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TWO-PERFECT FLUID INTERPRETATION OF AN ENERGY TENSOR

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A B S T R A C T :

The paper contains the necessary and sufficient conditions for a given energy tensor (to) be interpreted as a sum of two perfect fluids. Given a tensor of this class, the decomposition in two perfect fluids (which is determined up to a couple of real functions) is obtained.

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1. INTRODUCTION.

There are many topics in General Relativity where matter is represented by a mixture of two fluids. In fact, some astrophysical and cosmological situations need to be described by a^an^ energy tensor made with the sum of two or more perfect fluids rather that with only one. Dunn [1] has, recently, outlined some remarkable features of two-perfect fluid models in Gödel type spacetime, in which a fluid represents the matter and the other one the isotropic radiation in the universe. Letelier [2] studied two-perfect fluid solutions of the Einstein's equations when the velocities of both components are irrotational. Bayin [3] derived some analytic solutions for an anisotropic fluid and he argues the possibility that certain solutions could be interpreted as due to a pair of perfect fluids. Inhomogeneous cosmologies with two interacting and comoving fluids has been examined by Lima and Tiomno [4]; in this models, the fluids are material: one is taken as a FRW polytropic fluid and the other as an inhomogeneous dust.

However, at the present time, one knows few solutions of the Einstein's equations which describe the gravitational field associated with two noncomoving perfect fluids. We have obtained [5] a class of such solutions where the velocity of one of the fluids is geodesic, shear free and irrotational.

The goal of this paper is to analize the algebraic properties of the energy tensors which are the sum of two perfect fluids. Such a study seems interesting because it is useful to know if a given metric is a solution of the field equations with a mixture of two perfect fluids as source, or even to construct new solutions. Letelier [2] is the first, up to our knowledge, who has studied some algebraic aspects of this subject. The uniqueness problem has been considered by Hall and Negm [6].

In Section 2, we consider the class of symmetric tensors which have a spacelike 2-eigenplane. This class contains the tensors associated with the sum of two perfect fluids as a particular case.

In Section 3, we put the following question: if a tensor T can be interpreted as the sum of two perfect fluids, how many decompositions of T (in two fluids) are available?. We shown that, generically, there exists a two-parameter family of pairs $\{T_1, T_2\}$ of perfect fluids such that $T_1+T_2= T$. Also, we compute the several algebraic types (Segré types) which are compatible with this T.

Next, in Section 4, we obtain the expressions of the velocities, pressures and densities of T_1 and T_2 in terms of the eigenvalues and eigenvectors of T.

Finally, in Section 5, it is required that the energy tensor of each perfect fluid satisfies the Plebański energy conditions [7]. Then we give the invariant characterization, i.e. the necessary and sufficient requirements for a given tensor T splits in the sum of such two fluids. The results, without proof, of this section were comunicated [8] to the Spanish relativistic meeting E.R.E.-89.

2. SYMMETRIC TENSORS WITH A SPACELIKE 2-EIGENPLANE.

Let T be a real symmetric two-tensor on the space-time (V_4, g) with a spacelike 2-eigenplane Π, associated to the eigenvalue λ. The signature of the lorentzian metric g is taken to be $\{-+++\}$. In an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ adapted to Π, $\Pi\equiv\{e_2, e_3\}$, T can be written as

$$T = A\ e_0\otimes e_0 + B\ e_1\otimes e_1 + C\ e_0\tilde{\otimes}\ e_1 + \lambda\ (e_2\otimes e_2 + e_3\otimes e_3) \tag{1}$$

when $\tilde{\otimes}$ denotes the symmetrized tensorial product, and $\{e_0, e_1\}$ generates the timelike 2-plane $\Pi^{\perp}$ orthogonal to Π. Now, let $T_{\perp}$ be the restriction of T on $\Pi^{\perp}$; then, assuming that $\Pi^{\perp}$ is known, the tensor $T_{\perp}$ provides supplementary algebraic properties of T. The eigenvalues of $T_{\perp}$ are given by

$$\lambda_{\pm} = \frac{1}{2}\ (\ B - A \pm \sqrt{\delta}\ ) \tag{2}$$

with

$$\delta \equiv (A + B)^2 - 4C^2 = (\lambda_+ - \lambda_-)^2 \tag{3}$$

And introducing the invariant

$$\Delta \equiv (\lambda - \lambda_+)(\lambda - \lambda_-) = C^2 - (A + \lambda)(B - \lambda) \tag{4}$$

