

# A NEW PROOF OF THE EXISTENCE OF HIERARCHIES OF POISSON-NIJENHUIS STRUCTURES

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ABSTRACT. Given a Poisson-Nijenhuis manifold, a two-parameter family of Poisson-Nijenhuis structures can be defined. As a consequence we obtain a new and noninductive proof of the existence of hierarchies of Poisson-Nijenhuis structures.

## 1. INTRODUCTION

One of the main characteristics of the theory of Poisson-Nijenhuis structures is the possibility of constructing from a Poisson-Nijenhuis structure, a hierarchy of new ones. The different proofs of the existence of such a hierarchy that can be found in the literature all used proof by induction ([3],[9]).

The aim of this note is to obtain, from a single Poisson-Nijenhuis structure,  $(P, N)$ , a two-parameter family of Poisson-Nijenhuis structures  $(P_t, N_s)$ ,  $t, s \in \mathbb{R}$ . Such a family provides a noninductive way of proving the existence of the well known hierarchy of associated Poisson-Nijenhuis structures. In fact, we can say that the two-parameter family is a kind of integration of the hierarchy: all the structures of the hierarchy can be obtained as successive partial derivatives evaluated at  $(0, 0)$  of the two-parameter structures  $(P_t, N_s)$ .

In Section 2.2, we prove a consequence of this approach related to generating operators of Gerstenhaber brackets. Let  $(A, \llbracket \cdot, \cdot \rrbracket, \wedge)$  be a Gerstenhaber algebra. If  $\delta$  is a generating operator of a Gerstenhaber bracket and  $N$  is a degree 0 Nijenhuis endomorphism of the associative algebra  $(A, \wedge)$ , then  $[\delta, i_N]$  is a generating operator of the deformation by  $N$  of the Gerstenhaber bracket, where  $i_N$  denotes the extension, as a degree 0 derivation, of  $N$  to the whole algebra. This is Theorem 1. In Corollary 5 we apply the results of Section 2 to the different Gerstenhaber brackets which can be associated to a Poisson-Nijenhuis structure.

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## 2. NIJENHUIS TENSORS AND THEIR INTEGRAL FLOW

Let  $(E, [\cdot, \cdot])$  be a graded Lie algebra. In the applications that we shall give here,  $(E, [\cdot, \cdot])$  will be the vector space of smooth vector fields over a manifold  $M$ ,  $\mathfrak{X}(M)$ , together with the usual Lie bracket of vector fields, or the vector space of differential 1-forms,  $\Omega^1(M)$ , with the Lie bracket of 1-forms associated to a Poisson structure, or that of differential forms,  $\Omega(M)$ , together with the Koszul-Schouten bracket of differential forms on a Poisson manifold.

We can define the Frölicher-Nijenhuis bracket,  $[\cdot, \cdot]_{FN}$  of two degree 0 endomorphisms of  $E$ ,  $N, L$ , as

$$(2.1) \quad [N, L]_{FN}(X, Y) := [NX, LY] + [LX, NY] - N([LX, Y] + [X, LY]) - L([NX, Y] + [X, NY]) + (NL + LN)[X, Y],$$

for all  $X, Y \in E$ .

The Frölicher-Nijenhuis bracket of  $N$  with itself is called the Nijenhuis torsion of  $N$ , and  $N$  is said to be Nijenhuis if its Nijenhuis torsion vanishes.

**Definition 1.** *Let  $(E, [\cdot, \cdot]_\nu)$  be a graded Lie algebra. Given a degree 0 endomorphism,  $N$ , we can define the deformation of the Lie bracket,  $[\cdot, \cdot]_\nu$ , by means of  $N$  as*

$$[X, Y]_{N,\nu} = [NX, Y]_\nu + [X, NY]_\nu - N[X, Y]_\nu,$$

for all  $X, Y \in E$ .

If the Nijenhuis torsion of  $N$  vanishes, then  $[\cdot, \cdot]_{N,\nu}$  is a Lie bracket.

Occasionally, the deformed bracket will be simply denoted by  $[\cdot, \cdot]_N$ .

Let  $\Phi_t$  be a one-parameter group of graded automorphisms of degree 0 of the vector space  $E$  and let  $N$  be its infinitesimal generator,  $N = \frac{d}{dt}|_{t=0} \Phi_t$ .

Then

$$\frac{d}{dt}[X, Y]_{\Phi_t} = [\Phi_t NX, Y] + [X, \Phi_t NY] - \Phi_t N[X, Y]$$

and, in particular,

$$\frac{d}{dt}|_{t=0}[X, Y]_{\Phi_t} = [X, Y]_N.$$

So, we can think of the deformed bracket as the first derivative evaluated at  $t = 0$  of the one-parameter family of deformed brackets  $[\cdot, \cdot]_{\Phi_t}$ .

