

THE 199 CAUSAL CLASSES OF FRAMES
OF THE SPACE-TIME

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ABSTRACT:

It is shown that, from the causal point of view, the space-time admits 199, and only 199, different classes of frames.

I. INTRODUCTION

The space-time is usually described as a four dimensional lorentzian manifold. Its topology, its differentiable and metric structures, its asymptotic properties have been the object of many studies; from the formal point of view there is no doubt that the notion of space-time is at present well defined.

In spite of this fact, a good physical comprehension of this notion has not yet been attained. One of the points that contribute to this situation is our inability to conceive *directly*¹ local domains of the space-time.

The importance of this deficiency may be more sensibly explained by comparing the up to now evolution of the notion of space-time to the ancient elaboration of the notion of *space*.

For our purposes, this elaboration may be considered as achieved after the work by Aristarchus of Samos. Before it, in the historical evolution of the notion of space, one may find the following successive sketches²: For the Egyptians³, about 2.500-2.300 b.C., the earth is a flat disk crossed by the Nile and surrounded by the sea, and the heaven, separated from the earth by the atmosphere, is supported by eight columns⁴; About 2.300-1.500 b.C., the scenario is quite similar, but the heaven is supported by four columns⁵; For the Greeks, long before 600 b.C., the earth is a flat disk surrounded by the Ocean river, and the heaven is supported, at occident, by two columns⁶; About 600 b.C., there are no columns at all, the heaven is leaned upon the Ocean river forming the Vault of Heaven⁷; Anaximander, about 570 b.C., transforms the Vault to a Celestial Sphere and reduces the earth to a relatively small cylinder⁸; Aristarchus of Samos, about 270 b.C., affirms that the earth is spherical and turns around the sun⁹.

The important heuristic point here is that during the first steps, the space is constructed by elevation of the heaven *over* the earth, and so it appears as a *ground* \otimes *height* composed notion. This notion begins to change *asymptotically* with the Vault of Heaven, loses its *global* character with the cylinder of Anaximander and transforms its basic *ingredients* (ground and height) in simple local frame *parameters* with Aristarchus of Samos¹⁰. It is only after these changes that the comprehension of the notion of space is attained. Spatial objects are then thought *per se*, without reference to any support; this is the case, in particular, for the sphere, which is wanted as a realization of total symmetry¹¹.

Let us now come back to the notion of space-time. Once Copernicus reinvincates the forgotten theory of Aristarchus¹², the space-time of Galileo and Newton¹³ is constructed *over* the space and so it appears as a space \otimes time composed notion. The absolute character of its ingredients is lost in the Einstein Special Relativity, and their local character appear in its General Theory. The Penrose conformal infinity techniques allow to construct asymptotically some intrinsic concepts not related directly to them, but any time we need a precise physical interpretation, we are *still* constraint to locally decompose the space-time in its space \otimes time form even though we do not already conceive them as "ingredients" but better as a sort of "comfortable" parametrization.

If we try now to establish a parallelism between the evolution of both concepts, space and space-time, it appears that we are at present at a moment analog to some one situated *after* Anaximander, *but before* Aristarchus: we are not yet attained the analogous of the Aristarchus development. This last moment would correspond to a *direct* comprehension of the space-time, no matter what the decomposition of it be.

Is this analogy correct? Or, in other words, is it true that we have not sufficiently integrated the spatial and temporal parts of the space-time?¹⁴

The decomposition of the space-time in space \otimes time is intimately related to the use of three rods (space) and one clock (time) to locate space-time events. On the other hand, from a conceptual point of view,

clocks and rods are nothing but time-like and space-like projections of light beams¹⁵, which can be locally represented respectively by time-like, space-like and null directions; thus, the frames associated to the above decomposition of the space-time are constituted by one time-like and three space-like directions. Clearly, they form a *proper* subset of the set of all space-time frames, so that the fact that they are precisely the only frames usually called *physically admissible* is already a sign of the correctness of the above analogy.

Appart from the physically admissible frames, *how many classes of frames, causally different, does admit a space-time?* The fact that such a natural question has not been asked up to now seems to reinforce our idea that we are still submitted to the prejudice of the classical conception of the space-time. But it is the absolute lack of intuition about the answer that shows, we believe, the correctness of our analogy: *on the 4-dimensional space-time, there exist 199 causally different classes of frames.*

What is the interest of such a result?

We have choice the above historical analogy because, we believe, at the same time it delimites the unripen aspect of the present notion of space-time and it shows intuitively the direction at which it would be developed, which points to the comprehension of every space-time object *per se*, without reference to any spatial support. In order to acquire such an ability, it seems fair to try to develop the habit of regarding space-time objects from as many different view points as possible. Our table of the 199 different classes of frames appears thus as a basic device to this training.

It seems not too distant the possibility of using signals from satellites and planets to perform *solar frames*; this will constraint physicyists to study in detail some not so "physically admissible" frames as they have do until now. Our causal classification of them would help their study.

The analysis of the causal classes of frames may suggest new ways to measure the gravitational field. In this direction, among the unusual frames we have already considered, perhaps the more interesting ones are the natural frames attached to what we called¹⁶ *light-coordinates*. Roughly speaking, they are local charts such that their four coordinate lines are light-like geodesics. In principle, they may be constructed in the domain of intersection of four beams of laser light; the four frequencies and the six relative angles between the beams constitute a set of ten quantities which may be related to the ten components of the metric tensor, allowing to measure it.

Another domain in which the present causal analysis of frames is of interest is the classification of *symmetric frames*. The frames usually employed privilege some space-time directions (the time-like direction from the three space-like ones in the physically admissible frames, the two light-like directions from the two space-like ones in the null frames). Nevertheless, the Cosmological Principle suggest in part¹⁷ that some properties of the space-time would be best described in such frames that *no direction* be privileged. Such frames, constituted by metrically indistinguishable vectors, are called symmetric frames. They have been studied elsewhere¹⁸ both, from the points of view of *natural frames* and of *metric-concomitants frames*¹⁹.

Also, a direct, practical application of the present work is the *taxonomy of local charts*. It allows to label every local chart with a set of three numbers characterising the causal class of its associated natural frame²⁰.

Perhaps the more important incidence of the causal classification of frames will be found in the study of deformation of lorentzian metrics. Indeed, when one performs an arbitrary metric deformation, one obtains a mixed result: a wanted variation of the metric itself and a superfluous variation of the field of frames (gauge) with respect to which the metric is expressed. Our results allow to reduce the group of deformations by considering its "quotient" by the causal classes, that is to say, roughly speaking, by considering nothing but the "199th part of the group" which

transforms metrics but respects the causal class of the field of frames in which they are expressed.