There ~~It~~ results:

Lemma 2.1. Let T be a symmetric tensor with a spacelike 2-eigenplane; then, T is of Segré type[2]

a) {1,1(11)} iff $\delta > 0$ and $\Delta \neq 0$.

b) {1(111)} iff $\delta > 0$ and $\Delta = 0$.

c) either {(1,1)(11)} or {2(11)} iff $\delta = 0$ and $\Delta > 0$.

d) either {(1,111)} or {(211)} iff $\delta = \Delta = 0$.

e) {$z\bar{z}$(11)} iff $\delta < 0$.

Clearly, the Segré types {(21)1},{31} and {(31)} are forbidden. So, the case with a strict triple eigenvalue only corresponds to {1(111)}. In the cases c) and d), the Segré types can be distinguished by the minimal equation of $T_\perp$. So, $\tilde{T}_\perp=0$ characterizes {(1,1)11} and {(1,111)} types , and $\tilde{T}^2_\perp=0$ (with $\tilde{T}_\perp\neq 0$) the {2(11)} and {(211)} types, being $\tilde{T}_\perp$ the trace free part of T with respect the induced metric on $\Pi^\perp$.

From (3) and (4) we have

$$\delta + 4\,\Delta = (\,\lambda_+ + \lambda_- - 2\,\lambda\,)^2 \geq 0 \tag{5}$$

From this equation and according to c) and d), we have $\Delta \geq 0$ if $\delta = 0$, and $\Delta > 0$ if $\delta < 0$. On the other hand, if $\Delta < 0$ the Segré type of T is {1,1(11)}. We will use this result below dealing with two-fluids energy tensors.

From (2) and (3), $C^2 = (A + \lambda_+)(A + \lambda_-)$. Let us suppose $C \neq 0$, then the eigenvectors of T associated with $\lambda_\pm$ are given by

$$v_\pm = C\, e_0 + (A + \lambda_\pm)\, e_1 \tag{6}$$

whence

$$g(v_\pm, v_\pm) = \pm\,\frac{1}{2}\,\sqrt{\delta}\,(A + B \pm \sqrt{\delta}\,) \tag{7}$$

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[2] For a comprehensive ~~explication~~ explanation of Segré notation see, for exemple, [9].

The vectors $v_{\pm}$ are complex conjugate when $\delta < 0$ and they have a timelike real part and a spacelike imaginary part. When $\delta = 0$, then $v_{\pm}$ are collinear null vectors. Furthermore, from (3) and (7) ~~it~~ there results:

Lemma 2.2. For a tensor T as given by (1) and with $\delta > 0$, the sign of A+B is the same for any orthonormal basis. This sign ~~provides~~ determines the causal character of the eigenvectors $v_{\pm}$ of T according to

$$\mathrm{sgn.}[g(v_{\pm}, v_{\pm})] = \pm\, \mathrm{sgn.}(A+B) \tag{8}$$

~~The~~ Lemma 2.2. is useful ~~to study~~ in studying the Segré types {1,1(11)} and {1(111)}. In particular, it allows us to discriminate between the Segré subtypes {1,(111)} and {(1,11)1}.

3. TWO-PERFECT FLUID ENERGY TENSOR.

Henceforth, we consider the tensors which are obtained as the sum of two perfect fluids, $T_i = (\rho_i + p_i) u_i \otimes u_i + p_i g$ (i=1,2), that is

$$T = T_1 + T_2 = (\rho_1 + p_1)\, u_1 \otimes u_1 + (\rho_2 + p_2)\, u_2 \otimes u_2 + (p_1 + p_2)\, g \tag{9}$$

where ρ_i, p_i and u_i stand for the proper energy density, pressure and unit velocity of each fluid, respectively. No assumption about energy conditions is made in this Section, which will be devoted to study general properties of T.

