FACULTAT DE MATEMÀTIQUES
DEPARTAMENT DE MATEMÀTIQUES

Regular orbits of actions of finite soluble groups. Applications

TESI DOCTORAL PERTANYENT AL PROGRAMA DE DOCTORAT EN MATEMÀTIQUES

PRESENTADA PER HANGYANG MENG
DIRIGIDA PER ADOLFO BALLESTER BOLINCHES

Juny 2019

A mi esposa Xiaoying y a mi hijo Zhengze

Agradecimientos

Esta tesis es fruto de un trabajo de investigación realizado durante los últimos tres años en el Departament de Matemàtiques de la Universitat de València bajo la dirección de mi tutor Adolfo Ballester Bolinches. Me gustaría agradecer a Adolfo no solo por su consejo, ánimo y ayuda desinteresada, sino también por tolerar mi carácter chino y comprender mis problemas con los idiomas. En el aspecto académico, aprecio su profesionalismo para brindarme la mejor guía posible. Sus conversaciones y experiencias vividas juntos me han ayudado a crecer como persona. Estoy muy orgulloso de ser un estudiante suyo. No soy Ramanujan, pero él es mi Hardy sin duda.

Me gustaría agradecer al profesor Ramón Esteban Romero por su constante apoyo y ayuda en muchos aspectos del quehacer académico. Acompañados por Ramón, también han participado en los seminarios organizados por Adolfo: Enrique Cosme Llópez, Paz Jiménez Seral, Mari Carmen Pedraza Aguilera, Vicente Pérez Calabuig y Francesca Spagnuolo, con los que he podido compartir nuestra dedicación por las matemáticas. ¡Gracias a todos!

Durante la realización de mi trabajo de tesis, he disfrutado de la beca 201606890006 otorgada por China Scholarship Council, organismo que depende del Ministerio de Educación del gobierno de la República Popular China. Agradezco la gentileza que ha tenido dicho organismo al facilitarme esta ayuda económica.

Contents

Introducción

A lo largo de esta tesis, todos los conjuntos, grupos, cuerpos y módulos considerados se suponen finitos.

Consideremos un grupo G actuando sobre un conjunto no vacío Ω. Decimos que la órbita de un $w \in \Omega$ es *regular* si $\mathrm{C}_G(w) = \{g \in G : wg = w\} = 1$; en este caso, dicha órbita consta de $|G|$ elementos. El estudio de órbitas regulares de grupos lineales, es decir, órbitas regulares de acciones de subgrupos de $\mathrm{GL}(V)$, siendo V un espacio vectorial, es importante en el desarrollo de muchas ramas de la teoría de grupos, incluendo los grupos resolubles, teoría de representaciones y grupos de permutaciones. De hecho, la solución de algunos problemas importantes en el área como el problema $k(GV)$ ([22]) depende de la existencia de este tipo de órbitas. De esta forma, el problema de la existencia de órbitas regulares es un área de investigación activa e interesante de la teoría de grupos.

Espuelas ([7, Theorem 3.1]) demostró que si G es un grupo de orden impar y V es un G-módulo fiel y completamente reducible de característica impar, entonces G tiene una órbita regular en $V \oplus V$. Dolfi y Jabara ([6, Theorem 2]) extendieron el resultado de Espuelas al caso en el que los 2-subgrupos de Sylow del producto semidirecto $[V]G$ de V y el grupo resoluble G son abelianos, y Yang ([28]) demostró que el mismo resultado es cierto si 3 no divide el orden del grupo resoluble G. Wolf ([24, Theorem A]) demuestra un resultado similar en el case de que G es superresoluble. En el caso de que G sea nilpotente, dicho resultado se puede mejorar ([20]).

Dolfi ([5, Theorem 1.4]), utilizando técnicas de Seress ([23, Theorem 2.1]), demostró que cualquier grupo resoluble G tiene una órbita regular en $V \oplus V \oplus V$ y si $(|V|, |G|) = 1$ o G es de orden impar, entonces G también tiene una órbita regular en $V \oplus V$ ([5, Theorems 1.1 , 1.5]).

Más recientemente, Yang ([29]) extiende algunos de estos resultados para subgrupos H de un grupo resoluble G. Demuestra que si V es un G-módulo fiel y completamente reducible (posiblemente de característica mixta) y si H es nilpotente o 3 no divide el orden de H, entonces H tiene al menos tres órbitas regulares en $V \oplus V$. Si los 2-subgrupos de Sylow del producto semidirecto $[V]H$ son abelianos, entonces H tiene al menos dos órbitas regulares en $V \oplus V$.

El primer resultado importante de nuestro trabajo de tesis proporciona condiciones suficientes más generales para la existencia de órbitas regulares. La mayor parte de los resultados anteriores son consecuencias inmediatas del mismo.

Teorema A. *Consideremos un grupo resoluble G, y V un G-módulo fiel y completamente reducible (posiblemente de característica mixta). Supongamos que H es un subgrupo de G tal que el producto semidirecto $[V]H$ es S_4-libre. Entonces H tiene al menos dos órbitas regulares en $V \oplus V$. Además, si H es $\Gamma(2^3)$-libre y $\mathrm{SL}(2,3)$-libre, entonces H tiene al menos tres órbitas regulares en $V \oplus V$.*

Recordamos que si G y X son grupos, decimos que G es X-libre si X no se puede obtener como un cociente de un subgrupo de G.

Desgraciadamente, la supersolubilidad de un subgrupo H no implica que VH es S_4-libre en general. Por lo tanto, el teorema A extiende todos los resultados mencionados anteriormente, excepto el teorema de Wolf [24, Theorem A]. En consecuencia, la pregunta de si el teorema de Wolf se verifica para cada subgrupo superresoluble de un grupo resoluble completamente reducible G de $\mathrm{GL}(V)$ es pertinente e interesante.

El segundo resultado importante de nuestro trabajo responde afirmativamente a dicha pregunta.

Teorema B. *Consideremos un grupo resoluble G y V un G-módulo fiel y completamente reducible (posiblemente de característica mixta). Supongamos que H es un subgrupo superresoluble de G. Entonces H tiene al menos una órbita regular en $V \oplus V$.*

La primera aplicación importante los resultados anteriores se sitúa en el contexto de la conjetura de Gluck.

Consideremos un grupo G. Como es habitual, denotamos por $\mathrm{Irr}(G)$ el conjunto de todos los caracteres irreducibles complejos de G y consideramos $b(G) = \text{máx}\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, el mayor grado de un carácter irreducible de G.

Gluck [9] conjeturó que si G es resoluble, entonces

$$|G : \mathrm{F}(G)| \leq b(G)^2,$$

siendo $\mathrm{F}(G)$ el subgrupo de Fitting de G.

La conjetura de Gluck aún permanece todavía sin resolver y ha sido objeto de un muy exhaustivo estudio (ver [2, 6, 7, 24, 28]).

Nuestro tercer resultado principal incluye casi todas las aportaciones conocidas a la conjetura de Gluck como casos particulares, y podría ser muy útil para resolver dicha conjetura en el futuro.

Teorema C. *Consideremos un grupo resoluble que satisface una de las siguientes condiciones:*

1. G *es* S_4*-libre;*
2. $G/\mathrm{F}(G)$ *es* S_4*-libre y* $\mathrm{F}(G)$ *es de orden impar;*
3. $G/\mathrm{F}(G)$ *es* S_3*-libre;*
4. $G/\mathrm{F}(G)$ *es superresoluble.*

Entonces la conjetura de Gluck es cierta para G*.*

La segunda aplicación de nuestros teoremas sobre órbitas regulares se localiza en el estudio de intersecciones de distinguidos subgrupos de grupos resolubles.

Dolfi [5] demostró que si π es un conjunto de números primos, el mayor grupo normal π-subgroup $\mathrm{O}_\pi(G)$ de un grupo π-soluble G es la intersección de tres G-conjugados de un π-subgrupo de Hall H de G.

Este resultado extiende los teoremas anteriores de Passman [21] (caso $|\pi| = 1$) y Zenkov [30] (caso H es nilpotente). Por otra parte, como Mann hizo notar en [17], los resultados de Passman implican que el subrupo de Fitting de un grupo resoluble G es la intersección de tres G-conjugados de un inyector nilpotente H de G.

Teniendo en cuenta los resultados anteriores, y dada la importancia de los subgrupos de prefrattini y los normalizadores de sistemas en el estudio estructural de los grupos resolubles, las siguientes preguntas son naturales e interesantes:

Problema 1. [19, Kamornikov, Problem 17.55] *¿Existe una constante positiva* k *tal que el subgrupo Frattini* $\Phi(G)$ *de un grupo resoluble* G *es la intersección de* k G*-conjugados de cualquier subgrupo prefrattini* H *de* G*?*

Problema 2. [19, Shemetkov and Vasil'ev, Problem 17.39] *¿Existe una constante positiva* k *tal que el hipercentro de cualquier grupo resoluble* G *coincide con la intersección de* k G*-conjugados de los normalizadores de de sistemas de* G*? ¿Cuál es el número mínimo con esta propiedad?*

Nuestro último resultado principal proporciona soluciones generales a los problemas anteriores.

Teorema D. *Consideremos un grupo G y un subgrupo H de G. Supongamos que se cumple una de las siguientes afirmaciones.*

1. *H es un subgrupo $\mathfrak{F}$-prefrattini de G para alguna formación saturada $\mathfrak{F}$;*
2. *$\Phi(G) = 1$ y H es un normalizador de $\mathfrak{F}$ de G para alguna formación saturada $\mathfrak{F}$;*
3. *H es un inyector de $\mathfrak{F}$ de G para alguna clase de Fitting $\mathfrak{F}$.*

Entonces existen x, y, $z \in G$ tal que $H \cap H^x \cap H^y \cap H^z = \mathrm{Core}_G(H)$. Además, si G es S_4-libre o $\mathfrak{F}$ está formada por grupos S_3-libres, existen x, $y \in G$ tales que $H \cap H^x \cap H^y = \mathrm{Core}_G(H)$.

La tesis se organiza de la siguiente manera. En el capítulo 1, presentamos notación, terminología y resultados preliminares. Las demostraciones de los teoremas A y B fundamentan el capítulo 2. Nuestras aportaciones a la conjetura de Gluck se presentan en el capítulo 3, incluida la demostración del teorema C y sus consecuencias. El estudio de las intersecciones de subgrupos de prefrattini y normalizadores de sistemas (teorema D) se presenta en el capítulo 4.

Introduction

Throughout this thesis, all groups, fields and modules to be considered are finite, and we assume this without further comment.

Let G be a group and let Ω be a G-set. The element w in Ω is in a *regular* orbit if $\mathrm{C}_G(w) = \{g \in G : wg = w\} = 1$, i. e., the orbit of w is as large as possible and it has size $|G|$. The study of regular orbits of actions of linear groups, that is, regular orbits of actions of subgroups of $\mathrm{GL}(V)$ on a vector space V plays an important role in many branches of group theory, including the study of soluble groups, representation theory of finite groups and finite permutation groups. In fact, the solution of some well-known problems such as the so-called $k(GV)$-problem ([22]) depends on the existence of such orbits. Consequently, the problem of the existence of regular orbits has attracted the attention of several authors and it is an active and interesting research area in group theory.

In order to understand and motivate what is to follow it is convenient to use some previous results as a model.

Espuelas (see [7, Theorem 3.1]) proved that if G is a group of odd order and V is a faithful and completely reducible G-module of odd characteristic, then G has a regular orbit on $V \oplus V$. Dolfi and Jabara ([6, Theorem 2]) extended Espuelas' result to the case where the Sylow 2-subgroups of the semidirect product $[V]G$ of V and the soluble group G are abelian, and Yang ([28]) proved that the same is true if 3 does not divide the order of the soluble group G. A result of Wolf ([24, Theorem A]) shows that a similar result holds if G is supersoluble (see also [20] for an improved result when G is nilpotent).

Dolfi ([5, Theorem 1.4]), reproving a result of Seress ([23, Theorem 2.1]), proved that any soluble group G has a regular orbit on $V \oplus V \oplus V$ and if either $(|V|, |G|) = 1$ or G is of odd order, then G has also a regular orbit on $V \oplus V$ ([5, Theorems 1.1, 1.5]).

More recently, Yang ([29]) extend some of these results to the case when H is a subgroup of the soluble group G by proving that if V is a faithful completely reducible G-module (possibly of mixed characteristic) and if either H is nilpotent or 3 does not divide the order of H, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product $[V]H$ are abelian, then H has at least two regular orbits on $V \oplus V$.

We prove that almost all previous results are consequences of the following surprising theorem.

Theorem A. *Let G be a finite soluble group and V be a finite faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a subgroup of G such that the semidirect product VH is* S_4*-free. Then H has at least two regular orbits on $V \oplus V$. Furthermore, if H is* $\Gamma(2^3)$*-free and* $\mathrm{SL}(2,3)$*-free, then H has at least three regular orbits on $V \oplus V$.*

Recall that if G and X are groups, then G is said to be X-*free* if X cannot be obtained as a quotient of a subgroup of G; $\Gamma(2^3)$ denotes the semilinear group of the Galois field of 2^3 elements.

The S_4-free hypothesis in Theorem A is not superfluous (see [6, Example 1]).

Note that the supersolubility of H does not imply that VH is S_4-free in general. Hence Theorem A covers all the aforementioned results except the theorem of Wolf [24, Theorem A]. Thus the answer to the question of whether or not Wolf's theorem holds for every supersoluble subgroup of a finite completely reducible soluble subgroup G of $\mathrm{GL}(V)$, even if the supersoluble subgroup is not completely reducible, is a natural next objective. Our second main result gives a complete answer to this question.

Theorem B. *Let G be a finite soluble group and V be a finite faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a supersoluble subgroup of G. Then H has at least one regular orbit on $V \oplus V$.*

Our results have found an application to Gluck's conjecture about large character degrees. Let G be a finite group and let $\mathrm{Irr}(G)$ denote the set of all irreducible complex characters of G and write $b(G) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, so that $b(G)$ is the largest irreducible character degree of G.

Gluck [9] conjectured that if G is soluble, then

$$|G : \mathrm{F}(G)| \leq b(G)^2,$$

where $\mathrm{F}(G)$ is the Fitting subgroup of G. Gluck's conjecture is still open and has been studied extensively (see [2, 6, 7, 24, 28]). Our third main result is a significant step to the solution of Gluck's conjecture subsuming the earlier ones, and it could be very useful to solve Gluck's conjecture in the future.

Theorem C. *Let G be a soluble group satisfying one of the following conditions:*

1. *G is* S_4*-free;*
2. $G/\,F(G)$ *is* S_4*-free and* $F(G)$ *is of odd order;*
3. $G/\,F(G)$ *is* S_3*-free;*
4. $G/\,F(G)$ *is supersoluble.*

Then Gluck's conjecture is true for G.

Another interesting problem where the regular orbits play an important role is the study of intersections of canonical conjugate subgroups of finite soluble groups.

Dolfi [5] proved that if π is a set of primes, the largest normal π-subgroup $O_\pi(G)$ of a π-soluble group G is the intersection of three G-conjugates of a given Hall π-subgroup H of G. This result extends earlier theorems of Passman [21] (case $|\pi| = 1$) and Zenkov [30] (case H nilpotent). On the other hand, as Mann pointed out in [17], the results of Passman imply that the Fitting subgroup $F(G)$ of a soluble group G is the intersection of three G-conjugates of a nilpotent injector H of G.

Due to the above results and the important role played by the system normalisers and prefrattini subgroups in the structural study of soluble groups, the following questions turn out to be natural and interesting:

Problem 1. [19, Kamornikov, Problem 17.55] *Does there exist an absolute constant k such that the Frattini subgroup* $\Phi(G)$ *of a soluble group G is the intersection of k G-conjugates of any prefrattini subgroup H of G?*

Problem 2. [19, Shemetkov and Vasil'ev, Problem 17.39] *Is there a positive integer k such that the hypercentre of any finite soluble group coincides with the intersection of k system normalisers of that group? What is the least number with this property?*

Our fourth main result provides general answers to the above two questions.

Theorem D. *Let G be a finite soluble group and let H be a subgroup of G. Assume that one of the following statements holds.*

1. *H is an* $\mathfrak{F}$*-prefrattini subgroup of G for some saturated formation* $\mathfrak{F}$*;*
2. $\Phi(G) = 1$ *and H is a* $\mathfrak{F}$*-normaliser of G for some saturated formation* $\mathfrak{F}$*;*
3. *H is an* $\mathfrak{F}$*-injector of G for some Fitting class* $\mathfrak{F}$*.*

Then there exists $x, y, z \in G$ *such that* $H \cap H^x \cap H^y \cap H^z = \mathrm{Core}_G(H)$*, the largest normal subgroup of* G *contained in* H*. Furthermore, if* G *is* S_4*-free or* $\mathfrak{F}$ *is composed of* S_3*-free groups, there exists* $x, y \in G$ *such that* $H \cap H^x \cap H^y = \mathrm{Core}_G(H)$*.*

Chapter 1 contains the basic material we need about finite groups and their representations. In Chapter 2 we set the scence, giving the proofs of Theorems A and B. Chapter 3 is about Gluck's Conjecture and includes the proof of Theorem C. The study of intersections of some canonical conjugate subgroups and the proof of Theorem D are the main contents of Chapter 4.

Chapter 1

Preliminaries

In this chapter, we collect some definitions and basic results that are needed to prove our main theorems. For further details, background and undefined notation, we refer the reader to the books [1, 3, 11, 13, 12].

1.1 Actions and modules

We recall again that if a group G is acting on a non-empty set Ω, an element w of Ω is in a *regular* orbit if $\mathrm{C}_G(w) = \{g \in G : wg = w\} = 1$, i.e., the orbit of w is as large as possible and it has size $|G|$.

Let G be a group and Ω be a transitive G-set. Recall a subset $\Delta \subseteq \Omega$ is said to be a block if $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \varnothing$ holds for every $g \in G$. Clearly every transitive G-set Ω has a block Δ such that $1 \leq |\Delta| < |\Omega|$ if $|\Omega| \geq 2$. If we take such block Δ of the maximal size, then $\mathrm{Stab}_G(\Delta)$ is maximal in G. (see [1, Definition 1.1.1 and Proposition 1.1.2.]).

Let $\mathbb{F}$ be a field and V be a vector space over a field $\mathbb{F}$. Let G be a group and ϕ a representation of G on V. Then we make V into a $\mathbb{F}$ G-module by extending linearly to $\mathbb{F}$ G the following G-action: $v^g = v^{\phi(g)}$, where $g \in G$ and $v \in V$. In this case, we say that V is a G-module over $\mathbb{F}$, or G-module if $\mathbb{F}$ is understood.

We say that V is a G-module of mixed characteristic if $V = V_1 \oplus \cdots \oplus V_n$, where for each i there exists a field $\mathbb{F}_i$ such that V_i is a G-module over $\mathbb{F}_i$.

A G-module V is called *irreducible* if $V \neq 0$ and 0 and V are the only G-submodules of V; V is said *completely reducible* if it is the sum of some irreducible modules. In this case, V is actually a direct sum of irreducible modules.

The following lemma is elementary and it will be used without further reference.

Lemma 1. *Suppose that a group G acts on a non-empty set Ω. Then:*

1. *If $|\Omega| - |\bigcup_{1\neq g\in G} \mathrm{C}_\Omega(g)| > k|G|$ for some non-negative integer k, then G has at least $k+1$ regular orbits on Ω. In particular, if $k=0$, then G has at least one regular orbit on Ω.*

2. *If G has k regular orbits on Ω, then a subgroup H of G has at least $|G:H|k$ regular orbits on Ω.*

Let S be a permutation group on a set Ω. If K is a group, we denote by $K \wr S$ the wreath product of K with S with respect to the action of S on Ω, that is,

$$K \wr S = \{(f,\sigma) \mid f : \Omega \to K, \sigma \in S\}$$

with the product $(f_1,\sigma_1)(f_2,\sigma_2) = (g, \sigma_1\sigma_2)$, where $g(w) = f_1(w)f_2(w^{\sigma_1})$ for all $w \in \Omega$.

If Y is a subgroup of K, we set $Y^\natural = \{(f,1) \in K \wr S \mid f(w) \in Y$ for all $w \in \Omega\}$. It is clear that $Y^\natural$ is normalised by S and $Y^\natural S \cong Y \wr S$. In particular, $B = K^\natural$ is called the *base* group of $K \wr S$.

If W is a K-module, then we can consider $G \wr S$, where $G = [W]K$ is the semidirect product of W with K. In this case, $W^\natural$ is a $K \wr S$-module with the action given by $g^{(f,\sigma)}(w) = g(w^{\sigma^{-1}})^{f(w^{\sigma^{-1}})}$.

If H_1 and H_2 are permutation groups on the sets X_1 and X_2 respectively, then $H_1 \wr H_2 = \{(f,\sigma) \mid f : X_2 \to H_1; \sigma \in H_2\}$ is a permutation group on $X_1 \times X_2$ with the action $(i,j)^{(f,\sigma)} = (i^{f(j)}, j^\sigma)$ (see [11, Satz I.15.3].)

We are interested here in regular orbits of a group G on completely reducible G-modules V over finite fields. Note that if $\mathbb{K}$ is a subfield of the field $\mathbb{F}$ and V is a completely reducible G-module over $\mathbb{F}$, then V is a completely reducible G-module over $\mathbb{K}$. Therefore, in looking for regular orbits of G on V, we can assume without loss of generality that $\mathbb{F}$ is a prime field.

An irreducible G-module V over $\mathbb{F}$ is called *imprimitive* if there is non-trivial decomposition of V into a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ $(n > 1)$ such that the set $\{V_1, \ldots V_n\}$ is permuted transitively by G; otherwise it is called *primitive*. A linear group $G \leq \mathrm{GL}(d, p^k)$, p a prime, is said to be primitive if the natural G-module is primitive.

Let G be a group and let V be a faithful G-module. Assume that $V = V_1 \oplus \ldots \oplus V_m (m \geq 2)$ is a decomposition of V into a direct sum of subspaces

$\{V_1, ..., V_m\}$ which are permuted transitively by G. Write $L = \mathrm{N}_G(V_1)$. Then $|G : L| = m$. Let $g_1 = 1, ..., g_m$ be a right transversal of L in G. If $\Omega = \{1, ..., m\}$, there exists a homomorphism $\sigma : G \to S_\Omega$ such that $Lg_i g = Lg_{i^{\sigma(g)}}$ for any $g \in G$. Let $K = L/\mathrm{C}_G(V_1)$ and $S = \sigma(G)$. Consider the map:

$$\tau \colon G \to K \wr S; g \mapsto (h_g, \sigma_g),$$

where $h_g \in K^\Omega$ is defined by $h_g(i) = g_i g g_{i^{\sigma(g)}}^{-1} \mathrm{C}_G(V_1)$ for all $i \in \Omega$, and $\sigma_g = \sigma(g)$ for all $g \in G$. Write $\widehat{G} = K \wr S$. Then $V_1^\Omega = \{f \mid f \colon \Omega \to V_1$ a map$\}$ is a $\widehat{G}$-module. Moreover:

Lemma 2. *1. τ is a monomorphism.*

2. The actions of G on V and $\tau(G)$ on V_1^Ω are equivalent.

3. $\widehat{G} = K^\natural \tau(G)$.

4. If $W_1 = \{f \in V_1^\Omega \mid f(i) = 0, \forall i \neq 1\}$, then $\mathrm{N}_{\tau(G)}(W_1)/\,\mathrm{C}_{\tau(G)}(W_1) \cong K$.

Proof. 1. It is straightforward to verify that τ is a homomorphism. Let $g \in G$ such that $\tau(g) = (h_g, \sigma_g) = 1$. Then $g_i = g_{i^{\sigma(g)}}$. Since $h_g(i) = 1$ for each i, it follows that $g_i g = a(i, g) g_i$ for some $a(i, g) \in \mathrm{C}_G(V_1)$. Let $v \in V$ and assume that $v = \sum_i w_i g_i$, where $w_i \in V_1$, and $vg = \sum_i w_i(g_i g) = \sum_i w_i a(i, g) g_i = \sum_i w_i g_i = v$. This means that $g \in C_G(V) = 1$.

2. Let $v = \sum_i w_i g_i \in V$, where $w_i \in V_1$. If we set $\varphi \colon V \longrightarrow V_1^\Omega$, $v \longmapsto w$, where $w(i) = w_i$ for each $i \in \Omega$, it follows that φ is an isomorphism between the vector spaces V and V_1^Ω such that, for every $g \in G$,

$$\varphi(vg) = \varphi\left(\sum_i w_i g_i g\right) = \varphi\left(\sum_i w_i (g_i g g_{i^{\sigma(g)}}^{-1}) g_{i^{\sigma(g)}}\right) = w',$$

where $w'(i) = w_{i^{\sigma(g)^{-1}}}(g_{i^{\sigma(g)^{-1}}} g g_i^{-1})$. Bearing in mind the natural action of $\widehat{G}$ on V_1^Ω, we have that $\varphi(vg) = \varphi(v)\tau(g)$ for all $v \in V$ and $g \in G$.

3. Let $(f, \alpha) \in \widehat{G}$, $f \in K^\natural$, $\alpha \in S$. Since $S = \sigma(G)$, there exists $g \in G$ such that $\sigma_g = \alpha$. Then $(f, \alpha) = (f h_g^{-1}, 1)(h_g, \sigma_g) \in K^\natural \tau(G)$, as desired.

4. This follows directly from 2. □

Assume that V is a G-module as above. It is clear that if $V = V_1 \oplus \cdots \oplus V_m$ is a minimal decomposition of V into a direct sum of subspaces which are permuted transitively by G, it follows that L is a maximal subgroup of G and so S is a non-trivial primitive permutation group on Ω.