Any way, the surprise that the richness of the causal classes of frames has produced to all us, shows certainly that we are not yet attained the intellectual right to write the word *space-time* without its hyphen.

The paper is organized as follows: in Section 2 we consider, for the sake of simplicity, some general notions in arbitrary dimensions. In Section 3 we expose a set of arguments allowing to deduce the existence of the 199 classes announced. Finally, in Section 4, we present the corresponding table of causal classes and comment some applications.

The Table of the causal classes of frames was presented, without proof, to the spanish E.R.E. 88 and the french J.R. 89 annual relativistic meetings²¹.

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2. n-DIMENSIONAL ASPECTS.

a) Let r denote a *frame* of a linear space E_n , that is an *ordered* basis of vectors, $r \equiv \{e_\alpha\}$, $\alpha \in I_n \equiv \{1, \dots, n\}$, and let J_p be any of the $\binom{n}{p}$ combinations of p elements of I_n , $1 \leq p < n$. The p -planes Π_p of E_n generated by p elements of r , $\Pi_p \equiv \{\lambda^h e_h \mid h \in J_p\}$, $\lambda^h \in \mathbb{R}$, are the *adjoint p-planes* of r . Let Λr be a *homothetic deformation* of r , $\Lambda r \equiv \{e'_\alpha \mid e'_\alpha \equiv \lambda_\alpha e_\alpha\}$, $\lambda_\alpha \in \mathbb{R} - \{0\}$, and Θr a *permutation* of r , $\Theta r \equiv \{e_{\alpha'} \mid \alpha' = \theta(\alpha)\}$, $\theta(\alpha)$ being a permutation of I_n . Two frames r and r' have the same adjoint p -planes, for any p , if and only if $r' = \Theta \Lambda r$.

Let Π_s and Π'_s be two s -planes corresponding respectively to the combinations $J_s = \{\sigma_1, \dots, \sigma_s\}$ and $J'_s = \{\sigma'_1, \dots, \sigma'_s\}$, where $\sigma_1 < \dots < \sigma_s$ and $\sigma'_1 < \dots < \sigma'_s$. We shall say that Π_s *precedes* Π'_s if there exists t such that

$\sigma_1 = \sigma'_1, \dots, \sigma_{t-1} = \sigma'_{t-1}, \sigma_t < \sigma'_t$. Thus, the *adjoint set* of all the s-planes of r ,

$$\Pi(r) \equiv \bigcup_{s=1}^{n-1} \Pi_s$$

is an ordered set of $2^n - 2$ elements.

b) Suppose now E_n endowed with a hyperbolic metric g of arbitrary signature. The *causal type* of a s-plane is time-like, null or space-like if the restriction of g to it is respectively hyperbolic, degenerated or elliptic. The *causal character* of the adjoint set $\Pi(r)$ is the ordered sequence of the causal types of the adjoint s-planes of r . Let r and r' be two frames with adjoint sets $\Pi(r)$ and $\Pi(r')$, respectively.

Definition. The frames r and r' belong to the same *causal class* if there exists a permutation Θ such that $\Pi(r)$ and $\Pi(\Theta r')$ have the same causal character.

Denote by $[r]$ the causal class of r , let $\{\theta^\alpha\}$ be the algebraic dual co-frame of $r = \{e_\alpha\}$, $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$, and let $\theta_\alpha \equiv g(\theta^\alpha)$ the vectors associated to θ^α by g . The causal class $[r^*]$ of the frame $r^* = \{\theta_\alpha\}$ is called the *dual causal class* of $[r]$. If $[r^*] = [r]$, $[r]$ is said *self-dual*.

The adjoint s-plane of r^* associated to the combination $J_s = (\sigma_1, \dots, \sigma_s)$ is orthogonal to the $(n-s)$ -plane associated to the combination $J_{n-s} \equiv I - J_s$ and we have:

Proposition 1. The causal class $[r]$ of a frame r is determined by the sequence of the causal types of the adjoint s-planes of r and of the causal types of the adjoint s' -planes of r^* where $1 \leq s \leq n-k$ and $s' \leq k-1$ for any integer $k \leq n-1$.

c) It is known that the hyperbolic type (p,q) of a metric g , $p+q = n$, is determined by its signature, $\sigma(g) \equiv p-q$. In a similar way, we can associate to every frame r a *causal signature* $\sigma(r)$ which determines the number of vectors of the frame which are time-like, null or space-like.

Let r be a frame constituted by p time-like, q null and r space-like vectors, $p+q+r = n$; the triplet (p,q,r) is called the *causal type* of r . On the set of causal types, we define the following order: (p,q,r) *precedes* (p',q',r') if $p < p'$ or $p = p'$ and $q < q'$. The ordinal of the causal type will be called the *causal signature* σ of r . We have:

Proposition 2. The causal signature σ of a frame r of causal type (p,q,r) is given by

$$\sigma = \frac{1}{2} (p+q)(p+q+1) + p + 1$$

Conversely, the causal type can be obtained from the causal signature. Taking into account that, for a given σ , $s \equiv p+q$ is the highest integer verifying $s^2+s+2(1-\sigma) \leq 0$, it follows:

Proposition 3. Let σ be the causal signature of a frame; its causal type (p,q,r) is given by $p = \sigma-1-s(s+1)/2$, $q = s-p$, $r = n-s$ where

$$s = \mathbb{E} \left(\frac{1}{2} (\sqrt{8\sigma-7} - 1) \right)$$

\mathbb{E} being the integral part function.

It is clear that σ is an integer which verifies $1 \leq \sigma \leq \binom{n+2}{2}$. In particular, the causal signatures $\sigma = 1$, $\sigma = 1 + n(n+1)/2$ and $\sigma = (n+1)(n+2)/2$ correspond to frames whose vectors are, respectively, space-like, null and time-like, that is, of causal types $(0,0,n)$, $(0,n,0)$ and $(n,0,0)$, respectively. The *normal* frames of causal type $(1,0,n-1)$, which are the generalization to n dimensions of the physically admissible frames of the space-time, have causal signature $\sigma = 3$, and the *null* frames of causal type $(0,2,n-2)$ have $\sigma = 4$.