**2.1. The integral flow of a (1, 1)-tensor field.** As examples and because we will need them in the applications that we shall give later, let us recall how to construct the one-parameter groups of graded endomorphisms from their infinitesimal generators in some cases.

Let  $M$  be a manifold and let  $N$  be a (1, 1) tensor on  $M$ , i.e.,  $N$  is a bundle map  $N : TM \rightarrow TM$ . We shall denote its transpose by  $N^* : T^*M \rightarrow T^*M$ .

Let us consider the (1, 1)-tensor field defined by the formal series  $\exp(tN)$ . It has been previously used, for example in [3], page 41, as a way of justifying why the deformed bracket is called a deformed bracket. Such an expression,

$\exp(tN) = \sum_{i=0}^{\infty} \frac{1}{i!} t^i N^i$ , is in principle just a formal expression. But for each point  $m \in M$ ,  $N_m$  is an endomorphism of  $T_m M$ , and then, as is well known, the series  $\exp(tN_m)$  is always convergent. Therefore,  $\Phi_t = \exp(tN)$  is a well-defined automorphism of the vector bundle  $TM$  for all  $t \in \mathbb{R}$ .

Associated to the tensor field  $N$  we can define a zero-degree derivation of the algebra of differential forms on  $M$ ,  $\Omega(M)$ . This derivation is denoted by  $i_N$ , and it is defined as the extension as a derivation of the map,  $f \mapsto i_N f := 0$  for any smooth function  $f$  and for any differential 1-form  $\alpha$ ,  $\alpha \mapsto i_N \alpha := N^* \alpha$ .

The transpose of  $\Phi_t$  is  $\Phi_t^* = \exp(tN^*)$ , and it can be extended as an automorphism of  $\Omega(M)$  which we shall also denote by  $\Phi_t^*$ , in an abuse of notation. Note that this automorphism is the identity on  $\Omega^0(M) = C^\infty(M)$ . The derivative with respect to  $t$  of the automorphism  $\Phi_t^*$  gives rise to the derivation  $i_N$ . Note that the following conditions are satisfied

$$\begin{aligned} \Phi_t^{*'} &= \Phi_t^* \circ i_N = i_N \circ \Phi_t^*, \\ \Phi_0^* &= Id, \quad \Phi_{t+s}^* = \Phi_t^* \circ \Phi_s^*. \end{aligned}$$

It is in this sense that we can think of  $\Phi_t^*$  as the integral flow of the zero-degree derivation  $i_N$ . Analogous relations are valid for  $\Phi_t$ . Now we have all the ingredients to study what happens when  $N$  is a Nijenhuis tensor.

It is well known that if  $N$  is Nijenhuis, then all the powers of  $N$  are also Nijenhuis. Moreover, the Frölicher-Nijenhuis brackets  $[N^k, N^\ell]_{FN}$  vanish for all  $k, \ell \in \mathbb{N}$ . This is the so-called hierarchy of Nijenhuis operators.

Since  $N^k$  is the  $k^{\text{th}}$ -derivative at  $t = 0$  of the one-parameter group  $\Phi_t$ , it is natural to ask whether  $\Phi_t$  is also Nijenhuis. Below, we shall give a noninductive proof of the existence of the hierarchy of Nijenhuis operators. We shall obtain a direct proof of the following statement:  $N$  is Nijenhuis if and only if  $\Phi_t$  is Nijenhuis.

**Proposition 1.** *Let  $(E, [ , ])$  be a graded Lie algebra. Let  $\Phi_t$  be a one-parameter group of graded automorphisms of degree 0 of the vector space  $E$ , and let  $N$  its infinitesimal generator. Then  $N$  is Nijenhuis if and only if  $\Phi_t$  is Nijenhuis. In other words,*

$$(2.2) \quad [\Phi_t X, \Phi_t Y] = \Phi_t [X, Y]_{\Phi_t},$$

for all  $t \in \mathbb{R}$  if and only if the torsion of  $N$  vanishes.

*Proof.* First note that the second derivative of the Nijenhuis torsion of  $\Phi_t$  evaluated at  $t = 0$  is exactly the Nijenhuis torsion of  $N$ , up to a constant factor. Indeed,

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} [\Phi_t, \Phi_t]_{FN} &= 2 \frac{d}{dt} \Big|_{t=0} [\Phi_t, N \circ \Phi_t]_{FN} \\ &= 2([N, N]_{FN} + [Id, N^2]_{FN}) = 2[N, N]_{FN}. \end{aligned}$$

Therefore, if we suppose first that  $\Phi_t$  is Nijenhuis, then  $N$  also is Nijenhuis.