If V is a faithful imprimitive G-module, then we may assume further that V_1 is an irreducible L-module. Therefore if we are interested in regular orbits of the action of G on V, we may assume, by Lemma 2, that G is a subgroup of a wreath product $\widehat{G} = K \wr S$, where K is a group, W is a faithful K-module and S is a non-trivial primitive permutation group on a set Ω such that $\widehat{G} = K^{\natural}G$ and $V = W^{\Omega}$. In this context, a result of Wolf [25] that provides a formula to count the exact number of regular orbits $\widehat{G}$ on W^{Ω} is extremely useful.

Let $\Pi_l(\Omega, S)$ denote the set of all partitions of length l of Ω having the property that the subgroup $\{s \in S \mid \Delta_i^s = \Delta_i \text{ for all } i\}$ of S is trivial. Let k be the number of regular orbits of K on W. Then the number of regular orbits of $\widehat{G}$ on W^{Ω} is

$$\frac{1}{|S|} \sum_{2 \leq l \leq m} P(k, l) |\Pi_l(\Omega, S)|,$$

where $P(k, l) = k!/(k - l)!$ if $k \geq l$ and $P(k, l) = 0$ otherwise.

The following elementary result is also useful.

Lemma 3. *Let G be a group and V be a faithful G-module such that $V = W_1 \oplus ... \oplus W_s$, where W_i is G-module, $1 \leq i \leq s$. If $G/\operatorname{C}_G(W_i)$ has t_i regular orbits on $W_i \oplus W_i$, then G has at least $\prod_{i=1}^{s} t_i$ regular orbits on $V \oplus V$.*

Let V be the Galois field $\mathrm{GF}(p^n)$ for some prime p and integer n. Then V is also a vector space over $\mathrm{GF}(p)$ of dimension n. Denote semi-linear group of V,

$$\Gamma(V)(or\ \Gamma(q^n)) = \{x \to ax^{\sigma} | a \in \mathrm{GF}(p^n)^*, \sigma \in \mathrm{Gal}(\mathrm{GF}(p^n)/\,\mathrm{GF}(p))\}.$$

1.2 Soluble S_4-free groups

Let X be a group and recall that a group G is said to be X-*free* if X cannot be obtained as a quotient of a subgroup of G.

In this section, we show some useful characterizations of S_3-free and S_4-free groups, and introduce some known notations, definitions and results about classes of groups. Recall that a group G is said to be *p-nilpotent*, p a prime, if G has a normal Hall p'-subgroup.

Lemma 4. *Let G be a soluble group and let H be a* Hall $\{2, 3\}$*-subgroup of G. Then G is S_3-free if and only if H is* 3*-nilpotent.*

Proof. If H is 3-nilpotent, then every $\{2,3\}$-subgroup of any section of G is 3-nilpotent. Consequently, G is S_3-free. Conversely, assume, arguing by contradiction, that G is S_3-free but H is not 3-nilpotent. Then H has a non-3-nilpotent subgroup K of minimal order. Then every proper subgroup of K is 3-nilpotent. Applying [11, Satz IV. 5.4], K has a normal Sylow 3-subgroup P of exponent 3 and a Sylow 2-group Q of K is cyclic. Moreover, $\Phi(K) = \Phi(Q) \times \Phi(P)$, $P/\Phi(P) \cong P\Phi(K)/\Phi(K)$ and, by [3, Theorem VII.6.18], $Q\Phi(K)/\Phi(K)$ is a cyclic group of order 2 acting faithfully and irreducibly on $P/\Phi(P)$. It follows from [3, Theorem B.9.8] that $P/\Phi(P)$ is cyclic of order 3. Therefore $K/\Phi(K) \cong \mathrm{S}_3$. This contradiction means that H is 3-nilpotent, as desired. □

Corollary 5. *Let G be a soluble S_3-free group such that $\mathrm{O}_{3'}(G) = 1$. Then G is of odd order.*

Proof. Let H be a Hall $\{2,3\}$-subgroup of G and let X be a Hall $3'$-subgroup of G. Then $H \cap X$ is a Sylow 2-subgroup of G and $G = HX$ by [3, Lemma A.1.6]. Hence $H \cap X \trianglelefteq H$ by Lemma 4. Therefore

$$(H \cap X)^G = (H \cap X)^{HX} = (H \cap X)^X \leq X.$$

This implies that $(H\cap X)^G$ is a $3'$-subgroup of G and so $H\cap X \leq (H\cap X)^G \leq \mathrm{O}_{3'}(G) = 1$. Thus G is of odd order. □

Lemma 6. *Let G be a soluble group with $\mathrm{O}_{2'}(G) = 1$. Then G is S_3-free if and only if G is S_4-free.*

Proof. If G is S_3-free, then clearly G is S_4-free. Now assume that the converse is false and derive a contradiction. Let G be a counterexample of minimal order. Then G is S_4-free but not S_3-free.

Denote $X = \mathrm{O}_2(G)$. Then $X = \mathrm{F}(G)$ since $\mathrm{O}_{2'}(G) = 1$ and, by [3, Theorem A.10.6], $\mathrm{C}_G(X) \leq X$. Hence, for every subgroup S of G such that $X \leq S$, we have $\mathrm{O}_{2'}(S) = 1$ and so S satisfies the hypotheses of the lemma. The minimal choice of G implies that S is S_3-free provided that S is a proper subgroup of G. In particular, by Lemma 4, G is a $\{2,3\}$-group and every proper subgroup of G/X is 3-nilpotent. If G/X were 3-nilpotent, then G would be 3-nilpotent and so S_3-free by Lemma 4. This would contradict our assumption. Consequently, G/X is a minimal non-3-nilpotent group. Denote with bars the images in $\overline{G} = G/X$. Then, by [11, Satz IV. 5.4], $\overline{G} = \overline{PQ}$ has a normal Sylow 3-subgroup $\overline{P}$ of exponent 3 and a cyclic Sylow 2-subgroup $\overline{Q}$. Moreover, since $\Phi(\overline{Q}) \leq \mathrm{O}_2(\overline{G}) = 1$, we have $\Phi(\overline{G}) = \Phi(\overline{Q}) \times \Phi(\overline{P}) = \Phi(\overline{P})$ and $\overline{Q}$ is of order 2. As in Lemma 4, $\overline{P}/\Phi(\overline{P})$ is of order 3. Thus $\overline{P}$ is of order 3 and $\Phi(\overline{P}) = 1$ since the exponent of $\overline{P}$ is 3. Therefore $G/X \cong S_3$.

Note that $\mathrm{O}_{2'}(G/\Phi(G)) = 1$ and so $G/\Phi(G)$ satisfies the hypotheses of the lemma. Hence, if $\Phi(G) \neq 1$, then $G/\Phi(G)$ is S_3-free and so it is 3-nilpotent by Lemma 4. Since $\Phi(G)$ is a 2-group, it follows that G is 3-nilpotent and so it is S_3-free by Lemma 4. This contradiction yields $\Phi(G) = 1$. By [3, Theorem A.10.6], $X = \mathrm{Soc}(G)$ is an abelian subgroup of G and there exists a subgroup M of G such that $G = XM$ and $X \cap M = 1$. Assume that X_1 and X_2 are two different minimal normal subgroups of G. Let $T_i/X_i = \mathrm{O}_{2'}(G/X_i)$. Since G/T_i is 3-nilpotent by the minimal choice of G, and $T_1 \cap T_2 \leq \mathrm{O}_{2'}(G) = 1$, it follows that G is 3-nilpotent. This contradicts our assumption. Consequently, X can be regarded as a faithful and irreducible M-module over the field of 2-elements. Recall that $M \cong G/X \cong S_3$, in this case, $|X| = 4$ and $G \cong S_4$. This final contradiction completes the proof of the lemma. □

Corollary 7. *Let G be a soluble group and let V be a faithful G-module over a field $\mathbb{F}$ of characteristic 2. Then the semidirect product VG is S_4-free if and only if G is S_3-free.*

Proof. Observe that $\mathrm{O}_{2'}(VG) \leq \mathrm{C}_G(V) = 1$. Thus if VG is S_4-free, then G is S_3-free by Lemma 6. Assume that G is S_3-free and there exist subgroups $A \lhd B \leq VG$ such that $B/A \cong S_4$. Then $VB/VA \cong B/A(B \cap V)$ has a section isomorphic to S_3 since $A(B \cap V)/A \leq \mathrm{O}_2(B/A)$. This means that $G \cong GV/V$ is not S_3-free. This contradiction implies that VG is S_4-free, as desired. □

Remark 8. The above lemma does not hold in general for non-soluble groups. For example, $G = A_5$ has a subgroup $\langle(123)\rangle\langle(12)(45)\rangle \cong \mathrm{S}_3$. But G is S_4-free because clearly $|\mathrm{S}_4| = 24 \nmid |G| = 60$.

Denote by $l_p(G)$ the p-length of a group G for some prime p.

Lemma 9. *Let G be a soluble group and H is a* Hall *$\{2, 3\}$-subgroup of G. Then G is* S_4*-free if and only if $l_2(H) \leq 1$.*

Proof. Firstly assume that G is S_4-free. Then $G/\mathrm{O}_{2'}(G)$ is S_4-free, it follows from Lemma 6 that $G/\mathrm{O}_{2'}(G)$ is S_3-free. By Lemma 4, the Hall $\{2, 3\}$-subgroup $H\,\mathrm{O}_{2'}(G)/\mathrm{O}_{2'}(G)$ of $G/\mathrm{O}_{2'}(G)$ is 3-nilpotent. Observe that $H \cap \mathrm{O}_{2'}(G) \leq \mathrm{O}_{2'}(H)$, thus $H/\mathrm{O}_{2'}(H)$ is 3-nilpotent and clearly $l_2(H) \leq 1$.

Now assume that $l_2(H) \leq 1$. Assume that G has a section isomorphic to S_4. Then $A/B \cong \mathrm{S}_4$ for some $B \lhd A \leq G$. Without loss of generality, we may assume that $A \cap H$ is a Hall $\{2, 3\}$-subgroup of A. Then $(A \cap H)B = A$ since $|A : B|$ is $\{2, 3\}$-number. Then $A/B = (A \cap H)B/B \cong (A \cap H)/(B \cap H) \cong \mathrm{S}_4$. But $l_2(A \cap H/B \cap H) \leq l_2(H) \leq 1$, which is a contradiction. Thus we have G is S_4-free. □

Recall that a formation is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and subdirect products. Therefore every group G has an smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*residual* of G and denoted by $G^{\mathfrak{F}}$. We say that $\mathfrak{F}$ is *saturated* if it is closed under Frattini extensions.

A class of groups $\mathfrak{F}$ is said to be a *Fitting class* if $\mathfrak{F}$ is a class under taking subnormal subgroups and normal products. Therefore every group G has a largest normal $\mathfrak{F}$-subgroup called $\mathfrak{F}$-*radical* and denoted by $G_{\mathfrak{F}}$.

Let Σ_3 and Σ_4 be the classes of soluble S_3-free groups and S_4-free groups respectively. It is clear that they are closed under taking subgroups and epimorphic images. In fact we have.

Lemma 10. *Let Σ be the class of groups Σ_3 or Σ_4 and G a group. Then:*

1. *If $G/\operatorname{O}_{2'}(G) \in \Sigma_4$, then $G \in \Sigma_4$.*
2. *If $G/\operatorname{O}_{3'}(G) \in \Sigma_3$, then $G \in \Sigma_3$.*
3. *Suppose that $L, K \trianglelefteq G$ such that $K \leq \Phi(G)$ and $L/K \in \Sigma$. Then $L \in \Sigma$.*
4. *Σ is a saturated Fitting formation which is closed under taking subgroups.*

Proof. 1. Assume that $G/\operatorname{O}_{2'}(G) \in \Sigma_4$. Let H be a Hall $\{2,3\}$-subgroup of G. Then $H\operatorname{O}_{2'}(G)/\operatorname{O}_{2'}(G)$ is a Hall $\{2,3\}$-subgroup of $G/\operatorname{O}_{2'}(G)$. By Lemma 9, we have $l_2(H\operatorname{O}_{2'}(G)/\operatorname{O}_{2'}(G)) \leq 1$. Observe that $\operatorname{O}_{2'}(G) \cap H \leq \operatorname{O}_{2'}(H)$ and so $l_2(H/\operatorname{O}_{2'}(H)) \leq 1$. Thus $l_2(H) \leq 1$, which implies that $G \in \Sigma_4$ by Lemma 9.

2. Let H be a Hall $\{2,3\}$-subgroup of G. Then $H\operatorname{O}_{3'}(G)/\operatorname{O}_{3'}(G)$ is a Hall $\{2,3\}$-subgroup of $G/\operatorname{O}_{3'}(G)$. Then $H\operatorname{O}_{3'}(G)/\operatorname{O}_{3'}(G))$ is 3-nilpotent by Lemma 4, and so $H/\operatorname{O}_{3'}(H))$ is 3-nilpotent. Thus H is 3-nilpotent, which implies that $G \in \Sigma_3$ by Lemma 4.

3. Suppose that $L, K \trianglelefteq G$ such that $K \leq \Phi(G)$ and $L/K \in \Sigma$ but $L \notin \Sigma_3$ (resp. Σ_4). Choose such counterexample (G, L, K) such that $|G|+|L|+|K|$ is minimal. Let H be a Hall $\{2,3\}$-subgroup of L.

   Write $X = \operatorname{O}_{3'}(L)$ (resp. $\operatorname{O}_{2'}(L)$) and clearly $X \trianglelefteq G$. Denote with bars the images in $\overline{G} = G/X$. We have that $(\overline{G}, \overline{L}, \overline{K})$ satisfies the hypotheses of the lemma. Hence, if $X \neq 1$, it follows that $\overline{L} \in \Sigma_3$

(resp. Σ_4). By Statement 1 (resp. Statement 2), it follows that $L \in \Sigma_3$ (resp. Σ_4), which is a contradiction. Consequently, $X = 1$.

Since K is nilpotent, we have that K is a 3-group (resp. 2-group). Let $T/K = \mathrm{O}_{3'}(L/K)$ (resp. $\mathrm{O}_{2'}(L/K)$). Then $T \trianglelefteq G$, and $T = KT_1$, where T_1 is a Hall $3'$-subgroup (resp. Hall $2'$-subgroup) of T. By Frattini argument, we have that $G = \mathrm{N}_G(T_1)T = \mathrm{N}_G(T_1)K = \mathrm{N}_G(T_1)$ since $K \leq \Phi(G)$. Thus $T_1 \trianglelefteq G$ and so $T_1 \leq X = 1$. Hence $T = K$. By Corollary 5 (resp. Lemma 6), L/K is of odd order (resp. $L/K \in \Sigma_3$).

If L/K is of odd order and K is 3-group, we have that L is of odd order and so it is S_3-free. Assume that $L/K \in \Sigma_3$ and K is a 2-group. Let H be the Hall $\{2,3\}$-subgroup of L. Then H/K is the Hall $\{2,3\}$-subgroup of L/K. It follows from Lemma 4 that H/K is 3-nilpotent. As K is a 2-group, H is 3-nilpotent. By Lemma 4, $L \in \Sigma_3 \subseteq \Sigma_4$. This final contradiction proves Statement 3.

4. We prove first that Σ is closed under taking normal products. Assume that $G = N_1N_2$ is the product of its normal subgroups N_1 and N_2. Suppose that N_1 and N_2 belong to Σ. Let H be a Hall $\{2,3\}$-subgroup of N. Then $H_i = N_i \cap H$ is a Hall $\{2,3\}$-subgroup of N_i, $i = 1,2$, and $H = H_1H_2$ is the normal product of H_1 and H_2. Assume that $\Sigma = \Sigma_4$. By Lemma 9, $l_2(H_i) \leq 1$, $i = 1,2$. Applying [11, Hilfssatz VI.6.4(c)], it follows that $l_2(H) \leq \max\{l_2(H_1), l_2(H_2)\} \leq 1$. Therefore $G \in \Sigma_4$.

   If $\Sigma = \Sigma_3$, then H_1 and H_2 are 3-nilpotent by Lemma 4. Then H is 3-nilpotent too, and so G is a Σ_3-group by Lemma 4.

   This proves that Σ is a subgroup closed Fitting class. In particular, Σ is closed under direct products. Consequently, Σ is a formation as well. Applying Statement 3, it follows that Σ is saturated.

□

1.3 Normalisers, prefrattini subgroups and injectors

Let $\mathfrak{F}$ be a formation. A maximal subgroup M of a group G containing $G^{\mathfrak{F}}$ is called $\mathfrak{F}$-*normal* in G; otherwise, M is said to be $\mathfrak{F}$-*abnormal*.

Assume that $\mathfrak{F}$ is *saturated*. Then, by a well-known theorem of Gaschütz-Lubeseder-Schmid [3, Theorem IV.4.6], there exists a collection of formations $\mathrm{F}(p) \subseteq \mathfrak{F}$, one for each prime p, such that $\mathfrak{F}$ coincides with the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in \mathrm{F}(p)$ for all primes p dividing $|H/K|$. In this case, we say that H/K is $\mathfrak{F}$-*central* in G and $\mathfrak{F}$ is *locally defined* by the $\mathrm{F}(p)$. H/K is called $\mathfrak{F}$-*eccentric* if it is not $\mathfrak{F}$-central.

Note that a chief factor H/K supplemented by a maximal subgroup M is $\mathfrak{F}$-central in G if and only if M is $\mathfrak{F}$-normal in G.

Every group G has a largest normal subgroup such that every chief factor of G below it is $\mathfrak{F}$-central in G. This subgroup is called the $\mathfrak{F}$-*hypercentre* of G and it is denoted by $\mathrm{Z}_{\mathfrak{F}}(G)$ (see [3, Section IV.6].)

Every soluble group G has a conjugacy class of subgroups, called $\mathfrak{F}$-*injectors*, which are defined to be those subgroups I of G such that if S is a subnormal subgroup of G, then $I \cap S$ is $\mathfrak{F}$-maximal subgroup of S ([3, Theorem IX.1.4]). Note that, in this case, $\mathrm{Core}_G(I) = G_{\mathfrak{F}}$.

In the following, we shall give a review of the definitions of $\mathfrak{F}$-normaliser and $\mathfrak{F}$-prefrattini subgroup of a soluble group and their cores.

Let Σ be a Hall system of the soluble group G (see [3, Chapter I, Section 1.4]). Let S^p be the p-complement of G contained in Σ, and denote by $W^p(G)$ the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p ($W^p(G) = G$, if the set of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p is empty). Then $W(G, \Sigma, \mathfrak{F}) = \bigcap_{p\in\pi(G)} W^p(G)$ is called the $\mathfrak{F}$-*prefrattini subgroup* of G associated to Σ. The prefrattini subgroups of G form a characteristic class of G-conjugate subgroups (see [1, Section 4.3] for an exhaustive study of prefrattini subgroups).

The set all $\mathfrak{F}$-prefrattini subgroups of a group G is denoted by $\mathbf{Pref}_{\mathfrak{F}}(G)$. We recall some known properties about $\mathfrak{F}$-prefrattini subgroups. Recall that a subgroup X of a group G *covers* the section A/B of G if $A \leq XB$ and *avoids* A/B if $X \cap A \leq B$.

Lemma 11 ([1, 10]). *Let G be a soluble group and N a normal subgroup of G.*

1. $\mathbf{Pref}_{\mathfrak{F}}(G)$ *is a G-conjugacy class of subgroups of G.*
2. $\mathbf{Pref}_{\mathfrak{F}}(G/N) = \{HN/N : H \in \mathbf{Pref}_{\mathfrak{F}}(G)\}$.
3. *If $H \in \mathbf{Pref}_{\mathfrak{F}}(G)$, then H avoids every complemented $\mathfrak{F}$-eccentric chief factor of G and covers the rest.*

According to [1, Proposition 4.3.17], the intersection $\mathrm{L}_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-abnormal maximal subgroups of a soluble group G is the core of every $\mathfrak{F}$-prefrattini subgroup of G and $\mathrm{L}_{\mathfrak{F}}(G)/\Phi(G) = \mathrm{Z}_{\mathfrak{F}}(G/\Phi(G))$ for every group G.

The elementary properties of the subgroup $\mathrm{L}_{\mathfrak{F}}(G)$ are collected in the following.

Lemma 12. *If N is a normal subgroup of a group G, then the following conditions hold:*

1. $\mathrm{L}_{\mathfrak{F}}(G)N/N \leq \mathrm{L}_{\mathfrak{F}}(G/N)$.

2. *If* $N \leq \mathrm{L}_{\mathfrak{F}}(G)$, *then* $\mathrm{L}_{\mathfrak{F}}(G/N) = \mathrm{L}_{\mathfrak{F}}(G)/N$.

3. $\mathrm{L}_{\mathfrak{F}}(G/\mathrm{L}_{\mathfrak{F}}(G)) = 1$.

Let $\mathrm{F}(p)$ be a particular family of formations locally defining $\mathfrak{F}$ and such that $\mathrm{F}(p) \subseteq \mathfrak{F}$ for all primes p. Let $\pi = \{p : \mathrm{F}(p) \neq \varnothing\}$. For an arbitrary soluble group G and a Hall system Σ of G, choose for any prime p, the p-complement $K^p = S^p \cap G^{\mathrm{F}(p)}$ of the $\mathrm{F}(p)$-residual $G^{\mathrm{F}(p)}$ of G, where S^p is the p-complement of G in Σ. Then $D_{\mathfrak{F}}(\Sigma) = G_\pi \cap (\bigcap_{p\in\pi} \mathrm{N}_G(K^p))$, where G_π is the Hall π-subgroup of G in Σ, is the $\mathfrak{F}$-*normaliser* of G associated to Σ. The $\mathfrak{F}$-normalisers of G are a characteristic class of G-conjugate subgroups. There were introduced by Carter and Hawkes and coincide with the classical system normalisers of Hall when $\mathfrak{F}$ is the formation of all nilpotent groups (see [3, Sections V.2 and V.3] for details).

According to [1, Proposition 4.2.6], if D is an $\mathfrak{F}$-normaliser of G, then $\mathrm{Core}_G(D) = \mathrm{Z}_{\mathfrak{F}}(G)$.

Chapter 2

Main theorems

2.1 Primitive case

In attaining our objective, which is to prove Theorem A and Theorem B for primitive modules, the following lemmas are crucial. The first one concerns primitive soluble linear groups over a field of characteristic two.

Lemma 13. *Let G be a soluble group and V be a faithful primitive G-module over a field $\mathbb{F}$ of characteristic 2. Assume that VG is S_4-free, then G has at least three regular orbits on $V \oplus V$ unless $|V| = 2^3$ and $G = \Gamma(V)$. In this case, G has exactly two regular orbits on $V \oplus V$.*

Proof. Let A be an abelian normal subgroup of G. Since V is a primitive G-module and A is normal in G, then V_A is a faithful and homogeneous A-module by Clifford's Theorem (see [18, Theorem 0.1]). By [18, Lemma 0.5], A is cyclic. Then [18, Corollary 1.10] applies. Let $F = \mathrm{F}(G)$ be the Fitting subgroup of G. Then F is of odd order since V is faithful for F, and it is a central product $F = ET$ of two normal subgroups E and T of G such that $Z = E \cap T = \mathrm{Soc}(\mathrm{Z}(F))$ and $1 \neq T = \mathrm{C}_G(E)$ is cyclic. Hence $Z = \mathrm{Z}(E)$. Moreover, the Sylow subgroups of E are cyclic of prime order or extraspecial of prime exponent. Set $e^2 = |F/Z|$. Then 2 does not divide e.

Applying [29, Theorem 2.3], we have that G has at least four regular orbits on $V \oplus V$ unless $e = 1, 3, 9$.

Assume that $e = 1$. Then F is abelian. By [18, Corollary 2.3], G is isomorphic to a subgroup of $\Gamma(V) = \Gamma(2^n)$. If $n > 3$ and $0 \neq v \in V$, then $\mathrm{C}_G(v)$ has at least three regular orbits on V by [24, Proposition 9]. Hence G has at least three regular orbits on $V \oplus V$. If either $n = 1$ or G is of prime order, then G has at least three regular orbits on $V \oplus V$. Suppose that $1 \neq G$ is not of prime order. Then $n = 3$ since G is S_3-free and $\Gamma(2^2) \cong S_3$. In this case, $G \cong \Gamma(2^3)$ and so G has just two regular orbits on $V \oplus V$.

Suppose that either $e = 3$ or $e = 9$. Then every Hall $3'$-subgroup of E is contained in Z. Therefore $E/Z = LZ/Z$, where L is the Sylow 3-subgroup of E. Note that L is extra-especial since F is non-abelian.

Let $A = \mathrm{C}_G(T) \subseteq \mathrm{C}_G(Z)$. By [18, Corollary 1.10], E/Z is a completely reducible G/F-module and a faithful A/F-module over GF(3), the finite field of 3-elements. Hence $\mathrm{O}_3(A/F) = 1$. Let Q be a Sylow 2-subgroup of A. By Lemmas 4 and 6, every Hall $\{2, 3\}$-subgroup of G is 3-nilpotent. In particular, $QE/Z = E/Z \rtimes QZ/Z$ is nilpotent. Since $QF/F \leq A/F$ acts faithfully on E/Z, we have that $Q \leq F$. Consequently, $Q = 1$ and A is a $2'$-group. Furthermore, A preserves the non-degenerated symplectic form with respect to which E/Z is a symplectic space over GF(3) (see [11, Satz III.13.7]). Therefore A/F is either isomorphic to a completely reducible subgroup of $\mathrm{Sp}(2, 3) \cong \mathrm{SL}(2, 3)$ ($e = 3$) or a subgroup of $\mathrm{Sp}(4, 3)$ ($e = 9$). Applying [5, Lemma 3.2], we conclude that $|A/F|$ divides 3 or 5. In particular, $|A : F| \leq 5$.

Let W be an irreducible submodule of V_T. Then $V_T = sW$ for some positive integer s and $|G : A|$ divides $\dim W$ by [11, Hilfssatz II.3.11]. Since W is faithful for T and T is cyclic, we have that $|W| = 2^a$, where a is the smallest positive integer such that $|T| \mid 2^a - 1$ (see [18, Example 2.7]).