Let us note that the order we have assigned to the causal types induces an interesting property of *dimensional invariance*: as shown by proposition 2, the causal signature of a frame of causal type $(p,q,.)$ is independent of the dimension n of the space.

d) From now on, we consider E_n endowed with a *lorentzian* metric g ; we have the following simple lemmas:

Lemma 1. Let us consider s linearly independent directions. i) If they are space-like, they generate a s -plane that can be space-like, null or time-like. ii) If one of them is null and the others ones are space-like, they generate a s -plane that can be null or time-like. iii) If one direction is time-like or two of them are null, they generate a time-like s -plane.

Lemma 2. The null direction of a lorentzian frame of causal signature $\sigma = 2$ can not be orthogonal to the other $n-1$ space-like directions.

Consider now the dual frame $\mathbf{r}^* = \{\theta_\alpha\}$ of \mathbf{r} ; for every α , θ_α is orthogonal to the adjoint $(n-1)$ -plane Π_{n-1} of \mathbf{r} corresponding to the combination $I_n - \{\alpha\}$. Because the preceding lemmas and the fact that θ_α is respectively space-like, null or time-like according to the time-like, null or space-like character of Π_{n-1} , the causal type of \mathbf{r}^* is partially related to the causal type of \mathbf{r} . Thus, if $\sigma(\mathbf{r})=1$, all the adjoint hyperplanes of \mathbf{r} are generated by space-like vectors and \mathbf{r}^* may have any causal character. If $\sigma(\mathbf{r})=2$, only one adjoint hyperplane is generated by space-like vectors; the other ones are generated by $n-2$ space-like vectors and one null one and, from lemma 2, these $n-1$ hyperplanes are either one null and the others time-like, or they are all time-like. If $\sigma(\mathbf{r})=3$, there are $n-1$ time-like adjoint hyperplanes; the other one is generated by space-like vectors. If $\sigma(\mathbf{r})=4$, all the adjoint hyperplanes are non space-like and at most two of them are null. If $\sigma(\mathbf{r})=5$, the causal type of \mathbf{r} is $(1,1,n-2)$ and, consequently, there are $n-1$ time-like adjoint hyperplanes; the other one is either time-like or null. If $\sigma(\mathbf{r})=6$, there are two time-like vectors; and if $\sigma(\mathbf{r})>6$, at least three vectors are non space-like. Consequently, for $\sigma(\mathbf{r})>5$ all the adjoint hyperplanes are time-like and hence $\sigma(\mathbf{r}^*)=1$. We have thus shown:

Proposition 4. Let \mathbf{r} be a lorentzian frame in dimension n .

If $\sigma(\mathbf{r})=1$ then $\sigma(\mathbf{r}^*)=1,2,\dots,(n+1)(n+2)/2$

If $\sigma(\mathbf{r})=2$ then $\sigma(\mathbf{r}^*)=1,2,3,4,5$

If $\sigma(\mathbf{r})=3$ then $\sigma(\mathbf{r}^*)=1,2,3$

If $\sigma(\mathbf{r})=4$ then $\sigma(\mathbf{r}^*)=1,2,4$

If $\sigma(\mathbf{r})=5$ then $\sigma(\mathbf{r}^*)=1,2$

If $\sigma(\mathbf{r})>5$ then $\sigma(\mathbf{r}^*)=1$

e) When $\sigma(r)=\sigma(r^*)=2$ the frames are of causal type $(0,1,n-1)$, but they may present different properties:

α) Their space-like vectors generate a null hyperplane, all the other ones being time-like; in this case the null vectors of r and r^* can not be collinear.

β) Their space-like vectors generate a time-like hyperplane in such a way that $n-2$ of these vectors are orthogonal to the null vector of the frame; in this case the null vectors of r and r^* are necessarily collinear.

Except for these two cases all causal properties of the vectors and adjoints hyperplanes of r are completely determined by $\sigma(r)$ and $\sigma(r^*)$.

It is clear that two frames, r and r' , which differ by a permutation Θ , $r' = \Theta r$, belong to the same causal class. Let us denote by t, i, e , respectively, time-like, null, space-like vectors; by a permutation, we may associate, to every frame r of causal character (p,q,r) , an *ordered frame* $o(r) \equiv \{t_1, \dots, t_p, i_1, \dots, i_q, e_1, \dots, e_r\}$, t_i, i_j, e_k belonging to r . Obviously, the ordered frames associated to the frame r are determined up to a permutation Θ of the form $\Theta = \theta_p \times \theta_q \times \theta_r$, where θ_i denotes a permutation of i elements. In general, the dual $o^*(r)$ of an ordered frame $o(r)$ is not an ordered frame: $o^*(r) \neq o(r^*)$. But one can show that, apart from the frames belonging to the above case β , there exist ordered frames, which will be noted by $c(r)$, such that $c^*(r) = c(r^*)$. In the exceptional case β , one can always find ordered frames such that their duals are of the form $\{e_1, i, e_2, \dots, e_{n-1}\}$: they will also be noted by $c(r)$ and, like the ones verifying $c^*(r) = c(r^*)$, will be called *canonically ordered frames*.

From this considerations and proposition 4, it can be shown:

Proposition 5. For a n -dimensional lorentzian metric, the number N of pairs of canonically ordered frames $\{c(r), c(r^*)\}$ having different causal characters is given by $N = (n+1)(n+2)+9$.

f) Denoting, for short, by σ^* the causal signature of r^* , $\sigma^* \equiv \sigma(r^*)$, the table of the N different pairs $\{c(r), c(r^*)\}$ adopts the aspect indicated

in Table I. In it, one has $\ell, m \leq \binom{n+2}{2}$, and π_ℓ^m denotes the set of causal types of s-planes, $1 \leq s \leq n-1$. The table follows from propositions 1, 4 and 5. The above mentioned property of dimensional invariance induced by the chosen order is here clearly apparent: for increasing n , the occupied cases remain occupied, and the only occupied cases to be added are of the form π_ℓ^1 and π_1^m . Of course, the number of s-planes contained in every case depends, in general, of the dimension n . Its evaluation for $n=4$ will be our task in the next Section.

TABLE I

Remembering proposition 1, it follows directly from proposition 5 that:

Corollary. In dimension 3, there exist 29 causal classes of lorentzian frames.

3. THE CAUSAL CLASSIFICATION OF SPACE-TIME FRAMES.

a) For $n=4$, we can distinguish $N = 39$ causally different pairs $\{c(\mathbf{r}), c^*(\mathbf{r})\}$. From proposition 1, a complete causal study of the space-time frames still requires to specify the causal types of the adjoint 2-planes corresponding to every one of these 39 pairs.

