Spaces of Dirichlet Series

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I declare that this dissertation titled *Spaces of Dirichlet series* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a degree of Doctor in Mathematics at Valencia University.
- Where I have consulted the published works of others, this is always clearly attributed.
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Valencia, September 1th, 2019

Jaime Castillo Medina

We declare that this dissertation presented by **Jaime Castillo Medina** titled *Spaces of Dirichlet series* has been done under our supervision at Valencia University. We also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of Doctor in Mathematics.

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Domingo García **Manuel Maestre** **Pablo Sevilla**

A mi madre. 10287 y 10287 más.

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Resum

Aquest treball està dedicat a l'estudi de les series de Dirichlet múltiples i es focalitza en tres aspectes principals: convergència, espais de series de Dirichlet múltiples i els operadors de composició de dits espais.

Al Capítol 1 es tracta la convergència de series múltiples. Primer es mostra com la definició que s'obté de l'extensió natural al cas múltiple de la definició de convergència d'una successió no implica les propietats naturals que s'esperarien, com l'acotació de la successió o el càlcul del límit doble a través de límits iterats. Aquesta situació es repeteix òbviament al cas de les series múltiples, pel que s'ha de introduir una definició alternativa per la convergència de series múltiples.

Al 1917 Hardy va caracteritzar en [19] el conjunt de multiplicadors se les series convergents a l'espai de les successions com el conjunt de successions de *variació acotada*, les successions per les quals la serie que es defineix per les distàncies entre termes consecutius es finita. Quan intentà estendre aquesta caracterització al cas de les series dobles, Hardy va entendre la necessitat d'una definició de convergència estructuralment més forta per tal de replicar el seu resultat original en el cas doble. Aquest és cert en un context més general, doncs la definició usual de convergència per series dobles posa l'accent en una convergència combinada en ambdós índexs, però permet massa llibertat en les series que s'obtenen quan es fixa qualsevol dels índexs a un valor concret. Més endavant, Móricz va caracteritzar aquesta definició de convergència regular a les series

múltiples a través d'una condició més tècnica, que fa el paper de condició de tipus Cauchy per a la definició de convergència regular.

La caracterització que Hardy donà en [19] per als multiplicadors de series convergents es reprodueix ací, és a dir, s'inclouen les proves de l'equivalència entre els conceptes de factor de convergència i de successió de variació acotada. També es reprodueix el treball de Móricz, provant l'equivalència entre el concepte que va definir com convergència en sentit restringit i la convergència regular que va definir Hardy, on la condició més tècnica de convergència en sentit restringit es sovint més útil que la definició original de Hardy. A més, en la pròpia reproducció de la caracterització del factors de convergència en el cas doble s'utilitza la condició de Móricz, que de fet estava implícita en el treball de Hardy. Aquesta caracterització és especialment interessant en la seua aplicació al treball amb series de Dirichlet, perquè d'estendre aquests resultats per a les successions que es defineixen com *uniformement de variació acotada* s'obté una manera sistemàtica de treballar la convergència regular en series de Dirichlet múltiples.

Al Capítol 2 el treball es focalitza en els conceptes i resultats fonamentals de series de Dirichlet ordinàries d'una variable complexa. El primer assumpte que es tracta es el de la convergència de dites series, i no només la convergència, també la convergència absoluta i la convergència uniforme. Es podria dir que la teoria de series de potències està inclosa en la teoria de series de Dirichlet, ja que triant la freqüència $\lambda_n = n$ i el canvi de variables $z = e^{-s}$ s'obté que

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} a_n z^n.$$

En aquest cas particular es bàsic que si una serie de potències és convergent en un disc, aleshores convergeix aboluta i uniformement en qualsevol disc estrictament més menut. Utilitzant el canvi de variables invers s'arriba a que si una serie de Dirichlet amb freqüència $\lambda_n = n$ convergeix en el semiplà que queda a la dreta d'una certa abscissa, aleshores també convergeix absoluta i uniformement en qualsevol semipla estrictament a la dreta de dita abscissa. No obstant això, aquest no és el cas en general amb les series de Dirichlet ordinàries, per a les quals la freqüència és $\lambda_n = \log n$. En el cas ordinari es defineixen diferents abscisses, una per cada tipus de convergència: l'abscisa de convergència, l'abscissa de convergència absoluta i l'abscissa de convergència uniforme. L'amplitud de la banda vertical que separa cadascuna d'aquestes abscisses és un dels problemes clàssics en la teoria de series de Dirichlet, que va ser estudiat per Bohr i no va ser resolt finalment fins 1931 en [6] per Bohnenblust i Hille. De fet, hi existeix un altra abscissa més, l'abscissa d'acotació, que marca el màxim semiplà en el qual una serie de Dirichlet és convergent i acotada, i és d'especial rellevància per un resultat de Bohr publicat en 1913 en [7]. Aquest resultat, que es coneix com Teorema de Bohr, estableix que una serie de Dirichlet que té una extensió holomorfa acotada en un determinat semiplà, de fet coonvergeix uniformement a tal extensió en tot semiplà estrictament inclós en l'anterior, o el que és el mateix, que l'abscissa d'acotació i la de convergència uniforme coincideixen.

Una vegada s'han establert els diferents tipus de convergència per les series de Dirichlet, es recorden les diferents fórmules que poden utilitzar-se per calcular les abscisses corresponents, fórmules que són àmpliament conegudes i que van ser publicades juntes en el llibre fonamental sobre series de Dirichlet escrit per Hardy i Riesz, [20]. Aquestes fórmules en general només funcionen quan l'abscissa que es vol calcular es positiva, encara que també en el llibre de Hardy i Riesz es poden trobar modificacions de les fórmules que funcionen en els casos en que l'abscissa es negativa. Hi han, no obstant això, fórmules que no són tan conegudes i per les quals no és rellevant el signe de l'abscissa. La primera d'elles la va publicar Knopp en [23] per a l'abscissa de convergència de series de

Dirichlet ordinàries. Després, Kojima va estendre aquesta fórmula a les series de Dirichlet generals en [24], un article que va servir de preparació per un dels treballs més rellevants i extensos de Kojima, però que també seria l'últim que publicaria. En aquest article, publicat un any abans de que morira, Kojima confecciona una fórmula per al cas doble basada en la que publicà en [24], que li permetrà tractar el problema de la convergència regular de series de Dirichlet dobles. En la Secció 2.2 es reprodueix el treball de Kojima per a series de Dirichlet generals d'una variable complexa, que servirà de preparació per a l'estudi del seu treball en series de Dirichlet dobles.

La Secció 2.3 està dedicada a l'estudi de $\mathcal{H}_\infty(\mathbb{C}_+)$, l'espai de series de Drichlet que són convergents en $\mathbb{C}_+$, és a dir, que convergeixen en el semiplà de nombres complexos que tenen part real positiva; i que a més defineixen funcions acotades en eixe semiplà. Aquest espai va ser introduït per primera vegada probablement en [21], on es va obtindre com l'espai de multiplicadors de l'espai de series de Dirichlet els coeficients de les quals estan en ℓ^2, denotat per $\mathcal{H}^2$. No obstant això, $\mathcal{H}_\infty(\mathbb{C}_+)$ es veritablement rellevant pel Teorema de Bohr, ja que l'acotació en $\mathbb{C}_+$ implica la convergència uniforme de les sumes parcials a la funció límit en qualsevol semiplà estrictament inclòs en $\mathbb{C}_+$. L'última part d'aquest capítol es dedica a provar una versió quantitativa del Teorema de Bohr, necessària per provar que $\mathcal{H}_\infty(\mathbb{C}_+)$ és un àlgebra de Banach, i que serà fonamental per construir la inducció en el cas múltiple. A més, en [21] es prova que $\mathcal{H}_\infty(\mathbb{C}_+)$ és isomètricament isomorf a $H_\infty(B_{c_0})$, l'espai de funcions holomorfes i acotades definides en la bola unitat de c_0, i el Teorema de Bohr i les seues extensions als casos doble i múltiple seran fonamentals per estendre aquesta isometria a dits casos.

El Capítol 3 es dedica a l'estudi de la convergència de series de Dirichlet múltiples, on el primer objectiu és obtindre un teorema que imite l'estructura del teoremes originals donats per Jensen en [22] i Cahen en [8], però sobre convergència regular. La primera idea clau és que la successió harmònica múltiple és uniformement de variació acotada en qualsevol producte de regions angulars del tipus de les que apareixen en el resultat original de Cahen. Aquest fet es pot combinar amb l'extensió a la convergència regular de series múltiples del treball de Hardy en [19], que s'ha desenvolupat en l'última part de la Secció 1.3, per tal d'aconseguir el teorema cercat.

Tant la convergència absoluta com la uniforme també es poden estudiar per a les series de Dirichlet múltiples, i un teorema de convergència absoluta es pot donar seguint directament el d'una variable. No obstant això, el fet que diferents tipus de convergència donen lloc a diferents regions és d'especial interés en aquesta teoria, i en aquest capítol centra l'atenció en l'estudi d'eixos conjunts per a la convergència regular de series de Dirichlet dobles.

Una vegada s'ha donat un teorema de convergència regular per a series de Dirichlet dobles és clar que no es poden definir abscisses com en el cas d'una variable. Per tal de caracteritzar els conjunts de convergència regular en el cas doble, s'ha de tornar a [25] (ens centrem en el cas doble per simplicitat, però el treball en conjunts de convergència regular pt desenvolupar-se anàlogament per a sèries múltiples). En un treball extens i exhaustiu, fa ara un segle, Kojima va descriure els conjunts de convergència regular de les series de Dirichlet dobles com conjunts definits per corbes decreixents i convexes, la parametrització de les quals s'obté a través d'una fórmula inspirada pel seu treball anterior en [24]. Aquesta fórmula té l'avantatge de funcionar en el cas general, per a qualsevol parell de freqüències admissibles λ_m i μ_n, però el desavantatge de no ser especialment pràctica a l'hora de fer càlculs per obtindre de forma explícita la parametrització de la corba que és la frontera del conjunt de convergència regular d'una serie de Dirichlet doble particular. Recordant les fórmules que depenen del signe de l'abscissa que es vol calcular, s'han confeccionat extensions de la fórmula de Kojima al cas doble. D'aquestes dues noves fórmules s'ha de triar una o l'altra en cada cas en funció de si la series de Dirichlet doble convergeix regularment a l'origen, és a dir, de si la successió doble de coeficients convergeix regularment. No obstant això, encara que aquestes fórmules són més útils als casos pràctics, només s'ha pogut provar la seua validesa en el cas ordinari i en casos en què les freqüències creixen a un ritme molt paregut al que creix la freqüència $\lambda_n = \log n$. Es finalitza aquesta secció amb exemples no trivials de diferents tipus de conjunts de convergència regular per series de Dirichlet dobles ordinàries, mostrant així que els conjunts de convergència regular de les series de Dirichlet dobles componen una família més diversa, cosa que essencialment diferencia aquest cas del cas d'una variable.

Els resultats sobre convergència regular de series de Dirichlet múltiples d'aquest capítol han estat publicats a l'article:

- J. Castillo-Medina, D. García, and M. Maestre, Isometries between spaces of multiple Dirichlet series. *Journal of Mathematical Analysis and Applications*, 472 (1), 526 – 545, (2019)

Els resultats sobre fórmules de convergència d'aquest capítol poden han estat publicats a l'article:

- Convergence formulae for double Dirichlet series, to appear in *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, RACSAM.*

En el Capítol 4 es construeix de forma sistemàtica la teoria de series de Dirichlet dobles i múltiples des del punt de vista de l'anàlisi complexa i de l'anàlisi funcional, intentant imitar l'estructura de la teoria de series de Dirichlet ordinàries d'una variable complexa. Els principals objectius d'aquest capítol són dos: primer, estendre la definició d'espai de series de Dirichlet acotades al cas múltiple i provar que dit espai es de fet un àlgebra de Banach; i segon, provar que aquest àlgebra es isomètricament isomorfa al corresponent espai de funcions acotades en infinites variables. Per al primer d'aquestos objectius és convenient estudiar el cas doble abans que el múltiple, ja que la succesió de passos tècnics és més clara en el cas doble, i la intuïció que dirigeix dits passos es pot entendre més fàcilment.

En la Secció 4.1 es defineix $\mathcal{H}_\infty(\mathbb{C}_+^2)$ com l'espai de series de Dirichlet dobles i acotades que convergeixen regularment en $\mathbb{C}_+^2$, i immediatament s'obtenen les conseqüències del fet d'utilitzar la convergència regular. Si s'entenen les subseries fila i columna de la serie de Dirichlet doble com series de Dirichlet d'una varaible complexa, i notant que l'acotació de la serie de Dirichlet doble implica l'acotació d'aquestes subseries fila i columna, es pot desenvolupar un nou punt de vista per a les series de Dirichlet dobles: series de Dirichlet vectorials els coeficients de les quals són series de Dirichlet d'una variable complexa. Com els espais de les series de Dirichlet vectorials ja han sigut estudiats com espais de Banach en [11] i es té una versió vectorial del Teorema de Bohr, l'eina fonamental en el treball amb series de Dirichlet acotades, es tenen a l'abast els ingredients necessaris per aprofitar per complet els avantatges d'aquesta nova perspectiva. El primer pas serà obtindre una nova versió vectorial i *quantitativa* del Teorema de Bohr, i el següent pas serà formalitzar aquesta intuïció vectorial definint una isometria injectiva de l'espai de series de Dirichlet dobles i acotades a $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$, l'espai de series de Dirichlet vectorials els coeficients de les quals són series de Dirichlet acotades d'una varaible complexa. Aquest desenvolupament porta de forma natural a la següent qüestió: si una serie de Dirichlet doble pot ser vista com una serie de Dirichlet vectorial, siga bé per files o per columnes, són aquestes dues formes de veure-la equivalents? És a dir, es la isometria que s'ha definit també sobrejeciva? La resposta a aquesta pregunta serà afirmativa, i tant la versió escalar com la vectorial del

Teorema de Bohr apareixeran de forma més o menos inductiva en la prova d'aquest resultat (de fet fan el paper de allò que seria l'extensió del Teorema de Bohr quantitatiu al cas doble). La conseqüència final és que la isometria entre els espais $\mathcal{H}_\infty(\mathbb{C}_+^2)$ i $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ és bijectiva i que es disposa d'una versió doble del Teorema de Bohr, i per tant es té que qualsevol serie de Dirichlet doble en $\mathcal{H}_\infty(\mathbb{C}_+^2)$ es pot aproximar uniformement per les seues sumes parcials en qualsevol producte de semiplans que estiguen estrictament inclosos en $\mathbb{C}_+$.

En la segona part de la Secció 4.1 s'aconsegueix per a series dobles el primer objectiu d'aquest capítol. Els dos resultats principals que ací s'obtenen són que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ és un espai de Banach i que a més és un àlgebra. El treball per al primer d'aquests resultats ja està pràcticament finalitzat, doncs la prova de que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ es un espai de Banach seguiex el mateix esquema que la prova de que $\mathcal{H}_\infty(\mathbb{C}_+)$ és un espai de Banach, tenint en compte que en el cas doble es treballa amb convergència regular. La idea és senzilla: primer es nota que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ és un subespai de $H_\infty(\mathbb{C}_+^2)$, que és un espai de Banach, pel que tota successió de Cauchy en $\mathcal{H}_\infty(\mathbb{C}_+^2)$ convergeix a una determinada $f \in H_\infty(\mathbb{C}_+^2)$. Després es defineix formalment la serie de Dirichlet múltiple D que seria el límit hipotètic de la successió de Cauchy prenent els límits dels coeficients, i es prova que f i D coincideixen puntualment en un semiplà en el que totes les series que apareixen en la prova convergeixen absolutament. Es conclou aleshores que f es l'extensió holomorfa i acotada de D en $\mathbb{C}^2$, així que es pot utilitzar la versió doble del teorema de Bohr per estendre la convergencia de D i obtindre que en efecte pertany a $\mathcal{H}_\infty(\mathbb{C}_+^2)$.

El punt clau per obtindre la conclusió en el resultat anterior es l'extensió dels resultats previs necessaris a una versió vectorial. Aquest és exactament el mateix que s'ha de fer per provar que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ és un àlgebra: primer es prova que les series de Dirichlet vectorials també formen un àlgebra i a continuació en el cas doble es canvia a la perspectiva vectorial per utiltzar aquest mateix resultat. La clau en aquest tipus d'arguments és que la isometria entre $\mathcal{H}_\infty(\mathbb{C}_+^2)$ i $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ és bijectiva, i per tant es pot utilitzar una perspectiva o l'altra indistintament en la majoria dels casos.

En la Secció 4.2 es planteja obtindre l'extensió del resultats que s'han donat a la Secció 4.1, però ara per al cas múltiple. L'interès es centra en explicar les diferències entre el cas doble i el cas múltiple. Encara que la intuïció és la mateixa, utilitzar la perspectiva vectorial en el cas de les series de Dirichlet k-múltiples implica treballar amb series de Dirichlet $k-1$-múltiples, i aquesta és la raó per la qual molts dels resultats que s'obtenien de forma independent en el cas doble necessiten ser enunciats en una forma condensada. El Teorema 4.20 és un clar exemple d'aquesta situació, doncs conté les extensions de diversos resultats en el cas doble, però ha de ser enunciat de forma inductiva. Aquesta inducció és exactament la raó per la qual s'ha d'aconseguir una extensió precisa del Teorema de Bohr en la versió quantitativa, doncs una hipòtesi d'inducció quantitativa és necessària per poder provar aquesta extensió, al mateix temps que s'obtenen els principals ingredients per les proves de dos dels principals resultats d'aquesta secció: que $\mathcal{H}_\infty(\mathbb{C}_+^k)$ és un espai de Banach i que és isomètricament isomorf al seu equivalent vectorial, $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$.

És interessant notar que la manera en què es desenvolupa la inducció que ve de la perspectiva vectorial té un factor arbitrari, doncs es pot veure una serie de Dirichlet k-múltiple com una serie de Dirichlet vectorial els coeficients de la qual són series de Dirichlet $k-1$-múltiples, o també com una serie de Dirichlet $k-1$-múltiple i vectorial els coeficients de la qual són series de Dirichlet de una variable complexa. De fet, es pot triar qualsevol partició de les variables, doncs els espais $\mathcal{H}_\infty(\mathbb{C}_+^j, \mathcal{H}_\infty(\mathbb{C}_+^{k-j}))$ resulten ser tots isomètricament isomorfos a $\mathcal{H}_\infty(\mathbb{C}_+^k)$. Per al desenvolupament que ací es presenta s'ha triat l'opció més natural, en el sentit en que imita els esquemes de les proves del cas doble, esquemes en què normalment apareix la hipòtesi d'inducció en lloc del corresponent resultat escalar. Aquest esquema també es repeteix en la prova de que $\mathcal{H}_\infty(\mathbb{C}_+^k)$ és un àlgebra, resultat que també és cert en el cas múltiple.

En la Secció 4.3 s'estudia com estendre la isometria de [21] entre l'espai de series de Dirichlet acotades i el corresponent espai de funcions holomorfes en infinites variables, és a dir, entre $\mathcal{H}_\infty(\mathbb{C}_+^k)$ i $H_\infty(B_{c_0^k})$. No obstant això, provant aquesta isometria en realitat s'arriba a una conseqüència veritablement rellevant. Els espais $H_\infty(B_{c_0^k})$ són tots isomètricament isomorfos, independentment de la dimensió, pel que la isometria implicaria que els espais de series de Dirichlet múltiples acotades $\mathcal{H}_\infty(\mathbb{C}_+^k)$ són tots isomètricament isomorfos. Aquest és un fet sorprenent perquè en la teoria de funcions d'una variable complexa aquest fet no és té, ja que, per exemple, els espais no són $H_\infty(\mathbb{D})$ i $H_\infty(\mathbb{D}^2)$ isomètricament isomorofos.

La idea de d'aquesta isometria està basada en allò que es coneix com la transformada de Bohr, una aplicació que a priori es defineix entre les series de Dirichlet formals d'una variable complexa i les series de potències formals en infinites variables. La idea central és transformar els índexs de les series de Dirichlet, els nombres naturals, en multi-índexs de nombres enters no negatius a través de la descomposició en factors primers, separant els primers per convertir-los en un nombre infinit de variables. Més específicament, si $m \in \mathbb{N}$, siga la seua descomposició $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, i si es denota per $\mathfrak{p} = \{p_j\}_j$ la successió de nombres primers, aleshores $\mathfrak{p}^\alpha = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = m$, on $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots) \in c_{00}$ és un multi-índex, una successió finita de enters no negatius. Per tant,

$$\begin{aligned} a_m m^{-s} = a_m(\mathfrak{p}^\alpha)^{-s} &= a_m p_1^{-s\alpha_1} \cdots p_r^{-s\alpha_r} \\ &= a_m (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r} = c_\alpha z_1^{\alpha_1} \cdots z_r^{\alpha_r} = c_\alpha \mathbf{z}^\alpha, \end{aligned}$$

on $c_\alpha = a_m$, amb la identificació entre m i el multi-índex α que s'obté en la seua descomposició en factors primers, i on $z_j = p_j^{-s}$ són les noves variables complexes de la series de potències formal. Consequentment, el monomi de Dirichlet $a_m m^{-s}$ i el corresponent monomi en r variables complexes coincideixen puntualment, i el mateix es pot dir per a sumes finites. No només això, sinó que també els polinomis de Dirichlet, és a dir, les series finites, coincideixen en norma amb la corresponent series de potències formal. Aquest resultat, al que es coneix com Lema de Bohr, estabeix el següent:

$$\sup_{\mathrm{Re}\, s>0} \left| \sum_{m=1}^{M} \frac{a_m}{m^s} \right| = \sup_{\omega \in \mathbb{D}^{\pi(M)}} \left| \sum_{1 \leqslant \mathfrak{p}^\alpha \leqslant M} a_{\mathfrak{p}^\alpha} z^\alpha \right|,$$

on $\pi(M)$ és el primer més gran que és menor o igual que M. Aquest resultat és el primer pas per probar que $\mathcal{H}_\infty(\mathbb{C}_+^k)$ és isomètricament isomorf al espai corresponent de funcions holomofes i acotades en infinites variables, $H_\infty(B_{c_0^k})$.

En aquesta secció es donen dues proves diferents per a la isometria. La primera prova segueix l'esquema de [3, Theorem 2.5] i involucra els espais $\mathcal{A}(\mathbb{C}_+^k)$ de series de Dirichlet k-múltiples que són uniformement convergents en $\mathbb{C}_+^k$. Aquests espais són la clausura de l'espai que genera el conjunt de monomis de Dirichlet k-múltiples amb la norma suprem, doncs es prova que una serie de Dirichlet k-múltiple està en $\mathcal{A}(\mathbb{C}_+^k)$ si i només si és el límit uniforme d'una successió de polinomis de Dirichlet k-múltiples. Per tant, la extensió de les clausures del Lema de Bohr a les respectives clausures dona la isometria entre $\mathcal{A}(\mathbb{C}_+^k)$ i $\mathcal{A}_u(B_{c_0^k})$. Per donar el següent pas es necessita la versió per a series de Dirichlet múltiples del Teorema de Montel, un resultat molt útil en anàlisi complexa, l'anàleg del qual en la teoria de series de Dirichlet és part del treball de Bayart, [5, Lemma 18]. Una vegada s'ha obtés aquest anàleg per al cas k-múltiple, es pot utilitzar per estendre la isometria als espais $\mathcal{H}_\infty(\mathbb{C}_+^k)$ i $H_\infty(B_{c_0^k})$, finalitzant la prova.