Applying [18, Corollary 2.6], we have that $\dim V$ is divisible by $e \cdot \dim W$. Therefore, $|V| = 2^{eab}$ for some $b > 0$.

Suppose that $a \leq 3$. Then $a = 2$ since $3 \mid |T|$ and T is of order 3. If $|G/A| = 2$, there exists an element $g \in G \setminus A$ of order 2 such that $G = A\langle g\rangle$ since A is $2'$-group. Then $T\langle g\rangle \cong S_3$, contrary to assumption. Hence $G = A$ is a $2'$-group. By [4, Theorem 2.2], we have G has a regular orbit on V. Hence G has at least $|V| \geq |W| = 4$ regular orbits on $V \oplus V$.

Assume that $a \geq 4$. We next prove that F has at least a regular orbit on V. It is enough to prove that

$$|V \setminus \bigcup_{S \in \mathcal{P}} \mathrm{C}_V(S)| > 0,$$

where $\mathcal{P}$ be the set of all subgroups of prime order of F.

Let $S \in \mathcal{P}$. Note that T acts fixed point freely on V so that $\mathrm{C}_V(S) = \{0\}$ if $S \leq T$. If S is not contained in T, then $|\mathrm{C}_V(S)| \leq 2^{\frac{1}{2}aeb}$ by [27, Lemma 2.4]. Note that every subgroup in $\mathcal{P}$ not contained in T has order 3 and the number of such subgroups is 12 if $e = 3$ and 120 if $e = 9$. Since $2^{3ab} - 12 \cdot 2^{\frac{3}{2}ab} > 0$ and $2^{9ab} - 120 \cdot 2^{\frac{9}{2}ab} > 0$ if $a \geq 4$ and $b \geq 1$, it follows that F has a regular orbit on V. Hence $\mathrm{C}_G(v) \cap F = 1$ for some $v \in V$.

Let $C = \mathrm{C}_G(v)$. We may assume that $C \neq 1$. Note that $|C| \leq |G/F| = |G : A||A : F| \leq 5a$. Since $|(C \cap A)| = |(C \cap A)F/F| \leq |A/F|$ and $|A/F|$ is of prime order, we can apply [27, Lemma 2.4] to conclude that there exists at most one subgroup S contained in $C \cap A$ such that $|\,\mathrm{C}_V(S)| \leq 2^{\frac{3}{4}aeb}$. For a subgroup $S \subseteq C \setminus A$, we have $|\,\mathrm{C}_V(S)| \leq 2^{\frac{1}{2}aeb}$.

Since $a \geq 4, eb \geq 3$, we have that $2^{aeb-1} > (5a-1)2^{\frac{1}{2}aeb}$, $2^{aeb-2} > 2^{\frac{3}{4}aeb}$ and $2^{aeb-2} > 10a$. Therefore

$$|V| - (|C|-1)2^{\frac{1}{2}aeb} - 2^{\frac{3}{4}aeb} > 2|C|,$$

and then

$$|V \setminus \bigcup_{1 \neq g \in C} \mathrm{C}_V(g)| > 2|C|.$$

Consequently, $C = \mathrm{C}_G(v)$ has at least three regular orbits on V. This completes the proof of the lemma. □

Lemma 14. *Let G be a soluble primitive group of $\mathrm{GL}(d,p)$, p a prime number, and let V be the natural G-module. Assume that H is a subgroup of G such that the semidirect product VH is S_4-free. Then H has at least three regular orbits on $V \oplus V$ unless one of the following two cases occurs:*

1. *$d = 2$, $p = 3$ and $H = \mathrm{SL}(2,3)$.*

2. *$d = 3$, $p = 2$ and $H = \Gamma(V) \cong \Gamma(2^3)$.*

In both exceptional cases, H has just two regular orbits on $V \oplus V$.

Proof. Assume that p is odd. Then [5, Theorem 3.4] tells us that $H \leq G$ has at least $p \geq 3$ regular orbits on $V \oplus V$ unless one the following cases occurs:

1. $G = \mathrm{GL}(2,3)$. Then G has just one regular orbit on $V \oplus V$. Observe that $G/\mathrm{Z}(G) \cong \mathrm{PGL}(2,3) \cong S_4$, thus H is a proper subgroup of G since H is S_4-free. If $|G : H| \geq 3$, then H has at least three regular orbits on $V \oplus V$. Otherwise, $H = \mathrm{SL}(2,3)$ and the exceptional case 1 appears.

2. $G = \mathrm{SL}(2,3)$. Then G has just two regular orbits on $V \oplus V$. Hence if H is proper in G, H has at least four regular orbits on $V \oplus V$. Otherwise $H = G = \mathrm{SL}(2,3)$ and again the exceptional case 1 emerges.

3. $G = (Q_8 * Q_8)K \leq \mathrm{GL}(4,3)$, where K is isomorphic to a subgroup of index 1, 2 or 4 of $\mathrm{O}^+(4,2)$. If $\mathrm{O}_{2'}(H) = 1$, then H is 3-nilpotent by Lemmas 4 and 6. Using GAP, one can check that H has at least three regular orbits on $V \oplus V$.

If $O_{2'}(H) \neq 1$, then $O_{2'}(H)$ is isomorphic to C_3 or $C_3 \times C_3$. Then $H \leq N_G(O_{2'}(H))$. One checks by GAP that H has at least three regular orbits on $V \oplus V$.

Suppose that $p = 2$. If $H = G$, by Lemma 13, then H has at least three regular orbits on $V \oplus V$ unless $H = G = \Gamma(2^3) \leq \mathrm{GL}(3,2)$. In this exceptional case, H has just two regular orbits on $V \oplus V$.

Thus we can assume that H is a proper subgroup of G. By [5, Theorem 3.4], H has at least four regular orbits on $V \oplus V$ provided that G is not isomorphic to $\mathrm{GL}(2,2)$, $3^{1+2}.\mathrm{SL}(2,3)$ or $3^{1+2}.\mathrm{GL}(2,3)$.

If H is a proper subgroup of $G = \mathrm{GL}(2,2)$, then H is of prime order and there exists $v \in V$ such that $C_H(v) = 1$. Hence H has at least $|V| = 4$ regular orbits on $V \oplus V$.

Suppose that G is isomorphic to $3^{1+2}.\mathrm{SL}(2,3)$ or $3^{1+2}.\mathrm{GL}(2,3)$ (as a subgroup of $\mathrm{GL}(6,2)$). By Corollary 7, H is S_3-free. In this case, one checks by GAP that H has at least three regular orbits on $V \oplus V$. □

Lemma 15. *Let G be a soluble primitive group of $\mathrm{GL}(d,p)$, p an odd prime, and let V be the natural G-module. If H is a subgroup of G of odd order, then H has at least five regular orbits on $V \oplus V$.*

Proof. If G is of odd order, then G has at least five regular orbits on $V \oplus V$ by [6, Proposition 3 (a)] and so does H. Thus we may assume that 2 divides $|G|$. Then $|G : H| \geq 2$. By [5, Theorem 3.4] that G has at least $p \geq 3$ regular orbits on $V \oplus V$, and so H has at least six regular orbits on $V \oplus V$, unless G is isomorphic to $\mathrm{GL}(2,3)$, $\mathrm{SL}(2,3)$ or $(Q_8 * Q_8)K \leq \mathrm{GL}(4,3)$, where K is isomorphic to a subgroup of index 1, 2 or 4 of $O^+(4,2)$.

Assume that $G = \mathrm{GL}(2,3)$ or $\mathrm{SL}(2,3)$. Then G has at least one regular orbit on $V \oplus V$ and $|G : H| \geq 8$. It follows that H has at least eight regular orbits on $V \oplus V$.

Assume that $G = (Q_8 * Q_8)K \leq \mathrm{GL}(4,3)$, where K is isomorphic to a subgroup of index 1, 2 or 4 of $O^+(4,2)$. Then H is isomorphic to a subgroup of $C_3 \times C_3$. Using GAP, one can check that H has a regular orbit on V and so H has at least $|V| = 3^4$ regular orbits on $V \oplus V$.

The proof of the lemma is complete. □

Now we deal with the supersoluble primitive cases.

Lemma 16. *Let G be a supersoluble group and V be a faithful primitive G-module over $\mathrm{GF}(2)$. Then G has at least four regular orbits on $V \oplus V$ unless $G = \Gamma(V)$ and $|V| = 2^n$, $2 \leq n \leq 4$, and in these cases, G has exactly $n - 1$ regular orbits on $V \oplus V$.*

Proof. Let A be the maximal abelian normal subgroup of G and clearly $A \le \mathrm{C}_G(A) \trianglelefteq G$. If $A < \mathrm{C}_G(A)$, then we can take T/A is a chief factor of G such that $T \subseteq \mathrm{C}_G(A)$. Since G is supersoluble, T/A is cyclic and $T = \langle A, x\rangle$ for some $x \in \mathrm{C}_G(A)$. Then T is an abelian normal subgroup of G, contrary to the choice of A. Thus $A = \mathrm{C}_G(A)$. Since V is a primitive G-module, V_A is homogeneous. By [18, Lemma 2.2], V_A is irreducible. It follows from [18, Theorem 2.1] that $G \le \Gamma(V)$. Write $|V| = 2^n$ for some integer $n \ge 1$.

Firstly we assume that $G = \Gamma(V)$. Equivalently, if suffices to consider the regular orbits of $\Gamma(2^n)$ acting on the additional group of the field $\mathrm{GF}(2^n)$. Take the field automorphism $\sigma : \mathrm{GF}(2^n) \to \mathrm{GF}(2^n); u \mapsto u^2$, and the Galois group $\mathrm{Gal}(\mathrm{GF}(2^n)/\mathrm{GF}(2)) = \langle\sigma\rangle$ is of order n. Take $x = 1$, the identity element of the field $\mathrm{GF}(2^n)$ and clearly $\mathrm{C}_{\mathrm{GF}(2^n)}(x) = \langle\sigma\rangle$, denote by C.

For each prime p dividing n, $\langle\sigma^{\frac{n}{p}}\rangle$ is the unique subgroup of C with order p since C is cyclic. Then we have $\mathrm{C}_{\mathrm{GF}(2^n)}(\sigma^{\frac{n}{p}}) = \{u \in \mathrm{GF}(2^n) | u^{2^{\frac{n}{p}}} = u\}$ is a subfield of $\mathrm{GF}(2^n)$, which is isomorphic to $\mathrm{GF}(2^{\frac{n}{p}})$. Thus $|\mathrm{C}_{\mathrm{GF}(2^n)}(\sigma^{\frac{n}{p}})| = 2^{\frac{n}{p}}$.

In order to prove C has at least four regular orbits on $\mathrm{GF}(2^n)$ when $n \ge 5$, it suffices to show that

$$2^n - \sum_{p|n} 2^{\frac{n}{p}} > 3n$$

holds for $n \ge 5$. Observe that $\sum_{p|n} 2^{\frac{n}{p}} \le log_2 n \cdot 2^{\frac{n}{2}}$. It is not difficult to check that $2^n - \sum_{p|n} 2^{\frac{n}{p}} > 2^n - log_2 n \cdot 2^{\frac{n}{2}} > 3n$ for $n \ge 8$ and it is easy to find the inequality holds for $n = 5, 6, 7$.

Thus we have proved that $G \le \Gamma(V)$ has at least four regular orbits on $V \oplus V$ when $n \ge 5$. Now it suffice to discuss the following cases:

$n = 1$. $|V| = 2$ and $G = 1$. Then G has exactly four regular orbit on $V \oplus V$.

$n = 2$. $|V| = 2^2$ and $G \le \Gamma(V) \cong S_3$. If $G < \Gamma(V)$, then G has a regular orbit on V. Then G has at least $|V| = 4$ regular orbits on $V \oplus V$. If $G = \Gamma(V)$, in this case, G has exactly one regular orbit on $V \oplus V$.

$n = 3$. $|V| = 2^3$ and $G \le \Gamma(V) \cong [C_7]C_3$. If $G = \Gamma(V)$, then G has exactly two regular orbits on $V \oplus V$. Thus, if $G < \Gamma(V)$, G has at least four regular orbits on $V \oplus V$.

$n = 4$. $|V| = 2^4$ and $G \le \Gamma(V) \cong [C_{15}]C_4$. If $G = \Gamma(V)$, then G has exactly three regular orbits on $V \oplus V$. Thus, if $G < \Gamma(V)$, G has at least six regular orbits on $V \oplus V$.

Thus the lemma is proved completely. □

Lemma 17. *Let G be a soluble primitive group of* $\mathrm{GL}(d, 2)$, *and let V be the natural G-module. Assume that H is a supersoluble subgroup of G. Then*

H has at least three regular orbits on $V \oplus V$ unless one of the following two cases occurs:

(a) $d = 2$ and $H = \Gamma(V) \cong S_3$, has just one regular orbit on $V \oplus V$;

(b) $d = 3$ and $H = \Gamma(V) \cong \Gamma(2^3)$, has just two regular orbits on $V \oplus V$.

Furthermore if H is of odd order, then H has four regular orbits on $V \oplus V$ unless the case (b) occurs.

Proof. If $H = G$, then G is a supersoluble. It follows from Lemma 16 that the lemma is true. Now we may assume that $H < G$. Thus we can assume that H is a proper subgroup of G. By [5, Theorem 3.4], H has at least four regular orbits on $V \oplus V$ provided that G is not isomorphic to $\mathrm{GL}(2,2), 3^{1+2}.\mathrm{SL}(2,3)$ or $3^{1+2}.\mathrm{GL}(2,3)$.

If H is a proper subgroup of $G = \mathrm{GL}(2,2) \cong S_3$, then H is of prime order and there exists $v \in V$ such that $\mathrm{C}_H(v) = 1$. Hence H has at least $|V| = 4$ regular orbits on $V \oplus V$.

Suppose that G is isomorphic to $3^{1+2}.\mathrm{SL}(2,3)$ or $3^{1+2}.\mathrm{GL}(2,3)$ (as a subgroup of $\mathrm{GL}(6,2)$). In this case, one checks by GAP that H has at least three(four if $|H|$ is odd) regular orbits on $V \oplus V$. □

2.2 Regular orbits on power sets

The main goal of this section is to establish some results on regular orbits of permutation groups which play a crucial part in the proof of Theorem A.

Let S be a permutation group on a set Ω and consider the induced action of S on the power set $\mathcal{P}(\Omega)$ of Ω. Following [18, Chapter II, Section 5], we say that a regular orbit of S on $\mathcal{P}(\Omega)$ generated by $\Delta \subseteq \Omega$ is *strong* if the setwise stabilizer $\mathrm{Stab}_S(\Delta)$ is trivial, and $|\Delta| \neq \frac{|\Omega|}{2}$.

It is clear that a subset Δ of Ω generates a strong regular orbit of S on $\mathcal{P}(\Omega)$ if and only if so does $\Omega - \Delta$. Then we conclude that the number of the strong regular orbits of S on $\mathcal{P}(\Omega)$ is even.

Gluck (see [18, Theorem 5.6]) proved that a primitive soluble permutation group S acting on a set Ω has an strong regular orbit on $\mathcal{P}(\Omega)$ if $|\Omega| > 9$. Zhang [31] proves that in this case S has at least 8 regular orbits on $\mathcal{P}(\Omega)$.

As a consequence, if S is a group of odd order, then S has at least two strong regular orbits on $\mathcal{P}(\Omega)$. We can push these ideas a bit further to show the following:

Lemma 18. *Let S be a primitive soluble permutation group of odd order on a set Ω. Then S has at least 18 strong regular orbits on $\mathcal{P}(\Omega)$, unless one of the following cases occurs:*

1. $|\Omega| = 3$ *and* $S \cong A_3$*;*

2. $|\Omega| = 5$ *and* $S \cong C_5$*;*

3. $|\Omega| = 7$ *and* $S \cong \Gamma(2^3)$.

In the exceptional cases 1 and 3, S has exactly two strong regular orbits on $\mathcal{P}(\Omega)$ *and, in case 2, S has exactly* 6 *strong regular orbits on* $\mathcal{P}(\Omega)$.

Proof. Assume that S is a primitive soluble permutation group of odd order on Ω such that $(S, \Omega) \neq (A_3, 3), (C_5, 5), (\Gamma(2^3), 7)$. We shall prove that S has at least 18 strong regular orbits on $\mathcal{P}(\Omega)$.

Applying [11, Satz II.3.2], we conclude that S has a unique minimal normal subgroup, V say; $V = \mathrm{C}_S(V)$ and V is transitive and regular on Ω. Hence $|V| = |\Omega| = p^m$ for a prime p and a positive integer m. Moreover, if H is the stabilizer of an element of Ω, we have that $S = NH$ and $N \cap H = 1$. Furthermore, $|S| \leq \frac{1}{2}|\Omega|^{13/4}$ by [18, Corollary 3.6]. Let $n(g)$ be the number of cycles of $g \in S$ on Ω. Then $n(g) \leq 3|\Omega|/4$ by [18, Lemma 5.1] and g stabilizes exactly $2^{n(g)}$ subsets of Ω.

Next consider $X = \mathcal{P}(\Omega)$. We prove that

$$2^{|\Omega|} - \frac{1}{2}|\Omega|^{13/4} 2^{3|\Omega|/4} \geq 18 \cdot \frac{1}{2}|\Omega|^{13/4} \geq 18|S|.$$

It is rather easy to see that the inequality holds if $|\Omega| \geq 81$. In this case, S has at least 18 regular orbits on X. Hence we assume in the sequel that $|\Omega| \leq 80$.

Suppose that $|\Omega| = p$. Then S is isomorphic to a subgroup of $[C_p]C_{p-1}$. If S is cyclic of order p, then $p \geq 7$ because $(S, |\Omega|) \neq (A_3, 3)$ and $(C_5, 5)$. In this case, every non-empty proper subset of Ω generates a strong regular orbit on $\mathcal{P}(\Omega)$. Thus S has exactly $(2^p - 2)/p \geq 18$ strong regular orbits on $\mathcal{P}(\Omega)$. Assume that $1 \neq |H| \mid p-1$. Since $|S|$ is odd, we have $p \geq 7$. If $p = 7$, then $|H| = 3$ and so $G \cong [C_7]C_3 \cong \Gamma(2^3)$, contrary to assumption. Therefore $p \geq 11$. Let q be a prime different from p and let T be a subgroup of S of order q. Then T is contained in some conjugate of H, and T fixes exactly $2^{1+(p-1)/q}$ subsets of Ω. Since S contains exactly p subgroups of order q, it follows that the number of non-regular orbits of S is at most $p\sum_{q|(p-1)} 2^{1+(p-1)/q}$. Then we have

$$2^p - p \sum_{3 \leq q|(p-1)} 2^{1+(p-1)/q} > 17p(p-1) \geq 17|S|.$$

Therefore S has at least 18 regular orbits on X.

Suppose that $|\Omega| = p^2$. Then $p = 5$ or 7 since $|S|$ is odd. Assume that $p = 5$. Since V is a faithful H-module, H is isomorphic to a subgroup of

GL(2, 5). Hence $|H| \leq 15$ and so $|S| \leq 5^3 \cdot 3$. In this case, $n(g) \leq 15$ for any $g \in S - \{1\}$. Observe that

$$|X| - 2^{15} \cdot 5^3 \cdot 3 = 2^{25} - (2^{15} \cdot 5^3 \cdot 3) \geq 18 \cdot 5^3 \cdot 3 \geq 18|S|.$$

Now we assume that $p = 7$. Then $|S| \leq 3^2 \cdot 7^3$, $n(g) \leq 28$ for any $g \in S - \{1\}$, and

$$|X| - 2^{28} \cdot 3^2 \cdot 7^3 = 2^{49} - (2^{28} \cdot 3^2 \cdot 7^3) \geq 18 \cdot 3^2 \cdot 7^3 \geq 18|S|.$$

In both cases, S has at least 18 regular orbits on X.

Suppose that $|\Omega| = p^3 \leq 80$. Then $p = 3$ and H is isomorphic to an irreducible subgroup of GL(3, 3). By [18, Corollary 2.13], H can be considered as a subgroup of $\Gamma(3^3)$ or $C_2 \wr S_3$. Since H is of order odd and irreducible, the later case is impossible. Thus H is a subgroup of $\Gamma(3^3)$ and $|H| \leq 3 \cdot 13$. Then $|S| \leq 3^4 \cdot 13$. Let $g \in S - \{1\}$. Assume that g has not fixed points on Ω. Then g is either a product of a 13-cycle and some 3-cycles or a product of 3-cycles. Hence $n(g) \leq 27/3 = 9$. Suppose that g has at least one fixed point. Then g belongs to a conjugate of H. Since the action of H on Ω is equivalent to the action of H on V by conjugation, we have that the number of fixed points of g is just $|C_V(g)|$. If order of g is 3, then $|C_V(g)|$ and $n(g) \leq (27-3)/3+3 = 11$. If order of g is 13, then $n(g) \leq (27-1)/13+1 = 3$. Consequently, $n(g) \leq 11$ for any $g \in S - \{1\}$. Note that

$$|X| - 2^{11} \cdot 3^4 \cdot 13 = 2^{27} - (2^{11} \cdot 3^4 \cdot 13) \geq 18 \cdot 3^4 \cdot 13 \geq 18|S|.$$

Hence S has at least 18 regular orbits on X.

If $|\Omega| = 3$ and $S \cong A_3$, then S has exactly two regular orbits on X. If $|\Omega| = 7$ and $S \cong \Gamma(2^3)$, each element of order 7 in S is a 7-cycle and each element of order 3 in S is the product of two disjoint 3-cycles. Thus every two-element subset and every five-element subset of Ω generate a strong regular orbit on X and S has exactly two strong regular orbits on X. If $|\Omega| = 5$ and $S \cong C_5$, then S has exactly $(2^5 - 2)/5 = 6$ strong regular orbits on X. This completes the proof of the lemma. □

Lemma 19. *Let S be a primitive soluble permutation group on a set Ω. Assume that $S^* \leq S$ and S^* acts non-transitively on Ω. Then one of the following occurs:*

1. *S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$; or*
2. *for each S^*-orbit Δ on Ω with $|\Delta| > 4$, we have $O^{2'}(S^*)$ acts transitively on Δ and $|\Pi_3(\Delta, S^*)| \geq |S^*_\Delta|$, where S^*_Δ is the permutation group induced by the action of S^* on Δ.*

Proof. It is clear that we may assume that $|\Omega| \geq 5$ and $1 \neq S^*$ is a proper subgroup of S.

Since S is a primitive soluble permutation group on Ω, we can apply [11, Satz II.3.2] to conclude that S has a unique minimal normal subgroup, V say; $V = C_S(V)$ and V is transitive and regular on Ω. Moreover, if H is the stabilizer of an element $\alpha \in \Omega$, we have that $S = VH$ and $V \cap H = 1$. Moreover, the action of H on Ω is equivalent to the action of H on M by conjugation. In particular, if $\beta \in \Omega$, we have that $\mathrm{C}_H(\beta) := \mathrm{Stab}_H \beta = \mathrm{C}_H(v)$ for some $v \in V$.

Assume that $|V| = |\Omega|$ is a prime number, p say. Then V is a Sylow p-subgroup of S and so S^* is a p'-group. Without loss of generality, we may assume that S^* is contained in H. Let $\beta \in \Omega \setminus \{\alpha\}$. Then $\mathrm{C}_H(\beta) = \mathrm{C}_H(v)$ for some $1 \neq v \in V$. Therefore, $\mathrm{Stab}_H \beta = 1$. Then if $\Delta_1 = \{\beta\}$ and $\Delta_2 = \{\alpha, \beta\}$, it follows that $\mathrm{Stab}_{S^*} \Delta_i = 1$, $i = 1, 2$. Then Δ_1, Δ_2, $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$ are in different regular orbits of S^* on $\mathcal{P}(\Omega)$. Thus S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

Consequently, we may suppose that $|\Omega|$ is not a prime. If S has a strong regular orbit on $\mathcal{P}(\Omega)$, then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$ since $|S : S^*| \geq 2$. Then we may assume that S has no strong regular orbit on $\mathcal{P}(\Omega)$.

Therefore we only have to consider the exceptional cases (5) and (6) of [18, Theorem 5.6].

1. Suppose that $(S, |\Omega|) = (A\Gamma(2^3), 8)$.

   Since S^* is not transitive on Ω, the length of every orbit of S^* on Ω is at most 7.

   Assume that S^* has an orbit Δ on Ω such that $|\Delta| = 7$. Without loss of generality, we may suppose that α is fixed by all elements of S^* and so S^* is contained in H. By Lemma 18, $H \cong \Gamma(2^3)$ has a strong regular orbit on $\mathcal{P}(\Delta)$. Let Δ_1 is a two-element subset of Δ. Then $\mathrm{Stab}_{S^*}(\Delta_1) \leq \mathrm{Stab}_H(\Delta_1) = 1$. Denote $\Delta_2 = \{\alpha\} \cup \Delta_1$. Since $\mathrm{Stab}_{S^*}(\Delta_i)$=1 for $i = 1$, 2, it follows that Δ_1, Δ_2, $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$ lie in different regular orbits of S^* on $\mathcal{P}(\Omega)$. Thus S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

   Assume that S^* has an orbit Δ on Ω such that $|\Delta| = 6$. Then there exists $\beta \in \Delta$ with $|S^* : \mathrm{C}_{S^*}(\beta)| = 6$. Hence $|\mathrm{C}_{S^*}(\beta)|$ divides $2^2 \cdot 7$. On the other hand, $\mathrm{C}_{S^*}(\beta) \leq \mathrm{C}_S(\beta) \cong \Gamma(2^3)$. Thus $|\mathrm{C}_{S^*}(\beta)|$ divides 7. If $|\mathrm{C}_{S^*}(\beta)| = 7$, then $|S^*| = 2 \cdot 3 \cdot 7$. This is a contradiction since S has no subgroup of such order. Thus $\mathrm{C}_{S^*}(\beta) = 1$. Therefore if $\Delta_1 = \{\beta\}$ and $\Delta_2 = \{\gamma, \beta\}$ for some $\gamma \in \Omega \setminus \Delta$, it follows that $\mathrm{Stab}_{S^*} \Delta_i = 1$, $i = 1, 2$.