Clearly, T admits a spacelike 2-eigenplane Π of eigenvalue $\lambda = p_1 + p_2$. We can then write, without loss of generality, $u_i = \mathrm{ch}\phi_i\, e_0 + \mathrm{sh}\phi_i\, e_1$, where $\{e_0, e_1\}$ is an orthonormal basis on the 2-plane $\Pi^{\perp}$. By comparing (1) with (9), it follows

$$A + \lambda = Q_1 \mathrm{ch}^2\phi_1 + Q_2 \mathrm{ch}^2\phi_2 \tag{10a}$$

$$B - \lambda = Q_1 \mathrm{sh}^2\phi_1 + Q_2 \mathrm{sh}^2\phi_2 \tag{10b}$$

$$2C = Q_1 \mathrm{sh}\, 2\phi_1 + Q_2 \mathrm{sh}\, 2\phi_2 \tag{10c}$$

where $Q_i \equiv \rho_i + p_i$.

If A, B, C and λ are given, eqs. ~~the equations~~ (10) constitute a linear system in the unknown Q_1 and Q_2, with coefficients depending on ϕ_1 and ϕ_2. Thus~~,~~ there exist a solution if, and only if, the determinant of the extended matrix vanishes, that is to say, ϕ_1 and ϕ_2 satisfy the relation

$$(\mathrm{th}\phi_1 - \mathrm{th}\phi_2)\ [\ (A + \lambda)\ \mathrm{th}\phi_1 \mathrm{th}\phi_2 - C\ (\mathrm{th}\phi_1 + \mathrm{th}\phi_2) + B - \lambda\ ] = 0 \quad (11)$$

When both perfect fluids are "tilted" with ~~one~~ respect to each ~~the~~ other, that is noncomoving ($\phi_1 \neq \phi_2$), every solution to eq. (11) leads to a solution (Q_1, Q_2) to eqs. (10). Thus, we have the following:

Theorem 3.1. An energy tensor T can be interpreted as the sum of two noncomoving perfect fluids if, and only if, it is of the form (1) and the equation

$$(A + \lambda)\ \mathrm{th}\phi_1 \mathrm{th}\phi_2 - C\ (\mathrm{th}\phi_1 + \mathrm{th}\phi_2) + B - \lambda = 0 \qquad (12)$$

admits a solution (ϕ_1, ϕ_2) such that $\phi_1 \neq \phi_2$.

Theorem 3.2. Let T be an energy tensor sum of two noncomoving perfect fluids. Then~~,~~ every solution (ϕ_1, ϕ_2) to eq. (12), with $\phi_1 \neq \phi_2$, furnishes a one-parameter[3] family of pairs $\{T_1, T_2\}$ of perfect fluids such that $T_1 + T_2 = T$. The velocities of the fluids are given by[4]

$$u_i \propto e_0 + \mathrm{th}\phi_i\ e_1 \qquad (13)$$

and~~,~~ their pressures and energy densities are restricted by ~~submitted to the restrictions~~

$$p_1 + p_2 = \lambda \qquad\qquad \rho_i + p_i = Q_i \qquad (14)$$

where Q_i are given by

$$Q_i = \frac{1 - \mathrm{th}^2\phi_i}{\mathrm{th}\phi_j - \mathrm{th}\phi_i}\ [\ (A + \lambda)\ \mathrm{th}\phi_j - C\ ]\ , \qquad j \neq i \qquad (15)$$

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[3] Of course, when T is a tensor field, this parameter is a real function.

[4] Latin indices take values 1,2.

The one-parameter family ~~refered~~ referred to in the last theorem is generated by the transformations leaving invariant Eqs.(14). On the other hand, in Section 4 we will show that there also exists a one-parameter family of solutions (ϕ_1, ϕ_2) to Eq.(12), resulting ~~finaly~~ finally (Theorem 5.2) in a two-parameter family of two-perfect fluid interpretations. This multiplicity of physical interpretations has been treated previously in [6].

