Representations of Finite Groups: Blocks Relative to a Normal Subgroup

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A mi familia

Contents

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Burjassot, 20 de Febrero de 2019

Noelia Rizo Carrión

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Gabriel Navarro Ortega

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Resumen

Algunas de las principales conjeturas en Teoría de Representaciones de Grupos Finitos admiten refinamientos en términos de p-bloques de Brauer. Un ejemplo paradigmático de esto es la conjetura de Alperin-McKay propuesta por J. L. Alperin en [**Alp75**], que brinda una visión bloque-teórica a la afamada conjetura de McKay. Además, en general, los bloques dotan de más estructura a estos problemas.

Actualmente solo se vislumbra un camino para atacar este tipo de conjeturas: reducirlas a problemas de grupos simples y utilizar la Clasificación de los Grupos Finitos Simples para resolverlas. Entendemos por *reducir* a un problema de grupos simples que el problema tiene solución siempre que se comprueben una serie de condiciones para todos los grupos finitos simples. Por supuesto, los subgrupos normales (y sus caracteres irreducibles) juegan un papel fundamental en este proceso.

Una de las técnicas principales utilizadas en la reducción de ciertos problemas de teoría de caracteres a problemas de grupos simples es estudiar una *versión proyectiva* de los mismos. Con esto queremos decir lo siguiente: sea N un subgrupo normal de G, sea θ un caracter irreducible de N y sea $\mathrm{Irr}(G|\theta)$ el conjunto de constituyentes irreducibles del caracter inducido θ^G. Hacer una versión proyectiva de un problema es reformularlo en términos de $\mathrm{Irr}(G|\theta)$ en lugar de $\mathrm{Irr}(G)$, el grupo cociente G/N en lugar de G, clases de conjugación θ-buenas, en lugar de clases de conjugación, etc. En otras palabras, necesitamos entender totalmente la *teoría de caracteres sobre el caracter* θ. Un ejemplo de aplicación de este método es la reducción de la conjetura de McKay en [**IMN07**]. Además, cuando $N = 1$ debemos recuperar la conjetura o problema original. ¿Cuál es la ventaja de esta filosofía? En primer lugar, podemos proponer (y solucionar) problemas mucho más generales. En segundo lugar, de esta manera podemos utilizar una poderosa herramienta: inducción sobre $|G : N|$, que es una manera natural de introducir grupos simples en este tipo de problemas.

Siguiendo esta filosofía, si queremos atacar algunas conjeturas que involucran p-bloques, no solo necesitamos entender la teoría de caracteres sobre θ, sino también la *teoría de bloques sobre* θ. Esta es la motivación detrás de gran parte de esta tesis: definimos un conjunto de bloques canónicamente construidos *sobre* un caracter de un subgrupo normal. Estos bloques estarán definidos con respecto a un primo p y un caracter irreducible θ de

un subgrupo normal, y les llamaremos *θ-bloques* (no haremos referencia al primo p en la notación porque lo matendremos fijo). Definimos los θ-bloques utilizando representaciones proyectivas y la teoría de *character triples* introducida por I. M. Isaacs. Los θ-bloques están relacionados con los bloques de las *twisted group algebras*, pero nuestro acercamiento es totalmente caracter-teórico. Una parte no trivial de este trabajo es probar que los θ-bloques son una partición canónica del conjunto $\mathrm{Irr}(G|\theta)$, es decir, los θ-bloques son independientes de cualquier elección tomada a la hora de definirlos. Además, a cada θ-bloque le asociaremos una única clase de conjugación de p-subgrupos de G/N. A cada uno de estos grupos le llamaremos *θ-grupo defecto*. Veremos que, en general, los θ-grupos defecto se comportan como los grupos defecto de los p-bloques de Brauer clásicos.

Pero ¿por qué esta generalización? Primero, desde el punto de vista del subgrupo normal N y su caracter irreducible θ, en general los p-bloques de Brauer son demasiado grandes y por ello no captan por completo las sutilezas de la teoría de caracteres de G sobre θ. Probaremos que cada θ-bloque B_θ está contenido en $\mathrm{Irr}(B) \cap \mathrm{Irr}(G|\theta)$, donde B es un p-bloque de Brauer, pero en general, B_θ es mucho más pequeño, con lo que la partición en θ-bloques es más fina. La segunda razón es que usando nuestros θ-bloques podemos unificar resultados como no se había hecho antes en la literatura: por ejemplo, en nuestra Conjetura B, el teorema de Gluck-Wolf-Navarro-Tiep y la *Brauer's Height Zero Conjecture* (BHZC) aparecen unificados por primera vez. Este problema inspiró a G. Malle y G. Navarro quienes propusieron una versión proyectiva de la BHZC [**MN17**]. Más tarde, B. Sambale probó, utilizando la teoría de los sistemas de fusión, que esta conjetura es equivalente a la BHZC ([**Sam19**]). Con todo esto, hemos obtenido nueva información de los p-bloques de Brauer clásicos utilizando la idea de los θ-bloques. Pero esta no es la única ocasión en la que los θ-bloques nos han arrojado luz sobre los p-bloques clásicos. También probamos en el Teorema F que la matriz de descomposición clásica de un p-bloque de Brauer no se puede descomponer de cierta forma. La esperanza es que los θ-bloques puedan inspirar más resultados de este tipo.

En la primera parte de esta tesis probamos que la *conjetura $k(B)$ de Brauer* también admite una θ-versión. Esta conjetura es otro de los problemas abiertos fundamentales de R. Brauer de los 50, y no solo no se ha resuelto sino que ni siquiera se ha reducido a grupos finitos simples. Quizás nuestra θ-versión pueda ayudar a divisar tal reducción.

Otra parte importante del Capítulo 2 es la introducción de θ-caracteres de Brauer. En [**Nav00**], Navarro da una versión de los caracteres de Brauer relativos a un p-subgrupo normal N de G. Estos forman una base $\mathrm{IBr}(G|N)$ del espacio de funciones de clase definidas en $G^0 = \{x \in G | x_p \in N\}$. Esto le permitió definir números de descomposición $d_{\chi\varphi}$ para $\chi \in \mathrm{Irr}(G)$ y $\varphi \in \mathrm{IBr}(G|N)$. La importancia de esto es que los caracteres

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$$

eran los caracteres relativos indescomponibles N-proyectivos descubiertos previamente por B. Külshammer y G. R. Robinson en [**KR87**]. Aunque Navarro también dio una versión de este resultado para un subgrupo normal N arbitrario (es decir, no necesariamente un p-grupo) en [**Nav12**], no probó que la nueva N-base $\mathrm{IBr}(G|N)$ era canónica (tan canónica como lo son los caracteres de Brauer, es decir, salvo elección de un ideal maximal conteniendo p en el anillo de los enteros algebraicos). En esta tesis damos una base canónica para el espacio de funciones de clase definidas en G° con N arbitrario y demostramos que esta base coincide con la base canónica dada por Navarro en [**Nav00**].

A día de hoy, aún tenemos muchas preguntas sin contestar acerca de los θ-bloques. Por ejemplo, ¿pueden ser determinados a partir de la tabla de caracteres? ¿Podemos caracterizar los θ-bloques con un único caracter? No tenemos una respuesta completa a estas preguntas, y en otras seguimos trabajando.

Todos los resultados arriba mencionados son el Teorema A, la Conjetura B, el Teorema C, la Conjetura D, el Teorema E, el Teorema F y el Teorema G del Capítulo 2 de este trabajo. Excepto el Teorema G, todos aparecen en [**Riz18**]. El Teorema G aparecerá en [**Riz19**].

En el Capítulo 3 seguimos estudiando el conjunto $\mathrm{Irr}(G|\theta)$ y damos una generalización del conocido teorema de Howlett-Isaacs. En 1964, N. Iwahori y H. Matsumoto conjeturaron en [**IM64**] que si θ es G-invariante y $|\mathrm{Irr}(G|\theta)| = 1$, entonces G/N es resoluble (en este caso decimos que θ es *totalmente ramificado* en G/N). Este tipo de caracteres, como cualquier otra situación minimal en teoría de grupos, aparece con frecuencia en teoría de representaciones ordinarias (sobre cuerpos de característica 0) y modulares (sobre cuerpos de característica p). Por ejemplo, en la teoría de caracteres de los *chief factors* abelianos o en bloques con exactamente un caracter de Brauer. Fueron Isaacs y R. Howlett quienes resolvieron finalmente esta conjetura en [**HI82**], siendo este teorema una de las primeras aplicaciones de la Clasificación de los Grupos Finitos Simples a la teoría de caracteres. Cuando θ es totalmente ramificado en G/N y $N \subseteq M \lhd G$, por el teorema de Clifford tenemos que las constituyentes irreducibles de θ^M son todas G-conjugadas. Este hecho nos inspiró el siguiente resultado principal de esta tesis. El Teorema H es una generalización del aclamado teorema de Howlett y Isaacs, y de una manera un tanto más débil podemos enunciarlo así: si A actúa por automorfismos sobre G fijando N y algún caracter $\theta \in \mathrm{Irr}(N)$ G-invariante, y A permuta transitivamente los elementos de $\mathrm{Irr}(G|\theta)$, entonces G/N es resoluble. Es importante señalar que para la prueba de este teorema no utilizamos el teorema de Howlett y Isaacs. De hecho, creemos que nuestra prueba simplifica parte de la suya.

Nuestro siguiente resultado principal del Capítulo 3 versa de nuevo sobre el conjunto $\mathrm{Irr}(G|\theta)$, pero también sobre teoría de bloques. J. F. Humphreys conjeturó que si todos los caracteres en $\mathrm{Irr}(G|\theta)$ tienen el mismo grado, entonces G/N es resoluble. Esto, evidentemente, es una amplísima generalización del teorema de Howlett y Isaacs, así como del nuestro. En el momento de escritura de esta tesis no hay resultados parciales para esta conjetura. Nuestro resultado (Teorema I), es una caracterización grupo-teórica de qué ocurre cuando $N = \mathbf{O}_\pi(G)$ y G es π-separable (si π es un conjunto de primos, decimos que G es π-separable si sus factores de composición son bien π-grupos o bien π'-grupos). Como hemos dicho, este resultado está relacionado con la teoría de bloques y ahora explicamos por qué. Si π es el complemento de un primo p, un resultado muy conocido de P. Fong asegura que $\mathrm{Irr}(G|\theta)$ son los caracteres irreducibles de un p-bloque de Brauer (ver el Teorema 10.20 de [**Nav98a**], por ejemplo). Si todos los caracteres irreducibles de $\mathrm{Irr}(G|\theta)$ tienen el mismo grado, entonces estamos en la situación en que todos los caracteres irreducibles de un p-bloque tienen el mismo grado. Esta situación, sin la hipótesis de p-resolubilidad, fue caracterizada por T. Okuyama y Y. Tsushima en [**OT83**]. Podemos considerar, por tanto, nuestro teorema I como una π-versión de este resultado. Los Teoremas H e I aparecen en [**NR17**].

La parte final de este trabajo es de naturaleza un tanto distina. En esta última parte ya no trabajamos en términos de teoría de caracteres sobre un caracter de un subgrupo normal, sino que trabajamos con el conjunto completo de todos los caracteres irreducibles de G, $\mathrm{Irr}(G)$. La *tabla de caracteres* de G, $X(G)$, es una matriz cuadrada cuyas columnas están indexadas por las clases de conjugación de G, y cuyas filas están indexadas por los caracteres irreducibles de G. Uno de los problemas clásicos en teoría de caracteres es determinar qué propiedades de un grupo finito G podemos conocer a partir de su tabla de caracteres. Por ejemplo, la tabla de caracteres detecta si G es abeliano, nilpotente, superresoluble, resoluble o simple. En esta tesis, estamos interesados en qué sabe $X(G)$ sobre la p-estructura local de G, para un primo dado p, un problema mucho más complejo. En particular, nuestra motivación es la Pregunta 7 de [**Nav04**]: ¿determina la tabla de caracteres de G cuántos p-subgrupos de Sylow tiene G? En este trabajo damos una respuesta afirmativa a esta pregunta en algunos casos (Teorema J).

No obstante, más interesante que el resultado en sí, es quizás la manera de demostrarlo. Para probar este teorema necesitamos calcular el número de puntos fijos de la acción de un p-grupo sobre un grupo de orden coprimo con p. Para ello damos una fórmula (Teorema K) que permite calcular este número en términos de información que se puede obtener de la tabla de caracteres. Nuestra fórmula generaliza un resultado clásico de Brauer (y H.

Wielandt) para contar el número de puntos fijos de la acción de un 4-grupo de Klein sobre un grupo de orden impar. Los resultados del Capítulo 4, Teorema J y Teorema K, aparecen en [**NR16**].

Después de leer nuestra prueba de la fórmula del Teorema K, Isaacs y R. Lyons encontraron dos pruebas alternativas muy elegantes de la misma. Las reproducimos en este trabajo con su permiso.

Guión de la tesis

El Capítulo 1 brinda un breve repaso a la teoría de caracteres ordinarios y modulares (Secciones 1.1 y 1.3), cubriendo así los prerrequisitos para el resto de la tesis. Las referencias para la parte concerniente a caracteres ordinarios serán [**Isa76**] y [**Nav18**], mientras que para la parte relativa a caracteres de Brauer será [**Nav98a**]. Asimismo, hemos creído oportuno introducir brevemente resultados relativos a la teoría de *character triples* de Isaacs y representaciones proyectivas (Sección 1.2), pues constituyen la herramienta fundamental para definir los θ-bloques en el Capítulo 2.

En el Capítulo 2 empieza nuestro trabajo original. Si G es un grupo finito, N es un subgrupo normal de G, θ es un caracter irreducible de N G-invariante y p es un primo dado, definimos una partición del conjunto $\mathrm{Irr}(G|\theta)$ con respecto al primo p. A los elementos de esta partición los llamamos θ-bloques. Como hemos dicho, para definir los θ-bloques utilizamos representaciones proyectivas y la teoría de las *character triples*. Concretamente, asociamos a (G, N, θ) una representación proyectiva $\mathcal{P}$ que satisface ciertas propiedades y, utilizando esta representación proyectiva $\mathcal{P}$, construimos una *standard character triple* (G^*, N^*, θ^*) isomorfa a (G, N, θ), con N^* central en G^*. Que estas *character triples* sean isomorfas nos dice, entre otras cosas, que existe una biyección $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$, a la que llamaremos *standard bijection*. Decimos que un subconjunto no vacío $B_\theta \subseteq \mathrm{Irr}(G|\theta)$ es un θ-bloque si existe un p-bloque de G^*, B^*, tal que $(B_\theta)^* = \{\chi^* \mid \chi \in B_\theta\} = \mathrm{Irr}(B^*|\theta^*)$. A cada θ-bloque le asociamos una única clase de conjugación de p-subgrupos de G/N, y a cada uno de estos subgrupos le llamamos θ-grupo defecto. Como acabamos de ver, para la construcción de los θ-bloques (y la de los θ-grupos defecto) hacemos una elección de una representación proyectiva asociada a (G, N, θ). En la Sección 2.4 probamos que tanto los θ-bloques como los θ-grupos defecto están canónicamente definidos (Teorema A), es decir, son independientes de dicha elección. En la Sección 2.5 damos algunas propiedades de los θ-bloques. Probamos, por ejemplo, que para todo θ-bloque B_θ existe un p-bloque de G, B, tal que $B_\theta \subseteq \mathrm{Irr}(B|\theta) = \mathrm{Irr}(B) \cap \mathrm{Irr}(G|\theta)$. También probamos que si el subgrupo N es central, entonces los θ-bloques son exactamente los conjuntos $\mathrm{Irr}(B|\theta)$, donde B recorre los p-bloques de G, o que si G/N es un p-grupo, entonces solo hay un θ-bloque y un θ-grupo defecto, G (esto es el Teorema 2.10). Además, damos θ-*versiones* de algunos resultados conocidos de teoría de bloques. Por ejemplo, probamos que si $\chi \in \mathrm{Irr}(G|\theta)$, B_θ es el θ-bloque que contiene a χ y $(gN)_p$ no pertenece a ningún θ-grupo defecto de B_θ para algún $g \in G$, entonces $\chi(g) = 0$ (esto es el Teorema 2.12). También probamos que si B_θ es un θ-bloque y D_θ/N es un θ-grupo defecto de B_θ, entonces existe $xN \in G/N$ tal que $D_\theta/N \in \mathrm{Syl}_p(\mathbf{C}_{G/N}(xN))$ (Proposición 2.15). En la Sección 2.6, damos θ-versiones de las conocidas conjeturas *Brauer's Height Zero Conjecture* y *Brauer's $k(B)$ Conjeture* (Conjeturas B y D). Además probamos que nuestras θ-versiones son equivalentes a las conjeturas originales (Teoremas C y E). Como hemos dicho, para probar el Teorema C, necesitamos el resultado de Sambale en [**Sam19**]; mientras que para probar el Teorema E necesitamos un resultado nada trivial de Navarro en [**Nav17**]. En la Sección 2.7 probamos que la matriz de descomposición de un bloque no puede tener cierta forma (Teorema F). El ingrediente principal para probar este resultado es un teorema de R. Knörr. En la Sección 2.8, siguiendo las ideas de Navarro en [**Nav12**], damos una base canónica, $\mathrm{IBr}(G|N)$, del espacio de las funciones de clase definidas en G° y probamos que esta base coincide con la base canónica de Navarro en [**Nav00**] cuando N es un p-grupo. Si $\mathrm{cf}(G^\circ)$ es el espacio de las funciones de clase definidas en G° y Θ es un conjunto de representantes de las órbitas de la acción de G sobre $\mathrm{Irr}(N)$, Navarro prueba en [**Nav00**] que

$$\mathrm{cf}(G^\circ) = \bigoplus_{\theta \in \Theta} \mathrm{cf}(G^\circ|\theta).$$

Lo que hacemos es dar una base, $\mathrm{IBr}(G|\theta)$ de cada uno de estos espacios $\mathrm{cf}(G^\circ|\theta)$. A los elementos de esta base les llamamos θ-caracteres de Brauer. Por tanto, si $\chi \in \mathrm{Irr}(G|\theta)$ y χ° es la restricción de χ a G°, entonces podemos escribir

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi,$$

para ciertos enteros no-negativos $d_{\chi\varphi}$, unívocamente definidos. Llamamos a estos enteros θ-números de descomposición. Utilizando estos θ-números de descomposición se obtiene una partición del conjunto $\mathrm{Irr}(G|\theta)$ (esta partición ya fue estudiada por Navarro en [**Nav00**] y [**Nav12**]). Utilizando el Teorema F probamos que la partición de $\mathrm{Irr}(G|\theta)$ dada por los θ-números de descomposición y la partición de $\mathrm{Irr}(G|\theta)$ dada por los θ-bloques coincide (Teorema 2.30), relacionando así nuestro trabajo con el trabajo desarrollado por Külshammer y Robinson en [**KR87**]. En la Sección 2.9 definimos un *θ-linking* de la siguiente manera. Si $\chi, \psi \in \mathrm{Irr}(G|\theta)$, decimos que χ y ψ están *θ-linked* si

$$\sum_{x \in G^\circ} \chi(x)\overline{\psi(x)} \neq 0.$$

Probamos que si C es una componente conexa del grafo definido en $\mathrm{Irr}(G|\theta)$ mediante este nuevo *θ-linking*, entonces existe un θ-bloque B_θ tal que $C \subseteq B_\theta$. Sin embargo, la igualdad no se da en general, aunque sí bajo ciertas hipótesis de extendibilidad sobre θ (Teorema 2.35).

En el Capítulo 3 probamos una generalización del teorema de Howlett-Isaacs tomando en cuenta la acción de $\mathrm{Aut}(G)_{(N,\theta)}$ sobre $\mathrm{Irr}(G|\theta)$ (donde $\mathrm{Aut}(G)_{(N,\theta)}$ denota el subgrupo de $\mathrm{Aut}(G)$ que fija N y θ). Concretamente, probamos que si $\mathrm{Irr}(G|\theta)$ es una $\mathrm{Aut}(G)_{(N,\theta)}$-órbita, entonces G/N es resoluble. La prueba del teorema de Howlett y Isaacs tiene tres ingredientes principales: un teorema de DeMeyer y Janusz (ver el Teorema 8.3 de [**Nav18**], por ejemplo) que dice que si $\theta \in \mathrm{Irr}(N)$ es totalmente ramificado en G con $\chi \in \mathrm{Irr}(G)$ tal que $\theta^G = e\chi$, y P/N es un p-subgrupo de Sylow de G/N, entonces $\theta^P = e_p\eta$ para algún $\eta \in \mathrm{Irr}(P)$; la *correspondencia de Glauberman*, que afirma que si un grupo resoluble S actúa coprimamente sobre un grupo G, entonces existe una biyección natural de $\mathrm{Irr}_S(G)$ (los caracteres irreducibles de G fijados por la acción de S) en $\mathrm{Irr}(\mathbf{C}_G(S))$; y la Clasificación de los Grupos Finitos Simples. Para demostrar nuestra generalización necesitamos demostrar ciertas versiones de estos resultados.

En primer lugar, en la Sección 3.2 damos algunos resultados sobre acciones transitivas y p-subgrupos de Sylow. En particular, probamos que si P/N es un p-subgrupo de Sylow de G/N y A es un grupo que actúa sobre $\mathrm{Irr}(G|\theta)$ transitivamente y sobre $\mathrm{Irr}(P|\theta)$ satisfaciendo ciertas condiciones de compatibilidad, entonces $B \in \mathrm{Syl}_p(A)$ actúa transitivamente sobre $\mathrm{Irr}(P|\theta)$ (Teorema 3.2).

En segundo lugar, también necesitaremos propiedades no triviales de la *correspondencia de Glauberman*. Como hemos dicho, esta correspondencia afirma que si un grupo resoluble S actúa sobre un grupo G de orden coprimo con $|S|$, entonces existe una biyección natural de $\mathrm{Irr}_S(G)$ en $\mathrm{Irr}(\mathbf{C}_G(S))$. En particular necesitaremos un refinamiento bastante técnico de esta biyección, el cual probaremos en la Sección 3.4 utilizando resultados de A. Turull publicados en [**Tur08**], [**Tur09**] y [**Tur17**].

Por último, como hemos dicho anteriormente, el teorema de Howlett-Isaacs utiliza la Clasificación de los Grupos Finitos Simples (CGFS). También nosotros la necesitaremos para probar nuestra generalización. De hecho, necesitaremos el mismo resultado sobre grupos simples utilizado en [**HI82**]: si X es un grupo simple no abeliano, entonces existe un primo p tal que p divide a $|X|$, p no divide a $|M(X)|$ (el orden del *Schur multiplier* de X) y no existe un subgrupo de X resoluble con índice potencia de p.

En la parte final de este capítulo, en la Sección 3.6, probamos el Teorema I que caracteriza cuando todos los caracteres irreducibles sobre un caracter irreducible de un subgrupo normal tienen el mismo grado, en una situación específica. Para la prueba de este teorema necesitaremos tres resultados nada triviales: un teorema de U. Riese sobre inducción de caracteres irreducibles desde un subgrupo abeliano, un resultado profundo de S. Dolfi sobre órbitas regulares, y el teorema de Howlett-Isaacs.

Finalmente, en el Capítulo 4, contestamos parcialmente a la pregunta ¿sabe la tabla de caracteres de G cuántos p-subgrupos de Sylow tiene G? Como el número de p-subgrupos de Sylow de G es exactamente $|G : \mathbf{N}_G(P)|$, donde P es un p-subgrupo de Sylow de G, lo que nos estamos preguntando es si podemos calcular el orden del normalizador del p-subgrupo de Sylow a partir de la información que nos da la tabla de caracteres. Pensemos en el caso más simple, cuando G tiene un p-complemento normal N (un p-complemento normal de G es un subgrupo de G de orden coprimo con p e índice una potencia de p). En este caso $G = NP$ y $N \cap P = 1$, y puesto que $\mathbf{N}_N(P) = \mathbf{C}_N(P)$, tenemos que $|\mathbf{N}_G(P)| = |\mathbf{C}_N(P)||P|$. Por tanto, en esta situación minimal, para calcular $|\mathbf{N}_G(P)|$ nos bastaría calcular $|\mathbf{C}_N(P)|$. Resulta que el caso general (cuando G es p-resoluble) también reduce a una situación de este tipo. Como N es un subgrupo normal, tenemos que P actúa sobre N por conjugación y $|\mathbf{C}_N(P)|$ es precisamente el número de puntos fijos de esta acción.

En la Sección 4.2 damos una fórmula para calcular, en general, el número de puntos fijos por la acción de un p-grupo sobre un grupo de orden coprimo con p (Teorema K). La fórmula es la siguiente

$$|\mathbf{C}_N(P)| = \left(\prod_{x \in P} \frac{|\mathbf{C}_N(x)|}{|\mathbf{C}_N(x^p)|^{1/p}} \right)^{\frac{p}{(p-1)|P|}} .$$

Hemos llamado a esta fórmula la *fórmula de Brauer-Wielandt* porque fue Brauer el primero en obtener una fórmula de este tipo (en su caso, el grupo que actúa es un 4-grupo de Klein) y más tarde Wielandt dio la fórmula para el caso general. Sin embargo, no podemos utilizar la fórmula de Wielandt para nuestros propósitos, pues involucra términos que no se pueden leer de la tabla de caracteres. Esto es precisamente lo que hace más interesante nuestra fórmula: solo involucra los órdenes de los centralizadores de algunos elementos, y podemos encontrar esta información en la tabla de caracteres en algunos casos. En particular, en la Sección 4.3 aplicamos nuestra fórmula para obtener el orden del normalizador de un p-subgrupo de Sylow de un grupo p-resoluble, G, a partir de su tabla de caracteres, siempre que el p-subgrupo de Sylow sea abeliano o de exponente p (Teorema J). La prueba del caso en que el p-subgrupo de Sylow tiene exponente p es elemental, mientras que en el caso en que el p-subgrupo de Sylow es abeliano es mucho más complicada. En este último caso, la clave está en lo siguiente: si $\{y_1, \ldots, y_r\}$ son representantes de las clases de conjugación de los p-elementos de G (detectables en la tabla de caracteres gracias a un teorema de G. Higman), tenemos que determinar cuáles de estos elementos pertenecen a algún G-conjugado del subgrupo de Frattini de P, $\Phi(P)$ (el subgrupo de Frattini de P es la intersección de todos los subgrupos maximales de P). Para ello la clave es la utilización de cierto elemento del grupo de Galois $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, donde $\mathbb{Q}_n$ es la extensión de $\mathbb{Q}$ por una raíz n-ésima primitiva de la unidad.

A veces, cuando la tabla de caracteres parece no ser suficiente para resolver un problema, nos preguntamos si la tabla de caracteres más el *p-power map* lo es. Si $\{x_1, \dots, x_k\}$ son representantes de las clases de conjugación de G, el *p-power map* es la aplicación $f : \{1, \dots, k\} \to \{1, \dots, k\}$ tal que x_j^p pertenece a la clase de $x_{f(j)}$. Resulta que, utilizando nuestra fórmula, podemos determinar $|\mathbf{N}_G(P)|$ a partir de la tabla de caracteres y el *p-power map*, sin ninguna asunción sobre los p-subgrupos de Sylow de G (aunque manteniendo la hipótesis de p-resolubilidad sobre G).

Por último concluimos este capítulo con las pruebas de Isaacs y Lyons del Teorema K en la Sección 4.4.

Resum

Algunes de les principals conjectures en Teoria de Representacions de Grups Finits admeten refinaments en termes de p-blocs de Brauer. Un exemple paradigmàtic d'aquest fet és la conjectura d'Alperin-McKay proposada per J. L. Alperin en [**Alp75**], que dóna una visió bloc-teòrica a la coneguda conjectura de McKay. A més a més, en general, els blocs donen més estructura a aquests problemes.

Actualment la forma d'atacar aquest tipus de conjectures és la següent: reduir-les a problemes de grups simples i utilitzar la Classificació dels Grups Finits Simples per a resoldre-les. Entenem per *reduir* a un problema de grups simples que el problema té solució sempre que es verifiquen una sèrie de condicions per a tots els grups finits simples. Per descomptat, els subgrups normals (i els seus caracters irreductibles) juguen un paper fonamental en aquest procés.

Una de les tècniques principals utilitzades en la reducció de certs problemes de teoria de caracters a problemes de grups simples és estudiar una *versió projectiva* d'aquests problemes. Què volem dir amb açò? Siga N un subgrup normal de G, siga θ un caracter irreductible de N i siga $\mathrm{Irr}(G|\theta)$ el conjunt de constituents irreductibles del caracter induït θ^G. Fer una versió projectiva d'un problema és reformular-lo en termes de $\mathrm{Irr}(G|\theta)$ en lloc de $\mathrm{Irr}(G)$, el grup quocient G/N en lloc de G, classes de conjugació θ-bones, en lloc de classes de conjugació, etc. En altres paraules, necessitem entendre totalment la *teoria de caracters sobre el caracter* θ. Un exemple d'aplicació d'aquest mètode és la reducció de la conjectura de McKay en [**IMN07**]. A més a més, quan $N = 1$ hem de recuperar la conjectura o problema original. Però, quin és l'avantatge d'aquesta filosofia? En primer lloc, podem proposar (i resoldre) problemes molt més generals. En segon lloc, d'aquesta manera podem utilitzar una poderosa ferramenta: inducció sobre $|G : N|$, que és una manera natural d'introduir grups simples en aquest tipus de problemes.

Seguint aquesta filosofia, si volem atacar algunes conjectures que involucren p-blocs, no només necessitem entendre la teoria de caracters sobre θ, sinó també la *teoria de blocs sobre* θ. Aquesta és la motivació darrere de gran part d'aquesta tesi: definim un conjunt de blocs canònicament construïts *sobre* un caracter d'un subgrup normal. Aquests blocs estaran definits respecte d'un primer p i un caracter irreductible θ d'un subgrup normal, i els anomenarem θ-*blocs* (no farem referència al primer p en la notació perquè

romandrà fix). Definim els θ-blocs utilitzant representacions projectives i la teoria de *character triples* introduïda per I. M. Isaacs. Els θ-blocs estan relacionats amb els blocs de les *twisted group algebras*, però la nostra perspectiva és totalment caracter-teòrica. Una part no trivial d'aquest treball és provar que els θ-blocs són una partició canònica del conjunt $\mathrm{Irr}(G|\theta)$, és a dir, els θ-blocs són independents de qualsevol elecció feta a l'hora de definir-los. A més a més, a cada θ-bloc l'associarem una única classe de conjugació de p-subgrups de G/N. A cadascun d'aquests grups l'anomenarem *θ-grup defecte*. Veurem que, en general, els θ-grups defecte es comporten com els grups defecte dels p-blocs de Brauer clàssics.

Però, per què aquesta generalització? Primer, des del punt de vista del subgrup normal N i del seu caracter irreductible θ, en general els p-blocs de Brauer són massa grans i per aquesta raó no capten per complet les subtileses de la teoria de caracters de G sobre θ. Provarem que cada θ-bloc B_θ està contingut en $\mathrm{Irr}(B) \cap \mathrm{Irr}(G|\theta)$, on B és un p-bloc de Brauer, però en general, B_θ és molt més xicotet, per la qual cosa la partició en θ-blocs és més fina. La segona raó és que emprant els nostres θ-blocs podem unificar resultats i problemes com mai s'havien relacionat abans en la literatura: per exemple, en la nostra Conjectura B, el teorema de Gluck-Wolf-Navarro-Tiep i la *Brauer's Height Zero Conjecture* (BHZC) apareixen unificats per primera vegada. Aquest problema va inspirar G. Malle i G. Navarro per proposar una versió projectiva de la BHZC [**MN17**]. Més tard, B. Sambale va provar, utilitzant la teoria dels sistemes de fusió, que aquesta conjectura era equivalent a la BHZC ([**Sam19**]). Amb tot açò, per tant, hem obtingut nova informació dels p-blocs de Brauer clàssics utilitzant la idea dels θ-blocs. Però aquesta no és l'única ocasió en la que els θ-blocs ens han donat resultats sobre els p-blocs clàssics. També provem al Teorema F que la matriu de descomposició clàssica d'un p-bloc de Brauer no es pot descompondre de certa forma. L'esperança és que els θ-blocs puguen inspirar més resultats d'aquest estil.

En la primera part d'aquesta tesi provem que la *Conjectura $k(B)$ de Brauer* també admet una θ-versió. Aquesta conjectura és altre dels problemes oberts fonamentals de R. Brauer dels anys 50, i no només no s'ha resolt sinó que ni tan sols s'ha reduït a grups finits simples. Tal vegada la nostra θ-versió puga ajudar a divisar una reducció.

Altra part important del Capítol 2 és la introducció de θ-caracters de Brauer. En [**Nav00**], Navarro dóna una versió dels caracters de Brauer relatius a un p-subgrup normal N de G. Aquests formen una base $\mathrm{IBr}(G|N)$ de l'espai de funcions de classe definides en $G^0 = \{x \in G | x_p \in N\}$. Açò li va permetre definir nombres de descomposició $d_{\chi\varphi}$ per a $\chi \in \mathrm{Irr}(G)$ i $\varphi \in \mathrm{IBr}(G|N)$. La importància d'açò és que els caracters

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$$

eren els caracters relatius indescomponibles N-projectius descoberts prèviament per B. Külshammer y G. R. Robinson en [**KR87**]. Tot i que Navarro també va donar una versió d'aquest resultat per a un subgrup normal N arbitrari (és a dir, no necessàriament un p-grup) en [**Nav12**], no va provar que la nova N-base $\mathrm{IBr}(G|N)$ era canònica (tan canònica com ho són els caracters de Brauer, és a dir, llevat de l'elecció d'un ideal maximal que continga p a l'anell dels enters algebraics). En aquesta tesi donem una base canònica per a l'espai de funcions de classe definides en G° amb N arbitrari i demostrem que aquesta base coincideix amb la base canònica donada per Navarro en [**Nav00**].

Encara tenim moltes preguntes sense contestar sobre els θ-blocs. Per exemple, els podem determinar mitjançant la taula de caracters? Podem caracteriztar els θ-blocs amb un únic caracter? No tenim una resposta completa a aquestes preguntes, en altres seguim treballant.

Tots els resultats fins ara mencionats són el Teorema A, la Conjectura B, el Teorema C, la Conjectura D, el Teorema E, el Teorema F i el Teorema G del Capítol 2 d'aquest treball. Excepte el Teorema G, tots apareixen a [**Riz18**]. El Teorema G apareixerà a [**Riz19**].

Al Capítol 3 seguim estudiant el conjunt $\mathrm{Irr}(G|\theta)$ i donem una generalització del conegut teorema de Howlett-Isaacs. Al 1964, N. Iwahori i H. Matsumoto conjecturaren a [**IM64**] que si θ és G-invariant i $|\mathrm{Irr}(G|\theta)| = 1$, aleshores G/N és resoluble (en aquest cas diguem que θ és *totalment ramificat* en G/N). Aquest tipus de caracters, com qualsevol altra situació minimal en teoria de grups, apareixen amb freqüència en teoria de representacions ordinàries (sobre cossos de característica 0) i modulars (sobre cossos de característica p). Per exemple, en la teoria de caracters dels *chief factors* abelians o en blocs amb exactament un caracter de Brauer. Van ser Isaacs i R. Howlett qui resolgueren finalment aquesta conjectura en [**HI82**], en el que va ser una de les primeres aplicacions de la Classificació dels Grups Finits Simples a la teoria de caracters. Quan θ és totalment ramificat en G/N i $N \subseteq M \lhd G$, pel teorema de Clifford tenim que les constituents irreductibles de θ^M són totes G-conjugades. Aquest fet ens va inspirar el següent resultat principal d'aquesta tesi. El Teorema H és una generalització del famós teorema de Howlett i Isaacs, i d'una manera un tant més dèbil podem enunciar-lo així: si A actua per automorfismes sobre G fixant N i algun caracter $\theta \in \mathrm{Irr}(N)$ G-invariant, i A permuta transitivament els elements de $\mathrm{Irr}(G|\theta)$, aleshores G/N és resoluble. És important destacar que per a la prova d'aquest teorema no utilizem el teorema de Howlett i Isaacs. De fet, creem que la nostra prova simplifica part de la seua.

El següent resultat principal del Capítol 3 versa de nou sobre el conjunt $\mathrm{Irr}(G|\theta)$, però també involucra teoria de blocs. J. F. Humphreys va conjecturar que si tots els caracters en $\mathrm{Irr}(G|\theta)$ tenen el mateix grau, aleshores

G/N és resoluble. Açò, evidentment, és una ampla generalització del teorema de Howlett i Isaacs, així com del nostre. En el moment d'escriptura d'aquesta tesi no hi ha resultats parcials per a aquesta conjectura. El nostre resultat (Teorema I), és una caracterització en termes de teoria de grups de què ocorre quan $N = \mathbf{O}_\pi(G)$ i G és π-separable (si π és un conjunt de primers, diem que G és π-separable si els seus factors de composició són bé π-grups o bé π'-grups). Com hem dit, aquest resultat està relacionat amb la teoria de blocs i ara expliquem per què. Si π és el complement d'un primer p, un resultat molt conegut de P. Fong assegura que $\mathrm{Irr}(G|\theta)$ són els caracters irreductibles d'un p-bloc de Brauer (veure el Teorema 10.20 de [**Nav98a**], per exemple). Si tots els caracters irreductibles de $\mathrm{Irr}(G|\theta)$ tenen el mateix grau, aleshores estem en la situació en que tots els caracters irreductibles d'un p-bloc tenen el mateix grau. Aquesta situació, sense la hipòtesi de p-resolubilitat, va ser caracteritzada per T. Okuyama i Y. Tsushima en [**OT83**]. Podem considerar, per tant, el nostre Teorema I com una π-versió d'aquest resultat. Els Teoremes H i I apareixen en [**NR17**].

La part final d'aquest treball és de natura un tant distina. En aquesta darrera part ja no treballem en termes de teoria de caracters sobre un caracter d'un subgrup normal, sinó que treballem amb el conjunt complet de tots els caracters irreductibles de G, $\mathrm{Irr}(G)$. La *taula de caracters* de G, $X(G)$, és una matriu quadrada amb les columnes indexades per les classes de conjugació de G, i les files indexades pels caracters irreductibles de G. Un dels problemes clàssics en teoria de caracters és determinar quines propietats d'un grup finit G podem conèixer a partir de la seua taula de caracters. Per exemple, la taula de caracters detecta si G és abelià, nilpotent, superresoluble, resoluble o simple. En aquesta tesi, estem interessats en què sap $X(G)$ sobre la p-estructura local de G, per a un primer donat p, un problema molt més complicat. En particular, la nostra motivació és la Pregunta 7 de [**Nav04**]: determina la taula de caracters de G quants p-subgrups de Sylow té G? En aquest treball donem una resposta afirmativa a aquesta pregunta en alguns casos (Teorema J).

No obstant això, més interessant que el resultat en si mateix, es tal vegada la manera de demostrar-lo. Per tal de provar aquest teorema necessitem calcular el nombre de punts fixats per l'acció d'un p-grup sobre un grup d'ordre coprimer amb p. Per a això donem una fórmula (Teorema K) que permet calcular aquest nombre en termes de informació que pot ser obtinguda de la taula de caracters. La nostra fórmula generalitza un resultat clàssic de Brauer (i H. Wielandt) per a contar el nombre de punts fixats per l'acció d'un 4-grup de Klein sobre un grup d'ordre senar. Els resultats del Capítol 4, Teorema J i Teorema K, apareixen en [**NR16**].

Després de llegir la nostra prova de la fórmula del Teorema K, Isaacs i R. Lyons trobaren dos proves alternatives molt elegants d'aquesta fórmula. Les reproduïm en aquest treball amb el seu permís.

