A note on a result of Guo and Isaacs about p-supersolubility of finite groups

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Abstract

In this note, global information about a finite group is obtained by assuming that certain subgroups of some given order are S-semipermutable. Recall that a subgroup H of a finite group G is said to be S-semipermutable if H permutes with all Sylow subgroups of G of order coprime to $|H|$. We prove that for a fixed prime p, a given Sylow p-subgroup P of a finite group G, and a power d of p dividing $|G|$ such that $1 \leq d < |P|$, if $H \cap \mathrm{O}^p(G)$ is S-semipermutable in $\mathrm{O}^p(G)$ for all normal subgroups H of P with $|H| = d$, then either G is p-supersoluble or else $|P \cap \mathrm{O}^p(G)| > d$. This extends the main result of Guo and Isaacs in *Arch. Math. (Basel)*, 105, 215–222 (2015). We derive some theorems that extend some known results concerning S-semipermutable subgroups.

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1 Introduction

All groups mentioned are implicitly assumed to be finite.

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Recall that two subgroups H and K of a group G are said to *permute* if $HK = KH$, that is, HK is a subgroup of G. A subgroup H of a group G is said to be *S-permutable* ([5], see also [1, Section 1.2]) in G if H permutes with all Sylow subgroups of G, and is said to be *S-semipermutable* in G ([9]) if H permutes with all Sylow q-subgroups of G for the primes q not dividing $|H|$.

Skiba, in his seminal paper [8], introduced the following subgroup embedding property: a subgroup H of a group G is said to be *weakly S-permutable* in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the largest S-permutable subgroup of G contained in H. This embedding property of subgroups has a strong structural impact and generalises many other known properties.

Recently, Guo and Isaacs considered in [2] the condition $U \cap H \trianglelefteq U$, with $U = \mathrm{O}^p(G)$, for a subgroup H of order d, where $d > 1$ is a power of p such that d divides $|G|$ and p is a prime that we hold fixed. This condition is less restrictive than weak S-permutability [2, Lemma A] and allows them to prove the following result.

Theorem 1 ([2, Theorem B])**.** *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $1 \leq d < |P|$. Let $U = \mathrm{O}^p(G)$, and assume that $H \cap U \trianglelefteq U$ for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either G is p-supersoluble, or else $|P \cap U| > d$.*

An interesting idea of [2] is that in the hypothesis of the theorem, only the normal subgroups of order d are considered, not necessarily the set of all subgroups of order d, at the drawback of obtaining as a conclusion either p-supersolubility or a restriction on the order of the Sylow p-subgroup of $\mathrm{O}^p(G)$.

We prove an analogous result, but instead of assuming that all subgroups $H \cap U$ are normal in U, we assume that all of them are S-semipermutable in U. Our starting point is the following observation. Let A be a p-subgroup of a group G. By [1, Lemma 1.2.16], $A \cap \mathrm{O}^p(G) \trianglelefteq \mathrm{O}^p(G)$ if and only if $A \cap \mathrm{O}^p(G)$ is S-permutable in G. If A is contained in $\mathrm{O}^p(G)$, and q is a prime different from p, then every Sylow q-subgroup of G is contained in $\mathrm{O}^p(G)$. Hence A is S-semipermutable in G if and only if A is S-semipermutable in $\mathrm{O}^p(G)$.

We prove:

Theorem 2. *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $1 \leq d < |P|$. Assume that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either G is p-supersoluble, or else $|P \cap \mathrm{O}^p(G)| > d$.*

We present some applications of Theorem 2. They concern the structure of a group G in which the subgroups $H \cap \mathrm{O}^p(G)$ are S-semipermutable in G for all subgroups H of a fixed order d of a given Sylow p-subgroup of G, and can be regarded as S-semipermutable versions of [2, Corollary C and Corollary E]. They can be also considered as an improvement of the following result.

Theorem 3 ([7, Theorem 3.3])**.** *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $p \leq d < |P|$. Assume that every subgroup of P with order d and all cyclic subgroups of P of order 4 (if $d = 2$ and P is not abelian) are S-semipermutable in G. Then G is p-supersoluble.*

We consider first the case when d is a prime.