La segona versió de la prova de la isometria segueix l'esquema de [12, Theorem 3.8], i en comptes de construir des del Lema de Bohr topològicament, es pren la idea d'aproximar per sumes finites el cas de infinites variables utilitzant el resultat que els autors de [12] anomenen el criteri de Hilbert, una eina potent en anàlisi complexa en infinites variables que ve de establir condicions suficients i necessàries per fer que una serie de potències formal definisca una funció en $H_\infty(B_{c_0})$. Aquesta prova pot parèixer menys directa, però presenta un avantatge en comparació a l'anterior: no només s'obté l'isomorfisme isomètric entre $\mathcal{H}_\infty(\mathbb{C}_+^k)$ y $H_\infty(B_{c_0^k})$, sinó que també s'obté la igualtat puntual entre l'avaluació de la series de Dirichlet múltiple i de la seua imatge. És també interesant notar que l'extensió del resultat de Bayart, que és l'anàleg al Teorema de Montel per a series de Dirichlet k-múltiples, es pot obtindre com a corol · lari, mentre que per a l'altra prova era un resultat necessari. En aquest sentit es podria dir que aquesta versió del Teorema de Montel i la isometria entre $\mathcal{H}_\infty(\mathbb{C}_+^k)$ i $H_\infty(B_{c_0^k})$ són equivalents.

Els resultats sobre series de Dirichlet múltiples d'aquest capítol han estat publicats a l'article:

- J. Castillo-Medina, D. García, and M. Maestre, Isometries between spaces of multiple Dirichlet series. *Journal of Mathematical Analysis and Applications*, 472 (1), 526 – 545, (2019)

El Capítol 5 es dedica a l'estudi dels operadors de composició de'espais de series de Dirichlet, que són operadors que actuen per composició amb una funció, amb la notació $C_\phi(f) = f \circ \phi$, on ϕ s'anomena el *símbol* de l'operador de composició D_ϕ. La Secció 5.1 es focalitza en una revisió de la caracterització dels operadors de composició de $\mathcal{H}_\infty(\mathbb{C}_+)$, la Secció 5.2 es dedica a la caracterització dels operadors de composició de

$\mathcal{H}_\infty(\mathbb{C}_+^2)$, i finalment en la Secció 5.3 s'estudien breument els operadors de superposició d'espais de series de Dirichlet d'una variable complexa.

Els operadors de composició de series de Dirichlet van ser estudiats per primera vegada per Gordon i Hedenmalm en 1999 en [17], on es caracteritzen els operadors de composició de $\mathcal{H}^2$, l'espai de series de Dirichlet els coeficients de les quals estan en ℓ^2. Aquest article es pot dividir en dues parts. La primera d'elles es bàsicament la prova de [17, Theorem A], i conté els arguments més importants que s'utilitzen en el treball en operadors de composició d'espais de series de Dirichlet en quasi qualsevol circumstància, tant en la prova de la condició suficient com en la prova de la condició necessària. La segona part d'aquest article consisteix en el treball preliminar per a [17, Theorem B] i la prova d'aquest mateix resultat, i és en aquesta part on es tracten les particularitats del cas dels operadors de composició en l'espai $\mathcal{H}^2$. Gràcies al Teorema de Bohr no hi ha necessitat d'aquesta segona part en el treball en $\mathcal{H}_\infty(\mathbb{C}_+)$, ja que els operadors de composició sempre són acotats. Per aquest motiu es focalitza l'atenció en la prova de [17, Theorem A], que es reprodueix ací, afegint atenció especial als detalls que potser s'ometeren en [17], amb la intenció de que aquest capítol siga el més complet possible. També es reprodueixen ací alguns dels resultats complementaris sobre el rang del símbol del operador de composició que es basen en unes propietats particulars sobre el creixement de la norma de una serie de Dirichlet. A continuació es dona la prova de la caracterització dels operadors de composició de $\mathcal{H}_\infty(\mathbb{C}_+)$ que havia estat donada prèviament en el treball de Bayart en [5], i la qual Queffélec i Seip refinaren encara més en [29]. Es prova a més que es pot enunciar aquesta mateixa caracterització sense la necessitar de suposar a priori que el símbol ha de ser analític. Aquesta part serà clau en la prova del resultat principal de la Secció 5.2. Per finalitzar aquesta secció s'estableix una relació entre els operadors de composició de $\mathcal{H}_\infty(\mathbb{C}_+)$ i els de $H_\infty(B_{c_0})$, seguint la isometria de la

Secció 4.3, i es remarca que aquesta relació funciona només en un sentit, ja que qualsevol símbol s'un operador de composició de $\mathcal{H}_\infty(\mathbb{C}_+)$ defineix un símbol per a un operador de composició en $H_\infty(B_{c_0})$, però es donen exemples de que el contrari és fals.

La Secció 5.2 es divideix en diferents subseccions. En la primera es prova la condició suficient per a la caracterització dels operadors de composició de $\mathcal{H}_\infty(\mathbb{C}_+^2)$, estenent els arguments de la Secció 5.1. En la segona subsecció es dona la prova de condició necessària, que requereix nous arguments inductius que estan inspirats en les proves originals de [17, Theorem A]. Finalment es dona una caracterització completa que inclou la extensió dels últims refinaments que havien sigut donats per Queffélec i Seip en el cas d'una variable, i en aquest cas també es pot prescindir de la hipòtesi d'holomorfia del símbol, obtenint una extensió completa de la caracterització final que s'havia donat en la Secció 5.1. En la tercera subsecció s'estén satisfactòriament la relació unidireccional entre els símbols dels operadors de composició de $\mathcal{H}_\infty(\mathbb{C}_+)$ i de $H_\infty(B_{c_0})$ al cas doble, donant de nou exemples de com aquesta relació només funciona en un sentit.

Finalment, en la Secció 5.3 es recorda la caracterizaió dels operadors de superposició dels espais de Hardy en el disc que es dona en [9], y s'adapta per obtindre un resultat anàleg en els espais de Hardy de series de Dirichlet d'una variable complexa, finalitzant el capítol amb les diferències entre el cas en què p és finit i el cas $p = \infty$.

Els resultastas d'aquest capítol estan recopilats en l'article:

- F. Bayart, J. Castillo-Medina, D. García, M. Maestre and P. Sevila-Peris, Composition operators on spaces of double Dirichlet series, *en procés de ser publicat.* https://arxiv.org/abs/1903.08429

Resumen

Este trabajo está dedicado al estudio de las series de Dirichlet múltiples y se centra en tres aspectos principales: convergencia, espacios de series de Dirichlet múltiples acotadas y los operadores de composición de tales espacios.

En el Capítulo 1 se trata la convergencia de series múltiples. Primero se muestra como la definición que se obtiene de la extensión natural al caso múltiple de la definición de convergencia de una sucesión no implica las propiedades naturales que uno esperaría, como la acotación de la sucesión o el cálculo del límite doble a través de límites iterados. Esta situación se repite obviamente en el caso de las series múltiples, por lo que se tiene que introducir una definición alternativa para la convergencia de series múltiples.

En 1917 Hardy caracterizó en [19] el conjunto de multiplicadores de las series convergentes en el espacio de las sucesiones como el conjunto de sucesiones de *variación acotada*, las sucesiones para las cuales la serie que viene definida por las distancias entre términos consecutivos es finita. Cuando intentó extender esta caracterización al caso de las series dobles, Hardy entendió la necesidad de una definición de convergencia estructuralmente más fuerte para replicar su resultado original en el caso doble. Esto es cierto en un contexto más general, pues la definición usual de convergencia para series dobles pone el acento en una convergencia combinada en ambos índices, pero permite demasiada libertad en las series que se obtienen cuando se fija cualquiera de los índices a algún valor

concreto. Más adelante, Móricz extendió esta definición de convergencia regular a las series múltiples a través de una condición más técnica que juega el papel de condición de tipo Cauchy para la definición de convergencia regular.

La caracterización que Hardy da en [19] para los multiplicadores de series convergentes se reproduce aquí, es decir, se incluyen las pruebas de la equivalencia entre los conceptos de factor de convergencia y de sucesión de variación acotada. También se reproduce el trabajo de Móricz, probando la equivalencia entre el concepto que acuñó como convergencia en sentido restringido y la convergencia regular definida por Hardy, siendo que la condición más técnica de convergencia en sentido restringido normalmente es más útil que la definición original de Hardy. Es más, en la propia reproducción de la caracterización de los factores de convergencia en el caso doble se utiliza la condición de Móricz, que de hecho estaba implícita en el trabajo de Hardy. Esta caracterización es especialmente interesante en su aplicación al trabajo con series de Dirichlet porque, si se va un paso más allá y se extienden estos resultados para las sucesiones que a las que se llaman *uniformemente de variación acotada*, se obtiene una manera sistemática de trabajar en convergencia regular de series de Dirichlet múltiples.

El el Capítulo 2 el trabajo se centra en los conceptos y resultados fundamentales de series de Dirichlet ordinarias de una variable compleja. El primer asunto que se trata es el de la convergencia de tales series, y no solo la convergencia sino que también la convergencia absoluta y la convergencia uniforme. Se podría decir que la teoría de series de potencias está incluida en la teoría de series de Dirichlet, ya que escogiendo la frecuencia $\lambda_n = n$ y el cambio de variables $z = e^{-s}$ se tiene que

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} a_n z^n.$$

En este caso particular es básico que si una serie de potencias es convergente en un disco, entonces converge absoluta y uniformemente en cualquier disco estrictamente más pequeño. Utilizando el cambio de variables inverso se llega a que si una serie de Dirichlet con frecuencia $\lambda_n = n$ converge en el semiplano que queda a la derecha de una cierta abscisa entonces también converge absoluta y uniformemente en cualqier semiplano estrictamente a la derecha de dicha abscisa. Sin embargo, este no es el caso en general con las series de Dirichlet ordinarias, para las cuales la frecuencia es $\lambda_n = \log n$. En el caso oridnario se definen diferentes abscisas, una para cada tipo de convergencia: la abscisa de convergencia, la abscisa de convergencia absoluta y la abscisa de convergencia uniforme. La anchura de la banda vertical que separa cada una de estas abscisas es uno de los problemas clásicos en la teoría de series de Dirichlet, que fue estudiado por Bohr y no fue resuelto finalmente hasta 1931 en [6] por Bohnenblust y Hille. De hecho, existe otra abscisa más, la abscisa de acotación, que marca el máximo semiplano en la que una serie de Dirichlet es convergente y acotada, y es de especial relevancia por un resultado de Bohr publicado en 1913 en [7]. Este resultado, que se le conoce como Teorema de Bohr, dice que una serie de Dirichlet que tiene una extensión holomorfa acotada en un cierto semiplano, de hecho converge uniformemente a tal extensión en todo semiplano estrictamente incluido en el anterior, o lo que es lo mismo, que la abscisa de acotación y la abscisa de convergencia uniforme coinciden.

Una vez se han establecido los diferentes tipos de convergencia para series de Dirichlet, se recuerdan las diferentes fórmulas que pueden utilizarse para calcular las abscisas correspondientes, fórmulas que son ampliamente conocidas y que fueron publicadas juntas en el libro fundamental sobre series de Dirichlet escrito por Hardy y Riesz, [20]. Estas fórmulas suelen funcionar exclusivamente cuando la abscisa que se quiere calcular es positiva, aunque de nuevo en el libro de Hardy y Riesz se pueden encontrar modificaciones de las fórmulas que funcionan en los casos en que la abscisa es negativa. Hay, sin embargo, fórmulas que no son tan conocidas y para las cuales no es relevante el signo de la abscisa. La primera de ellas la publicó Knopp en [23] para la abscisa de convergencia de series de Dicihlet ordinarias. Después Kojima extendió esta fórmula a las series de Dirichlet generales en [24], un artículo que serviría de preparación para uno de los trabajos más relevantes y extensos de Kojima, pero también el último que publicaría. En este artículo, publicado un año antes de su muerte, Kojima extiende su fórmula al caso doble para tratar el problema de la convergencia regular de series de Dirchlet generales dobles. En la Sección 2.2 se reproduce el trabajo de Kojima para series de Dirichlet generales de una variable compleja, preparando el terreno para el estudio de su trabajo en series de Dirichlet dobles.

La Sección 2.3 está dedicada al estudio de $\mathcal{H}_\infty(\mathbb{C}_+)$, el espacio de series de Dirichlet que son convergentes en $\mathbb{C}_+$, es decir, que convergen en el semiplano de números complejos con parte real positiva; y que además definen funciones acotadas en tal semiplano. Este espacio fue probablemente introducido por primera vez en [21], donde se obtiene como el espacio de multiplicadores del espacio de series de Dirichlet cuyos coeficientes están en ℓ^2, que se denota por $\mathcal{H}^2$. Sin embargo, $\mathcal{H}_\infty(\mathbb{C}_+)$ es verdaderamente relevante por el Teorema de Bohr, ya que la acotación en $\mathbb{C}_+$ implica la convergencia uniforme de las sumas parciales a la función límite en cualquier semiplano estrictamente incluido en $\mathbb{C}_+$. La última parte de este capítulo se dedica a probar una versión cuantitativa del Teorema de Bohr, necesaria para probar que $\mathcal{H}_\infty(\mathbb{C}_+)$ es un álgebra de Banach, y que será fundamental para construir la inducción en el caso múltiple. Además, en [21] se prueba que $\mathcal{H}_\infty(\mathbb{C}_+)$ es isométricamente isomorfo a $H_\infty(B_{c_0})$, el espacio de funciones holomorfas y acotadas definidas en la bola unidad de c_0, y el Teorema de Bohr y sus extensiones a los casos doble y múltiple serán fundamentales para extender esta isometría a dichos casos.

El Capítulo 3 está dedicado al estudio de la convergencia de series de Dirichlet múltiples, siendo el primer objetivo obtener un teorema que imite la estructura de los teoremas originales dados por Jensen en [22] y Cahen en [8], pero sobre convergencia regular. La primera idea clave es que la sucesión armónica múltiple es uniformemente de variación acotada en cualquier producto de regiones angulares del tipo de la que aparece en el resultado original de Cahen. Este hecho se puede combinar con la extensión a la convergencia regular de series múltiples del trabajo de Hardy en [19], que se ha desarrollado en la última parte de la Sección 1.3, para conseguir el teorema que se busca.

Tanto la convergencia absoluta como la uniforme también se pueden estudiar para las series de Dirichlet múltiples, y un teorema de convergencia absoluta se puede dar siguiendo directamente el de una variable. Sin embargo, el hecho de que los diferentes tipos de convergencia producen diferentes regiones es un hecho de especial interés en esta teoría, y en este capítulo se centra la atención en el estudio de tales conjuntos para la convergencia regular de las series de Dirichlet dobles.

Una vez se ha dado un teorema de convergencia regular para series de Dirichlet múltiples, es claro que no se pueden definir abscisas como en el caso de una variable. Para caracterizar los conjuntos de convergencia regular en el caso doble, se ha de acudir a [25] (nos centramos en el caso doble por simplicidad, pero el trabajo en conjuntos de convergencia regular se puede desarrollar análogamente en el caso múltiple). En un trabajo extenso y exhaustivo, hace ahora un siglo Kojima describió los conjuntos de convergencia regular de las series de Dirchlet dobles como conjuntos definidos por curvas decrecientes y convexas, la parametrización de las cuales se obtiene a través de una fórmula inspirada por su anterior trabajo en [24]. Esta fórmula tiene la ventaja de que funciona en el caso general, para cualquier para de frecuencias admisibles λ_m y μ_n, pero la desventaja de que no es especialmente práctica a la hora de hacer cálculos para obtener de forma explícita la parametrización de la curva que es la frontera del conjunto de convergencia regular de una serie de Dirichlet doble particular. Recordando las fórmulas que dependen del signo de la absicsa que se quiere calcular, se han confeccionado extensiones de la fórmula de Kojima al caso doble. De estas dos nuevas fórmulas se ha de escoger en cada caso una o la otra en función de si la serie de Dirichlet doble converge regularmente en el origen, es decir, de si la sucesión doble de los coeficientes converge regularmente. Sin embargo, aunque estas fórmulas son más útiles en los casos prácticos, solo se ha podido probar su validez en el caso ordinario y en casos en los que las frecuencias crecen a un ritmo muy parecido al que crece la frecuencia $\lambda_n = \log n$. Se finaliza esta sección con ejemplos no triviales de diferentes tipos de conjuntos de convergencia regular para series de Dirichlet dobles ordinarias, demostrando así como los conjuntos de convergencia regular de las series de Dirichlet dobles conforman una familia más diversa, que esencialmente diferencia este caso del caso de una variable.

Los resultados sobre convergencia regular de series de Dirichlet múltiples de este capítulo han sido publicados en el artículo:

- J. Castillo-Medina, D. García, and M. Maestre, Isometries between spaces of multiple Dirichlet series. *Journal of Mathematical Analysis and Applications*, 472 (1), 526 – 545, (2019)

Los resultados sobre fórmulas de convergencia de este capítulo han sido publicados en el artículo:

- Convergence formulae for double Dirichlet series, aceptado en *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, RACSAM.*

En el Capítulo 4 se construye de forma sistemática la teoría de series de Dirichlet dobles y múltiples desde el punto de vista del análisis complejo y del análisis funcional, intentando imitar la estructura de la teoria de series de Dirichlet ordinarias de una varaible compleja. Los principales objetivos de este capítulo son dos: primero, extender la definición de espacio de series de Dirichlet acotadas al caso múltiple y probar que tal espacio es de hecho un álgebra de Banach; y segundo, probar que este álgebra es isométricamente isomorfa al correspondiente espacio de funciones acotadas en infinitas variables. Para el primero de estos objetivos es conveniente estudiar el caso doble antes que el múltiple, ya que la sucesión de pasos técnicos es más clara en el caso doble y la intuición que dirige dichos pasos se puede entender más fácilmente.

En la Sección 4.1 se define $\mathcal{H}_\infty(\mathbb{C}^2_+)$, el espacio de series de Dirichlet dobles y acotadas que convergen regularmente en $\mathbb{C}^2_+$, e inmediatamente se obtienen las consecuencias del hecho de utilizar la convergencia regular. Si se entienden las subseries fila y las subseries columna de la serie de Dirichlet doble como series de Dirichlet de una variable compleja, y viendo que la acotación de la serie de Dirichlet doble implica la acotación de estas subseries fila y columna, se puede desarrollar un nuevo punto de vista para las series de Diriclet dobles: series de Dirichlet vectoriales cuyos coeficientes son series de Dirichlet de una variable compleja. Como los espacios de series de Dirichlet vectoriales ya han sido estudiados como espacios de Banach en [11] y se tiene una version vectorial de Teorema de Bohr, la herramienta fundamental en el trabajo de series de Dirichlet acotadas, se disponen de los ingredientes necesarios para aprovechar por completo las ventajas de esta nueva perspectiva. El primer paso será obtener una versión vectorial y *cuantitativa* del Teorema de Bohr, y el siguiente paso será formalizar esta intuición vectorial definiendo una isometría inyectiva del epacio de series de Dirichlet dobles y acotadas a $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$, el espacio de series de Dirichlet vectoriales cuyos coeficientes son series de Dirichlet acotadas de una variable compleja. Este desarrollo lleva de forma natural a la siguiente pregunta: si una serie de Dirichlet doble puede ser vista como una serie de Dirichlet vectorial ya sea por filas o por columnas, ¿son estas dos maneras de verla equivalentes? Es decir, ¿es la isometría que se ha definido también sobreyectiva? La respuesta a esta pregunta será afirmativa, y tanto la versión escalar como la vectorial del Teorema de Bohr aparecerán de forma más o menos inductiva en la prueba de este resultado (de hecho jugando el papel de lo que sería la extensión del Teorema de Bohr cuantitativo al caso doble). La consecuencia final es que la isometría entre los espacios $\mathcal{H}_\infty(\mathbb{C}_+^2)$ y $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ es biyectiva y que se tiene una versión doble del Teorema de Bohr, lo que asegura que cualquier serie de Dirichlet doble en $\mathcal{H}_\infty(\mathbb{C}_+^2)$ se puede aproximar uniformemente por sus sumas parciales dobles en cualquier producto de semiplanos que estén estrictamente incluidos en $\mathbb{C}_+$.

En la segunda parte de la Sección 4.1 se consigue para series dobles el primer objetivo de este capítulo. Los dos resultados principales que aquí se obtienen son que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ es un espacio de Banach y que además es un álgebra. El trabajo para el primero de estos resultados ya está prácticamente finalizado pues la prueba de que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ es un espacio de Banach sigue el mismo esquema que la prueba de que $\mathcal{H}_\infty(\mathbb{C}_+)$ es un espacio de Banach, teniendo en cuenta que en el caso doble se trabaja con convergencia regular. La idea es sencilla: primero notar que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ es un subespacio de $H_\infty(\mathbb{C}_+^2)$, que es un espacio de Banach, por lo que toda sucesión de Cauchy en $\mathcal{H}_\infty(\mathbb{C}_+^2)$ converge a una cierta $f \in H_\infty(\mathbb{C}_+^2)$. Después se define formalmente la serie de Dirichlet múltiple D que sería el hipotético límite de la sucesión de Cauchy tomando los límites de los coeficientes, y se prueba que f y D coinciden puntualmente en un semiplano en el que todas las series que aparecen en la prueba convergen absolutamente. Se concluye entonces que f es la extensión holomorfa y acotada de D en $\mathbb{C}^2$, por lo que se puede usar la versión doble del teorema de Bohr para extender la convergencia de D y obtener que en efecto pertenece a $\mathcal{H}_\infty(\mathbb{C}_+^2)$.

Nótese que el punto clave para obtener la conclusión en el resultado anterior es la extensión de los resultados previos necesarios a su versión vectorial. Esto es exactamente lo mismo que se ha de hacer para probar que $\mathcal{H}_\infty(\mathbb{C}_+^2)$ es un álgebra: primero se prueba que las series de Dirichlet vectoriales también forman un álgebra y a continuación en el caso doble se cambia a la perspectiva vectorial para utilizar ese mismo resultado. La clave en este tipo de argumentos es que la isometría entre $\mathcal{H}_\infty(\mathbb{C}_+^2)$ y $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ es biyectiva, por lo que se puede utilizar una perspectiva u otra indistintamente en la mayoría de los casos.

En la Sección 4.2 se plantea obtener la extensión de los resultados que se han dado en la Sección 4.1 pero ahora para el caso múltiple. El interés se centra en explicar las diferencias entre el caso doble y el caso múltiple. Aunque la intuición es la misma, utilizar la perspectiva vectorial en el caso de las series de Dirichlet k-múltiples implica trabajar con series de Dirichlet $k-1$ múltiples, y esta es la razón de que muchos resultados que se obtenían de forma independiente en el caso doble necesiten ser enunciados en una forma condensada. El Teorema 4.20 es un claro ejemplo de esto, pues contiene las extensiones de varios resultados en el caso doble, pero ha de ser enunciado de forma inductiva. Esta inducción es exactamente la razón por la cual se ha de conseguir una extensión precisa del Teorema de Bohr en su versión cuantitativa, pues una hipótesis de inducción cuantitativa se necesita ara poder probar esta extensión, al mismo tiempo que se obtienen los principales ingredientes para las pruebas de dos de los principales resultados de esta sección: que $\mathcal{H}_\infty(\mathbb{C}_+^k)$ es un espacio de Banach y que es isométricamente isomorfo a su equivalente vectorial, $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$.

Nótese que la manera en la que se desarrolla la inducción que viene de la perspectiva vectorial tiene un factor arbitrario, pues se puede ver una serie de Dirichlet k-múltiple como una serie de Dirichlet vectorial cuyos coeficientes son series de Dirichlet $k-1$-múltiples, o también como una serie de Dirichlet $k-1$-múltiple y vectorial cuyos coeficientes son series de Dirichlet de una variable compleja. De hecho, se puede escoger cualquier partición de las variables, pues los espacios $\mathcal{H}_\infty(\mathbb{C}_+^j, \mathcal{H}_\infty(\mathbb{C}_+^{k-j}))$ resultan ser todos isométricamente isomorfos a $\mathcal{H}_\infty(\mathbb{C}_+^k)$. Para el desarrollo que aquí se presenta se ha escogido el enfoque más natural, en el sentido en el que es el que imita los esquemas de las pruebas del caso doble, esquemas en los que normalmente se reemplaza los resultados escalares por la correspondiente hipótesis de inducción. Este esquema también se repite en la prueba de que $\mathcal{H}_\infty(\mathbb{C}_+^k)$ es un álgebra, resultado que también se obtiene en el caso múltiple.