Guió de la tesi

El Capítol 1 és un breu repàs a la teoria de caracters ordinaris i modulars (Seccions 1.1 i 1.3), cobrint d'aquesta manera els prerrequisits per a la resta de la tesi. Les referències per a la part concernent a caracters ordinaris seran [**Isa76**] i [**Nav18**], mentre que per a la part relativa a caracters de Brauer serà [**Nav98a**]. D'altra banda, hem cregut oportú introduir breument resultats relatius a la teoria de *character triples* de Isaacs i representacions projectives (Secció 1.2), doncs constitueixen la ferramenta fonamental per a definir els θ-blocs al Capítol 2.

Al Capítol 2 comença el nostre treball original. Si G és un grup finit, N és un subgrup normal de G, θ és un caracter irreductible de N G-invariant i p és un primer donat, definim una partició del conjunt $\mathrm{Irr}(G|\theta)$ respecte del primer p. Als elements d'aquesta partició els anomenem θ-blocs. Com hem dit, per a definir els θ-blocs utilitzem representacions projectives i la teoria de les *character triples*. Concretament, associem a (G, N, θ) una representació projectiva $\mathcal{P}$ que satisfà certes propietats i, emprant aquesta representació projectiva $\mathcal{P}$, construim una *standard character triple* (G^*, N^*, θ^*) isomorfa a (G, N, θ), amb N^* central en G^*. Que aquestes *character triples* siguen isomorfes ens diu, entre altres coses, que existeix una bijecció $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$, a la qual anomenarem *standard bijection*. Diem que un subconjunt no buit $B_\theta \subseteq \mathrm{Irr}(G|\theta)$ és un θ-bloc si existeix un p-bloc de G^*, B^*, tal que $(B_\theta)^* = \{\chi^* \mid \chi \in B_\theta\} = \mathrm{Irr}(B^*|\theta^*)$. A cada θ-bloc li associem una única classe de conjugació de p-subgrups de G/N, i a cadascun d'aquests subgrups l'anomenem θ-grup defecte. Com acabem de veure, per a la construcció dels θ-blocs (i la dels θ-grups defecte) fem una elecció d'una representació projectiva associada a (G, N, θ). En la Secció 2.4 provem que tant els θ-blocs com els θ-grups defecte estan canònicament definits (Teorema A), és a dir, són independents d'aquesta elecció. A la Secció 2.5 donem algunes propietats dels θ-blocs. Provem, per exemple, que per a tot θ-bloc B_θ existeix un p-bloc de G, B, tal que $B_\theta \subseteq \mathrm{Irr}(B|\theta) = \mathrm{Irr}(B) \cap \mathrm{Irr}(G|\theta)$. També provem que si el subgrup N és central, aleshores els θ-blocs són exactament els conjunts $\mathrm{Irr}(B|\theta)$, on B recorre els p-blocs de G, o que si G/N és un p-grup, aleshores només hi ha un θ-bloc i un θ-grup defecte, G (açò es el Teorema 2.10). A més a més, donem *θ-versions* d'alguns resultats coneguts de la teoria de blocs. Per exemple, provem que si $\chi \in \mathrm{Irr}(G|\theta)$, B_θ és el θ-bloc que conté a χ i $(gN)_p$ no pertany a cap θ-grup defecte de B_θ per a algun $g \in G$, aleshores $\chi(g) = 0$ (açò és el Teorema 2.12). També provem que si B_θ és un θ-bloc i D_θ/N és un θ-grup defecte de B_θ, aleshores existeix $xN \in G/N$ tal que $D_\theta/N \in \mathrm{Syl}_p(\mathbf{C}_{G/N}(xN))$ (Proposició 2.15). A la Secció 2.6, donem θ-versions de les conegudes conjectures *Brauer's Height Zero Conjecture* i *Brauer's $k(B)$ Conjeture* (Conjectures B i D). A més a més, provem que les nostres θ-versions són equivalents a les conjectures originals (Teoremes C i E). Com hem dit, per a provar el Teorema C, necessitem el resultat de Sambale en [**Sam19**]; mentre que per a provar el Teorema E necessitem un resultat gens trivial de Navarro en [**Nav17**]. A la Secció 2.7 provem que la matriu de descomposició d'un bloc no pot tindre certa forma (Teorema F). L'ingredient principal per a provar aquest resultat és un teorema de R. Knörr. A la Sección 2.8, seguint les idees de Navarro en [**Nav12**], donem una base canònica, $\mathrm{IBr}(G|N)$, de l'espai de les funcions de classe definides en G° i provem que aquesta base coincideix amb la base canònica de Navarro en [**Nav00**] quan N és un p-grup. Si $\mathrm{cf}(G^\circ)$ és l'espai de les funcions de classe definides en G° i Θ és un conjunt de representants de les òrbites de l'acció de G sobre $\mathrm{Irr}(N)$, Navarro prova en [**Nav00**] que

$$\mathrm{cf}(G^\circ) = \bigoplus_{\theta\in\Theta} \mathrm{cf}(G^\circ|\theta).$$

El que fem és donar una base, $\mathrm{IBr}(G|\theta)$ de cadascun d'aquests espais $\mathrm{cf}(G^\circ|\theta)$. Als elements d'aquesta base els anomenem θ-caracters de Brauer. Per tant, si $\chi \in \mathrm{Irr}(G|\theta)$ i χ° és la restricció de χ a G°, aleshores podem escriure

$$\chi^\circ = \sum_{\varphi\in\mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi,$$

per a certs enters no-negatius $d_{\chi\varphi}$, unívocament definits. Anomenem a aquests enters θ-nombres de descomposició. Utilitzant aquests θ-nombres de descomposició s'obté una partició del conjunt $\mathrm{Irr}(G|\theta)$ (aquesta partició ja fou estudiada per Navarro en [**Nav00**] i [**Nav12**]). Utilitzant el Teorema F provem que la partició de $\mathrm{Irr}(G|\theta)$ donada pels θ-nombres de descomposició i la partició de $\mathrm{Irr}(G|\theta)$ donada pels θ-blocs coincideix (Teorema 2.30), relacionant així el nostre treball amb el treball desenvolupat per Külshammer i Robinson en [**KR87**]. A la Secció 2.9 definim un θ-*linking* de la següent manera. Si $\chi, \psi \in \mathrm{Irr}(G|\theta)$, diem que χ i ψ estan θ-*linked* si

$$\sum_{x\in G^\circ} \chi(x)\overline{\psi(x)} \neq 0.$$

Provem que si C és una component connexa del graf definit en $\mathrm{Irr}(G|\theta)$ mitjançant aquest nou θ-*linking*, aleshores existeix un θ-bloc B_θ tal que $C \subseteq B_\theta$. No obstant això, la igualtat no es dóna en general, tot i que sí sota certes hipòtesis d'extendibilitat sobre θ (Teorema 2.35).

Al Capítol 3 provem una generalització del teorema de Howlett-Isaacs considerant l'acció de $\mathrm{Aut}(G)_{(N,\theta)}$ sobre $\mathrm{Irr}(G|\theta)$ (on $\mathrm{Aut}(G)_{(N,\theta)}$ denota l'estabilitzador de N i θ sota l'acció de $\mathrm{Aut}(G)$ sobre $\mathrm{Irr}(N)$). La prova del teorema de Howlett i Isaacs té tres ingredients principals: un teorema de DeMeyer i Janusz (veure el Teorema 8.3 de [**Nav18**], per exemple) que diu que si $\theta \in \mathrm{Irr}(N)$ és totalment ramificat en G amb $\chi \in \mathrm{Irr}(G)$ tal que $\theta^G = e\chi$, i P/N és un p-subgrup de Sylow de G/N, aleshores $\theta^P = e_p\eta$ per a algun $\eta \in \mathrm{Irr}(P)$; la *correspondència de Glauberman*, que afirma que si un grup resoluble S actua coprimerament sobre un grup G, aleshores existeix una bijección natural de $\mathrm{Irr}_S(G)$ (els caracters irreductibles de G fixats per l'acció de S) en $\text{Irr}(\mathbf{C}_G(S))$; i la Classificació dels Grups Finits Simples. Per a demostrar la nostra generalització necessitem demostrar certes versions d'aquests resultats.

En primer lloc, a la Secció 3.2 donem alguns resultats sobre accions transitives i p-subgrups de Sylow. En particular, provem que si P/N és un p-subgrup de Sylow de G/N i A és un grup que actua sobre $\text{Irr}(G|\theta)$ transitivament i sobre $\text{Irr}(P|\theta)$ satisfent certes condicions de compatibilitat, aleshores $B \in \text{Syl}_p(A)$ actua transitivament sobre $\text{Irr}(P|\theta)$ (Teorema 3.2).

En segon lloc, també necessitarem propietats no trivials de la *correspondència de Glauberman*. Com hem dit abans, aquesta correspondència afirma que si un grup resoluble S actua sobre un grup G d'ordre coprimer amb $|S|$, aleshores existeix una bijecció de $\text{Irr}_S(G)$ en $\text{Irr}(\mathbf{C}_G(S))$. En particular necessitarem un refinament prou tècnic d'aquesta bijecció, el qual provarem a la Seccin 3.4 fent ús de resultats de A. Turull publicats en [**Tur08**], [**Tur09**] i [**Tur17**].

Com hem dit anteriorment, el teorema de Howlett-Isaacs empra la Classificació dels Grups Finits Simples (CGFS). També nosaltres la necessitarem per a provar la nostra generalització. De fet, necessitarem el mateix resultat sobre grups simples utilitzat en [**HI82**]: si X és un grup simple no abelià, aleshores existeix un primer p tal que p divideix a $|X|$, p no divideix a $|M(X)|$ (l'ordre del *Schur multiplier* de X) i no existeix un subgrup de X resoluble amb índex potència de p.

En la darrera part d'aquest capítol, provem el Teorema I que caracteritza quan tots els caracters irreductibles sobre un caracter irreductible d'un subgrup normal tenen el mateix grau, en una situació específica. Per a la prova d'aquest teorema necessitarem tres resultats gens trivials: un teorema de U. Riese sobre inducció de caracters irreductibles des d'un subgrup abelià, un resultat profund de S. Dolfi sobre òrbites regulars, i el teorema de Howlett-Isaacs.

Finalment, al Capítol 4, contestem parcialment a la pregunta: sap la taula de caracters de G quants p-subgrups de Sylow té G? Com el nombre de p-subgrups de Sylow de G és exactament $|G : \mathbf{N}_G(P)|$, on P és un p-subgrup de Sylow de G, el que ens estem preguntant és si podem calcular l'ordre del normalitzador del p-subgrup de Sylow a partir de la informació que ens dóna la taula de caracters. Pensem en el cas més simple, quan G té un p-complement normal N (un p-complement normal de G és un subgrup de G d'ordre coprimer amb p i índex una potència de p). En aquest cas $G = NP$ y $N \cap P = 1$, i com que $\mathbf{N}_N(P) = \mathbf{C}_N(P)$, tenim que $|\mathbf{N}_G(P)| = |\mathbf{C}_N(P)||P|$. Per tant, en aquesta situació minimal, per tal de calcular $|\mathbf{N}_G(P)|$ ens bastaria calcular $|\mathbf{C}_N(P)|$. Resulta que el cas general (quan G és p-resoluble) també redueix a una situació d'aquest tipus. Com N és un subgrup normal, tenim que P actua sobre N per conjugació i $|\mathbf{C}_N(P)|$ és precisament el nombre de punts fixes d'aquesta acció.

A la Secció 4.2 donem una fórmula per a calcular, en general, el nombre de punts fixes per l'acció d'un p-grup sobre un grup d'ordre coprimer amb p (Teorema K). La fórmula és la següent

$$|\mathbf{C}_N(P)| = \left(\prod_{x \in P} \frac{|\mathbf{C}_N(x)|}{|\mathbf{C}_N(x^p)|^{1/p}} \right)^{\frac{p}{(p-1)|P|}} .$$

Hem anomenat a aquesta fórmula la *fórmula de Brauer-Wielandt* perquè fou Brauer el primer en obtindre una fórmula d'aquest tipus (al seu cas, el grup que actua és un 4-grup de Klein) i més tard Wielandt va donar la fórmula per al cas general. No obstant això, no podem emprar la fórmula de Wielandt per als nostres propòsits, doncs involucra termes que no es poden llegir de la taula de caracters. Açò és precisament el que fa més interessant la nostra fórmula: només involucra els ordres dels centralitzadors d'alguns elements, i podem trobar aquesta informació en la taula de caracters en alguns casos. En particular, a la Secció 4.3 apliquem la nostra fórmula per a obtindre l'ordre del normalitzador d'un p-subgrup de Sylow d'un grup p-resoluble, G, a partir de la seua taula de caracters, sempre que el p-subgrup de Sylow siga abelià o d'exponent p (Teorema J). La prova del cas en que el p-subgrup de Sylow té exponent p és elemental, mentre que al cas en que el p-subgrup de Sylow és abelià és molt més complicada. En aquest últim cas, la clau està en açò: si $\{y_1, \ldots, y_r\}$ són representants de les classes de conjugació dels p-elements de G (detectables en la taula de caracters gràcies a un teorema de G. Higman), tenim que determinar quins d'aquests elements pertanyen a algun G-conjugat del subgrup de Frattini de P, $\Phi(P)$ (el subgrup de Frattini de P és la intersecció de tots els subgrups maximals de P). Per a açò la clau és la utilització de cert element del grup de Galois $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, on $\mathbb{Q}_n$ és l'extensió de $\mathbb{Q}$ per una arrel n-èssima primitiva de la unitat. Quan la taula de caracters pareix no ser suficient per a resoldre un problema, ens preguntem si la taula de caracteres més el *p-power map* ho és. Si $\{x_1, \ldots, x_k\}$ són representants de les classes de conjugació de G, el *p-power map* és l'aplicació $f : \{1, \ldots, k\} \to \{1, \ldots, k\}$ tal que x_j^p pertany a la classe de $x_{f(j)}$. Resulta que, utilitzant la nostra fórmula, podem determinar $|\mathbf{N}_G(P)|$ a partir de la taula de caracters i el *p-power map*, sense cap assumpció sobre els p-subgrups de Sylow de G (tot i que mantenint la hipòtesi de p-resolubilitat sobre G).

Per últim, concloem aquest capítol amb les proves alternatives de Isaacs i Lyons del Teorema K a la Secció 4.4.

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Introduction

Some of the main conjectures in the Representation Theory of Finite Groups admit refinements in terms of Brauer p-blocks. A paradigmatic example of this is the Alperin-McKay conjecture proposed by J. L. Alperin in [**Alp75**], which gives a block-version of the acclaimed McKay-conjecture. Blocks bring more structure into these problems.

It is a general belief today that the way to approach these conjectures is through a reduction of them to problems on simple groups, and then use the Classification of Finite Simple Groups to solve them. By *reducing* a problem to simple groups we mean that the problem has a positive solution provided that a specific set of conditions is checked for every simple group. Of course, normal subgroups (and their irreducible characters) play a fundamental role in this process.

One of the main techniques used in the reduction of character theory problems to simple groups is the study of *projective versions* of these conjectures. By a projective version we mean the following: let N be a normal subgroup of G, let θ be an irreducible character of N, and write $\mathrm{Irr}(G|\theta)$ for the set of the irreducible constituents of the induced character θ^G. We want to formulate the statement of our problem in terms of $\mathrm{Irr}(G|\theta)$ instead of $\mathrm{Irr}(G)$, the quotient group G/N instead of G, θ-good conjugacy classes instead of conjugacy classes of G, etc. In other words, we need to fully understand the *character theory over the character of a normal subgroup*. For instance, this is the idea behind the reduction theorem of the McKay conjecture in [**IMN07**]. When $N = 1$, we should recover our original problem. What is the advantage? First, not only are far more general results proposed (and proved), but also, a powerful tool is introduced in the problems: induction on $|G : N|$ usually brings simple groups into the picture.

Following this philosophy, if one wants to attack some of the conjectures involving blocks, one needs to understand not only the character theory over a character of a normal subgroup, but also the block theory. This motivates the main part of this thesis: we shall define a set of canonical blocks that are constructed *over* a character of a normal subgroup. These blocks are defined with respect to an irreducible character of a normal subgroup θ and a prime p, and we will call them *θ-blocks* (we are holding fixed our prime p for the rest of this thesis). The θ-blocks are defined by means of projective representations, using the theory of the character triples introduced by I.

M. Isaacs. They are related to blocks of twisted group algebras, but our approach is entirely character-theoretic. A non-trivial part of this work is to prove that they constitute a canonical partition of the set $\mathrm{Irr}(G|\theta)$, that is: θ-blocks are canonical and independent of any choice made in order to define them. Also, associated to every θ-block there is a uniquely defined G/N-conjugacy class of p-subgroups of G/N which we call the θ-defect groups. They behave as the defect groups of the classical Brauer p-blocks.

One can ask: why this generalization? First of all, from the point of view of the normal subgroup N and its irreducible character θ, it seems that Brauer p-blocks are in general too big and do not capture some of the subtleties of the character theory of G over θ. We do prove that each θ-block B_θ is contained in $\mathrm{Irr}(B) \cap \mathrm{Irr}(G|\theta)$, for some Brauer p-block B, but B_θ is in general much smaller. The second reason is that using θ-blocks we can unify statements that appear separated in the literature: for instance, in our Conjecture B, the Gluck-Wolf-Navarro-Tiep theorem and Brauer's Height Zero Conjecture (BHZC) are put together in a single statement for the first time. This statement inspired G. Malle and G. Navarro to propose a projective version of the BHZC [**MN17**]. This projective version was proved to be equivalent to the original BHZC by B. Sambale in [**Sam19**] by using fusion systems. Hence new information on classical Brauer blocks has been obtained by using the θ-blocks idea. This is not the only one. We shall also prove in Theorem F that the classical decomposition matrix of a Brauer p-block cannot be decomposed in a certain way. We hope that θ-blocks might inspire further results of this type.

In the first part of this thesis we shall prove that Brauer's $k(B)$ conjecture also admits a θ-version. Brauer's $k(B)$-conjecture is another of the famous open problems of R. Brauer from the 1950's, and remains unreduced to simple groups. Perhaps our θ-version might help to devise such a reduction.

Another important part of Chapter 2 is the introduction of θ-Brauer characters. In [**Nav00**], Navarro gave a version of Brauer characters relative to a normal p-subgroup N of G. These constituted a basis $\mathrm{IBr}(G, N)$ of the space of class functions defined on $G^0 = \{x \in G | x_p \in N\}$, and allowed him to define decomposition numbers $d_{\chi\varphi}$ for $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G, N)$. The significance of this was that the characters

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$$

were the relative N-projective indecomposable characters discovered previously by B. Külshammer and G. R. Robinson in [**KR87**]. Navarro also gave a version of this for an arbitrary normal subgroup N of G (not necessarily a p-group) in [**Nav12**], but he did not prove that this N-basis $\mathrm{IBr}(G, N)$ was canonical (as canonical as Brauer characters are, that is, up to a choice of a maximal ideal containing p in the ring of algebraic integers). We shall provide such a canonical basis.

Many questions remain open about θ-blocks. For instance, can θ-blocks be determined by the character table? How to characterize θ-blocks with a unique character? We do not have a full answer to these questions. On some others, we are currently working.

All the results mentioned above constitute Theorem A, Conjecture B, Theorem C, Conjecture D, Theorem E, Theorem F and Theorem G of Chapter 2 in this work. Except for Theorem G, they appear in [**Riz18**]. Theorem G will appear in [**Riz19**].

Continuing with the study of the set $\text{Irr}(G|\theta)$, we give a generalization of the renowned Howlett-Isaacs theorem in Chapter 3. In 1964, N. Iwahori and H. Matsumoto conjectured in [**IM64**] that if θ is G-invariant and $|\text{Irr}(G|\theta)| = 1$, then G/N is solvable (in this case, it is said that θ is *fully ramified* in G/N). Fully ramified characters, as any other minimal situation in group theory, appear quite often in character and modular representation theory (for instance, in the character theory of abelian chief factors, or in blocks with exactly one modular character). This conjecture was proven to be true by Isaacs and R. Howlett, in one of the first applications of the Classification of Finite Simple Groups to character theory. When θ is fully ramified in G/N and $N \subseteq M \lhd G$, by Clifford's theorem we have that the irreducible constituents of θ^M are all G-conjugate. This is what inspired the next main result of this thesis. Theorem H below is a generalization of the Howlett-Isaacs theorem, and in a weak form can be stated as this: if A acts via automorphisms on G fixing N and some G-invariant $\theta \in \text{Irr}(N)$, and A transitively permutes $\text{Irr}(G|\theta)$, then G/N is solvable. It is important to remark, that we do not use the Howlett-Isaacs theorem. We believe that our proof simplifies some parts of theirs.

Our next main result, also in Chapter 3, deals again with the set $\text{Irr}(G|\theta)$, but also with block theory. J. F. Humphreys conjectured that if all characters in $\text{Irr}(G|\theta)$ have the same degree, then G/N is solvable. This would be a far reaching generalization of the Howlett-Isaacs theorem (and of our Theorem H). There are no partial results for this conjecture. Our result, Theorem I, is a group characterization of when this happens if $N = \mathbf{O}_\pi(G)$, and G is a π-separable group (recall that if π is a set of primes, then a group is π-separable if its composition factors are either π-groups or π'-groups). We wrote that this is also related to block theory, and it really is. If π is the complement of a prime p, it is a well-known result of P. Fong that $\text{Irr}(G|\theta)$ constitutes the irreducible ordinary characters of a Brauer p-block (see Theorem 10.20 of [**Nav98a**], for instance). If all the irreducible characters in $\text{Irr}(G|\theta)$ have the same degree, then we have a situation of a p-block in which all characters have the same degree. This situation, with no p-solvability hypothesis, was characterized by T. Okuyama and Y. Tsushima in [**OT83**]. Our Theorem I can therefore be seen as a π-version of their result. Theorems H and I appear in [**NR17**].

The final part of this thesis has a different nature. In this part we do not work in terms of the character theory over a normal subgroup, but instead we work with $\mathrm{Irr}(G)$, the entire set of irreducible characters of G. The *character table* of G, $X(G)$, is the (square) matrix whose columns are indexed by the conjugacy classes of the group and whose rows are indexed by its irreducible characters. One of the classical problems in character theory is to determine which properties of a finite group G are encapsulated in its character table. For example, it is well-known that the character table detects if G is abelian, nilpotent, supersolvable, solvable, or simple. In this thesis we are interested in what $X(G)$ knows about the local p-structure of G for a given prime p, a much more complicated problem. In particular we aim to answer Question 7 in [**Nav04**], where it is asked if $X(G)$ determines the number of Sylow p-subgroups of G. We give a positive answer to this question in some specific cases (Theorem J).

Perhaps even more interesting than the result itself is the way it is obtained. It turns out that we need to compute the number of fixed points under the action of a p-group on a group of order coprime to p, and we give a formula (Theorem K) to compute this number in terms of information that can be collected from the character table. This result generalizes a classical result of Brauer (and H. Wielandt) on counting the number of fixed points of the action of a Klein 4-group on a group of odd order. The results of Chapter 4, Theorems J and K, appear in [**NR16**].

After reading the proof of our counting formula in Theorem K, Isaacs and R. Lyons wrote to us with two very nice different proofs of that. We reproduce them here with their kind permission.

Structure of the work

Chapter 1 is an expository chapter containing the background on ordinary and modular character theory needed for the rest of the work. Our references for the part concerning ordinary characters are [**Isa76**] and [**Nav18**], and for the part concerning modular (Brauer) characters is [**Nav98a**]. We also include a brief exposition of Isaacs' theory of character triples since this is the main tool needed to define the θ-blocks in Chapter 2.

In Chapter 2 we start our original work. If G is a finite group, N is a normal subgroup of G, θ is a G-invariant irreducible character of N, and p is a prime, we define a partition of the set $\mathrm{Irr}(G|\theta)$ with respect to the prime p. We call the elements of this partition the θ-blocks. To each θ-block we associate a unique conjugacy class of p-subgroups of G/N, and we call the elements of this conjugacy class the θ-defect groups. We prove here that both the θ-blocks and the θ-defect groups are canonically defined (Theorem A) and we give a θ-version of some results in block theory. For instance, we prove that if $\chi \in \mathrm{Irr}(G|\theta)$, B_θ is the θ-block containing χ, and $(gN)_p$ does not lie in a θ-defect group of B_θ for some $g \in G$, then

$\chi(g) = 0$ (this is Theorem 2.12). We also give θ-versions of the well-known Brauer's Height Zero conjecture and Brauer's $k(B)$-conjecture (Conjectures B and D), and in both cases we prove that our θ-version is equivalent to the original one (Theorems C and E). In this Chapter, we next introduce θ-Brauer characters, θ-decomposition numbers and θ-linking in Sections 2.8 and 2.9. Finally, we prove that our θ-blocks coincide with the blocks defined by Navarro in [**Nav00**] and [**Nav12**] (Theorem 2.30), relating our work with the work of Külshammer and Robinson in [**KR87**].

In Chapter 3 we give a generalization of the Howlett-Isaacs theorem taking into account the action of $\mathrm{Aut}(G)_{(N,\theta)}$ on $\mathrm{Irr}(G|\theta)$ (here $\mathrm{Aut}(G)_{(N,\theta)}$ is the subgroup of $\mathrm{Aut}(G)$ that fixes N and θ). In particular, we prove that if $\mathrm{Irr}(G|\theta)$ is an $\mathrm{Aut}(G)_{(N,\theta)}$-orbit, then G/N is solvable. To prove this we need some results on transitive actions which we prove in Section 3.2. We also need non-trivial properties of the *Glauberman correspondence*. The Glauberman correspondence asserts that if a solvable group S acts coprimely on a group G, there exists a natural bijection from $\mathrm{Irr}_S(G)$, the irreducible characters of G fixed by the action of S, onto $\mathrm{Irr}(\mathbf{C}_G(S))$. In particular we need a rather technical refinement of this bijection, that we prove in Section 3.4 using results of A. Turull in [**Tur08**], [**Tur09**] and [**Tur17**].

As we have mentioned before, the Howlett-Isaacs theorem uses the Classification of Finite Simple Groups (CFSG). We also need the CFSG to prove our generalization. In fact, we need the same result on simple groups that is used in [**HI82**]: if X is a non-abelian simple group, then there exists a prime p such that p divides $|X|$, p does not divide $|M(X)|$ (the size of the Schur multiplier of X) and there is no solvable subgroup of X having p-power index.

In the final part of this Chapter, we prove Theorem I on the characterization of when the irreducible characters over an irreducible character of a normal subgroup have the same degree in a specific situation. This theorem uses three non-trivial results: a theorem of U. Riese about inducing irreducible characters from an abelian subgroup, a deep result of S. Dolfi on regular orbits, and finally the Howlett-Isaacs theorem.

Finally, in Chapter 4, we give a formula to compute the number of fixed points of the action of a p-group on a group of order coprime to p (Theorem K), and we apply this formula to obtain the size of the normalizer of a Sylow p-subgroup of a finite p-solvable group G from its character table, provided that the Sylow p-subgroups of G are abelian or have exponent p (Theorem J). We call this formula a *Brauer-Wielandt formula*. Richard Brauer was the first to give a formula of this type in the case the group acting was a Klein 4-group and later Wielandt gave a formula for the general case. However, Wielandt's formula can not be used to obtain information from the character table, and this is what makes our formula interesting: it only involves centralizers of some elements and we can obtain that information from the character table in the cases we have just mentioned. We finish this Chapter with the alternative proofs of our formula given by Isaacs and Lyons.

CHAPTER 1

Preliminaries

In general, we follow the notation of [**Isa76**] and [**Nav18**] for characters, and the notation of [**Nav98a**] for blocks.

1.1. Preliminaries on ordinary characters

Let G be a finite group and let F be a field. Write $F[G]$ for the set of formal sums

$$F[G] = \{\sum_{g\in G} a_g g \mid a_g \in F\}.$$

If $c \in F$ and $\sum_{g\in G} b_g g$, define

$$c \cdot \sum_{g\in G} a_g g = \sum_{g\in G} (ca_g) g$$

and

$$\sum_{g\in G} b_g g + \sum_{g\in G} a_g g = \sum_{g\in G} (a_g + b_g) g.$$

It is easy to see that $F[G]$ has structure of F-vector space. Moreover, we can identify g with the element of $F[G]$ such that $a_g = 1$ and $a_h = 0$ for all $h \neq g$. This identification embeds G into $F[G]$ and in fact G is a basis for $F[G]$ under this identification. Now we can define a multiplication in $F[G]$ by extending linearly the multiplication in G. This makes $F[G]$ an F-algebra.

An *F-representation* of $F[G]$ is an F-algebra homomorphism $\mathfrak{X} : F[G] \to \mathrm{Mat}_n(F)$. The integer n is the *degree* of $\mathfrak{X}$. Two representations $\mathfrak{X}$, $\mathfrak{Y}$ are *similar* if there exists a non-singular matrix P such that $\mathfrak{X}(a) = P^{-1}\mathfrak{Y}(a)P$ for all $a \in F[G]$. If we restrict $\mathfrak{X}$ to G we obtain a group homomorphism $G \to \mathrm{GL}(n, F)$.

An *F-representation* of G is a group homomorphism $G \to \mathrm{GL}(n, F)$. Hence an F-representation of $F[G]$ determines an F-representation of G via restriction. The converse is also true, if $\mathfrak{X} : G \to \mathrm{GL}(n, F)$ is an F-representation of G, then $\mathfrak{X}$ determines an F-representation of $F[G]$, $\tilde{\mathfrak{X}}$, via

$$\tilde{\mathfrak{X}}(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \mathfrak{X}(g).$$

If $\mathfrak{X} : G \to \mathrm{GL}(n, F)$ is an F-representation of G, we say that $\mathfrak{X}$ is *irreducible* if it is not similar to a representation of G in block form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

An F-representation of G of degree n, consists of $n^2|G|$ elements of F, and it is clear that this is too much information (since we do not wish to distinguish between similar representations). In order to reduce this amount of information, we use *characters*, that is, traces of the representations. Over certain fields, characters essentially determine the representations.

Definition 1.1 (Character). If $\mathfrak{X} : G \to \mathrm{GL}(n, F)$ is an F-representation of G, the character afforded by $\mathfrak{X}$ is the function $\chi : G \to F$ given by $\chi(g) = \mathrm{tr}(\mathfrak{X}(g))$.

If χ is a character of G, the degree of χ is $\chi(1)$ (note that this is the degree of any F-representation affording χ). If $\chi(1) = 1$ it is said that χ is *linear*. We denote the set formed by linear characters of G as $\mathrm{Lin}_F(G)$.

Since the trace is invariant on similar matrices, we have that similar F-representations afford equal characters and that characters are *class functions*, that is, constant on the conjugacy classes of a group.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are representations of G of degrees n and m affording χ and ψ respectively, then the map $\mathfrak{Z} : G \to \mathrm{GL}(n, F)$ defined by

$$\mathfrak{Z}(g) = \begin{bmatrix} \mathfrak{X}(g) & 0 \\ 0 & \mathfrak{Y}(g) \end{bmatrix}$$

is also an F-representation of G. Since $\mathrm{tr}(\mathfrak{Z}(g)) = \mathrm{tr}(\mathfrak{X}(g)) + \mathrm{tr}(\mathfrak{Y}(g))$, we have that sum of characters are characters.

If χ, ψ are characters of G, we may define a new class function $\chi\psi$ on G by setting

$$(\chi\psi)(g) = \chi(g)\psi(g).$$

Now, if $\mathfrak{X}$ is an F-representation affording χ and $\mathfrak{Y}$ is an F-representation affording ψ, then $\mathfrak{X} \otimes \mathfrak{Y} : G \to \mathrm{GL}(nm, F)$, where $n = \chi(1)$ and $m = \psi(1)$, defined by

$$(\mathfrak{X} \otimes \mathfrak{Y})(g) = \mathfrak{X}(g) \otimes \mathfrak{Y}(g) = \begin{bmatrix} a_{11}\mathfrak{Y}(g) & \cdots & a_{1n}\mathfrak{Y}(g) \\ \vdots & \ddots & \vdots \\ a_{n1}\mathfrak{Y}(g) & \cdots & a_{nn}\mathfrak{Y}(g) \end{bmatrix},$$

where $\mathfrak{X}(g) = (a_{ij})$, is an F-representation of G affording $\chi\psi$. Hence profucts of characters are also characters (see Theorem 4.1 of [**Isa76**]). Moreover, $\mathrm{Lin}_F(G)$ is a group with this product.

We say that a character is *irreducible* if it is not the sum of two characters. We denote by $\mathrm{Irr}_F(G)$ the set of irreducible characters of G afforded by F-representations. Irreducible characters are afforded by irreducible F-representations.

From now on, we let $F = \mathbb{C}$, and we write $\mathrm{Irr}(G)$ for the set $\mathrm{Irr}_{\mathbb{C}}(G)$, $\mathrm{Lin}(G)$ for the set $\mathrm{Lin}_{\mathbb{C}}(G)$, etc. The following is called the Fundamental Theorem of Character Theory.

THEOREM 1.2 (Fundamental Theorem of Character Theory). *If G is a finite group, then $\mathrm{Irr}(G)$ is a basis of the vector space of complex class functions of G. In particular, $|\mathrm{Irr}(G)|$ is the number of conjugacy classes of G.*

PROOF. See Theorem 2.8 of [**Isa76**]. □

Hence if ψ is a complex class function of G, then we can write

$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} a_{\psi\chi}\chi,$$

for some uniquely determined complex numbers $a_{\psi\chi}$. It is also clear that ψ is a complex character of G if all $a_{\psi\chi}$ are non-negative integers (not all zero). If ψ is a character of G and $a_{\psi\chi} \neq 0$, we say that χ is an *irreducible constituent* of ψ.

The irreducible characters of G are usually presented in a table whose columns are indexed by the conjugacy classes of G and whose rows are indexed by its irreducible characters. This table is called the *character table* of G (which of course is uniquely determined up to permutation of rows and columns) and one of the main questions in character theory is to know how much information the character table of G contains about G.

There are two fundamental relations when we try to construct the character table of a group G.

THEOREM 1.3 (First Orthogonality Relation). *Suppose that $\chi, \psi \in \mathrm{Irr}(G)$. Then*

$$\frac{1}{|G|}\sum_{g \in G} \chi(g)\overline{\psi(g)} = \delta_{\chi\psi}.$$

PROOF. See Corollary 2.14 of [**Isa76**]. □

THEOREM 1.4 (Second Orthogonality Relation). *Let $g, h \in G$, then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = 0$$

if g is not G-conjugate to h. Otherwise, the sum is equal to $|\mathbf{C}_G(g)|$.

PROOF. See Theorem 2.18 of [**Isa76**]. □

As a consequence of the Second Orthogonality Relation, notice that

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2,$$

and hence the character table of G knows the order of G.

We can define an inner product in $\mathrm{cf}(G)$ (the complex vector space of the class functions) as follows:

DEFINITION 1.5. Let φ and η be class functions on a group G. Then

$$[\varphi, \eta] = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\eta(g)}$$

is the inner product of φ and η. In fact, this makes $\mathrm{cf}(G)$ into a finite dimensional Hilbert space.

As a consequence of the First Orthogonality Relation we have that $[\chi_i, \chi_j] = \delta_{ij}$ for $\chi_i, \chi_j \in \mathrm{Irr}(G)$. Hence, if $\varphi \in \mathrm{cf}(G)$, we have that

$$\varphi = \sum_{\chi \in \mathrm{Irr}(G)} [\chi, \varphi]\chi.$$

Moreover, if χ, ψ are characters then $[\chi, \psi] = [\psi, \chi]$ is a non-negative integer and χ is irreducible if and only if $[\chi, \chi] = 1$.

DEFINITION 1.6 (Kernel of a character). Let χ be a character of G. Then the *kernel* of χ is $\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$. If $\ker(\chi) = 1$, we say that χ is *faithful*.

If $\mathfrak{X}$ is a representation of G affording χ, we have that $g \in \ker(\mathfrak{X})$ if and only if $g \in \ker(\chi)$ (see Lemma 2.19 of [**Isa76**]), and hence $\ker(\chi)$ is a normal subgroup of G. Also, we have the following.

LEMMA 1.7. *Let* $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ *and let* χ *be a character of* G *with* $\chi = \sum_{i=1}^{k} n_i \chi_i$. *Then* $\ker(\chi) = \bigcap\{\ker(\chi_i) \mid n_i > 0\}$. *Also* $\bigcap\{\ker(\chi_i) \mid 1 \leqslant i \leqslant k\} = 1$.

PROOF. See Lemma 2.21 of [**Isa76**]. □

If N is a normal subgroup of G, one can prove that N is the intersection of the kernels of the irreducible characters of G that contain N in its kernel. It turns out that we can calculate $|N|$ from the character table: if $\{K_1, \ldots, K_r\}$ are the conjugacy classes of G contained in N, then $|N| = \sum_{i=1}^{r} |K_i| = \sum_{i=1}^{r} |G : \mathbf{C}_G(x_i)|$, where $x_i \in K_i$. Therefore simplicity, nilpotency or solvability can be easily read from the character table of G.

We have said that the character table of G knows the sizes of the normal subgroups of G. However, we can not construct the character table of N from the character table of G. What we can do, instead, is to obtain the character table of G/N.

LEMMA 1.8. *Let* $N \lhd G$.

(a) *If* χ *is a character of* G *and* $N \subseteq \ker(\chi)$, *then* χ *is constant on cosets of* N *in* G *and the function* $\hat{\chi}$ *on* G/N *defined by* $\hat{\chi}(Ng) = \chi(g)$ *is a character of* G/N.

(b) If $\hat{\chi}$ is a character of G/N, then the function χ defined by $\chi(g) = \hat{\chi}(Ng)$ is a character of G.
(c) In both (a) and (b), $\chi \in \mathrm{Irr}(G)$ iff $\hat{\chi} \in \mathrm{Irr}(G/N)$.

PROOF. See Lemma 2.22 of [**Isa76**]. □

Usually, we shall identify χ and $\hat{\chi}$ and see $\mathrm{Irr}(G/N)$ as a subset of $\mathrm{Irr}(G)$.

Let H be a subgroup of G. If χ is a character of G, then its restriction to H is a character of H. The dual process is called *induction*.

DEFINITION 1.9 (Induction of characters). Let $H \subseteq G$ and let φ be a class function of H. Then φ^G, the *induced class function* on G, is given by

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1}),$$

where $\varphi^\circ(h) = \varphi(h)$ if $h \in H$ and $\varphi^\circ(y) = 0$ if $y \notin H$.

The following is quite elementary but fundamental.

THEOREM 1.10 (Frobenius reciprocity). *Let $H \subseteq G$ and suppose that φ is a class function on H and that θ is a class function on G. Then*

$$[\varphi, \theta_H] = [\varphi^G, \theta].$$

PROOF. See Lemma 5.2 of [**Isa76**]. □

As a consequence of Theorem 1.10, we can see that if φ is a character of H, then φ^G is a character of G.

If $N \lhd G$, θ is a class function of N and $g \in G$, we define $\theta^g : N \to \mathbb{C}$ by $\theta^g(n) = \theta(gng^{-1})$. It is easy to see that if $\theta \in \mathrm{Irr}(N)$, then $\theta^g \in \mathrm{Irr}(N)$.

THEOREM 1.11 (Clifford). *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose that $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are the distinct G-conjugates of θ in G. Then*

$$\chi_N = e \sum_{i=1}^{t} \theta_i,$$

where $e = [\chi_N, \theta]$.

PROOF. See Theorem 6.2 of [**Isa76**]. □

As a consequence of Theorem 1.11 we have that if $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ is an irreducible constituent of χ_N, then $\theta(1)$ divides $\chi(1)$. The following is a much deeper result.

THEOREM 1.12. *Let $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$. Let θ be a constituent of χ_N, then $\chi(1)/\theta(1)$ divides $|G : N|$.*

Proof. See Corollary 11.29 of [**Isa76**]. □

Definition 1.13 (Stabilizer). Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$. Then

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

is the stabilizer of θ in G. It is also known as the *inertia group* of θ in G. We say that θ is G-*invariant* if $I_G(\theta) = G$.