Theorem 4. *Let $P \in \mathrm{Syl}_p(G)$. Suppose that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all subgroups $H \leq P$ with H cyclic of order p or 4 (if $p = d = 2$ and P is not abelian). Then G is p-supersoluble.*

Our last result concerns the case when $p < d < |P|$.

Theorem 5. *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $p < d < |P|$. Suppose that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all subgroups $H \leq P$ with $|H| = d$. Then G is p-supersoluble.*

We mention that Corollary C and Corollary E of [2] by Guo and Isaacs are immediate consequences of Theorems 4 and 5. Note that the proof of Corollary E in [2] is incomplete.

2 Proofs

Proof of Theorem 2. Assume the result is not true and let G be a counterexample of least order. Write $U = \mathrm{O}^p(G)$, $N = P \cap U$. Then $|N| \leq d$ and G is not p-supersoluble. In particular, $N \neq 1$ and $d \geq p$. Write $\mathcal{H} = \{H \trianglelefteq P \mid |H| = d\}$. By hypothesis, $H \cap U$ is S-semipermutable in G for each $H \in \mathcal{H}$.

Step 1. $\mathrm{O}_{p'}(G) = 1$.

Write $V = \mathrm{O}_{p'}(G)$ and assume that $V \neq 1$. Consider the factor group $\bar{G} = G/V$. Let $\bar{H}$ be a normal subgroup of $\bar{P}$ of order d. Then there is $H \in \mathcal{H}$ such that $\bar{H} = HV/V$. Since $H \cap U$ is S-semipermutable in G, we have $\bar{H} \cap \bar{U} = (H \cap U)V/V$ is S-semipermutable in G. Since $|\bar{U} \cap \bar{P}| \leq d$, it follows that $\bar{G}$ is p-supersoluble by the minimal choice of G. Hence G is p-supersoluble and this is contradiction. Thus $V = 1$, as required.

Step 2. N is S-semipermutable in G.

Since N is normal in P of order at most d, there is $H \in \mathcal{H}$ such that $N \leq H \leq P$. Then $N = H \cap U$ is S-semipermutable in G.

Step 3. Let Y be a maximal subgroup of P. Then $Y \cap U = Y \cap N$ is S-semipermutable in G.

Since $|N| \leq d$, there is an $H \in \mathcal{H}$ such that $Y \cap N \leq Y \cap U \leq H \leq Y$. Then $Y \cap N = Y \cap N \cap H = H \cap N = H \cap P \cap U = H \cap U$. Thus $Y \cap N = Y \cap U = H \cap U$ is S-semipermutable in G.

Step 4. Let $\langle N^G \rangle$ be the normal closure of N in G. Then $G/\langle N^G \rangle$ is p-nilpotent. In particular, G is p-soluble.

Since $\langle N^G \rangle \leq U$ and $|U : \langle N^G \rangle|$ divides the p'-number $|U : N|$, we have that $U/\langle N^G \rangle$ is a p'-group. Moreover, G/U is a p-group. It follows that $G/\langle N^G \rangle$ is p-nilpotent. On the other hand, we know that $G = PU$ and, by Step 2, N permutes with each Sylow q-subgroup of U for every prime $q \neq p$. By [4, Theorem A], the normal closure $\langle N^G \rangle$ of N in G is soluble. We conclude that G is p-soluble.

Step 5. A final contradiction.