Let us consider the 2-plane π_{xy} generated by the vectors x and y . The sign ε of the quantity $g(x,x)g(y,y)-[g(x,y)]^2$ does not depend neither of the choice of the basis on π_{xy} , nor of the sign of the signature of the lorentzian metric g . It will be called the *causal sign* of the 2-plane π_{xy} since we have $\varepsilon = +$, $\varepsilon = 0$ or $\varepsilon = -$, depending on whether π_{xy} is space-like, null or time-like.

Let $\pi_{\alpha\beta}$ be the adjoint 2-plane of the frame $\mathbf{r}=(e_\alpha)$ generated by e_α and e_β , and let $\varepsilon_{\alpha\beta}$ be its causal sign. Denoting by $\varepsilon_{\alpha\beta}^*$ the causal sign

of the adjoint 2-plane $\pi_{\alpha\beta}^*$ generated by the vectors θ_α and θ_β of r^* , it results:

Proposition 6. For any distinct values of the indices $\alpha, \beta, \gamma, \delta$, the causal signs $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\gamma\delta}^*$ are related by $\varepsilon_{\alpha\beta} = -\varepsilon_{\gamma\delta}^*$.

Thus, if $(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{23}, \varepsilon_{24}, \varepsilon_{34})$ is the ordered set of causal signs of the set Π of the adjoints 2-planes of r , the corresponding set of r^* is given by $(-\varepsilon_{34}, -\varepsilon_{24}, -\varepsilon_{23}, -\varepsilon_{14}, -\varepsilon_{13}, -\varepsilon_{12})$.

The invariance group, say Θ_c , of the pair $\{c(r), c^*(r)\}$ does not respect, in general, the order of the causal characters of the adjoint 2-planes; this is to say that the whole class of frames $c(r)$ is too large to be used to distinguish causal classes. From now on, we shall restrict $c(r)$ in such a way that those of the causal signs that are not invariant by the action of Θ_c be ordered not decreasingly (i.e.: -, 0, +). The set of adjoint 2-planes of $c(r)$ so ordered will be denoted by $c(\pi)$.

Now, we are able to study the different causal classes of frames. The method works in two steps: in the first one, one obtains the sets $c(\pi)$ associated to a given $c(r)$, and in the second one we check the sets $c(\pi)^*$ corresponding to all the possible $c^*(r)$.

b) Causal classes with $\sigma(r)=1$. Since $c(r)=\{eeee\}$, the adjoint 2-planes of r may have any causal type. If $\sigma(r^*)=1$, all the causal characters of $c(r)$ may be permuted: $\Theta_c \approx \theta_4$. Therefore, all the signs of any $c(\pi)$ may be ordered in a not decreasing way. The $c(\pi)$, considered as frames of the 6-dimensional bivector space, may be ordered by their causal signs; the result is:

{+++++}, {0++++}, {-++++}, {00++++}, {-0++++}, {--++++}, {000+++},
 {-00+++}, {-00+++}, {----+++}, {0000++}, {-000++}, {--00++}, {---0++},
 {-----+}, {00000+}, {-0000+}, {--000+}, {---00+}, {----0+}, {-----},
 {000000}, {-00000}, {--0000}, {---000}, {----00}, {-----0}, {-----}.

Thus, there are 28 causal classes with $\sigma(r)=\sigma(r^*)=1$. For $\sigma(r)=1$ and $\sigma(r^*) \geq 2$, the corresponding causal classes are the dual of the causal classes with $\sigma(r) \geq 2$ and $\sigma(r^*)=1$. These ones will be obtained below.

c) Causal classes with $\sigma(\mathbf{r})=2$. Now $c(\mathbf{r}) = \{ie_1e_2e_3\}$ and, from lemma 2, the three adjoint 2-planes (ie_1) , (ie_2) , (ie_3) can not be null at once. Either 1) they are time-like or 2) two of them are time-like and the other one is null, or 3) only one is time-like and the others are null. Denoting by Δ any causal character (that is, $\Delta = t, i, e$ for vectors, and $\Delta = -, 0, +$ for 2-planes), the cases 1) and 2) correspond to $c^*(\mathbf{r})=\{\Delta eee\}$ since the adjoint hyperplanes $\mathcal{H}_1 \equiv (ie_2e_3)$, $\mathcal{H}_2 \equiv (ie_1e_3)$ and $\mathcal{H}_3 \equiv (ie_1e_2)$ are time-like. For them we have $\theta_c \approx \theta_3$. Let us choice e_1 and e_2 in such a way that the first and the second adjoint 2-planes of \mathbf{r} be time-like. We have then $\{---\Delta\Delta\Delta\}$ for case 1) and $\{-0\Delta\Delta\Delta\}$ for case 2).

For case 1), no space-like vector of \mathbf{r} has been privileged so that we can take the 2-planes $[(e_1e_2)(e_1e_3)(e_2e_3)]$ of the hyperplane $\mathcal{H}_4 \equiv (e_1e_2e_3)$ with their causal signs in a non decreasing order:

$\{---\}$ $\{-0\}$ $\{-0+\}$ $\{-00\}$ $\{-0+\}$ $\{-++\}$ $\{000\}$ $\{00+\}$ $\{0++\}$ $\{+++\}$

For case 2), the vector e_3 remains privileged with respect to e_1 and e_2 , which are still interchangeable. For every causal type of the 2-plane (e_1e_2) , we can take the two 2-planes (e_1e_3) and (e_2e_3) with their causal signs in a non decreasing order. Now, in terms of theirs signs, the adjoint 2-planes of \mathcal{H}_4 are:

$\{\Delta--\}$ $\{\Delta-0\}$ $\{\Delta-+\}$ $\{\Delta00\}$ $\{\Delta0+\}$ $\{\Delta++\}$

For the cases 1) and 2), there are 28 different sets $c(\pi)$. The corresponding dual sets $c^*(\pi)$ are obtained from proposition 6. Now, taking into account lemma 1, it remains to check the sets $c^*(\pi)$ that are compatible with every one of the three sets $c^*(\mathbf{r})=\{\Delta eee\}$. Of course, if $c^*(\mathbf{r})=\{eeee\}$ there are no additional restrictions.

If $c^*(\mathbf{r})=\{teee\}$, the first three signs in $c^*(\pi)$ are negative. So, the last three signs in $c(\pi)$ are positive. The possible sets $c(\pi)$ are $\{---++\}$ and $\{-0++\}$.