It is obvious that $N \subseteq I_G(\theta)$ for all $\theta \in \mathrm{Irr}(N)$. Also, $|G : I_G(\theta)|$ is the size of the G-orbit of θ in $\mathrm{Irr}(N)$ and hence $t = |G : I_G(\theta)|$ in Theorem 1.11.

If N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, we write $\mathrm{Irr}(G|\theta)$ to denote the set of the irreducible characters of G having θ as an irreducible constituent of its restriction to N, that is

$$\mathrm{Irr}(G|\theta) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \theta] \neq 0\}.$$

Note that, using Frobenius reciprocity, we have that the elements of $\mathrm{Irr}(G|\theta)$ are exactly the irreducible constituents of the induced character θ^G.

The following is a key result in the character theory of normal subgroups.

Theorem 1.14 (Clifford correspondence). *Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$. Write $I = I_G(\theta)$. Then*

(a) If $\psi \in \mathrm{Irr}(I|\theta)$, then ψ^G is irreducible.
(b) The map $\psi \mapsto \psi^G$ from $\mathrm{Irr}(I|\theta)$ onto $\mathrm{Irr}(G|\theta)$ is a bijection.
(c) Let $\chi = \psi^G$ where $\psi \in \mathrm{Irr}(I|\theta)$. Then ψ is the unique irreducible constituent of χ_I which lies over θ.
(d) Let $\psi^G = \chi$ where $\psi \in \mathrm{Irr}(I|\theta)$. Then $[\psi_N, \theta] = [\chi_N, \theta]$.

Proof. See Theorem 6.11 of [**Isa76**]. □

We have said before that if χ is a character of G, then its restriction to H, χ_H is also a character. If χ is irreducible, χ_H need not be irreducible. When $\chi_H = \theta$ for some $\theta \in \mathrm{Irr}(H)$ we say that θ *extends* to G or that χ *extends* θ. Note that if θ extends to G, and θ is the character of a normal subgroup of G, then θ is G-invariant.

The following result, and its corollary (known as Gallagher's corollary) are frequently used when certain characters extend.

Theorem 1.15. *Let $N \lhd G$ and let $\varphi, \theta \in \mathrm{Irr}(N)$ be invariant in G. Assume $\varphi\theta$ is irreducible and that θ extends to $\chi \in \mathrm{Irr}(G)$. Then $\beta \mapsto \beta\chi$ defines a bijection of $\mathrm{Irr}(G|\varphi)$ onto $\mathrm{Irr}(G|\varphi\theta)$.*

Proof. See Theorem 6.16 of [**Isa76**]. □

Corollary 1.16 (Gallagher). *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of θ^G.*

Proof. See Corollary 6.17 of [**Isa76**]. □

Many times we need to know when an irreducible character of a normal subgroup extends to the whole group, and sometimes we need to know what the extension looks like. The following results are standard and quite useful in that sense.

If χ is a character of G, χ uniquely determines a linear character of G, called the *determinant* of χ, as follows: let $\mathfrak{X}$ be a representation of G affording χ and define $\det\chi : G \to \mathbb{C}$ as

$$(\det\chi)(g) = \det(\mathfrak{X}(g)).$$

The order of $\det\chi$ in the group of linear characters of G is known as the *determinantal order* of χ and it is denoted by $o(\chi)$.

THEOREM 1.17. *Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ with θ invariant in G. Suppose that $(|G : N|, o(\theta)\theta(1)) = 1$. Then θ has a unique extension, $\chi \in \mathrm{Irr}(G)$ with $(|G : N|, o(\chi)) = 1$. In fact, $o(\chi) = o(\theta)$. In particular this holds if $(|G : N|, |N|) = 1$.*

PROOF. See Corollary 8.16 of [**Isa76**]. □

1.2. Character triples and projective representations

Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. In Chapter 2, we introduce a canonical partition of the set $\mathrm{Irr}(G|\theta)$ into some subsets that we call θ-blocks. To define the θ-blocks we need some background on projective representations. We give that background now.

A complex *projective representation* of a finite group G is a map

$$\mathcal{P} : G \to \mathrm{GL}(n, \mathbb{C})$$

such that for every $x, y \in G$ there is some $\alpha(x, y) \in \mathbb{C}^\times$ satisfying

$$\mathcal{P}(x)\mathcal{P}(y) = \alpha(x, y)\mathcal{P}(xy).$$

The function $\alpha : G \times G \to \mathbb{C}^\times$ is called the *factor set* of $\mathcal{P}$.

If G is a finite group, $N \lhd G$, and $\theta \in \mathrm{Irr}(N)$ is G-invariant, then we say that (G, N, θ) is a *character triple*. The theory of character triples and their isomorphisms was developed by Isaacs, and we refer the reader to Chapter 11 of [**Isa76**] for a further insight of this theory. It turns out that character triples are associated to projective representations.

If (G, N, θ) is a character triple, we say that a projective representation of G is *associated* with θ if

(a) $\mathcal{P}_N$ is an ordinary representation of N affording θ, and
(b) $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$ and $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ for $g \in G$ and $n \in N$.

THEOREM 1.18. *Let (G, N, θ) be a character triple. There exists a projective representation of G associated with θ. Furthermore, if $\mathcal{P}_0$ is another projective representation of G associated with θ, then $\mathcal{P}_0(g) = \mathcal{P}(g)\xi(g)$ for some function $\xi : G \to \mathbb{C}^\times$, which is constant on cosets of N.*

Proof. See Theorem 11.2 of [**Isa76**]. □

Lemma 1.19. *Suppose that (G, N, θ) is a character triple, and let $\mathcal{P}$ be a projective representation of G associated with θ with factor set α. Then*

(a) $\alpha(1,1) = \alpha(g,n) = \alpha(n,g) = 1$ *for* $n \in N$, $g \in G$.
(b) $\alpha(xn, ym) = \alpha(x,y)$ *for* $x, y \in G$, $n, m \in N$.

Proof. This is Lemma 11.5 and Theorem 11.7 of [**Isa76**]. See also Lemma 5.3 of [**Nav18**]. □

An important fact about projective representations is that given a character triple (G, N, θ), there always exists a projective representation associated with θ such that its factor set has roots of unity values.

Theorem 1.20. *Let (G, N, θ) be a character triple. Then there exists a projective representation $\mathcal{P}$ associated with θ with factor set α such that*

$$\alpha(x,y)^{|G|\theta(1)} = 1$$

for all $x, y \in G$.

Proof. See for instance Theorem 8.2 of [**Isa73**] or Theorem 5.5 of [**Nav18**]. □

Using such a projective representation $\mathcal{P}$, it is possible to associate to each character triple (G, N, θ) a new finite group $\hat{G}$, a finite central extension of G which only depends on $\mathcal{P}$. This finite group $\hat{G}$ contains N as a normal subgroup, and an irreducible character $\tau \in \mathrm{Irr}(\hat{G})$ that extends θ. The next theorem explains exactly how to do this.

Theorem 1.21. *Let (G, N, θ) be a character triple and let $\mathcal{P}$ be a projective representation of G associated with θ such that the factor set α of $\mathcal{P}$ only takes roots of unity values. Let $Z \leqslant \mathbb{C}^\times$ be the subgroup generated by the values of α. Let $\hat{G} = \{(g,z) \mid g \in G, z \in Z\}$ with the multiplication given as follows:*

$$(x,a)(y,b) = (xy, \alpha(x,y)ab).$$

Then $\hat{G}$ is a finite group. Besides, if we identify N with $N \times 1$, Z with $1 \times Z$, and we let $\hat{N} = N \times Z$, then we have that the following hold.

(a) $N \lhd \hat{G}$, $Z \subseteq \mathbf{Z}(\hat{G})$, *and* $\hat{N} \lhd \hat{G}$. *Moreover, if $\pi : \hat{G} \to G$ is given by $(g,z) \mapsto g$, then π is an onto homomorphism with kernel Z. Also, if $N \subseteq \mathbf{Z}(G)$, then $\hat{N} \subseteq \mathbf{Z}(\hat{G})$.*
(b) *The function $\hat{\mathcal{P}}(g,z) = z\mathcal{P}(g)$ defines an irreducible linear representation of $\hat{G}$ whose character $\tau \in \mathrm{Irr}(\hat{G})$ extends θ. In fact, $\tau(n,z) = z\theta(n)$ for $n \in N$ and $z \in Z$. In particular, if $\hat{\theta} = \theta \times 1_Z \in \mathrm{Irr}(\hat{N})$, and $\hat{\lambda} \in \mathrm{Irr}(\hat{N})$ is defined by $\hat{\lambda}(n,z) = z^{-1}$, then $\hat{\lambda}$ is a linear $\hat{G}$-invariant character with $N = \ker(\hat{\lambda})$ and $\hat{\lambda}^{-1}\hat{\theta}$ extends to $\tau \in \mathrm{Irr}(\hat{G})$.*

Proof. See Theorem 11.28 of [**Isa76**] or Theorem 5.6 of [**Nav18**]. The properties of the factor set α that we have listed in Lemma 1.19 are essential to prove (a). □

We will call the group $\hat{G}$ a *representation group* for (G, N, θ) associated with $\mathcal{P}$.

In order to define the θ-blocks the notion of *character triple isomorphism* is essential. As we said before, this was first introduced by Isaacs (see Definition 11.23 of [**Isa76**]). However, for character triples, we shall frequently use the notation in [**Nav18**], so we reproduce here the definition of character triple isomorphism given there (see Definition 5.7 of [**Nav18**]).

Definition 1.22. Let (G, N, θ) and (G^*, N^*, θ^*) be character triples and let $^* : G/N \to G^*/N^*$ be an isomorphism of groups. If $N \leqslant U \leqslant G$, then we denote by U^* the unique subgroup $N^* \leqslant U^* \leqslant G^*$ such that $(U/N)^* = U^*/N^*$. Also, if β is a character of U/N, then β^* denotes the corresponding character of U^*/N^* via the isomorphism *. That is, β^* is the unique character of U^*/N^* satisfying $\beta^*(x^*) = \beta(x)$ for $x \in U/N$. Assume now that for every subgroup $N \leqslant U \leqslant G$, there is a bijection $^* : \mathrm{Irr}(U|\theta) \to \mathrm{Irr}(U^*|\theta^*)$ (which we extend linearly to $^* : \mathrm{Char}(U|\theta) \to \mathrm{Char}(U^*|\theta^*)$). It is said that * is a character triple isomorphism if for every $N \leqslant V \leqslant U \leqslant G$, $\chi \in \mathrm{Irr}(U|\theta)$ and $\beta \in \mathrm{Irr}(U/N)$, the following conditions hold:

(a) $(\chi_V)^* = (\chi^*)_{V^*}$, and
(b) $(\chi\beta)^* = \chi^*\beta^*$.

Notice that if (G, N, θ) and (G^*, N^*, θ^*) are isomorphic character triples, then $|\mathrm{Irr}(U|\theta)| = |\mathrm{Irr}(U^*|\theta^*)|$ for all $N \leqslant U \leqslant G$. Also, since $(\chi_N)^* = (\chi^*)_{N^*}$ we have that

$$\frac{\chi(1)}{\theta(1)} = \frac{\chi^*(1)}{\theta^*(1)}$$

for all $\chi \in \mathrm{Irr}(U|\theta)$. That is, character triple isomorphisms respect character degree ratios.

Theorem 1.23. *Let (G, N, θ) be a character triple and let $\mathcal{P}$ be a projective representation of G associated with θ such that its factor set has roots of unity values. Let $\hat{G}$ be a representation group for (G, N, θ) associated with $\mathcal{P}$, and let $\hat{N}$ and $\hat{\lambda}$ be as in Theorem 1.21. Then (G, N, θ) and $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ are isomorphic character triples.*

Proof. See Theorem 11.28 of [**Isa76**] or Corollary 5.9 of [**Nav18**]. □

We shall frequently use how this character triple isomorphism is constructed. Let $\chi \in \mathrm{Irr}(G|\theta)$. We show how to construct $\chi^* \in \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$. Let $\pi : \hat{G} \to G$ be the onto group homomorphism $(g, z) \mapsto g$, which has kernel Z. Since π induces an isomorphism $\hat{G}/Z \to G$, there is a unique $\chi^\pi \in \mathrm{Irr}(\hat{G})$ such that $\chi^\pi(g, z) = \chi(g)$ for all $g \in G, z \in Z$. Since χ lies over θ notice that

χ^π lies over $\hat{\theta} = \theta \times 1_Z$, and in particular over θ. Now by Theorem 1.21(b), the character τ extends θ. By Gallagher's Corollary (Corollary 1.16), there exists a unique $\chi^* \in \mathrm{Irr}(\hat{G}/N)$ such that $\chi^\pi = \chi^*\tau$. (Recall that we view the characters of H/N as characters of H that contain N in its kernel.) Now, evaluating in $(1, z)$ for $z \in Z$, we easily check that $\chi^* \in \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$. The fact that $\chi \mapsto \chi^*$ defines an isomorphism of character triples is the content of the proof of Theorem 1.23. (The same construction can be done for every subgroup $N \leqslant U \leqslant G$ instead of G.)

1.3. Preliminaries on blocks

Representation theory over a field of characteristic p is known as *modular representation theory*. The earliest work on modular representation theory is due to L. E. Dickson ([**Dic02**]) who showed that the representation theory of a finite group G over a field of characteristic p is quite similar to the representation theory over a field of characteristic 0 when the prime p does not divide $|G|$. The study of modular representations when the characteristic divides the order of the group was started by Richard Brauer ([**Bra35**]). He by himself essentially established modular representation theory as a main area in mathematics.

We denote by $\mathbf{R}$ the ring of algebraic integers in $\mathbb{C}$, and we choose a maximal ideal M of $\mathbf{R}$ containing $p\mathbf{R}$. Let $F = \mathbf{R}/M$, an algebraically closed field of characteristic p, and let $^*\colon \mathbf{R} \to F$ be the canonical ring homomorphism. Let

$$S = \{rs^{-1} \mid r \in \mathbf{R}, s \in \mathbf{R} - M\}.$$

Notice that the map * can be extended to S in a natural way. If $r \in \mathbf{R}$ and $s \in \mathbf{R} - M$, then

$$(rs^{-1})^* = r^*(s^*)^{-1}.$$

Richard Brauer introduced the notion of *Brauer characters* to understand the interplay between the representation theory in characteristic p and ordinary character theory. Let $\mathbf{U} \subseteq \mathbf{R}$ be the multiplicative group of roots of unity of order not divisible by p, so that

$$\mathbf{U} = \{\xi \in \mathbb{C} \mid \xi^k = 1 \text{ for some integer } k \text{ not divisible by } p\}.$$

Lemma 1.24. *The restriction of * to $\mathbf{U}$ defines an isomorphism $\mathbf{U} \to F^\times$ of multiplicative groups. Also F is an algebraically closed field of characteristic p.*

Proof. See Lemma 2.1 of [**Nav98a**]. □

We say that $g \in G$ is p-regular if p does not divide the order of g. We denote by $G^{p'}$ the set formed by the p-regular elements of G.

DEFINITION 1.25 (Brauer character). Suppose that $\mathfrak{X}\colon G \to \mathrm{GL}_n(F)$ is an F-representation of the group G. If $g \in G^{p'}$, then by Lemma 1.24, the eigenvalues of the matrix $\mathfrak{X}(g)$ are $\xi_1^*, \dots, \xi_n^* \in F^\times$ for uniquely determined $\xi_1, \dots, \xi_n \in \mathbf{U}$ (because F is algebraically closed). Then $\varphi\colon G^{p'} \to \mathbb{C}$ defined by $\varphi(g) = \xi_1 + \cdots + \xi_n$ is the *Brauer character* afforded by $\mathfrak{X}$. Notice that φ is uniquely determined (once the maximal ideal M has been chosen) by the equivalence class of the representation $\mathfrak{X}$.

As for ordinary characters, sums and products of Brauer characters are Brauer characters and we say that φ is an *irreducible* Brauer character if it is not the sum of two Brauer characters. We denote by $\mathrm{IBr}(G)$ the set of irreducible Brauer characters of G.

If $\varphi \in \mathrm{IBr}(G)$, we define the *kernel* of φ as $\ker(\varphi) = \{g \in G \mid \mathfrak{X}(g) = I_n\}$, where $\mathfrak{X} : G \to \mathrm{GL}(n, F)$ is an irreducible F-representation affording φ. Since $\mathfrak{X}$ is uniquely determined by φ up to similarity, this is well-defined.

The degree of φ is $\varphi(1)$, which is the degree of any F-representation affording φ and Brauer characters are constant on conjugacy classes. However, unlike ordinary characters, the degrees of the irreducible Brauer characters do not divide, in general, the order of the group ($\mathrm{PSL}_2(7)$ for $p = 7$ has an irreducible Brauer character of degree 5).

If $H \leqslant G$ and φ is a Brauer character of G, then we denote by φ_H the restriction of φ to $H^{p'}$. The function φ_H is a Brauer character of H.

Write $\mathrm{cf}(G^{p'})$ to denote the $\mathbb{C}$-vector space of class functions on $G^{p'}$ (functions $\theta\colon G^{p'} \to \mathbb{C}$ constant on the conjugacy classes contained in $G^{p'}$). Of course the dimension of $\mathrm{cf}(G^{p'})$ is equal to the number of conjugacy classes of p-regular elements of G.

As happens with ordinary characters, Brauer characters are non-negative integer linear combination of irreducible Brauer characters.

THEOREM 1.26. *Let G be a group. Then $\mathrm{IBr}(G)$ is a basis of $\mathrm{cf}(G^{p'})$. Moreover, $\psi \in \mathrm{cf}(G^{p'})$ is a Brauer character of G if and only if*

$$\psi = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi \varphi,$$

where $a_\varphi \in \mathbb{N}$, and not all a_φ are zero.

PROOF. See Corollary 2.10 and Theorem 2.3 of [**Nav98a**]. □

The non-negative integer a_φ in the decomposition of ψ in Theorem 1.26 is called the multiplicity of φ in ψ. If $a_\varphi \neq 0$, then we call φ an irreducible constituent of ψ.

If $\chi \in \mathrm{Irr}(G)$, we denote by $\chi^{p'}$ the restriction of χ to $G^{p'}$.

THEOREM 1.27. *If χ is an ordinary character of G, then $\chi^{p'}$ is a Brauer character of G.*

Proof. See Corollary 2.9 of [**Nav98a**]. □

Definition 1.28 (Decomposition numbers). Let $\chi \in \mathrm{Irr}(G)$, by Theorem 1.27, $\chi^{p'}$ is a Brauer character of G. Hence

$$\chi^{p'} = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

for suitable non-negative integers $d_{\chi\varphi}$. The non-negative integers $d_{\chi\varphi}$ in the above decomposition are called the *decomposition numbers* of χ.

One of Brauer's ideas was to distribute the irreducible characters and the irreducible Brauer characters into *p-blocks*. There are different ways to understand the p-blocks, but we follow a character theoretical approach.

Let K be a conjugacy class of G and let $\hat{K}$ be the class sum, that is

$$\hat{K} = \sum_{x \in K} x.$$

If $\mathrm{cl}(G)$ is the set formed by the conjugacy classes of G, the set $\{\hat{K} \mid K \in \mathrm{cl}(G)\}$ is a $\mathbb{C}$-basis of $\mathbf{Z}(\mathbb{C}G)$ (in fact, the set $\{\hat{K} \mid K \in \mathrm{cl}(G)\}$ is an R-basis of $\mathbf{Z}(RG)$ for any ring R). Now, if $\chi \in \mathrm{Irr}(G)$, χ uniquely determines an algebra homomorphism $\omega_\chi : \mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$, given by

$$\omega_\chi(\hat{K}) = \frac{|K|\chi(x_K)}{\chi(1)},$$

where $x_K \in K$. It is well known that $\omega_\chi(\hat{K})$ is an algebraic integer (see Theorem 3.7 of [**Isa76**]) and hence we can construct a map $\lambda_\chi : \mathbf{Z}(FG) \to F$ by setting

$$\lambda_\chi(\hat{K}) = (\omega_\chi(\hat{K}))^*$$

In fact the map λ_χ is also an algebra homomorphism.

In the same way, if $\varphi \in \mathrm{IBr}(G)$, we can associate to φ an algebra homomorphism $\lambda_\varphi : \mathbf{Z}(FG) \to F$. Let $\mathfrak{X} : FG \to \mathrm{Mat}(n, F)$ be an irreducible F-representation of G affording φ, then $\mathfrak{X}(\hat{K})$ is a scalar matrix for every $K \in \mathrm{cl}(G)$, and this scalar only depends on φ. Hence, the equality

$$\mathfrak{X}(\hat{K}) = \lambda_\varphi(\hat{K}) I_n$$

defines an algebra homomorphism $\lambda_\varphi : \mathbf{Z}(FG) \to F$.

Definition 1.29 (Brauer p-block). Let $\chi, \psi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$. Then χ and ψ *lie in the same p-block B* of G if

$$\lambda_\chi(\hat{K}) = \lambda_\psi(\hat{K})$$

for every conjugacy class K of G. In this case we write $\lambda_B = \lambda_\chi$.

The set formed by all the Brauer p-blocks of G is denoted by $\mathrm{Bl}(G)$. If $B \in \mathrm{Bl}(G)$, $\mathrm{Irr}(B)$ denotes the set of irreducible characters lying in B and $\mathrm{IBr}(B)$ denotes the set of irreducible Brauer characters lying in B. It turns out that

$$\mathrm{Irr}(G) = \bigcup_{B \in \mathrm{Bl}(G)} \mathrm{Irr}(B),$$

and

$$\mathrm{IBr}(G) = \bigcup_{B \in \mathrm{Bl}(G)} \mathrm{IBr}(B),$$

where the unions are disjoint.

Another way to understand the p-blocks is through the decomposition numbers.

THEOREM 1.30. *If $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G)$, are such that $d_{\chi\varphi} \neq 0$, then $\lambda_\chi = \lambda_\varphi$.*

PROOF. See Theorem 3.3 of [**Nav98a**]. □

If $\chi, \psi \in \mathrm{Irr}(G)$, we say that χ and ψ are *connected* if there exists $\varphi \in \mathrm{IBr}(G)$ such that

$$d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}.$$

The graph defined by connexion in $\mathrm{Irr}(G)$ is called the *Brauer graph*. It turns out that the connected components of the Brauer graph are exactly the sets $\mathrm{Irr}(B)$ such that B is a p-block of G (this is Theorem 3.9 of [**Nav98a**]).

There is a way to visualize the sets $\mathrm{Irr}(B)$ directly from the character table, without the need to choose a maximal ideal of $\mathbf{R}$ and computing in F.

DEFINITION 1.31. Let $\chi, \psi \in \mathrm{Irr}(G)$. We say that χ and ψ are *linked* if

$$\sum_{x \in G^{p'}} \chi(x)\overline{\psi(x)} \neq 0.$$

THEOREM 1.32. *The connected components of the graph in $\mathrm{Irr}(G)$ defined by linking are exactly the sets $\mathrm{Irr}(B)$ for $B \in \mathrm{Bl}(G)$.*

PROOF. See Theorem 3.19 of [**Nav98a**]. □

To each p-block, a unique conjugacy class of p-subgroups of G is associated, namely the *defect groups of B*. There are different ways to define them, but the approach we follow needs the concept of *defect class*. We need to talk about central idempotents in order to define a defect class.

Let $\chi \in \mathrm{Irr}(G)$ and let

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g \in \mathbb{C}G.$$

Then e_χ is a primitive idempotent of $\mathbf{Z}(\mathbb{C}G)$ (see Theorem 2.12 of [**Isa76**]).

Theorem 1.33. *Let*

$$f_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi.$$

Then

(a) $f_B \in \mathbf{Z}(SG)$.
(b) $f_B(\hat{K}) = 0$ *if* K *does not consist on* p*-regular elements.*

Proof. See Corollary 3.8 of [**Nav98a**]. □

The ring homomorphism $^* : S \to F$ extends to a ring homomorphism $^* : SG \to FG$ by setting

$$\left(\sum_{g \in G} s_g g \right)^* = \sum_{g \in G} s_g^* g.$$

Notice that * maps $\mathbf{Z}(SG)$ onto $\mathbf{Z}(FG)$. Since $f_B \in \mathbf{Z}(SG)$, we have that $f_B^* \in \mathbf{Z}(FG)$. Write

$$e_B = f_B^*,$$

e_B is the *block idempotent* of B.

Theorem 1.34. *Let* $B, B' \in \mathrm{Bl}(G)$*, then*

(a) $e_B e_{B'} = \delta_{BB'} e_B$.
(b) $1 = \sum_{B \in \mathrm{Bl}(G)} e_B$.
(c) $\lambda_B(e_{B'}) = \delta_{BB'}$.

Proof. Since $^* : \mathbf{Z}(SG) \to \mathbf{Z}(FG)$ is a ring homomorphism, (a) easily follows. Since $1 = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi$, we have (b). For (c) see Theorem 3.11 of [**Nav98a**]. □

Using (a) and (b) of Theorem 1.34 we have that

$$FG = \bigoplus_{B \in \mathrm{Bl}(G)} e_B FG.$$

It turns out that $e_B FG$ is an algebra with identity e_B. For many authors $B = e_B FG$ is the natural definition of p-block.

Since $e_B \in \mathbf{Z}(FG)$, we can write

$$e_B = \sum_{K \in \mathrm{cl}(G)} a_B(\hat{K})\hat{K},$$

with $a_B(\hat{K}) \in F$. Now, since $\lambda_B(e_B) = 1$, there exists $K \in \mathrm{cl}(G)$ such that

$$a_B(\hat{K}) \neq 0 \neq \lambda_B(\hat{K}).$$

If this happens, we say that K is a *defect class* of B.

PROPOSITION 1.35. *If B is a p-block and K is a defect class of B, K consists of p-regular elements.*

PROOF. This follows from Corollary 3.8 of [**Nav98a**]. □

DEFINITION 1.36 (Defect group of a class). Let $K \in \mathrm{cl}(G)$ be a conjugacy class of G. Let $x_K \in K$ and let $D_K \in \mathrm{Syl}_p(\mathbf{C}_G(x_K))$. The *defect groups of* K are

$$\{D_K^g \mid g \in G\}.$$

The set of defect groups of K is denoted by $\delta(K)$.

THEOREM 1.37. *If K and L are defect classes for B, then $\delta(K) = \delta(L)$.*

PROOF. See Corollary 4.5 of [**Nav98a**]. □

DEFINITION 1.38 (Defect group of a block). Let B be a p-block of G and let $K \in \mathrm{cl}(G)$ be a defect class of B. The defect groups of the block B are the defect groups of the class K. The set of defect groups of B is denoted by $\delta(B)$.

Recall that n_p is the largest power of p that divides the integer n. If $|D| = p^{d(B)}$ and $|G|_p = p^a$, it turns out that

$$p^{a-d(B)} = \min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\}.$$

The integer $d(B)$ is called the *defect* of B. It is clear that if $\psi \in \mathrm{Irr}(B)$, then $\psi(1)_p = p^{a-d(B)+h}$ for some non-negative integer h. The integer h is called the *height* of ψ, and if $h = 0$ it is said that ψ is a *height zero character*. The set of height-zero characters in a block B is usually denoted by $\mathrm{Irr}_0(B)$.

Two of the main conjectures in modular representation theory are due to Brauer in the 1950s ([**Bra56**] and [**Bra57**]). They are still open, although some spectacular advances have been achieved.

CONJECTURE 1.39 (Brauer's height zero conjecture). *Let B be a p-block of G and let D be a defect group of B, then all the characters in B have height zero if and only if D is abelian.*

The "if" direction was solved by R. Kessar and Malle in [**KM13**]. The "only if" direction is still open, but it was reduced to a question on simple groups by Navarro and B. Späth in [**NS14**].

CONJECTURE 1.40 (Brauer's $k(B)$-conjecture). *Let B be a p-block of G and let D be a defect group of B, then*

$$|\mathrm{Irr}(B)| \leqslant |D|.$$

The integer $|\mathrm{Irr}(B)|$ is usually referred as $k(B)$. This conjecture is not just open but also unreduced to simple groups. The solvable case of this conjecture is known, and its proof is very complicated ([**Nag62**] and [**GMRS**]). Brauer's Height Zero Conjecture is also known to be true for solvable groups, again, with a very complicated proof ([**GW84**]).

To end this brief introduction to blocks of finite groups we recall some results concerning normal subgroups. If $N \triangleleft G$, b is a block of N, and $g \in G$, the set $\{\psi^g \mid \psi \in \mathrm{Irr}(b) \cup \mathrm{IBr}(b)\}$ is a block of N, namely b^g. Hence G acts on $\mathrm{Bl}(N)$ by conjugation. If $\{b_1, \ldots, b_t\}$ is the G-orbit of b, we have that the idempotent $\sum_{i=1}^t f_{b_i}$ lies in $\mathbf{Z}(\mathbb{C}G)$ and there exist uniquely determined blocks $B_1, B_2, \ldots, B_s \in \mathrm{Bl}(G)$ such that

$$\sum_{i=1}^{t} f_{b_i} = \sum_{i=1}^{s} f_{B_i}$$

(for more details see discussion preceding Theorem 9.1 of [**Nav98a**]). In this case, we say that the block B_i covers b.

Theorem 1.41. *Suppose that $N \triangleleft G$. Let $b \in \mathrm{Bl}(N)$ and let $B \in \mathrm{Bl}(G)$. The following conditions are equivalent.*

(a) B covers b.
(b) If $\chi \in B$, then every irreducible constituent of χ_N lies in a G-conjugate of b.
(c) There is a $\chi \in B$ such that χ_N has an irreducible constituent in b.

Proof. See Theorem 9.2 of [**Nav98a**]. □

Theorem 1.42. *Suppose that $N \triangleleft G$ with G/N a p-group. If $b \in \mathrm{Bl}(N)$, then there is a unique $B \in \mathrm{Bl}(G)$ covering b.*

Proof. See Corollary 9.6 of [**Nav98a**]. □

Recall that we write $G^{p'}$ for the set of p-regular elements of G, and if $\chi \in \mathrm{cf}(G)$, we write $\chi^{p'}$ to denote the restriction of χ to $G^{p'}$. If N is normal in G and $\eta \in \mathrm{IBr}(G/N)$, then one can see that the class function $\varphi \in \mathrm{cf}(G^{p'})$ defined by $\varphi(g) = \eta(gN)$ is an irreducible Brauer character of G with $N \subseteq \ker(\varphi)$. Indeed, if $\mathfrak{X} : G/N \to \mathrm{GL}(n, F)$ affords η, then the irreducible representation $\mathfrak{Y} : G \to \mathrm{GL}(n, F)$ defined by $\mathfrak{Y}(g) = \mathfrak{X}(gN)$ affords φ.

On the other hand, if $\varphi \in \mathrm{IBr}(G)$ and $N \subseteq \ker(\varphi)$, then we can define η on $(G/N)^{p'}$ by $\eta(gN) = \varphi(g_{p'})$, and it is easy to see that η is an irreducible Brauer character of G/N. Hence, as happens with ordinary characters, we shall identify the functions φ and η and view $\mathrm{IBr}(G/N)$ as the set of irreducible Brauer characters of G having N in its kernel.

Now, if $\overline{\chi} \in \mathrm{Irr}(G/N)$ and $\chi \in \mathrm{Irr}(G)$ is the corresponding character of G (that is, $\chi(g) = \overline{\chi}(gN)$ for $g \in G$), then, for $x \in G^{p'}$, we have

$$\chi(x) = \overline{\chi}(xN) = \sum_{\overline{\varphi}\in \mathrm{IBr}(G/N)} d_{\overline{\chi}\overline{\varphi}}\overline{\varphi}(xN) = \sum_{\overline{\varphi}\in \mathrm{IBr}(G/N)} d_{\overline{\chi}\overline{\varphi}}\varphi(x),$$

and hence

$$d_{\overline{\chi}\overline{\varphi}} = d_{\chi\varphi}.$$

It follows that if $\overline{B}$ is a block of G/N and $\overline{\chi}, \overline{\psi} \in \mathrm{Irr}(\overline{B})$ are connected, then χ, ψ are also connected as characters of G. Then, there is a unique block B of G such that $\mathrm{Irr}(\overline{B}) \subseteq \mathrm{Irr}(B)$. Moreover, since $\overline{\varphi} \in \mathrm{IBr}(\overline{B})$ if and only if there exists $\overline{\chi} \in \mathrm{Irr}(\overline{B})$ such that $d_{\overline{\chi}\overline{\varphi}} \neq 0$, we have that $\mathrm{IBr}(\overline{B}) \subseteq \mathrm{IBr}(B)$.

THEOREM 1.43. *Let $N \lhd G$ and write $\overline{G} = G/N$.*

(a) Suppose that $\overline{B} \subseteq B$, where $\overline{B}$ is a block of $\overline{G}$ and B is a block of G. If $\overline{D}$ is a defect group of $\overline{B}$, then there is a defect group P of B such that $\overline{D} \subseteq PN/N$.

(b) If N is a p-group, then every block $B \in \mathrm{Bl}(G)$ contains a block $\overline{B} \in \mathrm{Bl}(\overline{G})$ such that $\delta(\overline{B}) = \{P/N \mid P \in \delta(B)\}$.

(c) If N is a p'-group and $\overline{B} \subseteq B$, where $\overline{B}$ is a block of $\overline{G}$ and B is a block of G, then $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$, $\mathrm{IBr}(B) = \mathrm{IBr}(\overline{B})$ and $\delta(\overline{B}) = \{PN/N \mid P \in \delta(B)\}$.

PROOF. See Theorem 9.9 of [**Nav98a**]. □

The following ends the preliminaries of this thesis.

THEOREM 1.44. *Suppose that G has a normal p-subgroup P such that $G/\mathbf{C}_G(P)$ is a p-group. Write $\overline{G} = G/P$. If $\overline{B} \in \mathrm{Bl}(\overline{G})$ and $B \in \mathrm{Bl}(G)$ is the unique block of G containing $\overline{B}$, then the map $\overline{B} \mapsto B$ is a bijection $\mathrm{Bl}(\overline{G}) \to \mathrm{Bl}(G)$. Also, $\mathrm{IBr}(\overline{B}) = \mathrm{IBr}(B)$, $\delta(\overline{B}) = \{D/P \mid D \in \delta(B)\}$ and the Cartan matrices of $\overline{B}$ and B are related by $C_B = |P|C_{\overline{B}}$.*

PROOF. See Theorem 9.10 of [**Nav98a**]. □

CHAPTER 2

p-blocks relative to a character of a normal subgroup

2.1. Introduction

As we said in the Introduction of this thesis, an important part of this work is devoted to the study of the set $\mathrm{Irr}(G|\theta)$ of irreducible constituents of the induced character θ^G, where θ is a G-invariant character of a normal subgroup N of G. In this Chapter we look at this set from a block-theoretical point of view. In particular, if p is our fixed prime number, we partition the set $\mathrm{Irr}(G|\theta)$ into smaller sets relative to the prime p, which look like Brauer p-blocks.

One of the motivations of the work in this Chapter comes from the Gluck-Wolf-Navarro-Tiep theorem ([**GW84**] and [**NT13**]), and the fact that its converse is not true. This result, quite deep, and crucial in the reduction of Brauer's Height Zero Conjecture (Conjecture 1.39) to a problem on simple groups, asserts the following.

THEOREM 2.1 (Gluck-Wolf, Navarro-Tiep). *Let G be a finite group, let N be a normal subgroup of G and let $\theta \in \mathrm{Irr}(N)$. Let p be a prime number and suppose that for all $\chi \in \mathrm{Irr}(G|\theta)$, p does not divide $\chi(1)/\theta(1)$. Then G/N has abelian Sylow p-subgroups.*

PROOF. See Theorem A of [**NT13**]. □

Of course, the converse is not true. For instance, if we let $G = S_3$, the symmetric group on three letters, $p = 2$ and $N = 1$, then we have that G has abelian Sylow 2-subgroups and there is $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 2$. The reason why the converse of this theorem is not true is that the set $\mathrm{Irr}(G|\theta)$ is too big. We want to reformulate this result, replacing $\mathrm{Irr}(G|\theta)$ by a smaller set, in order to have an if and only if.

To do so, we introduce a partition of $\mathrm{Irr}(G|\theta)$, associated to the prime number p which we have fixed, closely related to the classical partition of $\mathrm{Irr}(G)$ into Brauer p-blocks. We will call the members of this partition *θ-blocks*, and to each θ-block we will associate a conjugacy class of p-subgroups of G/N that we will call *θ-defect groups*. We will usually write B_θ to denote a θ-block and D_θ/N to denote a θ-defect group.

Of course, a natural candidate for a θ-block would be $\mathrm{Irr}(B|\theta) = \mathrm{Irr}(B) \cap \mathrm{Irr}(G|\theta)$, where B is any p-block. This would be the set of the irreducible characters in B lying over θ. Unfortunately, this set is, in general, too big. For instance, if G is a p-constrained group (that is, if $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$), then mimicking the proof of Corollary 15.40 of [**Isa76**], we have that G has only one p-block B and hence $\mathrm{Irr}(B|\theta) = \mathrm{Irr}(G|\theta)$ for any normal subgroup N of G and any $\theta \in \mathrm{Irr}(N)$. We want a finer partition.

Both the θ-blocks and their θ-defect groups are going to be defined in terms of some convenient central extensions of G and some projective representations of G associated with θ. In the first main result of this Chapter (and this thesis) we will show that the θ-blocks are independent of any choice that has been made in order to define them, as are their θ-defect groups. In other words, the partition given by θ-blocks and their θ-defect groups are *canonical.*

Theorem A. *Suppose that $N \lhd G$, and $\theta \in \mathrm{Irr}(N)$ is G-invariant. Then the θ-blocks of G are well defined. Furthermore, the set of θ-defect groups is a G/N-conjugacy class of p-subgroups of G/N.*

Using θ-blocks, we unify in the following statement both Brauer's Height Zero Conjecture and the Gluck-Wolf-Navarro-Tiep theorem. Recall that n_p is the largest power of p that divides the integer n.

Conjecture B. *Let (G, N, θ) be a character triple. Suppose that $B_\theta \subseteq \mathrm{Irr}(G|\theta)$ is a θ-block with θ-defect group D_θ/N. Assume that θ extends to D_θ. Then $(\chi(1)/\theta(1))_p = |G : D_\theta|_p$ for all $\chi \in B_\theta$ if and only if D_θ/N is abelian.*

If B is a p-block of G, recall that we denote by $\mathrm{Irr}(B|\theta)$ the subset of $\mathrm{Irr}(B)$ consisting on those characters lying over θ. As we shall show, for each θ-block B_θ there exists a unique p-block B of G such that $\mathrm{Irr}(B_\theta) \subseteq \mathrm{Irr}(B|\theta)$, and if D_θ/N is a θ-defect group of B_θ, we will see that there exists D, a defect group of B, such that $D_\theta/N \leqslant DN/N$. In the important case where N is central we have even more. In this case, we will prove that $\mathrm{Irr}(B_\theta) = \mathrm{Irr}(B|\theta)$ and $D_\theta/N = DN/N$. Using this, Conjecture B is then equivalent to a projective version of the Height Zero conjecture, noticed by Malle and Navarro in [**MN17**]. After being proposed, this projective version of the Brauer's Height Zero Conjecture has been proved to be equivalent to the original Brauer's conjecture by Sambale in [**Sam19**]. Using his work, we prove the following.

Theorem C. *Conjecture B and Brauer's Height Zero conjecture are equivalent.*

As we said, our definition of θ-blocks is related to projective representations, and therefore with blocks of twisted group algebras. Of course, these have been studied before by many authors (including S. B. Conlon [**Con64**], W. F. Reynolds [**Rey66**], J. F. Humphreys [**Hum77**], E. C. Dade [**Dad94**], and A. Laradji [**Lar15**] in the p-solvable case). However, our character theoretical approach is new and is specifically tailored to be used in the recent developments of the global-local counting conjectures.