Reciprocally, let us now suppose that  $N$  is Nijenhuis. The converse needs a kind of double integration process. We shall show as a first step that the Frölicher-Nijenhuis bracket of  $N$  with  $\Phi_t$  vanishes.

The first derivative of  $[N, \Phi_t]_{FN}$  is  $[N, N \circ \Phi_t]_{FN}$ . An easy computation using Eq. (2.1) shows that, for any  $X, Y \in E$ ,

$$\begin{aligned} [N, N \circ \Phi_t]_{FN}(X, Y) \\ = N \circ [N, \Phi_t]_{FN}(X, Y) + [N, N]_{FN}(\Phi_t X, Y) + [N, N]_{FN}(X, \Phi_t Y). \end{aligned}$$

Moreover,  $[N, \Phi_0]_{FN} = [N, Id]_{FN} = 0$ . Therefore, if  $N$  is Nijenhuis, we find that  $[N, \Phi_t]_{FN}$  is a solution of the first-order differential equation,  $\Psi'_t = N \circ \Psi_t$ , with the initial condition  $\Psi_0 = 0$ . But the trivial solution,  $\Psi_t = 0$ , is a solution of the same differential equation with the same initial condition, so, by uniqueness of solutions of first-order differential equations with identical initial conditions,  $[N, \Phi_t]_{FN} = 0$ .

Now, let us show that  $\Phi_t$  is Nijenhuis. We shall prove in fact that, for any  $t, s \in \mathbb{R}$ ,  $[\Phi_t, \Phi_s]_{FN} = 0$ . The first derivative of  $[\Phi_t, \Phi_s]_{FN}$  with respect to  $t$  is  $2[N \circ \Phi_t, \Phi_s]_{FN}$ . Once again, a simple computation using Eq. (2.1) shows that

$$\begin{aligned} [N \circ \Phi_t, \Phi_s]_{FN}(X, Y) &= N \circ [\Phi_t, \Phi_s]_{FN}(X, Y) + [N, \Phi_s]_{FN}(\Phi_t X, Y) \\ &\quad + [N, \Phi_s]_{FN}(X, \Phi_t Y) - \Phi_s \circ [N, \Phi_t]_{FN}(X, Y) - [\Phi_{t+s}, N]_{FN}(X, Y), \end{aligned}$$

where we have applied  $\Phi_t \circ \Phi_s = \Phi_{t+s}$ . Therefore, since  $[N, \Phi_t]_{FN} = 0$  for all  $t \in \mathbb{R}$ , we find that  $[\Phi_t, \Phi_s]_{FN}$  is a solution of  $\Psi'_t = N \circ \Psi_t$ . Moreover it satisfies the initial condition,  $[\Phi_0, \Phi_s]_{FN} = [Id, \Phi_s]_{FN} = 0$ . Using the same arguments as before, we obtain  $[\Phi_t, \Phi_s]_{FN} = 0$ . In particular,  $[\Phi_t, \Phi_t]_{FN} = 0$ .  $\square$

**Remark 1.** Note that we have shown that  $N$  is Nijenhuis if and only if

$$\Phi_{-t}[\Phi_t X, \Phi_t Y] = [X, Y]_{\Phi_t}.$$

In other words, the conjugation of the old Lie bracket by  $\Phi_t$  is precisely its deformation by  $\Phi_t$ .

**Corollary 1.** (The hierarchy of Nijenhuis tensors) If  $N$  is Nijenhuis, then  $[N^k, N^\ell]_{FN} = 0$  for any  $k, \ell \in \mathbb{N}$ .

*Proof.* Let us recall that in the proof of Proposition 1 we proved that if  $N$  is Nijenhuis then  $[\Phi_t, \Phi_s]_{FN} = 0$  for any  $t, s \in \mathbb{R}$ . Now taking successive partial derivatives with respect to  $t$  and  $s$  and evaluating them at  $t = 0$  and  $s = 0$ , we deduce that  $[N^k, N^\ell]_{FN} = 0$ .  $\square$

**2.2. Relationship with Gerstenhaber brackets.** We will show an application to the computation of a generating operator of a Gerstenhaber bracket.