If $c^*(\mathbf{r})=\{ieee\}$, the first three signs in $c^*(\pi)$ are non positive and simultaneously non zero; this implies the following possibilities for $c^*(\pi)$: $\{-00++\}$, $\{-0++\}$, $\{---++\}$, $\{-000++\}$, $\{-00++\}$, $\{00-0++\}$, $\{-0-0++\}$ and

{---0++}. But {-00+++} and {-000++} are forbidden by the following simple lemma,

Lemma 3. If $c^*(r)=(ieee)$ and $c^*(\pi)=(-00\Delta\Delta+)$ then $c(r)=(\Deltaiee)$.

Now $c^*(\pi)=(00-0++)$ is not compatible with $c^*(r)$ since the first two adjoint 2-planes being null, the fourth must be space-like. In consequence, if $c(r)=(ieee)$ and $c^*(r)=(ieee)$ then

$$c(\pi): \{-00-0+\} \quad \{-00-++\} \quad \{-00\Delta\Delta+\} \quad \{-0-0+\} \quad \{-0-0++\}$$

Finally, let us consider the case 3), that is, the case when one of the adjoint hyperplanes $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ is null (the others being necessarily time-like). The frame r^* also contains the null direction of r . Suppose $\mathcal{H}_3 \equiv (ie_2 e_3)$ is the null hyperplane; this fixes the first space-like vector of r : the adjoint 2-plane (ie_1) is time-like. The adjoint 2-plane $(e_2 e_3)$ being space-like, we have $c(\pi)=(-00\Delta\Delta+)$. The fourth and the fifth signs of $c(\pi)$ are interchangeable, that is $\Theta_c \approx \theta_2$; setting them in a non decreasing order, we have

$$c(\pi) : \{-00-0+\}, \{-00-0+\}, \{-00-++\}, \{-0000+\}, \{-000++\}, \{-00-0++\}$$

From proposition 6 it then follows:

$$c^*(\pi) : \{-++00+\}, \{-0+00+\}, \{-++00+\}, \{-0000+\}, \{-0-000+\}, \{-00-00+\}$$

which are all compatible with $c^*(r)=(ieee)$. If $c^*(r)=(tiee)$ only $c^*(\pi)=\{-00-00+\}$ is possible. If $c^*(r)=(iiee)$, the sets $c^*(\pi)$ neither contain the sign plus in the first three places nor are of the form $c(\pi)^* = \{-0000+\}$, because then the second vector of r would be null, in contradiction with $c(r)=(ieee)$. Consequently, the only possible $c^*(\pi)$ are $\{-00-00+\}$ and $\{-00-00+\}$.

d) Causal classes with $\sigma(r)=3$. We have $c(r)=(teee)$ and $c(\pi)=(-\Delta\Delta\Delta)$. From proposition 4, one has $\sigma(r^*)=1,2,3$, that is, $c^*(r)=(\Deltaeee)$ and consequently $\Theta_c \approx \theta_3$. Thus, the signs $[\Delta\Delta\Delta]$ can always be ordered in a non decreasing way,

$$c(\pi): \left\{ \begin{array}{l} \{-\Delta\Delta\Delta\} \quad \{-\Delta\Delta\Delta-0\} \quad \{-\Delta\Delta\Delta-+\} \quad \{-\Delta\Delta\Delta-00\} \quad \{-\Delta\Delta\Delta-0+\} \\ \{-\Delta\Delta\Delta-++\} \quad \{-\Delta\Delta\Delta-000\} \quad \{-\Delta\Delta\Delta-00+\} \quad \{-\Delta\Delta\Delta-0++\} \quad \{-\Delta\Delta\Delta-+++\} \end{array} \right\}$$

and their respective duals are

$$c(\pi^*): \left\{ \begin{array}{l} \{+++++\} \{0++++\} \{-++++\} \{00++++\} \{-0++++\} \\ \{-++++\} \{000++++\} \{-00++++\} \{-00++++\} \{-00++++\} \end{array} \right\}$$

If $c^*(r)=(eeee)$, all the above $c^*(\pi)$ are possible by lemma 1. If $c^*(r)=(teee)$, only $c^*(\pi)=\{-0++++\}$ is possible. And, if $c^*(r)=(ieee)$, the possible $c^*(\pi)$ are $\{-0++++\}$ and $\{-00++++\}$. In fact, the three first adjoint 2-planes of $c^*(r)$ are simply restricted to be not space-like, but $c^*(\pi)=\{000++++\}$ is forbidden by lemma 2 and $c^*(\pi)=\{-00++++\}$ cannot occur by lemma 3.

e) Causal classes with $\sigma(r)=4$. Now, $c(r)=(i_1 i_2 e_1 e_2)$ and the causal types of the adjoint 2-planes are of the form $\{-\circ\circ\circ\Delta\}$ where \circ stands for - or 0. There are the following possibilities:

- 1) $\{-0000\Delta\}$
- 2) $\{-000\Delta\}$, $\{-000\Delta\} (\theta_e)$, $\{-000\Delta\} (\theta_i)$, $\{-000\Delta\} (\theta_i \times \theta_e)$
- 3) $\{-000\Delta\}$, $\{-000\Delta\} (\theta_e)$
- 4) $\{-000\Delta\}$, $\{-000\Delta\} (\theta_i)$ or (θ_e)
- 5) $\{-000+\}$, $\{-000+\} (\theta_i)$
- 6) $\{-000+\}$, $\{-000+\} (\theta_e)$, $\{-000+\} (\theta_i)$, $\{-000+\} (\theta_i \times \theta_e)$
- 7) $\{-0000+\}$

Every one of these rows corresponds, for every value of Δ , to the same causal class. The first term on every row has already the correct order so that it is the causal sign representation of the corresponding $c(\pi)$. Also, we write the permutation going from every causal configuration of adjoint 2-planes to the ordered set $c(\pi)$; thus, θ_i (resp. θ_e) is the transposition of the null (resp. space-like) vectors of r . In 5), 6) and 7) the adjoint 2-plane $(e_1 e_2)$ is space-like since both vectors e_1 and e_2 are orthogonal to the same null direction. If $c^*(r)=(eeee)$, the adjoint hyperplanes $\mathcal{H}_1 \equiv (i_2 e_1 e_2)$ and $\mathcal{H}_2 \equiv (i_1 e_1 e_2)$ are both time-like, and the adjoint 2-planes are: $\{-0000\Delta\}$, $\{-000\Delta\}$, $\{-000\Delta\}$, $\{-000\Delta\}$. If $c^*(r)=(ieee)$, then \mathcal{H}_1 is null and \mathcal{H}_2 is time-like. In this case, there are two possibilities for