Assume that $d = p$. Then N is a normal Sylow p-subgroup of U by Steps 1 and 4. Hence G is p-supersoluble, against the choice of G. Thus $d > p$. Let T be a minimal normal subgroup of G contained in U. By Steps 1 and 4, $T \leq N$. Thus $|T| \leq d$. Suppose that $|T| < d$. We argue that G/T satisfies the hypotheses of the theorem. Clearly, $1 \leq d/|T| < |P/T|$. Let H/T be a normal subgroup of P/T of order $d/|T|$. Then $H \in \mathcal{H}$. It follows that $H/T \cap \mathrm{O}^p(G/T) = H/T \cap U/T = (H \cap U)/T$, which is S-semipermutable in G/T. This shows that G/T satisfies the hypotheses of the theorem, as claimed. Since $|P/T \cap U/T| = |(P \cap U)/T| \leq d/|T|$, it follows that G/T is p-supersoluble by minimality of G. By [3, Kapitel VI, Satz 8.6], we may suppose that $T \not\leq \Phi(P)$. Let Y be a maximal subgroup of P such that $T \not\leq Y$. By Step 3, $Y \cap N$ is S-semipermutable in G, and so $(Y \cap N)Q$ is a subgroup of G for every Sylow q-subgroup Q of G with $q \neq p$. Since T is a normal p-subgroup of G, it follows that $T \cap (Y \cap N)Q = T \cap Y \cap N = T \cap Y$, which is normal in $(Y \cap N)Q$. Then $T \cap Y$ is normalised by Q. It follows that $T \cap Y$ is normalised by U. Since $T \cap Y$ is a normal subgroup of P, we have that $T \cap Y \trianglelefteq G$. This implies that $T \cap Y = 1$, $|T| = p$, and so G is p-supersoluble, which is a contradiction.

Let $|T| = d$. Then $T = N$. By Step 4, G/T is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation by [3, Kapitel VI, Beispiele 7.6], T is a proper subgroup of P which is not contained in $\Phi(G)$. Let Y be a maximal subgroup of P such that $T \not\leq Y$. By Step 3, $N \cap Y = T \cap Y$ is S-semipermutable in G. Arguing as in the previous paragraph, we obtain that $Y \cap T = Y \cap N$ is normal in G. This implies that $T \cap Y = 1$ and $|T| = p$, against our assumption $|T| = d > p$. This final contradiction completes the proof. □

Proof of Theorem 4. We proceed by induction on $|G|$. Write $U = \mathrm{O}^p(G)$, $U_p = P \cap U$. We may suppose that $\mathrm{O}_{p'}(G) = 1$. By Theorem 2, we may suppose that $|U_p| > p$. If $G = U$, then each subgroup of order p or 4 is S-semipermutable in G. Applying Theorem 3, we get that G is p-supersoluble. Therefore, we may assume that U is a proper subgroup of G. Clearly $U = \mathrm{O}^p(U)$ and every subgroup of order p or 4 of U_p is S-semipermutable in U by [1, Lemma 1.2.7]. By the induction hypothesis, U is p-supersoluble. Since $\mathrm{O}_{p'}(U) = 1$, it follows that U_p is normal in U by [1, Lemma 1.2.16]. Hence U_p is a normal subgroup of G. Assume that $p = 2$. Then U is 2-nilpotent and so it is a 2-group. This implies that G is a 2-group. Therefore we may suppose that $p > 2$. Then every chief factor of G below U_p is cyclic, by [6, Theorem 3.3]. Since G/U_p has a normal Hall p'-subgroup, we conclude that G is p-supersoluble. □

The proof of Theorem 5 depends on the following lemmas. The first one is the S-semipermutable version of [2, Corollary C].

Lemma 6. *Let $P \in \mathrm{Syl}_p(G)$ with $|P| > p$. Suppose that, for every maximal subgroup H of P, $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G. Then G is p-supersoluble.*

Proof. With a contradiction in mind, assume that G is not p-supersoluble. Write $U = \mathrm{O}^p(G)$. By Theorem 2, $P \cap U = P$, that is, $P \leq U$, and so $G = U$. This means that every maximal subgroup of P is S-semipermutable in G. Applying [7, Theorem 3.2], we conclude that G is p-supersoluble. □

Lemma 7. *Let $P \in \mathrm{Syl}_p(G)$ and write $U = \mathrm{O}^p(G)$. Suppose that $U_p = P \cap U$ is a normal p-subgroup of G and that d is a power of p such that $p < d < |U_p|$. Suppose also that H is S-permutable in G for all subgroups $H \leq U_p$ with $|H| = d$. Then G is p-supersoluble.*

Proof. We proceed by induction on $|G|$. We may suppose that $\mathrm{O}_{p'}(G) = 1$. If $U = G$, we can apply Theorem 3 to conclude that G is p-supersoluble. Therefore we may assume that U is a proper subgroup of G. Since $U = \mathrm{O}^p(U)$ and every subgroup of order d of U_p is S-permutable in U by [1, Lemma 1.2.7], we have that U is p-supersoluble. If $p = 2$, we can argue as in Theorem 4 to conclude that G is a 2-group. Hence we may assume $p > 2$.