If  $\mathbf{A}$  is a  $\mathbb{Z}_2$ -graded commutative, associative algebra, then an *odd Poisson bracket* or a  *$\mathbb{Z}_2$ -Gerstenhaber bracket* on  $\mathbf{A}$  is, by definition, a bilinear map,  $[[, ]]$ , from  $\mathbf{A} \times \mathbf{A}$  to  $\mathbf{A}$ , satisfying, for any  $f, g, h \in \mathbf{A}$ ,

- $\llbracket f, g \rrbracket = -(-1)^{(|f|-1)(|g|-1)} \llbracket g, f \rrbracket$ , (skew-symmetry)
- $\llbracket f, \llbracket g, h \rrbracket \rrbracket = \llbracket \llbracket f, g \rrbracket, h \rrbracket + (-1)^{(|f|-1)(|g|-1)} \llbracket g, \llbracket f, h \rrbracket \rrbracket$ , (graded Jacobi identity)
- $\llbracket f, gh \rrbracket = \llbracket f, g \rrbracket h + (-1)^{(|f|-1)|g|} g \llbracket f, h \rrbracket$ , (Leibniz rule)
- $|\llbracket f, g \rrbracket| = |f| + |g| - 1 \pmod{2}$ .

An algebra  $\mathbf{A}$  together with a bracket satisfying the above conditions is called an *odd Poisson algebra* or a  *$\mathbb{Z}_2$ -Gerstenhaber algebra*.

A linear map of odd degree,  $\Delta : \mathbf{A} \rightarrow \mathbf{A}$ , such that, for all  $a, b \in \mathbf{A}$ ,

$$(2.3) \quad \llbracket f, g \rrbracket = (-1)^{|f|} \left( \Delta(fg) - (\Delta f)g - (-1)^{|f|} f(\Delta g) \right),$$

is called a *generator* or a *generating operator* of this bracket.

**Lemma 1.** *Assume that  $(A, \llbracket \cdot, \cdot \rrbracket, \wedge)$  is a Gerstenhaber algebra. Let  $\delta$  be a generator of the bracket. Let  $\Phi$  be an automorphism of the associative algebra  $(A, \wedge)$ .*

*Then, a generating operator of the conjugation of the bracket by the automorphism  $\Phi$  is the conjugation of the generating operator,  $\Phi^{-1} \circ \delta \circ \Phi$ .*

Let us suppose now that  $\Phi_t$  is a one-parameter group of automorphisms of the associative algebra  $(A, \wedge)$ . It is easy to check that now, the infinitesimal generator,  $N = \frac{d}{dt}|_{t=0} \Phi_t$ , is a derivation of  $(A, \wedge)$ .

**Theorem 1.** *Let  $(A, \llbracket \cdot, \cdot \rrbracket, \wedge)$  be a Gerstenhaber algebra. Let  $\delta$  be a generator of the Gerstenhaber bracket  $\llbracket \cdot, \cdot \rrbracket$ . Let  $\Phi_t$  be a one-parameter group of automorphisms of the associative algebra  $(A, \wedge)$ , and let  $N$  be its infinitesimal generator. If  $N$  is Nijenhuis, then the deformed Gerstenhaber bracket,  $\llbracket \cdot, \cdot \rrbracket_N$ , is generated by  $[\delta, N]$ .*

*Proof.* By Remark 1 we know that the deformed bracket  $\llbracket \cdot, \cdot \rrbracket_{\Phi_t}$ , agrees with the conjugation by  $\Phi_t$  of the bracket  $\llbracket \cdot, \cdot \rrbracket$ . Now, by lemma 1,  $\Phi_{-t} \circ \delta \circ \Phi_t$  is a generating operator of  $\llbracket \cdot, \cdot \rrbracket_{\Phi_t}$ . By taking derivatives with respect to  $t$  at  $t = 0$  we find that  $[\delta, N]$  is a generating operator of  $\llbracket \cdot, \cdot \rrbracket_N$ .  $\square$

Let us apply this result to a particular case: the deformation by a Nijenhuis tensor of the Schouten-Nijenhuis bracket of multivector fields.

Let  $M$  be a manifold and let us consider the Gerstenhaber algebra  $(\Gamma(\Lambda TM), \llbracket \cdot, \cdot \rrbracket_{SN}, \wedge)$ , where  $\llbracket \cdot, \cdot \rrbracket_{SN}$  denotes the Schouten-Nijenhuis bracket. Let  $\delta$  be a generating operator of this bracket (see [5]).

Let  $N$  be a Nijenhuis tensor with respect to the usual Lie bracket of vector fields. We know that the deformed bracket,  $\llbracket \cdot, \cdot \rrbracket_N$ , is also a Lie bracket on  $\mathfrak{X}(M)$ . It is then possible to define a Gerstenhaber bracket on the algebra of multivector fields, extending the deformed bracket,  $\llbracket \cdot, \cdot \rrbracket_N$ , as a biderivation on the algebra of multivector fields. We shall denote the resulting Gerstenhaber bracket by  $\llbracket \cdot, \cdot \rrbracket_{SN}^N$ .