Let us study now which Segré types admit a two-fluid interpretation. From (10), the invariants δ and Δ which were defined in (3), (4) may be written as

$$\delta = Q_1^2 + Q_2^2 + 2Q_1Q_2 \mathrm{ch}\, 2(\phi_1 - \phi_2) \tag{16}$$

$$\Delta = - Q_1Q_2 \mathrm{sh}^2(\phi_1 - \phi_2) \tag{17}$$

and taking into account Lemma 2.1, we obtain some remarkable consequences. Clearly, T is proportional to g when $\phi_1 = \phi_2$ and $Q_1 = - Q_2$ or when $\phi_1 \neq \phi_2$ and $Q_1 = Q_2 = 0$; ~~this~~ these cases are those for which $\delta = \Delta = 0$. In consequence,

Lemma 3.1. The sum of two perfect fluids with $Q_1 \neq 0$ is of type {(1,111)} if, and only if, the fluids are comoving and $Q_1 = - Q_2$.

Lemma 3.2. No tensor of type {(211)} can be obtained as the sum of two perfect fluids.

Suppose T be of type {(1,1)(11)}. Now, $\phi_1 \neq \phi_2$ because $\Delta > 0$. In a basis of eigenvectors of T, it is verified that C = 0 and A = - B, whence no solution exist to eq.(12). Therefore we get,

Lemma 3.3. No tensor of type {(1,1)(11)} can be obtained as the sum of two perfect fluids.

Since the regular electromagnetic field and pure radiation field are, respectively, of type {(1,1)(11)} and {(211)}, because of Lemmas 3.2 and 3.3, it follows:

Theorem 3.3. The energy tensor of the electromagnetic field (regular Maxwell field or pure radiation field) cannot be decomposed in the sum of two perfect fluids.

Besides, a triple eigenvalue for T is impossible when $Q_1 \neq 0$ and $\phi_1 \neq \phi_2$ because $\Delta \neq 0$. Therefore, it results:

Theorem 3.4. The energy tensor sum of two perfect fluids (with $Q_1 \neq 0$) is of type $\{1,1(11)\}$, $\{1,(111)\}$, $\{2(11)\}$ or $\{z\bar{z}(11)\}$. The type $\{1,(111)\}$ occurs if, and only if, the fluids are comoving.

The last assertion of Theorem 3.4. explains why in a two-fluid FRW models [10], either the fluids are comoving or one of them is an imperfect fluid.

We exclude the case $Q_i = 0$ because it correspond to a "degenerate fluid" $T_i \propto g$, and then $T_j + g = T$ $(i \neq j)$ is a perfect fluid too.

4. INVARIANT CHARACTERIZATION.

We will now ~~Below, we are going to~~ discuss separately each one of the three Segré types which are compatible with the sum of two noncomoving fluids.

a) Segré type {1,1(11)}.

Let T be of type $\{1,1(11)\}$ with single eigenvalues λ_0 and λ_1 associated respectively to normalized eigenvectors e_0 and e_1,

$$T = -\lambda_0 \, e_0 \otimes e_0 + \lambda_1 \, e_1 \otimes e_1 + \lambda \, (e_2 \otimes e_2 + e_3 \otimes e_3) \tag{18}$$

Then, eq. (12) may be written

$$\mathrm{th}\phi_1 \, \mathrm{th}\phi_2 = \frac{\lambda - \lambda_1}{\lambda - \lambda_0} \equiv -\Lambda \tag{19}$$

which admits a solution iff $|\Lambda| < 1$. Now, ϕ_1 and ϕ_2 give the relative velocity of each fluid with respect to e_o, and we have

Theorem 4.1. A symmetric tensor T of type {1,1(11)}, given by (18), admits a two-perfect fluid interpretation if, and only if, it ~~verifies~~ satisfies

$$|\lambda - \lambda_o| > |\lambda - \lambda_1|$$

The ~~n~~ velocities, pressures and energy densities of the fluids are then given by Theorem 3.2., with ϕ_1 and ϕ_2 given by

$$\mathrm{th}\phi_1 = r \ , \qquad \mathrm{th}\phi_2 = -\frac{\Lambda}{r} \ , \qquad |\Lambda| < |r| < 1$$

where

$$\Lambda \equiv \frac{\lambda_1 - \lambda}{\lambda - \lambda_o}$$

b) Segré type {2(11)}.

