q - 1)(3^q + 1)/24 = 3^{q-1}(3^q - 1)(3^q + 1)/8 \geq 3^2(3^3 - 1)(3^3 + 1)/8 = 819$.

Assume now that $G \cong \mathrm{PSL}_3(3)$. A calculation with GAP [4] shows that G possesses 1 093 non-supersoluble subgroups.

Finally, assume that $G \cong \mathrm{Sz}(2^q)$ with q an odd prime. The order of G is $2^{2q}(2^{2q}+1)(2^q-1)$ by [11]. According to [1, Table 8.16], G has a unique conjugacy class of maximal subgroups of G of type $[E_{2^q}^{1+1}]C_{2^q-1}$ and order $2^{2q}(2^q-1)$. Hence G has $2^{2q} + 1$ subgroups of this type and, therefore, the number of non-supersoluble subgroups of G is at least $(2^{2\cdot 3} + 1) + 1 = 66$. □

Proof of Theorem A. If $G \cong A_5$ or $G \cong \mathrm{SL}_2(5)$, then it is routine to check that G has exactly 22 non-nilpotent subgroups.

Conversely, assume that G has exactly 22 non-nilpotent subgroups. Let H be a maximal subgroup of G. If H is nilpotent, then H is certainly soluble. If H is non-nilpotent, then H has less than 22 non-nilpotent subgroups. By [13, Theorem A], H is soluble. It follows that G is a minimal non-soluble group, and so $G/\Phi(G)$ is a minimal simple group. Then, according to [13, Theorem A], $G/\Phi(G) \cong A_5$ and $G/\Phi(G)$ has exactly 22 non-nilpotent subgroups, and every second maximal subgroup of $G/\Phi(G)$ is nilpotent. Hence every second maximal subgroup of G is nilpotent. By [9, Satz], $G \cong A_5$ or $\mathrm{SL}_2(5)$, as desired. □

Proof of Theorem B. Assume that the number of non-supersoluble subgroups of a group G is less than 6. We prove that G is soluble by induction on $|G|$. Clearly, we may assume that every maximal subgroup of G is soluble. If G were not soluble, $G/\Phi(G)$ would be a minimal simple group with less than 6 non-supersoluble subgroups by Lemmas 2.1 and 2.2. This would contradict Lemma 2.3. Therefore G is soluble, as desired. □

Proof of Theorem C. Assume that G has exactly 6 non-supersoluble subgroups. Suppose, arguing by contradiction, that G is not isomorphic to A_5 or $\mathrm{SL}(2, 5)$. Let us choose G of least order. Since G is not soluble, G contains a minimal non-soluble subgroup S. By Lemma 2.2, $S/\Phi(S)$ is a minimal simple group. By Lemma 2.3, the only minimal simple group with at most 6 non-supersoluble subgroups is A_5. If $S < G$, then the number of non-supersoluble subgroups of S is less than the number of non-supersoluble subgroups of G, and so is the number of non-supersoluble subgroups of $S/\Phi(S)$. Hence $S = G$. By Lemma 2.3, $G/\Phi(G)$

is isomorphic to A_5. If $\Phi(G) = 1$, then $G \cong A_5$, contrary our supposition. Hence $\Phi(G) \neq 1$. Let $\Phi(G)/K$ be a chief factor of G. By [8, Kapitel III, Satz 3.6 and Satz 3.8], $\Phi(G)$ is a nilpotent $\{2, 3, 5\}$-group. By a result of Gaschütz [5], G/K is a quotient of a universal Frattini extension with elementary abelian kernel. Suppose that $\Phi(G)/K$ is a 3-group. Then $\Phi(G)/K$ is an irreducible module of dimension 4 for A_5 by [6, Example 1]. In this case, given a Sylow 5-subgroup C/K of G/K, $\Phi(G)C/K$ is a non-supersoluble subgroup of G/K. On the other hand, let $T/\Phi(G)$ be one of the 6 non-supersoluble subgroups of $G/\Phi(G)$. Then T/K is also a non-supersoluble subgroup of G/K. Moreover $\Phi(G)C/K$ cannot be obtained in this way because $\Phi(G)C/\Phi(G)$ is supersoluble. Hence G/K has more than 6 non-supersoluble subgroups, and the same can be said about G. Suppose that $\Phi(G)/K$ is a 5-group. Then $\Phi(G)/K$ is an irreducible module of dimension 3 for A_5 by [6, Example 1], namely, the head of the corresponding Frattini module. A Sylow 3-subgroup T/K of G/K does not centralise $\Phi(G)/K$ since $\Phi(G)/K$ is acted on faithfully by $G/\Phi(G)$. Therefore, by [3, Chapter A, Proposition 12.5], $[T/K, \Phi(G)/K]$ is a non-trivial normal subgroup of $\Phi(G)T/K$ on which T/K acts faithfully. In particular, $[T/K, \Phi(G)/K](T/K)$ cannot be supersoluble, since 3 does not divide $5 - 1$. Arguing as above, we conclude that G/K has more than 6 non-supersoluble subgroups. In particular, we can assume that $\Phi(G)$ is a 2-group. By [6, Example 1], the only possibility for $\Phi(G)/K$ is that $\Phi(G)/K$ has order 2, the head of the corresponding Frattini module. Therefore $G/K \cong \mathrm{SL}_2(5)$. Suppose that K/L is a chief 2-factor of G. Note that K/L is an irreducible module for $\mathrm{SL}_2(5)$ and so for A_5, by [3, Chapter B, Proposition 3.12]. By [8, Kapitel V, Satz 25.5], the Schur multiplier of $\mathrm{SL}_2(5)$ is trivial. It follows that K/L is not cyclic and so it has dimension 4 ([6, Example 1]). By considering a Sylow 5-subgroup C of G, we obtain that $\Phi(G)C/L$ is not supersoluble. As above, G/L has more than 6 non-supersoluble subgroups, and the same can be said about G, contrary to assumption. We conclude that $G \cong A_5$ or $G \cong \mathrm{SL}_2(5)$.

The converse is clear by Lemma 2.3. □

Acknowledgements

The first and second author are supported by the grant MTM2014-54707-C3-1-P from the Ministerio de Economía y Competitividad, Spain, and FEDER, European Union. The first author is supported by the National Natural Science Foundation of China (11271085) and a project of Natural Science Foundation of Guangdong Province (2015A030313791). The third author is supported by the National Natural Science Foundation of China (11461007), and the Guangxi Natural Science Foundation Program (2016GXNSFAA380156).

This research has been done during a visit of the third author to the Departament de Matemàtiques of the Universitat de València. He expresses his gratitude to this institution.

We thank the anonymous referee for his/her comments, that have helped us to improve the presentation of the paper.

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