A CHARACTERIZATION OF THE n-ARY MANY-SORTED CLOSURE OPERATORS AND A MANY-SORTED TARSKI IRREDUNDANT BASIS THEOREM

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Abstract. A theorem of single-sorted algebra states that, for a closure space (A, J) and a natural number n, the closure operator J on the set A is n-ary if, and only if, there exists a single-sorted signature Σ and a Σ-algebra $\mathbf{A}$ such that every operation of $\mathbf{A}$ is of an arity $\leq n$ and $J = \mathrm{Sg}_{\mathbf{A}}$, where $\mathrm{Sg}_{\mathbf{A}}$ is the subalgebra generating operator on A determined by $\mathbf{A}$. On the other hand, a theorem of Tarski asserts that if J is an n-ary closure operator on a set A with $n \geq 2$, and if $i < j$ with $i, j \in \mathrm{IrB}(A, J)$, where $\mathrm{IrB}(A, J)$ is the set of all natural numbers n such that (A, J) has an irredundant basis ($\equiv$ minimal generating set) of n elements, such that $\{i+1, \ldots, j-1\} \cap \mathrm{IrB}(A, J) = \varnothing$, then $j - i \leq n - 1$. In this article we state and prove the many-sorted counterparts of the above theorems. But, we remark, regarding the first one under an additional condition: the uniformity of the many-sorted closure operator.

1. Introduction.

A well-known theorem of single-sorted algebra states that, for a closure space (A, J) and a natural number $n \in \mathbb{N} = \omega$, the closure operator J on the set A is n-ary if, and only if, there exists a single-sorted signature Σ and a Σ-algebra $\mathbf{A}$ such that every operation of $\mathbf{A}$ is of an arity $\leq n$ and $J = \mathrm{Sg}_{\mathbf{A}}$, where $\mathrm{Sg}_{\mathbf{A}}$ is the subalgebra generating operator on A determined by $\mathbf{A}$. On the other hand, in [3], it was stated that, for an algebraic many-sorted closure operator J on an S-sorted set A, $J = \mathrm{Sg}_{\mathbf{A}}$ for some many-sorted signature Σ and some Σ-algebra $\mathbf{A}$ if, and only if, J is uniform. And, by using, among others, the just mentioned result, our first main result is the following characterization of the n-ary many-sorted closure operators: Let S be a set of sorts, A an S-sorted set, J a many-sorted closure operator on A, and $n \in \mathbb{N}$. Then J is n-ary and uniform if, and only if, there exists an S-sorted

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signature Σ and a Σ-algebra $\mathbf{A}$ such that $J = \mathrm{Sg}_{\mathbf{A}}$ and every operation of $\mathbf{A}$ is of an arity $\leq n$.

We turn next to Tarski's irredundant basis theorem for single-sorted closure spaces. But before doing that let us begin by recalling the terminology relevant to the case. Given an n in $\mathbb{N}$, a set A, and a closure operator J on A, the closure operator J is said to be an n-ary closure operator on A if $J = J^{\omega}_{\leq n}$, where $J^{\omega}_{\leq n}$ is the supremum of the family $(J^{m}_{\leq n})_{m\in\omega}$ of operators on A defined by recursion as follows: for $m = 0$, $J^{0}_{\leq n} = \mathrm{Id}_{\mathrm{Sub}(A)}$; for $m = k+1$, with $k \geq 0$, $J^{k+1}_{\leq n}(X) = J_{\leq n} \circ J^{k}_{\leq n}$, where $J_{\leq n}$ is the operator on A defined, for every $X \subseteq A$, as follows:

$$J_{\leq n}(X) = \bigcup \{J(Y) \mid Y \in \mathrm{Sub}_{\leq n}(X)\},$$

where $\mathrm{Sub}_{\leq n}(X)$ is $\{Y \subseteq X \mid \mathrm{card}(Y) \leq n\}$.

Alfred Tarski in [4], on pp. 190–191, proved, as reformulated by S. Burris and H. P. Sankappanavar in [2], on pp. 33–34, the following theorem. Given a set A and an n-ary closure operator J on A with $n \geq 2$, for every i, $j \in \mathrm{IrB}(A, J)$, where $\mathrm{IrB}(A, J)$ is the set of all natural numbers n such that (A, J) has an irredundant basis($\equiv$ minimal generating set) of n elements, if $i < j$ and $\{i+1, \ldots, j-1\} \cap \mathrm{IrB}(A, J) = \varnothing$, then $j - i \leq n - 1$. Thus, as stated by Burris and Sankappanavar in [2], on p. 33, the length of the finite gaps in $\mathrm{IrB}(A, J)$ is bounded by $n - 2$ if J is an n-ary closure operator. And our second main result is the proof of Tarski's irredundant basis theorem for many-sorted closure spaces.

2. Many-sorted sets, many-sorted closure operators, and many-sorted algebras.

In this section, for a set of sorts S in a fixed Grothendieck universe $\boldsymbol{\mathcal{U}}$, we begin by recalling some basic notions of the theory of S-sorted sets, e.g., those of subset of an S-sorted set, of proper subset of an S-sorted set, of delta of Kronecker, of cardinal of an S-sorted set, and of support of an S-sorted set; and by defining, for an S-sorted set A, the concepts of many-sorted closure operator on A and of many-sorted closure space. Moreover, for a many-sorted closure operator J on A, we define the notions of irredundant or independent part of A with respect to J, of basis or generator of A with respect to J, of irredundant basis of A with respect to J, and of minimal basis of A with respect to J. In addition, we state that the notion of irredundant basis of A with respect to J is equivalent to the notion of minimal basis of A with respect to J and, afterwards, for a many-sorted closure space (A, J), we define the subset $\mathrm{IrB}(A, J)$ of $\mathbb{N}$ as being formed by choosing those natural numbers which are the cardinal of an irredundant basis of A with respect to J. On the other hand, for a natural number n, we define the concept of n-ary many-sorted closure operator on A and provide a characterization of the n-ary many-sorted closure operators J on A, in terms of the fixed points of J. Besides, for a set of sorts S, we define the concept of S-sorted signature, and, for an S-sorted signature Σ, the notion of Σ-algebra and, for a Σ-algebra $\mathbf{A}$, the concept of subalgebra of $\mathbf{A}$ and the subalgebra generating many-sorted operator $\mathrm{Sg}_{\mathbf{A}}$ on A determined by $\mathbf{A}$. Subsequently, once defined the notion of finitely generated Σ-algebra, we state that, for a finitely generated Σ-algebra $\mathbf{A}$, $\mathrm{IrB}(A, \mathrm{Sg}_{\mathbf{A}}) \neq \varnothing$.

Definition 2.1. An *S-sorted set* is a function $A = (A_s)_{s\in S}$ from S to $\boldsymbol{\mathcal{U}}$.

Definition 2.2. Let S be a set of sorts. If A and B are S-sorted sets, then we will say that A is a *subset* of B, denoted by $A \subseteq B$, if, for every $s \in S$, $A_s \subseteq B_s$, and that A is a *proper* subset of B, denoted by $A \subset B$, if $A \subseteq B$ and, for some $s \in S$, $B_s - A_s \neq \varnothing$. We denote by $\mathrm{Sub}(A)$ the set of all S-sorted sets X such that $X \subseteq A$.