The canonical form of a tensor T of type {2(11)} in an orthonormal basis is [7]:

$$T = (\varkappa - \alpha)e_o \otimes e_o + (\varkappa + \alpha)e_1 \otimes e_1 + \varkappa\, e_o \tilde{\otimes} e_1 + \lambda(e_2 \otimes e_2 + e_3 \otimes e_3) \qquad (20)$$

where $e_o + e_1$ is the null eigenvector of T with eigenvalue α, and $\varkappa$ is exactly the sign of $T(e_o - e_1, e_o - e_1)$, $\varkappa = \pm 1$. Let us examine eq. (12). Comparing (20) with (1), $A = \varkappa - \alpha$, $B = \varkappa + \alpha$ and $C = \varkappa$. Thus, when $\alpha - \lambda = \varkappa$, eq. (12) becomes $\mathrm{th}\phi_1 + \mathrm{th}\phi_2 = 2$, which has ~~not any~~ no solution. However, when $\alpha - \lambda \neq \varkappa$, eq. (12) is of the form

$$x\, y - a\, (x + y) + b = 0 \qquad (21)$$

with $x = \mathrm{th}\phi_1$, $y = \mathrm{th}\phi_2$ and

$$a = \frac{\varkappa}{\varkappa + \lambda - \alpha} \qquad\qquad b = \frac{\varkappa - \lambda + \alpha}{\varkappa + \lambda - \alpha} \tag{22}$$

Clearly, $a^2 > b$ which says that the invariant Δ defined by (4) is positive.

Considering the intersection of the hyperbola (21) with the domain

$$\mathcal{R} \equiv \{(x,y) \in \mathbb{R}^2,\ |x| < 1 \text{ and } |y| < 1\}$$

we have the following:

Lemma 4.1. ~~The~~ Equation (21) with $a^2 > b$, admits a solution in $\mathcal{R}$ iff $|x_\pm| < 1$, being

$$x_\pm = a \pm \sqrt{a^2 - b} \tag{23}$$

and its solutions are given by

$$x \in [x_\pm, \pm 1) \quad \text{and} \quad y = H(x) \quad \text{if } |H(\pm 1)| \leq 1$$

$$x \in (\mp 1, x_\pm] \quad \text{and} \quad y = H(x) \quad \text{if } |H(\pm 1)| \geq 1 .$$

where

$$H(x) = \frac{b - ax}{a - x}$$

In particular, when a and b have the expressions (22) it ~~results~~ follows that

$$x_\pm = \frac{\varkappa \pm (\lambda - \alpha)}{\varkappa + \lambda - \alpha}$$

Now, $|x_+| = 1$, and we have $|x_-| < 1$ iff the sign of $\lambda - \alpha$ is equal to that of $\varkappa$ $\varkappa$.
In consequence, we have

Theorem 4.2. A symmetric tensor T of type {2(11)}, given by (20), admits a two-perfect fluid interpretation if, and only if, it verifies

$$\varkappa\ (\lambda - \alpha) > 0$$

Then the velocities, pressures and energy densities of the fluids are given by Theorem 3.2., with ϕ_1 and ϕ_2 given by

$$\mathrm{th}\phi_1 \equiv x \in (-1, x_-), \qquad \mathrm{th}\phi_2 = \frac{\varkappa\ (1 - x) + \alpha - \lambda}{\varkappa\ (1 - x) - \alpha + \lambda}$$

where

$$x_- = \frac{\varkappa + \alpha - \lambda}{\varkappa - \alpha + \lambda}$$

c) Segré type {z$\bar{z}$(11)}.