$c(\pi)$: $\{---00+\}$ and $\{-000+\}$. If $c^*(r)=\{i_2 i_1 ee\}$, then $c(\pi)=\{-0000+\}$ due to the fact that \mathcal{H}_1 and \mathcal{H}_2 are null. Note that the null directions of r and r^* are the same, but their order is interchanged.

f) Causal classes with $\sigma(r)=5$. Now, $c(r)=\{tie_1 e_2\}$ and the three first adjoint 2-planes of r are time-like. The adjoint hyperplane $\mathcal{H}_1 \equiv \{ie_1 e_2\}$ is null or time-like. If \mathcal{H}_1 is null then $c(\pi)=\{---00+\}$ and $c(r^*)=\{ieee\}$ (with same null direction as r). If \mathcal{H}_1 is time-like, that is $c(r^*)=\{eeee\}$, then the causal characters of the adjoint 2-planes $[(ie_1)(ie_2)(e_1 e_2)]$ are $[-\Delta]$ or $[-0\Delta]$. In the latter case we can set the two first signs in increasing order due to the fact that Θ_c is now the transposition of e_1 and e_2 .

g) Causal classes with $\sigma(r)>5$. From proposition 4, $c(r^*)=\{eeee\}$, and these causal classes are obtained directly as follows: If $\sigma(r)=6$, then $c(r)=\{ttee\}$ and $c(\pi)=\{-----\Delta\}$. If $\sigma(r)=7$, then $c(r)=\{iiie\}$ and $c(\pi)=\{--\circ-\circ\circ\}$. Because of $\Theta_c \approx \theta_3$ the signs $[\circ\circ\circ]$ can be placed in non decreasing order. This gives four causal classes. If $\sigma(r)=8$, then $c(r)=\{tiii\}$, $c(\pi)=\{-----\circ\}$ and there are three causal classes. If $\sigma(r)=9$, then $c(r)=\{ttie\}$ and $c(\pi)=\{-----\circ\}$. If $\sigma(r) = 10, 11, 12, 13, 14$ or 15 , then all the adjoint 2-planes of r are time-like, that is $c(\pi)=\{-----\}$.

h) Using the preceding results and counting the different possibilities, we have

THEOREM: The space-time admits 199, and only 199, causal classes of frames.

TABLE II

4. DISCUSSION AND COMMENTS.

a) The considerations of the preceding Section not only lead to the above Theorem but allow to construct explicitly the characterization of all the causal classes. This characterization is given in Table II. Table II differs from Table I in what it is a particularization to dimension $n=4$;

in what it explicits the notation of the causal character of the frames r and r^* (remember that, as shown by proposition 3, there is a *bijection* between the causal character of r and its causal signature); in what it splits of the cases corresponding to the pairs (r, r^*) (by giving them in convenient order we may identify r and $c(r)$); and in what it explicits the notation of the causal character of the adjoint 2-planes.

The natural reading of Table II begins by the left. For exemple, let be given a frame of causal type $\{iiee\}$; the corresponding row of the table indicates that it may belong to $3 \times 4 + 2 + 1 = 15$ causal classes. If, in addition, we know that its dual is of causal type $\{ieee\}$, the intersection of the row with the corresponding column of duals restricts the number of classes to two. They correspond to the only possibilities, for the adjoint 2-plane of the first and last vectors of r , of being time-like or null.

b) Thus, we can see some simple properties: *i)* the *impossible lorentzian frames* (blank cases of the table); for exemple, there is no frame $\{teee\}$ having as dual a co-frame $\{eiee\}$. *ii)* there exist only six causal classes which may be *univocally* determined by the causal character of the frame (the six last classes of the first column), *iii)* only the frames of causal character $\{eeee\}$ with dual $\{eeee\}$ can admit *any* of the 28 possible causal configurations of adjoint 2-planes. *iv)* the frames $\{ieee\}$ with dual $\{eeee\}$, or conversely, may belong also to 28 causal classes, but they do not correspond to the 28 causal configurations of adjoint 2-planes; the number 28 is attained by *different permutations* of some configurations: for exemple, for the first of these frames the first eight configurations as well as the 11th, 12th, 16th, 17th, 22th and 23th ones, are absent (this arrangement corresponds to that induced by the causal signature, and coincides with the arrangement shown in the first case); in fact, among the 28 causal classes, a half are permutations of the configurations of the other half.

c) As shown by Table II, a causal class is, generically, given by the triplet (r, π, r^*) involving $4+6+4=14$ symbols. The causal signature allows to condense them: $\sigma(r)$, the ordinal integer of the first column, stands for the four first causal symbols; and similarly works $\sigma(r^*)$, excepts for $r^* = \{eiee\}$, for which $\sigma(r^*)$ is noted 2_β (see Table II). Due to the

features indicated in the part iv) of the above paragraph, a set of indices (say, a, b, c,...) is need to indicate permutations of the same causal configurations of 2-planes. The notation we have adopted is given in Table III. A causal class may thus be indicated by three numbers, a sort of causal coordinates; for example, (4:26_d:1) stand for $r=\{iiee\}$, $\pi=\{-0-0-\}$, $r^*=\{eeee\}$.

TABLE III

d) In paragraph b) of Section 2 we defined self dual causal classes. We see now, from Table II, that the space-time admit 11 self dual classes; in causal signature notation they are (1:10:1), (1:13:1), (1:17:1), (1:22:1), (2:10:2), (2:13:2), (2:13_b:2), (2:13_d:2 _{β}), (2:17:2 _{β}), (3:10:3), and (4:17:4).

e) As was indicated in the introduction, we are now able to label coordinate systems from the causal point of view. For example, the coordinates (t,x,y,z) for the metric

$$ds^2 = dt^2 + \frac{1}{2} e^{2y} dx^2 - dy^2 - dz^2 + 2 e^y dt dx$$

(homothetic to the Gödel solution) are not physically admissible; its natural frame $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ belongs to the causal class (6:21:1). Similarly, for coordinates (u,r, θ , ϕ) for the metric

$$ds^2 = A du^2 + 2 du dr - r^2(d\theta^2 + \text{sen}^2\theta d\phi^2)$$

the natural frame $\{\partial_u, \partial_r, \partial_\theta, \partial_\phi\}$ belongs to the causal classes (5:19:2), (4:17:4) or (2:8:5) depending on whether $A>0$, $A=0$ or $A<0$ (the well known Vaidya solution corresponds to $A = 1 - 2m(u)/r$). This shows that the unusual character of a coordinate system may be quantitatively characterized.

