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Real Elements and p-Nilpotence of Finite Groups [1]

Adolfo Ballester-Bolinches — Ramón Esteban-Romero — Luis M. Ezquerro

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Dedicated to Professor H. Heineken on the occasion of his 80th birthday.

Abstract

Our first main result proves that every element of order 4 of a Sylow 2-subgroup S of a minimal non-2-nilpotent group G, is a real element of S. This allows to give a character-free proof of a theorem due to Isaacs and Navarro, (see [9, Theorem B]). As an application, the authors show a common extension of the p-nilpotence criteria proved in [3] and [9].

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1 Introduction

All groups considered in this paper will be finite.

Let p be a prime, that we hold fixed in the whole paper, and consider the following common situation: a p'-automorphism α acting on a p-group P. If $p \neq 2$ and α fixes all elements of order p in S, then α acts trivially on P. To obtain the corresponding conclusion for $p = 2$, an additional assumption

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would be required: for example, that every element of order 4 in P is also fixed by α (see [7, Kapitel IV, Satz 5.12]).

In [9, Theorem B], Isaacs and Navarro showed that, when $p = 2$, this result survives under the weaker assumption that α fixes all real elements of order 4 in S. Recall that an element g of a group G is said to be *real* if g is conjugate to its inverse g^{-1}.

Theorem 1 *Let α be a p'-automorphism of a p-group P. Assume that α centralises all elements of order p and all real elements of order 4 if $p = 2$. Then α acts trivially on P.*

The proof of this theorem given in [9] is character theoretic. As an application of Theorem 1, the authors improved a p-nilpotence criterion showed in [6, Main Theorem].

Recall that a group G is said to be a minimal non-p-nilpotent group if G is not p-nilpotent but all proper subgroups of G are p-nilpotent. The knowledge of the structure of minimal non-p-nilpotent groups provides a powerful tool to establish p-nilpotence criteria by direct arguments, basically because it can give some insight into what makes a group to be p-nilpotent. Assume that we want to prove that a subgroup-closed class $\mathfrak{L}$ is composed of p-nilpotent groups. If a non-p-nilpotent group G belongs to $\mathfrak{L}$, then G has a subgroup in $\mathfrak{L}$ which is a minimal non-p-nilpotent group. Therefore one has only to check that no minimal non-p-nilpotent group belongs to $\mathfrak{L}$.

These ideas have been succesfully applied in several papers (see [2], [3], [10]). In fact, the understanding of the structure of minimal non-p-nilpotent groups is crucial in the character-free proofs of Theorem 1 in [3], [10].

The principal aim in this paper is to present some results in this spirit. Our main result contains some useful information about the structure of a minimal non-2-nilpotent group. Applying [7, Kapitel IV, Satz 5.4], we have that the exponent of the Sylow 2-subgroup of a minimal non-2-nilpotent group is at most 4. Moveover, we have:

Theorem A *Let G be a minimal non-2-nilpotent group and let S be the Sylow 2-subgroup of G. If S has exponent 4, then every element of order 4 is a real element of S.*

We see that not only Theorem 1 follows directly as a consequence of Theorem A, but also it allow us to show a common extension of the p-nilpotence criteria proved in [3], [9].

If G is a group, we write $G^{\mathfrak{N}}$ to denote the nilpotent residual of G, i.e. the smallest normal subgroup of G with nilpotent quotient group. It is clear that $G^{\mathfrak{N}}$ is the last term of the lower central series of G.

Theorem B *Suppose that* S *is a Sylow* p*-subgroup of a group* G*. Then the following statements are pairwise equivalent.*

1. G *is* p*-nilpotent.*
2. *For every cyclic subgroup* P *of the focal subgroup* $S \cap G'$ *of* S *in* G*, such that* P *is generated by an element of order* p *or a real element of order* 4 *if* $p = 2$*,* S *controls fusion of* P *in* S*.*
3. *For every cyclic subgroup* P *of* $S \cap G^{\mathfrak{N}}$ *such that* P *is generated by an element of order* p *or a real element of order* 4 *if* $p = 2$*,* S *controls fusion of* P *in* S*.*

2 Proofs

Proof of Theorem A — Applying [7, Kapitel IV, Satz 5.4]), we obtain that G has order $2^t q^r$, where q is an odd prime, G has a normal Sylow 2-subgroup S of exponent at most 4, $\Phi(S)$ is elementary abelian, and the Sylow q-subgroups of G are cyclic. By a theorem of Gol′fand [5], G is an epimorphic image of a universal minimal non-2-nilpotent group G_0 of order $2^{a_0}q^r$, where $a_0 = a$ if a is odd and $a_0 = 3a/2$ if $a = 2m$ is even, and a is the order of 2 modulo q, i.e., a is the least positive integer such that $2^a \equiv 1 \pmod q$. A construction of the Gol′fand group is given in [1].