Then Δ_1, Δ_2, $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$ are in different regular orbits of S^* on $\mathcal{P}(\Omega)$. Thus S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

2. Suppose that $|\Omega| = 9$ and S is the semidirect product of $C_3 \times C_3$ with D_8, SD_{16}, $\mathrm{SL}(2,3)$ or $\mathrm{GL}(2,3)$.

   In this case, we may assume that $V = C_3 \times C_3$ and S is a subgroup of $\mathrm{AGL}(2,3)$, the semidirect product of $C_3 \times C_3$ with $\mathrm{GL}(2,3)$. In particular, H is a subgroup of $A = \mathrm{GL}(2,3)$.

   Since S^* is a $\{2,3\}$-group acting non-transitively on Ω and $|\Omega| = 9$, we have that the length of an orbit of S^* on Ω with more than 4 elements is either 6 or 8.

   Suppose that S^* has an orbit Δ on Ω such that $|\Delta| = 8$. Without loss of generality, we may suppose that α is fixed by all elements of S^* and so S^* is contained in H. If $\beta \in \Delta$, we have that $\mathrm{C}_A(\beta)$ has two fixed points, β, γ say, and a orbit Γ of length 6 on Δ. Let $\mu \in \Gamma$ and let $\Delta_1 = \{\beta\}, \Delta_2 = \{\gamma, \mu\}$ and $\Delta_3 = \Gamma \setminus \{\mu\}$. Observe that $\bigcap_i \mathrm{Stab}_{S^*}(\Delta_i) \leq \bigcap_i \mathrm{Stab}_H(\Delta_i) = 1$. Thus $|\Pi_3(\Delta, S^*)| \geq |S^*_\Delta|$. Since $|S^* : \mathrm{C}_{S^*}(\beta)| = 8$, we have $\mathrm{O}^{2'}(S^*)$ acts transitively on Δ. In this case, 2 holds.

   Suppose that S^* has an orbit Δ on Ω such that $|\Delta| = 6$. Put $\Gamma = \Omega \setminus \Delta$. Then S^* acts on Γ and $S^*/\mathrm{C}_{S^*}(\Gamma)$ is isomorphic to a subgroup of S_3. Note that $\mathrm{C}_{S^*}(\Gamma)$ is also isomorphic to a subgroup of S_3. Thus $|S^*|$ divides 36. Since $|\Delta| = 6$ divides $|S^*|$, we have that $|S^*| \in \{6, 12, 18, 36\}$.

   If $|S^*| = 6$ then S^* has a strong regular orbit on Δ, and so S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. If $|S^*| = 18$, one can check by GAP that S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. If $|S^*| = 12$ or 36, one can check by GAP that S^* satisfies Statement 2. □

Lemma 20. *Let S be a primitive soluble permutation group on a set Ω. Assume that $S^* \leq S$, S^* is transitive on Ω and S^* is S_4-free. Then either S^* has a strong regular orbit on $\mathcal{P}(\Omega)$ or S^* satisfies one of the following statements:*

1. *$|\Omega| = 2$ and $S^* \cong S_2$;*

2. *$\mathrm{O}^{2'}(S^*)$ acts transitively on the set Ω and there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_{S^*} \Delta_i = 1$.*

Proof. We may assume that $|\Omega| > 2$. If S has a strong regular orbit on $\mathcal{P}(\Omega)$, then so does S^*. Thus we may assume that $(S, |\Omega|)$ is one of the exceptional cases of [18, Theorem 5.6].

If $|\Omega| = 3$ and $S = S_3$, then either $S^* \cong C_3$ or $S^* \cong S_3$. If $S^* \cong C_3$, then S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. If $S^* \cong S_3$, then S^* satisfies Statement 2.

Assume that $|\Omega| = 4$ and $S = A_4$ or S_4. Since S^* is an S_4-free transitive subgroup of S, it follows that $S^* \cong A_4$, D_8 or $C_2 \times C_2$. Then $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω and there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of type $(1, 1, 2)$ of Ω such that $\bigcap_i \mathrm{Stab}_{S^*}(\Delta_i) = 1$. Thus Statement 2 holds.

Assume that $|\Omega| \in \{5, 7, 8, 9\}$. In this case, by [25, Theorem 3.1], there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_{S^*} \Delta_i = 1$.

Assume that $|\Omega| = 5$ and $S = F_{10}$ or F_{20}. Then $S^* \cong C_5$, F_{10} or F_{20}. If $S^* \cong C_5$, then S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. If $S^* \cong F_{10}, F_{20}$, then S^* satisfies Statement 2.

Assume that $|\Omega| = 7$ and $S = F_{42}$. Then $S^* \cong C_7, F_{21}$ or F_{42}. If $S^* \cong C_7$ or F_{21}, then S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. If $S^* \cong F_{42}$, then S^* satisfies Statement 2.

If $|\Omega| = 8$ and $S = A\Gamma(2^3)$, then one can check by GAP that $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω. Therefore S^* satisfies Statement 2.

Assume that $|\Omega| = 9$ and $S = \mathrm{AGL}(2, 3)$. If $\mathrm{O}^{2'}(S^*)$ is not transitive on Ω, then one can check by GAP that S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. □

Corollary 21. *Let S be a primitive soluble permutation group on a set Ω. Assume that $S^* \leq S$ is of odd order and S^* is transitive on Ω. Then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$, unless one of the following cases occurs:*

1. *$|\Omega| = 3$ and $S^* \cong A_3$;*
2. *$|\Omega| = 7$ and $S^* \cong \Gamma(2^3)$.*

In the exceptional cases, S^* has just two strong regular orbits on $\mathcal{P}(\Omega)$.

Proof. Assume that S has a strong regular orbit on $\mathcal{P}(\Omega)$. If S is of odd order, then by Lemma 18, then S has at least four strong regular orbits on $\mathcal{P}(\Omega)$ unless $(S, |\Omega|) = (A_3, 3)$ or $(\Gamma(2^3), 7)$. Then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$ unless $(S^*, |\Omega|) = (A_3, 3)$ or $(\Gamma(2^3), 7)$. If S is of order even, then $|S : S^*| \geq 2$. Since S has at least two strong regular orbits on $\mathcal{P}(\Omega)$, S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

If S has no strong regular orbit on $\mathcal{P}(\Omega)$, then $(S, |\Omega|)$ is one of exceptional cases (2)–(9) of [18, Theorem 5.6].

If $|\Omega| = 3$ and $S = S_3$, then $S^* \cong A_3$. We are in case (1). If $|\Omega| = 4$ and $S = A_4$ or S_4, then S has no odd order subgroups which are transitive on Ω. If $|\Omega| = 5$ and $S = F_{10}$ or F_{20}, then $S^* \cong C_5$ and S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

Assume that $|\Omega| = 7$ and $S = F_{42}$. Then $S^* \cong C_7$ or $\Gamma(2^3)$. If $S^* \cong C_7$, then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. If $S^* \cong \Gamma(2^3)$, we are in case (2). If $|\Omega| = 8$ and $S = A\Gamma(2^3)$, then S has no subgroup of odd order which is transitive on Ω.

Assume that $|\Omega| = 9$ and $S = \mathrm{AGL}(2,3)$. Then S^* is a subgroup of a Sylow 3-subgroup of S. It can be proved, using GAP, that S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. □

Lemma 22. *Let H_1 and H_2 be permutation groups on the sets X_1 and X_2 respectively. If H_1 has $2s$ strong regular orbits on $\mathcal{P}(X_1)$ and H_2 has $2t$ strong regular orbits on $\mathcal{P}(X_2)$. Then $H = H_1 \wr H_2$ has at least $2st$ strong regular orbits on $\mathcal{P}(X_1 \times X_2)$. If $s = 1$, then $H_1 \wr H_2$ has exactly $2t$ strong regular orbits on $\mathcal{P}(X_1 \times X_2)$.*

Proof. Assume that $\Delta_1, \ldots, \Delta_s$, $X_1 \setminus \Delta_1, \ldots, X_1 \setminus \Delta_s$ belong to different strong regular orbits of H_1 on $\mathcal{P}(X_1)$ and that $\Gamma_1, \ldots, \Gamma_t$, $X_2 \setminus \Gamma_1, \ldots, X_2 \setminus \Gamma_t$ belong to different strong regular orbits of H_2 on $\mathcal{P}(X_2)$. Let us denote

$$\Sigma_{ij} = \Delta_i \times \Gamma_j \bigcup (X_1 \setminus \Delta_i) \times (X_2 \setminus \Gamma_j),$$

for $1 \leq i \leq s, 1 \leq j \leq t$.

We prove first that $\mathrm{Stab}_H(\Sigma_{ij}) = 1$. Let $y \in X_2$, we denote $\varepsilon(y) = |\{(x_1, x_2) \in \Sigma_{ij} \mid x_2 = y\}|$. Since $|\Delta_i| \neq |X_1 \setminus \Delta_i|$, it is clear that $\varepsilon(y) = |\Delta_i|$ (respectively, $|X_1 \setminus \Delta_i|$) if and only if $y \in \Gamma_j$ (respectively, $y \in X_2 \setminus \Gamma_j$).

Let $(f, \sigma) \in \mathrm{Stab}_H(\Sigma_{ij})$ and $y \in \Gamma_j$. Then $(\Delta_i \times \{y\})^{(f,\sigma)} = \Delta_i^{f(y)} \times \{y^\sigma\} \subseteq \Sigma_{ij}$. Observe that $\varepsilon(y^\sigma) = |\Delta_i^{f(y)}| = |\Delta_i|$, which implies that $y^\sigma \in \Gamma_j$. Thus $\sigma \in \mathrm{Stab}_{H_2}(\Gamma_j) = 1$. We also have $\Delta_i^{f(y)} = \Delta_i$ and so $f(y) \in \mathrm{Stab}_{H_1}(\Delta_i) = 1$. Now we can argue similarly with $y \in X_2 \setminus \Gamma_j$ and conclude that $f = 1$. Thus $\mathrm{Stab}_H(\Sigma_{ij}) = 1$.

Observe that $|\Sigma_{ij}| \neq \frac{|X_1||X_2|}{2}$ and so Σ_{ij} generates a strong regular orbit of H on $\mathcal{P}(X_1 \times X_2)$.

Assume that there exists $(f, \sigma) \in H$ such that $\Sigma_{ij}^{(f,\sigma)} = \Sigma_{uv}$ for some indices $1 \leq i, u \leq s, 1 \leq j, v \leq t$. If $y \in X_2$, then $(\Delta_i \times \{y\})^{(f,\sigma)} = \Delta_i^{f(y)} \times y^\sigma \in \Sigma_{uv}$ and $\Delta_i^{f(y)} = \Delta_u$ or $X_1 \setminus \Delta_u$. This implies that $i = u$. Analogously, $j = v$. By using a similar argument, we can prove Σ_{ij} is not H-conjugate to $X_1 \times X_2 \setminus \Sigma_{uv}$. Thus $\Sigma_{ij}, X_1 \times X_2 \setminus \Sigma_{ij}$ belong to different strong regular orbits of H on $\mathcal{P}(X_1 \times X_2)$. Then we conclude that H has at least $2st$ strong regular orbits on $\mathcal{P}(X_1 \times X_2)$.

Assume that $s = 1$. We prove that the orbits generated by $\Sigma_{1j}, X_1 \times X_2 \setminus \Sigma_{1j}$ are exactly the strong regular orbits of H on $\mathcal{P}(X_1 \times X_2)$.

Let $\Phi \in \mathcal{P}(X_1 \times X_2)$ such that $\mathrm{Stab}_H(\Phi) = 1$. Then $\Phi = \bigcup_{y \in X_2} \Phi_y \times \{y\}$, where $\Phi_y = \{x \in X_1 \mid (x, y) \in \Phi\}$. Assume there exists $y_0 \in X_2$ such that $\mathrm{Stab}_{H_1}(\Phi_{y_0}) \neq 1$. Take $1 \neq u \in \mathrm{Stab}_{H_1}(\Phi_{y_0})$ and let $f \in H_1^{X_2}$ such that $f(y) = u$ if $y = y_0$ and $f(y) = 1$ otherwise. Then it follows that $1 \neq (f, 1) \in \mathrm{Stab}_H(\Phi) = 1$. This contradiction yields $\mathrm{Stab}_{H_1}(\Phi_y) = 1$ for each $y \in X_2$.

Since all H_1-regular orbits are generated by Δ_1 and $X_1 \setminus \Delta$, it follows that Φ_y is H_1-conjugate to Δ_1 or $X_1 \setminus \Delta_1$ for each $y \in X_2$. Let $B_1 = \{y \in X_2 \mid \Phi_y$ is H_1-conjugate to $\Delta_1\}$ and $B_2 = \{y \in X_2 \mid \Phi_y$ is H_1-conjugate to $X_1 \setminus \Delta_1\}$. Observe that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = X_2$.

For each $y \in X_2$, there exists $u_y \in H_1$ such that $\Phi_y^{u_y} = \Delta_1$ (if $y \in B_1$) or $= X_1 \setminus \Delta_1$ (if $y \in B_2$). Let $g \in H_1^{X_2}$ such that $g(y) = u_y$ for each $y \in X_2$. Write $\widetilde{\Phi} = \Phi^{(g,1)} = (\bigcup_{y \in B_1} \Phi_y^{g(y)} \times \{y\}) \cup (\bigcup_{y \in B_2} \Phi_y^{g(y)} \times \{y\}) = (\bigcup_{y \in B_1} \Delta_1 \times \{y\}) \cup (\bigcup_{y \in B_2} (X_1 \setminus \Delta_1) \times \{y\}) = \Delta_1 \times B_1 \bigcup (X_1 \setminus \Delta_1) \times B_2$.

Assume that $\mathrm{Stab}_{H_2}(B_1) \neq 1$, and let $1 \neq \sigma \in \mathrm{Stab}_{H_2}(B_1)$. Since $B_2 = X_2 \setminus B_1$, we have $\sigma \in \mathrm{Stab}_{H_2}(B_2)$. Thus $1 \neq (1, \sigma) \in \mathrm{Stab}_H(\widetilde{\Phi}) = 1$, which is a contradiction. Therefore B_1 generates a regular orbit of H_2 on X_2. Without loss of generality, we may assume that $B_1^\alpha = \Gamma_j$ for some $\alpha \in H_2$. Then $B_2^\alpha = (X_2 \setminus B_1)^\alpha = X_2 \setminus \Gamma_j$. So we have $\widetilde{\Phi}^{(1,\alpha)} = \Delta_1 \times \Gamma_j \bigcup (X_1 \setminus \Delta_1) \times (X_2 \setminus \Gamma_j) = \Sigma_{1j}$. Thus Φ is H-conjugate to Σ_{1j}, as desired. □

Remark 23. If $s \neq 1$, $H = H_1 \wr H_2$ has not exactly $2st$ strong regular orbits on the power set of $X_1 \times X_2$ in general. Let $(H_1 = \langle (1, 2, 3, 4, 5) \rangle$, $X_1 = \{1, 2, 3, 4, 5\})$ and $(H_2 = \langle (1, 2, 3) \rangle$, $X_2 = \{1, 2, 3\})$.

Note that the regular orbits generated by $\Delta_1 = \{1\}$, $\Delta_2 = \{1, 2\}$, $\Delta_3 = \{1, 3\}$, $X_1 \setminus \Delta_1$, $X_1 \setminus \Delta_2$, $X_1 \setminus \Delta_3$ are exactly the strong regular orbits of H_1 on $\mathcal{P}(X_1)$. It is also clear that H_2 has exactly two strong regular orbits on $\mathcal{P}(X_2)$, namely the ones generated by $\Gamma_1 = \{1\}$ and $X_2 \setminus \Gamma_1$.

According to Lemma 22, we have that the subsets $\Sigma_{i1} = \Delta_i \times \Gamma_j \cup (X_1 \setminus \Delta_i) \times (X_2 \setminus \Gamma_1)$, for $1 \leq i \leq 3$, generate 6 strong regular orbits of H on $\mathcal{P}(X_1 \times X_2)$. The subset

$$\Phi = \Delta_1 \times \{1\} \bigcup \Delta_2 \times \{2\} \bigcup \Delta_3 \times \{3\}$$

also generates a strong regular orbit on $\mathcal{P}(X_1 \times X_2)$ and Φ does not belong to the orbits generated by Σ_{i1}, $1 \leq i \leq 3$.

Definition 24. Let $\mathcal{K}$ denote the class of all pairs $(S, \mathrm{d}(S))$ satisfying the following conditions:

1. S is a permutation group of degree $\mathrm{d}(S)$, and

2. $S \cong H_1 \wr \cdots \wr H_n$, where H_i is either $H_i \cong A_3$ (of degree $\mathrm{d}(H_i) = |X_i| = 3$) or $H_i \cong \Gamma(2^3)$ (of degree $\mathrm{d}(H_i) = |X_i| = 7$) for each i, and $n \geq 1$.

Applying Lemmas 18 and 22, we have:

Corollary 25. *If S is a permutation group on Ω such that $(S, |\Omega|) \in \mathcal{K}$, then S has exactly two regular orbits on $\mathcal{P}(\Omega)$.*

2.3 The imprimitive case

Lemma 26. *Let K be a group and let W a faithful K-module over a field of prime characteristic, p say. Let S be a primitive soluble permutation group on an m-element set Ω, and assume that $S^* \leq S$ is transitive on Ω. Let $\widehat{G} = K \wr S^*$ and $V = W^\Omega$. Let G be a subgroup of $\widehat{G}$ such that $\widehat{G} = K^\natural G$ and VG is S_4-free. Then:*

1. *If K has at least five regular orbits on $W \oplus W$, then G has at least five regular orbits on $V \oplus V$.*

2. *If K is of even order, K has at least three regular orbits on $W \oplus W$ and $p \neq 2$, then G has at least three regular orbits on $V \oplus V$.*

3. *If K has at least three regular orbits on $W \oplus W$ and $p = 2$, then G has at least three regular orbits on $V \oplus V$.*

Proof. 1. It follows from [26, Proposition 3.2(3)] since G is a subgroups of $K \wr S$.

2. By [26, Proposition 3.2(2)], we may assume that $m \leq 4$. If S has a regular orbit on the power set of Ω, then $|\Pi_2(\Omega, S)| \geq |S|/2$. Thus, in this case, $K \wr S$ has at least three regular orbits on $V \oplus V$ by Wolf's formula and so does G. Therefore we may assume that S has not any regular orbit on $\mathcal{P}(\Omega)$ and so S is one of the first two exceptional cases of [18, Theorem 5.6]. Note that $S^* \cong \widehat{G}/K^\natural$ is isomorphic to a quotient of G. Hence S^* is S_4-free.

   Assume that $|\Omega| = 4$ and $S \cong A_4$ or S_4. Since S^* is a transitive on Ω, it follows that S^* is either isomorphic to a subgroup of A_4 or D_8. It suffices to consider that $S^* \cong A_4$ or D_8.

   If $S^* \cong A_4$, we have $|\Pi_3(\Omega, S^*)| = 6$. Thus $\widehat{G}$ (and so G) has at least three regular orbits on $V \oplus V$.

   If $S^* \cong D_8$, we have $|\Pi_3(\Omega, S^*)| = 4$. Thus $\widehat{G}$ (and so G) has at least three regular orbits on $V \oplus V$.

Assume that $|\Omega| = 3$ and $S \cong S_3$. Since S^* is transitive on Ω, it follows that $S \cong C_3$ or S_3. If $S^* \cong C_3$, we have $|\Pi_2(\Omega, S^*)| = 3$ and so H has at least three regular orbits on $V \oplus V$.

Assume that $S^* = S \cong S_3$. In this case, we have that $|\Pi_2(\Omega, S^*)| = 0$ and $|\Pi_3(\Omega, S^*)| = 1$. Thus $\widehat{G}$ has at least one regular orbit on $V \oplus V$.

Since K is of even order, $\widehat{G}$ has a subgroup isomorphic to $C_2 \wr S_3$ and so $\widehat{G}$ is not S_4-free. Since G is S_4-free, we have that G is a proper subgroup of $\widehat{G}$. Suppose that $|\widehat{G} : G| = 2$. Then $G \lhd \widehat{G}$ and $B = K^\natural$ is not contained in G. Let $N = B \cap G$. Then N is normal in $\widehat{G}$ and $|B : N| = 2$. In particular, there exists a direct factor K_1 of B which is not contained in N. Then $B = K_1N$ and $|K_1 : K_1 \cap N| = 2$. Note that $C = (K_1 \cap N)^\natural$ is a normal subgroup of $\widehat{G}$ contained in B such that $\widehat{G}/C \cong C_2 \wr S_3$. Thus there exists a normal subgroup L of $\widehat{G}$ contained in B such that $\widehat{G}/L \cong S_4$. Therefore $\widehat{G} = LG$ and $G/G \cap L \cong \widehat{G}/L \cong S_4$, contrary to assumption. Consequently, $|\widehat{G} : G| \geq 3$ and so G has at least three regular orbits on $V \oplus V$.

3. If $p = 2$, we have that G is S_3-free by Corollary 7. Arguing as in case 2, we conclude that G has at least three regular orbits on $V \oplus V$. □

Definition 27. Let G be a group and let V a G-module such that the action of G on V is equivalent to the action of a subgroup X of $U \wr S = U^\natural X$ on W^Ω, where U is a group, W is a U-module and S is a permutation group on a set Ω such that $(S, |\Omega|) \in \mathcal{K}$ (see Definition 24) or $(S, |\Omega|) = (1, 1)$.

1. We say that V of type (**I**) if $|W| = 2^3$ and $U = \Gamma(W)$.

2. V is said to be of type (**II**) if $|W| = 3^2$ and $U = \mathrm{SL}(2, 3)$.

Lemma 28. *Suppose that V is a G-module of type (**I**) or type (**II**) (see Definition 27). There exist $0 \neq x \in V$ and y_1, y_2, z_1, $z_2 \in V$ lying in different $\mathrm{C}_G(x)$-orbits satisfying the following conditions:*

1. *$\mathrm{C}_G(x) \cap \mathrm{C}_G(y_i) = 1$ for each i; and*

2. *$\mathrm{C}_G(x) \cap \mathrm{C}_G(z_i)$ is a 3-group for each i.*

Moreover, G has exactly two regular orbits on $V \oplus V$.

Proof. Without loss of generality, we may suppose that $G = U \wr S$ and $V = W^\Omega$, U is a group, W is a U-module and S is a permutation group on a set Ω such that $(S, |\Omega|) = (1, 1)$ or $(S, |\Omega|) \in \mathcal{K}$, and either $|W| = 2^3$ and $U = \Gamma(W)$ or $|W| = 3^2$ and $U = \mathrm{SL}(2, 3)$. Let $0 \neq w \in W$. Then $\mathrm{C}_U(w)$ is a 3-group and has exactly two regular orbits on W. Then we assume that u_1, u_2 belong to different regular orbits of $\mathrm{C}_U(w)$ on W. In particular, $\mathrm{C}_U(w)\cap\mathrm{C}_U(u_i)=1$ for each i. Write $v_1=0, v_2=w$. Then $\mathrm{C}_U(w)\cap\mathrm{C}_U(v_i)=\mathrm{C}_U(w)$ is a 3-group. Observe that u_1, u_2, v_1, v_2 belong to four different $\mathrm{C}_U(w)$-orbits. Thus the lemma holds when $(S,|\Omega|)=(1,1)$.

Now we may assume that $(S,|\Omega|)\in\mathcal{K}$. Applying Corollary 25, we get that S has exactly two strong regular orbits on $\mathcal{P}(\Omega)$. Hence, by Wolf's formula, G has exactly two regular orbits on $V\oplus V$. Let $\Delta\subseteq\Omega$ such that $\mathrm{Stab}_S(\Delta)=1$ and $x\in V=W^{\Omega}$ such that $x(i)=w$ for all $i\in\Omega$. Assume that y_1, y_2, z_1, $z_2\in V$ satisfy

$$\begin{aligned} y_1(i)&=u_1, \quad i\in\Delta; & y_1(i)&=u_2, \quad i\in\Omega\setminus\Delta;\\ y_2(i)&=u_2, \quad i\in\Delta; & y_2(i)&=u_1, \quad i\in\Omega\setminus\Delta;\\ z_1(i)&=v_1, \quad i\in\Delta; & z_1(i)&=v_2, \quad i\in\Omega\setminus\Delta;\\ z_2(i)&=v_2, \quad i\in\Delta; & z_2(i)&=v_1, \quad i\in\Omega\setminus\Delta. \end{aligned}$$

It is not difficult to see that y_1, y_2, z_1, z_2 belong to different regular orbits of $\mathrm{C}_G(x)$ on V. We first show that $\mathrm{C}_G(x)\cap\mathrm{C}_G(y_j)=1$ for each j. Let $(f,\sigma)\in\mathrm{C}_G(x)\cap C_G(y_j)$, where $f\in U^{\Omega}$ and $\sigma\in S$. Then

$$x(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})}=x(i); y_j(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})}=y_j(i), \forall i\in\Omega.$$

Hence $f(i)\in\mathrm{C}_U(w)$ for each i. Since u_1, u_2 lie in different orbits of $\mathrm{C}_U(w)$ on W, we have $\Delta^{\sigma}=\Delta$ and thus $\sigma\in\mathrm{Stab}_S(\Delta)=1$. Then $u_1^{f(i)}=u_1$ or $u_2^{f(i)}=u_2$ for each i and so $f(i)\in\mathrm{C}_U(w)\cap\mathrm{C}_U(u_1)=1$ or $f(i)\in\mathrm{C}_U(w)\cap\mathrm{C}_U(u_2)=1$. In any case, $f=1$, as desired.