Definition 2.3. Given a sort $t \in S$ and a set X we call *delta of Kronecker for* (t, X) the S-sorted set $\delta^{t,X}$ defined, for every $s \in S$, as follows:

$$\delta^{t,X}_s = \begin{cases} X, & \text{if } s = t; \\ \varnothing, & \text{otherwise.} \end{cases}$$

For a final set $\{x\}$, to abbreviate, we will write $\delta^{t,x}$ instead of the more accurate $\delta^{t,\{x\}}$.

We next define, for a set of sorts S, the concept of cardinal of an S-sorted set, for an S-sorted set A, the notion of support of A, and characterize the finite S-sorted sets in terms of its supports.

Definition 2.4. Let A be an S-sorted set. Then the *cardinal of* A, denoted by $\mathrm{card}(A)$, is the cardinal of $\coprod A$, where $\coprod A$, the coproduct of $A = (A_s)_{s\in S}$, is $\bigcup_{s\in S}(A_s \times \{s\})$. Moreover, $\mathrm{Sub}_{\mathrm{fin}}(A)$ denotes the set of all finite subsets of A, i.e., the set $\{X \subseteq A \mid \mathrm{card}(X) < \aleph_0\}$, and, for a natural number n, $\mathrm{Sub}_{\leq n}(A)$ denotes the set of all subsets of A with at most n elements, i.e., the set $\{X \subseteq A \mid \mathrm{card}(X) \leq n\}$. Sometimes, for simplicity of notation, we write $X \subseteq_{\mathrm{fin}} A$ instead of $X \in \mathrm{Sub}_{\mathrm{fin}}(A)$.

Definition 2.5. Let S be a set of sorts. Then the *support of* A, denoted by $\mathrm{supp}_S(A)$, is the set $\{\, s \in S \mid A_s \neq \varnothing \,\}$.

Proposition 2.6. *An S-sorted set A is finite if, and only if,* $\mathrm{supp}_S(A)$ *is finite and, for every* $s \in \mathrm{supp}_S(A)$, $\mathrm{card}(A_s) < \aleph_0$.

Definition 2.7. Let S be a set of sorts and A an S-sorted set. A *many-sorted closure operator on* A is a mapping J from $\mathrm{Sub}(A)$ to $\mathrm{Sub}(A)$, which assigns to every $X \subseteq A$ its *J-closure* $J(X)$, such that, for every $X, Y \subseteq A$, satisfies the following conditions:

(1) $X \subseteq J(X)$, i.e., J is extensive.

(2) If $X \subseteq Y$, then $J(X) \subseteq J(Y)$, i.e., J is isotone.
(3) $J(J(X)) = J(X)$, i.e., J is idempotent.

Given two many-sorted closure operators J and K on A, J is called *smaller than* K, denoted by $J \leq K$, if, for every $X \subseteq A$, $J(A) \subseteq K(A)$. A *many-sorted closure space* is an ordered pair (A, J) where A is an S-sorted set and J a many-sorted closure operator on A. Moreover, if $X \subseteq A$, then X is *irredundant* (or *independent*) *with respect to* J if, for every $s \in S$ and every $x \in X_s$, $x \notin J(X - \delta^{s,x})_s$, X is a *basis* (or a *generator*) *with respect to* J if $J(X) = A$, X is an *irredundant basis with respect to* J if X irreduntant and a basis with respect to J, and X is a *minimal basis with respect to* J if $J(X) = A$ and, for every $Y \subset X$, $J(Y) \neq A$.

We next state that the notion of irredundant basis of A with respect to J is equivalent to the notion of minimal basis of A with respect to J. Moreover, for a many-sorted closure space (A, J), we define $\mathrm{IrB}(A, J)$ as the intersection of the set of all natural numbers and the set of the cardinals of the irredundant basis of A with respect to J.

Proposition 2.8. *Let (A, J) be a many-sorted closure space and $X \subseteq A$. Then X is an irredundant basis with respect to J if, and only if, it is a minimal basis with respect to J.*

Definition 2.9. Let S be a set of sorts and (A, J) a many-sorted closure space. Then we denote by $\mathrm{IrB}(A, J)$ the subset of $\mathbb{N}$ defined as follows:

$$\mathrm{IrB}(A, J) = \mathbb{N} \cap \left\{ \mathrm{card}(X) \;\middle|\; \begin{array}{c} X \text{ is an irredundant basis} \\ \text{of } A \text{ with respect to } J \end{array} \right\}.$$

Later, in this section, after having defined, for a set of sorts S and an S-sorted signature Σ, the concept of Σ-algebra, for a Σ-algebra $\mathbf{A} = (A, F)$, the uniform algebraic many-sorted closure operator $\mathrm{Sg}_{\mathbf{A}}$ on A, called the subalgebra generating many-sorted operator on A determined by $\mathbf{A}$, and the notion of finitely generated Σ-algebra, we will state that, for a finitely generated Σ-algebra $\mathbf{A}$, $\mathrm{IrB}(A, \mathrm{Sg}_{\mathbf{A}}) \neq \varnothing$.

Definition 2.10. Let A be an S-sorted set, J a many-sorted closure operator on A, and n a natural number.

(1) We denote by $J_{\leq n}$ the many-sorted operator on A defined, for every $X \subseteq A$, as follows:
$$J_{\leq n}(X) = \bigcup \{ J(Y) \mid Y \in \mathrm{Sub}_{\leq n}(X) \}.$$
(2) We define the family $(J^m_{\leq n})_{m \in \mathbb{N}}$ of many-sorted operator on A, recursively, as follows:
$$J^m_{\leq n} = \begin{cases} \mathrm{Id}_{\mathrm{Sub}(A)}, & \text{if } m = 0; \\ J_{\leq n} \circ J^k_{\leq n}, & \text{if } m = k + 1, \text{ with } k \geq 0. \end{cases}$$

(3) We denote by $J^{\omega}_{\leq n}$ the many-sorted operator on A that assigns to an S-sorted subset X of A, $J^{\omega}_{\leq n}(X) = \bigcup_{m\in\mathbb{N}} J^{m}_{\leq n}(X)$.
(4) We say that J is *n-ary* if $J = J^{\omega}_{\leq n}$.

Remark. Let J be a many-sorted closure operator on A. Then J is 0-ary, i.e., $J = J^{\omega}_{\leq 0}$, if, and only if, for every $X \subseteq A$, we have that

$$J(X) = X \cup J(\varnothing^S),$$

where $\varnothing^S$ is the S-sorted set whose sth coordinate, for every $s \in S$, is $\varnothing$.

We next prove that J is 1-ary, i.e., that $J = J^{\omega}_{\leq 1}$, if and only if, for every $X \subseteq A$, we have that

$$J(X) = J(\varnothing^S) \cup \textstyle\bigcup_{s\in S, x\in X_s} J(\delta^{s,x}).$$

Let us suppose that, for every $X \subseteq A$, $J(X) = J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x})$. Then it is obvious that, for every $X \subseteq A$, $J(X) \subseteq J_{\leq 1}(X)$. Let us verify that, for every $X \subseteq A$, $J_{\leq 1}(X) = \bigcup\{J(Y) \mid Y \in \mathrm{Sub}_{\leq 1}(X)\} \subseteq J(X)$. Let Y be an element of $\mathrm{Sub}_{\leq 1}(X)$. Then $Y = \varnothing^S$ or $Y = \delta^{t,a}$, for some $t \in S$ and some $a \in X_t$. If $Y = \varnothing^S$, then

$$J(\varnothing^S) \subseteq J(\varnothing^S) \cup \textstyle\bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) = J(X).$$

If $Y = \delta^{t,a}$, then $J(\delta^{t,a}) \subseteq \bigcup_{s\in S, x\in X_s} J(\delta^{s,x})$, hence

$$J(\delta^{t,a}) \subseteq J(\varnothing^S) \cup \textstyle\bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) = J(X).$$

Thus $J_{\leq 1}(X) \subseteq J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) = J(X)$. Therefore $J = J_{\leq 1}$. Hence, for every $m \geq 1$, $J = J^{m}_{\leq 1}$. Consequently J is 1-ary.