Our second motivation to introduce θ-blocks is to have a better understanding of the celebrated Brauer's $k(B)$-conjecture (Conjecture 1.40). As it is well-known, this deep conjecture, that asserts that the number of ordinary characters in a block is less than or equal the size of its defect groups, remains not only unsolved but also unreduced to simple groups. We propose the following projective version.

Conjecture D. *Let (G, N, θ) be a character triple. Let B_θ be a θ-block and let D_θ/N be a θ-defect group of B_θ. Then*

$$|\mathrm{Irr}(B_\theta)| \leqslant |D_\theta/N|.$$

It is clear that Conjecture D implies Brauer's $k(B)$-conjecture (we just need to take $N = 1$). Using a result of Navarro in [**Nav17**] we can prove more.

Theorem E. *The $k(B)$-conjecture is true for every finite group if and only if Conjecture D is true for every character triple (G, N, θ).*

Let $\mathrm{cf}(G|\theta)$ be the $\mathbb{C}$-span of $\mathrm{Irr}(G|\theta)$ and let $G^\circ = \{x \in G \mid x_p \in N\}$. If $\delta \in \mathrm{cf}(G|\theta)$, write δ° for the restriction of δ to G°. Write $\mathrm{cf}(G|\theta)^\circ$ for the space consisting of the functions δ°, where $\delta \in \mathrm{cf}(G|\theta)$. It turns out that $\mathrm{cf}(G|\theta)^\circ$ is a $\mathbb{C}$-vector space and in [**Nav00**] and [**Nav12**], Navarro gives a basis (a priori not canonical) of $\mathrm{cf}(G|\theta)^\circ$. Using this basis, he gives a partition of $\mathrm{Irr}(G|\theta)$. The second part of this Chapter is devoted to proving that the partition given by Navarro and our partition into θ-blocks coincide. As a consequence of this, in the case that N is a p-group, we have that there is a direct relationship between our θ-blocks and the Külshammer-Robinson N-projective characters defined in [**KR87**].

In order to prove that our θ-blocks coincide with those found by Navarro, a certain new understanding of Brauer p-blocks is required. Specifically, we shall need to prove certain projective results like the following one.

Theorem F. *Suppose that Z is a central subgroup of G, and let $\theta \in \mathrm{Irr}(Z)$. Let B be a Brauer p-block of G. For $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G)$, denote by $d_{\chi\varphi}$ the classical decomposition number. Then the decomposition matrix $D_\theta = (d_{\chi\varphi})$, where $\chi \in \mathrm{Irr}(B|\theta)$ and $\varphi \in \mathrm{IBr}(B)$ is not of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

for any ordering of the rows and columns.

Moreover, using the θ-blocks, we will prove that the basis given by Navarro is canonical (up to choice of the maximal ideal M containing p in the ring of algebraic integers $\mathbf{R}$ of the complex numbers). This is the last main theorem of this Chapter.

THEOREM G. *Let (G, N, θ) be a character triple, then there exists a canonical basis (up to choice of the maximal ideal M) of* $\mathrm{cf}(G|\theta)^{\circ}$.

This Chapter is structured as follows. In Section 2.2, we give some results on ordinary blocks that we will use later on. In Section 2.3 we fix some notation. In Section 2.4 we give the definitions of θ-blocks and θ-defect groups and we prove Theorem A. In Section 2.5 we present some properties of the θ-blocks. In particular, we prove a θ-version of a classical theorem on blocks: if B_θ is a θ-block, D_θ/N is a θ-defect group of B_θ and $\chi \in \mathrm{Irr}(B_\theta)$, then $\chi(g) = 0$ if g_pN is not G/N-conjugate to any element of D_θ/N. In Section 2.6, we prove Theorems C and E, and we prove that Conjecture B implies the Gluck-Wolf Navarro-Tiep theorem. In Section 2.7, we prove Theorem F and in Section 2.8 we define canonical θ-Brauer characters, we prove Theorem G and we prove that our θ-blocks and the blocks defined by Navarro in [**Nav00**] and [**Nav12**] coincide. In Section 2.9, we define a θ-linking and we give some properties of it.

Part of the results in this Chapter appear in [**Riz18**].

2.2. Results on ordinary blocks

We shall need some basic facts on Brauer p-blocks which we prove in this section.

LEMMA 2.2. *Let B be a p-block of G and let μ be a linear character of G. Then*

$$\{\mu\chi \mid \chi \in \mathrm{Irr}(B)\}$$

is the set of irreducible ordinary characters lying in a p-block μB. Also, B and μB have the same defect groups.

PROOF. Since $\mu(x)$ is a root of unity for $x \in G$, then $\mu(x)\mu(x^{-1}) = 1$. Hence, it is clear that if $\chi, \psi \in \mathrm{Irr}(G)$, then $\lambda_\chi = \lambda_\psi$ if and only if $\lambda_{\mu\chi} = \lambda_{\mu\psi}$, and the first part follows. Now, let K be a defect class of B. Then $\lambda_B(\hat{K}) \neq 0$ and $a_B(\hat{K}) \neq 0$. Notice that $\lambda_{\mu B}(\hat{K}) = \overline{\mu(x_K)}\lambda_B(\hat{K})$, and $a_{\mu B}(\hat{K}) = \overline{\mu(x_K^{-1})}a_B(\hat{K})$, where $x_K \in K$. Since $\overline{\mu(x_K)} \neq 0$, the result follows from Theorem 1.37. □

If N is normal in G, recall that we view the (Brauer) characters of G/N as (Brauer) characters of G containing N in their kernel. We also know that every block of G/N is contained in a block of G. If $\chi \in \mathrm{Irr}(G)$, then we denote by $\mathrm{Bl}(\chi)$ the block of G containing χ.

LEMMA 2.3. *Let $Z \leqslant G$, with $Z = Z_p \times K$, where K is a normal p'-subgroup of G and Z_p is a central p-subgroup of G. Let $\alpha \in \mathrm{Irr}(G)$ with $Z \subseteq \ker(\alpha)$ and write $\overline{\alpha}$ for the character α viewed as a character of G/Z.*

(a) *Let $\beta \in \mathrm{Irr}(G)$ with $Z \subseteq \ker(\beta)$. Then $\mathrm{Bl}(\alpha) = \mathrm{Bl}(\beta)$ if and only if $\mathrm{Bl}(\overline{\alpha}) = \mathrm{Bl}(\overline{\beta})$, where $\overline{\beta}$ is the character β viewed as a character of G/Z.*

(b) *We have that*

$$\delta(\mathrm{Bl}(\overline{\alpha})) = \{PZ/Z \mid P \in \delta(\mathrm{Bl}(\alpha))\} = \{P/Z_p \mid P \in \delta(\mathrm{Bl}(\alpha))\}.$$

PROOF. (a) It is clear that if $\mathrm{Bl}(\overline{\alpha}) = \mathrm{Bl}(\overline{\beta})$, then $\mathrm{Bl}(\alpha) = \mathrm{Bl}(\beta)$. We need to prove the converse. We proceed by induction on $|G|$. By Theorem 1.43 we may assume that Z is a central p-group. The result follows by Theorem 1.44.

For (b), we proceed by induction on $|G|$. Let $\hat{\alpha}$ be the character α viewed as a character of G/K. By Theorem 1.43(c), we have that

$$\delta(\mathrm{Bl}(\hat{\alpha})) = \{PK/K \mid P \in \delta(\mathrm{Bl}(\alpha))\}.$$

If $K > 1$, since $G/Z \cong \frac{G/K}{Z/K}$, using induction we are done. Hence, we may assume that $K = 1$. In this case, Z is a central p-group. The result now follows by Theorem 1.44. □

Suppose $\alpha : \hat{G} \to G$ is a surjective group homomorphism with kernel Z. If $\psi \in \mathrm{Irr}(G)$, whenever is convenient, we denote by ψ^α the unique irreducible character of $\hat{G}$ such that $\psi^\alpha(x) = \psi(\alpha(x))$ for $x \in \hat{G}$. Notice that $Z \subseteq \ker(\psi^\alpha)$.

COROLLARY 2.4. *Suppose that $\alpha : \hat{G} \to G$ is an onto group homomorphism with $\ker(\alpha) = Z \subseteq \boldsymbol{Z}(\hat{G})$.*

(a) *If $\chi_i \in \mathrm{Irr}(G)$, then χ_1, χ_2 lie in the same block of G if and only if χ_1^α and χ_2^α lie in the same block of $\hat{G}$.*

(b) *Suppose that $L \leqslant G$ and let $\gamma \in \mathrm{Irr}(L)$. If $\hat{L} = \alpha^{-1}(L)$, let $\hat{\gamma} = \gamma^{\alpha_{\hat{L}}} \in \mathrm{Irr}(\hat{L})$. Then $[(\chi^\alpha)_{\hat{L}}, \hat{\gamma}] = [\chi_L, \gamma]$ for $\chi \in \mathrm{Irr}(G)$.*

(c) *Suppose that $\chi \in \mathrm{Irr}(G)$, let $B = \mathrm{Bl}(\chi)$, and let $\hat{B} = \mathrm{Bl}(\chi^\alpha)$. If $\hat{D}$ is a defect group of $\hat{B}$, then $\alpha(\hat{D})$ is a defect group of B.*

PROOF. Part (a) is a direct consequence of Lemma 2.3(a). Part (b) is straightforward. To prove part (c), define $\bar{\alpha} : \hat{G}/Z \to G$ to be the associated isomorphism. Since $Z \subseteq \ker(\chi^\alpha)$, by Lemma 2.3(b), we have that $\hat{D}Z/Z$ is a defect group of the block of χ^α viewed as a character of $\hat{G}/Z$. Since $\alpha(\hat{D}) = \bar{\alpha}(\hat{D}Z/Z)$, the result follows. □

We need the following result of [NS14].

Lemma 2.5. *Let $N \lhd G$, let $\theta \in \mathrm{Irr}(N)$ and suppose that $\tilde{\theta} \in \mathrm{Irr}(G)$ is an extension of θ. Write $\overline{G} = G/N$. Let $\overline{\eta} \in \mathrm{Irr}(\overline{G})$ and let $\eta \in \mathrm{Irr}(G)$ the corresponding character of G satisfying $\eta(g) = \overline{\eta}(gN)$ for all $g \in G$. If $x \in G$, let $H/N = \mathbf{C}_{\overline{G}}(\overline{x})$, where $\overline{x} = xN$. Let $K = x^G$, $L = x^H$ and $S = \overline{x}^{\overline{G}}$. Then,*

$$\lambda_{\tilde{\theta}\eta}(\hat{K}) = \lambda_{\tilde{\theta}_H}(\hat{L})\lambda_{\overline{\eta}}(\hat{S}).$$

Moreover, some defect group of $\mathrm{Bl}(\overline{\eta})$ is contained in DN/N, where D is a defect group of $\mathrm{Bl}(\tilde{\theta}\eta)$.

Proof. See Lemma 2.2 of [**NS14**] and Proposition 2.5(b) of [**NS14**]. □

Lemma 2.6. *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Suppose that $\chi_N = \theta \in \mathrm{Irr}(N)$. Let $\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta)$ and write $\chi_i = \beta_i\chi$, for $i = 1, 2$, where $\beta_i \in \mathrm{Irr}(G/N)$. Suppose that β_1 and β_2 lie in the same p-block of G/N. Then χ_1 and χ_2 lie in the same p-block of G. Also, if $\beta \in \mathrm{Irr}(G/N)$ and P/N is a defect group of $\mathrm{Bl}(\beta)$, then $P \subseteq DN$, for some defect group D of $\mathrm{Bl}(\beta\chi)$.*

Proof. Let K be a conjugacy class of G and let $x \in K$. Write $H/N = \mathbf{C}_{G/N}(xN)$, let L be the conjugacy class of H containing x, and let S be the conjugacy class of G/N containing xN. Then, by Lemma 2.5, we have that

$$\lambda_{\chi_1}(\hat{K}) = \lambda_{\chi\beta_1}(\hat{K}) = \lambda_{\chi_H}(\hat{L})\lambda_{\beta_1}(\hat{S}).$$

Since β_1 and β_2 lie in the same p-block of G/N, we have that

$$\lambda_{\beta_1}(\hat{S}) = \lambda_{\beta_2}(\hat{S}),$$

and hence, again by Lemma 2.5 we have that

$$\lambda_{\chi_1}(\hat{K}) = \lambda_{\chi_H}(\hat{L})\lambda_{\beta_1}(\hat{S}) = \lambda_{\chi_H}(\hat{L})\lambda_{\beta_2}(\hat{S}) = \lambda_{\chi\beta_2}(\hat{K}) = \lambda_{\chi_2}(\hat{K}).$$

Hence χ_1 and χ_2 lie in the same p-block of G. The second part follows straightforwardly from the second part of Lemma 2.5. □

2.3. The Notation

In this Section we give some notation that we will keep throughout this Chapter.

To define the θ-blocks and the θ-defect groups we need some background on projective representations and character triples. That background can be found in Section 1.2. In particular we are going to need Theorem 1.21. We recall here some of the notation and definitions needed.

Theorem 1.21 asserts that given a character triple (G, N, θ), there exists a projective representation $\mathcal{P}$ of G such that its factor set only takes roots of unity values, and a character triple $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ (depending on $\mathcal{P}$) isomorphic to (G, N, θ), with $\hat{N}/N$ central in $\hat{G}/N$. We say that $\hat{G}$ is a *representation group* for (G,N,θ), $(\hat{G}/N,\hat{N}/N,\hat{\lambda})$ is a *standard isomorphic character triple* given by $\mathcal{P}$, and the bijection $^*:\mathrm{Irr}(G|\theta)\mapsto\mathrm{Irr}(\hat{G}/N|\hat{\lambda})$ is the *standard bijection*. Notice that this bijection just depends on the choice of the projective representation $\mathcal{P}$ (see discussion after Theorem 1.23). We also say that $\tau\in\mathrm{Irr}(\hat{G})$ is the character of $\hat{G}$ *associated* with $\mathcal{P}$. By Theorem 1.21(b), we have that $\tau_N=\theta$.

If α is the factor set of $\mathcal{P}$, recall that we write Z for the subgroup of $\mathbb{C}^\times$ generated by the values of α (which are roots of unity values). Then $\hat{G}=\{(g,z)\,|\,g\in G, z\in Z\}$, with a suitable product involving α, and the map $\pi:\hat{G}\mapsto G$ given by $(g,z)\mapsto g$ is the canonical homomorphism with kernel Z.

2.4. θ-blocks and θ-defect groups

We are finally ready to define θ-blocks and their θ-defect groups.

Definition 2.7. Let (G,N,θ) be a character triple. Let $\hat{G}$ be a representation group for (G,N,θ) and let $\pi:\hat{G}\to G$ be the canonical homomorphism $(g,z)\mapsto g$ with kernel Z. Let $^*:\mathrm{Irr}(G|\theta)\to\mathrm{Irr}(\hat{G}/N|\hat{\lambda})$ be the associated standard bijection. We say that a non-empty subset $B_\theta\subseteq\mathrm{Irr}(G|\theta)$ is a *θ-block* of G if there exists a p-block $\hat{B}$ of $\hat{G}/N$ such that

$$B_\theta^*=\{\chi^*\mid\chi\in B_\theta\}=\mathrm{Irr}(\hat{B}|\hat{\lambda})\,.$$

If $\hat{D}/N$ is a defect group of $\hat{B}$, then we say that $\pi(\hat{D})/N$ is a *θ-defect group of B_θ*.

In the situation of Definition 2.7 we say that the θ-block B_θ *is afforded* by the p-block $\hat{B}$.

Of course, note that the definition of θ-blocks depends on the choice of the standard isomorphic character triple and therefore on the choice of the projective representation associated with θ. The same happens with the θ-defect groups. Our main result in this section is that the θ-blocks are in fact canonically defined, and that all the θ-defect groups are G/N-conjugate. The following result is key to proving that.

Theorem 2.8. *Let (G,N,θ) be a character triple. Let $\mathcal{P}_1,\mathcal{P}_2$ be projective representations of G associated with θ, with factor sets α_1 and α_2, respectively, whose values are roots of unity. Let $\hat{G}_i$ be the representation group associated with $\mathcal{P}_i$. Let $(\hat{G}_1/N,\hat{N}_1/N,\hat{\lambda}_1)$ and $(\hat{G}_2/N,\hat{N}_2/N,\hat{\lambda}_2)$ be the standard isomorphic character triples given by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. As usual, let Z_i be the subgroup of the multiplicative group $\mathbb{C}^\times$ generated by the values of α_i. Let $\hat{G}=G\times Z_1\times Z_2$ and define the product*

$$(g,z_1,z_2)(h,z_1',z_2')=(gh,\alpha_1(g,h)z_1z_1',\alpha_2(g,h)z_2z_2').$$

Then the following hold.

(a) *$\hat{G}$ is a finite group, $N \times 1 \times 1$ is a normal subgroup of $\hat{G}$ (which we identify with N), and $1 \times Z_1 \times Z_2$ is a central subgroup of $\hat{G}$ (which we identify with $Z_1 \times Z_2$). Also, $\hat{N} = N \times Z_1 \times Z_2$ is a normal subgroup of $\hat{G}$ and $\hat{N}/N$ is central in $\hat{G}/N$.*

(b) *The maps $\rho_1 : \hat{G} \to \hat{G}_1$ and $\rho_2 : \hat{G} \to \hat{G}_2$ given by $(g, z_1, z_2) \mapsto (g, z_1)$ and $(g, z_1, z_2) \mapsto (g, z_2)$ are surjective group homomorphisms with kernels Z_2 and Z_1, respectively.*

(c) *Suppose that $\tau_i \in \mathrm{Irr}(\hat{G}_i)$ is the character associated with $\mathcal{P}_i$, and let $\tau_i^{\rho_i} \in \mathrm{Irr}(\hat{G})$ be the corresponding character of $\hat{G}$. Then there exists a linear character $\beta \in \mathrm{Irr}(\hat{G}/N)$ such that*

$$\tau_1^{\rho_1} = \beta \tau_2^{\rho_2} .$$

(d) *Let $\chi \in \mathrm{Irr}(G|\theta)$ and let $\chi_i^* \in \mathrm{Irr}(\hat{G}_i/N|\hat{\lambda}_i)$ be the image of χ under the standard bijection. Let $\hat{\chi}_i = (\chi_i^*)^{\rho_i} \in \mathrm{Irr}(\hat{G}/N)$. Then $\beta\hat{\chi}_1 = \hat{\chi}_2$. As a consequence, if $\hat{B}_i$ is the block of $\hat{G}/N$ containing $\hat{\chi}_i$, then $\hat{B}_2 = \beta\hat{B}_1$.*

(e) *Let B_i^* be the block of $\hat{G}_i/N$ containing χ_i^*. Then the map $\psi \mapsto \psi^{\rho_i}$ is a bijection from $\mathrm{Irr}(B_i^*|\hat{\lambda}_i)$ to $\mathrm{Irr}(\hat{B}_i|\tilde{\lambda}_i)$, where $\tilde{\lambda}_1(n, z_1, z_2) = \hat{\lambda}_1(1, z_1) = z_1^{-1}$ and $\tilde{\lambda}_2(n, z_1, z_2) = \hat{\lambda}_2(1, z_2) = z_2^{-1}$ are linear characters of $\hat{N}/N$.*

(f) *The map $\psi \mapsto \beta\psi$ is a bijection from $\mathrm{Irr}(\hat{B}_1|\tilde{\lambda}_1)$ to $\mathrm{Irr}(\hat{B}_2|\tilde{\lambda}_2)$. In particular, $|\mathrm{Irr}(B_1^*|\hat{\lambda}_1)| = |\mathrm{Irr}(B_2^*|\hat{\lambda}_2)|$.*

(g) *Let $\pi_i : \hat{G}_i \to G$ be the canonical homomorphism $(g, z_i) \mapsto g$ with kernel Z_i. If $\hat{D}_i/N$ is defect group of B_i^*, then $\pi_1(\hat{D}_1)$ and $\pi_2(\hat{D}_2)$ are G-conjugate.*

Proof. Using Lemma 1.19, parts (a) and (b) are straightforward. We prove (c). Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are projective representations of G associated to θ, by Theorem 1.18 we know that there exists $\xi : G \to \mathbb{C}^\times$ with $\xi(1) = 1$, constant on the cosets of N, such that $\mathcal{P}_2 = \xi\mathcal{P}_1$, and the factor sets α_1 and α_2 are related in this way

$$\alpha_2(g, h) = \alpha_1(g, h)\xi(g)\xi(h)\xi(gh)^{-1},$$

for all $g, h \in G$.

Now $\tau_i \in \mathrm{Irr}(\hat{G}_i)$ is the character afforded by the irreducible representation $\hat{\mathcal{P}}_i$, which is defined by $\hat{\mathcal{P}}_i(g, z_i) = z_i\mathcal{P}_i(g)$, for $z_i \in Z_i$ and $g \in G$. Then, using that $\mathcal{P}_2 = \xi\mathcal{P}_1$, we have that

$$\tau_1(g, z_1) = z_1 z_2^{-1} \xi(g)^{-1} \tau_2(g, z_2)$$

for $g \in G$ and $z_i \in Z_i$. It is straightforward to prove that the function $\beta : \hat{G} \to \mathbb{C}^\times$ defined by

$$\beta(g, z_1, z_2) = z_1 z_2^{-1} \xi(g)^{-1}$$

is a linear character of $\hat{G}$ that contains N in its kernel.

By definition, we have that $\tau_1^{\rho_1}(g, z_1, z_2) = \tau_1(g, z_1)$ and $\tau_2^{\rho_2}(g, z_1, z_2) = \tau_2(g, z_2)$. Therefore $\tau_1^{\rho_1} = \beta\tau_2^{\rho_2}$, as desired. This proves (c).

Let us denote by $\pi_i : \hat{G}_i \to G$ the homomorphism $(g, z_i) \mapsto g$. Recall that, by definition, $\chi_i^* \in \mathrm{Irr}(\hat{G}_i/N)$ is the unique character satisfying $\chi^{\pi_i} = \chi_i^*\tau_i$. That is

$$\chi(g) = \chi^{\pi_i}(g, z_i) = \chi_i^*(g, z_i)\tau_i(g, z_i)$$

for $g \in G$ and $z_i \in Z_i$. By definition, we have that $\hat{\chi}_1(g, z_1, z_2) = \chi_1^*(g, z_1)$ and $\hat{\chi}_2(g, z_1, z_2) = \chi_2^*(g, z_2)$. In particular, $\hat{\chi}_i \in \mathrm{Irr}(\hat{G})$ contains N in its kernel. Notice that we have $\tau_1^{\rho_1}\hat{\chi}_1 = \tau_2^{\rho_2}\hat{\chi}_2$. Hence,

$$\beta\hat{\chi}_1\tau_2^{\rho_2} = \hat{\chi}_2\tau_2^{\rho_2} .$$

Since $\tau_2^{\rho_2}$ extends $\theta \in \mathrm{Irr}(N)$ and $\beta\hat{\chi}_1, \hat{\chi}_2 \in \mathrm{Irr}(\hat{G}/N)$, by Gallagher's Corollary (Corollary 1.16), we have that

$$\beta\hat{\chi}_1 = \hat{\chi}_2 .$$

Using Lemma 2.2, part (d) easily follows.

Next we prove part (e). Since $\rho_1(N) = N$, then ρ_1 uniquely defines an onto homomorphism $\tilde{\rho}_1 : \hat{G}/N \to \hat{G}_1/N$ with kernel $NZ_2/N \subseteq \mathbf{Z}(\hat{G}/N)$. Since $N \subseteq \ker(\chi_1^*)$, then notice that $\hat{\chi}_1 = (\chi_1^*)^{\tilde{\rho}_1}$. Now NZ_1/N is a subgroup of $\hat{G}_1/N$, and its inverse image under $\tilde{\rho}_1$ is $\hat{N}/N$. Also, the character corresponding to $\hat{\lambda}_1$ under $\tilde{\rho}_1$ is $\tilde{\lambda}_1$. By Corollary 2.4 (a) and (b), we have that $\psi \mapsto \psi^{\rho_i}$ is a bijection from $\mathrm{Irr}(B_i^*|\hat{\lambda}_i)$ to $\mathrm{Irr}(\hat{B}_i|\tilde{\lambda}_i)$. (Notice that $\psi^{\rho_i} = \psi^{\tilde{\rho}_i}$ because all of our characters have N in their kernel).

Now we prove part (f). By using their definitions (and the fact that $\xi(n) = 1$ for $n \in N$), we check that $\beta_{\hat{N}}\tilde{\lambda}_1 = \tilde{\lambda}_2$. Therefore, multiplication by the linear character β sends bijectively $\mathrm{Irr}(\hat{B}_1|\tilde{\lambda}_1) \to \mathrm{Irr}(\hat{B}_2|\tilde{\lambda}_2)$.

Finally, we prove part (g). As in part (e), we have that $\tilde{\rho}_i : \hat{G}/N \to \hat{G}_i/N$ is an onto homomorphism, with central kernel, such that the map $\psi \mapsto \psi^{\tilde{\rho}_i}$ is a bijection from $\mathrm{Irr}(B_i^*|\lambda_i)$ to $\mathrm{Irr}(\hat{B}_i|\tilde{\lambda}_i)$. Let E_i/N be a defect group of $\hat{B}_i$. Since $\hat{B}_2 = \beta\hat{B}_1$, we may assume that $E_i = E$ for $i = 1, 2$. by Lemma 2.2. By Corollary 2.4(c), we have that $\tilde{\rho}_i(E/N)$ is a defect group of B_i^*. Hence $\tilde{\rho}_i(E/N) = (\hat{D}_i/N)^{(g_i,1)}$ for some $g_i \in G$ (using that Z_i is central in $\hat{G}_i$). Now, since $\pi_i(N) = N$, we have that π_i uniquely determines an onto homomorphism $\tilde{\pi}_i : \hat{G}_i/N \to G/N$. We easily check that $\tilde{\pi}_1 \circ \tilde{\rho}_1 = \tilde{\pi}_2 \circ \tilde{\rho}_2$. Then

$$\pi_1(\hat{D}_1)^{g_1} = \pi_2(\hat{D}_2)^{g_2} ,$$

as desired. □

We can now prove the main result of this section. The following is Theorem A of the Introduction.

THEOREM 2.9. *Suppose that $N \lhd G$, and $\theta \in \mathrm{Irr}(N)$ is G-invariant. Then the θ-blocks of G are well defined. Furthermore, the set of θ-defect groups is a G/N-conjugacy class of p-subgroups of G/N.*

PROOF. Let (G, N, θ) be a character triple and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations associated with θ. Let $\hat{G}_1$ and $\hat{G}_2$ be representation groups for (G, N, θ) given by $\mathcal{P}_1$ and $\mathcal{P}_2$, and let $(\hat{G}_1/N, \hat{N}_1/N, \hat{\lambda}_1)$ and $(\hat{G}_2/N, \hat{N}_2/N, \hat{\lambda}_2)$ be the standard isomorphic character triples. Let $\pi_i : \hat{G}_i \to G$ be the homomorphism $\pi_i(g, z_i) = g$, and let $\tau_i \in \mathrm{Irr}(\hat{G}_i)$ be the character associated with $\mathcal{P}_i$. Recall that if $\chi \in \mathrm{Irr}(G|\theta)$, then $\chi^{\pi_i} = \chi_i^* \tau_i$, for some uniquely defined $\chi_i^* \in \mathrm{Irr}(\hat{G}_i/N)$. The map $\chi \mapsto \chi_i^*$ from $\mathrm{Irr}(G|\theta)$ to $\mathrm{Irr}(\hat{G}_i/N|\hat{\lambda}_i)$ is the standard bijection.

Let $A_1, A_2 \subseteq \mathrm{Irr}(G|\theta)$ be such that $A_1^* = \{\varphi_1^* \mid \varphi \in A_1\} = \mathrm{Irr}(B_1^*|\hat{\lambda}_1)$ and $A_2^* = \{\varphi_2^* \mid \varphi \in A_2\} = \mathrm{Irr}(B_2^*|\hat{\lambda}_2)$, where B_i^* is a block of $\hat{G}_i/N$. Suppose that $\chi \in A_1 \cap A_2$. We wish to to prove that $A_1 = A_2$.

In order to do so, we construct the group $\hat{G}$ as in Theorem 2.8, and consider the group homomorphisms $\rho_i : \hat{G} \to \hat{G}_i$, in Theorem 2.8(b). By Theorem 2.8(c), there is a linear character $\beta \in \mathrm{Irr}(\hat{G}/N)$ satisfying

$$\tau_1^{\rho_1} = \beta \tau_2^{\rho_2} .$$

As in Theorem 2.8(d), let $\hat{\chi}_i = (\chi_i^*)^{\rho_i} \in \mathrm{Irr}(\hat{G}/N)$, and let $\hat{B}_i$ be the block of $\hat{G}/N$ containing $\hat{\chi}_i$. By Theorem 2.8(d), we have that $\hat{B}_2 = \beta \hat{B}_1$. By Theorem 2.8(f), $|A_1^*| = |A_2^*|$, and therefore $|A_1| = |A_2|$. We only need to prove that $A_1 \subseteq A_2$, for instance.

Let $\varphi_1 \in A_1$. Now, $\varphi_1^* \in A_1^* = \mathrm{Irr}(B_1^*|\hat{\lambda}_1)$, and by Theorem 2.8(e) we have that $\hat{\varphi}_1 = (\varphi_1^*)^{\rho_1} \in \mathrm{Irr}(\hat{B}_1|\tilde{\lambda}_1)$. By Theorem 2.8(f), $\beta\hat{\varphi}_1 \in \mathrm{Irr}(\hat{B}_2|\tilde{\lambda}_2)$. By Theorem 2.8(e), let $\varphi_2 \in A_2$ be such that $\beta\hat{\varphi}_1 = (\varphi_2^*)^{\rho_2}$. We claim that $\varphi_1 = \varphi_2$. Recall that $\tau_i \varphi_i^* = \varphi_i^{\pi_i}$and that $\tau_1^{\rho_1} = \beta\tau_2^{\rho_2}$. If $g \in G$, then we have that

$$\begin{aligned}
\varphi_1(g) &= \varphi_1^{\pi_1}(g, 1) = \tau_1(g, 1)\varphi_1^*(g, 1) \\
&= \tau_1^{\rho_1}(g, 1, 1)\hat{\varphi}_1(g, 1, 1) \\
&= \beta(g, 1, 1)\tau_2^{\rho_2}(g, 1, 1)\hat{\varphi}_1(g, 1, 1) \\
&= \tau_2(g, 1)(\beta\hat{\varphi}_1)(g, 1, 1) \\
&= \tau_2(g, 1)(\varphi_2^*)^{\rho_2}(g, 1, 1) \\
&= \tau_2(g, 1)\varphi_2^*(g, 1) \\
&= \varphi_2^{\pi_2}(g, 1) \\
&= \varphi_2(g),
\end{aligned}$$

as desired. This completes the proof of the first part of the theorem. The second part easily follows from Theorem 2.8(g). It is elementary to show that the θ-defect groups are p-subgroups of G/N. □

2.5. Some properties of the θ-blocks

We collect in this Section some basic properties of the θ-blocks and its θ-defect groups.

THEOREM 2.10. *Let (G, N, θ) be a character triple. Let B_θ be a θ-block of G, and let D_θ/N be a θ-defect group of B_θ.*

(a) There is a p-block B of G such that B_θ is contained in the set $\mathrm{Irr}(B|\theta)$. Also, there is a defect group D of B such that $D_\theta \subseteq DN$.
(b) If $N \subseteq \mathbf{Z}(G)$, then there is a p-block B of G and a defect group D of B such that $B_\theta = \mathrm{Irr}(B|\theta)$, and $D_\theta = DN$.
(c) If θ has an extension $\chi \in \mathrm{Irr}(G)$, then there is a p-block $\bar{B}$ of G/N and a defect group $\bar{D}$ of $\bar{B}$ such that $B_\theta = \{\gamma\chi \mid \gamma \in \mathrm{Irr}(\bar{B})\}$ and $D_\theta/N = \bar{D}$.
(d) If G/N is a p-group, then $B_\theta = \mathrm{Irr}(G|\theta)$ and $D_\theta = G$.

PROOF. Let $\hat{G}$ be a representation group associated with (G, N, θ), with associated character $\tau \in \mathrm{Irr}(\hat{G})$. Recall that $\tau_N = \theta$. Let $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$ be the standard bijection. Let $\pi : \hat{G} \to G$ be the homomorphism $(g, z) \mapsto g$. Since $\pi(N) = N$, let $\hat{\pi} : \hat{G}/N \to G/N$ be the corresponding onto homomorphism. Notice that $\hat{G}/N$ is a central extension of G/N.

By definition, there is a Brauer p-block $\hat{B}$ of $\hat{G}/N$ such that $(B_\theta)^* = \mathrm{Irr}(\hat{B}|\hat{\lambda})$. Recall that $\chi^\pi = \chi^*\tau$ for $\chi \in \mathrm{Irr}(G|\theta)$.

Now, fix $\chi \in B_\theta$ and let B be the p-block of G containing χ. We claim that $B_\theta \subseteq \mathrm{Irr}(B|\theta)$. Indeed, let $\psi \in B_\theta$. Then $\chi^*, \psi^* \in \hat{B}$. Since $\tau_N = \theta$ and $\chi^\pi = \tau\chi^*$, $\psi^\pi = \tau\psi^*$, by Lemma 2.6 we have that χ^π and ψ^π lie in the same p-block of $\hat{G}$. By Corollary 2.4, χ, ψ lie in the same p-block of G. This proves the first part of (a). If $\hat{D}/N$ is a defect group of $\hat{B}$, by Lemma 2.6 we have that $\hat{D} \subseteq EN$ for some defect group E of the block of χ^π. Now, $\pi(E)$ is a defect group of the block of χ by Corollary 2.4(c), and $\pi(\hat{D}) \subseteq \pi(E)N$. This proves the second part of (a). Notice now that if N is central, then τ is linear and the defect groups of the block of $\chi^\pi = \tau\chi^*$ are the defect groups of the block of χ^* (multiplying by τ^{-1} and using Lemma 2.2). Since N is central in $\hat{G}$ by Theorem 1.21(a), we have that $\hat{D} = EN$ by Lemma 2.3(b).

Next, we complete the proof of part (b). Suppose that N is central and that $\gamma \in \mathrm{Irr}(B|\theta)$. In particular, τ is linear. Write $\gamma^\pi = \gamma^*\tau$, for some $\gamma^* \in \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$. Now, since γ and χ lie in the same p-block of G, we have that γ^π and χ^π lie in the same p-block of $\hat{G}$ by Corollary 2.4. Therefore $\gamma^*\tau$ and $\chi^*\tau$ lie in the same p-block of $\hat{G}$. By Lemma 2.2, multiplying by τ^{-1}, we have that γ^* and χ^* lie in the same p-block of $\hat{G}$. Now, $N \subseteq \mathbf{Z}(\hat{G})$, by Theorem 1.21(a). Thus γ^* and χ^* lie in the same p-block of $\hat{G}/N$ by Lemma 2.3 (a). Hence $\gamma^* \in \mathrm{Irr}(\hat{B}|\hat{\lambda})$, and therefore $\gamma \in B_\theta$. This proves (b). (The part on the defect groups follows from the previous paragraph.)

For part (c), notice that if $\mathcal{P}$ is a representation affording χ, then $\mathcal{P}$ is a projective representation associated with (G, N, θ) with trivial factor set. Hence $\hat{G} = G$ is a representation group for (G, N, θ) with associated character $\tau = \chi$. In this case the standard bijection is the map $\beta\chi \mapsto \beta$ from $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G/N)$ given by Gallagher's Corollary (Corollary 1.16), and part (c) easily follows.

Next we prove part (d). Let $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ be a standard isomorphic character triple and let $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$ be the standard bijection. Let $\hat{B}$ be the p-block of $\hat{G}/N$ such that $(B_\theta)^* = \mathrm{Irr}(\hat{B}|\hat{\lambda})$. By Theorem 1.41 and Theorem 1.42 we have that $\mathrm{Irr}(\hat{B}|\hat{\lambda}) = \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$. Therefore $|B_\theta| = |(B_\theta)^*| = |\mathrm{Irr}(\hat{B}|\hat{\lambda})| = |\mathrm{Irr}(\hat{G}/N|\hat{\lambda})| = |\mathrm{Irr}(G|\theta)|$, and the first part of (d) is proved. Let D_θ/N be a θ-defect group of B_θ and let $\hat{D}/N \leqslant \hat{G}/N$ be a defect group of $\hat{B}$ such that $\pi(\hat{D})/N = D_\theta/N$. Recall that $\hat{\pi} : \hat{G}/N \to G/N$ defined by $(g, z)N \mapsto gN$ is an onto homomorphism with $\ker(\hat{\pi}) = \hat{N}/N$. Write $G^* = \hat{G}/N$, $N^* = \hat{N}/N$ and $D^* = \hat{D}/N$, and write $\tilde{\pi} : G^*/N^* \to G/N$ for the induced isomorphism. Then $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$. Let $K^* \in \mathrm{cl}(G^*)$ be a defect class for $\hat{B}$. By Proposition 1.35, we know that K^* consists of p-regular elements. Since G^*/N^* is a p-group, we have that $K^* \subseteq N^* \subseteq \mathbf{Z}(G^*)$. Let $x^* \in K^*$ be such that $D^* \in \mathrm{Syl}_p(\mathbf{C}_{G^*}(x^*))$. Since K^* is central, we have that $D^* \in \mathrm{Syl}_p(G^*)$. Then $D^*N^*/N^* \in \mathrm{Syl}_p(G^*/N^*)$ and, since $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$ we have that $D_\theta/N \in \mathrm{Syl}_p(G/N)$. Since G/N is p-group, $D_\theta/N = G/N$.

□

Notice that in the proof of this Theorem we have seen that

$$\tilde{\pi}(D^*N^*/N^*) = D_\theta/N.$$

We will use this fact many times in this Chapter.

If (G, N, θ) is a character triple and $B_\theta \subseteq \mathrm{Irr}(G|\theta)$ is a θ-block, then, in general, B_θ is much smaller than the set $\mathrm{Irr}(B|\theta)$, where B is the Brauer p-block containing B_θ. For instance, suppose that G is a p-constrained group, that is, $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$. Then, as we have said in the Introduction of this Chapter, mimicking the proof of Corollary 15.40 of [**Isa76**], we have that G has only one p-block B and hence $\mathrm{IBr}(B|\theta) = \mathrm{IBr}(G|\theta)$ for any normal subgroup N of G and $\theta \in \mathrm{Irr}(N)$. Now, let $N \lhd G$ be such that p does not divide $|G/N|$, and let $\theta \in \mathrm{Irr}(N)$ such that θ extends to G. By Theorem 2.10(c), we have that the θ-blocks have size 1.

The following is a clasical result on blocks that explains further zeros in the character table.

Theorem 2.11. *Let $\chi \in \mathrm{Irr}(G)$ and let $g \in G$. If g_p is not contained in any defect group of the block of χ, then $\chi(g) = 0$.*

Proof. See Corollary 5.9 of [**Nav98a**]. □

The following is an analogue result for θ-blocks.

THEOREM 2.12. *Let (G, N, θ) be a character triple, let $\chi \in \mathrm{Irr}(G|\theta)$ and let B_θ be the θ-block containing χ. Let $g \in G$ and suppose that $(gN)_p$ is not G/N-conjugate to any element of D_θ/N, where D_θ/N is a θ-defect group of B_θ. Then $\chi(g) = 0$.*

PROOF. Let $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ be a standard isomorphic character triple of (G, N, θ). Write $\pi : \hat{G} \to G$ for the canonical onto homomorphism. Since $\pi(N) = N$, π induces a group homomorphism $\hat{\pi} : \hat{G}/N \to G/N$ with $\ker(\hat{\pi}) = \hat{N}/N$. Write $G^* = \hat{G}/N$ and $N^* = \hat{N}/N$ and write $\tilde{\pi} : G^*/N^* \to G/N$ for the induced isomorphism.