The canonical form of a tensor T of type {z$\bar{z}$(11)} in an orthonormal basis is [7]:

$$T = \mu(-e_0 \otimes e_0 + e_1 \otimes e_1) + \nu\ e_0 \tilde{\otimes} e_1 + \lambda\ (e_2 \otimes e_2 + e_3 \otimes e_3) \tag{24}$$

where $\nu > 0$ and $e_0 \pm ie_1$ are the eigenvectors of T associated with the conjugate complex eigenvalues $\lambda_\pm = \mu \pm i\nu$. Now, comparing (24) with (1), $-A = B = \mu$ and $C = \nu$. If $\lambda = \mu$, eq.(12) becomes $\mathrm{th}\phi_1 + \mathrm{th}\phi_2 = 0$, whose solutions are $\phi_2 = -\phi_1 \in (0, \infty)$. If $\lambda \neq \mu$, eq.(12) has again the form (21) with $a = \nu/(\lambda-\mu)$ and $b = -1$, and expression (23) gives

$$x_\pm = \frac{\nu \pm \sqrt{\nu^2 + (\lambda - \mu)^2}}{\lambda - \mu}$$

As $x_+ x_- = -1$ and $a^2 > b$, it results that $|x_-| < 1$ and $|x_+| > 1$. So ~~that~~, lemma 4.1 leads to the following:

Theorem 4.3. Any symmetric tensor T of type $\{z\bar{z}(11)\}$ admits a two-perfect fluid interpretation.

With the notation of (24), velocities, pressures and energy densities of the fluids are given by Theorem 3.2., with ϕ_1 and ϕ_2 given by

$$\mathrm{th}\phi_1 \equiv x \in (-1, x_-), \qquad \mathrm{th}\phi_2 = \frac{\lambda - \mu - \nu x}{\nu + (\mu - \lambda) x}$$

where

$$x_- = 0 \qquad \text{when } \lambda = \mu, \text{ and}$$

$$x_- = \frac{\nu - \sqrt{\nu^2 + (\lambda - \mu)^2}}{\lambda - \mu} \qquad \text{when } \lambda \neq \mu$$

Generically, there exists one-parameter family of solutions (ϕ_1, ϕ_2) to eq.(12) with $\phi_1 \neq \phi_2$. Thus, because of theorem 3.2., the three studied cases admet a two-parameter family of two-perfect fluid interpretations. When T is considered as a tensor field, these parameters are real functions. admit

5. ENERGY CONDITIONS.

In this section we require that each perfect fluid satisfy the Plebański energy conditions [7]. Generally, this conditions are assumed for macroscopic physics and they state that, for any observer, the energy density is non negative and the Poynting vector is non spacelike . Thus, a symmetric 2-tensor T satisfies the Plebański energy conditions (called in [11] the dominant energy condition) when

$$T(u,u) \geq 0 \quad \text{and} \quad T^2(u,u) \leq 0, \quad \forall\, u \text{ timelike.}$$

For a perfect fluid T_i the Plebański conditions are equivalents to the inequalities $Q_i \equiv \rho_i + p_i > 0$ and $\rho_i - p_i \geq 0$. From (17), $Q_i > 0$ implies that $\Delta < 0$, and ~~in~~ on account of lemma 2.1, we have, according to previous results [2][6]:

Lemma 5.1. If ~~a~~ an energy tensor T is the sum of two perfect fluids submitted to the Plebański energy conditions then T is of type {1,1(11)}.

Now, if T is of type {1,1(11)}, we search for the additional requirements in order to T may be decomposed in the sum of two perfect fluids submitted subject to the energy conditions. From Theorem 4.1. and expressions (15) and (19), ~~it results~~ it follows that

$$Q_1 = \frac{1 - r^2}{\Lambda + r^2}(\lambda_1 - \lambda), \qquad Q_2 = \frac{\Lambda^2 - r^2}{\Lambda + r^2}(\lambda_0 - \lambda)$$

so that, both Q_1 and Q_2 are positive iff $\lambda - \lambda_0 > \lambda_1 - \lambda > 0$. Also, we have $Q_1 + Q_2 = 2\lambda - \lambda_0 - \lambda_1$, and from (14) one gets

$$\rho_1 - p_1 = s \equiv 2\rho_1 - Q_1, \qquad \rho_2 - p_2 = -(\lambda_0 + \lambda_1) - s$$

whence, both $\rho_1 - p_1$ and $\rho_2 - p_2$ are positive iff $-(\lambda_0 + \lambda_1) \geq s \geq 0$. Thus We have ~~thus~~ the following theorems:

Theorem 5.1. A symmetric tensor T may be decomposed in the sum of two perfect fluids submitted to the Plebański energy conditions if, and only if, it is of type {1,1(11)} and its eigenvalues ~~verify~~ satisfy

$$\lambda_0 + \lambda_1 \leq 0, \qquad \frac{1}{2}(\lambda_0 + \lambda_1) < \lambda < \lambda_1$$

where λ_0 (resp. λ_1) is the simple eigenvalue of T which has associated with it a timelike (resp. spacelike) eigenvector, and λ is the double eigenvalue.