Reciprocally, let us suppose that J is 1-ary, i.e., that, for every $X \subseteq A$, $J(X) = \bigcup_{m\in\mathbb{N}} J^{m}_{\leq 1}(X)$. Then, obviously, we have that

$$J(X) \supseteq J(\varnothing^S) \cup \textstyle\bigcup_{s\in S, x\in X_s} J(\delta^{s,x}).$$

Let us verify that, for every $m \in \mathbb{N}$, $J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) \supseteq J^{m}_{\leq 1}$. Evidently $J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) \supseteq J^{0}_{\leq 1}(X) \cup J^{1}_{\leq 1}(X)$. Let k be ≥ 1 and let us suppose that $J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) \supseteq J^{k}_{\leq 1}(X)$. We will show that $J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}) \supseteq J^{k+1}_{\leq 1}(X)$. By definition we have that

$$J^{k+1}_{\leq 1}(X) = J_{\leq 1}(J^{k}_{\leq 1}(X)) = \textstyle\bigcup\{J(Z) \mid Z \in \mathrm{Sub}_{\leq 1}(J^{k}_{\leq 1}(X))\}.$$

Let Z be an element of $\mathrm{Sub}_{\leq 1}(J^{k}_{\leq 1}(X))$. Then $Z \subseteq J^{k}_{\leq 1}(X)$. But we have that $J^{k}_{\leq 1}(X) = \bigcup\{J(Y) \mid Y \in \mathrm{Sub}_{\leq 1}(J^{k-1}_{\leq 1}(X))\}$. Therefore, for some $Y \in \mathrm{Sub}_{\leq 1}(J^{k-1}_{\leq 1}(X))$, $Z \subseteq J(Y)$. Thus $J(Z) \subseteq J(J(Y)) = J(Y)$. But $J(Y) \subseteq J^{k}_{\leq 1}(X)$. Consequently $J(Z) \subseteq J^{k}_{\leq 1}(X)$. Whence, by the induction hypothesis, $J(Z) \subseteq J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x})$. From this, since Z was an arbitrary element of $\mathrm{Sub}_{\leq 1}(J^{k}_{\leq 1}(X))$, we infer that

$$J^{k+1}_{\leq 1}(X) = \textstyle\bigcup\{J(Z) \mid Z \in \mathrm{Sub}_{\leq 1}(J^{k}_{\leq 1}(X))\} \subseteq J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}).$$

Thus, for every $X \subseteq A$, we have that

$$J(X) = J(\varnothing^S) \cup \bigcup_{s\in S, x\in X_s} J(\delta^{s,x}).$$

Remark. Let n be ≥ 1, A an S-sorted set, $X \subseteq A$, and J a many-sorted closure operator on A. Then, for every $k \geq 0$ and every $Y \subseteq A$, if $Y \in \mathrm{Sub}_{\leq n}(J^k_{\leq n}(X))$, then $Y \in \mathrm{Sub}_{\leq n}(J^{k+1}_{\leq n}(X))$.

We next state, for a natural number $n \geq 1$ and a many-sorted closure operator J on an S-sorted set A, that the family of many-sorted operators $(J^m_{\leq n})_{m\in\mathbb{N}}$ on A is an ascending chain and that $J^\omega_{\leq n}$, which is the supremum of the above family, is the greatest n-ary many-sorted closure operator on A which is smaller than J.

Proposition 2.11. *For a natural number $n \geq 1$, an S-sorted set A, and a many-sorted closure operator J on A, the family of many-sorted operators $(J^m_{\leq n})_{m\in\mathbb{N}}$ on A is an ascending chain, i.e., for every $m \in \mathbb{N}$, $J^m_{\leq n} \leq J^{m+1}_{\leq n}$. Moreover, $J^\omega_{\leq n}$ is the greatest n-ary many-sorted closure operator on A such that $J^\omega_{\leq n} \leq J$.*

We next provide a characterization of the n-ary many-sorted closure operators J on an S-sorted set A in terms of the fixed points X of J and of its relationships with the J-closures of the subsets of X with at most n elements.

Proposition 2.12. *Let A be an S-sorted set, J a many-sorted closure operator on A, and n a natural number. Then J is n-ary if, and only if, for every $X \subseteq A$, if, for every $Z \in \mathrm{Sub}_{\leq n}(X)$, $J(Z) \subseteq X$, then $J(X) = X$ (i.e., if, and only if, for every $X \subseteq A$, X is a fixed point of J if X contains the J-closure of each of its subsets with at most n elements).*

Proof. If $n = 0$, then the result is obvious. So let us consider the case when $n \geq 1$. Let us suppose that J is n-ary and let X be a subset of A such that, for every $Z \in \mathrm{Sub}_{\leq n}(X)$, $J(Z) \subseteq X$. We want to show that $J(X) = X$. Because J is extensive, $X \subseteq J(X)$. Therefore it only remains to show that $J(X) \subseteq X$. Since, by hypothesis, $J(X) = \bigcup_{m\in\mathbb{N}} J^m_{\leq n}(X)$, to show that $J(X) \subseteq X$ it suffices to prove that, for every $m \in \mathbb{N}$, $J^m_{\leq n}(X) \subseteq X$.

For $m = 0$ we have that $J^0_{\leq n}(X) = X \subseteq X$.

Let us suppose that, for $k \geq 0$, $J^k_{\leq n}(X) \subseteq X$. Then we want to show that $J^{k+1}_{\leq n}(X) \subseteq X$. But, by definition, we have that

$$J^{k+1}_{\leq n}(X) = J_{\leq n}(J^k_{\leq n}(X)) = \bigcup\{J(Y) \mid Y \in \mathrm{Sub}_{\leq n}(J^k_{\leq n}(X))\}.$$

Hence what we have to prove is that, for every $Y \in \mathrm{Sub}_{\leq n}(J^k_{\leq n}(X))$, $J(Y) \subseteq X$. Let Y be a subset of $J^k_{\leq n}(X)$ such that $\mathrm{card}(Y) \leq n$. Since $J^k_{\leq n}(X) \subseteq X$, we have that $Y \subseteq X$ and $\mathrm{card}(Y) \leq n$, therefore $J(Y) \subseteq X$. Consequently, for every $X \subseteq A$, if, for every $Z \in \mathrm{Sub}_{\leq n}(X)$, $J(Z) \subseteq X$, then $J(X) = X$.