Now take $(f,\sigma)\in\mathrm{C}_G(x)\cap C_G(z_j)$ for each j. Arguing in a similar way, we have $f(i)\in\mathrm{C}_U(w)$ for each i and $\sigma=1$. Then $v_1^{f(i)}=y$ or $v_2^{f(i)}=z$ for each i and so $f(i)\in\mathrm{C}_U(w)\cap\mathrm{C}_U(v_1)$ or $f(i)\in\mathrm{C}_U(w)\cap\mathrm{C}_U(v_2)$. Note that $\mathrm{C}_U(w)\cap\mathrm{C}_U(v_1)$ is a 3-group. Then $(f,\sigma)=(f,1)$ is a 3-element and thus $\mathrm{C}_G(x)\cap C_G(z_j)$ is a 3-group for each j, as desired. □

Let G be a group and let V a faithful G-module. Assume that there $V=V_1\oplus\cdots\oplus V_m$ $(m\geq 2)$ is a direct sum of subspaces which are permuted transitively by G. Write $\Omega=\{1,\dots,m\}$, $L=\mathrm{N}_G(V_1)$ and $N=\mathrm{Core}_G(L)$. Then $m=|G:L|$ and $S=G/N$ is a permutation group on Ω induced by the action of G on a right transversal of L in G. We have:

Lemma 29. *Assume that G is soluble and VG is S_4-free. Assume further that V_1, as a $L/\mathrm{C}_G(V_1)$-module, is of type **(I)** or type **(II)** (see Definition 27).*

1. *Suppose that* $\mathrm{O}^{2'}(S)$ *acts transitively on* Ω *and there exists a* 3*-partition* $\{\Delta_1, \Delta_2, \Delta_3\}$ *of* Ω *such that* $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$*. Then* G *has at least three regular orbits on* $V \oplus V$*.*

2. *If* $m \leq 4$*, then* G *has at least three regular orbits on* $V \oplus V$ *unless* $m = 3$ *and* $G/N \cong C_3$*; in this case,* G *has at least two regular orbits on* $V \oplus V$*.*

Proof. Applying Lemma 2, we may assume without loss of generality G is a subgroup of $\widehat{G} = U \wr S$, where $U = L/\mathrm{C}_G(V_1)$. Moreover, we have that $\widehat{G} = U^\natural G$, $N = G \cap U^\natural$ and $\mathrm{N}_G(W_j)/\mathrm{C}_G(W_j) \cong U$, where $W_j = \{f \in V \mid f(i) = 0, \forall i \neq j\}$, $j \in \Omega$.

Applying Lemma 28 to the pair (U, V_1) allows us to conclude that there exists $0 \neq x \in V_1$ such that $\mathrm{C}_U(x)$ has four different orbits on V_1 with representatives y_1, y_2, z_1, z_2 satisfying $\mathrm{C}_U(x) \cap \mathrm{C}_U(y_i) = 1$ and $\mathrm{C}_U(x) \cap \mathrm{C}_U(z_i)$ is a 3-group for each i.

Assume that $\mathrm{O}^{2'}(S)$ acts transitively on Ω and there exists 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. Then $1 \neq \mathrm{O}^{2'}(S)$.

Our first goal will be to prove the following statement:

$(*)$ *Let* $(f, 1)$ *be a* 3*-element of* N *for* $f \in U^\natural$*. Suppose that* $f(i_0) = 1$ *for some* $i_0 \in \Omega$*. Then* $f = 1$*.*

Let $P \in \mathrm{Syl}_3(N)$ such that $(f, 1) \in P$. By the Frattini Argument, $G = N\,\mathrm{N}_G(P)$. Let $\rho \in S$ be a 2-element. Then ρ determines a 2-element $(g, \rho) \in \mathrm{N}_G(P)$. Let $T = N\langle (g, \rho) \rangle$.

We show that T is S_3-free. If U-module V_1 is of type (**I**), then $p = 2$ and G is S_3-free by Corollary 7. Hence T is S_3-free. If U-module V_1 is of type (**II**), then $p = 3$ and $\mathrm{O}_{2'}(U) = 1$. Since $N\,\mathrm{C}_G(W_j)/\mathrm{C}_G(W_j) \trianglelefteq \mathrm{N}_G(W_j)/\mathrm{C}_G(W_j) \cong U$, we have $\mathrm{O}_{2'}(N) \leq \cap_j \mathrm{C}_G(W_j) = \mathrm{C}_G(V) = 1$. Then we have $\mathrm{O}_{2'}(T) \leq \mathrm{O}_{2'}(N) = 1$ since T/N is a 2-group. By Lemma 6, T is S_3-free.

Since $P\langle (g, \rho) \rangle$ is $\{2, 3\}$-subgroup of T, we can apply Lemma 4 to conclude that $P\langle (g, \rho) \rangle$ is 3-nilpotent. Hence $(f, 1)(g, \rho) = (g, \rho)(f, 1)$, that is,

$$f(i)g(i) = g(i)f(i^\rho), \forall i \in \Omega,$$

Therefore $f(i) = 1$ if and only if $f(i^\rho) = 1$.

Since $\mathrm{O}^{2'}(S)$ acts transitively on Ω, it follows that for each $i \in \Omega$, there exist 2-elements $\rho_1, \ldots, \rho_s$ such that $i_0^{\rho_1 \cdots \rho_s} = i$. Since $f(i_0) = 1$, we have

$$1 = f(i_0) = f(i_0^{\rho_1}) = \cdots = f(i_0^{\rho_1 \cdots \rho_s}) = f(i),$$

thus $f(i) = 1$ for each $i \in \Omega$ and the statement is proved.

Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements u_1, u_2, $u_3 \in V$ defined by

$$\begin{array}{llllll} u_1(i) = z_1, & i \in \Delta_1; & u_1(i) = y_1, & i \in \Delta_2; & u_1(i) = y_2, & i \in \Delta_3; \\ u_2(i) = z_2, & i \in \Delta_1; & u_2(i) = y_1, & i \in \Delta_2; & u_2(i) = y_2, & i \in \Delta_3; \\ u_3(i) = z_1, & i \in \Delta_1; & u_3(i) = z_2, & i \in \Delta_2; & u_3(i) = y_2, & i \in \Delta_3. \end{array}$$

Then we will show $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for all $j \in \{1, 2, 3\}$, and u_1, u_2 and u_3 belong to different regular orbits of $\mathrm{C}_G(v)$ on V.

Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$, where $f \in U^\natural$ and $\sigma \in S$. Then

$$v(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = v(i); u_j(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = u_j(i), \forall i \in \Omega.$$

Hence $f(i) \in \mathrm{C}_U(x)$ for each i. Then we have $u_j(i^{\sigma^{-1}}), u_j(i)$ lie in the same orbit of $\mathrm{C}_U(x)$ on V_1, $\forall i \in \Omega$. Since y_1, y_2, z_1, z_2 lie in different orbits of $\mathrm{C}_U(x)$ on V_1, it implies that $\sigma \in \bigcap_i \mathrm{Stab}_S(\Delta_i) = 1$. For each $i \in \Omega$, $f(i) \in \mathrm{C}_U(x) \cap \mathrm{C}_U(y_i)$ or $\mathrm{C}_U(x) \cap \mathrm{C}_U(z_i)$ for $i = 1$ or 2. Thus $f(i)$ is a 3-element for each i and clearly $(f, \sigma) = (f, 1)$ is a 3-element. Let $i_0 \in \Delta_3$. Then $y_2^{f(i_0)} = y_2$, and so $f(i_0) \in \mathrm{C}_U(x) \cap \mathrm{C}_U(y_2) = 1$. Thus $f(i_0) = 1$.

Since $(f, \sigma) = (f, 1)$ is a 3-element of N and $f(i_0) = 1$ for some $i_0 \in \Omega$, it follows from Statement $(*)$ that $f = 1$. Thus $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$, $j = 1$, 2, 3. Similar arguments allows us to conclude that u_1, u_2 and u_3 belong to different regular orbits of $\mathrm{C}_G(v)$ on V. Consequently, G has at least three regular orbits on $V \oplus V$, and the Statement 1 holds.

Suppose that $|\Omega| \leq 4$. If $|\Omega| = 4$, then S is isomorphic to A_4, D_8, $C_2 \times C_2$ since S is transitive and S_4-free. In these cases, $\mathrm{O}^{2'}(S)$ acts transitively on Ω and S has a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of type $(1, 1, 2)$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. By Statement 1, G has at least three regular orbits on $V \oplus V$.

If $|\Omega| = 3$, we have that S is isomorphic to S_3 or C_3. Suppose that $S \cong C_3$. Then S has exactly two regular orbits on $\mathcal{P}(\Omega)$. Hence G has two regular orbits on $V \oplus V$. If $S \cong S_3$, it follows that $\mathrm{O}^{2'}(S)$ acts transitively on Ω and S has a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of type $(1, 1, 1)$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. By Statement 1, G has at least three regular orbits on $V \oplus V$.

If $|\Omega| = 2$, then $S \cong S_2$. Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements u_1, u_2, $u_3 \in V$ defined by

$$\begin{array}{ll} u_1(1) = z_1, & u_1(2) = y_1; \\ u_2(1) = z_2, & u_2(2) = y_1; \\ u_3(1) = y_2, & u_3(2) = y_1. \end{array}$$

With similar arguments to those used above, one can show that u_1, u_2 and u_3 belong to different regular orbits of $C_G(v)$ on V. Consequently, G has at least three regular orbits on $V \oplus V$. □

In the following, we consider the imprimitive case of Theorem B.

Lemma 30. *Let S be a supersoluble primitive permutation group on a finite set Ω. Then we have*

1. *If $|\Omega| \geq 3$, then there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\cap_i \operatorname{Stab}_S \Delta_i = 1$.*
2. *If $|\Omega| \geq 5$, then there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $|\Delta_1| \leq |\Delta_2| < |\Delta_3|$ and $\cap_i \operatorname{Stab}_S \Delta_i = 1$.*

Proof. Since S is a supersoluble and primitive, we have that $|\Omega|$ is a prime, p say, and S is a subgroup of $[C_p]C_{p-1}$. Now we may assume $p \geq 3$. Observe that S is a Frobenius group, that is, $C_S(x) \cap C_S(y) = 1$ for any distinct $x, y \in \Omega$. Fix two distinct $x, y \in \Omega$, and let $\Delta_1 = \{x\}; \Delta_2 = \{y\}$ and $\Delta_3 = \Omega - \{x, y\}$. Then clearly $\cap_i \operatorname{Stab}_S \Delta_i = 1$ and $|\Delta_1| \leq |\Delta_2| < |\Delta_3|$ if $p \geq 5$. Thus the lemma is proved. □

Lemma 31. *Let H be a group and S be a primitive permutation group on the finite set Ω with $|\Omega| \geq p$ for some prime p. Assume that G is a supersoluble group of $\widehat{G} = H \wr S$ such that $H^\natural G = \widehat{G}$. Write $N = H^\natural \cap G$ and assume that $O_p(N) = 1$. If f is a p-element of $H^\natural$ such that $(f, 1) \in N$ and $f(i_0) = 1$ for some $i_0 \in \Omega$. Then $f = 1$.*

Proof. Observe that $S \cong \widehat{G}/H^\natural \cong G/N$ is supersoluble. Since S is a primitive permutation group, we can conclude that S has the unique minimal normal subgroup X such that $|X| = |\Omega| = q$ for some prime q.

Let $P \in \operatorname{Syl}_p(N)$ such that $(f, 1) \in P$. By Frattini Argument, $G = N\,N_G(P)$ and consequently $\widehat{G} = H^\natural\,N_G(P)$. For any $\rho \in X$, we have that $\rho^q = 1$. Then it is not difficult to find a q-element $(g, \rho) \in N_G(P)$. Let $T = P\langle(g, \rho)\rangle$. Clearly $T \leq G$ is supersoluble and $[(f, 1), (g, \rho)] \in P$.

By hypothesis, $q \geq p$. If $q > p$, then $\langle(g, \rho)\rangle \lhd T$ since T has the supersoluble type Sylow Tower. Thus $[(f, 1), (g, \rho)] \in P \cap \langle(g, \rho)\rangle = 1$. If $p = q$, then T is a p-group. Observe that $G' \leq F(G)$ since G is supersoluble. Thus $T' \leq O_p(G)$. Then $[(f, 1), (g, \rho)] \in T' \cap N \leq O_p(G) \cap N = O_p(N) = 1$.

Thus we have $(f, 1)(g, \rho) = (g, \rho)(f, 1)$, that is, $f(i)g(i) = g(i)f(i^\rho), \forall i \in \Omega$. Therefore $f(i) = 1$ if and only if $f(i^\rho) = 1$.

Recall X acts transitively on Ω. For each $i \in \Omega$, there exists ρ_i(depending on i) $\in S$ such that $i_0^{\rho_i} = i$. Since $f(i_0) = 1$, we have $f(i) = f(i_0^{\rho_i}) = 1$. Thus $f(i) = 1$ for each $i \in \Omega$ and the statement is proved. □

Let G be a group and let V a faithful G-module.

We say G-module V satisfies that **Property I** if the following hypotheses hold.

(1) G is an odd order group and $\mathrm{O}_3(G) = 1$.

(2) there exists $0 \neq x \in V$ and $\mathrm{C}_G(x)$ has at least four different orbits on V with representatives y_1, y_2, z_1, z_2 satisfying $\mathrm{C}_G(x) \cap \mathrm{C}_G(y_i) = 1$ and $\mathrm{C}_G(x) \cap \mathrm{C}_G(z_i)$ is a 3-group for each i.

We say G-module V satisfies that **Property II** if the following hypotheses hold.

(1) G is an even order group with $\mathrm{O}_2(G) = 1$.

(2) there exists $0 \neq x \in V$ and $\mathrm{C}_G(x)$ at least three different orbits on V with representatives y, z_1, z_2 satisfying $\mathrm{C}_G(x) \cap \mathrm{C}_G(y) = 1$ and $\mathrm{C}_G(x) \cap \mathrm{C}_G(z_i)$ is a 2-group for each $1 \leq i \leq 2$.

Lemma 32. *Let G be a supersoluble group and V be a faithful G-module over* $\mathrm{GF}(2)$. *Assume that there $V = V_1 \oplus ... \oplus V_m (m \geq 1)$ is a direct sum of subspaces which are permuted transitively by G. Let $K = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and V_1 is a faithful K-module. Then we have:*

1. *If K has at least four regular orbits on $V_1 \oplus V_1$, then G has at least four regular orbits on $V \oplus V$.*

2. *If K is of even order, K has at least three regular orbits on $V_1 \oplus V_1$, then G has at least three regular orbits on $V \oplus V$.*

3. *If K-module V_1 satisfies **Property I** and G is of odd order, then G has at least four regular orbits on $V \oplus V$ or satisfies **Property I**.*

4. *If K-module V_1 satisfies **Property II**, then either G has three regular orbits on $V \oplus V$ or G-module V satisfies **Property II**.*

5. *If K-module V_1 satisfies **Property I**, then either G has three regular orbits on $V \oplus V$ or G-module V satisfies **Property I** or **Property II**.*

Proof. Work by induction on m. Clearly $(1) - (5)$ holds when $m = 1$. Now we assume that $m \geq 2$. Since G acts transitively on $\{V_1, ..., V_m\}$, we can take a block Δ of $\{V_1, ..., V_m\}$ such that $\mathrm{Stab}_G(\Delta)$ is maximal in G. Without loss of generality, we may assume that $\Delta = \{V_1, ..., V_s\} (s \geq 1)$.

Let $W_1 = \sum_{i=1}^{s} V_i$ and $L = \mathrm{N}_G(W_1)$. Then $L = \mathrm{Stab}_G(\Delta)$ is maximal in G. Assume that $\{g_1 = 1, g_2, ..., g_t\}$ is a right transversal of L in G with $t = |G : L| \geq 2$. Write $W_i = W_1 g_i$ for each i. Then $V = W_1 \oplus ... \oplus W_t$ and G/N acts faithfully and primitively on $\{W_1, ..., W_t\}$, where $N = \mathrm{Core}_G(L)$.

Write $H = L/\mathrm{C}_G(W_1)$. We argue that $\mathrm{O}_p(N) = 1$ if $\mathrm{O}_p(H) = 1$ for some prime p. Observe that $N \trianglelefteq L^{g_j} = \mathrm{N}_G(W_j)$ for each $1 \leq j \leq t$. Then $N\,\mathrm{C}_G(W_j)/\mathrm{C}_G(W_j) \trianglelefteq \mathrm{N}_G(W_j)/\mathrm{C}_G(W_j) \cong H$. Since $\mathrm{O}_p(H) = 1$, we have that $\mathrm{O}_p(N) \leq \cap_{j=1}^t \mathrm{C}_G(W_j) = 1$ since V is a faithful G-module. This argument will be useful below.

Then applying Lemma 2, we may assume that G is a subgroup of $\widehat{G} = H \wr S$ such that $\widehat{G} = H^\natural G$, $N = H^\natural \cap G$ and $V = W_1^\Omega$, where S is a primitive permutation group on $\Omega = \{1, ..., t\}$. Observe that $S \cong \widehat{G}/H^\natural \cong G/N$ is supersoluble. Thus t is a prime.

Denote $J = \mathrm{N}_H(V_1)/\mathrm{C}_H(V_1)$. Then $W_1 = V_1 \oplus ... \oplus V_s$ is a faithful H-module and by [1, Theorem 1.13], L(also H) acts transitively on $\Delta = \{V_1, ..., V_s\}$. Write $J' = \mathrm{N}_L(V_1)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1) \leq K$. It is not difficult to find that the action of J on V_1 is equivalent to the action of J' on V_1.

Now we will prove $(1)-(5)$ respectively. The main step is firstly to apply induction on (W_1, H, V_1, J) and then to calculate the number of regular orbits by Wolf's formula.

(1) By hypothesis, $J' \leq K$ has at least four regular orbits on $V_1 \oplus V_1$. Thus J has at least four regular orbits on $V_1 \oplus V_1$. Since $s = m/t < m$, by induction, H has at least four regular orbits on $W_1 \oplus W_1$.

If S has a regular orbit on the power set of Ω, then $|\Pi_2(\Omega, S)| \geq |S|/2$. Thus, in this case, $H \wr S$ has at least four regular orbits on $V \oplus V$ by Wolf's formula and so does G. Therefore we may assume that S has not any regular orbit on $\mathcal{P}(\Omega)$ and so S is one of exceptional cases of [18, Theorem II.5.6] and $3 \leq t \leq 9$. By [25, Theorem 3.1(iii)], we have $|\Pi_3(\Omega, S)| \geq |S|$ for $5 \leq t \leq 9$, which implies $G \leq H \wr S$ has at least four regular orbits on $V \oplus V$ by Wolf's formula. Thus we may assume that $t = 3$ since t is a prime. In this case, $S \cong S_3$. It is not difficult to calculate that $|\Pi_2(\Omega, S^*)| = 0$ and $|\Pi_3(\Omega, S^*)| = 1$. Thus $G(\leq \widehat{G})$ has at least four regular orbit on $V \oplus V$.

Thus the conclusion (1) is proved.

(2) If J is of order odd, then so is J'. Since K is of order even, $|K : J'| \geq 2$. Thus $J'($ also $J)$ has at least six regular orbits on $V_1 \oplus V_1$. Applying (a) on (W_1, H, V_1, J), H has at least four regular orbits on $W_1 \oplus W_1$. Applying **(a)** on (V, G, W_1, H) again, G has at least four regular orbits on $V \oplus V$, as desired.

Now we assume that J is of even order, by induction, H has at least three regular orbits on $W_1 \oplus W_1$. By [26, Proposition 3.2(2)] and Wolf's formula, we may assume that $t \leq 4$ and S has not any regular orbit on $\mathcal{P}(\Omega)$. Note t is a prime. Thus, by [18, Theorem II.5.6], we can conclude that $|\Omega| = 3$ and $S \cong S_3$. In this case, $|\Pi_2(\Omega, S^*)| = 0$ and $|\Pi_3(\Omega, S^*)| = 1$. Thus $\widehat{G}$ has at least one regular orbit on $V \oplus V$.

Observe that H is of even order since J is of even order. Then $\widehat{G}$ has a subgroup isomorphic to $C_2 \wr S_3$ and so $\widehat{G}$ is not supersoluble. Thus we have that G is a proper subgroup of $\widehat{G}$. Suppose that $|\widehat{G} : G| = 2$. Then $G \lhd \widehat{G}$ and $B = H^{\natural}$ is not contained in G. Let $N = B \cap G$. Then N is normal in $\widehat{G}$ and $|B : N| = 2$. In particular, there exists a direct factor $H_1 \cong H$ of B which is not contained in N. Then $B = H_1N$ and $|H_1 : H_1 \cap N| = 2$. Note that $C = (H_1 \cap N)^{\natural}$ is a normal subgroup of $\widehat{G}$ contained in B such that $\widehat{G}/C \cong C_2 \wr S_3$. Thus there exists a normal subgroup X of $\widehat{G}$ contained in B such that $\widehat{G}/X \cong S_4$. Therefore $\widehat{G} = XG$ and $G/G \cap X \cong \widehat{G}/X \cong S_4$, contrary to supposition. Consequently, $|\widehat{G} : G| \geq 3$ and so G has at least three regular orbits on $V \oplus V$. Thus the conclusion (2) is proved.

(3) Since K-module V_1 satisfies **Property I**, K has at least two regular orbits on $V_1 \oplus V_1$. If J' is proper in K, then J' has at least four regular orbits on $V_1 \oplus V_1$ and so does J. Applying **(a)** twice, G has at least four regular orbits on $V \oplus V$.

Then we may assume $J' = K$. Consequently $J(J')$-module V_1 satisfies **Property I**. By induction, H has at least four regular orbits on $W_1 \oplus W_1$ or H-module W_1 satisfies **Property I**. If H has at least four regular orbits on $W_1 \oplus W_1$, by **(a)**, G has at least four regular orbits on $V \oplus V$, as desired.

Now we assume that H-module W_1 satisfies **Property I**. By hypothesis, we have $\mathrm{O}_3(H) = 1$. Moreover, there exists there exists $0 \neq x \in V_1$ and $\mathrm{C}_H(x)$ has at least four different orbits on V_1 with representatives y_1, y_2, z_1, z_2 satisfying $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_i) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_i)$ is a 3-group for each i.

Since G is of odd order, we have S is of odd order. Consequently t is an odd prime and $t \geq 3$. By [18, Theorem II.5.6], S has a strong regular orbit on $\mathcal{P}(\Omega)$. We may assume that $\Delta \subseteq \Omega$ such that $\mathrm{Stab}_S(\Delta) = 1$ and $|\Delta| \neq |\Omega - \Delta|$. Take $v \in V = W_1^{\Omega}$ such that $v(i) = x$ for each $i \in \Omega$ and define $u_j, 1 \leq j \leq 4$ as follow:

$$u_1(i) = y_1, i \in \Delta; u_1(i) = y_2, i \in \Omega - \Delta;$$

$$u_2(i) = y_2, i \in \Delta; u_2(i) = y_1, i \in \Omega - \Delta;$$

$$u_3(i) = y_1, i \in \Delta; u_3(i) = z_1, i \in \Omega - \Delta;$$

$$u_4(i) = y_2, i \in \Delta; u_4(i) = z_2, i \in \Omega - \Delta;$$

It is not difficult to find that $u_j, 1 \leq j \leq 4$ lie different orbits of $\mathrm{C}_G(v)$ on V. Then we will show that $u_j, 1 \leq j \leq 4$ can generate regular orbits of $\mathrm{C}_G(v)$ on V.

Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ for $1 \leq j \leq 4$, where $f \in H^\natural$ and $\sigma \in S$. Then

$$v(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = v(i); u_j(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = u_j(i), \forall i \in \Omega.$$

Hence we have that $f(i) \in \mathrm{C}_H(x)$ for each $i \in \Omega$. Since y_1, y_2, z_1, z_2 lie in different $\mathrm{C}_H(x)$-orbit in V. Thus $\sigma \in \mathrm{Stab}_S(\Delta) = 1$.

Thus we conclude that $f(i) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_j)$ or $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_k)$ for each $i \in \Omega$, where $j, k = 1$ or 2. Thus $f(i)$ is a 3-element of H for each i. Moreover, take any $i_0 \in \Delta$, we have $f(i_0) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_1)$ or $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_2)$ and we can conclude that $f(i_0) = 1$.

Thus $(f, \sigma) = (f, 1) \in H^\natural \cap G = N$ is a 3-element and $f(i_0) = 1$ for some $i_0 \in \Delta$. Recall that $|\Omega| = t \geq 3$ and we can argue $\mathrm{O}_3(N) = 1$ since $\mathrm{O}_3(H) = 1$ by hypothesis. Applying Lemma 31(the case $p = 3$), we can conclude that $f = 1$. It implies that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for $1 \leq j \leq 4$, as desired. Thus G has at least four regular orbits on $V \oplus V$, as desired. Thus the conclusion (3) is proved.

(4) Since K-module V_1 satisfies **Property II**, we may assume that

- K is an even order group with $\mathrm{O}_2(K) = 1$.
- there exists $0 \neq x' \in V_1$ and three different $\mathrm{C}_K(x')$-orbits with representatives y', z'_1, z'_2 satisfying $\mathrm{C}_K(x') \cap \mathrm{C}_K(y') = 1$ and $\mathrm{C}_K(x') \cap \mathrm{C}_K(z'_i)$ is a 2-group for each i.

If J' is of odd order, then J' is proper in K. Then J' has at least two regular orbits on $V \oplus V$ and $\mathrm{C}_{J'}(x') \cap \mathrm{C}_{J'}(z'_i)$ is a 2-group for each i, which implies that J' has at least four regular orbits on $V_1 \oplus V_1$ and so is J. Applying (1) twice, G has at least four regular orbits on $V \oplus V$.