En la Sección 4.3 se estudia cómo extender la isometría de [21] entre el espacio de series de Dirichlet acotadas y el correspondiente espacio de funciones holomorfas en infinitas variables, es decir, entre $\mathcal{H}_\infty(\mathbb{C}_+^k)$ y $H_\infty(B_{c_0^k})$. Sin embargo, probando esta isometría en realidad se llega a una consecuencia verdaderamente relevante. Los espacios $H_\infty(B_{c_0^k})$ son todos isométricamente isomorfos, independientemente de la dimensión, por lo que la isometría implicaría que los espacios de series de Diriclet múltiples acotadas $\mathcal{H}_\infty(\mathbb{C}_+^k)$ son todos isométricamente isomorfos. Esto es sorprendente porque en la teoría de funciones de variable compleja esto no ocurre, ya que, por ejemplo, los espacios $H_\infty(\mathbb{D})$ y $H_\infty(\mathbb{D}^2)$ no son isométricamente isomorfos.

La idea de esta isometría está basada en lo que se conoce como la transformada de Bohr, una aplicación que a priori se define entre las series de Dirichlet formales de una variable compleja y las series de potencias formales en infinitas variables. La idea central es transformar los índices de las series de Dirichlet, los números naturales, en multi-índices de enteros no negativos a través de la descomposición en factores primos, separando los primos para convertirlos en un número infinito de variables. Más específicamente, si $m \in \mathbb{N}$, sea su descomposición en factores primos $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, y si se denota por $\mathfrak{p} = \{p_j\}_j$ la sucesión de números primos, entonces $\mathfrak{p}^\alpha = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = m$, donde $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots) \in c_{00}$ es un multi-índice, una sucesión finita de enteros no negativos. Por tanto,

$$\begin{aligned} a_m m^{-s} = a_m(\mathfrak{p}^\alpha)^{-s} &= a_m p_1^{-s\alpha_1} \cdots p_r^{-s\alpha_r} \\ &= a_m (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r} = c_\alpha z_1^{\alpha_1} \cdots z_r^{\alpha_r} = c_\alpha \mathbf{z}^\alpha, \end{aligned}$$

donde $c_\alpha = a_m$, con la identificación entre m y el multi-índice α que se obtiene en su descomposición en factores primos, y $z_j = p_j^{-s}$ son las nuevas variables complejas en la serie de potencias formal. Consecuentemente, el monomio de Dirichlet $a_m m^{-s}$ y el correspondiente monomio en r variables complejas coinciden puntualmente, y lo mismo puede decirse para sumas finitas. No sólo eso, sino que los polinomios de Dirichlet, es decir, las series finitas, también coinciden en norma con su correspondiente serie de potencias formal. Este resultado, al que se conoce como Lema de Bohr, establece lo siguiente:

$$\sup_{\operatorname{Re} s > 0} \left| \sum_{m=1}^{M} \frac{a_m}{m^s} \right| = \sup_{\omega \in \mathbb{D}^{\pi(M)}} \left| \sum_{1 \leqslant \mathfrak{p}^\alpha \leqslant M} a_{\mathfrak{p}^\alpha} z^\alpha \right|,$$

donde $\pi(M)$ es el primo más grande que es menor o igual que M. Este resultado es el primer paso para probar que $\mathcal{H}_\infty(\mathbb{C}_+^k)$ es isométricamente isomorfo al espacio correspondiente de funcionles holomorfas y acotadas en infinitas variables, $H_\infty(B_{c_0^k})$.

En esta sección se dan dos pruebas diferentes para la isometría. La primera prueba sigue el esquema de [3, Theorem 2.5] e involucra a los espacios $\mathcal{A}(\mathbb{C}_+^k)$ de series de Dirichlet k-múltiples que son uniformemente convergentes en $\mathbb{C}_+^k$. Estos espacios son la clausura del espacio que genera el conjunto de monomios de Dirichlet k-múltiples con la norma supremo, pues se prueba que una serie de Dirichlet k-múltiple está en $\mathcal{A}(\mathbb{C}_+^k)$ si y solo si es el límite uniforme de una sucesión de polinomios de Dirichlet k-múltiples, así que la extensión del Lema de Bohr a las respectivas clausuras da a isometría entre $\mathcal{A}(\mathbb{C}_+^k)$ y $\mathcal{A}_u(B_{c_0^k})$. Para dar el siguiente paso se necesita la versión para series de Dirichlet múltiples del Teorema de Montel, un resultado muy útil en análisis complejo cuyo análogo para series de Dirichlet de una variable compleja es parte del trabajo de Bayart, [5, Lemma 18]. Una vez se ha obtenido este análogo para el caso k-múltiple, se puede usar para extender la isometría a los espacios $\mathcal{H}_\infty(\mathbb{C}_+^k)$ y $H_\infty(B_{c_0^k})$, finalizando la prueba.

La segunda versión de la prueba de la isometría sigue el esquema de [12, Theorem 3.8], y en vez de construir desde el Lema de Borh topológicamente, se toma el enfoque de aproximar por sumas finitas el caso infinito-dimensional usando el resultado que los autores de [12] llaman el criterio de Hilbert, un herramienta potente en análisis complejo en infinitas variables que proviene de establecer condiciones suficientes y necesarias para que una serie de potencias (formal) defina una función en $H_\infty(B_{c_0})$. Esta prueba puede parecer más enrevesada, pero presenta una ventaja frente a la anterior: no solo se obtiene el isomorfismo isométrico entre $\mathcal{H}_\infty(\mathbb{C}_+^k)$ y $H_\infty(B_{c_0^k})$, sino que también se obtiene la igualdad puntual entre la evaluación de la serie de Dirichlet múltiple y la de su imagen. Es también interesante notar que la extensión del resultado de Bayart, que es el análogo al Teorema de Montel para series de Dirichlet k-múltiples, se puede obtener como corolario de esta prueba, mientras que para la otra prueba era un resultado necesario. En este sentido se podría decir que esta versión del Teorema de Montel y la isometría entre $\mathcal{H}_\infty(\mathbb{C}_+^k)$ y $H_\infty(B_{c_0^k})$ son equivalentes.

Los resultados sobre series de Dirichlet múltiples de este capítulo han sido publicados en el artículo:

- J. Castillo-Medina, D. García, and M. Maestre, Isometries between spaces of multiple Dirichlet series. *Journal of Mathematical Analysis and Applications*, 472 (1), 526 – 545, (2019)

El Capítulo 5 se dedica al estudio de operadores de composición de espacios de series de Dirichlet, que son operadores que actúan por composición con una función, con la notación $C_\phi(f) = f \circ \phi$, donde ϕ se conoce como el *símbolo* del operador de composición C_ϕ. La Sección 5.1 se centra en una revisión de la caracterización de los operadores de composición de $\mathcal{H}_\infty(\mathbb{C}_+)$, la Sección 5.2 se dedica a la caracterización de los operadores de composición de $\mathcal{H}_\infty(\mathbb{C}_+^2)$, y finalmente en la Sección 5.3 se estudian brevemente los operadores de superposición de espacios de series de Dirichlet de una variable compleja.

Los operadores de composición de series de Dirichlet se estudiaron por primera vez por Gordon y Hedenmalm en 1999 en [17], donde se caracterizan los operadores composición de $\mathcal{H}^2$, el espacio de series de Dirichlet cuyos coeficientes están en ℓ^2. Este artículo se puede dividir en dos partes. La primera de ellas es básicamente la prueba de [17, Theorem A], y contiene los argumentos más importantes que se usan en el trabajo en operadores de composición de espacios de series de Dirichlet en casi cualquier circunstancia, tanto en la prueba de la condición suficiente como en la prueba de la condición necesaria. La segunda parte de este artículo consiste del trabajo preliminar para [17, Theorem B] y la prueba de este mismo resultado, y es en esta parte donde se tratan las particularidades del caso de operadores de composición en el espacio $\mathcal{H}^2$. Gracias al Teorema de Bohr no hay necesidad de esta segunda parte en el trabajo en $\mathcal{H}_\infty(\mathbb{C}_+)$, ya que los operadores de composición siempre son acotados. Por este motivo se centra la atención en la prueba de [17, Theorem A], que se reproduce aquí, añadiendo atención especial a los detalles que pudieron ser omitidos en [17], con la intención de que este capítulo sea lo más completo posible. También se reproducen aquí algunos resultados complementarios sobre el rango del símbolo del operador de composición que se basan en unas propiedades particulares acerca del crecimiento de la norma de una serie de Dirichlet. A continuación se da la prueba de la caracterización de los operadores de composición de $\mathcal{H}_\infty(\mathbb{C}_+)$ que había sido dada previamente en el trabajo de Bayart en [5], y cuya versión más refinada fue aportada por Queffélec y Seip en [29]. Se prueba además que se puede enunciar esta misma caracterización sin la necesidad de asumir a priori que el símbolo haya de ser holomorfo. Esta parte será clave en la prueba del resultado principal de la Sección 5.2. Para finalizar esta sección se establece una relación entre los operadores de composición de $\mathcal{H}_\infty(\mathbb{C}_+)$ y los de $H_\infty(B_{c_0})$, siguiendo la isometría de la Sección 4.3, y se resalta que esta relación funciona solo en un sentido, ya que cualquier símbolo de un operador de composición de $\mathcal{H}_\infty(\mathbb{C}_+)$ genera un símbolo para un operador de composición en $H_\infty(B_{c_0})$,, pero se muestra ejemplos de que el converso no es cierto.

La Sección 5.2 se divide en diferentes subsecciones. En la primera se prueba la condición suficiente para la caracterización de los operadores de composición de $\mathcal{H}_\infty(\mathbb{C}_+^2)$ extendiendo los argumentos de la Sección 5.1. En la segunda subsección se da la prueba de la condición necesaria, que requiere de nuevos argumentos inductivos que están inspirados en las pruebas originales de [17, Theorem A]. Finalmente se da una caracterización completa que incluye la extensión de los últimos refinamientos que habían sido dados por Queffélec y Seip en el caso de una variable, y en este caso también se puede prescindir de la hipótesis de holomorfía del símbolo, terminando con una extensión completa de la caracterización final obtenida en la Sección 5.1. En la tercera subsección se extiende satisfactoriamente la relación unidireccional entre los símbolos de los operadores de composición de $\mathcal{H}_\infty(\mathbb{C}_+)$ y de $H_\infty(B_{c_0})$ al caso doble, dando de nuevo ejemplos de cómo esta relación solo funciona en un sentido y falla en el otro.

Finalmente, en la Sección 5.2 se recuerda la caracterizaión de los operadores de superposición de los espacios de Hardy en el disco que se da en [9], y se adapta para obtener un resultado análogo en los espacios de Hardy de series de Dirichlet de una variable compleja, finalizando el capítulo con las diferencias entre el caso en que p es finito y el caso $p = \infty$.

Los resultados de este capítulo están recopilados en el artículo:

- F. Bayart, J. Castillo-Medina, D. García, M. Maestre and P. Sevila-Peris, Composition operators on spaces of double Dirichlet series, *en proceso de ser publicado.* https://arxiv.org/abs/1903.08429

Abstract

This work is dedicated to the study of multiple Dirichlet series and it focuses on three main aspects: convergence, spaces of bounded multiple Dirichlet series and the composition operators of such spaces.

In Chapter 1 we deal with the issue of convergence of multiple series. We first show how the definition that results from a natural to the multiple case extension of the definition of convergence of a sequence does not imply the natural properties that one should expect, such as boundedness of the sequence or computation of the double limit via iterated limits. This situation is obviously repeated in the case of multiple series, so we have to introduce an alternative definition for convergence of multiple series.

In 1917 Hardy characterized in [19] the set of multipliers of convergent series in the space of sequences as the set of sequences of *bounded variation*, the sequences for which the series given by the sums of distances between consecutive terms is finite. When trying to extend this characterization to the case of double series he realized that a structurally stronger definition was needed to replicate his original result in the double case. This turns out to be true in a more general context, as the usual definition of convergence for double series only stresses the convergence in both indexes of the series, allowing too much freedom of choice for the situation in which one of those indexes is fixed but the other one is not. That is why, in the same paper, Hardy introduced his notion of regular convergence for double series, which asks also for

convergence of the series obtained when fixing any of the indexes to any given value. Later on, in [26], Móricz extended the definition of regular convergence for multiple series of k indexes and he characterized it via a technical condition that serves as a kind of Cauchy condition for this definition of regular convergence.

We reproduce here the characterizations given in [19] for the multipliers of convergent series, that is, we include Hardy's proofs of the equivalence between the concepts of convergence factor and sequence of bonded variation. We also reproduce Móricz's work, proving the equivalence between what he called convergence in a restricted sense and regular convergence, since this technical condition will be often much more useful than Hardy's original definition. Actually, when reproducing Hardy's characterization for the convergence factors in the double case, we will use Móricz's condition, as it was implicit in Hardy's work. This characterization is interesting for us because if we go one step further and we extend this result for what we call multiple sequences of uniform bounded variation, then we get a systematic way of working in regular convergence of multiple Dirichlet series.

In Chapter 2 we focus on the fundamentals on ordinary Dirichlet series of one complex variable. The first issue we deal with is convergence of such series, and not only convergence but also absolute and uniform convergence. One could say that the theory of general Dirichlet series includes the theory of power series, since choosing the frequency $\lambda_n = n$ and taking a simple change of variable by $z = e^{-s}$ one gets

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} a_n z^n.$$

In this particular case we know that if a power series is convergent in a disk, then it is absolutely and uniformly convergent in any disk strictly smaller. By using the inverse change of variables, this would imply that if the general Dirichlet series with frequency $\lambda_n = n$ were convergent in the half-plane to the right of an abscissa, then it would converge absolutely and uniformly in every halfplane that sits strictly to the right of the original one. However, this is not the case in ordinary Dirichlet series, where the frequency is $\lambda_n = \log n$. In the ordinary case one can define different abscissae which give the maximum half-plane for every each type of convergence: the abscissae of convergence, of absolute convergence and of uniform convergence. The width of one of the particular strips separating the abscissae was one of the classical problems in the theory of ordinary Dirichlet series that was tackled by Bohr and was not finally solved until 1931 in [6] by Bohnenblust and Hille. Actually, there is another abscissa, the abscissa of boundedness, which gives you the maximum half-plane in which a Dirichlet series is convergent and bounded, and its relevance comes through a fundamental result given by Bohr in [7] in 1913. This result, sometimes referred to as Bohr's Theorem, states that a Dirichlet series that has a bounded holomorphic extension defined in a certain half-plane will converge uniformly to that extension in every half-plane strictly smaller, or equivalently, that the abscissa of boundedness coincides with the abscissa of uniform convergence.

Once we have established the different kinds of convergence for Dirichlet series, we recall different formulae that can be used for computing the corresponding abscissae, which are widely known and were published together in 1915 in the fundamental book on Dirichlet series by Hardy and Riesz [20]. This formulae are generally given for positive abscissae, although there again in Hardy and Riesz's book one can find some modifications of the formulae that actually work for negative abscissae. There are, however, other formulae which are not generally known and which do not care for the sign of the abscissae. The first one was given by Knopp in [23] for the abscissa of convergence of ordinary Dirichlet series. This formula was then extended to general Dirichlet series by Kojima in [24], a paper which would be a preparatory work for one of Kojima's most remarkable and extensive pieces of work, but also his last. Is in this paper, published just one year before his death, he extended his formula to the double case to tackle the problem of regular convergence of general double Dirichlet series. In Section 2.2 we reproduce Kojima's work for general Dirichlet series of one complex variable, preparing the grounds for the study of his work in double Dirichlet series.

Section 2.3 is dedicated to the study of $\mathcal{H}_\infty(\mathbb{C}_+)$, the space of Dirichlet series that are convergent in $\mathbb{C}_+$, that is, that are convergent the half-plane of complex numbers with positive real part, and also define bounded functions there. This space was probably introduced for the first time in [21], where it was obtained as the space of multipliers of the space of Dirichlet series whose coefficients are in ℓ^2, denoted by $\mathcal{H}^2$. However, $\mathcal{H}_\infty(\mathbb{C}_+)$ is truly relevant thanks to Bohr's Theorem, since the boundedness in $\mathbb{C}_+$ implies the uniform convergence of the partial sums to the limit function in every strictly smaller half-plane $\mathbb{C}_\delta$, for $\delta > 0$. We dedicate the last part of this chapter to proving a quantitative version of Bohr's Theorem which is needed to prove that $\mathcal{H}_\infty(\mathbb{C}_+)$ is a Banach algebra and which will be fundamental to build the induction in the multiple case. Moreover, in [21] it was shown that $\mathcal{H}_\infty(\mathbb{C}_+)$ is isometrically isomorphic to $H_\infty(B_{c_0})$, the space of bounded holomorphic functions defined on the open unit ball of c_0, and Bohr's Theorem and its extension to the double and multiple cases will be fundamental to extend this isometry to such cases.

Chapter 3 is dedicated to the study of convergence of multiple Dirichlet series, being the first goal to give a theorem of convergence that replicates the structure of the original theorems given by Jensen in [22] and Cahen in [8], but now using regular convergence. The first key idea is that the harmonic multiple sequence is of uniform bounded variation in any product of angular regions such as the one appearing in Cahen's original result. This fact can be used altogether with the extension of Hardy's work of [19] to regularly convergent multiple series, which had been done in the last part of Section 1.3.

Absolute and uniform convergence can also be studied for multiple Dirichlet series, and a theorem of absolute convergence for multiple Dirichlet series can be given directly form the one variable case. However, as we pointed out before, the fact that different kinds of convergence produce different sets is a remarkable point of interest of this theory, and we turn our attention to the study of such sets for regular convergence of double Dirichlet series.

Once a theorem of regular convergence for multiple Dirichlet series has been given, it is clear that one cannot define abscissae as in the one variable case. To characterize the sets of regular convergence the double case, we have to refer to [25] (we focus on the double case for simplicity, but the work in sets of regular convergence can be developed in the multiple case analogously). In an extensive and exhaustive work, a century ago Kojima described the sets of regular convergence of double Dirichlet series as defined by particular decreasing convex curves, the parametrization of which can be obtained through a formula that is inspired by his previous work in [24]. This formula has the advantage that it works in the general case, for any pair of admissible frequencies λ_m and μ_n, but the disadvantage of not being very useful to perform actual calculations when trying to explicitly obtain the parametrization of the curve that is the boundary of the set of regular convergence of a particular double Dirichlet series. Recalling the formulae that depended on the sign of the abscissa we build extensions of Kojima's formula to the double case. From these two new formulae, one or the other has to be chosen depending on whether the double Dirichlet series converges regularly at the origin or, equivalently, on whether the double sequence of coefficients converges regularly. However, although easier to use for practical computations, these formulae can only be used in the ordinary case and for some particular frequencies in the general case that grow at a very similar path to the path at which the frequency $\lambda_n = \log n$ grows. We finish this section by giving some non-trival examples of different kinds of sets of regular convergence for ordinary double Dirichlet series, demonstrating how the sets of regular convergence for double Dirichlet series form a wider and more varied family than in the one variable case.

The results about regular convergence of multiple Dirichlet series from this chapter have been published in the paper:

- J. Castillo-Medina, D. García, and M. Maestre, Isometries between spaces of multiple Dirichlet series. *Journal of Mathematical Analysis and Applications*, 472 (1), 526 – 545, (2019)

The results about formulae of regular convergence from this chapter have been published in the paper:

- Convergence formulae for double Dirichlet series, to appear in *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, RACSAM.*

In Chapter 4 we systematically build the theory of double and multiple Dirichlet series from the complex and functional analysis point of view, trying to replicate the structure of the theory of ordinary Dirichlet series of one variable. The main aims of this chapter are two: first, to extend the definition of the space of bounded Dirichlet series to the multiple case and prove that such space is indeed a Banach algebra; and second, to show that this algebra is isometrically isomorphic to the corresponding space of bounded functions in (countable) infinitely many variables. For the first aim it is convenient to study the double case prior to the multiple one, as the succesion of technical steps is clearer in the double case and the intuition that leads those steps can be understood more easily.

In Section 4.1 we define $\mathcal{H}_\infty(\mathbb{C}_+^2)$, the space of bounded double Dirichlet series that are regularly convergent in $\mathbb{C}_+^2$, and we immediately obtain some consequences from using the regular convergence. Understanding the row and column subseries of a double Dirichlet series as Dirichlet series of one complex variable and how the boundedness of the double series implies the boundedness of these row and column subseries, we can develop a new point of view for double Dirichlet series: vector-valued Dirichlet series whose coefficients are Dirichlet series of one complex variable. Since spaces of vector-valued Dirichlet series had been studied before as Banach spaces in [11], and there is a vector-valued version of our fundamental tool, Bohr's Theorem, we have all the ingredients necessary to take full advantage of this new perspective. The first step is to obtain a vector-valued *quantitative* version of Bohr's Theorem, and the next step will be to formalize this vector-valued intuition defining an injective isometry form the space of bounded double Dirichlet series into $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$, the space of bounded vector-valued Dirichet series whose coefficients are bounded Dirichlet series. This development naturally brings out the next question: if a double Dirichlet series can be seen as a vector-valued Dirichlet series either by rows or by columns, are these two ways equivalent? This is to say, is the isometry we just defined also onto? The answer to this question will be positive, and both the scalar and the vector-valued version of Bohr's Theorem will appear in a sort of inductive way in the proof of this result, actually playing the role of what would be an extension of the quantitative Bohr's Theorem to the double case. The final consequence is then that there is a bijective isometry between the spaces $\mathcal{H}_\infty(\mathbb{C}_+^2)$ and $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ and that we have a double version of Bohr's Theorem, assuring that any double Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}_+^2)$ can be uniformly approximated by its partial double sums in every product of half-planes strictly smaller that $\mathbb{C}_+$.

In the second part of Section 4.1 the first aim of this chapter is finally met for double Dirichlet series. The two main results that are obtained here are that $\mathcal{H}_\infty(\mathbb{C}_+^2)$ is a Banach space, and that is also an algebra. The road to the first of those results has already been paved as the proof follows the same scheme as the proof of the analogous result for the one variable case, where now we have to take into account that we are dealing with regular convergence. The idea is simple: first note that $\mathcal{H}_\infty(\mathbb{C}_+^2)$ is a subspace of $H_\infty(\mathbb{C}_+^2)$, which is a complete space, so every Cauchy sequence in $\mathcal{H}_\infty(\mathbb{C}_+^2)$ must have a limit $f \in H_\infty(\mathbb{C}_+^2)$. Then, define formally the series D that would be the limit of the Cauchy sequence in $\mathcal{H}_\infty(\mathbb{C}_+^2)$ by taking the limits of the coefficients, and prove that both f and D coincide point-wise in a half-plane where all the series involved would converge absolutely. Then f would be the holomorphic bounded extension of D to $\mathbb{C}_+^2$, so the double version of Bohr's Theorem can be used to extend the convergence of D and get that it is indeed in $\mathcal{H}_\infty(\mathbb{C}_+^2)$.

Note how the key point in getting to the conclusion was to get to the extension of the necessary results using the vector-valued version of such results. This is also exactly what we need to do in order to prove that $\mathcal{H}_\infty(\mathbb{C}_+^2)$ is an algebra: first we prove that bounded vector-valued Dirichet series also form an algebra and then, in the double case, we switch to the vector-valued perspective to use said result. The key in these kind of arguments is that the isometry between $\mathcal{H}_\infty(\mathbb{C}_+^2)$ and $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ is bijective, so in practice we can go back and forth between the two perspectives.