**Corollary 2.** *If  $\delta$  is a generator of the Schouten-Nijenhuis bracket, then the Gerstenhaber bracket,  $\llbracket \cdot, \cdot \rrbracket_{SN}^N$ , is generated by  $[\delta, i_N]$ .*

*Proof.* Let  $\Phi_t$  be the one-parameter group of automorphism of  $TM$  having  $N$  as infinitesimal generator. Let us, by an abuse of language, also denote by  $\Phi_t$  the extension of  $\Phi_t : TM \rightarrow TM$  as an automorphism of the whole algebra of multivector fields. The one-parameter group of automorphisms of the algebra of multivector fields  $\Phi_t$  has the derivation  $i_N$  as infinitesimal generator (See subsection 2.1).

By Theorem 1, the deformation by  $i_N$  of the Schouten-Nijenhuis bracket is a Gerstenhaber bracket with  $[\delta, i_N]$  as a generating operator. Finally, it is easy to check that the deformation by  $i_N$  of the Schouten-Nijenhuis bracket agrees with the Gerstenhaber bracket,  $[\ , \ ]_{SN}^N$ . Indeed, one can check that they agree when acting on a pair of smooth functions and/or vector fields.  $\square$

### 3. POISSON-NIJENHUIS STRUCTURES

Let us first recall the definition of Poisson-Nijenhuis structures. Among all the equivalent definitions we prefer the one from [3].

**Definition 2.** *Given a Poisson bivector,  $P$ , on a differentiable manifold,  $M$ , we can define a Lie algebra bracket on  $\Omega(M)$  by*

$$\begin{aligned} [[\alpha, \beta]]_{\nu(P)} &= \mathcal{L}_{\#_P \alpha} \beta - \mathcal{L}_{\#_P \beta} \alpha - dP(\alpha, \beta), \\ [[\alpha, f]]_{\nu(P)} &= P(\alpha, df), \\ [[f, g]]_{\nu(P)} &= 0, \end{aligned}$$

for all  $\alpha, \beta \in \Omega^1(M)$  and  $f, g \in C^\infty(M)$ , where  $\#_P \alpha$  denotes the vector field defined by  $(\#_P \alpha)(f) = P(\alpha, df)$  for any  $f \in C^\infty(M)$ , and extending the Lie algebra bracket to the whole  $\Omega(M)$  by the Leibniz rule. This bracket is known as the Koszul-Schouten bracket associated to the Poisson bivector  $P$ .

The adjoint operator  $N^*$  can be seen as a  $C^\infty(M)$ -linear map  $N^* : \Omega^1(M) \rightarrow \Omega^1(M)$  as usual.

**Definition 3.** *A Nijenhuis tensor  $N$  and a Poisson tensor  $P$  on a manifold  $M$  are called compatible, that is, the pair  $(P, N)$  is called a Poisson-Nijenhuis structure, if*

$$i) \quad N \circ \#_P = \#_P \circ N^*$$

and if

$$ii) \quad [[\alpha, \beta]]_{\nu(NP)} = [[\alpha, \beta]]_{N^* \cdot \nu(P)},$$

for all  $\alpha, \beta \in \Omega(M)$ .

Note that the compositions  $N \circ \#_P$  and  $\#_P \circ N^*$  define two not necessarily skewsymmetric  $(2, 0)$ -tensor fields, denoted by  $NP$  and  $PN^*$ , such that  $N \circ \#_P = \#_{NP}$  and  $\#_P \circ N^* = \#_{PN^*}$ . The tensor fields are then

$$(NP)(\alpha, \beta) = P(\alpha, N^* \beta), \quad (PN^*)(\alpha, \beta) = P(N^* \alpha, \beta).$$

Thus, the first condition in the definition of a Poisson-Nijenhuis manifold can be written as  $NP = PN^*$ . This condition guarantees that

$$N \circ \#_P = \#_{NP} = \#_{PN^*} = \#_P \circ N^*.$$

In addition we can deduce that  $NP = PN^*$  is skewsymmetric.

The second condition can be expressed in another way. Let us define the concomitant  $C(P, N)$  by

$$C(P, N)(\alpha, \beta) = \llbracket \alpha, \beta \rrbracket_{\nu(NP)} - \llbracket \alpha, \beta \rrbracket_{N^* \cdot \nu(P)},$$

for all  $\alpha, \beta \in \Omega^1(M)$ . Because  $N \circ \#_P = \#_P \circ N^*$ ,  $C(P, N)$  is a tensor field of type  $(3, 0)$ . Thus the second condition is just the vanishing of  $C(P, N)$ .