Let T be a minimal normal subgroup of G contained in U_p. Assume that $|T| > d$. Let H be a normal subgroup of P such that $H \leq T$ and $|H| = d$. Since H is S-permutable in G, we have that $U \leq \mathrm{N}_G(H)$ by [1, Lemma 1.2.16]. Therefore $G = UP \leq \mathrm{N}_G(H)$, that is, H is a normal subgroup of G. Since T is a minimal normal subgroup of G, we conclude that $H = 1$, against our assumption $|H| = d > p$. Therefore $|T| \leq d$. We focus now on G/T and prove that it is p-supersoluble.

Assume that $|T| < d$. Then H/T is S-permutable in G/T for each $H/T \leq U_p/T$ with $|H/T| = d/|T|$. If $d/|T| > p$, then G/T is p-supersoluble by induction. If $d/|T| = p$, the conclusion follows from Theorem 4.

Assume now that $|T| = d$. Therefore we may assume that p is odd. Bearing in mind Lemma 6, we may suppose that $|U_p| > p|T| = pd$. Let K be a subgroup of U_p such that $|K/T| = p$. Since T is non-cyclic, there is a maximal subgroup L of K such that $K = TL$. Then L is of order d and so it is S-permutable in G. By [1, Lemma 1.2.7], K/T is S-permutable in G/T. By Theorem 4, G/T is p-supersoluble.

Since G/T is p-supersoluble and the class of all p-supersoluble groups is a saturated formation by [3, Kapitel VI, Hilfssatz 8.3], we may assume that T is the unique minimal normal subgroup of G contained in U_p and $\Phi(G) \cap U_p = 1$. By [3, Kapitel III, Satz 4.5], U_p is a direct product of some minimal normal subgroups of G. Thus $|U_p| = |T|$, against the hypothesis $|T| \leq d < |U_p|$. This final contradiction completes the proof. □

Proof of Theorem 5. We may suppose that $\mathrm{O}_{p'}(G) = 1$. By Theorem 2, we may suppose that $|P \cap U| > p$. If $G = U$, then each subgroup of order d is S-semipermutable in G. By Theorem 3, the conclusion follows. Suppose that $U < G$. By [1, Lemma 1.2.7], the hypotheses of the theorem hold in U. By induction, U is p-supersoluble. Since $\mathrm{O}_{p'}(U) = 1$, we can apply [1, Lemma 1.2.16] to conclude that $U_p = U \cap P$ is a normal subgroup of G. Therefore all subgroups of order d are S-permutable in G by [7, Lemma 2.2]. Applying Lemma 7, we conclude that G is p-supersoluble. □

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References

[1] A. Ballester-Bolinches, R. Esteban-Romero, and M. Asaad. *Products of finite groups*, volume 53 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter, Berlin, 2010.

[2] Y. Guo and I. M. Isaacs. Conditions on p-subgroups implying p-nilpotence or p-supersolvability. *Arch. Math. (Basel)*, 105:215–222, 2015.

[3] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[4] I. M. Isaacs. Semipermutable π-subgroups. *Arch. Math. (Basel)*, 102:1–6, 2014.

[5] O. H. Kegel. Sylow-Gruppen und Subnormalteiler endlicher Gruppen. *Math. Z.*, 78:205–221, 1962.

[6] Y. Li and B. Li. On weakly s-supplemented subgroups of finite groups. *J. Algebra Appl.*, 10:1–10, 2011.

[7] Y. Li, S. Qiao, N. Su, and Y. Wang. On weakly s-semipermutable subgroups of finite groups. *J. Algebra*, 371:250–261, 2012.

[8] A. N. Skiba. On weakly s-permutable subgroups of finite groups. *J. Algebra*, 315(1):192–209, 2007.

[9] L. Wang and Y. Wang. On s-semipermutable maximal and minimal subgroups of Sylow p-subgroups of finite groups. *Comm. Algebra*, 34:143–149, 2006.