In Section 4.2 we set ourselves to extend the results obtained in Section 4.1 for k-multiple Dirichlet series. We focus on explaining the main differences between the double case and the more general multiple case. Although the intuition is the same, going through the vector-valued perspective in the k-multiple case now implies working with $k-1$-multiple

Dirichlet series, and this is the reason why most of the results that can be obtained independently from one another in the double case need to be stated together in a condensed form. Theorem 4.20 is a clear example of this, as it contains the extensions of several different results of the double case, but needs to be stated inductively. This induction is exactly the reason why we need to build a precise extension of the quantitative version of Bohr's Theorem, since a quantitative induction hypothesis is needed in order to prove this extension at the same time that we obtain the main ingredients for the proofs of two of the main results of this section: that $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is a Banach space and that it is isometrically isomorphic to its vector-valued counterpart $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$.

Note that the way in which we develop the induction through the vector-valued perspective has an arbitrary factor, as we can choose to see k-multiple Dirichlet series as vector valued Dirichlet series whose coefficients are $k-1$-multiple Dirichlet series, or also as $k-1$-multiple vector-valued Dirichlet series whose coefficients are Dirichlet series of one complex variable. Actually, we could have chosen any partition of the variables, because the spaces $\mathcal{H}_\infty(\mathbb{C}_+^j, \mathcal{H}_\infty(\mathbb{C}_+^{k-j}))$ are actually all isometrically isomorphic to $\mathcal{H}_\infty(\mathbb{C}_+^k)$. The choice has been made for the more natural approach in the sense that it replicates the schemes of the proofs of the double case, usually replacing the scalar results by the corresponding inductive hypothesis. This is scheme is also repeated in the proof of $\mathcal{H}_\infty(\mathbb{C}_+^k)$ being an algebra, which is also the case for k-multiple Dirichlet series.

In Section 4.3 we study how to extend the isometry from [21] between bounded Dirichlet series and the corresponding space of bounded holomorphic functions in infinitely many variables, that is, between $\mathcal{H}_\infty(\mathbb{C}_+^k)$ and $H_\infty(B_{c_0^k})$. However, by proving this isometry one actually gets a very remarkable consequence. Since the spaces $H_\infty(B_{c_0^k})$ are all isometrically isomorphic independently from the value of k, this implies that the spaces of bounded multiple Dirichlet series $\mathcal{H}_\infty(\mathbb{C}_+^k)$ are all isometrically isomorphic, independently from the number of variables. This is striking because in the theory of funcions of a complex variable this is not the case, as, for example, $H_\infty(\mathbb{D})$ and $H_\infty(\mathbb{D}^2)$ are not isometrically isomorphic.

The idea of this isometry is based on what is called the Bohr Transform, a map a priori defined between formal Dirichlet series in one complex variable and formal power series in countably infinitely many variables. The central idea is to transform the indexes in which a formal Dirichlet series runs as a sum, the natural numbers, into multi-indexes of non-negative integers via the prime number decomposition, so we separate the prime numbers and we make them into a set of countably infinetly many varaibles. Let us see this in more detail. If $m \in \mathbb{N}$, let us take the decomposition in prime factors of m, $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, and let us denote by $\mathfrak{p} = \{p_j\}_j$ the sequence of prime numbers, so we write $\mathfrak{p}^\alpha = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = m$, where $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots) \in c_{00}$ is a multi-index, an eventually zero sequence of non-negative integers. Then,

$$\begin{aligned} a_m m^{-s} = a_m(\mathfrak{p}^\alpha)^{-s} &= a_m p_1^{-s\alpha_1} \cdots p_r^{-s\alpha_r} \\ &= a_m (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r} = c_\alpha z_1^{\alpha_1} \cdots z_r^{\alpha_r} = c_\alpha \mathbf{z}^\alpha, \end{aligned}$$

where $c_\alpha = a_m$ with the identification between m and the multi-index α obtained in its decomposition as primer factors, and $z_j = p_j^{-s}$ being the new complex variables in the formal power series. Therefore the Dirichlet monomial $a_m m^{-s}$ and the corresponding monomial in r complex variables coincide point-wise, and the same can be said for finite sums. Not only that, but finite sums coincide not only point-wise but also in norm. This very relevant result, usually referred to as Bohr's Lemma, states that

$$\sup_{\operatorname{Re} s > 0} \left| \sum_{m=1}^{M} \frac{a_m}{m^s} \right| = \sup_{\omega \in \mathbb{D}^{\pi(M)}} \left| \sum_{1 \leqslant \mathfrak{p}^\alpha \leqslant M} a_{\mathfrak{p}^\alpha} z^\alpha \right|,$$

where $\pi(M)$ is the greatest prime number less than or equal to M. This is the first step into proving that $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is isometrically isomorphic to its corresponding space of bounded holomorphic functions in countably infinitely many variables, $H_\infty(B_{c_0^k})$.

In this section we give two different proofs for the isometry we announced. The first one follows the scheme in [3, Theorem 2.5] and involves the spaces $\mathcal{A}(\mathbb{C}_+^k)$ of k-multiple Dirichlet series that are uniformly convergent in $\mathbb{C}_+^k$. This spaces are actually the closure of the k-multiple Dirichlet monomials with the supremum norm, as we prove that a k-multiple Dirichlet series is in $\mathcal{A}(\mathbb{C}_+^k)$ if and only if it is the uniform limit of a sequence of k-multiple Dirichlet polynomials, so the extension of Bohr's Lemma to the respective closures gives the isometry between $\mathcal{A}(\mathbb{C}_+^k)$ and $\mathcal{A}_u(B_{c_0^k})$. To go one step further we need the k-multiple version of Montel's Theorem, a very useful result in the theory of complex analysis whose analogous form for Dirichlet series of one complex varaible was given by Bayart in [5, Lemma 18]. Once we have built this extension of Bayart's result we can use it to extend the isometry to $\mathcal{H}_\infty(\mathbb{C}_+^k)$ and $H_\infty(B_{c_0^k})$, finishing the proof.

The second version of the proof of the isometry follows the scheme from [12, Theorem 3.8], and instead of building topologically from Bohr's Lemma, it takes the finite sums approach and approximates the infinitely many variables case using what the authors in [12] call Hilbert's criterion, a powerful tool in complex analysis in infinitely many variables that comes from establishing sufficient and necessary conditions for a (formal) power series to define a function in $H_\infty(B_{c_0})$. This proof may seem more involved, but it presents an advantage over the other one: we get that not only the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ and $H_\infty(B_{c_0^k})$ are isometrically isomorphic, but also that a k-multiple Dirichlet series and its image coincide point-wise, which was not obtained directly in the other proof. It is also interesting to note that our extension of Bayart's result, which works as a kind of

Montel's Theorem for k-multiple Dirichlet series, can now be obtained as a corollary from this proof, while for the other one it was a necessary previous result. In this sense one could say that this version of Montel's Theorem and the isometry between $\mathcal{H}_\infty(\mathbb{C}_+^k)$ and $H_\infty(B_{c_0^k})$ are equivalent results.

The results about multiple Dirichlet series from this chapter have been published in the paper:

- J. Castillo-Medina, D. García, and M. Maestre, Isometries between spaces of multiple Dirichlet series. *Journal of Mathematical Analysis and Applications*, 472 (1), 526 – 545, (2019)

Chapter 5 is dedicated to the study of composition operators of spaces of Dirichlet series, which are operators that act via composition with a function, using the notation $C_\phi(f) = f \circ \phi$, where ϕ is called the *symbol* of the composition operator C_phi. Section 5.1 centers around revisiting the characterization of the composition operators for $\mathcal{H}_\infty(\mathbb{C}_+)$ and Section 5.2 is dedicated to the characterization of the composition operators for $\mathcal{H}_\infty(\mathbb{C}_+^2)$, while in Section 5.3 we briefly study superposition operators of spaces of Dirichlet series of one complex variable.

Composition operators of Dirichlet series where studied first by Gordon and Hedenmalm in 1999 in [17], where they characterized the composition operators for $\mathcal{H}^2$, the space of Dirichlet series whose coefficients are in ℓ^2. This paper can be divided in two parts. The first one is basically the proof of [17, Theorem A] and contains the most important arguments that are used when working with composition operators of spaces of Dirichlet series in almost every circumstance, both in the proof of the sufficient condition and in the proof of the necessary one. The second one is the preparatory work for [17, Theorem B] and the proof of this result, which deals with the particularities of studying composition operators in $\mathcal{H}^2$. Thanks to Bohr's Theorem there is no need for this second part when working in $\mathcal{H}_\infty(\mathbb{C}_+)$, since a composition operator always is bounded. That is why we focus on the proof of [17, Theorem A], which we reproduce here, adding special attention to details that might have been omitted in [17] with the intention of having this chapter be self-contained. We also reproduce some other complementary results about the range of the symbol of the composition operator that rely on some particular properties on the growth of the norm of a Dirichlet series. Then we give a proof of the characterization of the composition operators on $\mathcal{H}_\infty(\mathbb{C}_+)$ that had been previously given by Bayart in [5], and the refined version that was given by Queffélec and Seip in [29]. We also prove that we can state the same characterization without assuming a priori that the symbol has to be holomorphic. This will be a key point for the proof of the main result in Section 5.2. We end this section by establishing a relationship between composition operators of $\mathcal{H}_\infty(\mathbb{C}_+)$ and composition operators in $H_\infty(B_{c_0})$, following the isometry from Section 4.3, and we note that this relationship is only one-sided, as every symbol of a composition operator in $\mathcal{H}_\infty(\mathbb{C}_+)$ generates a symbol for a composition operator in $H_\infty(B_{c_0})$, but we show examples and how the converse is not true.

We split Section 5.2 in different subsections. In the first one we prove the sufficient condition for the characterization of the composition operators of $\mathcal{H}_\infty(\mathbb{C}_+^2)$ extending the arguments of Section 5.1. In the second subsection we give the proof of the necessary condition, which requires of new inductive arguments that are also inspired in the original proof of [17, Theorem A]. We finally give a complete characterization that includes the extension of the latest refinements that had been given by Queffélec and Seip for the one variable case, and we are also able to drop the hypothsis of analiticity of the symbol, ending with a complete extension of the final characterization we had obtained in Section 5.1. In the third subsection we successfully extend the one sided relationship between the symbols of composition operators of $\mathcal{H}_\infty(\mathbb{C}_+)$

and of $H_\infty(B_{c_0})$ to the double case, giving again examples on how this relationship only works in one direction.

Finally, in Section 5.3 we recall the characterization of the superposition operators of the Hardy spaces of the disk that was given in [9] and we adapt it to get an analogous result in Hardy spaces of Dirichlet series of one complex varaible, and we finish the chapter by noting the striking differences between the case in which p is finite and the case of $p = \infty$.

The results in this chapter are collected in the paper:

- F. Bayart, J. Castillo-Medina, D. García, M. Maestre and P. Sevila-Peris, Composition operators on spaces of double Dirichlet series, *in the process of being published.* https://arxiv.org/abs/1903.08429

Contents

Chapter 1

Preliminaries in multiple series

In the study of multiple Dirichlet series, convergence is undoubtedly our first concern, so we dedicate this chapter to establishing a strong framework for convergence of multiple series. In the first section we will see how the ordinary definition of convergence presents some problems, so we will need to find a stronger definition of convergence for multiple series, which we will borrow form Hardy's work. In [19] Hardy tackles the problem of finding the set of multipliers of convergent series in the space of sequences, and he defines *regular convergence* of double series to solve this same problem for regularly convergent double series. We will reproduce his research and extend it to the k-multiple case, as it will truly simplify our work towards giving a convergence theorem for multiple Dirichlet series.

1.1 Convergence of multiple series.

We will give here some basic definitions about multiple sequences and multiple series, and some examples to illustrate the main difficulties that arise when dealing with the multiple case. We illustrate these problems by giving examples which, for the sake of clarity, will be of double sequences and series.

Definition 1.1. A k-multiple complex sequence

$$a = \{a_{m_1,\dots,m_k}\}_{m_1,\dots,m_k=1}^{\infty} \subset \mathbb{C}$$

is a map $a : \mathbb{N}^k \to \mathbb{C}$, and we write $a_{m_1,\dots,m_k} = a(m_1, \dots, m_k)$. We say that a k-multiple complex sequence is convergent to L if for every $\varepsilon > 0$ there exists $(M_1, \dots, M_k) \in \mathbb{N}^k$ such that

$$|a_{m_1,\dots,m_k} - L| < \varepsilon, \qquad \text{whenever } m_j \geqslant M_j \text{ for every } 1 \leqslant j \leqslant k.$$

Remark 1.2. Note that this definition of convergence can be stated in terms of the convergence of a net over $\mathbb{N}^k$ with the partial order defined as

$$(p_1, \dots, p_k) \leqslant (q_1, \dots, q_k) \leftrightarrow p_j \geqslant q_j \text{ for every } 1 \leqslant j \leqslant k.$$

Moreover, in the previous definition it is equivalent to say that there exists $M_0 \in \mathbb{N}$ such that

$$|a_{m_1,\dots,m_k} - L| < \varepsilon, \qquad \text{whenever } m_j \geqslant M_0 \text{ for every } 1 \leqslant j \leqslant k,$$

so we will use both of these conditions indistinctly.

Definition 1.3. Let $a = \{a_{m_1,\dots,m_k}\}_{m_1,\dots,m_k=1}^{\infty}$ be a k-multiple complex sequence. A k-multiple complex series is the k-multiple complex sequence of its partial sums

$$s_{p_1,\dots,p_k} = \sum_{m_1=1}^{p_1} \cdots \sum_{m_k=1}^{p_k} a_{m_1,\dots,m_k},$$

and we represent it by $\sum_{m_1,\ldots,m_k=1}^{\infty} a_{m_1,\ldots,m_k}$.
We say that the k-multiple complex series converges to a sum S if the k-multiple complex sequence of its partial sums converges to S, that is, if for every $\varepsilon > 0$ there exists $(q_1, \ldots, q_k) \in \mathbb{N}^k$ such that

$$|s_{p_1,\ldots,p_k} - S| < \varepsilon \quad \text{whenever } p_j \geqslant q_j \text{ for every } 1 \leqslant j \leqslant k.$$

From now own, we will just say “a k-multiple sequence” when referring to a k-multiple complex sequence, and we will just say “a k-multiple series” when referring to a k-multiple complex series.
Sometimes we will say that a k-multiple series is a k-dimensional series, emphasizing the fact that it is summed over k different indexes. With this idea, given a k-multiple series, one can fix one or more indexes at a certain value or values and one gets a multiple series in the remaining indexes, which we will call a *j-dimensional subseries* of the k-multiple series.

Definition 1.4. Given a k-multiple series $\sum_{m_1,\ldots,m_k=1}^{\infty} a_{m_1,\ldots,m_k}$ and $0 \leqslant j \leqslant k$, we say that a j-dimensional subseries is a series in just j indexes $m_{i_1}, \ldots, m_{i_j}$, where the other indexes $m_{l_1}, \ldots, m_{l_{k-j}}$ have been fixed. We represent a j-dimensional subseries by

$$\sum_{m_{i_1},\ldots,m_{i_j}=1}^{\infty} a_{m_1,\ldots,m_k}.$$

Notice that, in the previous definition, the improper cases $j = k$ and $j = 0$ are included, which refer respectively to the k-multiple series itself and the k-multiple sequence of terms of the series.
It is a natural question to ask if the convergence of k-multiple series implies the convergence of its j-dimensional subseries and, in that case, if the series can be summed iteratedly. This question can be better illustrated in the double case.

Definition 1.5. Let $\sum_{m,n=1}^{\infty} a_{m,n}$ be a double series, that is, a 2-multiple series. We call the 1-dimensional subseries of this double series the *row subseires* and the *column subseries*, depending on whether we are fixing one index or the other. More clearly,

- $\sum_{n=1}^{\infty} a_{m,n}$ is a row subseries.
- $\sum_{m=1}^{\infty} a_{m,n}$ is a column subseries.

The names *row subseries* and *column subseries* come from the interpretation of a double sequence as an infinite matrix, where the terms are organized as follows:

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

In this sense, in a *row subseries* we are summing all the terms in one row, because the index that indicates the row is fixed. Analogously in the *column subseries* we sum all the terms in a column. Now we are ready to define properly the iterated sums of a double series.

Definition 1.6. Let $\sum_{m,n=1}^{\infty} a_{m,n}$ be a double series. We call

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n} \right), \quad \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{m,n} \right)$$

the iterated series of the double series.

More generally, we can give the definition for k-multiple series.

Definition 1.7. Let $\sum_{m_1,\ldots,m_k=1}^{\infty} a_{m_1,\ldots,m_k}$ be a multiple series. Given an order for the indices $m_1,\ldots,m_k$ as $m_{i_1},\ldots,m_{i_k}$, the corresponding iterated sum is

$$\sum_{m_{i_1}}^{\infty}\sum_{m_{i_2}}^{\infty}\cdots\sum_{m_{i_k}}^{\infty} a_{m_1,\ldots,m_k}.$$

In the case of double series, the question stated above takes the following form: Does the convergence of a double series imply the existence of the iterated series? The answer to this question is negative, as it is shown in the following example from [27].

Example 1.8. Let $\{a_{m,n}\}$ be the double sequence defined by the matrix

1	-1	1	-1	1	-1	$\cdots$
-1	1	-1	1	-1	1	$\cdots$
1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\cdots$
-1	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\cdots$
1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\cdots$
-1	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Computing the partial sums of the double series we have that:

$$s_{p,q} = \begin{cases} \frac{2}{2+\min\{p+1,q+1\}}, & \text{if p and q are odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the double series is convergent, but the row subseries and the column subseries are all of them alternated series and therefore not convergent, so the iterated sum is not well defined in this case.

The other implication does not hold either: the existence of both iterated sums does not guarantee the convergence of the double series. Let us see it with the following example, again from [27].

Example 1.9. Let $\{b_{m,n}\}$ be the double sequence defined by the matrix

1	-1	0	0	0	0	$\cdots$
-1	1	0	0	0	0	$\cdots$
0	0	1	-1	0	0	$\cdots$
0	0	-1	1	0	0	$\cdots$
0	0	0	0	1	-1	$\cdots$
0	0	0	0	-1	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

It is obvious that every row subseries and every column subseries is convergent to 0, so the iterated sums both exists and are 0. However,

$$s_{p,\,p} = \begin{cases} 1, & \text{if p is odd,} \\ 0, & \text{if p is even.} \end{cases}$$

so the associated double series cannot be convergent.

This is not the only difficulty we encounter when dealing with convergence of double (and consequently, multiple) series. It is elementary that, in the one dimensional case, a convergent sequence is bounded, and for a convergent series the sequence of partial sums and the sequence of terms are bounded. However, this does not occur with double series, and the next example illustrates it.

Example 1.10. Let $\{c_{m,n}\}$ be the double sequence defined by the matrix

0	0	1	2	3	4	$\cdots$
0	0	-1	-2	-3	-4	$\cdots$
1	-1	0	0	0	0	$\cdots$
2	-2	0	0	0	0	$\cdots$
3	-3	0	0	0	0	$\cdots$
4	-4	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The double series associated with this double sequence is convergent as $s_{m,n} = 0$ when $m, n > 2$. However, it is obvious that the sequences $\{a_{m,n}\}$ ans $\{s_{m,n}\}$ are not bounded when $m \leqslant 2$ or $n \leqslant 2$.

1.2 Regular convergence of double and multiple series.

It has been shown that a new definition for convergence of multiple series is needed if we want to be able to give a theorem of convergence of multiple Dirichlet series that follows the same arguments of the theorem of convergence of Dirichlet series. This definition will be that of *regular convergence*, that we present now for multiple series.

Definition 1.11. We say that a k-multiple series is *regularly convergent* if it is convergent and all of its j-dimensional subseries are convergent for $1 \leqslant j \leqslant k$.

Note that this implies the convergence of the k-multiple series itself. A straightforward consequence of this definition is that it can be stated inductively: a k-multiple series is *regularly convergent* if and only if every $k-1$-dimensional subseries is *regularly convergent*.
This definition was given by Hardy on [19] for double series. Later on,

Móricz gave his definition for *convergence in a restricted sense* on [26], which he proved to be equivalent to Hardy's definition. We extend both the definition and the proof to multiple series, and from now on we will consider the condition of *convergence in a restricted sense* as a sort of Cauchy condition for *regular convergence*.

Definition 1.12. We say that a k-multiple series *converges in a restricted sense* if for every $\varepsilon > 0$ there exists $M_0 > 0$ such that

$$\left|\sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\dots,m_k}\right| < \varepsilon \quad \text{if } \max_{1\leqslant j\leqslant k} n_j \geqslant M_0, \quad p_j \geqslant n_j,\ 1 \leqslant j \leqslant k.$$

Remark 1.13. Note that, in the multiple case, the Cauchy condition of difference of multiple sums is not so easy to handle. With the aim of getting to a more confortable Cauchy condition we find that we need to use the condition in Definition 1.12 as a sort of Cauchy condition for regular convergence, instead of a more natural condition. Let us see why. Recall that, in the one dimensional case, the condition of convergence of a series is equivalent to its sequence of partial sums being Cauchy, that is, that given $\varepsilon > 0$ there exists N_0 such that $p \geqslant N_0$ implies $|s_q - s_p| = \left|\sum_{n=q+1}^{q} a_n\right| < \varepsilon$. Following the one index case one can extend the form of this Cauchy condition in the expectable way. If $\sum_{m_1,\dots,m_k=1}^{\infty} a_{m_1,\dots,m_k}$ is a convergent k-multiple series, given $\varepsilon > 0$ there exists M_0 such that, if $\min\{n_1, \dots, n_k\} \geqslant M_0$, then

$$\left|\sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\dots,m_k}\right| < \varepsilon, \quad p_j \geqslant n_j, \quad 1 \leqslant j \leqslant k. \tag{1.1}$$

Denoting by $s_{p_1,\dots,p_k}$ the corresponding partial sum, it is clear that the statement is true in the 2-dimensional case since by the Cauchy condition for double series there exists M_0 such that $p_j \geqslant n_j > M_0$, $j = 1, 2$, implies

$|s_{p_1,p_2} - s_{n_1,n_2}| < \frac{\varepsilon}{2^2}$, and, since

$$\sum_{m_1=n_1+1}^{p_1} \sum_{m_2=n_2+1}^{p_2} a_{m_1,m_2} = s_{p_1,p_2} - s_{n_1,p_2} - s_{p_1,n_2} + s_{n_1,n_2},$$

then

$$\left| \sum_{m_1=n_1+1}^{p_1} \sum_{m_2=n_2+1}^{p_2} a_{m_1,m_2} \right| \leqslant |s_{p_1,p_2} - s_{n_1,p_2}| + |s_{p_1,n_2} - s_{n_1,n_2}| < 2\frac{\varepsilon}{2^2} < \varepsilon.$$

This idea can be adapted to the k-dimensional case easily to obtain

$$\left| \sum_{m_1=n_1+1}^{p_1} \cdots \sum_{m_k=n_k+1}^{p_k} a_{m_1,\dots,m_k} \right| < 2^{k-1}\frac{\varepsilon}{2^k} < \varepsilon,$$

so (1.1) is a necessary condition for the convergence of a k-mutiple series. However, it is a not sufficient one. To see this it is enough to take the double series

$$a_{m,n} = \begin{cases} 1, & \text{if } m = 1 \text{ or } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The double series $\sum_{m,n} a_{m,n}$ trivially verifies (1.1) but it is obviously not convergent. This illustrates how the key point of Definition 1.12 is to take the maximum of the indexes and not the minimum, which intuitively means that we have some control over the j-dimensional subseries, $1 \leqslant j < k$. This intuition becomes formal in the following theorem.