Let S_0 be the Sylow 2-subgroup of G_0 and assume that z is a generator of the Sylow q-subgroup of G_0. If a is odd, then S_0 is elementary abelian. It follows that $a = 2m$ is even. Let $g \in S$ be an arbitrary element of order 4. Since $\Phi(S)$ has exponent 2, we have that $g \in S \setminus \Phi(S)$. We can now take an element $g_0 \in S_0 \setminus \Phi(S_0)$ of order 4 whose image in G under the above epimorphism is g. In the proof of Gol′fand's theorem given in [1], it is shown that $\Phi(S_0)$ can be generated by m elements of the form $u_i = [g_0, g_0^{z^i}]$ for $i \in \{s-m+1, \ldots, s\}$, where $q = 2s+1$. Since $g_0^2 \in \Phi(S_0)$, there exist $n_1, \ldots,$ $n_m \in \mathbb{Z}$ such that $g_0^2 = u_{s-m+1}^{n_1} \cdots u_s^{n_m}$. Since $S_0' \leqslant \Phi(S_0) \leqslant Z(S_0)$, by [4, Chapter A, 7.2] we have that $g_0^2 = [g_0, g_1]$, where $g_1 = \prod_{j=1}^{m} \left(g_0^{z^{s-m+j}}\right)^{n_j}$. Therefore $g_0^{-1} = g_0 g_0^2 = g_0[g_0, g_1] = g_0^{g_1}$, that is, g_0 is a real element of S_0. By taking images in G, we obtain that g is a real element of S. □

Proof of Theorem 1 — Let $G = [P]\langle\alpha\rangle$ be the semidirect product of P by $\langle\alpha\rangle$. Suppose that G is not p-nilpotent. Then, by [7, Kapitel IV, Satz 5.12], $p = 2$ and G possesses a minimal non-2-nilpotent subgroup X. Then X =

AB, where $A = X \cap P$ and, by considering a suitable conjugate, we can also assume that $B = \langle \beta \rangle \leqslant \langle \alpha \rangle$. By Theorem A, all elements of order 4 are real elements of P. The hypothesis of the theorem implies that B centralises A. This means that X is 2-nilpotent, a contradiction that proves the result. □

Proof of Theorem B — The arguments of the proof of [8, Theorem 5.25] prove that (1) implies (2) and (3).

Conversely, we assume, arguing by contradiction, that G is a non-p-nilpotent group which satisfies condition (3). Then G contains a minimal non-p-nilpotent subgroup C. By [7, Kapitel IV, Satz 5.4], $C = AB$, where A is a normal p-subgroup of C and $\exp A = p$ if p is odd, or $\exp A \leqslant 4$ if $p = 2$, and $B = \langle g \rangle$ is a cyclic Sylow q-subgroup of C, where $q \neq p$. Moreover, by Theorem A, every element of order 4 in A is real. The minimality of C implies that $A = [A, g]$. By [4, Chapter A, Corollary 12.4(b)], we have that $A = [A, g] = [A, g, g]$ and then $A \leqslant G^{\mathfrak{N}}$, by [7, Kapitel III, Satz 1.11]. Let S be a Sylow p-subgroup of G such that $A \leqslant S$. Let $a \in A$. The hypothesis on G implies that there exists $x_a \in S$ such that $a^{x_a} = a^g$. Hence $A \leqslant [A, S]$. Assume that the nilpotency class of S is n. Then

$$A \leqslant [A, S] \leqslant [A, S, S] \leqslant \ldots \leqslant [A, \underbrace{S, \ldots, S}_{n}] = 1.$$

Thus $A = 1$. Hence G cannot contain a minimal non-p-nilpotent subgroup and therefore G is p-nilpotent. □

REFERENCES

[1] A. Ballester-Bolinches – R. Esteban-Romero – D. J. S. Robinson: "On finite minimal non-nilpotent groups", *Proc. Amer. Math. Soc.* 133(12) (2005), 3455–3462.

[2] A. Ballester-Bolinches – L. M. Ezquerro – A. N. Skiba: "On subgroups of hypercentral type of finite groups", *Israel J. Math.* 199 (2014), 259–265.

[3] Y. Berkovich: "Sufficient conditions for 2-nilpotence of a finite group", *J. of Algebra App.* 12 (2013), 1250187.

[4] K. Doerk – T. Hawkes: "Finite Soluble Groups", *Walter de Gruyter, De Gruyter Expositions in Mathematics 4*, Berlin, New York (1992).

[5] Gol'fand, Ju. A.: "On groups all of whose subgroups are special", *Doklady Akad. Nauk SSSR* 60 (1948), 1313–1315.

[6] J. González-Sánchez: "A p-nilpotency criterion", *Arch. Math. (Basel)* 94 (2010), 201–205.

[7] B. Huppert: "Endliche Gruppen I", *volume 134 of Grund. Math. Wiss. Springer Verlag*, Berlin, Heildelberg, New York (1967).

[8] I. M. Isaacs: "Finite group theory", *volume 92 of AMS Graduate Studies in Mathematics 92*, Providence, RI, USA (2008).

[9] I. M. Isaacs and G. Navarro: "Normal p-complements and fixed elements", *Arch. Math. (Basel)* 95 (2010), 207–211.

[10] Z. Marciniak: "Fixed elements in 2-groups revisited", *Arch. Math. (Basel)* 97 (2011), 207–208.

Adolfo Ballester-Bolinches
Departament d'Àlgebra
Universitat de València
Dr. Moliner, 50, E-46100 Burjassot, València, Spain
e-mail: Adolfo.Ballester@uv.es

Ramón Esteban-Romero
Institut Universitari de Matemàtica Pura i Aplicada
Universitat Politècnica de València
Camí de Vera, s/n, E-46022 València, Spain
e-mail: resteban@mat.upv.es

Current address:
Departament d'Àlgebra
Universitat de València
Dr. Moliner, 50, E-46100 Burjassot, València, Spain
e-mail: Ramon.Esteban@uv.es

Luis M. Ezquerro
Departamento de Matemáticas - INAMAT
Universidad Pública de Navarra
Campus de Arrosadía, E-31006 Pamplona, Navarra, Spain
e-mail: ezquerro@unavarra.es