The concomitant  $C(P, N)$  can be also written as

$$\begin{aligned} (3.1) \quad C(P, N)(\alpha, \beta) &= \mathcal{L}_{P\alpha}(N^*\beta) - N^*\mathcal{L}_{P\alpha}\beta - \mathcal{L}_{P\beta}(N^*\alpha) + N^*\mathcal{L}_{P\beta}\alpha \\ &\quad + dNP(\alpha, \beta) - N^*dP(\alpha, \beta) \\ &= (\mathcal{L}_{P\alpha}N^*)\beta - (\mathcal{L}_{P\beta}N^*)\alpha + dNP(\alpha, \beta) - N^*dP(\alpha, \beta). \end{aligned}$$

Let us recall the definition of compatibility of Poisson structures.

**Definition 4.** *Poisson structures  $P_0$  and  $P_1$  on the same manifold  $M$  are compatible if the sum  $P_0 + P_1$ , is also a Poisson structure.*

**Remark 2.** *Let us recall that this is equivalent to*

$$\#_{P_0} \llbracket \alpha, \beta \rrbracket_{\nu(P_1)} + \#_{P_1} \llbracket \alpha, \beta \rrbracket_{\nu(P_0)} = [\#_{P_0}\alpha, \#_{P_1}\beta] + [\#_{P_1}\alpha, \#_{P_0}\beta]$$

for all  $\alpha, \beta \in \Omega^1(M)$ .

Let us recall the following

**Proposition 2.** *(see [3]) If  $(P, N)$  is a Poisson-Nijenhuis structure, then the  $(2, 0)$ -tensor,  $NP$ , defined by  $NP(\alpha, \beta) = P(\alpha, N^*\beta)$ , is a Poisson bivector that is compatible with  $P$ .*

#### 4. THE HIERARCHY OF POISSON-NIJENHUIS STRUCTURES

In this section we shall obtain a noninductive proof of the existence of the hierarchy of Poisson-Nijenhuis structures constructed from an initial one.

**Proposition 3.** *Let  $N$  be a  $(1, 1)$ -tensor field on  $M$ , and let  $\Phi_t = \exp(tN)$ . The pair  $(P, N)$  is a Poisson-Nijenhuis structure if and only if  $(P, \Phi_t)$  is a Poisson-Nijenhuis structure.*

*Proof.* We remark that we know that  $N$  is Nijenhuis if and only if  $\Phi_t$  is also Nijenhuis. So we need only prove that the compatibility conditions are satisfied.

Let us suppose first that  $(P, \Phi_t)$  is a Poisson-Nijenhuis structure. Then, by taking the first derivative at  $t = 0$  of the compatibility conditions between  $P$  and  $\Phi_t$ , we obtain those for  $P$  and  $N$ .

Reciprocally, let us suppose that  $(P, N)$  is a Poisson-Nijenhuis structure. We shall consider the tensor field  $\Phi_{-t}P\Phi_t^*$  defined by

$$\Phi_{-t}P\Phi_t^*(\alpha, \beta) := P(\Phi_{-t}^*\alpha, \Phi_t^*\beta).$$

The first derivative of  $\Phi_{-t}P\Phi_t^*$  is  $\Phi_{-t}(PN^* - NP)\Phi_t^* = 0$ . Therefore,  $\Phi_{-t}P\Phi_t^*$  is constant. But since its value at  $t = 0$  is  $P$ ,  $\Phi_{-t}P\Phi_t^* = P$ , or, equivalently,  $\Phi_tP = P\Phi_t^*$ .

Let us now study the second compatibility condition between  $P$  and  $\Phi_t$ .

The first derivative of  $C(P, \Phi_t)(\alpha, \beta)$  is  $C(P, \Phi_t \circ N)(\alpha, \beta)$ , and a simple computation using Eq. (3.1) shows that it is equal to

$$(4.1) \quad \Phi_t(C(P, N)(\alpha, \beta)) + C(P, \Phi_t)(\alpha, N^*\beta) + (\mathcal{L}_{PN\beta}\Phi_t^*)\alpha - (\mathcal{L}_{P\beta}\Phi_t^*)N^*\alpha.$$

The first term vanishes because we suppose that  $(P, N)$  is a Poisson-Nijenhuis structure. We shall see that the two last terms also vanish. Let us denote them by

$$H_t(\alpha, \beta) := (\mathcal{L}_{PN\beta}\Phi_t^*)\alpha - (\mathcal{L}_{P\beta}\Phi_t^*)N^*\alpha.$$

The first derivative of  $H_t$  with respect to  $t$  is

$$\begin{aligned} & (\mathcal{L}_{PN\beta}\Phi_t^*N^*)\alpha - (\mathcal{L}_{P\beta}\Phi_t^*N^*)N^*\alpha \\ &= H_t(N^*\alpha, \beta) + \Phi_t^*((\mathcal{L}_{PN\beta}N^*)\alpha - (\mathcal{L}_{P\beta}N^*)N^*\alpha). \end{aligned}$$

Now, let us recall that the following identity (See formula 7.13 [6])

$$\mathcal{L}_{NX}(N^*) = \mathcal{L}_X(N^*)N^*$$

is a condition equivalent to the vanishing of the Nijenhuis torsion of  $N$ . It is now clear that the two last terms vanish.