f) Table II may be considered as a sort of graphic representation of a *theorem of signature*. The hyperbolic character and the sign of signature of a regular matrix may be obtained from the table by analysing the sign of

the second order principal minors (they are nothing but the causal signs) and the compatibility of them with the signs of the terms of the principal diagonals of the matrix and its inverse. For example, if a matrix having $a_{\alpha\alpha} = 0$ is to be hyperbolic, their second order principal minors must all be strictly negative and then we know that all the elements $a^{\alpha\alpha}$ of its inverse will have the same sign ε ; it follows that the signature of the metric is $(-\varepsilon, \varepsilon, \varepsilon, \varepsilon)$, as corresponds to the case (iiii,-----,eeee) of the table.

g) Let us note that, except for the cases (3:10:3) and (4:17:4), the boundary of the impossible lorentzian frames is a "convex" stair; these two separated cases correspond to the physically admissible and null frames. What is the role of this "peninsular" isolation in our inertia to conceive physically other different frames?

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σ	1^*	2^*	2_β^*	3^*	4^*	5^*	6^*	...	ℓ^*	...
1	π_1^1	π_2^1		π_3^1	π_4^1	π_5^1	π_6^1	...	π_ℓ^1	...
2	π_1^2	π_2^2	$\pi_{2_\beta}^2$	π_3^2	π_4^2	π_5^2				
3	π_1^3	π_2^3		π_3^3						
4	π_1^4	π_2^4			π_4^4					
5	π_1^5	π_2^5								
6	π_1^6									
...	...									
m	π_1^m									
...	...									

TABLE I. Correspondence between the causal character of n-dimensional lorentzian frames and their duals. The frames are ordered according to their causal signature σ . The π_ℓ^m 's stand for the sets of adjoint s-planes of the frame, $1 < s < n-1$, and the column 2_β^* corresponds to the β exceptional case mentioned in the text.

$$\begin{aligned}
1 &= \{++++++\}, & 2_a &= \{0+++++\}, & 2_b &= \{+++0++\}, & 2_c &= \{+0++++\}, \\
3 &= \{-+++++\}, & 4_a &= \{00++++\}, & 4_b &= \{0++0++\}, & 4_c &= \{++00++\}, \\
4_d &= \{+0+0++\}, & 5_a &= \{-0++++\}, & 5_b &= \{-++0++\}, & 5_c &= \{+-0++\}, \\
6 &= \{---++++\}, & 7_a &= \{000+++ \}, & 7_b &= \{00+0++\}, & 7_c &= \{0+00++\}, \\
8_a &= \{-00+++ \}, & 8_b &= \{-0+0++\}, & 8_c &= \{-+00++\}, & 8_d &= \{0+-0++\}, \\
9_a &= \{--0+++ \}, & 9_b &= \{--+0++\}, & 9_c &= \{-+-0++\}, & 10 &= \{-----\}, \\
11 &= \{0000++\}, & 12_a &= \{-000++\}, & 12_b &= \{00-0++\}, & 13_a &= \{--00++\}, \\
13_b &= \{--0+0+\}, & 13_c &= \{-0-0++\}, & 13_d &= \{-00-++\}, & 14_a &= \{---0++\}, \\
14_b &= \{--0-++\}, & 14_c &= \{--0+--\}, & 15 &= \{-----\}, & 16 &= \{00000+\}, \\
17 &= \{-0000+\}, & 18_a &= \{--000+\}, & 18_b &= \{--0+00\}, & 18_c &= \{-00-0+\}, \\
19_a &= \{---00+\}, & 19_b &= \{--0-0+\}, & 19_c &= \{--00-+\}, & 19_d &= \{--0+-0\}, \\
19_e &= \{-00-++\}, & 20_a &= \{----0+\}, & 20_b &= \{--0-++\}, & 20_c &= \{--0+--\}, \\
21 &= \{-----\}, & 22 &= \{000000\}, & 23 &= \{-00000\}, & 24 &= \{--0000\}, \\
25_a &= \{---000\}, & 25_b &= \{--0-00\}, & 25_c &= \{--00-0\}, & 26_a &= \{----00\}, \\
26_b &= \{--0--0\}, & 26_c &= \{--00--\}, & 26_d &= \{--0-0-\}, & 27_a &= \{-----0\}, \\
27_b &= \{--0-0-\}, & 27_c &= \{----0-\}, & 28 &= \{-----\}.
\end{aligned}$$

TABLE III. Relation between the *causal signature* and the causal character of the adjoints planes. The indices differentiate the permutations of the adjoints planes that appear in Table II.

¹ "Directly" means here "without decomposing the local domain in its classical constituents *space and time*".

² The variety of mythologies, theologies and theories among Egyptians and Greeks concerning the space is so immense that, obviously, we do not pretend to resume them in a few lines. We have stripped these sketches from the abundant cosmogonic content in which the corresponding descriptions appear naturally immersed, and we have paid attention to present in them only those elements attached to the notion of space which have manifestly changed during the period concerned.

³ We start from Egyptian better than from Mesopotamian cosmologies because, for our purposes, they do not differ essentially (see *Histoire des croyances et des idées religieuses, I*, Mircea Eliade, Ed. Payot, Paris, 1976; *La Mesopotamie*, G. Roux, Ed. Seuil, Paris, 1985; *L'histoire commence à Summer*, S.N. Kramer, Ed. HCS, Paris, 1975).

⁴ These columns, although they frequently appear as divinities (see for exemple *La naissance du monde. Sources orientales*, S. Saumeron et al., Ed. Seuil, Paris, 1959), suggest strongly that the heaven is, like the earth from which it has been separated, a flat disc.

⁵ The collapse from eight to four columns seems, at the same time, theologically related to the association by pairs of the eight divinities (see *Histoire de l'Égypte Ancienne*, N. Grimal, Ed. Fayard, Paris, 1988; *Mythologies du monde entier*, V. Grigorieff, Ed. Marabout, Belgique 1987) and imaginarily related to the representations of the heaven by the trunk of the goddess Nut (Greenfield Papyrus, British Museum, London, Sarcophagus of Butehamon, Egyptian Museum, Turin) or of the Heavenly Cow (see *Mythology in ancient Egypt*, R. Anthes, in *Mythologies of the Ancient World*, Ed. S.N. Kramer, Doubleday & Co., Inc., N.Y., 1961); in these last cases, there is no doubt about the identity of the four columns.