Theorem 1.14. *A k-multiple series converges regularly if and only if it converges in a restricted sense.*

Proof. Let us suppose that a k-multiple series $\sum_{m_1,\dots,m_k=1}^{\infty} a_{m_1,\dots,m_k}$ converges in a restricted sense, and consider $\sum_{m_{i_1},\dots,m_{i_j}=1}^{\infty} a_{m_1,\dots,m_k}$ a j-dimensional subseries. To see that it converges, we check the Cauchy condition, so take $q_{i_l}, p_{i_l} \in \mathbb{N}$ for $1 \leqslant l \leqslant j$, and we can assume without loss of generality that $q_{i_l} \geqslant p_{i_l}$, $1 \leqslant l \leqslant j$. By splitting the q-sum by the p indexes we have

$$\left| \sum_{m_{i_1}=1}^{q_{i_1}} \cdots \sum_{m_{i_j}=1}^{q_{i_j}} a_{m_1,\ldots,m_k} - \sum_{m_{i_1}=1}^{p_{i_1}} \cdots \sum_{m_{i_j}=1}^{p_{i_j}} a_{m_1,\ldots,m_k} \right| \leqslant \sum_{(b_{i_l},r_{i_l})\in\Lambda} \left| \sum_{m_{i_1}=b_{i_1}}^{r_{i_1}} \cdots \sum_{m_{i_j}=b_{i_j}}^{r_{i_j}} a_{m_1,\ldots,m_k} \right|,$$

where Λ is the set of j-tuples of pairs $\{(b_{i_l}, r_{i_l})\}_{l=1}^{j}$ satisfying the following conditions:

(i) $b_{i_l} = 1$ or $b_{i_l} = p_{i_l} + 1$ for every $l \in \{1, \ldots, j\}$;

(ii) there exists $l_0 \in \{1, \ldots, j\}$ such that $b_{i_{l_0}} = p_{i_{l_0}} + 1$;

(iii) $b_{i_l} = 1 \Rightarrow r_{i_l} = p_{i_l}$ and $b_{i_l} = p_{i_l} + 1 \Rightarrow r_{i_l} = q_{i_l}$.

Given $\varepsilon > 0$, we can apply Definition 1.12 with $\frac{\varepsilon}{2^j}$ to each term in the Λ-sum. Therefore, there is a certain M_0, such that, if $q_{i_l} \geqslant p_{i_l} \geqslant M_0$, then each term in the Λ-sum is smaller than $\frac{\varepsilon}{2^j}$. Just taking into account that $|\Lambda| \leqslant 2^j - 1$ one gets the Cauchy condition for the j-dimensional subseries.

Now for the sufficient condition, we suppose that the k-multiple series converges regularly. Take $\varepsilon > 0$ and choose $M_0 \in \mathbb{N}$ from Remark 1.13 such that $\min\{n_1, \ldots, n_k\} \geqslant M_0$, $p_j \geqslant n_j$, $1 \leqslant j \leqslant k$ implies

$$\left| \sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\ldots,m_k} \right| < \frac{\varepsilon}{2}.$$

Remark 1.13 guarantees that for each $j \in \{1, \ldots, k\}$ we only need to consider a finite amount of the j-dimensional subseries, those in which the fixed indexes satisfy $1 \leqslant m_{l_1}, \ldots, m_{l_{k-j}} < M_0$. By using again Remark

1.13, there exists $M_{m_{l_1},\dots,m_{l_{k-j}}} \geqslant M_0$ such that if $\min\{n_{i_1},\dots,n_{i_j}\} \geqslant M_{m_{l_1},\dots,m_{l_{k-j}}}$ then

$$\left|\sum_{m_{i_1}=n_{i_1}}^{p_{i_1}} \cdots \sum_{m_{i_j}=n_{i_j}}^{p_{i_j}} a_{m_1,\dots,m_k}\right| < \frac{\varepsilon}{2M_0^k}, \qquad p_{i_r} \geqslant n_{i_r}, \quad 1 \leqslant r \leqslant j. \quad (1.2)$$

Now take $M_j = \max\{M_{m_{l_1},\dots,m_{l_{k-j}}} : (m_{l_1},\dots,m_{l_{k-j}}) \in \{1,\dots,M_0\}^{k-j}\}$ and $M = \max_{0\leqslant j<k} M_j$. For a choice of $(n_1,\dots,n_k)$, if $\max\{n_1,\dots,n_k\} > M$, then there are some indexes satisfying $n_{i_1},\dots,n_{i_j} \geqslant M \geqslant M_0$, where the rest of indexes satisfy $n_{l_1},\dots,n_{l_{k-j}} < M$. Now we can check the condition for the convergence in a restricted sense

$$\begin{aligned}
&\left|\sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\dots,m_k}\right| \\
&\qquad \leqslant \left|\sum_{m_{l_1}=M_0}^{p_{l_1}} \cdots \sum_{m_{l_{k-j}}=M_0}^{p_{l_{k-j}}} \left(\sum_{m_{i_1}=n_{i_1}}^{p_{i_1}} \cdots \sum_{m_{i_j}=n_{i_j}}^{p_{i_j}} a_{m_1,\dots,m_k}\right)\right| \\
&\qquad + \sum_{m_{l_1}=n_{l_1}}^{M_0-1} \cdots \sum_{m_{l_{k-j}}=n_{l_{k-j}}}^{M_0-1} \left|\sum_{m_{i_1}=n_{i_1}}^{p_{i_1}} \cdots \sum_{m_{i_j}=n_{i_j}}^{p_{i_j}} a_{m_1,\dots,m_k}\right| \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where the inequality regarding the first term is given by Remark 1.13 and the one regarding the second term is a consequence of (1.2). This completes the proof. □

Remark 1.15. Note that it is clear that absolute convergence implies regular convergence of any k-multiple series, since absolute convergence implies boundedness of any partial sum, including the ones that would be the partial sums for any j-dimensional subseries, which implies the convergence of such subseries.

Now it remains to check that *regular convergence* solved the problems that were found when using the usual convergence of multiple series. First, the definition of *regular convergence* demands the convergence of all the j-dimensional subseries, and in that situation it is not difficult to show that all iterated series that one can define converge to the same sum. We give here the result that illustrates this in the case of double series.

Proposition 1.16 ([16], Proposition 7.2, p. 372)**.** *Let $\{a_{m,n}\}$ be a double sequence, and let us suppose that there exists*

$$\lim_{m,n\to\infty} a_{m,n} = l.$$

If for every fixed $m \in \mathbb{N}$ there exists $\lim_{n\to\infty} a_{m,n}$, then there exists the iterated limit and its value is l, that is,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right) = l.$$

Proof. Let $F(m) = \lim_{n\to\infty} a_{m,n}$. Given $\varepsilon > 0$, take n_1 such that $|a_{m,n} - l| < \frac{\varepsilon}{2}$ if $m \geqslant n_1$ and $n \geqslant n_1$. For every fixed m, take n_2 such that $|F(m) - a_{m,n}| < \frac{\varepsilon}{2}$ if $n \geqslant n_2$. (Note here that n_2 depends on m and on ε).
Now, if $m \geqslant n_1$, take n_2 and fix n so that $n \geqslant n_1, n_2$. Then,

$$|F(m) - l| \leqslant |F(m) - a_{m,n}| + |a_{m,n} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and therefore $\lim_{m\to\infty} F(m) = l$.

□

Corollary 1.17. *Let $\{a_{m,n}\}$ be a double sequence, and let us suppose that the double series $\sum_{m,n=1}^{\infty} a_{m,n}$ is regularly convergent to a sum S. Then the iterated sums are well defined and are convergent to S.*

Proof. Let $F(m) = \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}$. Note that since $F(m)$ is well defined for all $m \in \mathbb{N}$ and $f(m) = \sum_{j=1}^{\infty} a_{m,j} = F(m) - F(m-1)$ for $m \geqslant 2$, $f(1) = F(1)$, then $f(m)$ is also well defined for all $m \in \mathbb{N}$. Now, using Proposition 1.16,

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n} \right) = \sum_{m=1}^{\infty} f(m) = \lim_{M\to\infty} \sum_{m=1}^{M} f(m)$$
$$= \lim_{M\to\infty} \sum_{m=1}^{M} F(m) - F(m-1) = \lim_{M\to\infty} F(M) = S.$$

The convergence of the other iterated series can be proven analogously. □

In addition to this, the boundedness problem is also fixed, as the next proposition shows.

Proposition 1.18. *If a k-multiple series $\sum_{m_1,\ldots,m_k=1}^{\infty} a_{m_1,\ldots,m_k}$ converges regularly, then the sets $\{a_{m_1,\ldots,m_k}\}$ and $\{s_{m_1,\ldots,m_k}\}$ are bounded.*

Proof. Using the Definition 1.12 for the series $\sum_{m_1,\ldots,m_k=1}^{\infty} a_{m_1,\ldots,m_k}$, given $\varepsilon = 1$, there exists $M_0 \in \mathbb{N}$ such that $\max\{n_1, \ldots, n_k\} \geqslant M_0$, $p_j \geqslant n_j$ for every $1 \leqslant j \leqslant k$ implies

$$\left| \sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\ldots,m_k} \right| < 1.$$

If $K = \max\{|a_{m_1,\ldots,m_k}| : m_j \leqslant M_0,\ 1 \leqslant j \leqslant k\}$, then, as

$$a_{q_1,\ldots,q_k} = \sum_{m_1=q_1}^{q_1} \cdots \sum_{m_k=q_k}^{q_k} a_{m_1,\ldots,m_k},$$

we have that $|a_{m_1,\ldots,m_k}| \leqslant \max(K, 1)$.

To prove the boundedness of $\{s_{m_1,\ldots,m_k}\}$, we split $s_{p_1,\ldots,p_k}$ into 2^k multiple sums and we bound them by either K or 1.

$$|s_{p_1,p_2}| = \left| \sum_{m_1=1}^{M_0} \sum_{m_2=1}^{M_0} a_{m_1,m_2} + \sum_{m_1=M_0+1}^{p_1} \sum_{m_2=1}^{M_0} a_{m_1,m_2} \right.$$
$$\left. + \sum_{m_1=1}^{M_0} \sum_{m_2=M_0}^{p_2} a_{m_1,m_2} + \sum_{m_1=M_0+1}^{p_1} \sum_{m_2=M_0+1}^{p_2} a_{m_1,m_2} \right| < M_0^2 K + 3.$$

The k-multiple case is analogous to the 2-dimensional case. □

1.3 Bounded variation sequences.

When trying to give a theorem of convergence for multiple Dirichlet series, the study of bounded variation sequences clarifies the work by allowing to build a more organized proof. We recall first a fundamental lemma which will be very useful when dealing with convergence of Dirichlet series.

Lemma 1.19 (Abel's Summation Lemma, [2], Theorem 8.27, p. 194)**.** *Let $\{a_n\}_n$, $\{b_n\}_n$ be complex sequences, then*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_{p,n}(b_n - b_{n+1}) + A_{p,q} b_q, \quad \textit{where } A_{p,n} = \sum_{k=p}^{n} a_k.$$

Proof.

$$\begin{aligned}
\sum_{n=p}^{q-1} A_{p,n}(b_n - b_{n+1}) &= \sum_{n=1}^{q-1}\left(\sum_{k=p}^{n} a_k\right)(b_n - b_{n+1}) = \sum_{n=p}^{q-1}\sum_{k=p}^{n} a_k(b_n - b_{n+1}) \\
&= \sum_{k=p}^{q-1} a_k \sum_{n=k}^{q-1}(b_n - b_{n+1}) = \sum_{k=p}^{q-1} a_k(b_k - b_q) \\
&= \sum_{k=p}^{q-1} a_k b_k - \sum_{k=1}^{q-1} a_k b_q = \sum_{k=p}^{q-1} a_k b_k - A_{p,q-1} b_q \\
&= \sum_{k=p}^{q-1} a_k b_k + a_q b_q - a_q b_q - A_{1,q-1} b_q \\
&= \sum_{k=p}^{q} a_k b_k - A_{p,q} b_q.
\end{aligned}$$

□

An immediate corollary is the following one.

Corollary 1.20. *Let* $\{a_n\}_n$, $\{b_n\}_n$ *be complex sequences. Then*

$$\sum_{n=p+1}^{q} a_n b_n = \sum_{n=p+1}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_p b_p, \quad \textit{where } A_n = \sum_{k=1}^{n} a_k.$$

This lemma can be generalized to double or multiple series, and that is what Hardy does in [18]. We give here the generalization for double series, but first let us introduce Hardy's notation, since it will be sed often throughout this section.

$$\Delta_i a_{i,q} := a_{i,q} - a_{i+1,q}, \quad \Delta_j a_{p,j} := a_{p,j} - a_{p,j+1},$$

$$\Delta_{i,j} a_{i,j} := a_{i,j} - a_{i+1,j} - a_{i,j+1} + a_{i+1,j+1}, \quad B_{m,i;n,j} := \sum_{k=m}^{i}\sum_{l=n}^{j} b_{k,l},$$

Lemma 1.21 ([18]). *Let* $\{a_{m,n}\}_{m,n}$*,* $\{b_{m,n}\}_{m,n}$ *be complex double sequences. Then*

$$\sum_{i=m}^{p}\sum_{j=n}^{q} a_{i,j}b_{i,j} = \sum_{i=m}^{p-1}\sum_{j=n}^{q-1} B_{m,i;n,j}\Delta_{i,j}a_{i,j} + \sum_{i=m}^{p-1} B_{m,i;n,q}\Delta_i a_{i,q} + \sum_{j=n}^{q-1} B_{m,p;n,j}\Delta_j a_{p,j} + B_{p,q}a_{p,q}.$$

Proof. This lemma can be checked by applying Lemma 1.19 succesively. □

Again we can give the following immediate corollary.

Corollary 1.22. *Let* $\{a_{m,n}\}_{m,n}$*,* $\{b_{m,n}\}_{m,n}$ *be complex sequences and* $B_{i,j} = \sum_{k=1}^{i}\sum_{l=1}^{j} b_{k,l}$ *then*

$$\begin{aligned}\sum_{i=m+1}^{p}\sum_{j=n+1}^{q} a_{i,j}b_{i,j} = &\sum_{i=m+1}^{p-1}\sum_{j=n+1}^{q-1} B_{i,j}\Delta_{i,j}a_{i,j} \\ &+ \sum_{i=m+1}^{p-1} (B_{i,q}\Delta_i a_{i,q} - B_{i,n}\Delta_{i,n}a_{i,n}) \\ &+ \sum_{j=n+1}^{q-1} (B_{p,j}\Delta_j a_{p,j} - B_{m,j}\Delta_{m,j}a_{m,j}) \\ &+ B_{p,q}a_{p,q} - B_{p,n}a_{p,n} - B_{m,q}a_{m,q} + B_{m,n}a_{m,n}.\end{aligned}$$

The study of bounded variation sequences arises form the following question: what conditions do we need ask to $\{a_m\}_m$ so that the series $\sum_m a_m u_m$ is convergent for every convergent series $\sum_{m=1}^{\infty} u_m$?

Definition 1.23 ([19]). We say that a sequence $\{a_m\}_m$ is a convergence factor if $\sum_{m=1}^{\infty} a_m u_m$ is convergent for every convergent series $\sum_{m=1}^{\infty} u_m$.

Now, the question can be restated in the following way: what conditions do we need ask to $\{a_m\}_m$ so that it is a convergence factor?

This question was answered in [19], where convergence factors where characterized as described below.

Definition 1.24 ([19]). We say that a sequence $\{a_m\}_m$ of real or complex numbers is of bounded variation if the series $\sum_{m=1}^{\infty} |a_m - a_{m+1}|$ is convergent.

Remark 1.25. Note that the previous definition implies the convergence of the sequence $\{a_m\}_m$ and then its boundedness. Indeed, taking $n < m$,

$$|a_n - a_m| \leqslant \sum_{k=n}^{m-1} |a_k - a_{k+1}|,$$

and the convergence of the series $\sum_{m=1}^{\infty} |a_m - a_{m+1}|$ implies that $\sum_{k=n}^{m-1} |a_k - a_{k+1}|$ goes to 0 when both indexes go to infinity.

Theorem 1.26 ([19]). *A sequence $\{a_m\}_m$ is of bounded variation if and only if it is a convergence factor.*

Proof of the sufficient condition of Theorem 1.26. By the hypothesis we get that $\sum_k |a_k - a_{k+1}|$ converges and by Remark 1.25 $\{a_m\}_m$ converges too, so there exists $K > 0$ such that $\sum_{k=n+1}^{m} |a_k - a_{k+1}| < K$ and $|a_m| < K$. Now, given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2K}$ and take $\sum_n u_n$ a convergent series. By the Cauchy condition for $\sum_n u_n$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geqslant n_0$ and $m > n$, then $\left|\sum_{k=n+1}^{m} u_k\right| < \delta$. Therefore, by Lemma 1.19,

$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k u_k\right| &\leqslant \sum_{k=n+1}^{m-1} \left|\sum_{j=n+1}^{j} u_k\right| |a_k - a_{k+1}| + |a_m| \left|\sum_{k=n+1}^{m} u_k\right| \\ &< \delta \left(\sum_{k=n+1}^{m-1} |a_k - a_{k+1}| + |a_m| \right) < 2K\delta = \varepsilon.\end{aligned}$$

□

An exhaustive development of these results was presented by Hardy in [19], where he also proved that being of bounded variation is not only a necessary but also a sufficient condition in the characterization of convergence factors. We reproduce this results here for the sake of completeness and to emphasize their relevance. First we will need some lemmas about divergent series.

Lemma 1.27 ([19, Lemma α])**.** *If $\sum_{n=1}^{\infty} c_n$ is a divergent series of non-negative terms, there exists a sequence of positive terms $\{\varepsilon_n\}$ convergent to 0 such that $\sum_{n=1}^{\infty} \varepsilon_n c_n$ is still divergent.*

Lemma 1.27 is a direct consequence of the following proposition.

Proposition 1.28. *If $\sum_{n=1}^{\infty} c_n$ is a divergent series of non-negative terms, then $\sum_{n=1}^{\infty} \frac{c_n}{s_{n-1}}$ is divergent, where $s_m = \sum_{n=1}^{m} c_n$ for $m \in \mathbb{N}$ and $s_0 = 1$.*

Proof. Fix $p \in \mathbb{N}$. Since $\lim_{q\to\infty} \frac{s_{p-1}}{s_q} = 0$, given $\varepsilon = \frac{1}{2}$ there exists $q_0 \in \mathbb{N}$ such that $q \geqslant q_0$ implies $\frac{s_{p-1}}{s_q} < \frac{1}{2}$. Now, if $q \geqslant q_0$

$$\sum_{n=p}^{q} \frac{c_n}{s_{n-1}} = \sum_{n=p}^{q} \frac{s_n - s_{n-1}}{s_{n-1}} \geqslant \frac{\sum_{n=p}^{q} s_n - s_{n-1}}{s_q} = \frac{s_q - s_{p-1}}{s_q} = 1 - \frac{s_{p-1}}{s_q} > \frac{1}{2}.$$

Then, if $p = 1$, there exists $q_1 \in \mathbb{N}$ such that $\sum_{n=1}^{q_1} \frac{c_n}{s_{n-1}} > \frac{1}{2}$. Again, if $p = q_1 + 1$, we can find some $q_2 \in \mathbb{N}$ such that $\sum_{n=q_1+1}^{q_2} \frac{c_n}{s_{n-1}} > \frac{1}{2}$ and therefore $\sum_{n=1}^{q_2} \frac{c_n}{s_{n-1}} > 1$. Inductively, suppose the existence of $q_1, \ldots, q_j$ and $\sum_{n=1}^{q_j} \frac{c_n}{s_{n-1}} > \frac{j}{2}$. Considering $p = q_j + 1$, there exists some $q_{j+1} \in \mathbb{N}$ such that $\sum_{n=q_j+1}^{q_{j+1}} \frac{c_n}{s_{n-1}} > \frac{1}{2}$ and therefore $\sum_{n=1}^{q_{j+1}} \frac{c_n}{s_{n-1}} > \frac{j+1}{2}$. This shows the divergence of $\{\sum_{n=1}^{q_j} \frac{c_n}{s_{n-1}}\}_{j=1}^{\infty}$, and therefore, as the terms $\frac{c_n}{s_{n-1}}$ are non-negative, the divergence of the series $\sum_{n=1}^{\infty} \frac{c_n}{s_{n-1}}$. □

Lemma 1.29 ([19, Lemma β])**.** *If $\sum_{n=1}^{\infty} c_n$ is a divergent series of non-negative terms, then we can find a sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ such that $\tilde{c}_{n_j} = 0$ with $\tilde{c}_n = c_n$ if $n \neq n_j$, $j \in \mathbb{N}$, and such that $\sum_{n=1}^{\infty} \tilde{c}_n$ is still divergent.*

Proof. As in Proposition 1.28, for every $p \in \mathbb{N}$ there exists some $q(p) \in \mathbb{N}$ such that $\sum_{n=p}^{q(p)} c_n > 1$. Then, repeating the argument of Proposition 1.28 and defining $n_1 = 1$, $n_{j+1} = q(n_j + 1) + 1$ for $j \in \mathbb{N}$,

$$\sum_{n=n_{j-1}+1}^{n_j-1} \tilde{c}_n > 1 \quad \text{for every } j \in \mathbb{N},\ j > 1.$$

Now define $\tilde{c}_n = c_n$ if $n \neq n_j$ for every $j \in \mathbb{N}$, and $\tilde{c}_{n_j} = 0$ for every $j \in \mathbb{N}$. Therefore

$$\sum_{n=1}^{n_j} \tilde{c}_n = \sum_{k=1}^{j} \left(\sum_{n=n_{k-1}+1}^{n_k-1} \tilde{c}_n + \tilde{c}_{n_k} \right) = \sum_{k=1}^{j} \sum_{n=n_{k-1}+1}^{n_k-1} c_n > j.$$

This shows the divergence of $\{\sum_{n=1}^{n_j} \tilde{c}_n\}_{j=1}^{\infty}$, and then the divergence of $\sum_{n=1}^{\infty} \tilde{c}_n$. □

Proof of the necessary condition of Theorem 1.26. By contradiciton, suppose $\{a_n\}_n$ is a convergence factor but also that it is not a sequence of bounded variation. That implies the divergence of $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$. As it is a series of non-negative terms, by Lemma 1.27 there exists a sequence of positive terms $\{\varepsilon_n\}$ convergent to 0 such that $\sum_{n=1}^{\infty} \varepsilon_n |a_n - a_{n+1}|$ is divergent. If we put $b_n = \varepsilon_n |a_n - a_{n+1}|$, then applying Lemma 1.29 to the divergent series $\sum_{n=1}^{\infty} b_n$ we can find a sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ such that $\tilde{b}_{n_j} = 0$ and $\tilde{b}_n = b_n$ if $n \neq n_j$ for every $j \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \tilde{b}_n$ is still divergent. Now define

$$U_m = \begin{cases} \frac{\tilde{b}_n}{a_n - a_{n+1}}, & \text{if } a_n \neq a_{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $U_{n_j} = 0$ for every $j \in \mathbb{N}$ and, if $n \neq n_j$, $U_n(a_n - a_{n+1}) = \tilde{b}_n$. Define also

$$u_m = \begin{cases} U_1, & \text{if } n = 1, \\ U_n - U_{n-1}, & \text{otherwise.} \end{cases}$$

Now, by Abel's Lemma 1.19,

$$\sum_{n=1}^{n_j} a_n u_n = \sum_{n=1}^{n_j} (a_n - a_{n+1}) U_n + a_{n_j} U_{n_j} = \sum_{n=1}^{n_j} \tilde{b}_n \xrightarrow{j\to\infty} +\infty.$$

As the terms $\tilde{b}_n$ are non-negative, that gives the divergence of the series $\sum_{n=1}^{n_j} a_n u_n$, while the series $\sum_{n=1}^{\infty} u_n$ is convergent because $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} u_k = \lim\limits_{n\to\infty} U_n$ and $|U_n| \leqslant \varepsilon_n \xrightarrow{n\to\infty} 0$. We had supposed that $\{a_n\}_n$ was a convergence factor, and we have reached a contradiction. Therefore $\{a_n\}_n$ has to be a sequence of bounded variation. □

With the objective of using the characterization of convergence factors as a tool to build a proof for a theorem of convergence of multiple Dirichlet series, our main aim now is to extend the results that have been just presented to multiple series. First we will reproduce the work that was done by Hardy in [19] with the aim of characterizing convergence factors for double series.