Thus we may assume J' is of even order. If $|K : J'| \geq 3$, then J'(also J) has at least three regular orbits on $V_1 \oplus V_1$. It follows from (2) that H has at least three regular orbits on $W_1 \oplus W_1$. Observe that $|H|$ is even since $|J|$ is even, applying (2) again, G has at least three regular orbits on $V \oplus V$. Now we may assume that $|K : J'| \leq 2$. Consequently $J' \lhd K$ and $\mathrm{O}_2(J') \leq \mathrm{O}_2(K) = 1$. Then $J(J')$-module V_1 satisfies **Property II**.

By induction, H has at least three regular orbits on $W_1 \oplus W_1$ or H-module W_1 satisfies **Property II**. If H has at least three regular orbits on $W_1 \oplus W_1$, since $|H|$ is even, then G has at least three regular orbits on $V \oplus V$ by (2), as desired.

Now we assume that H-module W_1 satisfies **Property II**.

- H is an even order group with $\mathrm{O}_2(H) = 1$.
- Then there exists $0 \neq x \in W_1$ and three different $\mathrm{C}_H(x)$-orbits with representatives y, z_1, z_2 satisfying $\mathrm{C}_H(x) \cap \mathrm{C}_H(y) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_i)$ is a 2-group for each i.

Firstly we consider the case $|\Omega| = t \geq 5$. By Lemma 30, there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_{i=1}^{3} \mathrm{Stab}_S(\Delta_i) = 1$ and $|\Delta_1| \leq |\Delta_2| < |\Delta_3|$. Take $v \in V = W_1^{\Omega}$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements $u_j \in V$, where $1 \leq j \leq 3$, defined by

$$u_1(i) = y, i \in \Delta_1; u_1(i) = z_2, i \in \Delta_2; u_1(i) = z_1, i \in \Delta_3;$$

$$u_2(i) = z_1, i \in \Delta_1; u_2(i) = y, i \in \Delta_2; u_2(i) = z_2, i \in \Delta_3;$$

$$u_3(i) = z_2, i \in \Delta_1; u_3(i) = z_1, i \in \Delta_2; u_3(i) = y, i \in \Delta_3;$$

Since y, z_1, z_2 lie in different orbits of $\mathrm{C}_H(x)$ on W_1 and $|\Delta_1| \leq |\Delta_2| < |\Delta_3|$, it is not difficult to find u_1, u_2 and u_3 lie in different orbits of $\mathrm{C}_G(v)$ on V. Then we will show that $u_j, 1 \leq j \leq 3$ can generate regular orbit of $\mathrm{C}_G(v)$ on V.

Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ for $1 \leq j \leq 3$, where $f \in H^{\natural}$ and $\sigma \in S$. Then

$$v(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = v(i); u_j(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = u_j(i), \forall i \in \Omega.$$

Hence we have that $f(i) \in \mathrm{C}_H(x)$ for each $i \in \Omega$. Since y, z_1, z_2 lie in different $\mathrm{C}_H(x)$-orbit in V. Thus $\sigma \in \mathrm{Stab}_S(\Delta) = 1$.

Thus we conclude that $f(i) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y)$ or $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_k)$ for each $i \in \Omega$, where $k = 1$ or 2. Thus $f(i)$ is a 2-element of H for each i. Moreover, Considering in u_j for a fixed j, we can find i_0(depending on j) $\in \Delta_j$ such that $f(i_0) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y)$. we can conclude that $f(i_0) = 1$.

Thus $(f, \sigma) = (f, 1) \in H^{\natural} \cap G = N$ is a 2-element and $f(i_0) = 1$ for some $i_0 \in \Omega$. Recall that $|\Omega| = t \geq 5$ and we can argue $\mathrm{O}_2(N) = 1$ since $\mathrm{O}_2(H) = 1$ by hypothesis. Applying Lemma 31(the case $p = 2$), we can conclude that $f = 1$. It implies that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for $1 \leq j \leq 3$, as desired. Thus G has at least three regular orbits on $V \oplus V$, as desired.

Recall that $|\Omega| = t$ is a prime. Thus we only consider the case $t = 2$ or 3.

Assume that $t = 3$. In this case, $S = S_3$ or $\langle (123) \rangle$. Take $v \in V = W^{\Omega}$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements $u_j \in V$, where $1 \leq j \leq 3$, defined by

$$u_1(1) = y, u_1(2) = z_1, u_1(3) = z_2;$$

$$u_2(1) = y, u_2(2) = y, u_2(3) = z_1;$$

$$u_3(1) = y, u_3(2) = y, u_3(3) = z_2;$$

With similar arguments to those used above, one can show that u_1, u_2 and u_3 belong to different orbits of $\mathrm{C}_G(v)$ on V and $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_1) = 1$. Now we will prove that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ is 2-group for $j = 2, 3$.

Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ for $j = 2, 3$, where $f \in H^{\natural}$ and $\sigma \in S$. Hence we have that $f(i) \in \mathrm{C}_H(x)$ for each $i \in \Omega$. Since y, z_1, z_2 lie in different

$C_H(x)$-orbit in V. Thus $3^\sigma = 3$ and consequently $\sigma \in \langle(12)\rangle$. Moreover, $f(1), f(2) \in C_H(x) \cap C_H(y) = 1$ and $f(3) \in C_H(x) \cap C_H(z_j)$ for some j is a 2-group. Thus $(f,\sigma)^2 = (g,1)$, where $g(1) = g(2) = 1$ and $g(3) = f(3)^2$ is a 2-element. Then we can conclude that (f,σ) is a 2-element, as desired.

Observe that $O_2(G/N) \cong O_2(S) = 1$ and consequently $O_2(G) \leq O_2(N) = 1$. Furthermore, G is of even order since H is of even order. Thus G-module V satisfies **Property II**, as desired.

Finally we assume that $|\Omega| = 2$ and $S \cong S_2$. Take $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements $u_1, u_2, u_3 \in V$ defined by

$$u_1(1) = z_1, u_1(2) = y;$$

$$u_2(1) = z_2, u_2(2) = y;$$

$$u_3(1) = z_1, u_3(2) = z_2;$$

With similar arguments to those used above, one can show that u_1, u_2 and u_3 belong to different orbits of $C_G(v)$ on V and $C_G(v) \cap C_G(u_j) = 1$ for $j = 1, 2$, $C_G(v) \cap C_G(u_3)$ is 2-group.

We will discuss it in the following two cases: $O_2(G) = 1$ or $O_2(G) \neq 1$. If $O_2(G) = 1$, then, in addition that G is of even order, we can conclude that G-module V satisfies **Property II**, as desired.

Now we assume that $O_2(G) \neq 1$. Then $O_2(G) \nleq N$ since $O_2(N) = 1$. Since now $G/N \cong S_2$, we have that $G = N\,O_2(G)$ and $N \cap O_2(G) = O_2(N) = 1$. Consequently $[N, O_2(G)] = 1$ and $\widehat{G} = H^\natural\, O_2(G)$.

Take $v' \in V$ such that $v'(1) = 0$ and $v'(2) = x$. Now we claim that $C_G(v') \cap C_G(u_1) = 1$. Let $(f,\sigma) \in C_G(v') \cap C_G(u_1)$, where $f \in H^\natural$ and $\sigma \in S$. Then $\sigma = 1$ and $(f,\sigma) = (f,1) \in G \cap H^\natural = N$. Consequently $f(2) \in C_H(x) \cap C_H(y) = 1$. Now take $\rho = (12) \in S$, we can find a element $(g,\rho) \in O_2(G)$ for some $g \in H^\natural$ since $\widehat{G} = H^\natural\, O_2(G)$. Since $[N, O_2(G)] = 1$, we have that $(f,1)(g,\rho) = (g,\rho)(f,1)$. Consequently $f(2^\rho) = g(2)^{-1} f(2) g(2) = 1$, that is, $f(1) = 1$. Thus we have that $f = 1$, as claimed.

We can observe that $(v, u_1), (v, u_2)$ and (v', u_1) lie in different regular orbits of G on $V \oplus V$, as desired. Thus the conclusion (4) is proved completely.

(5) Since K-module V_1 satisfies **Property I**, K has at least two regular orbits on $V_1 \oplus V_1$. If J' is proper in K, then J' has at least four regular orbits on $V_1 \oplus V_1$ and so does J. By (1), H has at least four regular orbits on $W_1 \oplus W_1$. Applying (1) again, G has at least four regular orbits on $V \oplus V$.Thus we may assume $J' = K$. Consequently $J(J')$-module V_1 satisfies **Property I**.

When H is of even order, by induction, H has at least three regular orbits on $W_1 \oplus W_1$ or H-module W_1 satisfies **Property I** or **Property II**. Since

H is of even order, clearly H-module W_1 does not satisfy **Property I**. If H has at least three regular orbits on $W_1 \oplus W_1$, then it follows from **(b)** that G has at least three regular orbits on $V \oplus V$, as desired. If H-module W_1 satisfies **Property II**, then we can conclude that G has at least three regular orbits on $V \oplus V$ or H-module W_1 satisfies **Property II**, as desired. When H is of odd order, applying (3) on (W_1, H, V_1, J), then we can conclude that H-module W_1 satisfies **Property I** or H has at least four regular orbits on $W_1 \oplus W_1$. If the latter case holds, then it follows from (1) that G has at least four regular orbits on $V \oplus V$, as desired.

Thus we only consider the case that H-module W_1 satisfies **Property I**. Then we have

- H is an odd order group and $\mathrm{O}_3(H) = 1$.
- There exists $0 \neq x \in W_1$ and three different $\mathrm{C}_H(x)$-orbits with representatives y_1, y_2, z_1, z_2 satisfying $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_i) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_i)$ is a 3-group for each i.

Firstly we consider the case $|\Omega| = t \geq 3$. By Lemma 30, there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_{i=1}^{3} \mathrm{Stab}_S(\Delta_i) = 1$.

Take $v \in V = W_1^{\Omega}$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements $u_j \in V$, where $1 \leq j \leq 3$, defined by

$$u_1(i) = y_1, i \in \Delta_1; u_1(i) = y_2, i \in \Delta_2; u_1(i) = z_1, i \in \Delta_3;$$

$$u_2(i) = y_1, i \in \Delta_1; u_2(i) = y_2, i \in \Delta_2; u_2(i) = z_2, i \in \Delta_3;$$

$$u_3(i) = y_1, i \in \Delta_1; u_3(i) = z_1, i \in \Delta_2; u_3(i) = z_2, i \in \Delta_3;$$

Since y_1, y_2, z_1, z_2 lie in different orbits of $\mathrm{C}_H(x)$ on W_1, it implies that u_1, u_2 and u_3 lie in different orbits of $\mathrm{C}_G(v)$ on V. Then we will show that $u_j, 1 \leq j \leq 3$ can generate regular orbits of $\mathrm{C}_G(v)$ on V. Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ for $1 \leq j \leq 3$, where $f \in H^{\natural}$ and $\sigma \in S$. Then we have that $f(i) \in \mathrm{C}_H(x)$ for each $i \in \Omega$. Since y_1, y_2, z_1, z_2 lie in different $\mathrm{C}_H(x)$-orbit in V, we have that $\sigma \in \mathrm{Stab}_S(\Delta) = 1$.

Thus we conclude that $f(i) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_l)$ or $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_k)$ for each $i \in \Omega$, where $k, l = 1$ or 2. Thus $f(i)$ is a 3-element of H for each $i \in \Omega$. Moreover, Take any element $i_0 \in \Delta_1$ such that $f(i_0) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_1) = 1$.

Thus $(f, \sigma) = (f, 1) \in H^{\natural} \cap G = N$ is a 3-element and $f(i_0) = 1$ for some $i_0 \in \Omega$. Recall that $|\Omega| = t \geq 3$ and we can argue $\mathrm{O}_3(N) = 1$ since $\mathrm{O}_3(H) = 1$ by hypothesis. Applying Lemma 31(the case $p = 3$), we can conclude that $f = 1$. It implies that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for $1 \leq j \leq 3$, as desired. Thus G has at least three regular orbits on $V \oplus V$, as desired.

Now we assume that $|\Omega| = 2$ and $S \cong S_2$. Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements $u_1, u_2, u_3 \in V$ defined by

$$u_1(1) = y_1, u_1(2) = y_2;$$

$$u_2(1) = y_1, u_2(2) = y_1;$$
$$u_3(1) = y_2, u_3(2) = y_2;$$

Clearly $u_j, 1 \leq j \leq 3$ lie in different orbits of $\mathrm{C}_G(v)$ on V. Then we will show that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_1) = 1$ and $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ is 2-group for $j = 2, 3$.

Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j), 1 \leq j \leq 3$. We have that $f(1), f(2) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_1)$ or $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_2)$. Thus we can conclude that $f(1) = f(2) = 1$ and consequently $f = 1$. When $j = 1$, $\sigma = 1$; and when $j = 2, 3$, we have $\sigma = (12)$. Thus $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_1) = 1$ and $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ is 2-group for $j = 2, 3$, as desired.

We will discuss it in the following two cases: $\mathrm{O}_2(G) = 1$ or $\mathrm{O}_2(G) \neq 1$. If $\mathrm{O}_2(G) = 1$, then, in addition that G is of even order since $G/N \cong S_2$, we can conclude that G-module V satisfies **Property II**, as desired.

Now we assume that $\mathrm{O}_2(G) \neq 1$. Clearly $\mathrm{O}_2(H) = 1$ since H is of odd order. Thus we can argue that $\mathrm{O}_2(N) = 1$. Since $G/N \cong S_2$, we have that $G = N\,\mathrm{O}_2(G)$ and $N \cap \mathrm{O}_2(G) = 1$. Consequently $[N, \mathrm{O}_2(G)] = 1$ and $\widehat{G} = H^\natural\,\mathrm{O}_2(G)$.

Take $v' \in V$ such that $v'(1) = x$ and $v'(2) = 0$. Define $u'_j \in V, 1 \leq j \leq 2$ as follows:

$$u'_1(1) = y_1, u'_1(2) = z_1;$$
$$u'_2(1) = y_1, u'_2(2) = z_2;$$

Now we claim that $\mathrm{C}_G(v') \cap \mathrm{C}_G(u'_j) = 1$, $1 \leq j \leq 2$. Let $(f, \sigma) \in \mathrm{C}_G(v') \cap \mathrm{C}_G(u_j)$, where $f \in H^\natural$ and $\sigma \in S$.

Then $\sigma = 1$ and $(f, \sigma) = (f, 1) \in G \cap H^\natural = N$. Consequently $f(1) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_1) = 1$.

Now take $\rho = (12) \in S$, we can find a element $(g, \rho) \in \mathrm{O}_2(G)$ for some $g \in H^\natural$ since $\widehat{G} = H^\natural\,\mathrm{O}_2(G)$. Since $[N, \mathrm{O}_2(G)] = 1$, we have that $(f, 1)(g, \rho) = (g, \rho)(f, 1)$. Consequently $f(1^\rho) = g(1)^{-1}f(1)g(1) = 1$, that is, $f(2) = f(1^\rho) = 1$. Thus we have that $f = 1$, as claim.

We can observe that $(v, u_1), (v', u'_1)$ and (v', u'_2) lie in different regular orbits of G on $V \oplus V$, as desired. Thus the conclusion (5) is proved completely. □

2.4 Proof of Theorem A

We prove:

Theorem 33 (Theorem A)**.** *Let G be a soluble group and let V be a faithful completely reducible G-module, possibly of mixed characteristic. Suppose that*

H is a subgroup of G such that the semidirect product $[V]H$ is S_4-free. Then H has at least two regular orbits on $V \oplus V$. Furthermore, if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then H has at least three regular orbits on $V \oplus V$.

Our proof depends heavily on some results which are of independent interest. The first one concerns the odd case.

Theorem 34. *Let G be a soluble group and let V be an irreducible and faithful G-module over $\mathrm{GF}(p)$, p an odd prime. If $H \leq G$ and H is of odd order, then H has at least five regular orbits on $V \oplus V$.*

Proof. We argue by induction on $|G|$. By Lemma 15, we may assume that V is an imprimitive G-module. Assume that $V = V_1 \oplus \cdots \oplus V_m$ $(m \geq 2)$ is a direct sum of subspaces which are permuted transitively by G. Write $\Omega = \{1, \ldots, m\}$, $L = \mathrm{N}_G(V_1)$ and $N = \mathrm{Core}_G(L)$. Then $m = |G : L|$ and $S = G/N$ is a permutation group on Ω induced by the action of G on a right transversal of L in G. By Lemma 2, we may assume without loss of generality G is a subgroup of $\widehat{G} = U \wr S$, where $U = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and $L = \mathrm{N}_G(V_1)$ is a maximal subgroup of G and $V = V_1^{\Omega}$. Since V is G-irreducible, we may also assume that V_1 is L-irreducible.

Let $A = (L \cap H)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1)$. Then the triple (L, A, V_1) satisfies the hypotheses of the theorem. By induction, A has at least five regular orbits on $V_1 \oplus V_1$.

Assume that $\{V_{11}, \ldots, V_{1t}\}$ is the H-orbit of V_1 in $\{V_1, \ldots, V_m\}$, $t = |H : L \cap H|$. Let $W = V_{11} \oplus \cdots \oplus V_{1t}$. It is clear that we may assume $t \geq 2$. Therefore, by Lemma 2, $H/\mathrm{C}_H(V_1)$ is isomorphic to a subgroup X of the wreath product $A \wr T = A^{\natural}X$, where T is a transitive permutation group on $\Omega_1 = \{1, \ldots, t\}$ and the action $H/\mathrm{C}_H(V_1)$ on W is equivalent to the action of X on $V_1^{\Omega_1}$. By [18, Corollary 5.7], T has an strong regular orbit on $\mathcal{P}(\Omega_1)$. By Lemma 26, H has at least five regular orbits on $W \oplus W$. Thus H has at least five regular orbits on $V \oplus V$. □

Lemma 35. *Let G be a soluble group and V be an irreducible and faithful G-module over $\mathrm{GF}(p)$, where p is a prime and $p \geq 5$. Then G has at least five regular orbits on $V \oplus V$.*

Proof. We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. If V is a primitive G-module, it follows from [5, Theorem 3.4] that either G has at least $p \geq 5$ regular orbits on $V \oplus V$. Now we assume V is an imprimitive G-module. Let $V = V_1 \oplus \cdots \oplus V_m$ $(m \geq 2)$ and G permutes $\{V_1, \ldots, V_m\}$. Without loss of generality, G is a subgroup of $\widehat{G} = U \wr S$, where $U = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and $L = \mathrm{N}_G(V_1)$ is a maximal subgroup of G, $S \cong G/N$ is a primitive permutation group on

$\Omega = \{1, \dots, m\}$, where $N = \mathrm{Core}_G(L)$, and $V = V_1^{\Omega}$. Moreover, V_1 is an irreducible and faithful U-module. By induction, U has at least five regular orbits on $V_1 \oplus V_1$. It follows from [26, Proposition 3.2(3)] that G has at least five regular orbits on $V \oplus V$. □

The following important result provides the key to prove Theorem A.

Theorem 36. *Let G be a soluble group and V be an irreducible and faithful, G-module over* GF(p)*. If $H \leq G$ and VH is S_4-free, then either H has at least three regular orbits on $V \oplus V$ or V, as H-module, is of type (**I**) or type (**II**) (see Definition 27).*

Proof. We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. If V is a primitive G-module, it follows from Lemma 14 that either H has at least three regular orbits on $V \oplus V$ or the H-module V of type (**I**) or type (**II**). This contradicts the choice of G. Consequently, V is an imprimitive G-module. Then, repeating the arguments of the first part of the proof of Theorem 34 and using the same notation, we may assume without loss of generality G is a subgroup of $\widehat{G} = U \wr S$, where $U = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and $L = \mathrm{N}_G(V_1)$ is a maximal subgroup of G, $S \cong G/N$, $N = \mathrm{Core}_G(L)$, and $V = V_1^{\Omega}$. Moreover, V_1 is an irreducible L-module.

Let $A = (L \cap H)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1)$. Then the triple (L, A, V_1) satisfies the hypotheses of the theorem. The minimal choice of G implies that either A at least three regular orbits on $V_1 \oplus V_1$ or V_1, as A-module, is of type (**I**) or type (**II**).

Let $\{V_{11}, \dots, V_{1t}\}$ be the H-orbit of V_1 in $\{V_1, \dots, V_m\}$, $t = |H : L \cap H|$. Let $W = V_{11} \oplus \cdots \oplus V_{1t}$. If we may assume $t \geq 2$, then, by Lemma 2, $H/\mathrm{C}_H(W)$ is isomorphic to a subgroup X of the wreath product $A \wr T = A^{\natural} X$, where T is a transitive permutation group on $\Omega_1 = \{1, \dots, t\}$ and the action $H/\mathrm{C}_H(W)$ on W is equivalent to the action of X on $V_1^{\Omega_1}$.

Write $S^* = HN/N \leq S$. Assume that S^* is not transitive on Ω. Our next aim is to prove that in this case S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. Suppose not. By Lemma 19, either $|\Omega_1| \leq 4$ or $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω_1 and $\Pi_3(\Omega_1, T) \geq |T|$.

If $|\Omega_1| = 1$, then $W = V_1$ and $H/\mathrm{C}_H(W)$ has at least two regular orbits on $W \oplus W$. Now we may assume that $|\Omega_1| = t \geq 2$.

If the A-module V_1 is of type (**I**) or type (**II**), then, by Lemma 29, we have that $H/\mathrm{C}_H(W)$ has at least two regular orbits on $W \oplus W$.

Assume that A at least three regular orbits on $V_1 \oplus V_1$. If $\Pi_3(\Omega_1, T) \geq |T|$, then $H/\mathrm{C}_H(W)$ has at least three regular orbits on $W \oplus W$ by Wolf's formula. Assume that $|\Omega_1| \leq 4$. If $p = 2$ or, $p \neq 2$ and A is of order even, then

$H/\mathrm{C}_H(W)$ has three regular orbits on $W \oplus W$ by Lemma 26. If $p \neq 2$ and A is of order odd, then $H/\mathrm{C}_H(W)$ has five regular orbits on $W \oplus W$ by Lemma 34.

Consequently, in both cases, $H/\mathrm{C}_H(W)$ has at least two regular orbits on $W \oplus W$. This implies that H has at least four regular orbits on $V \oplus V$, contrary to assumption.

Thus S^* has at least four regular orbits on $\mathcal{P}(\Omega)$. Let $L_i = \mathrm{N}_G(L_i)$ and $H_i = (L_i \cap H)\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ for all $i \in \{1, \dots, m\}$. Note that $A = H_1$ and $L = L_1$. Arguing as before, we conclude that H_i has at least two regular orbits on $V_i \oplus V_i$ for all $i \in \{1, \dots, m\}$.

Choose $u_i, v_i \in V_i \oplus V_i$ generating two different regular H_i-orbits on $V_i \oplus V_i$ for all $i \in \{1, \dots, m\}$. Note that these elements can be chosen to satisfy the following property: if $V_i = V_j^h$ for some $h \in H$, then $u_i = u_j^h$ and $v_i = v_j^h$. In particular, we have that u_i, v_j are not H-conjugate for all $i, j \in \{1, \dots, m\}$.

Assume that $\Delta \subseteq \Omega$ lies in a regular orbit of S^* on $\mathcal{P}(\Omega)$. This means that $\mathrm{Stab}_{S^*}(\Delta) = 1$. We may assume that $\Delta = \{1, \dots, s\}$, $s < m$. Let $x = u_1 + \cdots + u_s + v_{s+1} + \cdots + v_n$. Then $\mathrm{C}_H(x) \leq \mathrm{Stab}_H(\Delta) \leq N$ since $\mathrm{Stab}_{S^*}(\Delta) = 1$. This implies that $\mathrm{C}_H(x) \leq \mathrm{C}_N(u_i) \leq \mathrm{C}_H(V_i)$, $1 \leq i \leq s$, and $\mathrm{C}_H(x) \leq \mathrm{C}_N(v_j) \leq \mathrm{C}_H(V_j)$, $s+1 \leq j \leq m$. Hence $\mathrm{C}_H(x) \subseteq \bigcap_i \mathrm{C}_G(V_i) = 1$ and x lies in an H-regular orbit on $V \oplus V$.

Therefore every regular orbit of S^* on $\mathcal{P}(\Omega)$ determines a regular orbit of H on $V \oplus V$. In particular, H has at least four regular orbits on $V \oplus V$. This contradicts the choice of G.

Consequently, S^* acts transitively on Ω. Then $\Omega = \Omega_1$, $S^* = T$, $V = W$. We may assume that $X = H$ and so H is a subgroup of $\widehat{H} = A \wr T = A^\natural H$.

If A had at least three regular orbits on $V_1 \oplus V_1$, then H would have at least three regular orbits on $V \oplus V$ by Lemmas 26 and 34. This would contradict the choice of G. Therefore, V_1 is an A-module of type (**I**) or (**II**).

Assume that T has a strong regular orbit on $\mathcal{P}(\Omega)$. Since, by Lemma 28, A has two regular orbits on $V_1 \oplus V_1$, it follows that $\widehat{H}$ has at least two regular orbits on $V \oplus V$ by Wolf's formula. If $|\widehat{H} : H| \geq 2$, then H would have at least four regular orbits on $V \oplus V$, against the choice of G. Thus $H = \widehat{H}$.