Reciprocally, let us suppose that, for every $X \subseteq A$, if, for every $Z \in \mathrm{Sub}_{\leq n}(X)$, $J(Z) \subseteq X$, then $J(X) = X$. We want to show that J is n-ary, i.e., that $J = J^{\omega}_{\leq n}$. Let X a subset of A. Then it is obvious that $J^{\omega}_{\leq n}(X) = \bigcup_{m\in\mathbb{N}} J^{m}_{\leq n}(X) \subseteq J(X)$. We now proceed to prove that $J(X) \subseteq J^{\omega}_{\leq n}(X)$. Since J is isotone and, by the definition of $J^{\omega}_{\leq n}$, $X \subseteq J^{\omega}_{\leq n}(X)$, we have that $J(X) \subseteq J(J^{\omega}_{\leq n}(X))$. Therefore to prove that $J(X) \subseteq J^{\omega}_{\leq n}(X)$ it suffices to prove that $J(J^{\omega}_{\leq n}(X)) = J^{\omega}_{\leq n}(X)$. But the just stated equation follows from the supposition because, as we will prove next, for every $Z \in \mathrm{Sub}_{\leq n}(J^{\omega}_{\leq n}(X))$, we have that $J(Z) \subseteq J^{\omega}_{\leq n}(X)$. Let Z be a subset of $J^{\omega}_{\leq n}(X)$ such that $\mathrm{card}(Z) \leq n$. Then, for some $\ell \in \mathbb{N}$, $\mathrm{supp}_S(Z) = \{s_0, \dots, s_{\ell-1}\}$ and, for every $\alpha \in \ell$, there exists an $n_\alpha \in \mathbb{N} - 1$ such that $Z_{s_\alpha} = \{z_{\alpha,0}, \dots, z_{\alpha,n_\alpha-1}\}$. Therefore, for every $\alpha \in \ell$ and every $\beta \in n_\alpha$ there exists an $m_{\alpha,\beta} \in \mathbb{N}$ such that that $z_{\alpha,\beta} \in J^{m_{\alpha,\beta}}_{\leq n}(X)_{s_\alpha}$. Since it may be helpful for the sake of understanding, let us represent the situation just described by the following figure:

$$\begin{array}{ccc}
z_{0,0} \in J^{m_{0,0}}_{\leq n}(X)_{s_0} & \dots & z_{0,n_0-1} \in J^{m_{0,n_0-1}}_{\leq n}(X)_{s_0} \\
\vdots & \ddots & \vdots \\
z_{\ell-1,0} \in J^{m_{\ell-1,0}}_{\leq n}(X)_{s_{\ell-1}} & \dots & z_{\ell-1,n_{\ell-1}-1} \in J^{m_{\ell-1,n_{\ell-1}-1}}_{\leq n}(X)_{s_{\ell-1}}
\end{array}$$

Hence, for every $\alpha \in \ell$ there exists a $\beta_\alpha \in n_\alpha$ such that $Z_{s_\alpha} \subseteq J^{m_{\alpha,\beta_\alpha}}_{\leq n}(X)_{s_\alpha}$. On the other hand, since the family of many-sorted operators $(J^{m}_{\leq n})_{m\in\mathbb{N}}$ on A is an ascending chain, there exists an m in the set $\{m_{\alpha,\beta_\alpha} \mid \alpha \in \ell\}$ such that, for every $\alpha \in \ell$, $J^{m_{\alpha,\beta_\alpha}}_{\leq n} \leq J^{m}_{\leq n}$. Thus $Z \subseteq J^{m}_{\leq n}(X)$. Therefore, since, in addition, $\mathrm{card}(Z) \leq n$, we have that $Z \in \mathrm{Sub}_{\leq n}(J^{m}_{\leq n}(X))$. Thus

$$J(Z) \subseteq J^{m+1}_{\leq n}(X) = J_{\leq n}(J^{m}_{\leq n}(X)) = \bigcup\{J(K) \mid K \in \mathrm{Sub}_{\leq n}(J^{m}_{\leq n}(X))\}.$$

Consequently $J(Z) \subseteq J^{\omega}_{\leq n}(X)$. Hence $J(X) \subseteq J^{\omega}_{\leq n}(X)$. Whence $J = J^{\omega}_{\leq n}$, which completes the proof. $\square$

We next recall the notion of free monoid on a set and, for a set of sorts S, we define, by using the the just mentioned notion, the concept of S-sorted signature and, for an S-sorted signature Σ, the concept of Σ-algebra.

Definition 2.13. Let S be a set of sorts. The *free monoid on* S, denoted by $\mathbf{S}^\star$, is $(S^\star, \curlywedge, \lambda)$, where $S^\star$, the set of all *words on* S, is $\bigcup_{n\in\mathbb{N}} \mathrm{Hom}(n, S)$, the set of all mappings $w\colon n \longrightarrow S$ from some $n \in \mathbb{N}$ to S, $\curlywedge$, the *concatenation* of words on S, is the binary operation on $S^\star$ which sends a pair of words (w, v) on S to the mapping $w \curlywedge v$ from $|w| + |v|$ to S, where $|w|$ and $|v|$ are the lengths ($\equiv$ domains) of the mappings w and v, respectively, defined as follows:

$$w \curlywedge v \begin{cases} |w|+|v| \longrightarrow S \\ \quad i \quad \longmapsto \begin{cases} w_i, & \text{if } 0 \leq i < |w|; \\ v_{i-|w|}, & \text{if } |w| \leq i < |w|+|v|, \end{cases} \end{cases}$$

and λ, the *empty word on* S, is the unique mapping $\lambda\colon \varnothing \longrightarrow S$.

Definition 2.14. Let S be a set of sorts. Then an S-*sorted signature* is a function Σ from $S^\star \times S$ to $\boldsymbol{\mathcal{U}}$ which sends a pair $(w,s) \in S^\star \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w, *sort* (or *coarity*) s, and *rank* (or *biarity*) (w,s).

Definition 2.15. Let Σ be an S-sorted signature and A an S-sorted set. The $S^\star \times S$-sorted set of the *finitary operations on* A is the family $(\mathrm{Hom}(A_w, A_s))_{(w,s)\in S^\star\times S}$, where, for every $w \in S^\star$, $A_w = \prod_{i\in|w|} A_{w_i}$. A *structure of* Σ-*algebra on* A is an $S^\star \times S$-mapping $F = (F_{w,s})_{(w,s)\in S^\star\times S}$ from Σ to $(\mathrm{Hom}(A_w, A_s))_{(w,s)\in S^\star\times S}$. For a pair $(w,s) \in S^\star \times S$ and a formal operation $\sigma \in \Sigma_{w,s}$, in order to simplify the notation, the operation from A_w to A_s corresponding to σ under $F_{w,s}$ will be written as F_σ instead of $F_{w,s}(\sigma)$. A Σ-*algebra* is a pair (A,F), abbreviated to $\mathbf{A}$, where A is an S-sorted set and F a structure of Σ-algebra on A.

Since it will be used afterwards, we next define, for a set of sorts S and an S-sorted set A, the notions of algebraic and of uniform many-sorted closure operator on A.

Definition 2.16. A many-sorted closure operator J on an S-sorted set A is *algebraic* if, for every $X \subseteq A$, $J(X) = \bigcup_{K\subseteq_{\mathrm{fin}} X} J(K)$, and is *uniform* if, for every X, $Y \subseteq A$, if $\mathrm{supp}_S(X) = \mathrm{supp}_S(Y)$, then $\mathrm{supp}_S(J(X)) = \mathrm{supp}_S(J(Y))$.

We next prove that, for a many-sorted closure operator, the property of being n-ary is stronger than that of being algebraic.

Proposition 2.17. *Let n be a natural number. If a many-sorted closure operator J on an S-sorted set A is n-ary, then J is an algebraic many-sorted closure operator on A.*

Proof. Let J be an n-ary many-sorted closure operator on an S-sorted set A and let X be a subset of A. Then, obviously, $\bigcup_{K\subseteq_{\mathrm{fin}} X} J(K) \subseteq J(X)$. Since $J(X) = J^{\omega}_{\leq n}(X) = \bigcup_{m\in\mathbb{N}} J^{m}_{\leq n}(X)$, to prove that $J(X) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}} X} J(K)$ it suffices to prove that, for every $m \in \mathbb{N}$, $J^{m}_{\leq n}(X) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}} X} J(K)$.