Theorem 5.2. Let T be as in the previous theorem. Then, there exist a two-parameter family of pair $\{T_1, T_2\}$ of perfect fluids submitted to the energy Plebański conditions such that $T_1 + T_2 = T$.

Velocities, energy densities and pressures of the fluids are given by

$$u_1 \propto e_0 + r\, e_1, \qquad u_2 \propto e_0 - \frac{\Lambda}{r}\, e_1$$

$$\rho_1 = \frac{1}{2}\,(Q + s)\ , \qquad \rho_2 = \lambda - \lambda_0 - \lambda_1 - \frac{1}{2}\,(Q + s)$$

$$p_1 = \frac{1}{2}\,(Q - s)\ , \qquad p_2 = \lambda - \frac{1}{2}\,(Q - s)$$

where e_0 (resp. e_1) is the unit eigenvector associated with λ_0 (resp. λ_1) and the parameters r and s taking the values:

$$r \in (\Lambda, 1)\ , \qquad s \in [0, -\lambda_0 - \lambda_1]$$

with

$$\Lambda \equiv \frac{\lambda_1 - \lambda}{\lambda - \lambda_0}\ , \qquad Q \equiv \frac{1 - r^2}{\Lambda + r^2}\,(\lambda_1 - \lambda)$$

In Theorem 5.1. we have given the invariant characterization of the class of tensors which admit a macroscopic two-fluid interpretation. From this result, and taking into account lemma 2.2, and expressions (2), (3), (2)-(4)
(4), one obtains a practical characterization in terms of the components of T in an orthonormal basis adapted to the spacelike 2-eigenplane:

Corollary 5.1. ~~A~~ An energy tensor T may be interpreted as a sum of two perfect fluids submitted to the Plebański energy conditions if, and only if, T has a spacelike 2-eigenplane and, with the notation of (1), T ~~verifies:~~ satisfies

$$-A < B \le A\ , \qquad A - B + 2\lambda > 0 \qquad \text{and} \qquad C^2 < (A + \lambda)(B - \lambda)$$

6. SUMMARY AND CONCLUSIONS.

We have presented a general study of the algebraic properties of energy tensors which admit an interpretation as the mixture of two perfect fluids. We have shown that only Segré types {1,(111)}, {1,1(11)}, {2(11)}, and $\{z\bar{z}(11)\}$ are possible; the first one if, and only if, the fluids are comoving (theorem 3.4). For every type, we give the invariant characterization (only in terms of its eigenvalues and eigenvectors) and the family (depending of two real functions) of possible interpretations (theorems 4.1, 4.2 and 4.3). In this part of the work no energy conditions have been imposed because, as it is known [7], these conditions may not be applicable in some microphysic situations. Finally, the case of fluids submitted to the Plebański energy conditions has been considered (theorems 5.1 and 5.2). subject

It follows ~~of~~ from this study that there exist two degrees of freedom in the splitting of an energy tensor T as sum of two perfect fluids. This propert~~ie~~y was shown in [6]. In our paper we give, for every interpretation, the explicit expressions of the densities, pressures and velocities. In the case of macroscopic fluids the degrees of freedom are given by two functions r and s (see theorem 5.2) taking values in bounded real intervals. The first function has a kinematic meaning and it determines the relative velocity of a fluid with respect to the other, $\beta = (r + \Lambda/r)/(1 + \Lambda)$; the other one is thermodynamic and it fixes the transformations leaving invariant $p_1 + p_2$ and $\rho_1 + p_1$. Both freedoms may be useful in the research of two-perfect fluid solutions submitted to given kinematic or thermodynamic properties (equation of state of each fluid, law which describes their~~s~~ interaction, particular movement for one or both fluids, ~~...~~ etc).