Then, we find that  $H_t$  satisfies the equation  $H_t'(\alpha, \beta) = H_t(N^*\alpha, \beta)$  with initial condition  $H_0(\alpha, \beta) = (\mathcal{L}_{PN\beta}Id)\alpha - (\mathcal{L}_{P\beta}Id)\alpha = 0$ . Therefore,  $H_t = 0$  for all  $t \in \mathbb{R}$ .

Let us return to  $C(P, \Phi_t)$ . By Eq. (4.1),  $C(P, \Phi_t)$  satisfies  $C(P, \Phi_t)'(\alpha, \beta) = C(P, \Phi_t)(\alpha, N^*\beta)$ , and the initial condition,  $C(P, \Phi_0) = C(P, Id) = 0$ . Therefore,  $C(P, \Phi_t) = 0$  for all  $t \in \mathbb{R}$ .  $\square$

**Corollary 3.** *If  $(P, N)$  is a Poisson-Nijenhuis structure, then, for any  $t, s \in \mathbb{R}$ ,*

- (1)  $(\Phi_sP, \Phi_t)$  is a Poisson-Nijenhuis structure, and
- (2)  $\Phi_sP$  and  $\Phi_tP$  are compatible Poisson bivectors.

*Proof.* The first statement is just a consequence of the following relation, which can be obtained from equation (3.1),

$$C(P, \Phi_t)'(\alpha, \beta) = C(P, N)(\Phi_t^*\alpha, \beta) + C(P, N)(\alpha, \Phi_t^*\beta) - C(\Phi_tP, N)(\alpha, \beta).$$

Then, if  $(P, N)$  is a Poisson-Nijenhuis structure, both  $C(P, N)$  and  $C(P, \Phi_t)$  vanish, and  $C(\Phi_tP, N) = 0$ . This means that  $(\Phi_tP, N)$  is a Poisson-Nijenhuis structure (Note that by Propositions 2 and 3,  $\Phi_tP$  is a Poisson bivector). By applying Proposition 3 to  $(\Phi_tP, N)$  we find that  $(\Phi_tP, \Phi_s)$  is a Poisson-Nijenhuis structure for all  $t, s \in \mathbb{R}$ .

The second statement is a consequence of the first.  $\square$



**Remark 3.** In [10] it is observed that if  $(P, N)$  is a Poisson-Nijenhuis structure, then not only the elements of the hierarchy are again Poisson-Nijenhuis structures, but so is any structure of the kind  $((\sum_{i=0}^{\infty} a_i N^i) \circ P, \sum_{j=0}^{\infty} b_j N^j)$ , where the series involved are convergent power series with constant coefficients. So, the first statement of the corollary is a particular case of this observation. The novelty here is that we have obtained it before proving the existence of the hierarchy, whereas in [10], it is a consequence of the existence of such a hierarchy. In fact, Corollary 3 is a condensed way of writing the hierarchy, as the next corollary will show.

**Remark 4.** What we have found is a kind of surface in the set of all Poisson-Nijenhuis structures. If we write  $x(t, s) = (\Phi_t P, \Phi_s)$  then

$$\begin{aligned} x(0, 0) &= (P, Id), & x_t(0, 0) &= (NP, Id), & x_s(0, 0) &= (P, N), \\ x_{tt}(0, 0) &= (N^2 P, Id), & x_{ts}(0, 0) &= (NP, N), & x_{ss}(0, 0) &= (P, N^2). \end{aligned}$$

**Corollary 4.** (The hierarchy of Poisson-Nijenhuis structures.) If  $(P, N)$  is a Poisson-Nijenhuis structure, then, for any  $k, \ell \in \mathbb{N}$ ,

- (1)  $(N^k P, N^\ell)$  is a Poisson-Nijenhuis structure, and
- (2)  $N^k P$  and  $N^\ell P$  are compatible Poisson bivectors.