Definition 1.30. We say that a double sequence $\{a_{m,n}\}_{m,n}$ is a convergence factor if $\sum_{m,n=1}^{\infty} a_{m,n} u_{m,n}$ is regularly convergent for every regularly convergent double series $\sum_{m,n=1}^{\infty} u_{m,n}$.

Definition 1.31. We say that a double sequence $\{a_{m,n}\}$ is of bounded variation in (m,n) if:

(i) For each fixed value of $n \in \mathbb{N}$, $\{a_{m,n}\}$ is of bounded variation in m.

(ii) For each fixed value of $m \in \mathbb{N}$, $\{a_{m,n}\}$ is of bounded variation in n.

(iii) The double series $\sum_{m,n} |a_{m,n} - a_{m+1,n} - a_{m,n+1} + a_{m+1,n+1}|$ converges.

Remark 1.32. The previous definition is the natural extension from the one in the case of simple sequences. However, assuming condition (iii), it is enough to ask that $\{a_{m,1}\}_m$ and $\{a_{1,n}\}_n$ are of bounded variation to get conditions (i) and (ii) respectively. Indeed, with the notation introduced before Lemma 1.21,

$$\sum_{l=1}^{n-1} |\Delta_l a_{m,l}| \leqslant \sum_{l=1}^{n-1} |\Delta_l a_{1,l}| + \sum_{k=1}^{m-1} \sum_{l=1}^{n-1} |\Delta_{k,l} a_{k,l}|. \tag{1.3}$$

This is clarified in the proposition below.

Proposition 1.33. *If $\{a_{m,n}\}_{m,n}$ is a double sequence of bounded variation, then for every $m \in \mathbb{N}$ there exists $a_m = \lim_{n\to\infty} a_{m,n}$ and for every $n \in \mathbb{N}$ there exists $a_n = \lim_{m\to\infty} a_{m,n}$. Moreover, the sequences $\{a_n\}_n$, $\{a_m\}_m$ are of bounded variation.*

Proof. Fix $n \in \mathbb{N}$ and let us see that $\{a_n\}_n$ is of bounded variation. Note that

$$a_{m,n} = a_{1,n} + \sum_{k=1}^{m-1} a_{k+1,n} - a_{k,n} = a_{1,n} - \sum_{k=1}^{m-1} \Delta_k a_{k,n},$$

so $\lim_{m\to\infty} a_{m,n} = a_{1,n} - \lim_{m\to\infty} \sum_{k=1}^{m-1} \Delta_k a_{k,n}$, which exists because $\{a_{m,n}\}_{m,n}$ is a double sequence of bounded variation. Hence $\{a_n\}_n$ is well defined and

$$a_n - a_{n+1} = a_{1,n} - a_{1,n+1} - \sum_{k=1}^{\infty} \Delta_k a_{k,n} + \sum_{k=1}^{\infty} \Delta_k a_{k,n+1} = \Delta_n a_{1,n} - \sum_{k=1}^{\infty} \Delta_{k,n} a_{k,n}.$$

Therefore,

$$\sum_{l=1}^{n} |a_l - a_{l+1}| \leqslant \sum_{l=1}^{n} |\Delta_l a_{1,l}| + \sum_{l=1}^{n} \sum_{k=1}^{\infty} |\Delta_{k,l} a_{k,l}|,$$

where the sums in the right hand side are partial sums of convergent series, so $\sum_{l=1}^{n} \Delta_n a_n$ converges (To be completely clear, the fact that these series are of positive terms and Remark 1.15 imply the convergence of the double sum at the right-hand side of the inequality above). ☐

Theorem 1.34 ([19, Theorems 10 and 12])**.** *The double sequence* $\{a_{m,n}\}_{m,n}$ *is of bounded variation if and only if it is a convergence factor.*

Proof of the sufficient condition of Theorem 1.34. As $\{a_{m,n}\}_{m,n}$ is of bounded variation we have that there are some positive constants satisfying

- As $\sum_{i,j} |\Delta_{i,j} a_{i,j}|$ converges, $\sum_{i=1}^{p} \sum_{j=1}^{q} |\Delta_{i,j} a_{i,j}| < K_0$ for every $p, q \in \mathbb{N}$,
- As $\sum_i |\Delta_i a_{i,1}|$ converges, $\sum_{i=1}^{p} |\Delta_i a_{i,1}| < K_1$ for every $p \in \mathbb{N}$,
- As $\sum_j |\Delta_j a_{1,j}|$ converges, $\sum_{j=1}^{q} |\Delta_j a_{1,j}| < K_2$ for every $q \in \mathbb{N}$.

Now, using (1.3), $\sum_i |\Delta_i a_{i,j}| < K_0 + K_1$ for every $j \in \mathbb{N}$. Analogously, $\sum_j |\Delta_i a_{i,j}| < K_0 + K_2$ for every $i \in \mathbb{N}$. Moreover, for every $i, j \in \mathbb{N}$,

$$|a_{i,j}| \leqslant \sum_{k=i}^{\infty} |a_{k,j} - a_{k+1,j}| \leqslant \sum_{k=1}^{\infty} |\Delta_k a_{k,j}| < K_0 + K_1.$$

Take $K = K_0 + K_1 + K_2$. Since $\sum_{m,n} u_{m,n}$ converges regularly, given $\varepsilon > 0$, if we take $\delta = \frac{\varepsilon}{4K}$ there exists n_0 such that $\max\{m, n\} \geqslant n_0$ implies

$$\left| \sum_{i=m}^{p} \sum_{l=n}^{q} u_{i,j} \right| < \delta, \quad \text{for } p \geqslant m,\ q \geqslant n.$$

Therefore, if $\max\{m, n\} \geqslant n_0$, $p \geqslant m$, $q \geqslant n$, by Lemma 1.21,

$$\begin{aligned} &\left| \sum_{i=m}^{p} \sum_{j=n}^{q} a_{k,l} u_{k,l} \right| \\ &\quad \leqslant \sum_{i=m}^{p-1} \sum_{j=n}^{q-1} |\Delta_{i,j} a_{i,j}| \left| \sum_{k=m}^{i} \sum_{l=n}^{j} u_{i,j} \right| + \sum_{i=m}^{p-1} |\Delta_i a_{i,j}| \left| \sum_{k=m}^{i} \sum_{l=n}^{q} u_{i,j} \right| \\ &\quad + \sum_{j=n}^{q-1} |\Delta_j a_{i,j}| \left| \sum_{k=m}^{p} \sum_{l=n}^{j} u_{i,j} \right| + |a_{p,q}| \left| \sum_{k=m}^{p} \sum_{l=n}^{q} u_{i,j} \right| \end{aligned}$$

$$< \delta \left\{ \sum_{i=m}^{p-1} \sum_{j=n}^{q-1} |\Delta_{i,j} a_{i,j}| + \sum_{i=m}^{p-1} |\Delta_i a_{i,j}| + \sum_{j=n}^{q-1} |\Delta_j a_{i,j}| + |a_{p,q}| \right\}$$
$$< 4\delta K = \varepsilon,$$

and the condition for the regular convergence of $\sum_{m,n} a_{m,n} u_{m,n}$ is satisfied. □

The path to the proof of the necessity condition is again longer and we will need the following lemmas.

Lemma 1.35 ([19, Lemma ε]). *If $\sum_{m,n=1}^{\infty} c_{m,n}$ is a divergent double series of non-negative terms, we can find a double sequence $\{\varepsilon_{m,n}\}$ such that*

(i) $\{\varepsilon_{m,n}\}$ is decreasing in both indexes separately, that is

$$\varepsilon_{m+1,n} \leqslant \varepsilon_{m,n}, \quad \varepsilon_{m,n+1} \leqslant \varepsilon_{m,n}, \quad \forall m, n \in \mathbb{N}.$$

(ii) $\lim_{m,n\to\infty} \varepsilon_{m,n} = 0, \quad \lim_{m\to\infty} \varepsilon_{m,n} = 0, \quad \lim_{n\to\infty} \varepsilon_{m,n} = 0.$

(iii) The series $\sum_{m,n=1}^{\infty} \varepsilon_{m,n} c_{m,n}$ is divergent.

Proof. Given the divergence of $\sum_{m,n=1}^{\infty} c_{m,n}$ there are two distinct possibilities:

(1) Suppose that at least one row or column of the original series, say the $k-$th row $\sum_{n=1}^{\infty} c_{k,n}$, is divergent. By Lemma 1.27 we can choose a decreasing sequence $\{\eta_n\}$, with limit zero, so that $\sum_{n=1}^{\infty} \eta_n c_{k,n}$ is divergent. Take

$$\varepsilon_{m,n} = \eta_n \text{ if } m \leqslant k, \quad \varepsilon_{m,n} = 0 \text{ if } m > k,$$

and then conditions (i) and (ii) are plainly satisfied and, as all the terms are positive,

$$\sum_{m,n=1}^{\infty} \varepsilon_{m,n} c_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{k} \eta_n c_{m,n} \geqslant \sum_{n=1}^{\infty} \eta_n c_{k,n},$$

so the divergence of the latter gives the divergent of the former.

(2) Suppose that every row and column is convergent and let

$$\alpha_m = \sum_{n=1}^{\infty} c_{m,n}, \quad \beta_n = \sum_{m=1}^{\infty} c_{m,n}.$$

Then $\sum_{n=1}^{\infty} \beta_n$ is divergent. We choose a decreasing sequence $\{\eta_n\}$ with limit zero so that $\sum_{n=1}^{\infty} \eta_n \beta_n$ is divergent. As every column subseries is convergent, $\eta_n \beta_n = \sum_{m=1}^{\infty} \eta_n c_{m,n}$ for every $n \in \mathbb{N}$, and then the divergence of $\sum_{n=1}^{\infty} \eta_n \beta_n$ implies the divergence of $\sum_{m,n=1}^{\infty} \eta_n c_{m,n}$ as a consequence of Corollary 1.17. Using the same argument, $\tilde{\alpha}_m = \sum_{n=1}^{\infty} \eta_n c_{m,n}$ converges for every $m \in \mathbb{N}$, and $\sum_{m=1}^{\infty} \tilde{\alpha}_m$ diverges. Now, choose a decreasing sequence $\{\zeta_n\}$ with limit zero so that $\sum_{m=1}^{\infty} \zeta_m \tilde{\alpha}_m$ is divergent. Taking $\varepsilon_{m,n} = \eta_n \zeta_m$, conditions (i) and (ii) are satisfied and again by Corollary 1.17 $\sum_{m,n=1}^{\infty} \varepsilon_{m,n} c_{m,n}$ cannot be convergent, so it is divergent. □

Lemma 1.36 ([19, Lemma ζ]). *If $\sum_{m,n=1}^{\infty} c_{m,n}$ is a divergent double series of non-negative terms, we can find a double sequence of integers $\{(m_j, n_j)\}$ tending to infinity with j, so that the series $\sum_{m,n=1}^{\infty} \tilde{c}_{m,n}$ is divergent, where $\tilde{c}_{m,n} = 0$ if $m = m_j$, $n \leqslant n_j$, or $m \leqslant m_j$, $n = n_j$, and $\tilde{c}_{m,n} = c_{m,n}$ otherwise.*

Proof. The modification to be made in the series is effected by drawing perpendiculars on to the axes form the points (m_j, n_j), and annulling all the terms which correspond to points on these perpendiculars. Define $S_k = \sum_{m=1}^{k} \sum_{n=1}^{k} c_{m,n}$, then $\sum_{k=1}^{\infty} S_k$ is divergent. It is enough to apply Lemma 1.29 to $\sum_{k=1}^{\infty} S_k$ to get the construction required, with $m_j = n_j$ for every $j \in \mathbb{N}$. □

Proof of the necessary condition of Theorem 1.34. In the first place it follows from Theorem 1.26 that $a_{m,n}$ is, for every value of n (resp. m), of bounded variation in m (resp. n). It remains only to show that

$$\sum_{m,n=1}^{\infty} |\Delta_{m,n} a_{m,n}|$$

is convergent. Suppose, on the contrary, that it is divergent. By Lemma 1.35, we can choose a double sequence of positive numbers $\{\varepsilon_{m,n}\}$ such that

(i) $\{\varepsilon_{m,n}\}$ is decreasing in both indexes separately, that is

$$\varepsilon_{m+1,n} \leqslant \varepsilon_{m,n}, \quad \varepsilon_{m,n+1} \leqslant \varepsilon_{m,n}, \quad \forall m, n \in \mathbb{N}.$$

(ii) $\lim_{m,n\to(\infty,\infty)} \varepsilon_{m,n} = 0, \quad \lim_{m\to\infty} \varepsilon_{m,n} = 0, \quad \lim_{n\to\infty} \varepsilon_{m,n} = 0.$

(iii) The series $\sum_{m,n=1}^{\infty} \varepsilon_{m,n} |\Delta_{m,n} a_{m,n}|$ is divergent.

We can then modify the series $\sum_{m,n=1}^{\infty} \varepsilon_{m,n} |\Delta_{m,n} a_{m,n}|$ as in Lemma 1.36 without losing its divergence. Now let

$$U_{m,n} = \sum_{k=1}^{m} \sum_{l=1}^{n} u_{k,l},$$

and suppose that $U_{m,n} = 0$ if if $m = m_j$, $n \leqslant n_j$, or $m \leqslant m_j$, $n = n_j$, and otherwise

$$U_{m,n} = \varepsilon_{m,n} \frac{|\Delta_{m,n} a_{m,n}|}{\Delta_{m,n} a_{m,n}}$$

if $\Delta_{m,n} a_{m,n} \neq 0$, $U_{m,n} = 0$ when $\Delta_{m,n} a_{m,n} = 0$. This equations define $\{u_{m,n}\}$ uniquely for all values of m and n, and

$$\lim_{m,n\to\infty} U_{m,n} = 0, \quad \lim_{m\to\infty} U_{m,n} = 0, \quad \lim_{n\to\infty} U_{m,n} = 0,$$

so $\sum_{m,n=1}^{\infty} u_{m,n}$ is regularly convergent. On the other hand, by Lemma 1.21,

$$\sum_{m=1}^{m_j}\sum_{n=1}^{n_j} a_{m,n}u_{m,n} = \sum_{m=1}^{m_j-1}\sum_{n=1}^{n_j-1} \Delta_{m,n}a_{m,n}U_{m,n} = \sum_{m=1}^{m_j-1}\sum_{n=1}^{n_j-1} \varepsilon_{m,n}|\Delta_{m,n}a_{m,n}|,$$

which tends to infinity with j, which proves that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}u_{m,n}$ is divergent. This contradicts our hypothesis of $\{a_{m,n}\}_{m,n}$ being a convergence factor, so we have reached a contradiction, which gives that $\{a_{m,n}\}$ must be of bounded variation. □

We go back now to the main aim of this section, that is, to use bounded variation sequences as a tool to provide an organized proof for a theorem of convergence of multiple Dirichlet series. In order to do that, we are going to make a generalization of the definitions given above and the subsequent theorems, focusing exclusively on the pieces we will need for later work.

Definition 1.37. We say that a sequence of bounded functions $\{a_m\}$, where $a_m : \Omega \subset \mathbb{C} \to \mathbb{C}$, is of uniform bounded variation in Ω if $\sum_m |a_m(s) - a_{m+1}(s)|$ is uniformly convergent in Ω.

Theorem 1.38. *Let $\{a_m(s)\}$ be a functional sequence of uniform bounded variation on a certain set Ω. Then, if $\sum u_m$ converges, $\sum a_m(s)u_m$ converges uniformly on Ω.*

Proof. Since a uniformly convergent series of bounded functions is uniformly bounded, there exists $K > 0$:

$$\sum_{k=1}^{\infty} |a_k(s) - a_{k+1}(s)| \leqslant K, \quad \text{for every } s \in \Omega.$$

Moreover, since $\sum_m |a_m(s) - a_{m+1}(s)|$ being convergent implies that $\lim_{k\to\infty} a_k(s) = 0$ for any fixed $s \in \Omega$, then for each $m \in \mathbb{N}$ and each

$s \in \Omega$ we have that

$$|a_m(s)| \leqslant \sum_{k=m}^{\infty} |a_k(s) - a_{k+1}(s)| \leqslant \sum_{k=1}^{\infty} |a_k(s) - a_{k+1}(s)| \leqslant K.$$

Now, given $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{2K}$, and there exists n_0 such that $n \geqslant n_0$ and $m > n$ imply $\left|\sum_{k=n+1}^{m} u_k\right| < \delta$. Therefore,

$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k(s)u_k\right| &\leqslant \sum_{k=n+1}^{m-1} \left|\sum_{j=n+1}^{k} u_j\right| |a_k(s) - a_{k+1}(s)| + |a_m(s)| \left|\sum_{k=n+1}^{m} u_k\right| \\ &< \delta \left(\sum_{k=n+1}^{m-1} |a_k(s) - a_{k+1}(s)| + |a_m(s)|\right) < 2K\delta = \varepsilon,\end{aligned}$$

which gives us the uniform convergence of $\sum a_m(s)u_m$ on Ω. □

Definition 1.39. We say that a double sequence of bounded functions $\{a_{m,n}\}$ with $a_{m,n} : \Omega \subset \mathbb{C}^2 \to \mathbb{C}$ is of uniform bounded variation on a certain set Ω if the series

$$\sum_{m=1}^{\infty} |a_{m,n}(s,t) - a_{m+1,n}(s,t)|, \quad \sum_{n=1}^{\infty} |a_{m,n}(s,t) - a_{m,n+1}(s,t)|,$$

$$\sum_{m,n=1}^{\infty} |a_{m,n}(s,t) - a_{m+1,n}(s,t) - a_{m,n+1}(s,t) + a_{m+1,n+1}(s,t)|,$$

are uniformly convergent in Ω.

Theorem 1.40. *Let $\{a_{m,n}\}$ be a functional double sequence with $a_{m,n} : \Omega \subset \mathbb{C}^2 \to \mathbb{C}$ of uniform bounded variation on Ω. Then, if $\sum_{m,n} u_{m,n}$ converges regularly, $\sum_{m,n} a_{m,n}(s,t)u_{m,n}$ converges regularly and uniformly with respect to s and t on Ω.*

Proof. As $\{a_{m,n}(s,t)\}$ is of bounded variation uniformly we have that there are some positive constants satisfying

- $\sum_{i=1}^{p}\sum_{j=1}^{q}|\Delta_{i,j}a_{i,j}(s,t)| < K_0$ for every $p,q\in\mathbb{N}$ and every $(s,t)\in\Omega$,
- $\sum_{i=1}^{p}|\Delta_i a_{i,1}(s,t)| < K_1$ for every $p\in\mathbb{N}$ and for every $(s,t)\in\Omega$,
- $\sum_{j=1}^{q}|\Delta_j a_{1,j}(s,t)| < K_2$ for every $q\in\mathbb{N}$and for every $(s,t)\in\Omega$.

Now, using (1.3), $\sum_i|\Delta_i a_{i,j}(s,t)| < K_0+K_1$ for every $j\in\mathbb{N}$ and every $(s,t)\in\Omega$. Analogously, $\sum_j|\Delta_i a_{i,j}(s,t)| < K_0+K_2$ for every $i\in\mathbb{N}$ and every $(s,t)\in\Omega$. Moreover, for every $i,j\in\mathbb{N}$,

$$|a_{i,j}(s,t)| \leqslant \sum_{k=i}^{\infty}|\Delta_k a_{k,j}(s,t)| \leqslant \sum_{k=1}|\Delta_k a_{k,j}(s,t)| < K_0+K_1.$$

Take $K=K_0+K_1+K_2$. Since $\sum_{m,n}u_{m,n}$ converges regularly, given $\varepsilon>0$, we put $\delta=\frac{\varepsilon}{4K}$, and there exists n_0 such that $\max\{m,n\}\geqslant n_0$ implies

$$\left|\sum_{i=m}^{p}\sum_{l=n}^{q}u_{i,j}\right| < \delta, \quad \text{for } p\geqslant m,\ q\geqslant n.$$

Therefore, on the same conditions for the subindexes,

$$\begin{aligned}
&\left|\sum_{i=m}^{p}\sum_{j=n}^{q}a_{k,l}(s,t)u_{k,l}\right| \\
&\quad\leqslant \sum_{i=m}^{p-1}\sum_{j=n}^{q-1}|\Delta_j a_{i,j}(s,t)|\left|\sum_{k=m}^{i}\sum_{l=n}^{j}u_{i,j}\right| + \sum_{i=m}^{p-1}|\Delta_i a_{i,j}(s,\ t)|\left|\sum_{k=m}^{i}\sum_{l=n}^{q}u_{i,j}\right| \\
&\quad+\sum_{j=n}^{q-1}|\Delta_j a_{i,j}(s,t)|\left|\sum_{k=m}^{p}\sum_{l=n}^{j}u_{i,j}\right| + |a_{p,q}(s,t)|\left|\sum_{k=m}^{p}\sum_{l=n}^{q}u_{i,j}\right| \\
&\quad<\delta\left\{\sum_{i=m}^{p-1}\sum_{j=n}^{q-1}|\Delta_{i,j}a_{i,j}(s,t)| + \sum_{i=m}^{p-1}|\Delta_i a_{i,j}(s,t)|\right. \\
&\quad\left.+\sum_{j=n}^{q-1}|\Delta_j a_{i,j}(s,t)| + |a_{p,q}(s,t)|\right\} < 4\delta K = \varepsilon.
\end{aligned}$$

That gives us the condition for the regular convergence of the series $\sum_{m,n} a_{m,n}(s,t)u_{m,n}$ uniformly on Ω. □

What remains now is to prove a theorem which characterizes convergence factors for k-multiple series, in order to get a theorem of convergence of multiple Dirichlet series. The k-dimensional case is analogous to the 2-dimensional case, so we give now the definition of a k-multiple sequence of bounded variation to characterize them later as the convergence factors for regularly convergent k-multiple sequences.

Definition 1.41. We say that a k-multiple sequence $\{a_{m_1,\dots,m_k}\}_{m_1,\dots,m_k}$ is a convergence factor if $\sum_{m_1,\dots,m_k} a_{m_1,\dots,m_k} u_{m_1,\dots,m_k}$ is regularly convergent for all $\sum_{m_1,\dots,m_k} u_{m_1,\dots,m_k}$ regularly convergent series.

Following the line of arguments above the next step would be to obtain the extension of Abel's summation Lemma to the k-multiple case, but before we do that we are going to introduce some notation. If $\{b_{m_1,\dots,m_k}\}_{m_1,\dots,m_k=1}^{\infty}$ is a k-multiple sequence we write the difference of two consecutive terms in a given index i as follows,

$$\Delta_i b_{m_1,\dots,m_i,\dots,m_k} = b_{m_1,\dots,m_i,\dots,m_k} - b_{m_1,\dots,m_i+1,\dots,m_k}.$$

We can combine these differences when they involve two different indexes,

$$\begin{aligned}\Delta_{i_1 i_2} b_{m_1,\dots,m_k} &= \Delta_{i_1}(\Delta_{i_2} b_{m_1,\dots,m_k}) \\ &= \Delta_{i_1}(b_{m_1,\dots,m_{i_2},\dots,m_k}) - \Delta_{i_1}(b_{m_1,\dots,m_{i_2+1},\dots,m_k}).\end{aligned}$$

If $p = (p_1,\dots,p_k) \in \mathbb{N}^k$ and $\{u_{m_1,\dots,m_k}\}_{m_1,\dots,m_k=1}^{\infty}$ is a k-multiple sequence, we will write $U_{p_{i_1},\dots,p_{i_j}}|_p$ to represent the partial sum of that sequence on those indexes, where the indexes that do not appear in the sum are set to the corresponding p_j,

$$U_{p_{i_1},\dots,p_{i_j}}|_p = \left(\sum_{m_{i_1}=1}^{p_{i_1}} \cdots \sum_{m_{i_j}=1}^{p_{i_j}} u_{m_1,\dots,m_k} \right)\Bigg|_p .$$

With this notation Abel's Lemma for k-multiple series can be stated in the following way.