For $m = 0$, since $J^{0}_{\leq n}(X) = X$, we have that $J^{0}_{\leq n}(X) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}} X} J(K)$.

Let m be $k+1$ with $k \geq 0$ and let us suppose that $J^{k}_{\leq n}(X) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}} X} J(K)$. We want to prove that $J^{k+1}_{\leq n}(X) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}} X} J(K)$. However, by definition, $J^{k+1}_{\leq n}(X) = \bigcup\{J(Z) \mid Z \in \mathrm{Sub}_{\leq n}(J^{k}_{\leq n}(X))\}$. Thus it suffices to prove that, for every $Z \in \mathrm{Sub}_{\leq n}(J^{k}_{\leq n}(X))$, $J(Z) \subseteq$

$\bigcup_{K\subseteq_{\mathrm{fin}}X} J(K)$. Let Z be a subset of $J^k_{\leq n}(X)$ such that $\mathrm{card}(Z) \leq n$. Then, since, by the induction hypothesis, $J^k_{\leq n}(X) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}}X} J(K)$, we have that $Z \subseteq \bigcup_{K\subseteq_{\mathrm{fin}}X} J(K)$ and, in addition, that $\mathrm{card}(Z) \leq n$. Hence, for some $\ell \in \mathbb{N}$, $\mathrm{supp}_S(Z) = \{s_0, \ldots, s_{\ell-1}\}$ and, for every $\alpha \in \ell$, there exists an $n_\alpha \in \mathbb{N} - 1$ such that $Z_{s_\alpha} = \{z_{\alpha,0}, \ldots, z_{\alpha,n_\alpha-1}\}$. Therefore, for every $\alpha \in \ell$ and every $\beta \in n_\alpha$ there exists a $K^{\alpha,\beta} \subseteq_{\mathrm{fin}} X$ such that that $z_{\alpha,\beta} \in J(K^{\alpha,\beta})_{s_\alpha}$. Since it may be helpful for the sake of understanding, let us represent the situation just described by the following figure:

$$\begin{array}{ccc} z_{0,0} \in J(K^{0,0})_{s_0} & \cdots & z_{0,n_0-1} \in J(K^{0,n_0-1})_{s_0} \\ \vdots & \ddots & \vdots \\ z_{\ell-1,0} \in J(K^{\ell-1,0})_{s_{\ell-1}} & \cdots & z_{\ell-1,n_{\ell-1}-1} \in J(K^{\ell-1,n_{\ell-1}-1})_{s_{\ell-1}} \end{array}$$

Then, for every $\alpha \in \ell$, $Z_{s_\alpha} \subseteq J(\bigcup_{\beta\in n_\alpha} K^{\alpha,\beta})_{s_\alpha}$, where $\bigcup_{\beta\in n_\alpha} K^{\alpha,\beta} \subseteq_{\mathrm{fin}} X$. So, for $L = \bigcup_{\alpha\in\ell}\bigcup_{\beta\in n_\alpha} K^{\alpha,\beta}$, we have that $L \subseteq_{\mathrm{fin}} X$ and $Z \subseteq J(L)$. Therefore $J(Z) \subseteq J(J(L)) = J(L) \subseteq \bigcup_{K\subseteq_{\mathrm{fin}}X} J(K)$. $\square$

We next define when a subset X of the underlying S-sorted set A of a Σ-algebra $\mathbf{A}$ is closed under an operation F_σ of $\mathbf{A}$, as well as when X is a subalgebra of $\mathbf{A}$.

Definition 2.18. Let $\mathbf{A}$ be a Σ-algebra and $X \subseteq A$. Let σ be a formal operation in $\Sigma_{w,s}$. We say that X is *closed under the operation* $F_\sigma\colon A_w \longrightarrow A_s$ if, for every $a \in X_w$, $F_\sigma(a) \in X_s$. We say that X is a *subalgebra* of $\mathbf{A}$ if X is closed under the operations of $\mathbf{A}$. We denote by $\mathrm{Sub}(\mathbf{A})$ the set of all subalgebras of $\mathbf{A}$ (which is an algebraic closure system on A).

Definition 2.19. Let $\mathbf{A}$ be a Σ-algebra. Then we denote by $\mathrm{Sg}_{\mathbf{A}}$ the many-sorted closure operator on A defined as follows:

$$\mathrm{Sg}_{\mathbf{A}} \begin{cases} \mathrm{Sub}(A) \longrightarrow \mathrm{Sub}(A) \\ \quad X \quad\longmapsto \bigcap\{\, C \in \mathrm{Sub}(\mathbf{A}) \mid X \subseteq C \,\}, \end{cases}.$$

We call $\mathrm{Sg}_{\mathbf{A}}$ the *subalgebra generating many-sorted operator on* A *determined by* $\mathbf{A}$. For every $X \subseteq A$, we call $\mathrm{Sg}_{\mathbf{A}}(X)$ the *subalgebra of* $\mathbf{A}$ *generated by* X. Moreover, if $X \subseteq A$ is such that $\mathrm{Sg}_{\mathbf{A}}(X) = A$, then we say that X is an S-sorted set of *generators* of $\mathbf{A}$, or that X *generates* $\mathbf{A}$. Besides, we say that $\mathbf{A}$ is *finitely generated* if there exists an S-sorted subset X of A such that X *generates* $\mathbf{A}$ and $\mathrm{card}(X) < \aleph_0$.

Proposition 2.20. *Let* $\mathbf{A}$ *be a* Σ*-algebra. Then the many-sorted closure operator* $\mathrm{Sg}_{\mathbf{A}}$ *on* A *is algebraic, i.e., for every* S*-sorted subset* X *of* A*,* $\mathrm{Sg}_{\mathbf{A}}(X) = \bigcup_{K\subseteq_{\mathrm{fin}}X} \mathrm{Sg}_{\mathbf{A}}(K)$.

For a Σ-algebra $\mathbf{A}$ we next provide another, more constructive, description of the algebraic many-sorted closure operator $\mathrm{Sg}_{\mathbf{A}}$, which, in addition, will allow us to state a crucial property of $\mathrm{Sg}_{\mathbf{A}}$. Specifically, that $\mathrm{Sg}_{\mathbf{A}}$ is uniform.

Definition 2.21. Let Σ be an S-sorted signature and $\mathbf{A}$ a Σ-algebra.

(1) We denote by $\mathrm{E}_{\mathbf{A}}$ the many-sorted operator on A that assigns to an S-sorted subset X of A, $\mathrm{E}_{\mathbf{A}}(X) = X \cup \big(\bigcup_{\sigma \in \Sigma_{\cdot,s}} F_\sigma[X_{\mathrm{ar}(\sigma)}] \big)_{s \in S}$, where, for $s \in S$, $\Sigma_{\cdot,s}$ is the set of all many-sorted formal operations σ such that the coarity of σ is s and for $\mathrm{ar}(\sigma) = w \in S^\star$, the arity of σ, $X_{\mathrm{ar}(\sigma)} = \prod_{i \in |w|} X_{w_i}$.