⁶ Of course, the central role played by the Nile is, for the Greeks, implicitly assumed by the Mediterranean sea. The two columns indicated in this sketch condense a variety of descriptions involving Atlas and Heracles. Atlas appears bearing the heaven with the aid of columns or directly on his shoulders or on his neck or playing the role of a mountain or even of the Titan who separated the waters of the heaven from those of the earth. Heracles replaces temporarily Atlas in his task during his Hesperides Garden labor, or constructs himself two columns in souvenir of his capture of the oxen of Geryon (*Les mythes grecs*, I, R. Graves, Ed. Fayard, 1967; *Dieux et mythes de la Grèce ancienne*, J. Desautels, Ed. Presses Université Laval, Quebec, 1988; *Mythologie et géographie*, J. Ramin, Ed. Les Belles Lettres, Paris, 1979). Curiously, all these "devices" to maintain the heaven seem located at Occident: Atlas mountains, Hesperides Garden, Heracles Columns. The mythic and symbolic character of these pictures is magnifically analysed in *Le soleil et le Tartare: l'image mythique du monde en Grèce archaïque*, A. Ballabriga, Ed. E.H.E.S.S., Paris, 1986. Their scientific absurdity and their lack of balance are, perhaps, at the basis of the conception of the Vault.

⁷ This is the conception of the world known by Thales and, philosophically modified by him, transmitted to his pupil Anaximander.

⁸ He is the first to speak about antipodes.

⁹ In *On the sizes and distances of the sun and moon*, see *Aristarchus of Samos, the Ancient Copernicus*, T. Heath, Ed. Dover Pub., Inc., N.Y., 1981.

¹⁰ The vanishing of the central role played by the earth in the construction of the space is made progressively; thus, the earth turns around a central fire for Philolaus (about 420 b.C.), and around its axis for Heraclides (about 320 b.C.).

¹¹ It is interesting to note that, after Anaximander, the earth is thought as a sphere because of a philosophical need of symmetry, not for observational evidence. The role played by the concept of sphere in the social structure is considered in *Mythe et pensée chez les grecs*, J.P. Vernant, Ed. La Découverte, Paris, 1988.

¹² The modesty of Copernicus honour him; in his *De revolutionibus Caelestibus*, he explicitly refers to Greek astronomers, and specially to Aristarchus. See *Civilisation greque, III*, A. Bonnard, Ed. La Guilde du Livre, Lausanne, 1959.

¹³ We consider here only the cinematological aspects of them, so that they are identical.

¹⁴ The "integration" we are speaking here concerns the direct feeling of the *hyperbolic* space-time. It has nothing to do with the ability to use covariant, intrinsic or fourdimensional formalisms. These formalisms are originated by a more or less direct transcription of the *elliptic* formalism of Riemannian geometry and, in spite of their unquestionable interest, mask (almost) completely the specific features of hyperbolicity.

¹⁵ Think to the present definitions of the units of time and length.

¹⁶ B. Coll, *Coordenadas luz en Relatividad*, communication to the spanish E.R.E 85 relativistic meeting, Pub. E.T.S.E.I.B., Barcelona, Spain, 1985.

¹⁷ This is to approach to the greek exigence of sphericity from symmetry considerations; see Reference quoted in Note 11.

¹⁸ B. Coll and J.A. Morales, C.R. Acad. Sc. Paris, 306, Sér. I, p. 791-4, 1988 and *Repères symétriques lorentziens*, Journées Relativistes 1988, p. 143-8, Univ. Genève, CH-1211 Genève 4, Switzerland.

¹⁹ A symmetric frame is said to be of order s if its vectors are indistinguishable for the metric and its differential concomitants of order s .

²⁰ We give some examples in section 4.

²¹ B. Coll and J.A. Morales, *Las 199 clases causales de referenciales de espacio-tiempo*, in Proceedings E.R.E. 88 meeting, p. 171-80, Pub. Univ. Salamanca, Salamanca, Spain, 1988, and Annales de Physique, Coll. n° 1, 14, p. 65-69, 1989.

²² In the two first cases, the paper was directly refused by its editors, L. C. Biedenharn and L. S. Brown respectively.

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199 Causal Classes of Space-Time Frames

Bartolomé Coll¹ and Juan Antonio Morales²

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It is shown that from the causal point of view Minkowskian space-time admits 199, and only 199, different classes of frames.

1. INTRODUCTION

Space-time is usually described as a four-dimensional Lorentzian manifold. Its topology, its differentiable and metric structures, and its asymptotic properties have been the object of many studies; from the formal point of view there is no doubt that the notion of space-time is at present well defined.

In spite of this fact, a good physical comprehension of this notion has not yet been attained. A point that contributes to this situation is our inability to conceive *directly*³ local domains of space-time. The importance of this deficiency may be clarified by comparing the evolution of the notion of space-time up to now to the ancient elaboration of the notion of *space*.

For our purposes, this elaboration may be considered as having been achieved after the work of Aristarchus of Samos. The earlier history of the notion of space can be sketched successively as follows⁴: For the Egyptians⁵

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³"Directly" means here "without decomposing the local domain into its classical constituents *space and time*."

⁴The variety of mythologies, theologies, and theories among Egyptians and Greeks concerning space is so immense that, obviously, we do not pretend to summarize them in a few lines. We have stripped these sketches from the abundant cosmogonic content in which the corresponding descriptions appear naturally immersed, and we present only those elements attached to the notion of space which have manifestly changed during the period concerned.

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the natural frame $\{\partial_u, \partial_r, \partial_\theta, \partial_\phi\}$ belongs to the causal classes (5:19:2), (4:17:4), or (2:8:5), depending on whether $A > 0$, $A = 0$, or $A < 0$ [the well-known Vaidya solution corresponds to $A = 1 - 2m(u)/r$]. This shows that the unusual character of a coordinate system may be quantitatively characterized.

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(g) Let us note that, except for the cases (3:10:3) and (4:17:4), the boundary of the impossible Lorentzian frames is a "convex" stair; these two distinct cases correspond to the physically admissible and null frames. What is the role of this "peninsular" isolation in our inertia to conceive physically other different frames?

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