Lemma 1.42 ([18], p. 125, (A')).

$$\sum_{m_1=1}^{p_1} \cdots \sum_{m_k=1}^{p_k} b_{m_1,\dots,m_k} u_{m_1,\dots,m_k} = \sum_{j\leqslant k} \sum_{i_1<\dots<i_j} \left(\sum_{m_{i_1}=1}^{p_1} \cdots \sum_{m_{i_j}=1}^{p_j} U_{m_{i_1},\dots,m_{i_j}} \Delta_{i_1,\dots,i_j} b_{m_1,\dots,m_k} \right)\Bigg|_p .$$

Proof. The proof follows with a straightforward induction using Lemma 1.19 both as the case $k=1$ and as the link between the thesis and the induction hypothesis. □

As we are presenting Lemma 1.42 in its version for k-multiple series, the notation can be difficult to understand. With the sum $\sum_{i_1<\dots<i_j}$ we mean that the accumulated sum that comes after is written for all possible combination of j indexes $i_1 < \dots < i_j$.

Example 1.43. If $k=3$, the term corresponding to the indexes m_1, m_3 in Abel's Lemma 1.42 is the following one:

$$(U_{m_1,m_3} \Delta_{m_1,m_3} a_{m_1,m_2,m_3})\,|_p =$$
$$(a_{m_1,p_2,m_3} - a_{m_i+1,p_2,m_3} - a_{m_1,p_2,m_3+1} + a_{m_1+1,p_2,m_3+1}) \sum_{m_1=1}^{p_1} \sum_{m_3=1}^{p_3} u_{m_1,p_2,m_3}$$

Definition 1.44. We say that a k-multiple sequence $\{a_{m_1,\ldots,m_k}\}_{m_1,\ldots,m_k=1}^{\infty}$ is of bounded variation if, for any number of subindexes $j \leqslant k$

$$\sum_{m_{i_1},\ldots,m_{i_j}} \left|\Delta_{m_{i_1},\ldots,m_{i_j}} a_{m_1,\ldots,m_k}\right|$$

converges.

Theorem 1.45. *If the k-multiple sequence $\{a_{m_1,\ldots,m_k}\}_{m_1,\ldots,m_k=1}^{\infty}$ is of bounded variation then it is a convergent factor.*

Proof. Suppose $\sum_{m_1,\ldots,m_k=1}^{\infty} u_{m_1,\ldots,m_k}$ is a regularly convergent k-multiple series. We are going to apply Abel's summation Lemma 1.42 to check the condition of convergence in a restricted sense. Given $\varepsilon > 0$, there exist $M_0 > 0$ such that, if $\max\{n_1,\ldots,n_k\} \geqslant M_0$, then

$$\left|\sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} u_{m_1,\ldots,m_k}\right| < \frac{\varepsilon}{2^k C}$$

where C is a bound for every sum of the kind

$$\sum_{m_{i_1},\ldots,m_{i_j}} \left|\Delta_{m_{i_1},\ldots,m_{i_j}} a_{m_1,\ldots,m_k}\right|.$$

This bound can be obtained in applying the k-dimensional analogue of (1.3) to every series of that kind. Then,

$$\begin{aligned}
&\left|\sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\ldots,m_k} u_{m_1,\ldots,m_k}\right| \\
&\qquad = \left|\sum_{j\leqslant k}\left(\sum_{i_1<\ldots<i_j} \left(U_{m_{i_1},\ldots,m_{i_j}} \Delta_{i_1,\ldots,i_j} a_{m_1,\ldots,m_k}\right)\Big|_p\right)\right| \\
&\qquad \leqslant \underbrace{\sum_{j\leqslant k}\sum_{i_1<\ldots<i_j}}_{2^k-1 \text{ terms}} \frac{\varepsilon}{2^k C} \left|\Delta_{i_1,\ldots,i_j} a_{m_1,\ldots,m_k}\Big|_p\right| < \varepsilon.
\end{aligned}$$

□

Definition 1.46. Let $\{a_{m_1,\dots,m_k}(s_1,\dots,s_k)\}_{m_1,\dots,m_k=1}^{\infty}$ be a k-multiple functional sequence defined on $\Omega \subset \mathbb{C}^k$. We say that it is of uniform bounded variation if, for any number of sub-indexes $1 \leqslant j \leqslant k$ the series

$$\sum_{m_{i_1},\dots,m_{i_j}} \left|\Delta_{m_{i_1},\dots,m_{i_j}} a_{m_1,\dots,m_k}(s_1,\dots,s_k)\right|$$

converges uniformly in Ω.

Corollary 1.47. *Let $\{a_{m_1,\dots,m_k}(s_1,\dots,s_k)\}_{m_1,\dots,m_k=1}^{\infty}$ be a k-multiple sequence of bounded functions defined on $\Omega \subset \mathbb{C}^k$. If it is of uniform bounded variation then the k-multiple functional series*

$$\sum_{m_1,\dots,m_k} a_{m_1,\dots,m_k}(s_1,\dots,s_k) u_{m_1,\dots,m_k}$$

converges regularly and uniformly for any $\sum_{m_1,\dots,m_k} u_{m_1,\dots,m_k}$ a regularly convergent k-multiple series.

Proof. The argument of the proof follows the same ideas of the proof of Theorem 1.40 and the structure of Theorem 1.45. □

Chapter 2

Preliminaires on Dirichlet series

In this chapter we deal with the fundamentals of Dirichlet series which will be needed for the study of spaces of multiple Dirichlet series. In the first section the issue of convergence is treated using the results form Chapter 1, where we finally give a theorem of convergence for Dirichlet series of one complex variable. The consequences of this theorem and the definition of the absicssae of convergence are treated in the second section of this chapter, while the third one is dedicated to the definition of the space $\mathcal{H}_\infty(\mathbb{C}_+)$ and to the proof of one of most important tools that we can use when working with this space, Bohr's Theorem 2.20. Before all that, however, we need our very fundamental definition.

Definition 2.1. An ordinary Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $\{a_n\}_n \subset \mathbb{C}$ is the sequence of coefficients of the series, and $s \in \mathbb{C}$ is a complex variable.

2.1 Convergence of Dirichlet series.

The characterization of convergence factors as sequences of bounded variation is the key that is behind the traditional proof of Cahen's Theorem of convergence (see [8]), which was an improvement over the first theorem of convergence for Dirichlet series of one complex variable, published by Jensen in [22] in 1884. We use this approach to prove Cahen's theorem, so first we will need the following lemma.

Lemma 2.2. *With the notation $\sigma = \operatorname{Re} s > 0$, the the sequence $\{\frac{1}{n^s}\}$ is of uniform bounded variation on the angular region $\mathcal{S}_{0,a} = \{s \in \mathbb{C} : |\operatorname{Arg} s| < a\}$ for any $0 < a < \frac{\pi}{2}$.*

Proof.

$$\begin{aligned}\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| &= \left|\left[x^{-s}\right]_{x=n+1}^{x=n}\right| = \left|-s\int_n^{n+1} x^{-s-1}dx\right| \\ &\leqslant |s|\int_n^{n+1}\left|x^{-s-1}\right|dx = |s|\int_n^{n+1} x^{-\sigma-1}dx \\ &= |s|\left[\frac{x^{-\sigma}}{-\sigma}\right]_{x=n}^{x=n+1} = \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right).\end{aligned}$$

Therefore, for every $s \in \mathcal{S}_{0,a}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| &\leqslant \frac{|s|}{\sigma}\sum_{n=1}^{\infty}\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \\ &= \frac{|s|}{\sigma} = \frac{1}{\cos(\operatorname{Arg}(s))} < \frac{1}{\cos(a)} < 1.\end{aligned}$$

□

Theorem 2.3 ([8])**.** *Let $D(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ a Dirichlet series which converges at $s_0 \in \mathbb{C}$. Then $D(s)$ converges uniformly on the angular region $\mathcal{S}_{s_0,a} = \{s \in \mathbb{C} : |\operatorname{Arg}(s - s_0)| \leqslant a < \frac{\pi}{2}\}$.*

Proof. Let $t = s - s_0$, then the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}} \frac{1}{n^t}$$

converges at $t = 0$, that is, $\sum_{n=1} \frac{a_n}{n^{s_0}}$ is convergent. As $\{\frac{1}{n^t}\}_{n=1}^{\infty}$ is of uniform bounded variation on $\mathcal{S}_{0,a}$, by Theorem 1.38 $\sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}} \frac{1}{n^t}$ converges uniformly on $\mathcal{S}_{0,a}$, and therefore $D(s)$ converges uniformly on $\mathcal{S}_{s_0,a}$. □

A natural corollary of this theorem is Jensen's Theorem of convergence, which was actually published a decade before.

Corollary 2.4 ([22])**.** *Let* $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *a Dirichlet series which converges at* s_0*. Then* $D(s)$ *converges on* $\mathbb{C}_{\operatorname{Re} s_0}$*.*

Proof. Take $s \in \mathbb{C}_{\operatorname{Re} s_0}$, then $\operatorname{Arg}(s - s_0) < \frac{\pi}{2}$, so we can choose a such that $|\operatorname{Arg}(s - s_0)| \leqslant a < \frac{\pi}{2}$ and apply Theorem 2.3 to get the convergence of $D(s)$. □

These results and the ones that follow in this section are fundamental for the theory of Dirichlet series and can be found for example in [20]. Theorem 2.3 not only provides more information than Corollary 2.4, but this information turns out to be essential as it gives that Dirichlet series are uniformly convergent in compact sets which are included in a half-plane of convergence of the series. This fact, together with Weierstrass convergence Theorem, gives that Dirichlet series are holomorphic at any point in every half-plane in which the series is convergent. This is fundamental, since it gives rise to the study of spaces of Dirichlet series as spaces of holomorphic functions.

Corollary 2.4 also has the following immediate consequence: for a Dirichlet series which converges somewhere there is a maximum half-plane of convergence, which is given by we is called the *abscissa of convergence* of the series.

Definition 2.5. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ a Dirichlet series, we define its abscissa of convergence as

$$\sigma_c(D) = \inf\{\operatorname{Re} s \in \mathbb{C} : D(s) \text{ is convergent}\}.$$

Remark 2.6. If $\operatorname{Re} s > \sigma_c(D)$ the Dirichlet series D converges, and if $\operatorname{Re} s < \sigma_c(D)$ the Dirichlet series diverges. The convergence at the vertical line $\{s \in \mathbb{C} : \operatorname{Re} s = \sigma_c(D)\}$ cannot be determined in general, as there are Dirichlet series which converge in the whole line, while others converge in just one point or nowhere at all in that line (see [20], page 5, for examples).

We can get analogous results with respect to absolute convergence, but with much simpler proofs.

Proposition 2.7. *Let* $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be a Dirichlet series which converges absolutely at* s_0*. Then* $D(s)$ *converges absolutely on* $\mathbb{C}_{\operatorname{Re} s_0}$*.*

Proof. Take $s \in \mathbb{C}_{\operatorname{Re} s_0}$, then

$$\sum_{n=p}^{q} \left| \frac{a_n}{n^s} \right| = \sum_{n=p}^{q} \frac{|a_n|}{n^{\operatorname{Re} s}} = \sum_{n=p}^{q} \frac{|a_n|}{n^{\operatorname{Re} s_0}} \frac{1}{n^{\operatorname{Re}(s-s_0)}} \leqslant \sum_{n=p}^{q} \frac{|a_n|}{n^{\operatorname{Re} s_0}},$$

so the absolute convergence of $D(s)$ follows from the absolute convergence of $D(s_0)$. □

This last proposition motivates the definition of the *abscissa of absolute convergence*. Given that these abscissae establish maximum half-planes of certain types of convergence, it is natural to also define in the same manner the *abscissa of uniform convergence* and compare it with the other abscissae.

Definition 2.8. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ a Dirichlet series, we define its abscissa of absolute convergence as

$$\sigma_a(D) = \inf\{\operatorname{Re} s \in \mathbb{C} : D(s) \text{ is absolutely convergent}\}.$$

Definition 2.9. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ a Dirichlet series, we define the abscissa of uniform convergence as

$$\sigma_u(D) = \inf\{\sigma \in R : D(s) \text{ is uniformly convergent in } \mathbb{C}_\sigma\}.$$

There are some relations between these abscissae. First, it follows quite trivially that for any Dirichlet series D,

$$\sigma_c(D) \leqslant \sigma_u(D) \leqslant \sigma_a(D). \tag{2.1}$$

We finish this section by proving the rest of the classical inequalities involving these abscissae, leaving for the third section of this chapter the most important statement about abscissae of Dirichlet series.

Proposition 2.10. *Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ a Dirichlet series with $\{a_n\}_n$ a bounded sequence. Then D converges absolutely in $\mathbb{C}_1$.*

Proof. Suppose $|a_n| \leqslant K$ for every $n \in \mathbb{N}$ and for a certain $K > 0$. Then

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\operatorname{Re} s}} \leqslant \sum_{n=1}^{\infty} \frac{K}{n^{\operatorname{Re} s}} = K \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}},$$

where the sum on the right-hand side is convergent for every $s \in \mathbb{C}_1$. □

Corollary 2.11. *If D is any Dirichlet series then $\sigma_a(D) - \sigma_c(D) \leqslant 1$.*

Proof. Let $\delta > 0$ and consider $\tau = \sigma_c(D) + \frac{\delta}{2}$ and $s \in \overline{\mathbb{C}_\tau}$. Then

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\operatorname{Re} s+1}} \leqslant \sum_{n=1}^{\infty} \frac{|a_n|}{n^\tau} \frac{1}{n^{1+\frac{\delta}{2}}} \leqslant K \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\delta}{2}}} < \infty,$$

where the convergence of D at $\tau > \sigma_c(D)$ gives that $\{\frac{a_n}{n^\tau}\}$ is bounded by some positive constant K. □

Proposition 2.12. *If* $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *is any Dirichlet series then* $\sigma_a(D) - \sigma_u(D) \leqslant \frac{1}{2}$.

Proof. The is mainly a consequence of the Cauchy-Schwarz inequalitiy. Let $\sigma > \sigma_u(D)$ and assume $\operatorname{Re} s > \sigma + \frac{1}{2}$, so there exists $\delta > 0$ such that $\operatorname{Re} s > \sigma + \frac{1}{2} + \delta$. Then

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\operatorname{Re} s}} \leqslant \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma+\delta}}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+2\delta}}\right)^{\frac{1}{2}},$$

where $\sum_{n=1}^{\infty} n^{-1-2\delta}$ in convergent, so it is enough to check that $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}$ converges. Since $\sigma > \sigma_u(D)$, there exists $K > 0$ such that $\left|\sum_{n=1}^{N} \frac{a_n}{n^{\sigma+it}}\right| \leqslant K$ for all $N \in \mathbb{N}$, $t \in \mathbb{R}$, so

$$\begin{aligned} K^2 &\geqslant \left|\sum_{n=1}^{N} \frac{a_n}{n^{\sigma+it}}\right|^2 = \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\sigma+it}}\right)\left(\sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{\sigma-it}}\right) \\ &= \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + \sum_{1\leqslant m<n\leqslant N} \frac{a_m\overline{a_n}}{m^{\sigma+it}n^{\sigma-it}} + \sum_{1\leqslant n<m\leqslant N} \frac{a_m\overline{a_n}}{m^{\sigma+it}n^{\sigma-it}} \\ &= \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + 2\operatorname{Re} \sum_{1\leqslant m<n\leqslant N} a_m\overline{a_n}(mn)^{-\sigma}\left(\frac{n}{m}\right)^{it}. \end{aligned}$$

Using that $\lim_{T\to\infty} \int_{-T}^{T} \left(\frac{n}{m}\right)^{it} dt = \delta_{m,n}$ is enough to get that $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \leqslant K^2$ and therefore in converges, which completes the proof. □

2.2 The formulas for the abscissae

In 1915, Hardy and Riesz published in [20] the formulas for the abscissae of convergence obtained previously by Cahen, Dedekind and Jensen, and also proved by Bohr after them. These formulas let you compute the different abscissae for an ordinary Dirichlet series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ in the case in which it is satisfied that $0 < \sigma_c(D)$ in the following way:

$$\begin{aligned} \sigma_c(D) &= \limsup_{N\to\infty} \frac{\log \left|\sum_{n=1}^{N} a_n\right|}{\log N}, \\ \sigma_u(D) &= \limsup_{N\to\infty} \frac{\log \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} a_n n^{-it}\right|}{\log N}, \\ \sigma_a(D) &= \limsup_{N\to\infty} \frac{\log \sum_{n=1}^{N} |a_n|}{\log N}. \end{aligned} \tag{2.2}$$

The proofs of these formulas can be consulted for instance in [11, Proposition 1.6] or in Section 4.2 of [28]. As Hardy and Riesz point out in a footnote on page 8 of [20], in the case in which $\sigma_c(D) < 0$ there is another formula

$$\sigma_c(D) = \limsup_{N\to\infty} \frac{\log \left|\sum_{n=N}^{\infty} a_n\right|}{\log N}, \tag{2.3}$$

obtained by Pincherle, and also Knopp and Schnee. However, a formula was obtained which is valid independently from the sign of the abscissa. This formula, currently quite unknown, was obtained first by Knopp in 1911 in [23], and in 1914 it was extended to general Dirichlet series by Kojima in [24]. Although more complicated at first, Kojima's result contains the Cauchy-Hadamard formula for the radius of convergence of a power series, which endows it of special relevance. For this reason we reproduce it here, making slight modifications on the proof to try to organize it better. Before we do that, we should introduce formally the definition of general Dirichlet series and state a proper theorem of convergence for those series.

Definition 2.13. A general Dirichlet series is a series of the form

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s},$$

where $\{\lambda_m\}_m$ is an increasing sequence of real numbers diverging to $+\infty$ and s is a complex variable.

Remark 2.14. In some instances the frequency of a general Dirichlet series, that is, the sequence $\{\lambda_n\}$, is supposed to be of positive terms, since it is eventually of positive terms, given that it is increasing and divergent. Whenever we deal with general Dirichlet series throughout this text we will suppose this assumption on the frequency if necessary.

The theorem of convergence of general Dirichlet series is follows exactly the same structure that the one of the ordinary case. However, we will need an analogue of Lemma 2.2 for the general case.

Lemma 2.15. *If $\{\lambda_m\}_m$ is an increasing sequence of real numbers diverging to $+\infty$, the sequence $\{e^{-\lambda_m s}\}_m$ is of uniform bounded variation on $\mathcal{S}_a = \{s \in \mathbb{C} : |\operatorname{Arg} s| \leqslant a\}$ for every $0 < a < \frac{\pi}{2}$.*

Proof. Let $\sigma = \operatorname{Re} s$, then

$$\begin{aligned} \left|e^{-\lambda_m s} - e^{-\lambda_{m+1} s}\right| &= \left|\left[e^{-xs}\right]_{x=\lambda_m}^{x=\lambda_{m+1}}\right| = \left|-s \int_{\lambda_m}^{\lambda_{m+1}} e^{-xs} dx\right| \\ &\leqslant |s| \int_{\lambda_m}^{\lambda_{m+1}} \left|e^{-xs}\right| dx = |s| \int_{\lambda_m}^{\lambda_{m+1}} e^{x\sigma} dx \\ &= |s| \left[\frac{e^{-x\sigma}}{-\sigma}\right]_{\lambda_m}^{\lambda_{m+1}} = \frac{|s|}{\sigma} \left(e^{-\lambda_m \sigma} - e^{-\lambda_{m+1}\sigma}\right). \end{aligned}$$

□

With essentially the same proof of Theorem 2.3, we can give an analogous theorem of convergence of general Dirichlet series now using Lemma 2.15.

Theorem 2.16. *Let* $D(s) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}$ *be a general Dirichlet series which converges at* s_0*. Then* $D(s)$ *converges uniformly on the angular region* $\mathcal{S}_{s_0,a} = \{s \in \mathbb{C} : |\operatorname{Arg}(s - s_0)| \leqslant a < \frac{\pi}{2}\}$*.*

Corollary 2.17. *Let* $D(s) = \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}$ *be a general Dirichlet series which converges at* s_0*. Then* $D(s)$ *converges on* $\mathbb{C}_{\operatorname{Re} s_0}$*.*

This results also give the definition of the abscissae in the same way that have been defined for ordinary Dirichlet series. The basic inequality $\sigma_c(D) \leqslant \sigma_u(D) \leqslant \sigma_a(D)$ is still satisfied, and when these abscissae are positive we have the analogous formulae to (2.2),

$$\begin{aligned} \sigma_c(D) &= \limsup_{N\to\infty} \frac{\log \left|\sum_{n=1}^{N} a_n\right|}{\lambda_N}, \\ \sigma_u(D) &= \limsup_{N\to\infty} \frac{\log \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} a_n e^{-\lambda_N it}\right|}{\lambda_N}, \\ \sigma_a(D) &= \limsup_{N\to\infty} \frac{\log \sum_{n=1}^{N} |a_n|}{\lambda_N}. \end{aligned} \tag{2.4}$$

As before, we have another formula for when it is negative,

$$\sigma_c(D) = \limsup_{N\to\infty} \frac{\log |\sum_{n=N}^{\infty} a_n|}{\log N}. \tag{2.5}$$

This dichotomy is solved, as we announced previously, by Kojima's formula, which we prove below.

Theorem 2.18. *Let* $D(s) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ *be a general Dirichlet series. Then*

$$\sigma_c(D) = \limsup_{x\to\infty} \frac{\log \left|\sum_{[x]}^{x} a_j\right|}{x},$$

where $[x]$ *denotes the integer part of* x *and*

$$\sum_{[x]}^{x} a_j = \sum \{a_j : [x] \leqslant \lambda_j \leqslant x\}.$$

Proof. Let

$$\sigma = \limsup_{x\to\infty} \frac{\log\left|\sum_{[x]}^{x} a_j\right|}{x}, \tag{2.6}$$

and suppose that $\sigma \in \mathbb{R}$. First we will show that $\sigma_c(D) \leqslant \sigma$, that is, $\mathbb{C}_\sigma \subset \mathbb{C}_{\sigma_c(D)}$.