*Proof.* As before, we need only take partial derivatives with respect to  $t$  and  $s$  of the compatibility conditions between  $\Phi_s P$  and  $\Phi_t$  and evaluate them at  $t = 0$  and  $s = 0$ .  $\square$

**Corollary 5.** If  $(P, N)$  is a Poisson-Nijenhuis structure, then

$$\llbracket \alpha, \beta \rrbracket_{\nu(\Phi_t P)} = \llbracket \alpha, \beta \rrbracket_{\Phi_t^* \cdot \nu(P)} = \Phi_{-t}^* \llbracket \Phi_t^* \alpha, \Phi_t^* \beta \rrbracket_{\nu(P)}.$$

*Proof.* It is a consequence of the fact that  $(P, \Phi_t)$  is a Poisson-Nijenhuis structure and of the fact that, for any Poisson-Nijenhuis structure,  $(P, N)$ ,  $N^*$  is Nijenhuis with respect to the bracket  $\llbracket \cdot, \cdot \rrbracket_{\nu(P)}$  (see [3] lemma 4.2), and then of Formula 2.2.  $\square$

This last result has an interpretation in terms of generating operators. Let us recall that a generating operator of the Koszul-Schouten bracket,  $\llbracket \cdot, \cdot \rrbracket_{\nu(P)}$ , is  $\mathcal{L}_P = [i_P, d]$ , where  $d$  denotes the exterior derivative (see [5]). Now, as a consequence of Lemma 1, we can state the following

**Corollary 6.** If  $(P, N)$  is a Poisson-Nijenhuis structure then,

$$\mathcal{L}_{\Phi_t P} = \Phi_{-t}^* \circ \mathcal{L}_P \circ \Phi_t^*.$$

*Proof.* Two generating operators of the same Gerstenhaber bracket differ by a derivation of degree  $-1$ . In this case, it is easy to check that  $\mathcal{L}_{\Phi_t P} - \Phi_{-t}^* \circ \mathcal{L}_P \circ \Phi_t^*$  is the null derivation.  $\square$

**Remark 5.** We have worked here with the definition of Poisson-Nijenhuis manifolds given in [3], but the same results can be obtained for similar, but not fully equivalent, definitions, for example, the one given in [9]. The key point is to observe that the statement in Proposition 1 is also valid in the

following form: Let  $F \subset E$  be a vector subspace, then  $[N, N]_{FN}$  vanishes on  $F$  if and only if  $[\Phi_t, \Phi_t]_{FN}$  vanishes on  $F$ ,  $\Phi_t$  being a one-parameter group of graded automorphisms and  $N$  is its infinitesimal generator.

**Remark 6.** Recently, the notion of Jacobi-Nijenhuis structure has also been studied, see for example [7], [8] or [2]. It is not difficult to see that a proof of the existence of hierarchies of Jacobi-Nijenhuis manifolds can also be obtained using similar arguments to those advanced in this note.

## 5. AN EXAMPLE

The initial Poisson bivector, taken from [1] for  $n = 5$ , is

$$P = \begin{pmatrix} 0 & 0 & -a_1 & a_1 & 0 \\ 0 & 0 & 0 & -a_2 & a_2 \\ a_1 & 0 & 0 & 0 & 0 \\ -a_1 & a_2 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 & 0 \end{pmatrix},$$

where  $\{a_1, a_2, b_1, b_2, b_3\}$  are the coordinates for  $\mathbb{R}^5$ .

Let us consider the tensor fields given by

$$N = \begin{pmatrix} f & 0 & 0 & 0 & 0 \\ 0 & L(a_2, b_3) & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & f & f - L(a_2, b_3) \\ 0 & 0 & 0 & 0 & L(a_2, b_3) \end{pmatrix},$$

where  $f$  is a constant and  $L$  a function depending on the variables  $a_2, b_3$ . An easy computation shows that  $(P, N)$  is a Poisson-Nijenhuis structure.

The integral flow of  $N$  is

$$\Phi_t = \begin{pmatrix} e^{ft} & 0 & 0 & 0 & 0 \\ 0 & e^{L(a_2, b_3)t} & 0 & 0 & 0 \\ 0 & 0 & e^{ft} & 0 & 0 \\ 0 & 0 & 0 & e^{ft} & e^{ft} - e^{L(a_2, b_3)t} \\ 0 & 0 & 0 & 0 & e^{L(a_2, b_3)t} \end{pmatrix}.$$

And the 1-parameter family,  $\Phi_s P$  of compatible Poisson bivectors is

$$\begin{pmatrix} 0 & 0 & -a_1 e^{fs} & a_1 e^{fs} & 0 \\ 0 & 0 & 0 & -a_2 e^{L(a_2, b_3)s} & a_2 e^{L(a_2, b_3)s} \\ a_1 e^{fs} & 0 & 0 & 0 & 0 \\ -a_1 e^{fs} & a_2 e^{L(a_2, b_3)s} & 0 & 0 & 0 \\ 0 & -a_2 e^{L(a_2, b_3)s} & 0 & 0 & 0 \end{pmatrix}.$$

By our results, each pair  $(\Phi_s P, \Phi_t)$  is a Poisson-Nijenhuis structure for all  $t, s \in \mathbb{R}$ .

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