(2) If $X \subseteq A$, then we define the family $(\mathrm{E}^n_{\mathbf{A}}(X))_{n \in \mathbb{N}}$ in $\mathrm{Sub}(A)$, recursively, as follows:

$$\begin{aligned} \mathrm{E}^0_{\mathbf{A}}(X) &= X, \\ \mathrm{E}^{n+1}_{\mathbf{A}}(X) &= \mathrm{E}_{\mathbf{A}}(\mathrm{E}^n_{\mathbf{A}}(X)),\ n \geq 0. \end{aligned}$$

(3) We denote by $\mathrm{E}^\omega_{\mathbf{A}}$ the many-sorted operator on A that assigns to an S-sorted subset X of A, $\mathrm{E}^\omega_{\mathbf{A}}(X) = \bigcup_{n \in \mathbb{N}} \mathrm{E}^n_{\mathbf{A}}(X)$.

Proposition 2.22. *Let $\mathbf{A}$ be a Σ-algebra and $X \subseteq A$, then $\mathrm{Sg}_{\mathbf{A}}(X) = \mathrm{E}^\omega_{\mathbf{A}}(X)$.*

In [3], on pp. 82, we stated the following proposition (there called Proposition 2.7).

Proposition 2.23. *Let $\mathbf{A}$ be a Σ-algebra and $X, Y \subseteq A$. Then we have that*

(1) *If $\mathrm{supp}_S(X) = \mathrm{supp}_S(Y)$, then, for every $n \in \mathbb{N}$, $\mathrm{supp}_S(\mathrm{E}^n_{\mathbf{A}}(X)) = \mathrm{supp}_S(\mathrm{E}^n_{\mathbf{A}}(Y))$.*

(2) $\mathrm{supp}_S(\mathrm{Sg}_{\mathbf{A}}(X)) = \bigcup_{n \in \mathbb{N}} \mathrm{supp}_S(\mathrm{E}^n_{\mathbf{A}}(X))$.

(3) *If $\mathrm{supp}_S(X) = \mathrm{supp}_S(Y)$, then $\mathrm{supp}_S(\mathrm{Sg}_{\mathbf{A}}(X)) = \mathrm{supp}_S(\mathrm{Sg}_{\mathbf{A}}(Y))$.*

Therefore the algebraic many-sorted closure operator $\mathrm{Sg}_{\mathbf{A}}$ is uniform.

Proposition 2.24. *If $\mathbf{A}$ is a finitely generated Σ-algebra, then every S-sorted set of generators of $\mathbf{A}$ contains a finite S-sorted subset which also generates $\mathbf{A}$.*

Corollary 2.25. *If $\mathbf{A}$ is a finitely generated Σ-algebra, then we have that $\mathrm{IrB}(A, \mathrm{Sg}_{\mathbf{A}})$ is not empty.*

3. A characterization of the n-ary many-sorted closure operators.

A theorem of Birkhoff-Frink (see [1]) asserts that every algebraic closure operator on an ordinary set arises, from some algebraic structure on the set, as the corresponding generated subalgebra operator. However, for many-sorted sets such a theorem is not longer true without qualification. In [3], on pp. 83–84, Theorem 3.1 and Corollary 3.2, we characterized the corresponding many-sorted closure operators as precisely the uniform algebraic operators. We next recall the just mentioned characterization since it will be applied afterwards to provide a characterization of the n-ary many-sorted closure operators on an S-sorted set.

Let us notice that in what follows, for a word $w\colon |w| \to S$ on S, with $|w|$ the lenght of w, and an $s \in S$, we denote by $w^{-1}[s]$ the set $\{i \in |w| \mid w(i) = s\}$, and by $\mathrm{Im}(w)$ the set $\{w(i) \mid i \in |w|\}$

Theorem 3.1. *Let J be an algebraic many-sorted closure operator on an S-sorted set A. If J is uniform, then $J = \mathrm{Sg}_{\mathbf{A}}$ for some S-sorted signature Σ and some Σ-algebra $\mathbf{A}$.*

Proof. Let $\Sigma = (\Sigma_{w,s})_{(w,s)\in S^\star\times S}$ be the S-sorted signature defined, for every $(w,s) \in S^\star \times S$, as follows:

$$\Sigma_{w,s} = \{\, (X,b) \in \textstyle\bigcup_{X\in\mathrm{Sub}(A)}(\{X\}\times J(X)_s) \mid \forall\, t \in S\,(\mathrm{card}(X_t) = |w|_t)\,\},$$

where for a sort $s \in S$ and a word $w\colon |w| \to S$ on S, with $|w|$ the lenght of w, the number of occurrences of s in w, denoted by $|w|_s$, is $\mathrm{card}(w^{-1}[s])$.

Before proceeding any further, let us remark that, for $(w,s) \in S^\star \times S$ and $(X,b) \in \bigcup_{X\in\mathrm{Sub}(A)}(\{X\} \times J(X)_s)$, the following conditions are equivalent:

(1) $(X,b) \in \Sigma_{w,s}$, i.e., for every $t \in S$, $\mathrm{card}(X_t) = |w|_t$.
(2) $\mathrm{supp}_S(X) = \mathrm{Im}(w)$ and, for every $t \in \mathrm{supp}_S(X)$, $\mathrm{card}(X_t) = |w|_t$.

On the other hand, for the index set $\Lambda = \bigcup_{Y\in\mathrm{Sub}(A)}(\{Y\}\times\mathrm{supp}_S(Y))$ and the Λ-indexed family $(Y_s)_{(Y,s)\in\Lambda}$ whose (Y,s)-th coordinate is Y_s, precisely the s-th coordinate of the S-sorted set Y of the index $(Y,s) \in \Lambda$, let f be a choice function for $(Y_s)_{(Y,s)\in\Lambda}$, i.e., an element of $\prod_{(Y,s)\in\Lambda} Y_s$.

Moreover, for every $w \in S^\star$ and $a \in \prod_{i\in|w|} A_{w(i)}$, let $M^{w,a} = (M^{w,a}_s)_{s\in S}$ be the finite S-sorted subset of A defined as $M^{w,a}_s = \{a_i \mid i \in w^{-1}[s]\}$, for every $s \in S$.

Now, for $(w,s) \in S^\star \times S$ and $(X,b) \in \Sigma_{w,s}$, let $F_{X,b}$ be the many-sorted operation from $\prod_{i\in|w|} A_{w(i)}$ into A_s that to an $a \in \prod_{i\in|w|} A_{w(i)}$ assigns b, if $M^{w,a} = X$ and $f(J(M^{w,a}),s)$, otherwise.

We will prove that the Σ-algebra $\mathbf{A} = (A,F)$ is such that $J = \mathrm{Sg}_{\mathbf{A}}$. But before doing that it is necessary to verify that the definition of the many-sorted operations is sound, i.e., that for every $(w,s) \in S^\star \times S$, $(X,b) \in \Sigma_{w,s}$ and $a \in \prod_{i\in|w|} A_{w(i)}$, it happens that $s \in \mathrm{supp}_S(J(M^{w,a}))$, and for this it suffices to prove that $\mathrm{supp}_S(M^{w,a}) = \mathrm{supp}_S(X)$, because, by hypothesis, J is uniform and, by definition, $b \in J(X)_s$.

If $t \in \mathrm{supp}_S(M^{w,a})$, then $M^{w,a}_t$ is nonempty, i.e., there exists an $i \in |w|$ such that $w(i) = t$. Therefore, because $(X,b) \in \Sigma_{w,s}$, we have that $0 < |w|_t = \mathrm{card}(X_t)$, hence $t \in \mathrm{supp}_S(X)$.