For example, when both components are formed by dust ($p_i = 0$), one has necessarily $\lambda = 0$ and, then s depends of r which take values in $(-\lambda_1/\lambda_0, 1)$; in this case there exist a degree of freedom. On the other hand, when an isotropic radiative fluid ($\rho_1 = 3p_1$) and a dust ($p_2 = 0$) are considered, s and r are uniquely determined and then the interpretation is ~~unic~~. unique An approach concerning more general and kinematic restrictions will be considered elsewhere.

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Two-Perfect Fluid Interpretation of an Energy Tensor

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The paper contains the necessary and sufficient conditions for a given energy tensor to be interpreted as a sum of two perfect fluids. Given a tensor of this class, the decomposition in two perfect fluids (which is determined up to a couple of real functions) is obtained.

1. INTRODUCTION

There are many topics in General Relativity where matter is represented by a mixture of two fluids. In fact, some astrophysical and cosmological situations need to be described by an energy tensor made up of the sum of two or more perfect fluids rather than that with only one. Dunn [1] has recently outlined some remarkable features of two-perfect fluid models in Gödel type space-time, in which one fluid represents the matter and the other one the isotropic radiation in the universe. Letelier [2] studied two-perfect fluid solutions of the Einstein equations when the velocities of both components are irrotational. Bayin [3] derived some analytic solutions for an anisotropic fluid and he argues the possibility that certain solutions could be interpreted as due to a pair of perfect fluids. Inhomogeneous cosmologies with two interacting and comoving fluids have been examined by Lima and Tiomno [4]; in these models the fluids are material: one is taken as a FRW polytropic fluid and the other as an inhomogeneous dust.

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However, at the present time we know few solutions of the Einstein equations which describe the gravitational field associated with two noncomoving perfect fluids. We have obtained [5] a class of such solutions where the velocity of one of the fluids is geodesic, shear free and irrotational.

The goal of this paper is to analyze the algebraic properties of the energy tensors which are the sum of two perfect fluids. Such a study seems interesting because it is useful to know whether a given metric is a solution of the field equations with a mixture of two perfect fluids as source, or even to construct new solutions. Letelier [2] is the first, to our knowledge, to have studied some algebraic aspects of this subject. The uniqueness problem has been considered by Hall and Negm [6].

In Section 2, we consider the class of symmetric tensors which have a spacelike 2-eigenplane. This class contains the tensors associated with the sum of two perfect fluids as a particular case.

In Section 3, we put the following question: if a tensor T can be interpreted as the sum of two perfect fluids, how many decompositions of T (in two fluids) are available? We show that, generically, there exists a two-parameter family of pairs $\{T_1, T_2\}$ of perfect fluids such that $T_1 + T_2 = T$. Also, we compute the several algebraic type (Segré types) which are compatible with this T.

Next, in Section 4, we obtain the expressions of the velocities, pressures and densities of T_1 and T_2 in terms of the eigenvalues and eigenvectors of T.

Finally, in Section 5, it is required that the energy tensor of each perfect fluid satisfy the Plebanski energy conditions [7]. Then we give the invariant characterization, i.e. the necessary and sufficient requirements for a given tensor T split in the sum of two such fluids. The results, without the proof, of this section were communicated to the Spanish relativity meeting E.R.E.-89 [8].

2. SYMMETRIC TENSORS WITH A SPACELIKE 2-EIGENPLANE

Let T be a real symmetric two-tensor on the space-time (V_4, g) with a spacelike 2-eigenplane Π, associated to the eigenvalue λ. The signature of the lorentzian metric g is taken to be $\{-+++\}$. In an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ adapted to Π, $\Pi \equiv \{e_2, e_3\}$, T can be written as

$$T = Ae_0 \otimes e_0 + Be_1 \otimes e_1 + Ce_0 \tilde{\otimes} e_1 + \lambda(e_2 \otimes e_2 + e_3 \otimes e_3) \tag{1}$$

where $\tilde{\otimes}$ denotes the symmetrized tensorial product, and $\{e_0, e_1\}$ generates the timelike 2-plane $\Pi^{\perp}$ orthogonal to Π. Now, let $T_{\perp}$ be the restriction