Let $gN \in G/N$ and let $g^*N^* \in G^*/N^*$ such that $\tilde{\pi}(g^*N^*) = gN$. Let $\hat{B}$ be the p-block of G^* such that $(B_\theta)^* = \mathrm{Irr}(\hat{B}|\hat{\lambda})$, where $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\hat{\lambda})$ is the standard bijection. Let $D^* = \hat{D}/N$ be the defect group of $\hat{B}$ such that $\pi(\hat{D})/N = D_\theta/N$. Notice that $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$.

Since $(gN)_p$ is not G/N-conjugate to any element of D_θ/N, we have that $(g^*)_pN^*$ is not G^*/N^*-conjugate to any element of D^*N^*/N^*. Hence $(g^*)_p$ is not contained in any defect group of the block of χ^*. By Theorem 2.11 we have that $\chi^*(g^*) = 0$. Recall that $\chi^\pi = \tau\chi^*$, where $\tau \in \mathrm{Irr}(\hat{G})$ is the character associated to $\mathcal{P}$. Since $\tilde{\pi}(((g,1)N)(\hat{N}/N)) = \hat{\pi}((g,1)N) = gN$, we have that $g^* = (g,1)N$ and then

$$\chi(g) = \chi^\pi(g,1) = \tau(g,1)\chi^*((g,1)N) = \tau(g,1)\chi^*(g^*) = 0.$$

□

If B is a p-block of G, D is a defect group of B, and K is a defect class of B, we know that $D \in \mathrm{Syl}_p(\mathbf{C}_G(x_K))$, for some $x_K \in K$. Next, we give an analogue for θ-blocks. To do so, we need the following lemma. Recall that we say that $x \in G$ is θ-good if every extension of θ to $N\langle x\rangle$ is S-invariant, where $S/N = \mathbf{C}_{G/N}(Nx)$. It is not hard to see that if x is θ-good, then so is every G-conjugate to x. In this case, we say that the conjugacy class of x is θ-good.

LEMMA 2.13. *Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If $g \in G$ is not θ-good, then $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(G|\theta)$. Moreover, if θ is linear and faithful, then g is θ-good if and only if $\mathbf{C}_{G/N}(Ng) = \mathbf{C}_G(g)/N$.*

PROOF. See Lemma 5.13 of [**Nav18**]. □

LEMMA 2.14. *Let (G, N, θ) be a character triple and let B be a block of G such that $\mathrm{Irr}(B|\theta)$ is non-empty. Let K be a defect class of B. Then K is θ-good.*

PROOF. Let $x_K \in K$ and $\chi \in \mathrm{Irr}(B|\theta)$. Since K is a defect class of B we have that

$$0 \neq \lambda_B(\hat{K}) = \omega_\chi(\hat{K})^* = \left(\frac{\chi(x_K)|K|}{\chi(1)}\right)^*.$$

Then, $\chi(x_K) \neq 0$ and x_K is θ-good by Lemma 2.13. □

Proposition 2.15. *Let (G, N, θ) be a character triple, let B_θ be a θ-block and let D_θ/N be a θ-defect group of B_θ. Then there is $xN \in G/N$ such that $D_\theta/N \in \mathrm{Syl}_p(\mathbf{C}_{G/N}(xN))$.*

Proof. Let $\hat{D}/N \leqslant \hat{G}/N$ be a defect group of $\hat{B}$ such that $\hat{\pi}(\hat{D}/N) = D_\theta/N$. Write $G^* = \hat{G}/N$ and $N^* = \hat{N}/N$. We know that $\ker(\hat{\pi}) = \hat{N}/N = N^*$. Denote by $\tilde{\pi} : G^*/N^* \to G/N$ the isomorphism induced by $\hat{\pi}$. Write $D^* = \hat{D}/N$. Then we know that $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$.

Since D^* is a defect group of $\hat{B}$ and K is a defect class of $\hat{B}$, we know that there exists $x_K \in K$ such that $D^* \in \mathrm{Syl}_p(\mathbf{C}_{G^*}(x_K))$. Since $N^* \subseteq \mathbf{Z}(G^*)$, we have that $N^* \subseteq \mathbf{C}_{G^*}(x_K)$ and hence $D^*N^*/N^* \in \mathrm{Syl}_p(\mathbf{C}_{G^*}(x_K)/N^*)$. By Lemma 2.14 and Lemma 2.13 we have that $\mathbf{C}_{G^*}(x_K)/N^* = \mathbf{C}_{G^*/N^*}(N^*x_K)$. Hence

$$D_\theta/N = \tilde{\pi}(D^*N^*/N^*) \in \mathrm{Syl}_p(\tilde{\pi}(\mathbf{C}_{G^*/N^*}(N^*x_K))) = \mathrm{Syl}_p(\mathbf{C}_{G/N}(\hat{\pi}(x_K))).$$

□

2.6. Projective conjectures

We start this Section by proving Theorem E of the introduction. Recall that Brauer's $k(B)$-conjecture asserts that if B is a block with defect group D, then $k(B) = |\mathrm{Irr}(B)| \leqslant |D|$.

The key is the following result of Navarro.

Theorem 2.16. *Suppose that Z is a central p-subgroup of G, and let $\lambda \in \mathrm{Irr}(Z)$. Let B be a p-block of G, and let $\overline{B}$ be the unique p-block of G/Z contained in B. Then*

$$k(B|\lambda) \leqslant k(\overline{B})$$

where $k(B|\lambda)$ is the number of irreducible characters in B lying over λ.

Proof. See Theorem C of [**Nav17**]. □

The following is Theorem E of the Introduction.

Theorem 2.17. *The $k(B)$-conjecture is true for every finite group if and only if for every character triple (G, N, θ), we have that every θ-block B_θ has size less than or equal the size of any of its θ-defect groups.*

Proof. Let (G, N, θ) be a character triple and let $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ be a standard isomorphic character triple. Write $G^* = \hat{G}/N$, $N^* = \hat{N}/N$ and $\theta^* = \hat{\lambda}$. Let B_θ be a θ-block and let D_θ/N be a θ-defect group of B_θ. Suppose first that the $k(B)$-conjecture holds for every finite group. Write $N^* = N^*_p \times N^*_{p'}$ where $N^*_p \in \mathrm{Syl}_p(N^*)$, and write $\theta^* = \theta^*_p \times \theta^*_{p'}$, with $\theta^*_p \in \mathrm{Irr}(N^*_p)$ and $\theta^*_{p'} \in \mathrm{Irr}(N^*_{p'})$. From the definition of the θ-blocks, we have that there exists a p-block B^* of G^* such that $B^*_\theta = \mathrm{Irr}(B^*|\theta^*)$, where $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ is the standard bijection. Thus

$$|B_\theta| = |\mathrm{Irr}(B^*|\theta^*)|\,.$$

Now by Theorem 1.41, and using that $\mathrm{Irr}(B^*|\theta^*)$ is not empty (by the definition of θ-blocks), we have that $\mathrm{Irr}(B^*|\theta^*) = \mathrm{Irr}(B^*|\theta^*_p)$. By Theorem 2.16 we have that $|\mathrm{Irr}(B^*|\theta^*_p)| \leqslant |\mathrm{Irr}(\overline{B^*})|$, where $\overline{B^*}$ is the unique p-block of G^*/N^*_p contained in B^*. Let D^* be a defect group of B^*. By Theorem 1.44 we have that D^*/N^*_p is a defect group of $\overline{B^*}$. Since the $k(B)$ conjecture holds for G^*/N^*_p we have that $|\mathrm{Irr}(\overline{B^*})| \leqslant |D^*/N^*_p| = |D^*N^*/N^*|$. Now, write $D^* = \hat{D}/N$, for some subgroup $\hat{D}$ of $\hat{G}$, and by definition, recall that $\pi(\hat{D})/N = D_\theta/N$ is a θ-defect group of B_θ, where $\pi : \hat{G} \to G$ is the onto homomorphism $(g, z) \mapsto g$. It is then enough to show that $|D^*N^*/N^*| = |D_\theta/N|$. Notice that $\hat{\pi} : G^* \to G/N$ defined by $(g, z)N \mapsto gN$ is an onto group homomorphism with kernel N^*. Write $\tilde{\pi} : G^*/N^* \to G/N$ for the isomorphism induced by $\hat{\pi}$, and notice that $\tilde{\pi}(D^*N^*/N^*) = \hat{\pi}(D^*) = \pi(\hat{D})/N = D_\theta/N$. Then D_θ/N and D^*N^*/N^* are isomorphic, and $|D_\theta/N| = |D^*N^*/N^*|$, as desired.

For the converse, simply take $N = 1$ and apply Theorem 2.10(b). □

As we said in Chapter 1, the $k(B)$-conjecture is known to be true for p-solvable groups (it was reduce to the so-called "$k(GV)$-problem" in [**Nag62**] and it was finally proved in [**GMRS**]). Notice that if (G, N, θ) is a character triple and G/N is p-solvable, then we have that the θ-version of the $k(B)$-conjecture holds, that is, we have that $|B_\theta| \leqslant |D_\theta/N|$ for any θ-block B_θ and θ-defect group of B_θ, D_θ/N. Indeed, let (G^*, N^*, θ^*) be a standard isomorphic character triple, let $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ be the standard bijection, let B^* be the p-block of G^* such that $(B_\theta)^* = \mathrm{Irr}(B^*|\theta^*)$ and let D^* be a defect group of B^*. Then G^*/N^* is isomorphic to G/N and hence, since $N^* \subseteq \mathbf{Z}(G^*)$, we have that G^* is p-solvable. Then G^*/N^*_p is p-solvable, where $N^*_p \in \mathrm{Syl}_p(N^*)$ and hence the $k(B)$-conjecture holds for G^*/N^*_p. Now, if $\overline{B^*}$ is the unique p-block of G^*/N^*_p contained in B^*, arguing as in the proof of Theorem 2.17, we have that

$$|B_\theta| = |\mathrm{Irr}(B^*|\theta^*)| = |\mathrm{Irr}(B^*|\theta^*_p)| \leqslant |\mathrm{Irr}(\overline{B^*})| \leqslant |D^*N^*/N^*| = |D_\theta/N|.$$

Next we prove Theorem C of the introduction. To do so we first introduce the projective version of the Height Zero conjecture due to Malle and Navarro (see [**MN17**]). Recall that if B is a p-block, $\mathrm{Irr}_0(B)$ denotes the set of irreducible characters of B of height zero. Analogously, if $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}_0(B|\theta)$ denotes the set of characters in $\mathrm{Irr}(B|\theta)$ of height 0.

Conjecture 2.18 (Projective version of BHZC). *Let G be a finite group, let p be a prime, and let B be a p-block of G with defect group D. Suppose that $Z \leqslant G$ is a central p-subgroup of G, and let $\lambda \in \mathrm{Irr}(Z)$. Then the following are equivalent:*

(a) $\mathrm{Irr}(B|\lambda) = \mathrm{Irr}_0(B|\lambda)$.

(b) D/Z is abelian and λ extends to D.

We first prove that Conjecture B is equivalent to Conjecture 2.18. Recall that Conjecture B asserts the following: if $B_\theta \subseteq \mathrm{Irr}(G|\theta)$ is a θ-block with θ-defect group D_θ/N and θ extends to D_θ, then $(\chi(1)/\theta(1))_p = |G : D_\theta|_p$ for all $\chi \in B_\theta$ if and only if D_θ/N is abelian.

We need the following well-known result.

Theorem 2.19. *If B is a p-block of G, then $\mathbf{O}_p(G)$ is contained in every defect group of B.*

Proof. See Theorem 4.8 of [**Nav98a**]. □

Theorem 2.20. *Conjecture B and Conjecture 2.18 are equivalent.*

Proof. Let (G, N, θ) be a character triple, and let B_θ be a θ-block with θ-defect group D_θ/N. As in the proof of Theorem 2.17, let (G^*, N^*, θ^*) be a standard isomorphic triple, with standard bijection $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$, and suppose that B^* is the block of G^* such that $(B_\theta)^* = \mathrm{Irr}(B^*|\theta^*)$. Recall that $N^* \subseteq \mathbf{Z}(G^*)$. We have shown above that D_θ/N is isomorphic to D^*N^*/N^*, where D^* is a defect group of B^*. (In fact, we have shown that if $\tilde{\pi} : G^*/N^* \to G/N$ is the group isomorphism induced by $\pi : \hat{G} \to G$, then $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$.) Notice that, since N^* is central in G^*, we have that the Sylow p-subgroup of N^* is contained in D^* (Theorem 2.19), and therefore $|D^*N^* : D^*|_p = 1$. Then

$$|G : D_\theta|_p = |G/N : D_\theta/N|_p = |G^* : D^*N^*|_p = |G^* : D^*|_p\,.$$

Now, as is well-known, character triple isomorphisms preserve ratios of character degrees (see Definition 1.22), that is $\chi(1)/\theta(1) = \chi^*(1)/\theta^*(1) = \chi^*(1)$ for $\chi \in \mathrm{Irr}(G|\theta)$. In particular, if $\chi \in B_\theta$, then

$$(\chi(1)/\theta(1))_p = \chi^*(1)_p = |G^* : D^*|_p p^{h(\chi^*)} = |G : D_\theta|_p p^{h(\chi^*)}\,,$$

where $0 \leqslant h(\chi^*)$ is the height of χ^* in B^*.

By the properties of character triple isomorphisms, notice that θ extends to D_θ if and only if θ^* extends to D^*N^*. □

As a Corollary we obtain Theorem C. The key is Theorem 3 of [**Sam19**] that asserts that Brauer's Height Zero conjecture and Conjecture 2.18 are equivalent. We should point out that Sambale's theorem uses the theory of fusion systems. We have not been able to find a proof of this without this theory.

Corollary 2.21. *Conjecture B and Brauer's Height Zero conjecture are equivalent.*

We have said in the Introduction of this Chapter that Conjecture B generalizes the Gluck-Wolf-Navarro-Tiep theorem. We prove that now.

Proposition 2.22. *Conjecture B implies the Gluck-Wolf-Navarro-Tiep theorem.*

Proof. Suppose that for all $\chi \in \mathrm{Irr}(G|\theta)$, p does not divide $\chi(1)/\theta(1)$. Let B_θ be a θ-block and let $\chi \in B_\theta$. Since $(\chi(1)/\theta(1))_p = |G : D_\theta|_p p^{h(\chi^*)}$, we have that $|G : D_\theta|$ is not divisible by p. Hence D_θ/N is a Sylow p-subgroup of G/N. Since p does not divide $\chi(1)/\theta(1)$, and all the irreducible constituents of χ_{D_θ} lie over θ, it follows that there is some irreducible constituent $\gamma \in \mathrm{Irr}(D_\theta|\theta)$ such that p does not divide $\gamma(1)/\theta(1)$. By Theorem 1.12, we have that $\gamma_N = \theta$. Since we are assuming that Conjecture B holds, we have that D_θ/N is abelian. Since $D_\theta/N \in \mathrm{Syl}_p(G/N)$, we have that G/N has abelian Sylow p-subgroups. □

2.7. Theorem F

In this section we prove Theorem F of the introduction. We will need the following result, which is essentially a result of R. Knörr. Recall that if (G, N, θ) is a character triple, then $xN \in G/N$ is θ-good if θ has a D-invariant extension to $N\langle x\rangle$, where $D/N = \mathbf{C}_{G/N}(xN)$. The θ-good conjugacy classes of G/N (those consisting of θ-good elements) play the role of the conjugacy classes of G when we are working with characters of G over θ. For instance, it is a theorem of P. X. Gallagher that $|\mathrm{Irr}(G|\theta)|$ is the number of conjugacy classes of G/N consisting of θ-good elements (see Theorem 5.16 of [**Nav18**]).

Theorem 2.23. *Suppose that $Z \subseteq \boldsymbol{Z}(G)$ and let $\theta \in \mathrm{Irr}(Z)$. Suppose that gZ and hZ are not G/Z-conjugate. Then*

$$\sum_{\chi\in\mathrm{Irr}(G|\theta)} \chi(g)\chi(h^{-1}) = 0.$$

Also

$$\sum_{\chi\in\mathrm{Irr}(G|\theta)} |\chi(g)|^2 = |\mathbf{C}_{G/Z}(gZ)|$$

if g is θ-good.

Proof. The first part is a special case of Corollary 7 of [**Kno06**]. The second part is an unpublished result of Isaacs. For a proof see Theorem 5.21 of [**Nav18**]. □

Recall that we write $G^{p'}$ for the set of p-regular elements of the finite group G, that is, those elements whose order is not divisible by p. If $\chi \in \mathrm{Irr}(G)$, we denote by $\chi^{p'}$ the restriction of χ to $G^{p'}$. We know that we can write

$$\chi^{p'} = \sum_{\varphi\in\mathrm{IBr}(G)} d_{\chi\varphi}\varphi$$

for uniquely determined non-negative integers $d_{\chi\varphi}$ called the decomposition numbers. The matrix $D = (d_{\chi\varphi})$ is called the decomposition matrix of G. The following is Theorem F of the introduction.

Theorem 2.24. *Suppose that Z is a central subgroup of G, and let $\theta \in \mathrm{Irr}(Z)$. Let B be a Brauer p-block of G such that $\mathrm{Irr}(B|\theta)$ is not empty. Let $D_{B,\theta} = (d_{\chi\varphi})$, where $\chi \in \mathrm{Irr}(B|\theta)$ and $\varphi \in \mathrm{IBr}(B)$. Then $D_{B,\theta}$ is not of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

for any ordering of the rows and columns.

Proof. Let $D = (d_{\chi\varphi})$ be the decomposition matrix of G and let M_θ be the submatrix of D whose rows are indexed by the characters in $\mathrm{Irr}(G|\theta) = \{\chi_1, \ldots, \chi_k\}$. Let $\{x_1, x_2, \ldots, x_l\}$ be a set of representatives of the p-regular conjugacy classes of G. Let $X_\theta = (\chi_i(x_j))$ be the submatrix of the character table of G with rows indexed by elements in $\mathrm{Irr}(G|\theta)$ and columns indexed by the representatives of the p-regular conjugacy classes of G. Let $Y = (\varphi_i(x_j))$ be the Brauer character table of G, where $\mathrm{IBr}(G) = \{\varphi_1, \ldots, \varphi_l\}$. Then we have that $X_\theta = M_\theta Y$.

We first assume that Z is a p-group. Suppose that $g \in G$ is p-regular. We claim that g is θ-good. First we prove that $\mathbf{C}_{G/Z}(gZ) = \mathbf{C}_G(g)/Z$. Indeed, let $xZ \in \mathbf{C}_{G/Z}(gZ)$. Then,

$$gZ = (gZ)^{xZ} = g^x Z,$$

and therefore $g^x = gz$ for some $z \in Z$. Since z is a central p-element and g^x and g are p-regular elements, we have that $z = 1$ and therefore $x \in \mathbf{C}_G(g)$. Now let η be an extension of θ to $\langle Z, g\rangle$. We need to prove that η is $\mathbf{C}_G(g)$-invariant. But this is clear since $\mathbf{C}_G(g) \subseteq \mathbf{C}_G(x)$ for all $x \in \langle Z, g\rangle$. Hence g is θ-good and the claim is proven.

Note that if x_i and x_j are not G-conjugate p-regular elements, then x_iZ and x_jZ are not G/Z-conjugate. Indeed, suppose that there exists $gZ \in G/Z$ such that $x_iZ = (x_jZ)^{gZ} = x_j^g Z$, hence $x_i = x_j^g z$ for some $z \in Z$. Again, since Z is a central p-group and x_i and x_j^g are p-regular elements, we have that $z = 1$ and hence $x_i = x_j^g$. Let $E \in \mathrm{Mat}_l(\mathbb{C})$ be the diagonal matrix with diagonal entries $|\mathbf{C}_{G/Z}(x_iZ)|$. By Theorem 2.23 we have that

$$E = X_\theta^t \overline{X}_\theta = Y^t (M_\theta)^t M_\theta \overline{Y}.$$

What we have done until now holds for every $\theta \in \mathrm{Irr}(Z)$. If $\theta = 1_Z$ is the trivial character of Z, notice that M_{1_Z} is the decomposition matrix of G/Z, since $\mathrm{Irr}(G|1_Z) = \mathrm{Irr}(G/Z)$ and $\mathrm{IBr}(G/Z) = \mathrm{IBr}(G)$ by Theorem 1.44. By the previous equation, for $\theta = 1_Z$, we have

$$E = X_{1_Z}^t \overline{X}_{1_Z} = Y^t (M_{1_Z})^t M_{1_Z} \overline{Y},$$

and since Y is a regular matrix, we conclude that

$$C = M_{1_Z}^t M_{1_Z} = M_\theta^t M_\theta,$$

where C is the Cartan matrix of G/Z. Until now, our ordering of $\mathrm{Irr}(G|\theta) = \{\chi_1, \ldots, \chi_k\}$ and $\mathrm{IBr}(G) = \{\varphi_1, \ldots, \varphi_l\}$ was arbitrary. Now let $B_1, \ldots, B_r$

be the different p-blocks of G, and order $\mathrm{Irr}(G|\theta)$ and $\mathrm{IBr}(G)$ by blocks (so that the first characters are in B_1, and so on). Since Z is a central p-group, by Theorem 1.44 we have that there exists a unique p-block $\overline{B}_i$ of G/Z contained in B_i. Let $C_{\overline{B}_i}$ be the Cartan matrix of $\overline{B}_i$. We have that $C = \mathrm{diag}(C_{\overline{B}_1}, \ldots, C_{\overline{B}_r})$ and $M_\theta = \mathrm{diag}(M_{B_1,\theta}, \ldots, M_{B_r,\theta})$ are block diagonal matrices. Then,

$$M_\theta^t M_\theta = \mathrm{diag}(M_{B_1,\theta}^t M_{B_1,\theta}, \ldots, M_{B_r,\theta}^t M_{B_r,\theta}).$$

Since $C = M_\theta^t M_\theta$, we necessarily have that $C_{\overline{B}_i} = M_{B_i,\theta}^t M_{B_i,\theta}$ for every i. Now if $M_{B,\theta}$ is of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

so is $C_{\overline{B}}$. By Problem 3.4 of [**Nav98a**] this is a contradiction.

This ends the proof of the case where Z is a p-group. We prove now the general case. Write $Z = Z_p \times Z_{p'}$, where Z_p is the Sylow p-subgroup of Z and write $\theta = \theta_p \times \theta_{p'}$, with $\theta_p \in \mathrm{Irr}(Z_p)$ and $\theta_{p'} \in \mathrm{Irr}(Z_{p'})$. By assumption, there is $\chi \in \mathrm{Irr}(B)$ over θ, and therefore over $\theta_{p'}$. Now, B covers the block $\{\theta_{p'}\}$, and by Theorem 1.41, we have that $\mathrm{Irr}(B|\theta_{p'}) = \mathrm{Irr}(B)$. We conclude that $D_{B,\theta} = D_{B,\theta_p}$, and we are done by the central p-group case. □

Easy examples show that in Theorem F, we cannot replace Z central by a G-invariant character of an abelian $Z \lhd G$. For instance, if $G = \mathsf{A}_4$, $p = 2$, $Z \lhd G$ is the Klein subgroup, and $\theta = 1_Z$, then G has a unique 2-block, and the matrix $D_{B,\theta}$ is the identity.

2.8. θ-Brauer characters

Suppose again that N is a normal subgroup of G. It is natural to consider the normal set $G^0 = \{x \in G \,|\, x_p \in N\}$ and the complex space $\mathrm{cf}(G^0)$ of complex class functions defined on G^0. If $\delta \in \mathrm{cf}(G)$, we denote by δ^0 the restriction of δ to G^0. The space $\mathrm{cf}(G^0)$ can naturally be decomposed as a direct sum of subspaces. Indeed, for a given $\theta \in \mathrm{Irr}(N)$, we define $\mathrm{cf}(G|\theta)$ to be the $\mathbb{C}$-span of $\mathrm{Irr}(G|\theta)$, and we let

$$\mathrm{cf}(G^0|\theta) = \mathrm{cf}(G|\theta)^0 = \{\delta^0 \,|\, \delta \in \mathrm{cf}(G|\theta)\}\,.$$

Of course, $\mathrm{cf}(G^0|\theta) = \mathrm{cf}(G^0|\theta^g)$ for $g \in G$. It is easy to prove (see Lemma 2.1 of [**Nav00**]) that if Θ is a complete set of representatives of the G-action on $\mathrm{Irr}(N)$, we have that

$$\mathrm{cf}(G^0) = \bigoplus_{\theta \in \Theta} \mathrm{cf}(G^0|\theta)\,.$$

Hence, the strategy is now to fix $\theta \in \mathrm{Irr}(N)$ and focus on $\mathrm{cf}(G^0|\theta)$. The next natural step is to prove that if $T = G_\theta$ is the stabilizer of θ in G, then induction $\psi \mapsto \psi^G$ defines a linear isomorphism

$$\mathrm{cf}(T^0|\theta) \to \mathrm{cf}(G^0|\theta)\,.$$

This is done in Lemma 2.2 of [**Nav00**]. So, using induction, we are left with a G-invariant θ, that is, with a character triple (G, N, θ).

Suppose now that a set $\mathrm{IBr}(G)$ of irreducible Brauer characters of G is given. (Or that we have chosen a maximal ideal M containing p in the ring of algebraic integers $\mathbf{R}$ of the complex numbers.) If N is a p-group, Navarro constructed in [**Nav00**] a natural basis of $\mathrm{cf}(G^0)$ only depending on $\mathrm{IBr}(G)$ (or M). Let us review Navarro's construction.

Suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant. We define $\hat{\theta} \in \mathrm{cf}(G^0|\theta)$ as follows. If $x \in G^0$, then $x_p \in N$ and $N\langle x\rangle/N$ is a p'-group. Since N is a p-group, there is a canonical extension $\hat{\theta}_x \in \mathrm{Irr}(N\langle x\rangle)$ by Theorem 1.17. This is the unique extension of θ to $N\langle x\rangle$ whose determinantal order is a power of p. Now we define $\hat{\theta}(x) = \hat{\theta}_x(x)$. If η is any class function defined on the p-regular elements of G, we define

$$(\theta \star \eta)(x) = \hat{\theta}(x)\eta(x_{p'})$$

for $x \in G^0$. We are finally ready to define Navarro's N-Brauer characters. These are

$$\mathrm{IBr}(G|\theta) = \{\theta \star \eta \mid \eta \in \mathrm{IBr}(G)\}\,.$$

One of the main results in [**Nav00**] (Theorem 4.3) is that $\mathrm{IBr}(G|\theta)$ is a basis of $\mathrm{cf}(G^0|\theta)$ and that if $\chi \in \mathrm{Irr}(G|\theta)$, then

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi$$

for some (uniquely defined) non-negative integers $d_{\chi\varphi}$. This integers are closely related to the work of Külshammer and Robinson on N-projective modules in [**KR87**]. To understand this relation we need a bit more. As we said before, using Lemma 2.1 and Lemma 2.2 of [**Nav00**], we can construct a basis

$$\mathrm{IBr}(G, N) = \{\varphi^G \mid \varphi \in \mathrm{IBr}(G|\theta), \theta \in \Theta\}$$

of $\mathrm{cf}(G^\circ)$ such that if $\chi \in \mathrm{Irr}(G)$ then

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G,N)} d_{\chi\varphi}\varphi$$

(this is Theorem A of [**Nav00**]). Now, if we let

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi,$$

where $\varphi \in \mathrm{IBr}(G, N)$, then Theorem B of [**Nav00**] asserts that the set $\{\Phi_\varphi \mid \varphi \in \mathrm{IBr}(G, N)\}$ is the Külshammer-Robinson $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $M_{p'}(G, N)$ generated by the characters of all the N-projective RG-modules for certain complete discrete valuation ring R.

What happens when N is not necessarily a p-group? This case is solved (in a non-canonical way) in [**Nav12**]. Let us explain how. Suppose first that N is central. Hence $N = N_p \times N_{p'}$, where $N_p \in \mathrm{Syl}_p(N)$. Therefore we can write $\theta = \alpha \times \beta$, where $\alpha \in \mathrm{Irr}(N_p)$ and $\beta \in \mathrm{Irr}(N_{p'})$. If η is a class function defined on the p-regular elements of G, let us define now

$$(\theta \star \eta)(x) = \theta(x_p)\eta(x_{p'}) = (\alpha \star \eta)(x)$$

for $x \in G^0$. (Notice that the last equation makes sense, since $x_p \in N$ if and only if $x_p \in N_p$). Now, in [**Nav12**] it is proved that

$$\mathrm{IBr}(G|\theta) = \{\alpha \star \eta \mid \eta \in \mathrm{IBr}(G|\beta)\}$$

is a basis of $\mathrm{cf}(G^0|\theta)$, and that if $\chi \in \mathrm{Irr}(G|\theta)$, then

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi$$

for some (uniquely defined) non-negative integers $d_{\chi\varphi}$. (One might think that the notation is quite confusing, since if $\varphi \in \mathrm{IBr}(N)$, $\mathrm{IBr}(G|\varphi)$ usually denotes the set of irreducible Brauer characters of G lying over φ. But there is no such confusion: if $\theta \in \mathrm{IBr}(N)$ (that is, if N is a p'-group), then $\beta = \theta$, $G^\circ = G^{p'}$, and $\theta \star \eta = \eta$ for all $\eta \in \mathrm{IBr}(G)$).

Finally, it is shown in Lemma 2.1 of [**Nav12**], that if (G, N, θ) and (G^*, N^*, θ^*) are isomorphic character triples then there is a natural isomorphism of the vector spaces

$$^* : \mathrm{cf}(G^0|\theta) \to \mathrm{cf}((G^*)^0|\theta^*)$$

such that

$$(\chi^*)^0 = (\chi^0)^*$$

for $\chi \in \mathrm{cf}(G|\theta)$. This easily shows that, if N^* is central in G^*, then the inverse image of

$$\mathrm{IBr}(G^*|\theta^*) = \{\theta^*_p \star \eta^* \mid \eta \in \mathrm{IBr}(G^*|\theta^*_{p'})\}$$

where $\theta^* = \theta^*_p \times \theta^*_{p'}$, is a basis of $\mathrm{cf}(G^0|\theta)$. Since this basis depends on the choice of the isomorphic character triple (G^*, N^*, θ^*), we will denote it by $\mathcal{B}_{(G^*,N^*,\theta^*)}$. Hence, whenever $\chi \in \mathrm{Irr}(G|\theta)$, we can write

$$\chi^0 = \sum_{\varphi \in \mathcal{B}_{(G^*,N^*,\theta^*)}} d_{\chi\varphi}\varphi$$

for some uniquely determined non-negative integers $d_{\chi\varphi}$. The problem with this construction is that there is no known way of choosing a canonical (G^*, N^*, θ^*) with N^* central that is isomorphic to (G, N, θ).

Our main theorem in this section asserts that if we choose two standard isomorphic triples, then the corresponding basis that is obtained through this process does not change.

Assume that (G^*, N^*, θ^*) is any character triple isomorphic to (G, N, θ), with N^* central, and again write $N^* = N^*_p \times N^*_{p'}$, where $N^*_p \in \mathrm{Syl}_p(N^*)$ and $\theta^* = \theta^*_p \times \theta^*_{p'}$, with $\theta^*_p \in \mathrm{Irr}(N^*_p)$ and $\theta^*_{p'} \in \mathrm{Irr}(N^*_{p'})$. By Theorem 2.4 of [**Nav12**], the set

$$\mathrm{IBr}(G^*|\theta^*) = \{\theta^* \star \varphi^* \mid \varphi^* \in \mathrm{IBr}(G^*|\theta^*_{p'})\}$$

is a basis of $\mathrm{cf}((G^*)^0|\theta^*)$.

Since $\mathrm{cf}(G^*|\theta^*)$ is the $\mathbb{C}$-span of $\mathrm{Irr}(G^*|\theta^*)$, for each $\varphi^* \in \mathrm{IBr}(G^*|\theta^*_{p'})$ we can write

$$\theta^* \star \varphi^* = \sum_{\chi^* \in \mathrm{Irr}(G^*|\theta^*)} a_{\varphi^*\chi^*} (\chi^*)^\circ$$

for some complex numbers $a_{\varphi^*\chi^*} \in \mathbb{C}$. (Notice that these numbers are not necessarily unique, but for our purposes this is not going to matter.) Since (G, N, θ) and (G^*, N^*, θ^*) are isomorphic character triples, we know that there exists a bijection $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ and by Lemma 2.1 of [**Nav12**], the map $\Psi^\circ \mapsto (\Psi^*)^\circ$ from $\mathrm{cf}(G|\theta)^\circ \to \mathrm{cf}(G^*|\theta^*)^\circ$ is an isomorphism of vector spaces. Hence the basis of $\mathrm{cf}(G|\theta)^\circ$ described in [**Nav12**] is

$$\mathcal{B}_{(G^*,N^*,\theta^*)} = \{ \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi^*\chi^*} \chi^\circ \mid \varphi^* \in \mathrm{IBr}(G^*|\theta^*_{p'})\}.$$

Our goal now is to prove Theorem G. To do so, we need to prove that $\mathcal{B}_{(G_1,N_1,\theta_1)} = \mathcal{B}_{(G_2,N_2,\theta_2)}$ whenever (G_1, N_1, θ_1) and (G_2, N_2, θ_2) are standard isomorphic character triples. We need the following easy observation.

Lemma 2.25. *Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ linear, and $\alpha, \beta \in \mathrm{cf}(G^{p'})$, then*

(a) $\theta \star (\alpha + \beta) = \theta \star \alpha + \theta \star \beta$,
(b) if $\theta \star \alpha = \theta \star \beta$ then $\alpha = \beta$.

Proof. Let $x \in G^\circ$, then

$$\begin{aligned}(\theta \star (\alpha + \beta))(x) &= \theta(x_p)(\alpha + \beta)(x_{p'}) = \theta(x_p)(\alpha(x_{p'}) + \beta(x_{p'})) \\ &= \theta(x_p)\alpha(x_{p'}) + \theta(x_p)\beta(x_{p'}) = (\theta \star \alpha)(x) + (\theta \star \beta)(x),\end{aligned}$$

and (a) is proven. Now suppose that $\theta \star \alpha = \theta \star \beta$, and take $y \in G^{p'}$. Notice that $y \in G^\circ$. Since θ is linear, we have that $\alpha(y) = \theta(1)\alpha(y) = (\theta \star \alpha)(y) = (\theta \star \beta)(y) = \theta(1)\beta(y) = \beta(y)$ and (b) follows. □

We will use also the following non-trivial result of [**Nav12**]. If $\chi \in \mathrm{cf}(G) \cup \mathrm{cf}(G)^{p'}$, we define $\chi_{p'} \in \mathrm{cf}(G)$ as $\chi_{p'}(g) = \chi(g_{p'})$ for all $g \in G$.

Theorem 2.26. *Suppose that $N \subseteq \boldsymbol{Z}(G)$, and let $\theta = \alpha\beta \in \mathrm{Irr}(N)$, where $\alpha \in \mathrm{Irr}(N)$ has p-power order and $\beta \in \mathrm{Irr}(N)$ has p'-order. Let $\chi \in \mathrm{cf}(G|\theta)$, then $\chi^\circ = \alpha \star \chi_{p'}$.*

Proof. See Theorem 2.4 of [**Nav12**]. □

Recall that if $\alpha : \hat{G} \to G$ is a surjective group homomorphism with kernel Z and $\psi \in \mathrm{Irr}(G)$, we denote by ψ^α the unique irreducible character of $\hat{G}$ such that $\psi^\alpha(x) = \psi(\alpha(x))$ for $x \in \hat{G}$. Now, note that if $x \in \hat{G}^{p'}$, then $\alpha(x) \in G^{p'}$. Hence, if $\varphi \in \mathrm{IBr}(G)$, we denote by φ^α the unique irreducible Brauer character of $\hat{G}$ such that $\varphi^\alpha(x) = \varphi(\alpha(x))$ for $x \in \hat{G}^{p'}$. Notice that $Z \subseteq \ker(\varphi^\alpha)$.

THEOREM 2.27. *Let (G, N, θ) be a character triple and let (G_1, N_1, θ_1) and (G_2, N_2, θ_2) be standard isomorphic character triples. Then*

$$\mathcal{B}_{(G_1,N_1,\theta_1)} = \mathcal{B}_{(G_2,N_2,\theta_2)}.$$

PROOF. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations associated with θ arising (G_1, N_1, θ_1) and (G_2, N_2, θ_2) respectively (that is, $G_i = \hat{G}_i/N$, $N_i = \hat{N}_i/N$, and $\theta_i = \hat{\lambda}_i$ in the notation of Theorem 1.23). If $\chi \in \mathrm{Irr}(G|\theta)$, we write $\chi_i \in \mathrm{Irr}(G_i|\theta_i)$ for the image of χ through the respective standard bijections. Now,

$$\begin{aligned}\mathrm{IBr}(G_1|\theta_1) &= \{\theta_1 \star \varphi_1 \mid \varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})\}\\ &= \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ \mid \varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})\}.\end{aligned}$$

and

$$\begin{aligned}\mathrm{IBr}(G_2|\theta_2) &= \{\theta_2 \star \varphi_2 \mid \varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})\}\\ &= \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_2\chi_2}\chi_2^\circ \mid \varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})\}.\end{aligned}$$

Hence, in the notation we have just introduced,

$$\mathcal{B}_{(G_1,N_1,\theta_1)} = \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi^\circ \mid \varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})\}$$

and

$$\mathcal{B}_{(G_2,N_2,\theta_2)} = \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_2\chi_2}\chi^\circ \mid \varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})\}$$

Since $\mathcal{B}_{(G_1,N_1,\theta_1)}$ and $\mathcal{B}_{(G_2,N_2,\theta_2)}$ are basis of $\mathrm{cf}(G|\theta)^\circ$, we have that $|\mathcal{B}_{(G_1,N_1,\theta_1)}| = |\mathcal{B}_{(G_2,N_2,\theta_2)}|$. Therefore we just need to prove that

$$\mathcal{B}_{(G_1,N_1,\theta_1)} \subseteq \mathcal{B}_{(G_2,N_2,\theta_2)}.$$

Let

$$\psi = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi^\circ \in \mathcal{B}_{(G_1,N_1,\theta_1)}.$$

In order to prove that $\psi \in \mathcal{B}_{(G_2,N_2,\theta_2)}$ we need to show that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_2^\circ \in \mathrm{IBr}(G_2|\theta_2).$$

In other words we need to prove that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_2^\circ = \theta_2 \star \varphi_2$$

for some $\varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})$. By Theorem 2.26 we have that $\chi_2^\circ = \theta_2 \star (\chi_2)_{p'}$. Then, by Lemma 2.25 (a),

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_2^\circ = \theta_2 \star \left(\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}(\chi_2)_{p'} \right).$$

Write

$$\varphi_2 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}((\chi_2)_{p'})^{p'}.$$

Then $\varphi_2 \in \mathrm{cf}(G_2^{p'})$ and since $((\chi_2)_{p'})^{p'}(x) = (\chi_2)_{p'}(x)$ for all $x \in G_2^{p'}$ and for all $\chi_2 \in \mathrm{Irr}(G_2)$, we have for $g \in G_2^\circ$ that

$$(\theta_2 \star \varphi_2)(g) = \theta_2(g_p)\varphi_2(g_{p'}) = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_2^\circ(g).$$

To end we just need to prove that $\varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})$.