Assume that T has even order. If V_1 is of type (**I**), 3 divides $|A|$ and so H has a subgroup isomorphic to $C_3 \wr C_2$. In particular, H is not S_3-free. This contradicts our assumption since H is S_3-free by Lemma 7. If V_1 is of type (**II**), then H has a subgroup isomorphic to $\mathrm{SL}(2,3) \wr C_2$ which has a section isomorphic to S_4, which is not the case. Therefore $|T|$ is odd. In this case, we can apply Corollary 21 to conclude that T has at least four strong regular orbits on $\mathcal{P}(\Omega)$, and so H has at least four regular orbits on $V \oplus V$ by Wolf's formula, unless $(T, \mathrm{d}(T)) = (A_3, 3)$ or $(\Gamma(2^3), 7)$. In any case we have $(T, \mathrm{d}(T)) \in \mathcal{K}$ and the H-module V is of type (**I**) or (**II**), a conclusion which contradicts our choice of G.

Consequently, T has not strong regular orbits on $\mathcal{P}(\Omega)$. By Lemma 20, either $|\Omega| = 2$, $T \cong S_2$ or $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω and there exists 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. We can then apply Lemma 29 to conclude that H has at least three regular orbits on $V \oplus V$. This is the desired contradiction. □

The Proof of Theorem A. We argue by induction on $|G| + |H| + |V|$. Assume that V is not an irreducible G-module. Then there exist non-zero G-submodules V_1 and V_2 such that $V = V_1 \oplus V_2$. Clearly, V_i is a faithful, completely reducible $G/\mathrm{C}_G(V_i)$-module, $i = 1, 2$. Since $H\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ satisfies the hypotheses of the theorem, we conclude that $H\,\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ has at least two regular orbits on $V_i \oplus V_i$, $i = 1, 2$. Moreover, if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then $H\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ has three regular orbits on $V_i \oplus V_i$ for each i. Therefore we may assume that V is an irreducible G-module over $\mathrm{GF}(p)$ for some prime p. Applying Theorem 36 we conclude that either H has at least three regular orbits on $V \oplus V$ or V, as H-module, is of type (**I**) or type (**II**). In the latter case, H has at least two regular orbits on $V \oplus V$ by Lemma 28. Note that if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then H-module V is not of type (**I**) or type (**II**), and so H has at least three regular orbits on $V \oplus V$ by Theorem 36.

We now draw a series of conclusions from Theorem A.

Corollary 37 ([29])**.** *Let G be a soluble group acting completely reducibly and faithfully on an odd order module V. Suppose that H is a subgroup of G. If H is nilpotent or $3 \nmid |H|$, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroup of the semidirect product VH is abelian, then H has at least two regular orbits on $V \oplus V$.*

Corollary 38 (see [5, Theorem 1.1])**.** *Let G be a soluble group and V be a faithful completely reducible G-module. Suppose that $(|G|, |V|) = 1$. Then G has at least two regular orbits on $V \oplus V$.*

Proof. Arguing by induction on $|V| + |G|$, we may assume that V is an irreducible and faithful G-module over $\mathrm{GF}(p)$ for some prime p.

Applying Lemma 35, we may assume that $p = 2$ or 3. In both cases, VG is S_4-free. From Theorem 36, G has at least two regular orbits on $V \oplus V$ when $p = 2$, 3. □

Our next corollary shows that Theorem A of [24] holds for supersoluble subgroups of a soluble group provided that $|V|$ is odd.

Corollary 39. *Let G be a soluble group acting completely reducibly and faithfully on an odd order module V. If H is a supersoluble subgroup of G, then H has at least two regular orbits on $V \oplus V$.*

Proof. Note that H is S_4-free. Since V is of odd order, HV is S_4-free. By Theorem A, H has at least two regular orbits on $V \oplus V$. □

2.5 Proof of Theorem B

We prove:

Theorem 40 (Theorem B)**.** *Let G be a finite soluble group and V be a finite faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a supersoluble subgroup of G. Then H has at least one regular orbit on $V \oplus V$.*

The following theorem is crucial.

Theorem 41. *Let G be a soluble group and let V be an irreducible and faithful G-module over* GF(2)*. If H is a odd order supersoluble group of G, then H has at least four regular orbits on $V \oplus V$ or H-module V satisfies* ***Property I****.*

Proof. We argue by induction on $|G|$. By Lemma 17, H has four regular orbits on $V \oplus V$ or $|V| = 2^3, H = \Gamma(V) \cong [C_7]C_3$. The later case satisfies **Property I**, as desired. Now we may assume that V is an imprimitive G-module. Assume that there $V = V_1 \oplus ... \oplus V_m (m \geq 2)$ is a direct sum of subspaces which are permuted transitively by G. If we do this so that m is as small as possible, then we can assume that $L = \mathrm{N}_G(V_1)$ is maximal in G, and we observe also that L acts irreducibly on V_1. Write $U = L/\mathrm{C}_G(V_1)$ and V_1 is a faithful, irreducible U-module.

Assume that $\Omega_1, ..., \Omega_s (s \geq 1)$ are the all H-orbit in $\{V_1, ..., V_m\}$. Write $W_j = \Sigma_{W \in \Omega_j} W$. Firstly We claim that $H/\mathrm{C}_H(W_j)$ has at least four regular orbits on $W_j \oplus W_j$ or $H/\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** for each j.

Without loss of generality, we only consider the case $j = 1$ and assume that $\Omega_1 = \{V_1, ..., V_t\}$, $t = |H : L \cap H|$. Write $W = W_1$, $K = H/\mathrm{C}_H(W_1)$ and $J = \mathrm{N}_K(V_1)/\mathrm{C}_K(V_1)$.

Now we claim that K has at least four regular orbits on $W \oplus W$ or K-module W satisfies **Property I**. Observe that the action of J on V_1 is equivalent to the action of $(L \cap H)\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1)$(*denote by* A) $\leq U$ on V_1. Then the triple (U, A, V_1) satisfies the hypotheses of the theorem. By induction, A(also J) has at least four regular orbits on $V_1 \oplus V_1$ or A(also J) -module V_1 satisfies **Property I**. If J has at least four regular orbits on $V_1 \oplus V_1$, then it follows from Lemma 32(a) that K has at least four regular orbits on $W \oplus W$, as claim. If J-module V_1 satisfies **Property I**, since $|H|$ is odd, then it follows from Lemma 32(c) that K has at least four regular orbits on $W \oplus W$ or K-module W satisfies **Property I**, as claim.

Thus $H/\,\mathrm{C}_H(W_j)$ has at least two regular orbits on $W_j \oplus W_j$ for each $1 \le j \le s$. If $s \ge 2$, then H has at least four regular orbits on $V \oplus V$ by Lemma 3, as desired. Now we may assume that $s = 1$, that is, H acts transitively on $\{V_1, ..., V_m\}$. Thus $H = K$ and $W = V$, and consequently H has at least four regular orbits on $V \oplus V$ or H-module V satisfies **Property I**. The theorem is proved. □

Theorem 42. *Let G be a soluble group and V be an irreducible and faithful, G-module over* GF(2). *If H is a supersoluble subgroup of G, then either H has at least three regular orbits on $V \oplus V$ or V, as H-module, satisfies* ***Property I*** *or* ***Property II***.

Proof. Work by induction on $|GV|$. If V is a primitive G-module, it follows from Lemma 16 that either H has at least three regular orbits on $V \oplus V$ or H-module V satisfies

(1) $|V| = 2^2$ and $H = \Gamma(V) \cong S_3$; or

(2) $|V| = 2^3$ and $H = \Gamma(V) \cong [C_7]C_3$.

It is not difficult to find that the case (1) satisfies **Property II** and the case (2) satisfies **Property I**, as desired. Consequently, we assume that V is an imprimitive G-module. Then there $V = V_1 \oplus ... \oplus V_m (m \ge 2)$ is a direct sum of subspaces which are permuted transitively by G. If we do this so that m is as small as possible, then we can assume that $L = \mathrm{N}_G(V_1)$ is maximal in G, and we observe also that L acts irreducibly on V_1. Write $U = L/\,\mathrm{C}_G(V_1)$ and V_1 is a faithful, irreducible U-module.

Assume that $\Omega_1, ..., \Omega_s (s \ge 1)$ are the all H-orbit in $\{V_1, ..., V_m\}$. Write $W_j = \Sigma_{W \in \Omega_j} W$.

Firstly We claim that $H/\,\mathrm{C}_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$ or $H/\,\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** or **Property II** for each j.

Without loss of generality, we only consider the case $j = 1$ and assume that $\Omega_1 = \{V_1, ..., V_t\}$, $t = |H : L \cap H|$. Write $W = W_1$, $K = H/\,\mathrm{C}_H(W_1)$ and $J = \mathrm{N}_K(V_1)/\,\mathrm{C}_K(V_1)$. Then W is a faithful H-module. Observe that the action of J on V_1 is equivalent to the action of $(L \cap H)\,\mathrm{C}_G(V_1)/\,\mathrm{C}_G(V_1)$(denoted by$A$) $\le U$ on V_1. Then the triple (U, A, V_1) satisfies the hypotheses of the theorem. By induction, either A(also J)

has at least three regular orbits on $V_1 \oplus V_1$ or $A($ also$J)$-module V_1 satisfies **Property I** or **Property II**.

If J-module V_1 satisfies **Property I**, it follows from Lemma 32(5) that our claim holds. If J-module V_1 satisfies **Property II**, it follows from Lemma 32(4) that our claim holds. Now we assume that J has at least three regular orbits on $V_1 \oplus V_1$. If J is of even order, then K has at least three regular orbits on $W \oplus W$ by Lemma 32(2). If J is of odd order, then A is of odd order and the triple (U, A, V_1) satisfies the hypotheses of Theorem 41. Thus A(also J) has at least four regular orbits on $V_1 \oplus V_1$ or A(also J) -module V_1 satisfies **Property I**. If J has at least four regular orbits on $V_1 \oplus V_1$, then K has at least four regular orbits on $W \oplus W$ by Lemma 32(1), as claim. If J-module V_1 satisfies **Property I**, then, by Lemma 32(5) again, our claim holds.

Now we have proven that $H/\operatorname{C}_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$ or $H/\operatorname{C}_H(W_j)$-module W_j satisfies **Property I** or **Property II** for each $1 \leq j \leq s$. In particular, $H/\operatorname{C}_H(W_j)$ has at least one regular orbits on $W_j \oplus W_j$ for each $1 \leq j \leq s$. If there exists some $j \in \{1, ..., s\}$ such that $H/\operatorname{C}_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$, then we can conclude that H has at least three regular orbits on $V \oplus V$ by Lemma 3, as desired.

Now we can assume that $H/\operatorname{C}_H(W_j)$-module W_j satisfies **Property I** or **Property II** for each $1 \leq j \leq s$. Thus if $s = 1$, then V, as H-module, satisfies **Property I** or **Property II**, as desired. Thus $s \geq 2$.

Take $\mathcal{C} = \{1 \leq j \leq s : H/\operatorname{C}_H(W_j)$-module W_j satisfies **Property II**$\}$;

Firstly we assume that $\mathcal{C} = \varnothing$. Then $H/\operatorname{C}_H(W_j)$-module W_j satisfies **Property I** for each $1 \leq j \leq s$. It implies that $H/\operatorname{C}_H(W_j)$ has at least two regular orbits on $W_j \oplus W_j$. Since $s \geq 2$, then we can conclude that H has at least four regular orbits on $V \oplus V$ by Corollary 3, as desired.

Now we assume that $\mathcal{C} \neq \varnothing$, then, without loss of generality, we may assume that $\mathcal{C} = \{1, ..., l\}$ for some $1 \leq l \leq s$.

Write $K_j = H/\operatorname{C}_H(W_j)$. For $j = 1$, we have

• K_1 is an even order group and $\operatorname{O}_2(K_1) = 1$.

• Then there exists $0 \neq x_1 \in V_1$ and $\operatorname{C}_{K_1}(x_1)$ has three different orbits on V_1 with representations y_1, z_1, z_2 such that $\operatorname{C}_{K_1}(x_1) \cap \operatorname{C}_{K_1}(y_1) = 1$ and $\operatorname{C}_{K_1}(x_1) \cap \operatorname{C}_{K_1}(z_i)$ is a 2-group for $i = 1, 2$.

Recall that K_j has at least one regular orbit on $V_j \oplus V_j$ for each $2 \leq j \leq s$. We can assume that $\operatorname{C}_{K_j}(x_j) \cap \operatorname{C}_{K_j}(y_j) = 1$ for some $x_j, y_j \in V_j$.

Thus we can conclude that $\operatorname{C}_H(x_j) \cap \operatorname{C}_H(y_j) \subseteq \operatorname{C}_H(W_j)$ for each $1 \leq j \leq s$ and $X_i/\operatorname{C}_H(W_1)$ is a 2-group, where $X_i = \operatorname{C}_H(x_1) \cap \operatorname{C}_H(z_i)$ for $i = 1, 2$.

Write $v = \sum_{i=1}^{s} x_i$, $u = \sum_{i=1}^{s} y_i$, $w_1 = z_1 + \sum_{i=2}^{s} y_i$ and $w_2 = z_1 + \sum_{i=2}^{s} y_i$. It is not difficult to find that u, w_1, w_2 lie in different orbits of $\operatorname{C}_H(v)$ on V.

Moreover, we have

$$\mathrm{C}_H(v) \cap \mathrm{C}_H(u) = \bigcap_{j=1}^{s} \mathrm{C}_H(x_j) \cap \mathrm{C}_H(y_j) \subseteq \bigcap_{j=1}^{s} \mathrm{C}_H(W_j) = 1;$$

and

$$\mathrm{C}_H(v) \cap \mathrm{C}_H(w_i) \subseteq X_i \cap \bigcap_{j=2}^{s} \mathrm{C}_H(W_j) \cong (X_i \cap \bigcap_{j=2}^{s} \mathrm{C}_H(W_j))\, \mathrm{C}_H(W_1)/\, \mathrm{C}_H(W_1)$$

is a 2-group for $i = 1, 2$.

On the other hand, H is of even order since $H/\,\mathrm{C}_H(W_j)$ is of even order. Moreover, for each $j \in \mathcal{C}$, we have that $H/\,\mathrm{C}_H(W_j)$ is an even order group and $\mathrm{O}_2(H/\,\mathrm{C}_H(W_j)) = 1$; and for each $j \in \{1, ..., s\} - \mathcal{C}$, we have that $H/\,\mathrm{C}_H(W_j)$ is an odd order group. Thus $\mathrm{O}_2(H) \leq \cap_{i=1}^{s} \mathrm{C}_H(W_j) = 1$. Thus H-module V satisfies **Property II**, as desired. Thus the theorem is proved completely. □

The Proof of Theorem B. Assume that the theorem is false and let (G, H, V) be the counterexample such that $|G|+|H|+|V|$ minimal. Firstly we can claim that V is an irreducible G-module. Assume not; let $V = V_1 \oplus V_2$, where $0 \neq V_i$ is a G-module. Then V_i is a faithful, completely reducible $G/\,\mathrm{C}_G(V_i)$-module. Observe that $H\,\mathrm{C}_G(V_i)/\,\mathrm{C}_G(V_i)$ satisfies the hypotheses. Thus by the choice of (G, H, V), $H\,\mathrm{C}_G(V_i)/\,\mathrm{C}_G(V_i)$ has at least one regular orbit on $V_i \oplus V_i$. Thus H has at least one regular orbits on $V \oplus V$, against the choice of (G, H, V). Thus V is an irreducible G-module over the field of characteristic p for some prime p. Then V is a completely reducible G-module over $\mathrm{GF}(p)$, the filed of p elements. Arguing as above, we may assume that V is an irreducible, faithful G-module over $\mathrm{GF}(p)$. If p is odd, then it follows from Corollary 39 that H has at least two regular orbits on $V \oplus V$. Thus we may assume that $p = 2$. It follows from Theorem 42 that H has at least three regular orbits on $V \oplus V$, or H-module V satisfies **Property I** or **Property II**. In all these cases above, we can conclude that H has at least one regular orbit on $V \oplus V$ and the main theorem is completely proved.

Chapter 3

Application I: On Gluck's conjecture

Suppose that a group H acts on an abelian group A. Then H acts on the set $A^\star = \mathrm{Irr}(A)$ of all complex characters of A: for any $\chi \in A^\star$ and $h \in H$, χ^h is defined by setting $\chi^h(a) = \chi(a^{h^{-1}}), a \in A$.

Lemma 43. *Suppose that a group H acts on an abelian group A. Then*

1. $\mathrm{C}_H(A) = \mathrm{C}_H(A^\star)$.
2. *If $A = A_1 \times \cdots \times A_n$ and A_i is H-invariant, then $(A_1)^\star \times \cdots \times (A_n)^\star$ i and $A^\star$ are H-isomorphic.*
3. *If A is a completely reducible H-module, then $A^\star$ is a completely reducible H-module.*

Proof. 1. Let $h \in \mathrm{C}_H(A)$ and any $\chi \in A^\star$. Then $\chi^h(a) = \chi(a^{h^{-1}}) = \chi(a), \forall a \in A$. Thus $\chi^h = \chi$ and so $h \in \mathrm{C}_H(A^\star)$. On the other hand, for any $h \in \mathrm{C}_H(A^\star)$ and any $a \in A$, we have

$$\begin{aligned}\chi(a^{h^{-1}}a^{-1}) &= \chi(a^{h^{-1}})\chi(a)^{-1} \\ &= \chi^h(a)\chi(a)^{-1} = \chi(a)\chi(a)^{-1} = 1, \forall \chi \in A^\star.\end{aligned}$$

Thus $a^{h^{-1}}a^{-1} \in \bigcap_{\chi \in A^\star} \mathrm{Ker}\,\chi = 1$ and so $h \in \mathrm{C}_H(A)$.

2. Let

$$\varphi\colon (A_1)^\star \times \cdots \times (A_n)^\star \longrightarrow A^*; (\chi_1, \ldots, \chi_n) \mapsto \chi(a) = \Pi_{i=1}^n \chi_i(a_i),$$

where $a = \Pi_{i=1}^n a_i \in A$ and $a_i \in A_i$. It is not difficult to verify that φ is a group-isomorphism. Now we show that it is an H-isomorphism. For any $(\chi_1, \dots, \chi_n) \in (A_1)^\star \times \cdots \times (A_n)^\star$, $h \in H$, $a = \Pi_{i=1}^n a_i \in A$ and $a_i \in A_i$, we have

$$\varphi((\chi_1, \dots, \chi_n)^h)(a) = \prod_{i=1}^n \chi_i(a_i^{h^{-1}}) = \varphi((\chi_1, \dots, \chi_n))(a^{h^{-1}})$$
$$= \varphi((\chi_1, \dots, \chi_n))^h(a).$$

Then we have $\varphi((\chi_1, \dots, \chi_n)^h) = \varphi((\chi_1, \dots, \chi_n))^h$, as desired.

3. Suppose that $A = A_1 \times \cdots \times A_n$ for some irreducible H-modules A_i, for each $1 \leq i \leq n$. Then, by (2), $(A_1)^\star \times \cdots \times (A_n)^\star \cong_H A^\star$. Since A_i is an irreducible H-module, we have $(A_i)^\star$ is an irreducible H-module by [18, Proposition 12.1]. Thus $A^\star$ is a completely reducible H-module.

□

Lemma 44. *Assume that a group X acts on an abelian group U and let $G = [U]X$ be the corresponding semidirect product. Then $|X : \mathrm{C}_X(\lambda)| \leq b(G)$ for each $\lambda \in U^\star$.*

Proof. For each $\lambda \in U^\star = \mathrm{Irr}(U)$, given $\chi \in \mathrm{Irr}(G, \lambda)$ we have that $\chi(1) \geqslant |G : \mathrm{C}_G(\lambda)|$ by Theorem [12, Theorem 19.3]. Since U is abelian, we have $U \subseteq \mathrm{C}_G(\lambda)$ and so $\mathrm{C}_G(\lambda) = U\,\mathrm{C}_X(\lambda)$. Thus $|X : \mathrm{C}_X(\lambda)| = |G : \mathrm{C}_G(\lambda)| \leq \chi(1) \leq \mathrm{b}(G)$, as desired. □

We are now ready to prove our third main result.

Theorem 45 (Theorem C)**.** *Let G be a soluble group satisfying one of the following conditions:*

1. *G is S_4-free;*
2. *$G/\mathrm{F}(G)$ is S_4-free and $\mathrm{F}(G)$ is of odd order;*
3. *$G/\mathrm{F}(G)$ is S_3-free;*
4. *$G/\mathrm{F}(G)$ is supersoluble.*

Then Gluck's conjecture is true for G.

Proof. Set $U = \mathrm{F}(G)/\Phi(G)$ and $V = U^\star = \mathrm{Irr}(U)$. According to [3, Theorem A.10.6], there exists a subgroup X of $\overline{G} = G/\Phi(G)$ such that $\overline{G} = UX$ and $U \cap X = 1$ and U is a faithful completely reducible X-module. By Lemma 43, V is a faithful completely reducible X-module. Let U_1 be the Hall $2'$-subgroup and let U_2 be the Sylow 2-subgroup of U. Then $U = U_1 \times U_2$.

Applying Lemma 43, we have that $W = W_1 \oplus W_2$, where $W_i = (U_i)^\star$, is X-isomorphic to V, and $\mathrm{C}_X(W_i) = \mathrm{C}_X(U_i)$, $i = 1, 2$.

With the above observations in mind, the burden lies in proving that X has a regular orbit on $W \oplus W$.

Assume that G is S_4-free. Since $U_2X/\mathrm{C}_X(U_2)$ is S_4-free, we have that $X/\mathrm{C}_X(W_2) = X/\mathrm{C}_X(U_2)$ is S_3-free by Corollary 7. Applying again this corollary, we have that $W_2X/\mathrm{C}_X(W_2)$ is S_4-free. Since $X/\mathrm{C}_X(W_1)$ is S_4-free, we have $W_1X/\mathrm{C}_X(W_1)$ is S_4-free as W_1 is $2'$-group. By Theorem A that $X/\mathrm{C}_X(W_i)$ has at least two regular orbits on $W_i \oplus W_i$, $i = 1, 2$. This implies that X has a regular orbit on $W \oplus W$.

If G satisfies Statement (2), then $W_2 = 1$ and $W = W_1$. Since X is S_4-free, WX is S_4-free. It follows from Theorem A that X has a regular orbit on $W \oplus W$.

Assume that G satisfies Statement (3). Since X is S_3-free, we have $X/\mathrm{C}_X(W_2)$ is S_3-free. It follows from Corollary 7 that $W_2X/\mathrm{C}_X(W_2)$ is S_4-free. Since $X/\mathrm{C}_X(W_1)$ is S_4-free, we have $W_1X/\mathrm{C}_X(W_1)$ is S_4-free since W_1 is $2'$-group. Thus $W_iX/\mathrm{C}_X(W_i)$ is S_4-free for both $i = 1, 2$. It follows from Theorem A that $X/\mathrm{C}_X(W_i)$ has at least two regular orbits on $W_i \oplus W_i$, $i = 1, 2$. Thus X has a regular orbit on $W \oplus W$.

Assume that $G/\mathrm{F}(G)$ is supersoluble. Then X is supersoluble and V is a faithful, completely reducible X-module. By Theorem B, X has a regular orbit on $V \oplus V$.

Thus X has a regular orbit on $V \oplus V$ in all cases. Then there exists $\lambda \in V$ such that $|\mathrm{C}_X(\lambda)| \leq |X|^{1/2}$. Consequently $|X|^{1/2} \leq |X : \mathrm{C}_X(\lambda)|$. By Lemma 44, we have that $|X|^{1/2} \leq \mathrm{b}(G/\Phi(G))$. Thus $|G : \mathrm{F}(G)| = |X| \leq \mathrm{b}(G)^2$. □

We derive now some results related to Gluck's conjecture. The first one is part of [2, Theorem 7].

Corollary 46. *Let G be a soluble group and let H be a π-Hall subgroup of G, where $\pi = \pi(\mathrm{F}(G))$. Then $|G : H| \leq \mathrm{b}(G)^2$.*

Proof. Let K be a Hall π'-subgroup of G. Since $(|K|, |U|) = 1$, we have $\mathrm{C}_K(U) \leq \mathrm{C}_K(\mathrm{F}(G)) \leq K \cap \mathrm{F}(G) = 1$. Thus U and V are faithful completely reducible K-modules. By Lemma 44, $|G : H| = |K| \leq \mathrm{b}(G)^2$. □

The second one is part of [2, Corollary 2].

Corollary 47. *Let G be a soluble group. If $|G/\mathrm{F}(G)|$ is not divisible by 6, then Gluck's conjecture holds.*

Corollary 48. [29, Theorem 4.6] *Let G be a soluble group. Then $|G : \mathrm{F}(G)|_{3'} \leq b(G)^2$.*

Proof. Let K be a $3'$-Hall subgroup of X. Clearly KV is S_4-free, by Theorem A, K has a regular orbit on $V \oplus V$. Thus there exists $\lambda \in V$ such that $|\operatorname{C}_K(\lambda)| \leq |K|^{\frac{1}{2}}$. By Lemma 44,

$$|K|^{\frac{1}{2}} \leq |K : \operatorname{C}_K(\lambda)| \leq |X : \operatorname{C}_X(\lambda)| \leq b(G/\Phi(G)).$$

Consequently $|G : \operatorname{F}(G)|_{3'} = |K| \leq b(G)^2$. □

Our last result of this chapter generalises a theorem of T. M. Keller and Y. Yang [16, Theorem 1.2] by replacing the nilpotent residual by the residual with respect to the saturated formation Σ_3 of all S_3-free groups.

Theorem 49. *Let G be a soluble group and V a faithful completely reducible G-module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V. Then*

$$|G : G^{\Sigma_3}| \leq M^2.$$

Proof. Let H be a Σ_3-projector of G. Then $G^{\Sigma_3}H = G$ and $H \in \Sigma_3$. Then H is S_3-free and clearly HV is S_4-free. By Theorem A, H has a regular orbit on $V \oplus V$. It implies that $|\operatorname{C}_H(v)| \leq |H|^{1/2}$ for some $v \in V$. Let M_H be the largest orbit size of H on V. Then it follows that $|H| \leq |H : \operatorname{C}_H(v)|^2 \leq M_H^2$. Hence clearly $|G/G^{\Sigma_3}| \leq |H| \leq M_H^2 \leq M^2$, as desired. □

Chapter 4

Application II: Intersections of subgroups

This chapter has as its main theme the study of intersections of normalisers and prefrattini subgroups of finite soluble groups associated to saturated formations and intersections of injectors associated to Fitting classes. It provides answers to two questions raised by Kamornikov and Shemetkov and Vasil'ev in the Kourovka Notebook [19].