Reciprocally, if $t \in \mathrm{supp}_S(X)$, $|w|_t > 0$, and there is an $i \in |w|$ such that $w(i) = t$, hence $a_i \in A_t$, and from this we conclude that $M^{w,a}_t \neq \varnothing$, i.e., that $t \in \mathrm{supp}_S(M^{w,a})$. Therefore, $\mathrm{supp}_S(M^{w,a}) = \mathrm{supp}_S(X)$ and, by the uniformity of J, $\mathrm{supp}_S(J(M^{w,a})) = \mathrm{supp}_S(J(X))$. But, by definition, $b \in J(X)_s$, so $s \in \mathrm{supp}_S(J(M^{w,a}))$ and the definition is sound.

Now we prove that, for every $X \subseteq A$, $J(X) \subseteq \mathrm{Sg}_{\mathbf{A}}(X)$. Let X be an S-sorted subset of A, $s \in S$ and $b \in J(X)_s$. Then, because J is algebraic, $b \in J(Y)_s$, for some finite S-sorted subset Y of X. From such an Y we will define a word w_Y in S and an element a_Y of $\prod_{i \in |w_Y|} A_{w_Y(i)}$ such that

1. $Y = M^{w_Y, a_Y}$,
2. $(Y, b) \in \Sigma_{w_Y, s}$, i.e., $b \in J(Y)_s$ and, for all $t \in S$, $\mathrm{card}(Y_t) = |w_Y|_t$, and
3. $a_Y \in \prod_{i \in |w_Y|} X_{w_Y(i)}$,

then, because $F_{Y,b}(a_Y) = b$, we will be entitled to assert that $b \in \mathrm{Sg}_{\mathbf{A}}(X)_s$.

But given that Y is finite if, and only if, $\mathrm{supp}_S(Y)$ is finite and, for every $t \in \mathrm{supp}_S(Y)$, Y_t is finite, let $\{\, s_\alpha \mid \alpha \in m \,\}$ be an enumeration of $\mathrm{supp}_S(Y)$ and, for every $\alpha \in m$, let $\{\, y_{\alpha,i} \mid i \in p_\alpha \,\}$ be an enumeration of the nonempty s_α-th coordinate, Y_{s_α}, of Y. Then we define, on the one hand, the word w_Y as the mapping from $|w_Y| = \sum_{\alpha \in m} p_\alpha$ into S such that, for every $i \in |w_Y|$ and $\alpha \in m$, $w_Y(i) = s_\alpha$ if, and only if, $\sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1$ and, on the other hand, the element a_Y of $\prod_{i \in |w_Y|} A_{w_Y(i)}$ as the mapping from $|w_Y|$ into $\bigcup_{i \in |w_Y|} A_{w_Y(i)}$ such that, for every $i \in |w_Y|$ and $\alpha \in m$, $a_Y(i) = y_{\alpha, i - \sum_{\beta \in \alpha} p_\beta}$ if, and only if, $\sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1$. From these definitions follow (1), (2) and (3) above. Let us observe that (1) is a particular case of the fact that the mapping M from $\bigcup_{w \in S^\star}(\{w\} \times \prod_{i \in |w|} A_{w(i)})$ into $\mathrm{Sub}_{\mathrm{fin}}(A)$ that to a pair (w, a) assigns $M^{w,a}$ is surjective.

From the above and the definition of $F_{Y,b}$ we can affirm that $F_{Y,b}(a_Y) = b$, hence $b \in \mathrm{Sg}_{\mathbf{A}}(X)_s$. Therefore $J(X) \subseteq \mathrm{Sg}_{\mathbf{A}}(X)$.

Finally, we prove that, for every $X \subseteq A$, $\mathrm{Sg}_{\mathbf{A}}(X) \subseteq J(X)$. But for this, by Proposition 2.22, it is enough to prove that, for every subset X of A, we have that $\mathrm{E}_{\mathbf{A}}(X) \subseteq J(X)$. Let $s \in S$ be and $c \in \mathrm{E}_{\mathbf{A}}(X)_s$. If $c \in X_s$, then $c \in J(X)_s$, because J is extensive. If $c \notin X_s$, then, by the definition of $\mathrm{E}_{\mathbf{A}}(X)$, there exists a word $w \in S^\star$, a many-sorted formal operation $(Y, b) \in \Sigma_{w,s}$ and an $a \in \prod_{i \in |w|} X_{w(i)}$ such that $F_{Y,b}(a) = c$. If $M^{w,a} = Y$, then $c = b$, hence $c \in J(Y)_s$, therefore, because $M^{w,a} \subseteq X$, $c \in J(X)_s$. If $M^{w,a} \neq Y$, then $F_{Y,b}(a) \in J(M^{w,a})_s$, but, because $M^{w,a} \subseteq X$ and J is isotone, $J(M^{w,a})$ is a subset of $J(X)$, hence $F_{Y,b}(a) \in J(X)_s$. Therefore $\mathrm{E}_{\mathbf{A}}(X) \subseteq J(X)$. □

The just stated theorem together with Proposition 2.23 entails the following corollary.

Corollary 3.2. *Let J be an algebraic many-sorted closure operator on an S-sorted set A. Then $J = \mathrm{Sg}_{\mathbf{A}}$ for some S-sorted signature Σ and some Σ-algebra $\mathbf{A}$ if, and only if, J is uniform.*

We next prove that for a natural number n, an S-sorted signature Σ, and a Σ-algebra $\mathbf{A}$, under a suitable condition on Σ related to n, the uniform algebraic many-sorted closure operator $\mathrm{Sg}_{\mathbf{A}}$ is an n-ary many-sorted closure operator on A.

Proposition 3.3. *Let Σ be an S-sorted signature, $\mathbf{A}$ a Σ-algebra, and $n \in \mathbb{N}$. If Σ is such that, for every $(w, s) \in S^{\star} \times S$, $\Sigma_{w,s} = \varnothing$ if $|w| > n$—in which case we will say that every operation of $\mathbf{A}$ is of an arity $\leq n$—, then the uniform algebraic many-sorted closure operator $\mathrm{Sg}_{\mathbf{A}}$ is an n-ary many-sorted closure operator on A, i.e., $\mathrm{Sg}_{\mathbf{A}} = (\mathrm{Sg}_{\mathbf{A}})^{\omega}_{\leq n}$.*

Proof. It follows from $\mathrm{Sg}_{\mathbf{A}}(X) = \mathrm{E}^{\omega}_{\mathbf{A}}(X)$ and from the fact that, for every $X \subseteq A$, $\mathrm{E}_{\mathbf{A}}(X) \subseteq (\mathrm{Sg}_{\mathbf{A}})_{\leq n}(X) \subseteq \mathrm{Sg}_{\mathbf{A}}(X)$. The details are left to the reader. However, we notice that it is advisable to split the proof into two cases, one for $n = 0$ and another one for $n \geq 1$. □

Proposition 3.4. *Let A be an S-sorted set, J a many-sorted closure operator on A, and $n \in \mathbb{N}$. If J is n-ary (hence, by Proposition 2.17, algebraic) and uniform, then there exists an S-sorted signature Σ' and a Σ'-algebra $\mathbf{A}'$ such that $J = \mathrm{Sg}_{\mathbf{A}'}$ and every operation of $\mathbf{A}'$ is of an arity $\leq n$.*

Proof. If we denote by $\mathbf{A} = (A, F)$ the Σ-algebra associated to J constructed in the proof of Theorem 3.1, then taking as Σ' the S-sorted signature defined, for every $(w, s) \in S^{\star} \times S$, as: $\Sigma'_{w,s} = \Sigma_{w,s}$, if $|w| \leq n$; and $\Sigma'_{w,s} = \varnothing$, if $|w| > n$, and as $\mathbf{A}' = (A', F')$ the Σ'-algebra defined as: $A' = A$, and $F' = F \circ \mathrm{inc}^{\Sigma',\Sigma}$, where $\mathrm{inc}^{\Sigma',\Sigma} = (\mathrm{inc}^{\Sigma',\Sigma}_{w,s})_{(w,s)\in S^{\star}\times S}$ is the canonical inclusion of Σ' into Σ, then one can show that $J = \mathrm{Sg}_{\mathbf{A}'}$. □

From the just stated proposition together with Proposition 3.3 it follows immediately the following corollary, which is an algebraic characterization of the n-ary and uniform many-sorted closure operators.