Write $\hat{G} = G \times Z_1 \times Z_2$ as in Theorem 2.8, and let $\rho_1 : \hat{G} \to \hat{G}_1$ and $\rho_2 : \hat{G} \to \hat{G}_2$ be the maps defined in Theorem 2.8(b). Write $\eta_i = \chi_i^{\rho_i} \in \mathrm{Irr}(\hat{G})$ and let β be the linear character of $\hat{G}/N$ such that $\beta\eta_1 = \eta_2$ (Theorem 2.8(c)). Since β is linear and $N \subseteq \ker(\beta)$ we have that $(\beta_{p'})^{p'} = \beta^{p'} \in \mathrm{IBr}(\hat{G}/N)$. If $\varphi \in \mathrm{IBr}(G_1|(\theta_1)_{p'})$, we have that $\varphi^{\rho_1} \in \mathrm{IBr}(\hat{G}/N)$ and then $(\beta_{p'})^{p'}\varphi^{\rho_1} \in \mathrm{IBr}(\hat{G}/N)$ by Problem 2.13 of [**Nav98a**]. Since $\psi \in \mathcal{B}_{(G_1,N_1,\theta_1)}$ we have that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ \in \mathrm{IBr}(G_1|\theta_1),$$

and hence

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ = \theta_1 \star \varphi_1$$

for some $\varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})$. Now, again by Theorem 2.26 and Lemma 2.25(a) we have that

$$\theta_1 \star \varphi_1 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ = \theta_1 \star \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}(\chi_1)_{p'}.$$

Since θ_1 is linear, by Lemma 2.25(b) we have that

$$\varphi_1 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1 \chi_1} ((\chi_1)_{p'})^{p'}.$$

Hence

$$\varphi_1^{\rho_1} = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1 \chi_1} ((\chi_1^{\rho_1})_{p'})^{p'} = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1 \chi_1} ((\eta_1)_{p'})^{p'}.$$

Now, since $\beta\eta_1 = \eta_2$, we have that

$$\begin{aligned} \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1 \chi_1} ((\eta_2)_{p'})^{p'} &= \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1 \chi_1} (\beta_{p'} (\eta_1)_{p'})^{p'} \\ &= (\beta_{p'})^{p'} \varphi_1^{\rho_1} \in \mathrm{IBr}(\hat{G}/N). \end{aligned}$$

Hence $\varphi_2 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1 \chi_1} ((\chi_2)_{p'})^{p'} \in \mathrm{IBr}(G_2|(\theta_2)_{p'})$ and the proof is concluded. □

From now on, if (G, N, θ) is a character triple we define $\mathrm{IBr}(G|\theta) = \mathcal{B}_{(G^*, N^*, \theta^*)}$, where (G^*, N^*, θ^*) is a standard isomorphic character triple. What we have just proved is that this basis is independent of the choice of isomorphic character triples and hence it is canonical, once we have fixed $\mathrm{IBr}(G^*)$. This is exactly Theorem G of the introduction.

We call *θ-Brauer characters* the elements of this basis.

If $\chi \in \mathrm{Irr}(G|\theta)$ and $\varphi \in \mathrm{IBr}(G|\theta)$, we denote the coefficient of φ in χ° by $d_{\chi\varphi}$. In this thesis we call the numbers $d_{\chi\varphi}$ the *θ-decomposition numbers.*

First, we prove that these θ-decomposition numbers are the same that Navarro gives in [**Nav00**]. But this is easy. First, it is proved in Theorem 3.1 of [**Nav00**] that if (G, N, θ) is a character triple and N is a p-group, then there exists a standard isomorphic character triple (G^*, N^*, θ^*) with N^* a p-group. Now, it is proved in Theorem 4.3 of [**Nav00**] that in this case,

$$\hat{\theta}^*(x)\eta^*(x_{p'}) = \theta^*(x_p)\eta^*(x_{p'}),$$

for all $x \in (G^*)^\circ$, $\eta^* \in \mathrm{IBr}(G^*)$ (this follows from the fact that $N^*\langle x\rangle = N^* \times \langle x_{p'}\rangle$ and in this case the unique extension of θ^* to $N^*\langle x\rangle$ with p-power order is $\theta^* \times 1_{\langle x_{p'}\rangle}$).

Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. We say that χ and ψ are *θ-connected* if there exists $\varphi \in \mathrm{IBr}(G|\theta)$ such that

$$d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}.$$

The connected components of the graph defined by θ-connection define a partition in $\mathrm{Irr}(G|\theta)$. We call the elements of this partition the *blocks defined by θ-decomposition numbers.* We shall prove that these blocks are, in fact, the θ-blocks.

Recall that if χ and ψ are irreducible characters of G, we say that χ and ψ are connected if there exists $\varphi \in \mathrm{IBr}(G)$ such that

$$d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi},$$

where $d_{\chi\varphi}$ and $d_{\psi\varphi}$ are the classical decomposition numbers. (We hope this notation does not confuse the reader. As before, if N is a p'-group, then $\theta \in \mathrm{IBr}(N)$ and the classical decomposition numbers and the θ-decomposition numbers coincide).

Lemma 2.28. *Let (G, N, θ) be a character triple with $N \subseteq \mathbf{Z}(G)$. Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. Then χ and ψ are connected if and only if they are θ-connected.*

Proof. As before, write $N = N_p \times N_{p'}$, with $N_p \in \mathrm{Syl}_p(N)$, and $\theta = \theta_p \times \theta_{p'}$, with $\theta_p \in \mathrm{Irr}(N_p)$ and $\theta_{p'} \in \mathrm{Irr}(N_{p'})$. Since $(\chi_{p'})^{p'} = \chi^{p'}$, we have by Theorem 2.26 that $\chi^\circ = \theta \star \chi_{p'} = \theta \star \chi^{p'}$. Since $\chi \in \mathrm{Irr}(G|\theta_{p'})$, it is clear that all the Brauer irreducible constituents of $\chi^{p'}$ lie over $\theta_{p'}$. Now, using Lemma 2.25 we have that

$$\chi^{p'} = \sum_{\varphi \in \mathrm{IBr}(G|\theta_{p'})} d_{\chi\varphi}\varphi,$$

if and only if

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G|\theta_{p'})} d_{\chi\varphi}(\theta \star \varphi).$$

□

Note that from Lemma 2.28 we deduce that in the case that N is central, the θ-decomposition numbers and the classical decomposition numbers coincide. This agrees with the results obtained by J. Zeng in [**Zen03**].

Using Lemma 2.28 and Theorem F we easily obtain the following.

Theorem 2.29. *Let (G, N, θ) be a character triple with $N \subseteq \mathbf{Z}(G)$. Then the blocks defined by θ-decomposition numbers are exactly the sets* $\mathrm{Irr}(B|\theta)$ *where B runs over the p-blocks of G.*

Proof. Let B_θ be a block defined by θ-decomposition numbers. By Lemma 2.28 we know that $B_\theta \subseteq \mathrm{Irr}(B|\theta)$ for some p-block B. We prove now that $\mathrm{Irr}(B|\theta) \subseteq B_\theta$.

Let $D = (d_{\chi\varphi})$ be the decomposition matrix of G and write D_θ for the submatrix of D whose rows and columns are indexed by elements in $\mathrm{Irr}(B|\theta)$ and $\mathrm{IBr}(B)$ respectively. By Theorem F, we know that D is not of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

for any ordering of the rows and columns. Hence if $\chi, \psi \in \mathrm{Irr}(B|\theta)$ there exist $\chi = \chi_1, \chi_2, \ldots, \chi_k = \psi$ and $\varphi_1, \varphi_2, \ldots, \varphi_{k-1}$ with $\chi_i \in \mathrm{Irr}(B|\theta)$ and $\varphi_i \in \mathrm{IBr}(B)$ such that

$$d_{\chi_i\varphi_i} \neq 0 \neq d_{\chi_{i+1}\varphi_i}.$$

By Lemma 2.28, the θ-decomposition numbers are the classical decomposition numbers. This completes the proof. □

THEOREM 2.30. *Let (G, N, θ) be a character triple. The blocks defined by θ-decomposition numbers are exactly the θ-blocks of G.*

PROOF. Let (G^*, N^*, θ^*) be a standard isomorphic character triple and let $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ be the standard bijection. Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. By Lemma 2.1 of [**Nav12**] we have that the map $\Xi^\circ \mapsto (\Xi^*)^\circ$ from $\mathrm{cf}(G|\theta)^\circ \to \mathrm{cf}(G^*|\theta^*)^\circ$ is an isomorphism of vector spaces (see discussion preceding Lemma 2.25). Therefore, $\chi, \psi \in \mathrm{Irr}(G|\theta)$ lie in the same block defined by θ-decomposition numbers if and only if χ^*, ψ^* lie in the same block defined by θ^*-decomposition numbers. Since $N^* \subseteq \mathbf{Z}(G^*)$, using Theorem 2.29 we have that χ^*, ψ^* lie in the same block defined by θ^*-decomposition numbers if and only if χ^*, ψ^* lie in the same p-block of G^*, that is, if and only if χ and ψ lie in the same θ-block. □

2.9. θ -linking

One of the ways to define the Brauer classical p-blocks is through a linking (see Definition 1.31). We aim to do the same with the θ-blocks.

DEFINITION 2.31 (θ-linking). Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. We say that χ and ψ are *θ-linked* if

$$\sum_{x \in G^\circ} \chi(x)\overline{\psi(x)} \neq 0,$$

where $G^\circ = \{x \in G \mid x_p \in N\}$.

We would like to say, as in the classical theory, that the connected components of the graph defined by θ-linking in $\mathrm{Irr}(G|\theta)$ are the θ-blocks. Unfortunately, we will see that this is not true in general, unless we impose some extendibility condition on θ. On the other hand, it is always true that each connected component of the graph defined by θ-linking is contained in a unique θ-block. To prove it, we need the following.

PROPOSITION 2.32. *Let (G, N, θ) and (G^*, N^*, θ^*) be isomorphic character triples, and write $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ for the associated bijection. Then, $\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta)$ are θ-linked if and only if χ_1^*, χ_2^* are θ^*-linked.*

PROOF. Write

$$G^\circ = Nx_1 \cup \ldots \cup Nx_t$$

and

$$(G^*)^\circ = N^*x_1^* \cup \ldots \cup N^*x_t^*,$$

as disjoint unions, where $(Nx)^* = N^*x^*$.

Since $N\langle x_j\rangle/N$ is cyclic, θ extends to $N\langle x_j\rangle$. Let $\tilde{\theta}_j \in \mathrm{Irr}(N\langle x_j\rangle)$ extending θ. Now, by Gallagher's Corollary (Corollary 1.16), every irreducible constituent of $\theta^{N\langle x_j\rangle}$ is of the form $\psi\tilde{\theta}_j$ with $\psi \in \mathrm{Irr}(N\langle x_j\rangle/N)$. Then

$$(\chi_1)_{N\langle x_j\rangle} = \sum_{\psi\in\mathrm{Irr}(N\langle x_j\rangle/N)} e_\psi\psi\tilde{\theta}_j = \psi_{1,j}\tilde{\theta}_j.$$

In the same way, we can write $(\chi_2)_{N\langle x_j\rangle} = \psi_{2,j}\tilde{\theta}_j$, with $\psi_{2,j}$ a character of $N\langle x_j\rangle/N$. Now,

$$\begin{aligned}
\sum_{x\in G^\circ} \chi_1(x)\overline{\chi_2(x)} &= \sum_{j=1}^{t}\sum_{n\in N} \chi_1(x_jn)\overline{\chi_2(x_jn)} \\
&= \sum_{j=1}^{t}\sum_{n\in N} \psi_{1,j}(x_jn)\tilde{\theta}_j(x_jn)\overline{\psi_{2,j}(x_jn)\tilde{\theta}_j(x_jn)} \\
&= \sum_{j=1}^{t} \psi_{1,j}(x_j)\overline{\psi_{2,j}(x_j)}\sum_{n\in N} \tilde{\theta}_j(x_jn)\overline{\tilde{\theta}_j(x_jn)} \\
&= |N|\sum_{j=1}^{t} \psi_{1,j}(x_j)\overline{\psi_{2,j}(x_j)}.
\end{aligned}$$

where the last equality holds by Lemma 8.14 of [**Isa76**].

Now, for all $N \subseteq H \subseteq G$ and $\beta \in \mathrm{Char}(H/N)$, define $\beta^\tau \in \mathrm{Char}(H^*/N^*)$ by $\beta^\tau(x^*N^*) = \beta(xN)$, where $(xN)^* = x^*N^*$. Then, $((\chi_i)_{N\langle x_j\rangle})^* = \psi^\tau_{i,j}\tilde{\theta}_j^{\,*}$, for $i = 1, 2$. Reasoning as above,

$$\begin{aligned}
\sum_{x^*\in(G^*)^\circ} \chi_1^*(x^*)\overline{\chi_2^*(x^*)} &= |N^*|\sum_{j=1}^{t} \psi^\tau_{1,j}(x_j^*)\overline{\psi^\tau_{2,j}(x_j^*)} \\
&= |N^*|\sum_{j=1}^{t} \psi^\tau_{1,j}(x_j^*N^*)\overline{\psi^\tau_{2,j}(x_j^*N^*)} \\
&= |N^*|\sum_{j=1}^{t} \psi_{1,j}(x_jN)\overline{\psi_{2,j}(x_jN)} \\
&= |N^*|\sum_{j=1}^{t} \psi_{1,j}(x_j)\overline{\psi_{2,j}(x_j)}.
\end{aligned}$$

This completes the proof. □

PROPOSITION 2.33. *Let $N \lhd G$ with $N \subseteq \mathbf{Z}(G)$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and let $\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta)$. Then, χ_1 and χ_2 are θ-linked if and only if they are linked in the sense of the Definition 1.31.*

Proof. Write $N = N_p \times N_{p'}$, where $N_p \in \mathrm{Syl}_p(N)$, and $\theta = \theta_p \times \theta_{p'}$, where $\theta_p \in \mathrm{Irr}(N_p)$ and $\theta_{p'} \in \mathrm{Irr}(N_{p'})$. Now, $G^\circ = \{x \in G \mid x_p \in N\} = \{x \in G \mid x_p \in N_p\}$. Write $G^{p'} = \{x \in G \mid p \nmid o(x)\}$. Since $N \subseteq \mathbf{Z}(G)$, if $n \in N_p$, $y \in G$ and $\chi \in \mathrm{Irr}(G|\theta)$, we have that $\chi(ny) = \theta_p(n)\chi(y)$. Now,

$$\begin{aligned}\sum_{x \in G^\circ} \chi_1(x)\overline{\chi_2(x)} &= \sum_{n \in N_p} \sum_{y \in G^{p'}} \chi_1(ny)\overline{\chi_2(ny)} \\ &= \sum_{n \in N_p} \sum_{y \in G^{p'}} \theta_p(n)\chi_1(y)\overline{\theta_p(n)\chi_2(y)} \\ &= \sum_{n \in N_p} \sum_{y \in G^{p'}} \chi_1(y)\overline{\chi_2(y)} \\ &= |N_p| \sum_{y \in G^{p'}} \chi_1(y)\overline{\chi_2(y)}.\end{aligned}$$

This completes the proof. □

As an immediate corollary we obtain the following.

Corollary 2.34. *Let $N \lhd G$, let $\theta \in \mathrm{Irr}(N)$ be G-invariant and let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. If χ and ψ are θ-linked then they lie in the same θ-block.*

Proof. Let (G^*, N^*, θ^*) be a standard isomorphic character triple and write $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ for the standard bijection. By Proposition 2.32, we have that χ^* and θ^* are θ^*-linked and hence, by Proposition 2.33 we have that they are linked in the sense of Definition 1.31. Therefore χ^* and ψ^* lie in the same p-block and hence χ, ψ lie in the same θ-block. □

It is clear now that if C is a connected component of the graph defined by θ-linking in $\mathrm{Irr}(G|\theta)$, then there exists a θ-block B_θ such that $C \subseteq \mathrm{Irr}(B_\theta)$. However, it is possible for a θ-block to contain more than one connected component of this graph, as the following example illustrates.

Example. Take $p = 2$, $G = \mathrm{SL}(2,3)$ and $N = \mathbf{Z}(G)$. Let θ be the non-trivial character of N. Then, $\mathrm{Irr}(G|\theta) = \{\chi_1, \chi_2, \chi_3\}$, where the values of χ_1, χ_2, χ_3 are given in the following table:

Class:	1	2	4	3_1	3_2	6_1	6_2
$\lvert\mathbf{C}_G(g)\rvert$:	24	24	4	6	6	6	6
$\lvert\mathrm{Cl}(g)\rvert$:	1	1	6	4	4	4	4
χ_1	2	-2	0	-1	-1	1	1
χ_2	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
χ_3	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω

where $\omega = e^{2\pi i/3}$ is a primitive $3th$ root of unity.

Now, if $G^\circ = \{x \in G \mid x_p \in N\}$, we have that G° contains all of the conjugacy classes of G except for the class consisting of the elements of order 4. Since $\chi_1(y) = \chi_2(y) = \chi_3(y) = 0$ for all $y \in G$ of order 4, we have that

$$\sum_{x \in G^\circ} \chi_i(x)\overline{\chi_j(x)} = \sum_{x \in G} \chi_i(x)\overline{\chi_j(x)} = 0, \quad i, j \in \{1, 2, 3\}, i \neq j.$$

Hence χ_1, χ_2, χ_3 lie in distinct connected components of the graph defined by θ-linking in $\mathrm{Irr}(G|\theta)$. Now, G has just one p-block B and hence $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}(B|\theta)$. Since N is central in G, by Theorem 2.10(b), we have that χ_1, χ_2, χ_3 belong to the same θ-block.

However, as we said before, if we add some extendibility hypothesis on θ, we can see the θ-blocks in terms of θ-linking.

THEOREM 2.35. *Let (G, N, θ) be a character triple, and let $P/N \in \mathrm{Syl}_p(G/N)$. Suppose that θ extends to P. Then the θ-blocks are the connected components of the graph defined by θ-linking in $\mathrm{Irr}(G|\theta)$.*

To prove this we need the following.

LEMMA 2.36. *Let B be a p-block of G and let $\chi \in \mathrm{Irr}(B)$ be of height zero. Let $\psi \in \mathrm{Irr}(B)$. Then*

$$\sum_{x \in G^{p'}} \chi(x)\overline{\psi(x)} \neq 0.$$

PROOF. See Corollary 3.25 of [**Nav98a**]. □

PROPOSITION 2.37. *Let (G, N, θ) be a character triple, let B_θ be a θ-block and let D_θ/N be a θ-defect group for B_θ. Suppose that there exists $\chi \in \mathrm{Irr}(B_\theta)$ with $(\chi(1)/\theta(1))_p = |G : D_\theta|_p$. Let $\psi \in \mathrm{Irr}(B_\theta)$. Then, χ and ψ are θ-linked.*

PROOF. Let (G^*, N^*, θ^*) be a standard isomorphic character triple and let $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ be the standard bijection. As always, let χ^* denote the image of χ through *. Let B be the p-block of G^* containing χ^*. In the proof of Theorem 2.20, we show that

$$(\chi(1)/\theta(1))_p = |G : D_\theta|_p p^{h(\chi^*)},$$

where $h(\chi^*)$ is the height of χ^* in B. By hypothesis, we have that

$$(\chi(1)/\theta(1))_p = |G : D_\theta|_p$$

and hence χ^* has height zero in B. Since χ^* and ψ^* lie in B^*, by Lemma 2.36 and Proposition 2.33, we have that χ^* and ψ^* are θ^*-linked. By Proposition 2.32, we have that χ and ψ are θ-linked. □

The key to proving Theorem 2.35 is the following result of M. Murai.

THEOREM 2.38. *Let (G, N, θ) be a character triple. Let b be the p-block of N containing θ and suppose that θ has height zero in b. Let B be a p-block of G covering b and let D be a defect group of B. Suppose that θ extends to DN, then there exists $\chi \in \mathrm{Irr}(B|\theta)$ of height zero.*

PROOF. See Theorem 4.4 of [**Mur94**]. □

THEOREM 2.39. *Let (G, N, θ) be a character triple, let B_θ be a θ-block and let D_θ/N be a θ-defect group of B_θ. Suppose that θ extends to D_θ. Let $\chi, \psi \in B_\theta$. Then, χ and ψ lie in the same connected component of the graph defined by θ-linking in $\mathrm{Irr}(G|\theta)$.*

PROOF. Let (G^*, N^*, θ^*) be a standard isomorphic character triple, with standard bijection $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$, and suppose that B^* is the block of G^* such that $(B_\theta)^* = \mathrm{Irr}(B^*|\theta^*)$. Recall that we have shown in Theorem 2.10 that D_θ/N is isomorphic to D^*N^*/N^*, where D^* is a defect group of B^*.

Since θ extends to D_θ, we have that θ^* extends to D^*N^*, by the properties of character triple isomorphisms. Since θ^* is linear, we have that θ^* has height zero and hence by Theorem 2.38 applied to (G^*, N^*, θ^*), we have that there exists $\xi^* \in \mathrm{IBr}(B^*|\theta^*)$ of height zero. Now, let $\xi \in \mathrm{Irr}(G|\theta)$ be the pre-image of ξ^* under * (notice that $\xi \in \mathrm{Irr}(B_\theta)$). As in the proof of Theorem 2.20, using that N^* is central in G^* and ξ^* has height zero, we have that

$$(\xi(1)/\theta(1))_p = \xi^*(1)_p = |G^* : D^*|_p p^{h(\xi^*)} = |G : D_\theta|_p \,.$$

By Proposition 2.37 we have that χ and ξ are θ-linked, and ψ and ξ are θ-linked. □

Notice that if P/N is a Sylow p-subgroup of G/N and $\theta \in \mathrm{Irr}(N)$ is G-invariant and extends to P, then θ extends to every p-subgroup of G/N. Therefore, as a direct consequence of Theorem 2.39, we get Theorem 2.35.

CHAPTER 3

On the Howlett-Isaacs theorem

3.1. Introduction

Let N be a normal subgroup of G and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. In Chapter 2, we were interested in the study of $\mathrm{Irr}(G|\theta)$ in terms of what we defined as θ-blocks. In this Chapter our interest in $\mathrm{Irr}(G|\theta)$ continues but from a different point of view. In particular, we are interested in the case where the elements of $\mathrm{Irr}(G|\theta)$ form an orbit under the action of certain elements of $\mathrm{Aut}(G)$.

Suppose that there is just one irreducible character of G lying over θ, that is $\mathrm{Irr}(G|\theta) = \{\chi\}$. In this case it is easy to see that $\chi_N = e\theta$ where $e^2 = |G : N|$. (See Lemma 8.2 in [**Nav18**].) In this situation, we say that χ and θ are *fully ramified* with respect to G/N. Fully ramified characters are essential in both ordinary and modular representation theory, and they appear naturally. For instance, if K/L is an abelian chief factor of G and $\psi \in \mathrm{Irr}(K)$ is invariant in G, then one of the following holds: ψ_L is irreducible, $\psi_L = \sum_{i=1}^t \varphi_i$, where $\varphi_i \in \mathrm{Irr}(L)$ are distinct and $t = |K : L|$, or $\psi_L = e\varphi$ for some $\varphi \in \mathrm{Irr}(L)$ and $e^2 = |K : L|$, that is ψ and φ are fully ramified with respect to K/L. (See Theorem 6.18 of [**Isa76**]).

If χ and θ are fully ramified with respect to G/N and $N = \mathbf{Z}(G)$, we say that G is *of central type*. In 1964, Iwahori and Matsumoto [**IM64**] proposed a conjecture: if G is of central type, then G is solvable. This conjecture was claimed to be solved by R. Liebler and J. Yellen in [**LY79**], but there was a gap in their proof. Later, Howlett and Isaacs filled that gap and proved the conjecture in their celebrated paper [**HI82**]. Now, this is known as the Howlett-Isaacs theorem. This theorem is one of the first applications of the Classification of Finite Simple Groups to Representation Theory.

Our first main result in this Chapter is the following generalization of the Howlett-Isaacs theorem where we weaken the hypothesis of $\mathrm{Irr}(G|\theta)$ having just one element by introducing the action of $\mathrm{Aut}(G)$.

Theorem H. *Suppose that $Z \lhd G$, and let $\lambda \in \mathrm{Irr}(Z)$. Assume that if $\chi, \psi \in \mathrm{Irr}(G)$ are irreducible constituents of the induced character λ^G, then there exists $a \in \mathrm{Aut}(G)$ stabilizing Z, such that $\chi^a = \psi$. If T is the stabilizer of λ in G, then T/Z is solvable.*

In a different language of projective representations, Theorem H was obtained by R. J. Higgs under some solvability conditions in [**Hig88**]. His proof is mostly sketched, among other reasons because he uses some of the arguments in [**HI82**] or [**LY79**] or in some other papers by the author. Here, we choose to present a complete proof of Theorem H, in the language of character theory, and by doing so we shall adapt several arguments in all these papers. We would like to acknowledge this now.

Theorem H is one case of a more general problem, which seems intractable for now: if all the irreducible characters of G over some G-invariant $\lambda \in \mathrm{Irr}(Z)$ have the same degree, then G/Z is solvable. (See Conjecture 11.1 of [**Nav10**].)

In the second main result of this Chapter, we study this latter situation under some special hypothesis.

THEOREM I. *Suppose G is π-separable and let $N = \mathbf{O}_\pi(G)$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Then all members of $\mathrm{Irr}(G|\theta)$ have equal degrees if and only if G/N is an abelian π'-group.*

Theorem I has a block theory flavour, as we shall explain now. Suppose that π is the set of primes different from a prime p, and assume the hypothesis of Theorem I. By Theorem 10.20 of [**Nav98a**], we have that there is a unique Brauer p-block B such that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$. Hence we are studying blocks all of whose irreducible characters have the same degree. These were studied by Okuyama and Tsushima in [**OT83**]. They showed that these blocks were exactly the blocks with abelian defect group and inertial index one. (See Proposition 1 and Theorem 3 of [**OT83**].) Our Theorem I can be seen as a π-separable version of the Okuyama-Tsushima theorem.

All the results in this section are published in [**NR17**]. The proof of Theorem I that we presented there was an improvement by Isaacs of an earlier version. We reproduce this improved version of the proof here with his kind permission.

3.2. Transitive Actions

If $Z \lhd G$, $\lambda \in \mathrm{Irr}(Z)$ is G-invariant and $P/Z \in \mathrm{Syl}_p(G/Z)$, in this Section we explore the connection between $\mathrm{Irr}(G|\lambda)$ and $\mathrm{Irr}(P|\lambda)$.

LEMMA 3.1. *Suppose that $Z \lhd G$, and let $\lambda \in \mathrm{Irr}(Z)$ be G-invariant. Assume that all characters in $\mathrm{Irr}(G|\lambda)$ have the same degree $d\lambda(1)$. Let $P/Z \in \mathrm{Syl}_p(G/Z)$. Then $d_p\lambda(1)$ is the minimum of $\{\delta(1) \mid \delta \in \mathrm{Irr}(P|\lambda)\}$ and $|\mathrm{Irr}(P|\lambda)| \leqslant |\mathrm{Irr}(G|\lambda)|_p$.*

PROOF. Notice that (G, Z, λ) is a character triple. By Theorem 1.23, we can construct an isomorphic character triple (G^*, Z^*, λ^*) with Z^* central in G^*. Notice that $|\mathrm{Irr}(G|\lambda)| = |\mathrm{Irr}(G^*|\lambda^*)|$ and if $P/Z \in \mathrm{Syl}_p(G/Z)$, then $P^*/Z^* \in \mathrm{Syl}_p(G^*/Z^*)$ and $|\mathrm{Irr}(P|\lambda)| = |\mathrm{Irr}(P^*|\lambda^*)|$. Recall that if $\chi \in \mathrm{Irr}(G|\lambda)$ and $\chi^* \in \mathrm{Irr}(G^*|\lambda^*)$ is the image of χ under the isomorphism of character triples, then $\chi(1)/\lambda(1) = \chi^*(1)$. Hence if all the irreducible characters in $\mathrm{Irr}(G|\lambda)$ have degree $d\lambda(1)$, then all the irreducible characters in $\mathrm{Irr}(G^*|\lambda^*)$ have degree $d = d\lambda^*(1)$. Finally, if d_p is the minimum of $\{\delta^*(1)\,|\,\delta^* \in \mathrm{Irr}(P^*|\lambda^*)\}$, then $d_p\lambda(1)$ is the minimum of $\{\delta(1)\,|\,\delta \in \mathrm{Irr}(P|\lambda)\}$. Therefore, we may assume that λ is linear and Z is central.

Write $\mathrm{Irr}(G\,|\,\lambda) = \{\chi_j \mid 1 \leqslant j \leqslant s\}$, and observe that the multiplicity of χ_j in λ^G is $\chi_j(1)$ by Frobenius reciprocity (Theorem 1.10). Since by hypothesis, all of the degrees $\chi_j(1)$ are equal, we can write $\lambda^G = d\sum_j \chi_j$, where $d = \chi_j(1)$ for all j. Also, since $\lambda^G(1) = |G:Z|\lambda(1) = |G:Z|$, we have that $sd^2 = |G:Z|$. Write $\mathrm{Irr}(P\,|\,\lambda) = \{\delta_i \mid 1 \leqslant i \leqslant t\}$. Then $(\delta_i)_Z = \delta_i(1)\lambda$ and hence, again by Frobenius reciprocity, $\lambda^P = \sum_i d_i\delta_i$, where $d_i = \delta_i(1)$ and $\sum {d_i}^2 = |P:Z|$. Note that if $\chi \in \mathrm{Irr}(G|\delta_i)$, then $\chi \in \mathrm{Irr}(G|\lambda)$ and hence we can write

$$\delta_i^G = \sum_{j=1}^{s} d_{ij}\chi_j \,,$$

where we allow d_{ij} to be zero. It follows that d divides $\delta_i^G(1) = |G:P|d_i$ for all $i = 1, \ldots, t$, and since $|G:P|$ is a p'-number, the p-part d_p of d divides d_i for all i. If e is the greatest common divisor of $\{\delta_i(1)\,|\,i = 1, \ldots, t\}$, then we conclude that d_p divides e.

Since $(\chi_j)_Z = \chi_j(1)\lambda$ for all $j = 1, \ldots, s$, if $\delta \in \mathrm{Irr}(P)$ lies under χ_j, then $\delta \in \mathrm{Irr}(P|\lambda)$. Then, by Frobenius reciprocity, we also have that

$$(\chi_j)_P = \sum_{i=1}^{t} d_{ij}\delta_i,$$

and thus e divides $\chi_j(1) = d$. Since e is a p-power, we see that e divides d_p, and thus $e = d_p$. Then we have that

$$|P:Z| = \sum_{i=1}^{t} {d_i}^2 \geqslant e^2 t = (d_p)^2 t\,.$$

Taking p-parts in $sd^2 = |G:Z|$, we obtain that $s_p \geqslant t$. Finally, since d_i is p-power for all $i = 1, \ldots, t$, we have that $e = d_p$ is the minimum of $\{\delta_i(1)\,|\,i = 1, \ldots, t\}$, and we are done. □

The following result is a character-theoretical version of Theorem 1.2 of [**Hig88**].

THEOREM 3.2. *Suppose that $Z \lhd G$, $\lambda \in \mathrm{Irr}(Z)$ is G-invariant, p is a prime and $P/Z \in \mathrm{Syl}_p(G/Z)$. Let $\mathcal{A} = \mathrm{Irr}(G|\lambda)$ and $\mathcal{B} = \mathrm{Irr}(P|\lambda)$. Suppose that A is a finite group acting on $\mathcal{A}$ and $\mathcal{B}$ in such a way that*

$$[(\chi^a)_P, \delta^a] = [\chi_P, \delta]$$

for all $\chi \in \mathcal{A}$, $\delta \in \mathcal{B}$ and $a \in A$. Assume further that $\chi^a(1) = \chi(1)$ for all $\chi \in \mathcal{A}$ and $a \in A$. Let $B \in \mathrm{Syl}_p(A)$. If A acts transitively on $\mathcal{A}$, then B acts transitively on $\mathcal{B}$ and $|\mathcal{A}|_p = |\mathcal{B}|$.

Proof. Write $\mathcal{A} = \{\chi_1, \dots, \chi_s\}$ and $\mathcal{B} = \{\delta_1, \dots, \delta_t\}$. By hypothesis, we have that all the characters in $\mathcal{A}$ have the same degree $d\lambda(1)$ and we can write $(\chi_i)_Z = d\lambda$ for all $i = 1, \dots, s$. By Frobenius reciprocity (Theorem 1.10), we have that $\lambda^G = d\sum_{i=1}^s \chi_i$ and therefore

$$|G : Z| = d^2 s\,.$$

By Lemma 3.1, we have that $d_p\lambda(1)$ is the minimum of the degrees in $\mathcal{B}$ and that $t \leqslant s_p$. Write

$$(\chi_i)_P = \sum_{j=1}^{t} d_{ij}\delta_j$$

so that

$$(\delta_j)^G = \sum_{i=1}^{s} d_{ij}\chi_i$$

by Frobenius reciprocity. Let B be a Sylow p-subgroup of A. Let δ_j be such that $\delta_j(1) = d_p\lambda(1)$.

Now, let $S = I_B(\delta_j)$ be the stabilizer of δ_j in B. Since

$$[(\chi^a)_P, \delta^a] = [\chi_P, \delta]$$

for all $a \in A$, we have that $S \subseteq A$ acts on the set $\mathrm{Irr}(G|\delta_j)$ of irreducible constituents of δ_j^G. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the set of S-orbits. Let $\psi_i \in \mathcal{O}_i$. We may write

$$(\delta_j)^G = \sum_{k=1}^{r} b_k (\sum_{\xi \in \mathcal{O}_k} \xi)\,.$$

If $\xi \in \mathcal{O}_k$, then there exists $a \in S$ such that $\xi = \psi_k^s$ and hence $\xi(1) = \psi_k(1)$ by hypothesis. Then,

$$|G : P| d_p\lambda(1) = (\delta_j)^G(1) = \sum_{k=1}^{r} b_k|\mathcal{O}_k|\psi_k(1) = d\lambda(1)\sum_{k=1}^{r} b_k|\mathcal{O}_k|$$

and therefore p does not divide

$$\sum_{k=1}^{r} b_k|\mathcal{O}_k|\,.$$

Therefore there is k such that $|\mathcal{O}_k|$ is not divisible by p. Since

$$|\mathcal{O}_k| = |S : I_S(\psi_k)|$$

is a power of p, we have that $|\mathcal{O}_k| = 1$. Hence ψ_k is S-fixed and then $S \subseteq I_B(\psi_k) \subseteq B$. Also $I_B(\psi_k) \subseteq R$ for some Sylow p-subgroup R of $I_A(\psi_k)$. Since A acts transitively on $\mathrm{Irr}(G|\lambda)$, we have $s = |A : I_A(\psi_k)|$. Thus

$$
\begin{aligned}
s_p = \frac{|A|_p}{|I_A(\psi_k)|_p} = \frac{|A|_p}{|R|} = \frac{|B|}{|R|} \leqslant \frac{|B|}{|I_B(\psi_k)|} \\
\leqslant \frac{|B|}{|S|} = |B : I_B(\delta_j)| \leqslant t \leqslant s_p
\end{aligned}
$$

Thus $t = s_p$, $|B : I_B(\delta_j)| = t$ and everything follows. □

3.3. Auxiliary results

Of course, if A acts by automorphisms on G, then A also acts on $\mathrm{Irr}(G)$. If $\chi \in \mathrm{Irr}(G)$ and $a \in A$, then $\chi^a \in \mathrm{Irr}(G)$ is the unique character satisfying that $\chi^a(g^a) = \chi(g)$ for $g \in G$.

We are frequently using the following hypotheses, so we state them separately:

HYPOTHESES 3.3. *Suppose that $Z \subseteq N \lhd G$, where $Z \lhd G$. Let $\lambda \in \mathrm{Irr}(Z)$. Suppose that if $\tau_i \in \mathrm{Irr}(N|\lambda)$ for $i = 1, 2$, then there exists $g \in G$ such that $\tau_1^g = \tau_2$.*

We say in this case that (G, N, Z, λ) satisfies Hypothesis 3.3. We need the following technical lemma.

LEMMA 3.4. *Suppose that (G, N, Z, λ) satisfies Hypotheses 3.3. Let $Z \subseteq K \subseteq N$, where $K \lhd G$. Then the following holds.*

(a) Let $\tau_i \in \mathrm{Irr}(K|\lambda)$ for $i = 1, 2$. Then there exists $g \in G$ such that $\tau_1^g = \tau_2$. That is, (G, K, Z, λ) satisfies Hypotheses 3.3.
(b) Suppose that $L \lhd G$ is contained in K. Let $\epsilon \in \mathrm{Irr}(L)$. Suppose that $\gamma_i \in \mathrm{Irr}(I_K(\epsilon)|\epsilon)$ are such that γ_i^K lie over λ for $i = 1, 2$. Then there is $g \in I_G(\epsilon)$ such that $\gamma_1^g = \gamma_2$.
(c) Let $\tau \in \mathrm{Irr}(K|\lambda)$. Let $\gamma_i \in \mathrm{Irr}(I_N(\tau)|\tau)$ for $i = 1, 2$. Then there exists $g \in I_G(\tau)$ such that $\gamma_1^g = \gamma_2$. That is, $(I_G(\tau), I_N(\tau), K, \tau)$ satisfies Hypotheses 3.3.

PROOF. (a) Let $\gamma_i \in \mathrm{Irr}(N)$ over τ_i. By hypothesis, we have that $\gamma_1^x = \gamma_2$ for some $x \in G$. We have that τ_1^x and τ_2 are under γ_2, so by Clifford's theorem (Theorem 1.11) there is $n \in N$ such that $\tau_1^{xn} = \tau_2$. Set $g = xn$.

(b) By Clifford's correspondence (Theorem 1.14), $\gamma_i^K \in \mathrm{Irr}(K|\epsilon)$ and hence, by hypothesis, $\gamma_i^K \in \mathrm{Irr}(K|\lambda)$. By part (a), there is $x \in G$ such that $(\gamma_1^K)^x = \gamma_2^K$. Now, ϵ^x and ϵ are under $(\gamma_2)^K$, so again by Clifford's theorem, there exists $k \in K$ such that $\epsilon^{xk} = \epsilon$. Then $g = xk \in I_G(\epsilon)$. Now, $\gamma_1^g, \gamma_2 \in \mathrm{Irr}(I_K(\epsilon)|\epsilon)$ and $(\gamma_1^g)^K = (\gamma_1^K)^g = (\gamma_1^K)^x = \gamma_2^K$. By the uniqueness in the Clifford correspondence, we deduce that $\gamma_1^g = \gamma_2$.

(c) By the Clifford correspondence, we have that $\gamma_i^N \in \mathrm{Irr}(N)$ lies over λ. By hypotheses, there is $g \in G$ such that $(\gamma_1^N)^g = \gamma_2^N$. Now, τ^g and τ are N-conjugate by Clifford's theorem, so by replacing g by gn, for some $n \in N$, we may assume that $\tau^g = \tau$. Notice now that $g \in I_G(\tau)$. Also, $\gamma_1^g = \gamma_2$, by the uniqueness in the Clifford correspondence. □

We are going to need the following result of Isaacs.

Theorem 3.5. *Let $N \lhd G$ and $K \leqslant G$ with $G = NK$ and $N \cap K = M$. Let $\theta \in \mathrm{Irr}(N)$ be invariant in G and assume $\theta_M = \varphi$ is irreducible. Then restriction defines a bijection $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(K|\varphi)$.*

Proof. See Corollary 4.2 of [**Isa84**]. □

Theorem 3.6. *Assume that (G, N, Z, λ) satisfies Hypotheses 3.3, with $Z \subseteq \mathbf{Z}(N)$. Let $U \subseteq N$, with $U \lhd G$. Suppose that q is a prime dividing $|U|$. Then q divides $|Z \cap U|$.*

Proof. Let $K = UZ \lhd G$. If q does not divide $|K : Z| = |U : U \cap Z|$ then we are done. Let $1 \neq Q/Z \in \mathrm{Syl}_q(K/Z)$. Since $Z \subseteq \mathbf{Z}(N)$ and $K \subseteq N$, we have that λ is K-invariant and hence $I_K(\lambda) = K$. By Lemma 3.4(b) (taking $L = Z$ and $\epsilon = \lambda$), we know that $G_\lambda = I_G(\lambda)$ acts transitively on $\mathrm{Irr}(K|\lambda)$. By the Frattini argument, we have that $G_\lambda = K\mathbf{N}_{G_\lambda}(Q)$. Notice then that $A = \mathbf{N}_{G_\lambda}(Q)$ acts transitively on $\mathrm{Irr}(K|\lambda)$. Also A acts on $\mathrm{Irr}(Q|\lambda)$ and $[(\chi^a)_Q, \delta^a] = [\chi_Q, \delta]$ for $a \in A$, $\chi \in \mathrm{Irr}(K|\lambda)$ and $\delta \in \mathrm{Irr}(Q|\lambda)$. By Theorem 3.2, we have that A acts transitively on $\mathrm{Irr}(Q|\lambda)$.