Problem 1. [19, Kamornikov, Problem 17.55] *Does there exist an absolute constant k such that the Frattini subgroup $\Phi(G)$ of a soluble group G is the intersection of k G-conjugates of any prefrattini subgroup H of G?*

Problem 2. [19, Shemetkov and Vasil'ev, Problem 17.39] *Is there a positive integer k such that the hypercentre of any finite soluble group coincides with the intersection of k system normalisers of that group? What is the least number with this property?*

The main results of the chapter can be summarised in the following theorem.

Theorem 50 (Theorem D)**.** *Let G be a soluble group and let H be a subgroup of G. Assume that one of the following statements holds.*

1. *H is an $\mathfrak{F}$-prefrattini subgroup of G for some saturated formation $\mathfrak{F}$;*
2. *$\Phi(G) = 1$ and H is a $\mathfrak{F}$-normaliser of G for some saturated formation $\mathfrak{F}$;*
3. *H is an $\mathfrak{F}$-injector of G for some Fitting class $\mathfrak{F}$.*

Then there exists $x, y, z \in G$ such that $H \cap H^x \cap H^y \cap H^z = \mathrm{Core}_G(H)$, the largest normal subgroup of G contained in H. Furthermore, if G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, there exists $x, y \in G$ such that $H \cap H^x \cap H^y = \mathrm{Core}_G(H)$.

Corollary 51 ([17])**.** *If I is a nilpotent injector of a soluble group G, then $(G, I, \mathrm{F}(G))$ is a 3-conjugate system.*

4.1 Background results

In the sequel, $\mathfrak{F}$ will be a saturated formation. We begin with an elementary observation which will be used throughout the chapter.

Lemma 52 ([3, Lemma A.16.3])**.** *Let $G = NH$ be a semidirect product of a normal subgroup N with a subgroup H.*
(*a*) *If $n \in N$, then $H \cap H^n = \mathrm{C}_H(n)$,*
(*b*) $\mathrm{Core}_G(H) = \mathrm{C}_H(N)$.

Our next lemmas turn out to be crucial in the proof of our results about prefrattini subgroups.

Lemma 53. *Let N be a minimal normal subgroup of a soluble group G. Assume that M is an $\mathfrak{F}$-abnormal maximal subgroup of G complementing N in G. Then $\mathbf{Pref}_{\mathfrak{F}}(G) = \bigcup_{g \in G} \mathbf{Pref}_{\mathfrak{F}}(M^g)$.*

Proof. Since $\mathfrak{F}$-prefrattini subgroups of G are conjugate in G, it suffices to show that $\mathbf{Pref}_{\mathfrak{F}}(M) \subseteq \mathbf{Pref}_{\mathfrak{F}}(G)$.

Let $H = W(M, \Sigma_M, \mathfrak{F})$ be the $\mathfrak{F}$-prefrattini subgroup of M associated to the Hall system Σ_M of M. Let p be the prime dividing the order of N and let P be the Sylow p-subgroup of M in Σ. Then $\Sigma = \Sigma_M \cup \{PN\}$ is a Hall system of G.

Let $1 = A_0 \leq A_1 \leq \cdots \leq A_n = M$ be a chief series of M, and let $\{A_i/A_{i-1} \mid i \in I\}$ be the set of all complemented $\mathfrak{F}$-eccentric chief factors in this series. By [1, Proposition 4.3.6], $H = W(\Sigma) = \bigcap_{i \in I} M_i$, where M_i is a maximal subgroup of M, complementing A_i/A_{i-1} in G, into which the Hall system Σ_M reduces, $i \in I$. Consider the following chief series of G:

$$1 \leq N = A_0N \leq A_1N \leq \cdots \leq A_nN = MN = G$$

Then $A_iN/A_{i-1}N$ is a complemented $\mathfrak{F}$-eccentric chief factor of G if and only if A_i/A_{i-1} is a complemented $\mathfrak{F}$-eccentric chief factor of M. Moreover, N is an $\mathfrak{F}$-eccentric chief factor of G which is complemented by M, and Σ

reduces into M. Thus $\{N, A_i/A_{i-1} \mid i \in I\}$ is the set of all complemented $\mathfrak{F}$-eccentric chief factors in the above chief series.

On the other hand, M_iN is a maximal subgroup of G complementing $A_iN/A_{i-1}N$ in G and Σ reduces into M_iN for all $i \in I$. Applying [1, Proposition 4.3.6], $M \cap (\bigcap_{i\in I} M_iN) = \bigcap_{i\in I} M_i(M \cap N) = \bigcap_{i\in I} M_i = H$ is the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ. □

Remark 54. Under the hypotheses of Lemma 53, $(H \cap H^m)N = HN \cap H^mN$ for all $m \in M$.

Proof. $HN \cap H^mN = (H \cap H^mN)N = (H \cap M \cap H^mN)N$ and $M \cap H^mN = H^m(M \cap N) = H^m$. □

Lemma 55. *Let N be a minimal normal subgroup of a soluble group G. Assume that M is an $\mathfrak{F}$-abnormal maximal subgroup of G complementing N in G. Then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{C}_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$.*

Proof. By Lemma 53, we have:

$$\begin{aligned}\mathrm{L}_{\mathfrak{F}}(G) &= \bigcap\{H : H \in \mathbf{Pref}_{\mathfrak{F}}(G)\}\\ &= \bigcap_{g\in G}\bigcap\{H : H \in \mathbf{Pref}_{\mathfrak{F}}(M^g)\}\\ &= \bigcap_{g\in G} \mathrm{L}_{\mathfrak{F}}(M)^g = \mathrm{Core}_G(\mathrm{L}_{\mathfrak{F}}(M)).\end{aligned}$$

Since $\mathrm{L}_{\mathfrak{F}}(G) \cap N \leq M \cap N = 1$, we have $\mathrm{L}_{\mathfrak{F}}(G) \leq \mathrm{C}_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$. On the other hand, since $\mathrm{C}_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$ is normalised by M and centralised by N, we have that $\mathrm{C}_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$ is normal in G and hence $\mathrm{C}_{\mathrm{L}_{\mathfrak{F}}(M)}(N) \leq \mathrm{Core}_G(\mathrm{L}_{\mathfrak{F}}(M)) = \mathrm{L}_{\mathfrak{F}}(G)$. □

Lemma 56 ([5, Theorem 1.4])**.** *Let G be a soluble group and V a finite faithful G-module. If V is completely reducible (possibly of mixed characteristic), then there exist $v_1, v_2, v_3 \in V$ such that $\mathrm{C}_G(v_1) \cap \mathrm{C}_G(v_2) \cap \mathrm{C}_G(v_3) = 1$.*

4.2 Main results

We have considered convenient to give the following definition.

Definition 57. A 3-tuple (G, X, Y) is said to be a *k-conjugate system* if G is a group, X, Y are subgroups of G with $Y = \mathrm{Core}_G(X)$, and there exist k elements $g_1, \dots, g_k$ such that $Y = \bigcap_{i=1}^{k} X^{g_i}$.

Assume we are trying to prove a result of the following type: Let G be a soluble group and let H be an $\mathfrak{F}$-prefrattini subgroup of G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a k-conjugate system.

Assume the statement is false. Thus there would exist a counterexample G of minimal order. Let H be an $\mathfrak{F}$-prefrattini subgroup of G such that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is not a k-conjugate system. Then:

(i) $\mathrm{L}_{\mathfrak{F}}(G) = 1$. In particular, $\Phi(G) = 1$.

For suppose that X is a minimal normal subgroup of G contained in $\mathrm{L}_{\mathfrak{F}}(G)$. Then H/X is an $\mathfrak{F}$-prefrattini subgroup of G by Lemma 11. Therefore, because $|G/X| < |G|$, it follows that $(G/X, H/X, \mathrm{L}_{\mathfrak{F}}(G/X))$ is a k-conjugate system. Since $\mathrm{L}_{\mathfrak{F}}(G/X) = \mathrm{L}_{\mathfrak{F}}(G)/X$ by Lemma 12, we have that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a k-conjugate system, giving a contradiction. Thus Statement (i) must hold.

Also (ii) There exists a minimal normal subgroup N and an $\mathfrak{F}$-abnormal maximal subgroup M containing H of G such that $G = MN$ and $M \cap N = 1$ and $(M, H, \mathrm{L}_{\mathfrak{F}}(M))$ is a k-conjugate system.

Let N be the minimal normal subgroup of G. Then N is a p-group for some prime p. By Statement (i), N is not contained in $\mathrm{L}_{\mathfrak{F}}(G) = 1$ and so there exists an $\mathfrak{F}$-abnormal maximal subgroup of M such that $G = NM$ and $N \cap M = 1$. By Lemma 53, we may assume that H is an $\mathfrak{F}$-prefrattini subgroup of M. Again by choice of G, $(M, H, \mathrm{L}_{\mathfrak{F}}(M))$ is a k-conjugate system and therefore there exist $m_1, \ldots, m_k \in M$ such that $\bigcap_{i=1}^{k} H^{m_i} = \mathrm{L}_{\mathfrak{F}}(M)$.

(iii) Assume that N is a p-group for some prime p and $L = \mathrm{L}_{\mathfrak{F}}(M)$. Then N is a faithful completely reducible L-module over $\mathrm{GF}(p)$, the finite field of p-elements.

Clearly N is an irreducible M-module over $\mathrm{GF}(p)$. By [3, Theorem B.7.3], N is a completely reducible L-module. By Lemma 55 and Statement (i), $\mathrm{C}_L(N) = 1$ and so N is faithful for L.

Let $T = LN$. Then $\mathrm{Core}_T(L) = 1$. Moreover:

(iv) $(T, L, 1)$ is not a k-conjugate system.

Assume that $(T, L, 1)$ is a k-conjugate system. Let $n_1, \ldots, n_k \in N$ such that $\bigcap_{i=1}^{k} L^{n_i} = 1$. We consider the subgroup $D = \bigcap_{i=1}^{k} H^{m_i n_i}$. Then

$$D \leq \bigcap_{i=1}^{k} H^{m_i n_i} N = \bigcap_{i=1}^{k} H^{m_i} N = (\bigcap_{i=1}^{k} H^{m_i}) N = LN$$

by Remark 54. Then

$$D = D \cap LN = \bigcap_{i=1}^{k} H^{m_i n_i} \cap LN$$
$$= \bigcap_{i=1}^{k} (H^{m_i} \cap LN)^{n_i} = \bigcap_{i=1}^{k} L^{n_i} = 1 = \mathrm{L}_{\mathfrak{F}}(G).$$

Therefore $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a k-conjugate system, against our supposition.

The next two theorems subsume the main result of [15] and give a complete answer to a general version of Question 1.

Theorem 58. *Let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 4-conjugate system.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order such that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is not a 4-conjugate system. Then Statements (i)–(iv) hold for $k = 4$. By Statement (iii), N is a faithful completely reducible L-module over $\mathrm{GF}(p)$ for some prime p. By Lemma 56, there exist $v_1, v_2, v_3 \in N$ such that $\mathrm{C}_L(v_1) \cap \mathrm{C}_L(v_2) \cap \mathrm{C}_L(v_3) = 1$. It implies that $L \cap L^{v_1} \cap L^{v_2} \cap L^{v_3} = 1$ by Lemma 52. Thus $(T, \mathrm{L}, 1)$ is a 4-conjugate system, contrary to Step (iv). □

Theorem 59. *Let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Assume that either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Proof. Suppose, arguing by contradiction, that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is not a 3-conjugate system. Let us choose G a counterexample of least order. Then Statements (i)–(iv) hold for $k = 3$. By Statement (iii), $L \cap N = 1$ and N is a faithful completely reducible L-module over $\mathrm{GF}(p)$ for some prime p. If G is S_4-free, then LN is S_4-free. Assume that $\mathfrak{F}$ is composed of S_3-free groups. Recall that $L = \mathrm{L}_{\mathfrak{F}}(M)$, by [1, Proposition 4.3.17], $L/\Phi(M) = \mathrm{Z}_{\mathfrak{F}}(M/\Phi(M))$. Let $\mathfrak{X}$ be the class of all soluble S_3-free groups. By Lemma 10, $\mathfrak{X}$ is a subgroup-closed saturated formation. Since $\mathfrak{F} \subseteq \mathfrak{X}$ by hypothesis, it follows that $\mathrm{Z}_{\mathfrak{F}}(M/\Phi(M)) \leq \mathrm{Z}_{\mathfrak{X}}(M/\Phi(M))$. By [3, Theorem IV.6.15], $\mathrm{Z}_{\mathfrak{X}}(M/\Phi(M)) \in \mathfrak{X}$. Thus $L/\Phi(M) = \mathrm{Z}_{\mathfrak{F}}(M/\Phi(M))$ is S_3-free. Then, by Lemma 10, L is S_3-free. If p is odd, then LN is S_4-free by Lemma 6 and if $p = 2$, then LN is S_4-free by Corollary 7. In both cases, we can apply Theorem A to conclude that there exist $v_1, v_2 \in N$ such that $\mathrm{C}_L(v_1) \cap \mathrm{C}_L(v_2) = 1$. Thus, by Lemma 52, $(T, L, 1)$ is a 3-conjugate system, contrary to Statement (iv). □

If $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{L}(G)$ is the intersection of all self-normalising maximal subgroups of G. It is a characteristic nilpotent subgroup of G that was introduced by Gaschütz in [8]. If $\mathfrak{F}$ is the trivial formation, then $\mathrm{L}_{\mathfrak{F}}(G) = \Phi(G)$, the Frattini subgroup of G. Hence:

Corollary 60 ([14]). *If G is soluble and H is an $\mathfrak{N}$-prefrattini subgroup of G, then $(G, H, \mathrm{L}(G))$ is a 3-conjugate system.*

Corollary 61 ([15]). *If G is soluble and H is a prefrattini subgroup of G, then $(G, H, \Phi(G))$ is a 3-conjugate system.*

The proof of our next theorem depends on a nice result about factorisations of prefrattini subgroups proved in [10, Theorem 4.1] (see [1, Theorem 4.3.32]).

Lemma 62. *If D is an $\mathfrak{F}$-normaliser and W is a prefrattini subgroup of a soluble group G, both associated to the Hall system Σ of G, then D and W permute and DW is the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ.*

Theorem 63. *Let D be an $\mathfrak{F}$-normaliser of a soluble group G. If $\Phi(G) = 1$, then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system.*

Proof. Let D be the $\mathfrak{F}$-normaliser of G associated to the Hall system Σ. Assume that H is the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ. Then, by Lemma 62, we have $D \leq H$. Since $\Phi(G) = 1$, it follows by [1, Proposition 4.3.17] that $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$. By Theorem 58, we have that $(G, H, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Hence

$$\begin{aligned}\mathrm{Z}_{\mathfrak{F}}(G) &\leq D \cap D^x \cap D^y \cap D^z \\ &\leq H \cap H^x \cap H^y \cap H^z \\ &= \mathrm{Z}_{\mathfrak{F}}(G).\end{aligned}$$

Thus $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system. □

Theorem 64. *Let D be an $\mathfrak{F}$-normaliser of a soluble subgroup G such that $\Phi(G) = 1$. Assume that either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups. Then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Proof. Assume that Σ is the Hall system of G to which D is associated. Let H be the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ. By Theorem 59, $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 3-conjugate system. Since $D \leq H$ by Lemma 62 and $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ by [1, Proposition 4.3.17], it follows that $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system. □

Recall that if $\mathfrak{F} = \mathfrak{N}$ is the formation of all nilpotent groups, then the $\mathfrak{N}$-normalisers of a soluble group G are exactly the system normalisers of G and $\mathrm{Z}_{\mathfrak{N}}(G) = \mathrm{Z}_\infty(G)$ is the hypercentre of G. Therefore the answer of Question 2 for groups with trivial Frattini subgroup is contained in the following:

Corollary 65. *Let G be a soluble group with $\Phi(G) = 1$. If D is a system normaliser of G, then $(G, D, \mathrm{Z}_\infty(G))$ is a 3-conjugate system.*

Our next example shows that $(G, D, \mathrm{Z}_\infty(G))$ is not a 2-conjugate system in general.

Example 1. Let D be the dihedral group of order 8. Then D has an irreducible and faithful module V of dimension 2 over the field of 3-elements such that $\mathrm{C}_D(v) \neq 1$ for all $v \in V$. Let $G = V \rtimes D$ be the corresponding semidirect product. Then D is a system normaliser of G and $Z_\infty(G) = 1$. By [3, Lemma A.16.3], $D \cap D^v = \mathrm{C}_D(v) \neq 1$ for all $v \in V$. Hence $(G, D, Z_\infty(G))$ is not a 2-conjugate system.

Our last theorem has Mann's result ([17]) as starting point and analyses the intersections of injectors associated to Fitting classes of soluble groups.

Theorem 66. *Let $\mathfrak{F}$ be a Fitting class and let I be an $\mathfrak{F}$-injector of a soluble group G. Then $(G, I, G_{\mathfrak{F}})$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, I, G_{\mathfrak{F}})$ is a 3-conjugate system.*

Proof. Let $R = \mathrm{Core}_G(I) = G_{\mathfrak{F}}$. We prove that (G, I, R) is a 4-conjugate system by induction on the order of G. Let F be the normal subgroup of G such that $F/R = \mathrm{F}(G/R)$, the Fitting subgroup of G/R. Clearly, $F \cap I$ is contained in R. Hence $F \cap I = R$. On the other hand, by [3, Theorem IX.1.5], I is an $\mathfrak{F}$-injector of FI. Thus $R \leq S = (FI)_{\mathfrak{F}}$ is contained in I. Assume that R is a proper subgroup of S and let N/R be a minimal normal subgroup of FI/R contained in S/R. Then N belongs to $\mathfrak{F}$ and so N is contained in R. This is a contradiction yields $S = R$. If FI were a proper subgroup of G, (FI, I, R) would be a 4-conjugate system. Hence (G, I, R) would be a 4-conjugate system and the result would follow. Therefore we may assume that $G = FI$. Let M be the normal subgroup of G such that $M/R = \Phi(G/R)$. Then $G/M = (IM/M)(F/M)$. Applying [3, Theorem A.10.6], $F/M = \mathrm{Soc}(G/M)$ is a self-centralising normal subgroup of G/M. In particular, F/M is a completely reducible G/M-module (possibly of mixed characteristic). By Lemma 56, there exist $v_1M, v_2M, v_3M \in F/M$ such that $\mathrm{C}_{IM/M}(v_1M) \cap \mathrm{C}_{IM/M}(v_2M) \cap \mathrm{C}_{IM/M}(v_3M) = 1$. It implies that $I \cap I^{v_1} \cap I^{v_2} \cap I^{v_3} \leq R$ by Lemma 52. Thus (G, I, R) is a 4-conjugate system.

Assume that either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups. If G is S_4-free, then $G/M = (IM/M)(F/M)$ is S_4-free. By Theorem A, there exist $v_1M, v_2M \in F/M$ such that $\mathrm{C}_{IM/M}(v_1M) \cap \mathrm{C}_{IM/M}(v_2M) = 1$.

Suppose that $\mathfrak{F}$ is composed of S_3-free groups. Denote with bars the images in $\overline{G} = G/M = \overline{IF}$. Since $\overline{I} \in \mathfrak{F}$, $\overline{I}$ is S_3-free. Let A be the Hall $2'$-subgroup of $\overline{F}$. It follows that $\overline{I}A$ is S_4-free. Let B be the Sylow 2-subgroup of $\overline{F}$. By Corollary 7, $\overline{I}B/\mathrm{C}_{\overline{I}}(B)$ is S_4-free. Then we can apply Theorem A to conclude that there exist $a_1M, a_2M \in A$ and $b_1M, b_2M \in B$ such that $\mathrm{C}_{\overline{I}}(a_1M) \cap \mathrm{C}_{\overline{I}}(a_2M) \subseteq \mathrm{C}_{\overline{I}}(A)$ and $\mathrm{C}_{\overline{I}}(b_1M) \cap \mathrm{C}_{\overline{I}}(b_2M) \subseteq \mathrm{C}_{\overline{I}}(B)$. Let $v_i = a_i + b_i$, $i = 1, 2$. Then $\mathrm{C}_{\overline{I}}(v_1M) \cap \mathrm{C}_{\overline{I}}(v_2M) \subseteq \mathrm{C}_{\overline{I}}(A) \cap \mathrm{C}_{\overline{I}}(B) = \mathrm{C}_{\overline{I}}(\overline{F}) = 1$.

In both cases, we conclude that (G, I, R) is a 3-conjugate system by Lemma 52. This completes the proof of the theorem. □

Corollary 67 ([17])**.** *If I is a nilpotent injector of a soluble group G, then $(G, I, \mathrm{F}(G))$ is a 3-conjugate system.*

Bibliography

[1] A. Ballester-Bolinches and L. M. Ezquerro. *Classes of Finite Groups*, volume 584 of *Mathematics and Its Applications*. Springer, Dordrecht, 2006.

[2] J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen. On a conjecture of Gluck. *Math. Z.*, 279:1067–1080, 2015.

[3] K. Doerk and T. Hawkes. *Finite soluble groups*, volume 4 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1992.

[4] S. Dolfi. Intersections of odd order Hall subgroups. *Bull. London Math. Soc.*, 37(1):61–66, 2005.

[5] S. Dolfi. Large orbits in coprime actions of solvable groups. *Trans. Amer. Math. Soc.*, 360(1):135–152, 2008.

[6] S. Dolfi and E. Jabara. Large character degrees of solvable groups with abelian Sylow 2-subgroups. *J. Algebra*, 313(2):687–694, 2007.

[7] A. Espuelas. Large character degrees of groups of odd order. *Illinois J. Math.*, 35(3):499–505, 1991.

[8] W. Gaschütz. Über die Φ-Untergruppe endlicher Gruppen. *Math. Z.*, 58:160–170, 1953.

[9] D. Gluck. The largest irreducible character degree of a finite group. *Canad. J. Math.*, 37(3):442–415, 1985.

[10] T. Hawkes. Analogues of Prefrattini subgroups. In *Proc. Internat. Conf. Theory of Groups (Canberra, 1965)*, pages 145–150. Gordon and Breach, New York, 1967.

[11] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[12] B. Huppert. *Character theory of finite groups*, volume 25 of *De Gruyter Expositions in Mathematics.* Walter de Gruyter, 1998.

[13] B. Huppert and N. Blackburn. *Finite Groups II*, volume 242 of *Grund. Math. Wiss.* Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[14] S. F. Kamornikov. One characterization of the Gaschütz subgroup of a finite soluble group. *Fundam. Prikl. Mat.*, 20:65–75, 2015. Russian.

[15] S.F. Kamornikov. Intersections of prefrattini subgroups in finite soluble groups. *Int. J. Group Theory*, 6(2):1–5, 2017.

[16] T. M. Keller and Y. Yang. Abelian quotients and orbit sizes of solvable linear groups. *Israel J. Math.*, 211:23–44, 2016.

[17] A. Mann. The intersection of Sylow subgroups. *Proc. Amer. Math. Soc.*, 53(2):262–264, 1975. Addendum: ibidem Vol. 62, No. 1, p. 188 (1977).

[18] O. Manz and T.R. Wolf. *Representations of Solvable Groups*, volume 185 of *Mathematical Society Lecture Note Series.* Cambridge University Press, London, 1993.

[19] V. D. Mazurov and E. I. Khukhro, editors. *Unsolved problems in Group Theory: The Kourovka Notebook.* Russian Academy of Sciences, Siberian Branch, Institute of Mathematics, Novosibirsk, Russia, 17 edition, 2010.

[20] A. Moretó and T. R. Wolf. Orbit sizes, character degrees and Sylow subgroups. *Adv. Math.*, 184(1):18–36, 2004. Erratum: *ibid.*, no. 2, page 409.

[21] D. S. Passman. Groups with normal, solvable Hall p'-subgroups. *Trans. Amer. Math. Soc.*, 123(1):99–111, 1966.

[22] P. Schmid. *The solution of the $k(GV)$-problem*, volume 4 of *ICP Advanced Texts in Mathematics.* Imperial College Press, London, 2007.

[23] Á. Seress. The minimal base size of primitive solvable permutation groups. *J. London Math. Soc.*, 53(2):243–255, 2006.

[24] T. R. Wolf. Large orbits of supersolvable linear groups. *J. Algebra*, 215:235–247, 1999.

[25] T. R. Wolf. Regular orbits of induced modules of finite groups. In C. Y. Ho, P. Sin, P. H. Tiep, and A. Turull, editors, *Finite Groups 2003. Proceedings of the Gainesville conference on finite groups. March 6–12,*

2003. In honour of John Thompson to this 70th birthday, pages 389–399, Berlin, 2004. Walter de Gruyter.

[26] Y. Yang. Orbits of the actions of finite solvable groups. *J. Algebra*, 321:2012–2021, 2009.

[27] Y. Yang. Regular orbits of finite primitive solvable groups. *J. Algebra*, 323:2735–2755, 2010.

[28] Y. Yang. Large character degrees of solvable $3'$-groups. *Proc. Amer. Math. Soc.*, 139(9):3171–3173, 2011.

[29] Y. Yang. Large orbits of subgroups of solvable linear groups. *Israel J. Math.*, 199(1):345–362, 2014.

[30] V. I. Zenkov. Intersections of nilpotent subgroups in finite groups. *Fundam. Prikl. Mat.*, 2(1):1–92, 1996.

[31] J. Zhang. Finite groups with few regular orbits on the power set. *Algebra Colloq.*, 4(4):471–480, 1997.