Corollary 3.5. *Let J be a many-sorted closure operator on an S-sorted set A and $n \in \mathbb{N}$. Then J is n-ary and uniform if, and only if, there exists an S-sorted signature Σ and a Σ-algebra $\mathbf{A}$ such that $J = \mathrm{Sg}_{\mathbf{A}}$ and every operation of $\mathbf{A}$ is of an arity $\leq n$.*

4. The irredundant basis theorem for many-sorted closure spaces.

We next show Tarski's irredundant basis theorem for many-sorted closure spaces.

Theorem 4.1 (Tarski's irredundant basis theorem for many-sorted closure spaces)**.** *Let (A, J) be a many-sorted closure space. If J is an*

n-ary many-sorted operator on the S-sorted set A, with $n \geq 2$, and if $i < j$ with $i, j \in \mathrm{IrB}_J(A)$ such that

$$\{i+1, \ldots, j-1\} \cap \mathrm{IrB}_J(A) = \varnothing,$$

then $j - i \leq n - 1$. In particular, if $n = 2$, then $\mathrm{IrB}_J(A)$ is a convex subset of $\mathbb{N}$.

Proof. Let $Z \subseteq A$ be an irredundant basis with respect to J such that $\mathrm{card}(Z) = j$ and $\mathcal{K} = \{ X \in \mathrm{IrB}_J(A) \mid \mathrm{card}(X) \leq i \}$. Since J is n-ary, we can assert that $J(Z) = A = \bigcup_{m \in \mathbb{N}} J^m_{\leq n}(Z)$, so, for every $s \in S$, $J(Z)_s = A_s = \bigcup_{m \in \mathbb{N}} J^m_{\leq n}(Z)_s$. Let X be an element of $\mathcal{K}$. Then there exists a $k \in \mathbb{N} - 1$ such that $X \subseteq J^k_{\leq n}(Z)$. The natural number k should be strictly greater than 0, because if $k = 0$, $X \subseteq J^0_{\leq n}(Z) = Z$, but $\mathrm{card}(X) = i < j = \mathrm{card}(Z)$, so Z would not be an irredundant basis. So that, for every $X \in \mathcal{K}$, $\{ k \in \mathbb{N} - 1 \mid X \subseteq J^k_{\leq n}(Z) \} \neq \varnothing$. Therefore, for every $X \in \mathcal{K}$, we can choose the least element of such a set, denoted by $d_Z(X)$, and there is fulfilled that $d_Z(X)$ is greater than or equal to 1. For $d_Z(X) - 1$ we have that $X \nsubseteq J^{d_Z(X)-1}_{\leq n}(Z)$. So we conclude that there exists a mapping $d_Z \colon \mathcal{K} \longrightarrow \mathbb{N} - 1$ that to an $X \in \mathcal{K}$ assigns $d_Z(X)$. The image of the mapping d_Z, which is a nonempty part of $\mathbb{N} - 1$, is well-ordered, hence it has a least element, which is, necessarily, non zero, $t + 1$, therefore, since $\mathcal{K}/\mathrm{Ker}(d_Z)$ is isomorphic to $\mathrm{Im}(d_Z)$, by transport of structure, it will also be well-ordered, then we can always choose an $X \in \mathcal{K}$ such that, for every $Y \in \mathcal{K}$, $d_Z(X) \leq d_Z(Y)$, e.g., an X such that its equivalence class corresponds to the minimum $t + 1$ of $\mathrm{Im}(d_Z)$. Moreover, among the X which have the just mentioned property, we choose an X^0 such that, for every $Y \in \mathcal{K}$ with $Y \subseteq J^{t+1}_{\leq n}(Z)$, it happens that

$$\mathrm{card}(X^0 \cap (J^{t+1}_{\leq n}(Z) - J^t_{\leq n}(Z))) \leq \mathrm{card}(Y \cap (J^{t+1}_{\leq n}(Z) - J^t_{\leq n}(Z))).$$

By the method of election we have that $X^0 \subseteq J^{t+1}_{\leq n}(Z)$ but $X^0 \nsubseteq J^t_{\leq n}(Z)$. Of the latter we conclude that there exists an $s_0 \in S$ such that $X^0_{s_0} \nsubseteq J^t_{\leq n}(Z)_{s_0}$, therefore

$$(J^{t+1}_{\leq n}(Z)_{s_0} - J^t_{\leq n}(Z)_{s_0}) \cap X^0_{s_0} \neq \varnothing.$$

Let $a_0 \in (J^{t+1}_{\leq n}(Z)_{s_0} - J^t_{\leq n}(Z)_{s_0}) \cap X^0_{s_0}$ be. Then $a_0 \in X^0_{s_0}$, $a_0 \in J^{t+1}_{\leq n}(Z)_{s_0}$ but $a_0 \notin J^t_{\leq n}(Z)_{s_0}$. However, $J^{t+1}_{\leq n}(Z) = J_{\leq n}(J^t_{\leq n}(Z))$, by definition, hence there exists a part F of $J^t_{\leq n}(Z)$ such that $\mathrm{card}(F) \leq n$ and $a_0 \in J(F)_{s_0}$. Let X^1 be the part of A defined as follows:

$$X^1_s = \begin{cases} X^0_s \cup F_s, & \text{if } s \neq s_0; \\ (X^0_{s_0} - \{a_0\}) \cup F_{s_0}, & \text{if } s = s_0. \end{cases}$$

It holds that $X^0 \subseteq J(X^1)$. Therefore $J(X^0) \subseteq J(X^1)$, but $J(X^0) = A$, hence $J(X^1) = A$, i.e., X^1 is a finite generator with respect to J, thus X^1 will contain a minimal generator X^2 with respect to J. It holds that $\mathrm{card}(X^2) \leq \mathrm{card}(X^1) < \mathrm{card}(X^0) + n$. It cannot happen that $\mathrm{card}(X^0)+n \leq j$. Because if $\mathrm{card}(X^0)+n \leq j$, then $\mathrm{card}(X^2) < j$, hence, since

$$\{i+1, \ldots, j-1\} \cap \mathrm{IrB}(A, J) = \varnothing,$$

$X^2 \in \mathcal{K}$, but $X^2 \subseteq J^{t+1}_{\leq n}(Z)$ and, moreover, it happens that

$$\mathrm{card}(X^2 \cap (J^{t+1}_{\leq n}(Z) - J^{t}_{\leq n}(Z))) < \mathrm{card}(X^0 \cap (J^{t+1}_{\leq n}(Z) - J^{t}_{\leq n}(Z))),$$

because $a_0 \notin X^2_{s_0}$ but $a_0 \in X^0_{s_0}$, which contradicts the choice of X^0. Hence $\mathrm{card}(X^0) + n > j$. But $\mathrm{card}(X^0) \leq i$, therefore $j - i < n$, i.e., $j - i \leq n - 1$. □

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