Suppose now that q does not divide $|Z \cap U|$. Let $\nu = \lambda_{Z\cap U}$. Then $o(\nu)$ is a q'-number. Since $|(Q \cap U)/(Z \cap U)|$ is a power of q, we have that ν has a canonical extension $\hat{\nu} \in \mathrm{Irr}(Q \cap U)$ of q'-order, by Theorem 1.17. By Theorem 3.5, we know that restriction defines a natural bijection

$$\mathrm{Irr}(Q|\lambda) \to \mathrm{Irr}(Q \cap U|\nu)\,.$$

Let $\rho \in \mathrm{Irr}(Q|\lambda)$ be such that $\rho_{Q\cap U} = \hat{\nu}$. In particular, ρ is linear. Also $\rho_Z = \lambda$. Let $a \in A$. Then a fixes λ, and therefore ν. Now, a normalizes Q and U, so a normalizes $U \cap Q$. By the uniqueness of the extension in Theorem 1.17, we have that $(\hat{\nu})^a = \hat{\nu}$. Thus $\rho^a = \rho$. Since A acts transitively on $\mathrm{Irr}(Q|\lambda)$, it follows that $\mathrm{Irr}(Q|\lambda) = \{\rho\}$. Since $\rho_Z = \lambda$, by Gallagher Corollary (Corollary 1.16), we know that $|\mathrm{Irr}(Q|\lambda)| = |\mathrm{Irr}(Q/Z)|$. We conclude that $Q = Z$. This contradiction shows that q divides $|Z \cap U|$. □

3.4. The Glauberman Correspondence

If a solvable group S acts coprimely on a group G, then there exists a bijection from $\mathrm{Irr}_S(G)$, the irreducible characters of G fixed by the action of S, onto $\mathrm{Irr}(\mathbf{C}_G(S))$. This map is known as the *Glauberman correspondence* (see Definition 13.20 of [**Isa76**] for more details).

In the particular case where the group acting is a p-group, the Glauberman correspondence has a very nice and easy expression. If Q is a p-group that acts by automorphisms on a p'-group L, and $C = \mathbf{C}_L(Q)$ then the Glauberman correspondence is a bijection $^* : \mathrm{Irr}_Q(L) \to \mathrm{Irr}(C)$, such that for every $\chi \in \mathrm{Irr}_Q(L)$, we have that

$$\chi_C = e\chi^* + p\Delta\,,$$

where p does not divide the integer e and Δ is a character of C (or zero). That is, the *Glauberman correspondent* χ^* of χ is the unique irreducible constituent of χ_C with $[\chi_C, \chi^*] \not\equiv 0 \mod p$ (in fact, $[\chi_C, \chi^*] \equiv \pm 1 \mod p$, see Theorem 13.14 of [**Isa76**]). In particular, we easily check that the Glauberman correspondence * commutes with the action of $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, where $\mathbb{Q}_{|G|} = \mathbb{Q}(\xi)$, where ξ is a primitive $|G|$-th root of unity, and with the action of the group of automorphisms of the semidirect product LQ that fix Q. In particular, we have that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$. (We give more details of the action of $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ on $\mathrm{Irr}(G)$ in Chapter 4).

The idea to introduce the Glauberman correspondence in the Iwahori-Matsumoto conjecture (page 145 of [**IM64**]) appears in [**HI82**]. As we shall see in the proof of our main theorem, we need to do the same here, in a more sophisticated way.

The next deep result is key in character theory. Its proof, in the case where $Z = 1$, is due to Dade ([**Dad80**]). (Other proofs are due to L. Puig [**Pui86**], see also Section 7.9 of [**Lin18**]). The following useful strengthening is due to Turull, who we thank for useful conversations on this subject.

THEOREM 3.7. *Suppose that G is a finite group, $LQ \lhd G$, where $L \lhd G$, $(|L|, |Q|) = 1$, and Q is a p-group for some prime p. Suppose that $LQ \subseteq N \lhd G$, and $Z \lhd G$, is contained in Q and in $\mathbf{Z}(N)$. Let $\lambda \in \mathrm{Irr}(Z)$. Let $H = \mathbf{N}_G(Q)$ and $C = \mathbf{C}_L(Q)$. Then for every $\tau \in \mathrm{Irr}_Q(L)$ there is a bijection*

$$\pi(N, \tau) : \mathrm{Irr}(N|\tau) \to \mathrm{Irr}(N \cap H|\tau^*)\,,$$

where $\tau^* \in \mathrm{Irr}(C)$ is the Glauberman correspondent of τ, such that:

(a) For $\gamma \in \mathrm{Irr}(N|\tau)$, $h \in H$ we have that

$$\pi(N, \tau^h)(\gamma^h) = (\pi(N, \tau)(\gamma))^h\,.$$

(b) $\rho \in \mathrm{Irr}(N|\tau)$ lies over λ if and only if $\pi(N, \tau)(\rho)$ lies over λ.

PROOF. The theorem follows from the proofs of Theorem 7.12 of [**Tur09**] and Theorem 6.5 of [**Tur08**]. Specifically, we make $\psi = \theta$ in Theorem 7.12 of [**Tur09**], and G, H, θ in Theorem 7.12 of [**Tur09**], correspond to G, L, τ; while G', H', θ' correspond to H, C and τ^*, respectively. Now, Theorem 7.12 of [**Tur09**] (1) and (2) predicts a bijection

$$' : \bigcup_{x \in H} \mathrm{Irr}(N|\tau^x) \to \bigcup_{x \in H} \mathrm{Irr}(N \cap H|(\tau^*)^x)\,,$$

which commutes with the action of H by part (7) of this Theorem. By parts (4), (1) and (2) of the same theorem, writing $R = L$ and $S = N$, we have that $\gamma \in \mathrm{Irr}(N|\tau)$ if and only if $\gamma' \in \mathrm{Irr}(N \cap H|\tau^*)$. We call $\pi(N, \tau)$ the restriction of the map $'$ to $\mathrm{Irr}(N|\tau)$. Part (b) follows from Theorem 10.1 of [**Tur17**]. □

Let Q be a p-group acting on a p'-group L and write $^* : \mathrm{Irr}_Q(L) \to \mathrm{Irr}(\mathbf{C}_L(Q))$ for the Glauberman correspondence. If N is a normal subgroup of L and N is Q-invariant, then Q acts on $\mathrm{Irr}(N)$ and there is a Glauberman correspondence $(^*, N) : \mathrm{Irr}_Q(N) \to \mathrm{Irr}(\mathbf{C}_N(Q))$.

We need the following property of the Glauberman correspondence.

THEOREM 3.8. *Let Q be a p-group acting on a p'-group L, let $C = \mathbf{C}_L(Q)$ and let $^* : \mathrm{Irr}_Q(L) \to \mathrm{Irr}(C)$ be the Glauberman correspondence. Let $N \lhd L$ be Q-invariant. Let $\chi \in \mathrm{Irr}_Q(L)$, let $\theta \in \mathrm{Irr}_Q(N)$, and write $\theta^{(*,N)}$ for the Glauberman correspondent of θ under the Glauberman map $(^*, N) : \mathrm{Irr}_Q(N) \to \mathrm{Irr}(C \cap N)$. Then $[\theta^G, \chi] \neq 0$ if and only if $[(\theta^{(*,N)})^C, \chi^*] \neq 0$.*

PROOF. See Theorem 13.29 of [**Isa76**]. □

We also need the following easy observation.

LEMMA 3.9. *Suppose that $LQ \lhd G$, where $L \lhd G$, $(|L|, |Q|) = 1$, and Q is a p-group for some prime p. Suppose that $\tau \in \mathrm{Irr}_Q(L)$, and let $\tau^* \in \mathrm{Irr}(C)$ be the Glauberman correspondent, where $C = \mathbf{C}_L(Q)$. Suppose that $Z \lhd G$ is contained in C. Let $\lambda \in \mathrm{Irr}(Z)$ be L-invariant. Let $H = \mathbf{N}_G(Q)$. Suppose that*

$$\lambda^L = f(\tau^{h_1} + \cdots + \tau^{h_s}),$$

for some $h_i \in H$, and some integer f. Then

$$\lambda^C = f^*((\tau^*)^{h_1} + \cdots + (\tau^*)^{h_s}),$$

for some integer f^*.

PROOF. We know by Theorem 3.8 that if $\nu \in \mathrm{Irr}_Q(L)$, then ν^* lies above λ if and only if ν lies above λ. Let $\rho \in \mathrm{Irr}(C|\lambda)$. Then $\rho = \nu^*$ for some $\nu \in \mathrm{Irr}(L|\lambda)$. Thus $\nu = \tau^h$ for some $h \in H$, by hypothesis. Then

$$\rho = \nu^* = (\tau^h)^* = (\tau^*)^h,$$

because H commutes with Glauberman correspondence. Since λ is C-invariant, then we easily conclude the proof of the lemma. □

3.5. Theorem H

It is well-known that each finite simple group S uniquely determines (up to isomorphism) a perfect finite group $\tilde{S}$ with $\tilde{S}/\mathbf{Z}(\tilde{S}) \cong S$, such that whenever G is perfect and $G/\mathbf{Z}(G) \cong S$ then $G \cong \tilde{S}/Z$, for some $Z \subseteq \mathbf{Z}(\tilde{S})$. $\mathbf{Z}(\tilde{S})$ is called the *Schur multiplier* of S and it is usually written as $M(S)$ (see discussion after Corollary 5.4 of [**Isa08**] or Appendix B of [**Nav18**] for further details, or for a character theoretical approach see Chapter 11 of [**Isa76**]).

We have mentioned before that the Howlett-Isaacs Theorem was one of the first applications of the Classification of Finite Simple Groups (CFSG) to representation theory. In particular, the CFSG is needed in the following result of [**HI82**], which we shall use later on.

THEOREM 3.10. *Let X be a non-abelian simple group. Then there exists a prime p such that p divides $|X|$, p does not divide $|M(X)|$, and there is no solvable subgroup of X having p-power index.*

PROOF. This is Theorem (2.1) of [**HI82**]. □

THEOREM 3.11. *Suppose that $H/\mathbf{Z}(H) = S_1 \times S_2 \times \cdots \times S_k$, where S_i is simple and there exists a prime p such that, for all i, p does not divide the order of the Schur multiplier $M(S_i)$. Then p does not divide $|H' \cap \mathbf{Z}(H)|$.*

PROOF. See Corollary 7.2 of [**HI82**]. □

THEOREM 3.12. *Assume (G, N, Z, λ) satisfies Hypothesis 3.3. Then $I_N(\lambda)/Z$ is solvable.*

PROOF. We argue by induction on $|N : Z|$. Let S/Z be the largest solvable normal subgroup of N/Z. Let $T = I_G(\lambda)$ be the stabilizer of λ in G.

Step 1. We may assume that λ is G-invariant.

By Lemma 3.4(c) (with $K = Z$), we have that $(I_G(\lambda), I_N(\lambda), Z, \lambda)$ satisfies Hypothesis 3.3. Hence, by working in $I_G(\lambda)$, we see that it is no loss to assume that λ is invariant in G. Hence, we wish to prove that N/Z is solvable, that is, that $S = N$.

Step 2. If $Z \leqslant K < N$, with $K \lhd G$, then K/Z is solvable. Also N/S is isomorphic to a direct product of a non-abelian simple group X.

By Lemma 3.4 (a) and induction, we have that if $Z \leqslant K < N$, with $K \lhd G$, then K/Z is solvable. Then N/S is a chief factor of G/Z, and it is isomorphic to a direct product of a non-abelian simple group X by Lemma 9.6 of [**Isa08**].

Step 3. We may assume that Z is central and that λ is faithful. Hence we may assume that Z is cyclic.

Since λ is G-invariant by Step 1, we have that (G, Z, λ) is a character triple. Now, let (G^*, Z^*, λ^*) be an isomorphic character triple with Z^* central in G^* and λ^* faithful. Let $N^*/Z^* = (N/Z)^*$. It is easy to see that $(G^*, N^*, Z^*, \lambda^*)$ satisfies Hypothesis 3.3. Hence we may assume that Z is central and λ is faithful. Now, by Theorem 2.32 of [**Isa76**] we have that we may assume that Z is cyclic.

Step 4. If $Z < K \subseteq N$ is a normal subgroup of G, and $\tau \in \mathrm{Irr}(K|\lambda)$, then $I_N(\tau)/K$ is solvable. Also $S > Z$.

The first part is a direct consequence of Lemma 3.4(c) and induction. If $S = Z$, then by Step 2, we have that N/Z is a minimal normal non-abelian subgroup of G/Z. Then N/Z is a direct product of non-abelian simple groups isomorphic to X, and $Z = \mathbf{Z}(N)$. Also, $N'Z = N$. By Theorem 3.10, there is a prime p dividing $|X|$ such that p does not divide $|M(X)|$. By Theorem 3.11, we have that p does not divide $|N' \cap Z|$. Since p divides $|N'|$, this contradicts Theorem 3.6 with $U = N'$.

Step 5. $\mathbf{F}(N) = S$.

Let $F = \mathbf{F}(N)$. It is clear that $F \subseteq S$. Suppose that $F < S$ and let R/F be a solvable chief factor of G inside N. Thus R/F is a q-group for some prime q. Let L be the Sylow q-complement of F. Let $Z_{q'} = L \cap Z$. Let Q be a Sylow q-subgroup of R, so that $R = LQ$. Let $Z_q = Q \cap Z$, so that $Z = Z_{q'} \times Z_q$. We have that $G = LH$, where $H = \mathbf{N}_G(Q)$, by the Frattini argument. Let $C = \mathbf{C}_L(Q)$.

Write $\lambda = \lambda_{q'} \times \lambda_q$, where $\lambda_{q'} = \lambda_{Z_{q'}}$, and $\lambda_q = \lambda_{Z_q}$. By coprime action and counting, we see that Q fixes some $\tau_{q'} \in \mathrm{Irr}(L|\lambda_{q'})$. Let $\tau = \tau_{q'} \times \lambda_q \in \mathrm{Irr}(LZ)$. By hypothesis and Lemma 3.4(a), we can write

$$\lambda^{LZ} = f(\tau^{h_1} + \cdots + \tau^{h_s}),$$

where $h_i \in H$, and $\lambda^{h_i} = \lambda$, because λ is G-invariant. Hence

$$\lambda_{q'}^L = f(\tau_{q'}^{h_1} + \cdots + \tau_{q'}^{h_s}).$$

By Lemma 3.9, we have that

$$\lambda_{q'}^C = f^*((\tau_{q'}^*)^{h_1} + \cdots + (\tau_{q'}^*)^{h_s}).$$

By Theorem 3.7, we know that there is a bijection

$$\pi(N, \tau_{q'}) : \mathrm{Irr}(N|\tau_{q'}) \to \mathrm{Irr}(\mathbf{N}_N(Q)|\tau_{q'}^*))$$

that commutes with H-action.

We claim that $\mathbf{N}_N(Q) < N$. If $\mathbf{N}_N(Q) = N$, we would have that $N \subseteq \mathbf{N}_G(Q)$ and hence $Q \subseteq F$. But then $R = LQ \subseteq LF = F$ and therefore $R = F$, which implies that $S = F$, a contradiction. Hence the claim is proven.

Next we claim that $(\mathbf{N}_G(Q), \mathbf{N}_N(Q), \lambda)$ satisfies Hypothesis 3.3. If this is the case, since $\mathbf{N}_N(Q) < N$, we will have that $|\mathbf{N}_N(Q) : Z| < |N : Z|$, and by induction, we will conclude that $\mathbf{N}_N(Q)/Z$ is solvable. This implies that N/Z is solvable, and the proof of the theorem would be complete. Suppose now that $\psi_i \in \mathrm{Irr}(\mathbf{N}_N(Q)|\lambda)$ for $i = 1, 2$. We are going to show that there exists $x \in H$ such that $\psi_1^x = \psi_2$. Since ψ_i lies over $\lambda_{q'}$, then we have that ψ_1 lies over some $(\tau_{q'}^*)^{h_j}$, and ψ_2 lies over some $(\tau_{q'}^*)^{h_k}$ for some $h_j, h_k \in H$. Conjugating by h_j^{-1} and by h_k^{-1}, we may assume that ψ_1 and ψ_2 lie over $\tau_{q'}^*$.

Now, we know that there exists $\mu_i \in \mathrm{Irr}(N|\tau_{q'})$ such that $\pi(N, \tau_{q'})(\mu_i) = \psi_i$. In fact, since ψ_i lies over λ_q, we have that $\mu_i \in \mathrm{Irr}(N|\lambda_q)$ by Theorem 3.7(b) (with the role of λ in that theorem being played now here by λ_q), and therefore $\mu_i \in \mathrm{Irr}(N|\tau) \subseteq \mathrm{Irr}(N|\lambda)$. By hypothesis, there is $h \in H$ such that $\mu_1^h = \mu_2$. Now, $\tau_{q'}^h$ and $\tau_{q'}$ are below μ_2, so there is $h_1 \in N \cap H$ such that $\tau_{q'}^{hh_1} = \tau_{q'}$. Replacing h by hh_1, we may assume that $(\tau_{q'})^h = \tau_{q'}$. Now

$$\psi_1^h = \pi(N, \tau_{q'})(\mu_1)^h = \pi(N, \tau_{q'}^h)(\mu_1^h) = \pi(N, \tau_{q'})(\mu_2) = \psi_2 \,,$$

as desired. By induction, $N \cap H$ is solvable, so N is solvable. This proves Step 5.

Step 6. If p divides $|F : Z|$, then N has a solvable subgroup of p-power index. Therefore, so do the simple groups factors in the direct product of N/S.

Suppose that Q/Z is a non-trivial normal p-subgroup of G/Z, where Q is contained in N. Then the irreducible constituents of λ^Q all have the same degree by Lemma 3.4(a), for instance. So we can write

$$\lambda^Q = f(\tau_1 + \cdots + \tau_k) \,,$$

where $\tau_i \in \mathrm{Irr}(Q|\lambda)$ are all the different constituents. Write $\tau = \tau_1$. Notice that $f = \tau(1)$. Thus we deduce that k is a power of p. Now, since G acts on $\Omega = \{\tau_1, \ldots, \tau_k\}$ transitively by conjugation by Lemma 3.2(a), we have that $|G : I_G(\tau)| = k$ is a power of p. Hence, $|N : I_N(\tau)|$ is a power of p. If $Q > Z$, then we know by induction that $I_N(\tau)/Q$ is solvable. In this case, we deduce that that N has a solvable subgroup with p-power index. The same happens for factors of N.

Step 7. Final contradiction.

We know by Step 2 that N/S is isomorphic to a direct product of a non-abelian simple group X. By Theorem 3.10, there exists a prime q dividing $|X|$, such that q does not divide the order of the Schur multiplier of X, and such that no solvable subgroup of X has q-power index. By Step 6, we have that q does not divide $|F : Z|$. Let W be the normal q-complement of F. Hence $F = WZ$. Also $F/W = \mathbf{Z}(N/W)$. By Corollary 7.2 of [HI], we have that q does not divide $|(N/W)' \cap F/W|$. But F/W is a q-group, so $(N/W)' \cap F/W = W/W$. In particular, $N' \cap F \subseteq W$. Thus q does not divide $|N' \cap F|$. Thus q does not divide $|N' \cap Z|$. Since N/F is perfect, we have that $N'F = N$, so that q divides $|N'|$. But this contradicts Theorem 3.6 with $U = N'$. □

Next is Theorem H.

Corollary 3.13. *Suppose that* $Z \lhd G$, *and let* $\lambda \in \mathrm{Irr}(Z)$. *Assume that if* $\chi, \psi \in \mathrm{Irr}(G|\lambda)$, *then there exists* $a \in \mathrm{Aut}(G)$ *stabilizing* Z *such that* $\chi^a = \psi$. *If* T *is the stabilizer of* λ *in* G, *then* T/Z *is solvable.*

Proof. Let $A = \mathrm{Aut}(G)_Z$ be the group of automorphisms of G that stabilize Z. Let $\Gamma = GA$ be the semidirect product. We have that $Z \lhd \Gamma$. By hypothesis, (Γ, G, Z, λ) satisfies Hypothesis 3.3. By Theorem 3.12, we have that T/Z is solvable. □

3.6. Theorem I

The main theorem of this Section uses several non-trivial results on character theory and regular orbits. First, we are going to review these results.

We start with the following elementary observation.

Lemma 3.14. *Let* $H \subseteq G$ *and* $\alpha \in \mathrm{Irr}(H)$. *Suppose that* $\alpha^G = \chi \in \mathrm{Irr}(G)$ *and that every irreducible constituent of* χ_H *has degree equal to* $\alpha(1)$. *Then* χ *vanishes on* $G - H$.

Proof. By hypothesis, χ_H is the sum of $\chi(1)/\alpha(1) = |G : H|$ irreducible characters, and thus $[\chi_H, \chi_H] \geqslant |G : H|$. Then $|H|[\chi_H, \chi_H] \geqslant |G|[\chi, \chi]$, and so χ vanishes on $G - H$, as claimed. □

We shall use the following theorem of Riese ([**Rie98**]).

Theorem 3.15. *Let* $A \subseteq G$, *where* A *is abelian, and assume that* λ^G *is irreducible, where* $\lambda \in \mathrm{Irr}(A)$. *Then* $A \lhd \lhd G$.

Proof. See Theorem 6.15 of [**Nav18**]. □

Corollary 3.16. *Let* $\theta \in \mathrm{Irr}(N)$, *where* $N \lhd G$ *and* θ *is* G*-invariant. Let* $N \subseteq A \subseteq G$, *where* A/N *is abelian, and suppose that* θ *has an extension* $\varphi \in \mathrm{Irr}(A)$ *such that* φ^G *is irreducible. Then* A *is subnormal in* G.

Proof. By using character triple isomorphisms we can assume that θ is linear and faithful. Then φ is linear and $A' \subseteq N \cap \ker(\varphi) = \ker(\theta) = 1$. Then A is abelian, and since φ^G is irreducible, Theorem 3.15 yields the result. □

We need the well-known Hall-Higman Lemma 1.2.3.

Theorem 3.17 (Hall-Higman 1.2.3). *Let* G *be a* π*-separable group, and assume that* $\mathbf{O}_{\pi'}(G) = 1$. *Then* $\mathbf{C}_G(\mathbf{O}_\pi(G)) \subseteq \mathbf{O}_\pi(G)$.

Proof. See Theorem 3.21 of [**Isa08**], for instance. □

A deep result on regular orbits is also needed.

Theorem 3.18. *Let G be a solvable group acting coprimely and faithfully on a finite group K. Then there exist $x, y \in K$ such that $\mathbf{C}_G(x) \cap \mathbf{C}_G(y) = 1$.*

Proof. This is Theorem 1.1 of [**Dol08**]. □

We shall also need a "large orbit" result.

Theorem 3.19. *Let P be a non-trivial p-group that acts faithfully on group H of order not divisible by p. Then there is an element $x \in H$ such that*

$$|\mathbf{C}_P(x)| \leqslant (|P|/p)^{\frac{1}{p}} .$$

Proof. This is Theorem A of [**Isa99**]. □

Finally, we are ready to prove an extension of Theorem I (which is recovered by setting $N = \mathbf{O}_\pi(G)$). Since the proof of this Theorem uses the Howlett-Isaacs theorem, the Classification of Finite Simple Groups is implicitely used.

Theorem 3.20. *Let $N \lhd G$. Suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant and that $o(\theta)\theta(1)$ is a π-number. Assume that G/N is π-separable and that $\mathbf{O}_\pi(G/N) = 1$. Then all members of $\mathrm{Irr}(G|\theta)$ have equal degrees if and only if G/N is an abelian π'-group.*

Proof. If G/N is an abelian π'-group, then θ extends to G by Theorem 1.17, and we are done by Gallagher's Corollary (Corollary 1.16). To prove the converse, we argue by induction on $|G/N|$ and assume that $|G/N| > 1$. We argue first that the common degree d of the characters in $\mathrm{Irr}(G|\theta)$ is a π-number. To see this, let $q \in \pi'$ and let $Q/N \in \mathrm{Syl}_q(G/N)$. Then θ extends to Q, and the induction to G of such an extension has degree $\theta(1)|G : Q|$, which is a q'-number. Since this degree is a multiple of d, it follows that d is a q'-number, and since $q \in \pi'$ was arbitrary, we see that d is a π-number.

Let $U/N = \mathbf{O}_{\pi'}(G/N)$ and note that $U > N$. All degrees of characters in $\mathrm{Irr}(U|\theta)$ divide d, and so are π-numbers. But since U/N is a π'-number, it follows that all degrees of characters in $\mathrm{Irr}(U|\theta)$ equal $\theta(1)$, and so all of these characters extend θ. It follows that U/N is abelian by Gallagher Corollary (Corollary 1.16). If $U = G$, we are done, and so we suppose that $U < G$ and we let $V/U = \mathbf{O}_\pi(G/U)$. Note that $V > U$. By Theorem 1.17, there exists a unique extension $\hat{\theta} \in \mathrm{Irr}(U)$ of θ with determinantal π-order. By uniqueness, $\hat{\theta}$ is G-invariant. Now, let $\varphi \in \mathrm{Irr}(V|\hat{\theta})$. Since V/U is a π-group, φ_U is a multiple of $\hat{\theta}$ and $o(\hat{\theta})$ is a π-number, we easily have that $o(\varphi)$ is a π-number. Write $T = G_\varphi$ for the stabilizer of φ in G. Then all members of $\mathrm{Irr}(T|\varphi)$ induce irreducibly to G, yielding characters of degree d, and thus these characters all have degree $d/|G : T|$. We claim that T satisfies the hypotheses of the theorem with respect to the character φ and the normal subgroup V. To see this, we need to check that $\mathbf{O}_\pi(T/V)$ is trivial.

Let $W/V = \mathbf{O}_{\pi'}(G/V)$. We argue that W stabilizes φ. This is because the G/V-orbit of φ has size dividing d, and so is a π-number, and W/V is a normal π'-subgroup of G/V. Thus $W \subseteq T$ and $\mathbf{O}_\pi(T/V)$ centralizes the normal π'-subgroup $W/V = \mathbf{O}_{\pi'}(G/V)$. But $\mathbf{O}_\pi(G/V)$ is trivial, and Hall-Higman Lemma 1.2.3 (3.17) applies to show that $\mathbf{O}_\pi(T/V) = 1$, as wanted.

By the inductive hypothesis, we conclude that T/V is a π'-group. Also, by the Clifford correspondence (Theorem 1.14), $|G : T|$ divides d, which we know is a π-number. Thus T/V is a full Hall π'-subgroup of G/V. Also, φ extends to T, and so $\varphi(1) = d/|G : T| = d/|G/V|_\pi$ is constant for $\varphi \in \mathrm{Irr}(V|\theta)$. It follows that the hypotheses are satisfied in the group V with respect to θ. If $V < G$, the inductive hypothesis yields that V/N is a π'-group, and this is a contradiction.

It follows that $V = G$ and G/U is a π-group. Also, G/U acts faithfully on U/N because $\mathbf{O}_\pi(G/N)$ is trivial. Now let $\lambda \in \mathrm{Irr}(U/N)$, so that λ is linear. Let $S = G_\lambda$, and note that λ extends to S since S/U is a π-group. Write $a = |G : S|$.

Note that S is the stabilizer of $\lambda\hat{\theta}$ in G, and thus all characters in $\mathrm{Irr}(S|\lambda\hat{\theta})$ have degree d/a. If r is the number of such characters, this yields $r(d/a)^2 = |S : U|\theta(1)^2$. Also, since λ extends to S, by Theorem 1.15 there is a degree-preserving bijection between $\mathrm{Irr}(S|\lambda\hat{\theta})$ and $\mathrm{Irr}(S|\hat{\theta})$, and hence the latter set contains exactly r characters, and each one has degree d/a. Each of these must therefore induce irreducibly to G, and it follows that each member of $\mathrm{Irr}(G|\hat{\theta})$ is induced from a member of $\mathrm{Irr}(S|\hat{\theta})$.

Note that the number of different members of $\mathrm{Irr}(S|\hat{\theta})$ that can have the same induction to G is at most $|G : S| = a$.

Now let $t = |\mathrm{Irr}(G|\hat{\theta})|$ so that $td^2 = |G : U|\theta(1)$. If we divide this equation by our previous one, we get $ta^2/r = |G : S| = a$, and so $t = r/a$. It follows that each of the t members of $\mathrm{Irr}(G|\hat{\theta})$ is induced from exactly a characters in $\mathrm{Irr}(S|\hat{\theta})$. In other words, if $\chi \in \mathrm{Irr}(G|\hat{\theta})$, then χ_S has exactly a distinct irreducible constituents, each with degree d/a, and so by Lemma 3.14, it follows that χ vanishes on $G - S$. In other words, the only elements of G on which χ can have a nonzero value lie in the stabilizer of λ for every linear character λ of U/N. But G/U acts faithfully on this set of linear chararacters, and thus χ vanishes on $G - U$. In other words, $\hat{\theta}$ is fully ramified in G. It follows that $d = \theta(1)|G : U|^{1/2}$.

Also, $a\theta(1)$ divides d, and so a must divide $|G : U|^{1/2}$. Write $s = |S : U|$, so that $as = |G : U|$. Then a^2 divides as, and thus a divides s. In particular, we have $a \leqslant s$, so $|G : S| \leqslant |S : U|$. Thus

$$|G : U| = |G : S||S : U| \leqslant |S : U|^2 .$$

Now, by the Howlett-Isaacs theorem we have that G/U is solvable. This group acts faithfully on the group of linear characters of U/N, and so by Theorem 3.18, there exist character stabilizers T and R such that $T \cap R = U$. By the result of the previous paragraph, each of T/U and R/U has order at least $|G : U|^{1/2}$. Now

$$|G : U| = |G : T||T : U| \geqslant |R : U||T : U| \geqslant |G : U| .$$

Then $TR = G$, and then each of $|T : U|$ and $|R : U|$ has order $|G : U|^{1/2}$. Therefore all characters in $\mathrm{Irr}(T|\hat{\theta})$ are extensions of $\hat{\theta}$ and induce irreducibly to G. In particular, T/U is abelian, and similarly R/U is abelian.

By Corollary 3.16, it follows that R is subnormal in G, and since R/U is abelian, $R/U \subseteq \mathbf{F}(G/U)$. Similarly, $T/U \subseteq \mathbf{F}(G/U)$ and thus G/U is nilpotent. But then, since G/U acts faithfully on the group of linear characters of U/N, it follows that if G/U is non-trivial, then some character $\lambda \in \mathrm{Irr}(U/N)$ has a stabilizer S in G such that

$$|S : U| < |G : U|^{1/2}$$

by Theorem 3.19. But then $|G : U| = |G : S||S : U| \leqslant |S : U|^2 < |G : U|$. This contradiction completes the proof. □

CHAPTER 4

A Brauer-Wielandt formula

4.1. Introduction

One of the classical problems in character theory is to determine which properties of a finite group G are encapsulated by its character table. For example, we know that

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2,$$

and hence the character table of G ($X(G)$ from now on) determines the order of G. We also know that G is abelian if and only if all of its irreducible characters are linear. We can also determine the normal subgroups of G from its character table (they are the kernels of the irreducible characters of G and their intersections), and hence $X(G)$ determines if G is nilpotent or solvable, for instance. Furthermore, if N is a normal subgroup of G, we can construct $X(G/N)$ from $X(G)$.

In this chapter we are interested in what $X(G)$ knows about the p- local structure of G, for a given prime p. We are specially interested in Question 7 in [**Nav04**], where it is asked if the character table of G determines $|\mathbf{N}_G(P)|$, where P is a Sylow p-subgroup of G. Note that this is the same as asking if $X(G)$ "knows" the number of Sylow p-subgroups of G.

So far, some partial answers to this Question have been given. For instance, a positive answer has been found in the cases that P is cyclic ([**Nav04**]), or if G is nilpotent-by-nilpotent([**KK15**]). Although for solvable groups no answer is known yet, it is shown in [**IN02**] that in this case $X(G)$ determines the set of primes dividing $|\mathbf{N}_G(P)|$.

Our aim in this section is to prove some more cases.

Theorem J. *Let p be a prime and let G be a finite p-solvable group. If $P \in \mathrm{Syl}_p(G)$ is abelian or has exponent p, then the character table of G determines $|\mathbf{N}_G(P)|$.*

Notice that $X(G)$ knows if G has abelian Sylow p-subgroups (in [**NT14**] and [**NST15**] an easy algorithm is given, although it was previously proved in [**KS95**] indirectly and an algorithm was given in [**CH80**] for the prime $p = 2$.) However, $X(G)$ fails to determine whether a Sylow p-subgroup of G has exponent p (the smallest counterexamples are of order 27).

It turns out that the key to prove Theorem J is to be able to compute $|\mathbf{C}_N(P)|$ from the character table of G, when $N = \mathbf{O}_{p'}(G)$. To do so, we need to generalize a classical result of Brauer. If P is a Klein 4-group acting on a group of odd order N, Brauer's celebrated formula to count the number of fixed points of the action of P on N is the following:

$$|\mathbf{C}_N(P)| = \sqrt{\frac{|\mathbf{C}_N(x)||\mathbf{C}_N(y)||\mathbf{C}_N(xy)|}{|N|}},$$

where $P = \langle x\rangle \times \langle y\rangle$. Brauer stated this formula at a conference in Tübingen in 1958, but it first appeared in the literature in [**Wie60**].

Wielandt generalized this result in [**Wie60**], giving a formula for the number of fixed points of a p-group P acting on a p'-group N. If $|P| = p^\alpha$, the formula of Wielandt is the following:

$$|\mathbf{C}_N(P)|^{p^{\alpha-1}} = \frac{\prod_{S\in\mathcal{S}} |\mathbf{C}_N(S)|}{|N|^{\frac{p^{\alpha-1}-1}{p-1}}},$$

where $\mathcal{S}$ is the set formed by all the non-trivial cyclic subgroups of P. Since we can not distinguish between elements generating different groups from the character table, we can not use this formula for our purposes. Instead, we give an alternative formula.

Theorem K. *Suppose that P is a p-group acting via automorphisms on a p'-group G. Then*

$$|\mathbf{C}_G(P)| = \left(\prod_{x\in P} \frac{|\mathbf{C}_G(x)|}{|\mathbf{C}_G(x^p)|^{1/p}}\right)^{\frac{p}{(p-1)|P|}}.$$

As we have mentioned before, Theorem K is used in the proof of Theorem J. A complete answer to Question 7 of [**Nav04**] seems still far from being given. When the character table does not seem sufficient to determine a group theoretical invariant, it is common to ask if the character table plus the p-power maps are. (See the Brauer's survey [**Bra63**].) If $\{x_1, \ldots, x_c\}$ are representatives of the conjugacy classes of G, then the p-power map is the function $f : \{1, \ldots, c\} \to \{1, \ldots, c\}$ such that x_j^p lies in the class of $x_{f(j)}$. After reading our proof of Theorem J, Lyons and R. Solomon pointed out the following.

Theorem 4.1. *Let p be a prime and let G be a finite p-solvable group. Then the character table of G together with the p-power map determine $|\mathbf{N}_G(P)|$.*

The main results of this Chapter have been published in [**NR16**].

4.2. Proof of the formula

In this Section, we give a proof of Theorem K which is independent of Wielandt's proof. We need some well-known results on coprime action. The first of them is due to Hartley and Turull.

THEOREM 4.2 (Hartley-Turull). *Let A act via automorphisms on G, where A and G are finite groups, and suppose that $(|A|, |G|) = 1$. Then A acts via automorphisms on some abelian group H in such a way that every subgroup $B \subseteq A$ has equal numbers of fixed points on G and on H. Also, there is a size-preserving bijection from the set of A-orbits on G to the set of A-orbits on H.*

PROOF. See Theorem 3.31 of [**Isa08**], for instance. □

Next is the so-called "fixed points come from fixed points" theorem.

THEOREM 4.3. *Let A act via automorphisms on G, where A and G are finite groups, and let $N \lhd G$ be A-invariant. Assume that $(|A|, |N|) = 1$. Then,*

$$\mathbf{C}_{G/N}(A) = \mathbf{C}_G(A)N/N.$$

PROOF. See Corollary 3.28 of [**Isa08**]. □

We also need the following well-known fact. Recall that if q is a prime, we say that a q-group H is an *elementary abelian q-group* if it is abelian and all the non-trivial elements of H have order q. In other words, $H \cong C_q \times C_q \times \cdots \times C_q$. If H is an elementary abelian q-group and $|H| = q^n$, then we can view H as an n-dimensional vector space over the field $GF(q)$ of q elements by simply writing the group operation in H as addition.

Now, if P is a group acting via automorphisms on H, it is easy to see that H is a $GF(q)[P]$-module. Notice that the subgroups of H are exactly its subspaces, and hence, the P-invariant subgroups of H are exactly its $GF(q)$-submodules.

Finally, recall that if A is an algebra and V is an A-module, we say that V is *completely reducible* if for every A-submodule W of V, there exists an A-submodule U of V such that $V = W + U$, where the sum is direct. We say that V is *irreducible* if it has no proper A-submodules.

We need the following classical result of Maschke.

THEOREM 4.4 (Maschke). *Let P be a finite group and F be a field whose characteristic does not divide $|P|$. Then every $F[P]$-module is completely reducible.*

PROOF. See Theorem 1.9 of [**Isa76**]. □

Theorem 4.5. *Suppose that P is a p-group acting via automorphisms on a p'-group G. Then*

$$|\mathbf{C}_G(P)| = \left(\prod_{x\in P} \frac{|\mathbf{C}_G(x)|}{|\mathbf{C}_G(x^p)|^{1/p}}\right)^{\frac{p}{(p-1)|P|}} .$$

Proof. By Theorem 4.2, there exists an abelian group H on which P acts such that $|\mathbf{C}_G(R)| = |\mathbf{C}_H(R)|$ for every subgroup R of P. In particular, $|H| = |G|$, whence H is a p'-group and it suffices to prove the theorem for H. So, we may and shall assume henceforth that G is abelian. We prove the theorem by induction on $|G||P|$.

Step 1. We may assume that P acts faithfully.

Let $Q = \{x \in P \mid g\cdot x = g \text{ for all } g \in G\}$, that is, the kernel of the action of P on G, which is normal in P. Let $x \in P$ and $y \in Q$. Since $g\cdot(xy) = (g\cdot x)\cdot y = g\cdot x$, we have that $\mathbf{C}_G(xy) = \mathbf{C}_G(x)$. Moreover, since Q acts trivially on G, we have that P/Q acts on G and $\mathbf{C}_G(P) = \mathbf{C}_G(P/Q)$. If we write $P = Qx_1 \cup \ldots \cup Qx_t$ as a disjoint union, we have that

$$\prod_{x\in P} \frac{|\mathbf{C}_G(x)|}{|\mathbf{C}_G(x^p)|^{1/p}} = \prod_{j=1}^{t}\prod_{x\in Q} \frac{|\mathbf{C}_G(xx_j)|}{|\mathbf{C}_G((xx_j)^p)|^{1/p}} = \left(\prod_{j=1}^{t} \frac{|\mathbf{C}_G(Qx_j)|}{|\mathbf{C}_G((Qx_j)^p)|^{1/p}}\right)^{|Q|} .$$

Now, if $1 < Q$, by induction we have that

$$\left(\prod_{j=1}^{t} \frac{|\mathbf{C}_G(Qx_j)|}{|\mathbf{C}_G((Qx_j)^p)|^{1/p}}\right)^{|Q|} = |\mathbf{C}_G(P/Q)|^{\frac{|Q|(p-1)|P:Q|}{p}} = |\mathbf{C}_G(P)|^{\frac{(p-1)|P|}{p}}$$

and the result follows. Hence we may assume that $Q = 1$.

Step 2. We may assume that G is an irreducible $GF(q)[P]$-module.

If N is a P-invariant subgroup of G and $R \leqslant P$, we have that R acts on G/N. Since $(|R|, |N|) = 1$, by Theorem 4.3 we have that

$$\mathbf{C}_{G/N}(R) = \mathbf{C}_G(R)N/N \cong \mathbf{C}_G(R)/\mathbf{C}_N(R),$$

and hence

$$|\mathbf{C}_G(R)| = |\mathbf{C}_{G/N}(R)||\mathbf{C}_N(R)|.$$

If $1 < N$, the theorem again follows by induction. Thus we may assume that G has not proper P-invariant normal subgroups. In particular, if q is a prime dividing G, since G is abelian we have that G is a q-group. Since the Frattini subgroup of G, $\Phi(G)$, is a characteristic subgroup, we have that $\Phi(G) = 1$ and hence G is an elementary abelian q-group.

By the discussion preceding this proof, G is a $GF(q)[P]$-module and, since $\mathrm{char}(GF(q)) = q$ does not divide $|P|$, by Maschke's Theorem 4.4 we have that it is irreducible.

Step 3. We may assume that P is not abelian.

Suppose that P is abelian and let $x \in P$. Then $\mathbf{C}_G(x)$ is P-invariant and by Step 2, we have that either $\mathbf{C}_G(x) = G$ or $\mathbf{C}_G(x) = 1$. If $x \neq 1$, by Step 1 we have that $\mathbf{C}_G(x) = 1$. This means that P acts as a Frobenius complement on G. By Theorem 6.21 of [**Isa08**] we have that P is cyclic. Then there are just p elements in P satisfying $x^p = 1$ and therefore,

$$\prod_{x \in P} \frac{|\mathbf{C}_G(x)|}{|\mathbf{C}_G(x^p)|^{1/p}} = 1 = |\mathbf{C}_G(P)|\,,$$

and the result follows. Thus we may assume that P is not abelian.

Step 4. Final Step.

Since $\mathbf{C}_G(P)$ is a P-invariant subgroup of G we have that $\mathbf{C}_G(P) = 1$ by Steps 1 and 2. Our goal now is to prove that

$$\prod_{x \in P} |\mathbf{C}_G(x)|^p = \prod_{x \in P} |\mathbf{C}_G(x^p)|\,.$$

If $1 < N$ is a proper normal subgroup of P, again we know that $\mathbf{C}_G(N)$ is a P-invariant subgroup of G and hence $\mathbf{C}_G(N) = 1$ by Steps 1 and 2. By induction, we have that

$$\prod_{x \in N} |\mathbf{C}_G(x)|^p = \prod_{x \in N} |\mathbf{C}_G(x^p)|\ .$$

Let $R = \Phi(P)$. Since P is not abelian by Step 3, we know that $\Phi(P) > 1$ and hence we have that $\mathbf{C}_G(R) = 1$. Hence $\mathbf{C}_G(J) = 1$ for $R \subseteq J \subseteq P$. Write

$$(P/R)^{\#} = \langle Rx_1 \rangle^{\#} \cup \ldots \cup \langle Rx_k \rangle^{\#}$$

as a disjoint union, where $X^{\#}$ is the set of the non-identity elements of the group X.

Suppose that P/R is cyclic. Since P/R is an elementary abelian p-group, we have that $P/R = \langle Rx \rangle$ with $o(Rx) = p$. Since $|P : R| = p$, we have that $P = R\langle x \rangle$. If $\langle x \rangle < P$, let M be a maximal subgroup of P such that $\langle x \rangle \leqslant M < P$. Then $R \leqslant M$ and hence $P = R\langle x \rangle \leqslant M\langle x \rangle = M$. This contradiction shows that $P = \langle x \rangle$, and hence P is cyclic. But this is not possible by Step 3 and hence P/R is not cyclic. Thus, $\langle R, x_j \rangle$ is proper in P. Also, $\langle R, x_j \rangle = R \cup Rx_j \cup Rx_j^2 \cup \cdots \cup Rx_j^{p-1}$, because R has index p in $\langle R, x_j \rangle$. By induction and using that $|\mathbf{C}_G(R)| = |\mathbf{C}_G(\langle R, x_j \rangle)| = 1$ for

$j \geqslant 1$, we obtain

$$\prod_{x\in\langle R,x_j\rangle} |\mathbf{C}_G(x)|^p = \prod_{x\in\langle R,x_j\rangle} |\mathbf{C}_G(x^p)| = \prod_{u\in R} |\mathbf{C}_G(u^p)| \prod_{u\in R}\prod_{i=1}^{p-1} |\mathbf{C}_G((ux_j^i)^p)|$$
$$= \prod_{u\in R} |\mathbf{C}_G(u)|^p \prod_{u\in R}\prod_{i=1}^{p-1} |\mathbf{C}_G((ux_j^i)^p)|\,.$$

Now, since

$$P = R \cup Rx_1 \cup Rx_1^2 \cup \ldots \cup Rx_1^{p-1} \cup \ldots \cup Rx_k \cup Rx_k^2 \cup \ldots \cup Rx_k^{p-1}$$

is a disjoint union, we have that

$$\prod_{g\in P} |\mathbf{C}_G(g)|^p = \frac{\prod_{j=1}^k \left(\prod_{x\in\langle R,x_j\rangle} |\mathbf{C}_G(x)|^p\right)}{\left(\prod_{x\in R} |\mathbf{C}_G(x)|^p\right)^{k-1}}$$
$$= \frac{\prod_{j=1}^k \left(\prod_{u\in R} |\mathbf{C}_G(u)|^p \prod_{u\in R}\prod_{i=1}^{p-1} |\mathbf{C}_G((ux_j^i)^p)|\right)}{\left(\prod_{x\in R} |\mathbf{C}_G(x)|^p\right)^{k-1}}$$
$$= \left(\prod_{x\in R} |\mathbf{C}_G(x)|^p\right) \prod_{j=1}^k \left(\prod_{u\in R}\prod_{i=1}^{p-1} |\mathbf{C}_G((ux_j^i)^p)|\right)$$
$$= \left(\prod_{x\in R} |\mathbf{C}_G(x^p)|\right) \prod_{j=1}^k \left(\prod_{u\in R}\prod_{i=1}^{p-1} |\mathbf{C}_G((ux_j^i)^p)|\right)$$
$$= \prod_{g\in P} |\mathbf{C}_G(g^p)|\,,$$

and this proves the theorem. □

4.3. Proof of Theorem J

As we said before, the main ingredient for the proof of Theorem J is our Brauer-Wielandt formula. The one other ingredient we require is the following result of Navarro, which was proved using Isaacs π-character theory in [**Nav98b**]. In [**NR16**] we give an alternative elementary proof provided by Gordon Keller, which we reproduce here for the reader's convenience. We need the following well-known result of G. Higman.

THEOREM 4.6 (Higman). *Let $K_1, \ldots, K_k$ be the conjugacy classes of G. Then the character table of G determines the set of primes dividing $o(g_i)$ for $g_i^G = K$.*

PROOF. See 8.21 of [**Isa76**]. □

Note that, as a consequence of Theorem 4.6, it is easy to see that the conjugacy classes of p-elements are detectable in the character table of G.

THEOREM 4.7. *Let p be a prime, G a finite p-solvable group, P a Sylow p-subgroup of G, and $K = x^G$ the G-conjugacy class of $x \in G$. Then the character table of G determines $|K \cap P|$.*

PROOF. If N is a normal subgroup of G, then let $\overline{P}$ and $\overline{K}$ denote the images of P and K respectively in $\overline{G} = G/N$.

First, by Theorem 4.6, we know that the character table of G determines the set of primes that divide the common order of the elements in K, and hence, we may assume that K consists of p-elements (otherwise, $|K \cap P|$ is zero).

Let $N = \mathbf{O}_{p'}(G)$ and suppose that $N > 1$. We claim that

$$|\overline{P} \cap \overline{K}| = |P \cap K| \, .$$

Since N is a p'-group, we have that the map $g \to Ng$ is a bijection from P to $\bar{P}$. Hence, the map $g \to Ng$ defines a one-to-one map from $P \cap K$ into $\overline{P} \cap \overline{K}$. Next we show that it is surjective. Suppose that $Ny \in \overline{P} \cap \overline{K}$. We may assume that $y \in P$. Let $z \in K \cap Ny$. As $z \in K$, z is a p-element of G. Now, $N\langle y\rangle = N\langle z\rangle$. It follows that $\langle y\rangle$ and $\langle z\rangle$ are Sylow p-subgroups of $N\langle y\rangle$, and hence are N-conjugate. Hence there exists $n \in N$ such that $y^n = z^i$ for some integer i. Since $Nz = Ny = Ny^n = Nz^i$ we deduce that $z^{i-1} = z^i z^{-1} \in N$. Since z is a p-element, we have that $i = 1$ and hence $y^n = z$. In particular, $y \in P \cap K$. Hence the map $g \to Ng$ defines a bijection between $P \cap K$ and $\overline{P} \cap \overline{K}$, as desired. Since we can obtain the character table of G/N from the character table of G, we are done by induction.

Thus, we may assume that $\mathbf{O}_{p'}(G) = 1$, and then, since G is p-solvable, we have that $\mathbf{O}_p(G) > 1$. Let $N = \mathbf{O}_p(G)$.We claim that

$$|P \cap K| = \frac{|\overline{P} \cap \overline{K}||N||\mathbf{C}_{\bar{G}}(\bar{x})|}{|\mathbf{C}_G(x)|} \, .$$

To prove this, we compute the value of the induced character $(1_P)^G$ on x. If $\Omega = \{y \in G \,|\, x^y \in K \cap P\}$, by the definition of the induced character, we have that

$$(1_P)^G(x) = \frac{|\Omega|}{|P|} \, .$$

Now, write $G = \mathbf{C}_G(x)y_1 \cup \ldots \cup \mathbf{C}_G(x)y_n$ as a disjoint union, where $n = |G : \mathbf{C}_G(x)| = |K|$. Let $\{y_1, \ldots, y_r\} = \{y_1, \ldots, y_n\} \cap \Omega$. Then it is easy to see that the map $\mathbf{C}_G(x)y_i \to x^{y_i}$ is a bijection from $\{\mathbf{C}_G(x)y_1, \ldots, \mathbf{C}_G(x)y_r\}$ to $K \cap P$. Then $r = |K \cap P|$ and Ω is the disjoint union of $|P \cap K|$ right cosets of $\mathbf{C}_G(x)$ in G. It follows that

$$(1_P)^G(x) = \frac{|P \cap K||\mathbf{C}_G(x)|}{|P|} \, .$$

Now, since N is a normal p-subgroup of G, we have that N is contained in the kernel of $(1_P)^G$, and therefore

$$(1_P)^G(x) = (1_{\bar{P}})^{\bar{G}}(\bar{x}) = \frac{|\bar{P} \cap \bar{K}||\mathbf{C}_{\bar{G}}(\bar{x})|}{|\bar{P}|},$$

from which the desired equality easily follows.

By induction, $|\bar{P} \cap \bar{K}|$ can be read off from the character table of G/N which can be read off from the character table of G. Since the character table of G detects the normal subgroups of G we can also obtain $|N|$ from the character table. Finally, the size of the centralizers of elements can also be found from the character table by means of the second orthogonality relation. This ends the theorem. □

The proof of Theorem J is far more complicated in the case P abelian than in the case P of exponent p. When P is abelian we need the following auxiliary results in order to prove it. The first of them is a very elementary fact.

LEMMA 4.8. *If P is an abelian p-group, then the map $\varphi : P \to \Phi(P)$ given by $x \mapsto x^p$ is an onto group homomorphism with kernel $\Omega_1(P) = \{x \in P | x^p = 1\}$.*

PROOF. Let $x \in P$, since $P/\Phi(P)$ is elementary abelian, $(\Phi(P)x^p) = (\Phi(P)x)^p = \Phi(P)$ and then $x^p \in \Phi(P)$, which proves that φ is well defined. Since P is abelian, it is clear that φ is an homomorphism. We just need to prove that it is onto. Note that if $x \in P$, we have that $(\varphi(P)x)^p = \varphi(P)x^p = \varphi(P)$ and hence $P/\varphi(P)$ is elementary abelian. Since P is a p-group, the Frattini subgroup of P is the unique normal subgroup of P minimal with the property that the factor group is elementary abelian, and hence $\Phi(P) \subseteq \varphi(P) \subseteq \Phi(P)$. Therefore φ is onto. □

Before stating the second auxiliary result we need to introduce the notion of *Galois conjugate* of a character. Let $\chi \in \mathrm{Irr}(G)$, the *field of values* of χ is

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G),$$

that is, the smallest subfield of $\mathbb{C}$ containing the values of χ. If n is the exponent of G, then we know that $\chi(g)$ is a sum of n-th roots of unity for all $g \in G$ and therefore $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$, where $\mathbb{Q}_n$ is the n-th cyclotomic field, that is $\mathbb{Q}_n = \mathbb{Q}(\xi)$, where ξ is a primitive n-th root of unity. In particular, $\mathbb{Q}(\chi)/\mathbb{Q}$ is a normal extension. Now, if $\mathbb{Q}(\chi) \subseteq F \subseteq \mathbb{C}$ is any field and $\sigma : F \to F$ is a field automorphism, then $\sigma(\mathbb{Q}(\chi)) = \mathbb{Q}(\chi)$ by elementary Galois theory. Thus we may define the Galois conjugate function $\chi^\sigma : G \to \mathbb{C}$ by letting

$$\chi^\sigma(x) = \sigma(\chi(g)).$$

The following are basic properties of the Galois action:

PROPOSITION 4.9. *Let G be a finite group and let $\chi \in \mathrm{Char}(G)$. Let $\mathbb{Q}(\chi) \subseteq F \subseteq \mathbb{C}$ be any field. Let $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. Then,* *(a)* $\chi^\sigma \in \mathrm{Char}(G)$ *and* $\mathbb{Q}(\chi^\sigma) = \mathbb{Q}(\chi)$. *Moreover,* χ *is irreducible if and only if* χ^σ *is irreducible.*

(b) If $\psi \in \mathrm{Char}(G)$ *and* $\mathbb{Q}(\psi) \subseteq F$, *we can define* ψ^σ *and*

$$[\chi^\sigma, \psi^\sigma] = [\chi, \psi].$$

Proof. Part (a) is Theorem 3.1 of [**Nav18**]. Part (b) follow easily since

$$[\chi^\sigma, \psi^\sigma] = \frac{1}{|G|}\sum_{g\in G}\chi^\sigma(g)\overline{\psi^\sigma(g)} = \left(\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)}\right)^\sigma = [\chi,\psi]^\sigma = [\chi,\psi]$$

□

Now we can prove the following, which is essential in the proof of Theorem J for P abelian.

Lemma 4.10. *Let* G *be a finite group,* $N \triangleleft G$ *a* p'*-group and let* P *be a Sylow* p*-subgroup of* G. *Suppose that* P *is abelian and* $NP \triangleleft G$. *Let* n *be the exponent of* G *and let* $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ *be the Galois automorphism that fixes* p'*-roots of unity and sends* p*-power roots of unity* ξ *to* ξ^{p+1}. *Then*

$$N\Phi(P) = \bigcap_{\substack{\chi\in \mathrm{Irr}(G/N)\\ \chi^\sigma=\chi}} \ker(\chi).$$

Proof. First of all it is straightforward to check that σ has p-power order. Let $\chi \in \mathrm{Irr}(G/N)$ be σ-fixed, and let λ be an irreducible constituent of χ_{NP}. Since $\lambda \in \mathrm{Irr}(NP/N)$ and NP/N is abelian, we have that λ is linear. Since χ is σ-fixed, by Proposition 4.9 we have that

$$0 \neq [\chi_{NP}, \lambda] = [\chi^\sigma_{NP}, \lambda^\sigma] = [\chi_{NP}, \lambda^\sigma]$$

and hence λ^σ is an irreducible constituent of χ_{NP}. Then, there exists $g \in G$ such that $\lambda^\sigma = \lambda^g$. It is easy to see that $\lambda^{\sigma^m} = \lambda^{g^m}$ for all $m \in \mathbb{Z}$ and hence $\lambda^{g^{o(\sigma)}} = \lambda$. Then $g^{o(\sigma)}NP \in I_G(\lambda)/NP$. Since G/NP is a p'-group and $o(\sigma)$ is a power of p, we have that $\langle gNP\rangle = \langle g^{o(\sigma)}NP\rangle$. Therefore, $gNP \in I_G(\lambda)/NP$ and $\lambda^\sigma = \lambda^g = \lambda$.

Since NP/N is a p-group and λ is linear, $\lambda(xN)$ is a p-power root of unity for all $x \in NP$, and

$$\lambda(xN) = \lambda^\sigma(xN) = \sigma(\lambda(xN)) = (\lambda(xN))^{p+1} = \lambda(xN)(\lambda(xN))^p.$$

Then the order of λ divides p. Let $z \in \Phi(P)$. By Lemma 4.8, we know that $z = x^p$ for some $x \in P$ and then

$$\lambda(z) = \lambda(x^p) = \lambda(x)^p = 1,$$

and $\Phi(P) \subseteq \ker(\lambda)$. Therefore $N\Phi(P) \subseteq \ker(\lambda)^x$ for all $x \in G$. Hence $N\Phi(P)$ is contained in $\ker(\chi)$ by Lemma 1.7.

It remains to prove that if $\chi \in \mathrm{Irr}(G)$ has $N\Phi(P)$ in its kernel, then χ is σ-fixed. Let $\lambda \in \mathrm{Irr}(NP)$ be an irreducible constituent of χ_{NP}, then

$N\Phi(P) \subseteq \ker(\lambda)$ and $\lambda \in \mathrm{Irr}(NP/N\Phi(P))$. Since $NP/N\Phi(P)$ is a p-group, again we have that, for every $x \in NP$,

$$\lambda^\sigma(x) = \lambda^\sigma(xN\Phi(P)) = \lambda^{p+1}(xN\Phi(P)) = \lambda(xN\Phi(P)) = \lambda(x),$$

and λ is σ-fixed. By Theorem 1.17, λ has a canonical extension $\hat{\lambda} \in \mathrm{Irr}(I_G(\lambda))$. This canonical extension satisfies that it is the unique extension of λ with the property that $(|I_G(\lambda) : NP|, o(\hat{\lambda})) = 1$, and in fact, $o(\hat{\lambda}) = o(\lambda)$, which divides p. Since λ is σ-fixed, again by Proposition 4.9, we have that

$$[(\hat{\lambda}^\sigma)_{NP}, \lambda] = [(\hat{\lambda}^\sigma)_{NP}, \lambda^\sigma] = [\hat{\lambda}_{NP}, \lambda] \neq 0,$$

and λ^σ lies over λ. Moreover, $\hat{\lambda}^\sigma(1) = \sigma(\hat{\lambda}(1)) = \sigma(1) = 1$ and therefore $\hat{\lambda}^\sigma$ extends λ. Now, for all $x \in I_G(\lambda)$, we have

$$(\hat{\lambda}^\sigma)^{o(\lambda)}(x) = (\hat{\lambda}^\sigma(x))^{o(\lambda)} = (\sigma(\hat{\lambda}(x))^{o(\lambda)} = \sigma(\hat{\lambda}^{o(\lambda)}(x)) = 1.$$

Thus $o(\hat{\lambda}^\sigma)$ divides p and $(|I_G(\lambda) : NP|, o(\hat{\lambda}^\sigma)) = 1$ and by the uniqueness of $\hat{\lambda}$ we have that $\hat{\lambda}$ is σ-fixed.

Now, let $\psi \in \mathrm{Irr}(I_G(\lambda)|\lambda)$ be the Clifford correspondent of χ (Theorem 1.14), that is, $\chi = \psi^G$. By Gallagher's Corollary (Corollary 1.16) we have that $\psi = \beta\hat{\lambda}$ for some $\beta \in \mathrm{Irr}(I_G(\lambda)/NP)$. Since $I_G(\lambda)/NP$ is a p'-group, $\beta(x)$ is a sum of $\beta(1)$ p'-roots of unity, and hence it is σ-fixed. Therefore, ψ is σ-fixed and $\chi = \psi^G$ is σ-fixed. □

The following includes Theorems J and 4.1.

Theorem 4.11. *Let p be a prime and G a finite p-solvable group. Let $P \in \mathrm{Syl}_p(G)$. If P is abelian or has exponent p, then the character table of G determines $|\mathbf{N}_G(P)|$. Otherwise, the character table of G and the p-power map (on the conjugacy classes of p-elements of G) determines $|\mathbf{N}_G(P)|$.*

Proof. Recall that to know the p-power map of a character table is to know the following. If $\{x_1, \dots, x_c\}$ are representatives of the conjugacy classes of G (columns in the character table), then the p-power map is the function $f : \{1, \dots, c\} \to \{1, \dots, c\}$ such that x_j^p lies in the class of $x_{f(j)}$. (In fact, we shall only need to know this function on the classes of p-elements of G.) We've already said that the character table of G determines the character table of G/N. It is also true that the p-power map of G determines the p-power map of G/N. Both the conditions that P is abelian or has exponent P are inherited by quotients of G.

As usual, $X(H)$ will denote the character table of the group H. We argue by induction on $|G|$ that if P is abelian or has exponent p, then $X(G)$ determines $|\mathbf{N}_G(P)|$. (Essentially the same proof is going to show the assertion about character tables and p-power maps, until the very end. Then we will make a comment.)

If $\mathbf{O}_p(G) > 1$, then $X(G)$ determines $X(G/\mathbf{O}_p(G))$, and hence by induction, we know

$$|\mathbf{N}_{G/\mathbf{O}_p(G)}(P/\mathbf{O}_p(G))| = \frac{|\mathbf{N}_G(P)|}{|\mathbf{O}_p(G)|}.$$

Since $|\mathbf{O}_p(G)|$ can also be determined by $X(G)$, we are done. Then we may assume that $\mathbf{O}_p(G) = 1$.

Let $N = \mathbf{O}_{p'}(G) > 1$. By induction, we know that $X(G)$ determines

$$|\mathbf{N}_G(P)N/N| = \frac{|\mathbf{N}_G(P)|}{|\mathbf{C}_N(P)|}.$$

Hence, in order to prove the theorem, we need to show that the character table of G determines $|\mathbf{C}_N(P)|$.

Now, let $x \in P$, and $K = x^G$. By Theorem 4.7, we know that the character table of G determines $|K \cap P|$. Moreover, by Theorem 4.3 (since N is normal in G, P acts via automorphism by conjugation on N), we have that

$$\mathbf{C}_{G/N}(Nx) = \mathbf{C}_G(x)N/N.$$

Since $X(G)$ determines $X(G/N)$, we have that the character table of G determines

$$|\mathbf{C}_{G/N}(Nx)| = |\mathbf{C}_G(x)N/N| = |\mathbf{C}_G(x)|/|\mathbf{C}_N(x)|\,,$$

by the second orthogonality relation. Since the character table of G determines $|\mathbf{C}_G(x)|$, we deduce that the character table of G determines $|\mathbf{C}_N(x)|$.

If P is abelian, then the map $\varphi : P \to \Phi(P)$ given by $x \mapsto x^p$ is an onto group homomorphism with kernel $\Omega_1(P) = \{x \in P |\, x^p = 1\}$ by Lemma 4.8. We can restate the formula in Theorem K in the following ways:

$$|\mathbf{C}_N(P)| = \left(\frac{\prod_{x\in P}|\mathbf{C}_N(x)|}{\prod_{x\in\Phi(P)}|\mathbf{C}_N(x)|^{|P:\Phi(P)|/p}}\right)^{\frac{p}{(p-1)|P|}},$$

if P is abelian, or

$$|\mathbf{C}_N(P)| = \left(\frac{\prod_{x\in P}|\mathbf{C}_N(x)|}{|N|^{|P|/p}}\right)^{\frac{p}{(p-1)|P|}},$$

if P has exponent p. (This latter formula was known to Wielandt, see [**Wie60**].)

Now, let $\{y_1, \ldots, y_k\}$ be representatives of the G-conjugacy classes of the p-elements of G (which are detectable in the character table by Theorem 4.6), and write $L_i = P \cap (y_i)^G$. Then

$$P = L_1 \cup \ldots \cup L_k$$

is a disjoint union.

By Theorem 4.7, we know how to compute $|L_i|$ in the character table. Furthermore $|\mathbf{C}_N(z)|$ is constant in L_i. This shows that

$$\prod_{x\in P} |\mathbf{C}_N(x)| = \prod_{j=1}^{k} |\mathbf{C}_N(y_j)|^{|L_j|}$$

is always computable from the character table of G. In particular, $|\mathbf{C}_N(P)|$ is computable from the character table of G if the exponent of P is p, and the Theorem is proven for groups with exponent p.

If P is abelian, then we need to calculate $|P : \Phi(P)|$, and determine which y_j lie on some G-conjugate of $\Phi(P)$.

First we claim that $NP \lhd G$. Indeed, since $\mathbf{O}_{p'}(G/N) = 1$, by the Hall-Higman Lemma 1.2.3 (Theorem 3.17), we have that $\mathbf{C}_{G/N}(\mathbf{O}_p(G/N)) \subseteq \mathbf{O}_p(G/N)$. Since P is abelian, PN/N is abelian and then $\mathbf{O}_p(G/N) \subseteq PN/N \subseteq \mathbf{C}_{G/N}(\mathbf{O}_p(G/N)) \subseteq \mathbf{O}_p(G/N)$. Then, PN/N is normal in G/N and the claim follows.

By Lemma 4.10 we know that $N\Phi(P)$ is the intersection of the kernels of the σ-fixed irreducible characters of G having N in its kernel, where σ is the Galois automorphism sending p-power roots of unity ξ to ξ^{p+1} and fixing p'-roots of unity. We deduce that y_j lies in some G-conjugate of $\Phi(P)$ if and only if y_j is in the kernel of the σ-invariant irreducible characters that contain N in their kernel. Indeed, if $y_j \in \Phi(P)^x$ for some $x \in G$, then

$$y_j \in N\Phi(P)^x = (N\Phi(P))^x = N\Phi(P) = \bigcap_{\substack{\chi\in\mathrm{Irr}(G/N)\\ \chi^\sigma=\chi}} \ker(\chi).$$

On the other hand, if y_j is in the kernel of the σ-invariant irreducible characters of G/N, then $y_j \in N\Phi(P)$. Since y_j is a p-element and $\Phi(P)$ is a Sylow p-subgroup of $N\Phi(P)$, we have that $y_j \in \Phi(P)^x$ for some $x \in N\Phi(P)$. This ends the case P abelian.

In order to show that the character table and the p-power map determine $|\mathbf{N}_G(P)|$, the same arguments of this proof show that we only need to be able to calculate

$$\prod_{x\in P} |\mathbf{C}_N(x^p)|$$

from the character table and then use Theorem K. If we know the p-power map, then we know the integers $1 \leqslant f_1, \ldots, f_k \leqslant k$ such that y_i^p is G-conjugate to y_{f_i}. Then

$$\prod_{x\in P} |\mathbf{C}_N(x^p)| = \prod_{j=1}^{k} |\mathbf{C}_N(y_{f_j})|^{|L_j|},$$

and the proof of the theorem is complete. □

4.4. Alternative proofs

To end this Chapter, we give alternative proofs of Theorem K. These nice proofs were given to us by Isaacs and Lyons, and the author would like to thank them for sharing them with her.

We need the following well-known results. The first of them is the module theoretic version of Clifford's theorem (Theorem 1.11).

THEOREM 4.12 (Clifford). *Let F be an arbitrary field, P a finite group, $Q \lhd P$ and let V be an irreducible $F[P]$-module. Let W be any irreducible $F[Q]$-submodule of V. Then*

(a) $V = W_1 \oplus \cdots \oplus W_k$, where the W_i are irreducible $F[Q]$-modules, and $W_i \cong W$.
(b) P acts transitively on $\{W_1, \ldots, W_k\}$.
(c) Viewed as an $F[Q]$-module, V is completely reducible.

PROOF. See Theorem 6.5 and Corollary 6.6 of [**Isa76**]. □

LEMMA 4.13. *Let A be an abelian group and suppose that there exists a faithful irreducible module W of $F[A]$, where F is an arbitrary field. Then A is cyclic.*

PROOF. See Lemma 0.5 of [**MW93**]. □

LEMMA 4.14. *Let P be a p-group in which every normal abelian subgroup is cyclic. Then:*

(a) If $p > 2$, then P is cyclic.
(b) If $p = 2$, P is dihedral, generalized quaternion or semidihedral.

In both cases, P has a cyclic normal subgroup of index p.

PROOF. See Theorem 6.12 of [**Isa08**] or Theorem III.7.6 of [**Hup67**]. □

We also need the notion of *primitive module*. Let F be a field, G a finite group and V an $F[G]$-module. Suppose that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

where the sum is direct and the W_i are subspaces of V which are transitively permuted by G. Then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ is a *imprimitive decomposition* of V. If V is irreducible and there is no such decomposition with $k > 1$, then V is a primitive $F[G]$-module. See Definition 5.7 of [**Isa76**] for more details.

ISAACS' PROOF. Isaacs' approach is similar to ours. As in our proof, we can reduce to the case where G is an elementary abelian q-group for some prime $q \neq p$, and $\mathbf{C}_G(P) = 1 = \mathbf{C}_P(G)$. In this case, we must show that

$$\prod_{x \in P} \frac{|\mathbf{C}_G(x)|^p}{|\mathbf{C}_G(x^p)|} = 1.$$

If G is not irreducible as a P-module, the result follows easily as in our proof, so we can assume the action is irreducible, that is, G has no proper P-invariant normal subgroups. Also, the result follows easily if $|P| = p$, so we can assume $|P| > p$.

Suppose first that G is a primitive $GF(q)[P]$-module and let Q be an abelian normal subgroup of P. Then Q acts on G and the action is faithful. We claim that G is irreducible as $GF(q)[Q]$-module. If not, we can write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$ with $k > 1$, G_i irreducible $GF(q)[Q]$-submodules of G and P acts transitively on $\{G_1, G_2, \ldots, G_k\}$ by Theorem 4.12. Hence G is not primitive, a contradiction. Thus G is a faithful irreducible $GF(q)[Q]$-module and by Lemma 4.13 we have that Q is cyclic. By Lemma 4.14, P has a cyclic normal subgroup N of index p. Since N is abelian, the action of N on G is Frobenius. Then, since N is cyclic we have that

$$\prod_{x \in N} \frac{|\mathbf{C}_G(x)|^p}{|\mathbf{C}_G(x^p)|} = 1.$$

Let $x \in P - N$, so $x^p \in N$. If $x^p \neq 1$, since N acts Frobenius on G, we have that $\mathbf{C}_G(x) = 1 = \mathbf{C}_G(x^p)$. Hence we may assume that $x^p = 1$. If $p > 2$, by Lemma 4.14 we know that P is cyclic and hence $x \in N$, a contradiction. Hence we may assume that $p = 2$ and $x^2 = 1$. Since in a generalized quaternion group there is just one involution, we have that P is either dihedral or semidihedral. Let $|P| = 2^n$ and write $N = \langle a \rangle$. Since $|P| = 2^n$, we have that $|N| = 2^{n-1}$. Now, write $y = a^{2^{n-2}}$ so $\mathbf{Z}(P) = \langle y \rangle$, and let $K = \langle x, y \rangle$. Then K is a 4-Klein group acting on G and we can apply Brauer's classical formula.

$$|\mathbf{C}_G(K)| = \sqrt{\frac{|\mathbf{C}_G(x)||\mathbf{C}_G(y)||\mathbf{C}_G(xy)|}{|G|}}.$$

Since $y \in N$ and the action of N on G is Frobenius, we have that $\mathbf{C}_G(y) = 1$, and hence $\mathbf{C}_G(K) \subseteq \mathbf{C}_G(y) = 1$. Hence

$$|\mathbf{C}_G(x)||\mathbf{C}_G(xy)| = |G|$$

and it follows that

$$\prod_{x \in P-N} \frac{|\mathbf{C}_G(x)|^p}{|\mathbf{C}_G(x^p)|} = 1.$$

We now assume that the action is imprimitive. Then

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_p,$$

and the stabilizer of all the G_i is a subgroup N with index p in P. Also $\mathbf{C}_G(N) = 1$, so

$$\prod_{x \in N} \frac{|\mathbf{C}_G(x)|^p}{|\mathbf{C}_G(x^p)|} = 1.$$

Let $x \in P - N$. We have that $(G_i)^x = G_{i+1}$, and $G_p^x = G_1$. Let $g \in G$ and write $g = g_1 g_2 \cdots g_p$ with $g_i \in G_i$. Note that g is fixed by x if and only if $g_i^x = g_{i+1}$ and $g_p^x = g_1$. Hence $g_1^{x^p} = g_1$ and

$$|\mathbf{C}_G(x)| = |\mathbf{C}_{G_1}(x^p)|.$$

Also, $\mathbf{C}_G(x^p)$ is the product of $\mathbf{C}_{G_i}(x^p)$, and these are conjugate under x. Thus

$$|\mathbf{C}_G(x^p)| = \prod_{i=1}^{p} |\mathbf{C}_{G_i}(x^p)| = |\mathbf{C}_{G_1}(x^p)|^p = |\mathbf{C}_G(x)|^p.$$

It follows that

$$\prod_{x \in P-N} \frac{|\mathbf{C}_G(x)|^p}{|\mathbf{C}_G(x^p)|} = 1.$$

and the theorem is proven. □

Lyons' proof follows a character theoretical approach, entirely different from ours. The key is the following result, kindly provided to us by Lyons.

THEOREM 4.15 (Lyons). *Let $F = GF(q)$, let P be a p-group with $p \neq q$, and let V be an $F[P]$-module. Then*

$$\dim_F(\mathbf{C}_V(P)) = \frac{p}{(p-1)|P|} \sum_{x \in P} \left(\dim_F(\mathbf{C}_V(x)) - \frac{1}{p} \dim_F(\mathbf{C}_V(x^p)) \right).$$

PROOF. Let $\tilde{V} = V \otimes_F \overline{F}$, where $\overline{F}$ is the algebraic closure of F. Then $\tilde{V}$ is an $\overline{F}[P]$-module and it is easy to check that $\dim_F(\mathbf{C}_V(H)) = \dim_{\overline{F}}(\mathbf{C}_{\tilde{V}}(H))$, for all $H \leqslant P$. Hence we may assume that $\overline{F} = F$.

Let χ be the Brauer character of P afforded by V. By Theorem 2.12 of [**Nav98a**], we have that χ is an ordinary character of P. We claim that $\dim_F(\mathbf{C}_V(H)) = [1_H, \chi_H]$ for all $H \leqslant P$. Indeed, let $H \leqslant P$ and notice that V is an $F[H]$-module. Since $\mathrm{char}(F) = q$ does not divide $|H|$, we have by Maschke's Theorem (Theorem 4.4) that V is completely reducible. Write $V = (V_1 \oplus V_2 \oplus \cdots \oplus V_r) \oplus (V_{r+1} \oplus \cdots \oplus V_k)$, with V_i is an irreducible FH-module, and $W = \mathbf{C}_V(H) = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. Now if $v \in V_i$ for $i = 1, \ldots, r$, we have that $v \cdot h = v$ for all $h \in H$, and hence V_i affords 1_H. Conversely, if V_i affords 1_H, then $v \cdot h = v$ for all $v \in V_i$, and hence $V_i \leqslant W$. Therefore, $\dim_F(W) = [1_H, \chi_H]$ and the claim follows.

Now let $\nu : P \to \mathbb{C}$ be the function defined as follows

$$\nu = \sum_{x \in P - \{1\}} \frac{1}{|\langle x \rangle|} (1_{\langle x \rangle} - 1_{\langle x^p \rangle}) + \frac{p-1}{p} \mu_{\{1\}},$$

where $\mu_{\{1\}}$ is the characteristic function of the singleton $\{1\}$. We claim that

$$\nu = \frac{p-1}{p} 1_P.$$

First note that $\nu(1) = \frac{p-1}{p}$ and if $1 \neq y \in P$, it is easy to see that

$$\nu(y) = \sum_{\substack{x \in P - \{1\} \\ y \in \langle x \rangle - \langle x^p \rangle}} \frac{1}{|\langle x \rangle|}.$$

Now note that $y \in \langle x \rangle - \langle x^p \rangle$ if and only if $\langle y \rangle = \langle x \rangle$. Indeed, if $o(y) < o(x)$, we would have that $o(y) \mid o(x^p)$ since $o(x^p) = o(x)/p$, and then $y \in \langle x^p \rangle$, a contradiction. Therefore $o(y) = o(x)$ and $\langle x \rangle = \langle y \rangle$. The converse is trivial. Then, if $o(y) = p^\alpha$, we have that

$$\nu(y) = \sum_{\substack{x \in P - \{1\} \\ \langle y \rangle = \langle x \rangle}} \frac{1}{|\langle x \rangle|} = \frac{(p-1)p^{\alpha-1}}{p^\alpha} = \frac{p-1}{p},$$

where the second equality follows from the fact that the number of generators of $\langle y \rangle$ is $(p-1)p^{\alpha-1}$. Hence

$$\nu = \frac{p-1}{p} 1_P$$

and the claim is proven. Hence,

$$\dim_F(\mathbf{C}_V(P)) = [1_P, \chi] = \frac{p}{(p-1)}[\nu, \chi].$$

On the other hand, since $|\langle x \rangle| = p|\langle x^p \rangle|$, we have that

$$\begin{aligned} |P|[\nu, \chi] &= \sum_{x \in P - \{1\}} \left([1_{\langle x \rangle}, \chi_{\langle x \rangle}] - \frac{1}{p}[1_{\langle x^p \rangle}, \chi_{\langle x^p \rangle}] \right) + \frac{p-1}{p}\chi(1) \\ &= \sum_{x \in P - \{1\}} \left(\dim_F(\mathbf{C}_V(x)) - \frac{1}{p}\dim_F(\mathbf{C}_V(x^p)) \right) + \frac{p-1}{p}\chi(1) \end{aligned}$$

Since $\chi(1) = \dim_F(V) = \dim_F(\mathbf{C}_V(1))$, we have that

$$\frac{p-1}{p}\chi(1) = \chi(1) - \frac{1}{p}\chi(1) = \dim_F(\mathbf{C}_V(1)) - \frac{1}{p}\dim_F(\mathbf{C}_V(1))$$

and therefore

$$[\nu, \chi] = \frac{1}{|P|} \sum_{x \in P} \left(\dim_F(\mathbf{C}_V(x)) - \frac{1}{p}\dim_F(\mathbf{C}_V(x^p)) \right).$$

This concludes the proof. □

As a consequence we obtain Theorem K.

Lyons' proof. As in our proof, we can reduce to the case where G is an elementary abelian q-subgroup for some prime $q \neq p$. Hence G is a $GF(q)[P]$-module and then using Theorem 4.15 we have that

$$\begin{aligned}|\mathbf{C}_G(P)| &= q^{\dim_{GF(q)}(\mathbf{C}_G(P))}\\ &= q^{\frac{p}{(p-1)|P|}\sum_{x\in P}\left(\dim_F(\mathbf{C}_G(x))-\frac{1}{p}\dim_F(\mathbf{C}_G(x^p))\right)}\\ &= \left(\prod_{x\in P}\frac{q^{\dim_F(\mathbf{C}_G(x))}}{q^{\frac{1}{p}\dim_F(\mathbf{C}_G(x^p))}}\right)^{\frac{p}{(p-1)|P|}}\\ &= \left(\prod_{x\in P}\frac{|\mathbf{C}_G(x)|}{|\mathbf{C}_G(x^p)|^{\frac{1}{p}}}\right)^{\frac{p}{(p-1)|P|}},\end{aligned}$$

as desired. □

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