Suppose $s \in \mathbb{C}_\sigma$, and it is enough to consider the case in which $s \in \mathbb{R}$ since convergence of the Dirichlet series in s implies convergence in $\mathbb{C}_{\operatorname{Re} s}$. Let $\delta = s - \sigma > 0$. Since $\lim_{x\to\infty} \frac{x}{[x]} = 1 = \lim_{x\to\infty} \frac{x}{[x]+1}$, we can modify the definition of σ (2.6) to

$$\sigma = \limsup_{x\to\infty} \frac{\log\left|\sum_{[x]}^{x} a_j\right|}{[x]}, \quad \text{or} \quad \sigma = \limsup_{x\to\infty} \frac{\log\left|\sum_{[x]}^{x} a_j\right|}{[x]+1}.$$

These other formulas for σ imply that there exists some $X > 0$ such that $x > X$ gives

$$\left|\sum_{[x]}^{x} a_j\right| \leqslant e^{[x](\sigma+\frac{\delta}{2})}, \tag{2.7}$$

and

$$\left|\sum_{[x]}^{x} a_j\right| \leqslant e^{([x]+1)(\sigma+\frac{\delta}{2})}. \tag{2.8}$$

Since the sequence $\{\lambda_j\}_j$ in an increasing sequence diverging to $+\infty$ there exists some $j_0 \in \mathbb{N}$ such that $j \geqslant j_0$ implies $\lambda_j > 0$. Moreover, there exists a $j_1 \in \mathbb{N}$ such that $j \geqslant j_1$ implies $\lambda_j > X$. Now we build a partition of the sequence $\{\lambda_j\}_j$ in the well-ordered sets $B_k = \{\lambda_j : [\lambda_j] = k\} = \{\lambda_{k,1}, \ldots, \lambda_{k,r_k}\}$. With this notation, $\lambda_{k,1}$ is the smallest of the λ_j whose integer part is k, and λ_{k,r_k} is the greatest of them. This notation for the λ_j induces a notation for $\mathbb{N}$, and often throughout this proof we will represent the natural number j with the induced notation (k, r_k). By the Cauchy condition for convergence of series, it will be enough to see that given $\varepsilon > 0$ there exists some $J \in \mathbb{N}$ such that $p \geqslant J$ implies

$$\left| \sum_{j=p}^{q} a_j e^{-\lambda_j s} \right| < \varepsilon \quad \text{for every } q \geqslant p,$$

to get that the Dirichlet series converges in s. We will deal first with a particular case of this kind of sum in which $\{\lambda_p, \dots, \lambda_q\} \subset B_k$ for some $k \in \mathbb{N}$, that is, $\{\lambda_p, \dots, \lambda_q\} = \{\lambda_{k,m}, \dots, \lambda_{k,M}\}$ with $1 \leqslant m \leqslant M \leqslant r_k$. Using Lemma 1.19,

$$\begin{aligned} &\sum_{j=m}^{M} a_{k,j} e^{-\lambda_{k,j} s} \\ &\qquad = \sum_{j=m}^{M-1} \left(\sum_{n=m}^{j} a_{k,n} \right) \left(e^{-\lambda_{k,j} s} - e^{-\lambda_{k,j+1} s} \right) + \left(\sum_{j=m}^{M} a_{k,j} \right) e^{-\lambda_{k,M} s}. \end{aligned}$$

Hence,

$$\left| \sum_{j=m}^{M} a_{k,j} e^{-\lambda_{k,j} s} \right| = \sum_{j=m}^{M-1} \left| \sum_{j=m}^{n} a_{k,j} \right| \left| e^{-\lambda_{k,j} s} - e^{-\lambda_{k,j+1} s} \right| + \left| \sum_{j=m}^{M} a_{k,j} \right| e^{-\lambda_{k,M} s}.$$

Notice that we can find $x > 0$ such that $\sum_{[x]}^{x} a_j = \sum_{j=m}^{n} a_{k,j}$ for all $m \leqslant n \leqslant M$, so if $\sigma > 0$ then $s = \sigma + \delta > 0$, so using (2.7),

$$\begin{aligned} \left| \sum_{j=m}^{M} a_{k,j} e^{-\lambda_{k,j} s} \right| &= \sum_{j=m}^{M-1} \left| \sum_{j=m}^{n} a_{k,j} \right| \left(e^{-\lambda_{k,j} s} - e^{-\lambda_{k,j+1} s} \right) + \left| \sum_{j=m}^{M} a_{k,j} \right| e^{-\lambda_{k,M} s} \\ &\leqslant e^{k(\sigma + \frac{\delta}{2})} \left[\sum_{j=m}^{M-1} (e^{-\lambda_{k,j} s} - e^{-\lambda_{k,j+1} s}) + e^{-\lambda_{k,M} s} \right] \\ &= e^{k(\sigma + \frac{\delta}{2})} \left[e^{-\lambda_{k,m} s} - e^{-\lambda_{k,M} s} + e^{-\lambda_{k,M} s} \right] \\ &= e^{k(\sigma + \frac{\delta}{2})} e^{-\lambda_{k,m} s} \leqslant e^{k(\sigma + \frac{\delta}{2})} e^{-ks} = e^{-k(s - (\sigma + \frac{\delta}{2}))} \leqslant (e^{-\frac{\delta}{2}})^k. \end{aligned}$$

If $\sigma < 0$, it is enough to consider only the case in which $s = \sigma + \delta < 0$. Using (2.8),

$$\left|\sum_{j=m}^{M} a_{k,j} e^{-\lambda_{k,j} s}\right| \leqslant e^{(k+1)(\sigma+\frac{\delta}{2})} \left[\sum_{j=m}^{M-1} (e^{-\lambda_{k,j+1} s} - e^{-\lambda_{k,j} s}) + e^{-\lambda_{k,M} s}\right]$$

$$\begin{aligned} &= e^{(k+1)(\sigma+\frac{\delta}{2})} \left[e^{-\lambda_{k,M} s} - e^{-\lambda_{k,m} s} + e^{-\lambda_{k,M} s}\right] \\ &\leqslant 2e^{(k+1)(\sigma+\frac{\delta}{2})} e^{-\lambda_{k,M} s} \leqslant 2e^{k(\sigma+\frac{\delta}{2})} e^{-(k+1)s} = 2e^{-(k+1)(s-(\sigma+\frac{\delta}{2}))} \\ &\leqslant 2(e^{-\frac{\delta}{2}})^{(k+1)} \leqslant 2(e^{-\frac{\delta}{2}})^{k}. \end{aligned}$$

In either case,

$$\left|\sum_{j=m}^{M} a_{k,j} e^{-\lambda_{k,j} s}\right| \leqslant 2(e^{-\frac{\delta}{2}})^{k}.$$

Now, if $p, q \in \mathbb{N}$, $p \leqslant q$, wirte $k = [\lambda_p]$, $k + l = [\lambda_q]$ where $l \geqslant 0$, and then

$$\begin{aligned} \left|\sum_{j=p}^{q} a_j e^{-\lambda_j s}\right| &= \left|\sum_{n=0}^{l} \sum_{j=m_n}^{M_n} a_{k+n,j} e^{-\lambda_{k+n,j} s}\right| \\ &\leqslant \sum_{n=0}^{l} \left|\sum_{j=m_n}^{M_n} a_{k+n,j} e^{-\lambda_{k+n,j} s}\right| \leqslant \sum_{n=k}^{k+l} 2(e^{-\frac{\delta}{2}})^{n}. \end{aligned}$$

Since for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $k_0 \geqslant k$ implies $\sum_{n=k}^{k+l} 2(e^{-\frac{\delta}{2}})^n < \varepsilon$, and there exists a $j_2 \in \mathbb{N}$ such that $j \geqslant j_2$ implies $[\lambda_j] \geqslant k_0$, it is enough to ask that $j \geqslant \max(j_0, j_1, j_2)$ to get that $\left|\sum_{j=p}^{q} a_j e^{-\lambda_j s}\right| < \varepsilon$. Finally, this means that the Dirichlet series is convergent in $\mathbb{C}_\sigma$, implying that $\sigma_c(D) \leqslant \sigma$.

To get the other inequality, suppose $s \geqslant \sigma_c(D)$ so the general Dirichlet series $D(s) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ is convergent and therefore its partial sums are bounded, that is, there exists some $K > 0$ such that

$$|S_m| = \left|\sum_{j=1}^{m} a_j e^{-\lambda_j s}\right| < K \quad \text{for every } m \in \mathbb{N}.$$

With the alternative notation for the natural subindices, that is,

$$S_{k,j} = \sum_{n=1}^{k-1} \sum_{l=1}^{r_n} a_{n,l} e^{-\lambda_{n,l} s} + \sum_{l=1}^{j} a_{k,l} e^{-\lambda k,ls}, \quad S_{k,0} = S_{k-1,r_{k-1}},$$

and taking $x > 0$ such that $[x] = k$ and such that the set $\{j : [x] \leqslant \lambda_j \leqslant x\}$ is not empty, the for some $1 \leqslant M \leqslant r_k$,

$$\begin{aligned}\sum_{[x]}^{x} a_j = \sum_{j=1}^{M} a_{k,j} &= \sum_{j=1}^{M} (S_{k,j} - S_{k,j-1}) e^{\lambda_{k,j} s} \\ &= \sum_{j=1}^{M-1} S_{k,j} (e^{\lambda_{k,j} s} - e^{\lambda_{k,j+1} s}) + S_{k,M} e^{\lambda_{k,M} s} - S_{k,0} e^{\lambda_{k,1} s}.\end{aligned}$$

If $s < 0$, choose $\delta > 0$ such that $s + \delta < 0$, and applying the equality above to $s + \delta$,

$$\begin{aligned}\left|\sum_{[x]}^{x} a_j\right| &\leqslant K[\sum_{j=1}^{M-1} (e^{\lambda_{k,j}(s+\delta)} - e^{\lambda_{k,j+1}(s+\delta)}) + e^{\lambda_{k,M}(s+\delta)} + e^{\lambda_{k,1}(s+\delta)}] \\ &= 2K e^{\lambda_{k,1}(s+\delta)} \leqslant 2K e^{(x-1)(s+\delta)} = e^{x(s+\delta)} (2K e^{-(s+\delta)}).\end{aligned}$$

If $s \geqslant 0$, $s + \delta > 0$ and again by the equality above

$$\begin{aligned}\left|\sum_{[x]}^{x} a_j\right| &\leqslant K[\sum_{j=1}^{M-1} (e^{\lambda_{k,j+1}(s+\delta)} - e^{\lambda_{k,j}(s+\delta)}) + e^{\lambda_{k,M}(s+\delta)} + e^{\lambda_{k,1}(s+\delta)}] \\ &= 2K e^{\lambda_{k,M}(s+\delta)} \leqslant 2K e^{x(s+\delta)} = e^{x(s+\delta)} (2K).\end{aligned}$$

In any case, if $C = 2K\max(1, e^{-(s+\delta)})$, $\left|\sum_{[x]}^{x} a_j\right| \leqslant Ce^{x(s+\delta)}$, so

$$\sigma = \limsup_{x\to\infty} \frac{\log\left|\sum_{[x]}^{x} a_j\right|}{x} \leqslant \limsup_{x\to\infty} \frac{\log C}{x} + s + \delta = s + \delta,$$

where $\delta > 0$ is small enough but arbitrary, so $\sigma \leqslant s$ for all $s \in \mathbb{C}_{\sigma_c(D)}$, giving that $\sigma \leqslant \sigma_c(D)$. □

As we said before, we can deduce from Theorem 2.18 the Cauchy-Hadamard formula for power series. Indeed, if we choose $\lambda_m = m$ for every $m \in \mathbb{N}$, then the general Dirichlet series that you get is $D(s) = \sum_{m=1}^{\infty} a_m (e^{-s})^m$. Writing $z = e^{-s}$, we can see $D(s)$ as $f(z) = \sum_{m=1}^{\infty} a_m z^m$. For this particular choice of λ_m it is clear that, for any $x > 0$, $\sum_{[x]}^{x} a_j = a_{[x]}$, so

$$\begin{aligned}\sigma_c(D) &= \limsup_{m\to\infty} \frac{\log|a_m|}{m}\\ &= \limsup_{m\to\infty} \log\sqrt[m]{|a_m|} = \log\limsup_{m\to\infty}\sqrt[m]{|a_m|}.\end{aligned}$$

Therefore, since

$$\begin{aligned}\sup\{|z| : f(z) \text{ is convergent}\} &= \sup\{e^{-\operatorname{Re} s} : D(s) \text{ is convergent}\}\\ &= e^{\inf\{s: D(s) \text{ is convergent}\}} = e^{\sigma_c(D)},\end{aligned}$$

then the radius of convergence is

$$\rho(f) = e^{\sigma_c(D)} = \limsup_{m\to\infty}\sqrt[m]{a_m}.$$

2.3 The Banach algebra $\mathcal{H}_\infty(\mathbb{C}_+)$.

This section is dedicated to the study of the space $\mathcal{H}_\infty(\mathbb{C}_+)$, so first of all we give its definition.

Definition 2.19. $\mathcal{H}_\infty(\mathbb{C}_+)$ is the space of all Dirichlet D series which are convergent on $\mathbb{C}_+$ and define a bounded holomorphic function there, where $\mathbb{C}_+$ denotes the half-plane of complex numbers with positive real part.

Endowed with the norm

$$||D||_\infty = \sup_{\operatorname{Re} s>0} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right|,$$

the space $\mathcal{H}_\infty(\mathbb{C}_+)$ is a Banach algebra which was introduced in [21] by Hedenmalm, Lindqvist and Seip as the space of multipliers of $\mathcal{H}^2$, where $\mathcal{H}^2$ is the space of Dirichlet series whose sequence of coefficients is in ℓ^2. The fact that $\mathcal{H}_\infty(\mathbb{C}_+)$ is a Banach space is a consequence of the fundamental result due to Bohr ([7] or [28, Theorem 6.2.3, p. 145]) which states that every Dirichlet series bounded on $\mathbb{C}_+$ is uniformly aproximated by its partial sums in $\mathbb{C}_\delta$ for every $\delta > 0$. The main theorem of this section is a quantitative version of that result, explicitly found in [12] (see Remark 1.22), which is an improvement of [11, Lemma 6]. Before we give this theorem, we recall the space $H_\infty(\mathbb{C}_+)$ of bounded holomorphic functions f defined on the positive half-plane $\mathbb{C}_+$, which is also a Banach space with the norm

$$||f||_\infty = \sup_{\operatorname{Re} s>0} |f(s)|.$$

Since we have said earlier, Dirichlet series define holomorphic functions in their half-plane of convergence, so $\mathcal{H}_\infty(\mathbb{C}_+)$ is a subspace of $H_\infty(\mathbb{C}_+)$, and Bohr's result is the key to prove it is a Banach subspace.
In the version of the theorem we present below we wanted to put special emphasis on the fact that we can approximate the limit function of a Dirichlet series by its partial sums with an error that depends on the norm of the limit function, a suitable term regarding the index of

the partial sum and a factor that only depends on the distance to the imaginary axis, ε.

Theorem 2.20. *Let* $D(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ *be a Dirichlet series such that* $D \in \mathcal{H}_\infty(\mathbb{C}_+)$. *Then, for every* $\varepsilon > 0$ *there exists* $c_\varepsilon > 0$, *only depending on* ε, *such that, if* $\operatorname{Re} s > \varepsilon$,

$$\left| f(s) - \sum_{n=1}^{M} \frac{b_n}{n^s} \right| \leqslant c_\varepsilon \frac{\log M}{M^\varepsilon} ||f||_\infty$$

Proof. Viewing s, with $\operatorname{Re} s > \varepsilon$, and M as fixed for the moment, consider integrating

$$\frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}$$

as a function of z around the rectangular contour shown (Figure 2.1), considering that $\sigma_a \leqslant 1$, so if $\operatorname{Re} s \geqslant \varepsilon + 1$ the series is absolutely convergent. Since f is analytic in $\mathbb{C}_+$ and $\operatorname{Re} s - \varepsilon > 0$, we have that by Cauchy's integral formula this integral is exactly $2\pi f(s)$.

We will bound now:
(I) The integral over the left-hand edge.

$$\begin{aligned}\int_{s-\varepsilon-iM^3}^{s-\varepsilon+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s} dz &= \int_{-M^3}^{M^3} \frac{f(s-\varepsilon+it)(M+\frac{1}{2})^{s-\varepsilon+it-s}}{s-\varepsilon+it-s} idt \\ &= \int_{-M^3}^{M^3} \frac{f(s-\varepsilon+it)(M+\frac{1}{2})^{-\varepsilon+it}}{-\varepsilon+it} idt.\end{aligned}$$

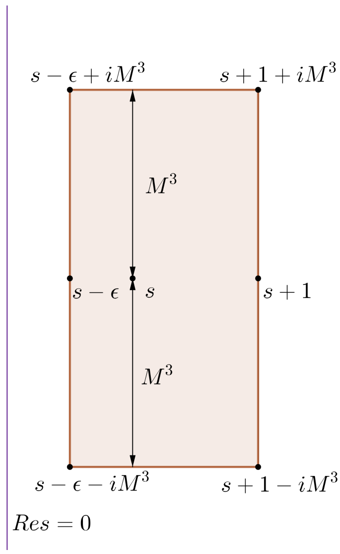

Figure 2.1 Integration contour

Then,

$$\left|\int_{s-\varepsilon-iM^3}^{s-\varepsilon+iM^3}\frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz\right|\leqslant\int_{-M^3}^{M^3}\frac{|f(s-\varepsilon+it)|(M+\frac{1}{2})^{\mathrm{Re}(-\varepsilon+it)}}{|-\varepsilon+it|}dt$$
$$\leqslant\sup\{|f(z)|:z\in[s-\varepsilon-iM^3,s-\varepsilon+iM^3]\}\int_{-M^3}^{M^3}\frac{(M+\frac{1}{2})^{\mathrm{Re}(-\varepsilon)}}{\sqrt{t^2+\varepsilon^2}}dt$$
$$\leqslant||f||_\infty M^{-\varepsilon}\int_{-M^3}^{M^3}\frac{dt}{\sqrt{t^2+\varepsilon^2}};$$

where, taking $M \geqslant 3$,

$$
\begin{aligned}
\int_{-M^3}^{M^3} \frac{dt}{\sqrt{t^2+\varepsilon^2}} &= 2\int_0^{M^3} \frac{dt}{\sqrt{t^2+\varepsilon^2}} \leqslant 2\sqrt{2}\int_0^{M^3} \frac{dt}{t+\varepsilon} \\
&\leqslant 2\sqrt{2}\left[\int_0^{\varepsilon} \frac{dt}{t+\varepsilon} + \int_{\varepsilon}^{M^3} \frac{dt}{t}\right] = 2\sqrt{2}\left[\log 2\varepsilon + \log M^3 - 2\log\varepsilon\right] \\
&= 2\sqrt{2}\left[\log 2 + 3\log M - \log\varepsilon\right] = 2\sqrt{2}\left[\log\frac{2}{\varepsilon} + 3\log M\right] \\
&\leqslant 2\sqrt{2}\left[\log\frac{2}{\varepsilon}\log M + 3\log M\right] = 2\sqrt{2}\log M\left[\log\frac{2}{\varepsilon} + 3\right].
\end{aligned}
$$

Defining $K_1(\varepsilon) := 2\sqrt{2}\left[\log\frac{2}{\varepsilon} + 3\right]$, which depends only on ε, we get that

$$
\left|\int_{s-\varepsilon-iM^3}^{s-\varepsilon+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz\right| \leqslant ||f||_\infty \frac{\log M}{M^\varepsilon} K_1(\varepsilon).
$$

(II) The top and bottom edges.

$$
\begin{aligned}
&\int_{s-\varepsilon+iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz \\
&\qquad = \int_{-\varepsilon+\operatorname{Re} s}^{1+\operatorname{Re} s} \frac{f(t+i(M^3+\operatorname{Im} s))(M+\frac{1}{2})^{t+i(M+\operatorname{Im} s)-s}}{t+i(M^3+\operatorname{Im} s)-s}dt \\
&\qquad = \int_{-\varepsilon+\operatorname{Re} s}^{1+\operatorname{Re} s} \frac{f(t+i(M^3+\operatorname{Im} s))(M^3+\frac{1}{2})^{t+i(M^3+\operatorname{Im} s)-s}}{t+i(M^3+\operatorname{Im} s)-s}dt.
\end{aligned}
$$

Then;

$$\begin{aligned}
&\left|\int_{s-\varepsilon+iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz\right| \\
&\qquad \leqslant \int_{-\varepsilon+\operatorname{Re}s}^{1+\operatorname{Re}s} ||f||_\infty \frac{(M+\frac{1}{2})^{t-\operatorname{Re}s}}{\sqrt{(t-\operatorname{Re}s)^2+(M^3)^2}}dt \\
&\qquad \leqslant ||f||_\infty \int_{-\varepsilon+\operatorname{Re}s}^{1+\operatorname{Re}s} \frac{(M+\frac{1}{2})^{t-\operatorname{Re}s}}{M^3}dt \\
&\qquad = ||f||_\infty M^{-3} \int_{-\varepsilon+\operatorname{Re}s}^{1+\operatorname{Re}s} (M+\frac{1}{2})^{t-\operatorname{Re}s}dt;
\end{aligned}$$

where, again assuming $M \geqslant 3$,

$$\begin{aligned}
\int_{-\varepsilon+\operatorname{Re}s}^{1+\operatorname{Re}s} (M+\frac{1}{2})^{t-\operatorname{Re}s}dt &= \int_{-\varepsilon+\operatorname{Re}s}^{1+\operatorname{Re}s} e^{(t-\operatorname{Re}s)\log(M+\frac{1}{2})}dt \\
&= \frac{e^{\log(M+\frac{1}{2})}}{\log(M+\frac{1}{2})} - \frac{e^{-\varepsilon\log(M+\frac{1}{2})}}{\log(M+\frac{1}{2})} \\
&\leqslant \frac{(M+\frac{1}{2})}{\log(M+\frac{1}{2})} \leqslant M+\frac{1}{2};
\end{aligned}$$

so we get that

$$\left|\int_{s-\varepsilon+iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz\right| \leqslant ||f||_\infty M^{-3}(M+\frac{1}{2}) \leqslant ||f||_\infty \frac{2}{M^{-2}}.$$

Proceeding analogously we get the same bound for the bottom side of the integral. Now, using the Residue Theorem,

$$\begin{aligned}
2\pi i f(s) = &\int_{s-\varepsilon-iM^3}^{s+1-iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz + \int_{s+1-iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz \\
&+ \int_{s+1+iM^3}^{s-\varepsilon+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz + \int_{s-\varepsilon+iM^3}^{s-\varepsilon-iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz,
\end{aligned}$$

and therefore

$$\left|2\pi i f(s) - \int_{s+1-iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz\right|$$
$$\leqslant ||f||_\infty \left\{4M^{-2} + K_1(\varepsilon)\frac{\log M}{M^\varepsilon}\right\}$$
$$\leqslant ||f||_\infty \frac{\log M}{M^\varepsilon}\left\{\frac{4}{M^{2-\varepsilon}\log M} + K_1(\varepsilon)\right\}.$$

As $\lim_{M\to\infty} \frac{4}{M^{2-\varepsilon}\log M} = 0$, the sequence is bounded by a certain $K_2(\varepsilon)$, which only depends on ε, so

$$\begin{aligned}\left|2\pi i f(s) - \int_{s+1-iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz\right| \\ \leqslant ||f||_\infty \frac{\log M}{M^\varepsilon}\left\{K_2(\varepsilon) + K_1(\varepsilon)\right\}.\end{aligned} \tag{2.9}$$

As it was stated before, the Dirichlet series $\sum_{n=1}^\infty \frac{b_n}{n^s}$ is absolutely convergent for $s \in [s+1-iM^3, s+1+iM^3]$, so it is uniformly convergent there and

$$\int_{s+1-iM^3}^{s+1+iM^3} \frac{f(z)(M+\frac{1}{2})^{z-s}}{z-s}dz = \int_{s+1-iM^3}^{s+1+iM^3} \sum_{n=1}^\infty \frac{b_n}{n^{z+s-s}} \frac{(M+\frac{1}{2})^{z-s}}{z-s}dz$$
$$= \sum_{n=1}^\infty \frac{b_n}{n^s} \int_{s+1-iM^3}^{s+1+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz.$$

We will now work with this expression, differentiating when $n \geqslant M+1$ and $n \leqslant M$, and we will apply in each case the Residue Theorem as we did before.

First case: $n \geqslant M+1$.

We are going to integrate over the following rectangular contour from

Figure 2.2, and we will later take the limit when $T \to \infty$.
Using the Residue Theorem, as $g(z) = \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}$ is analytic on the contour on which we are integrating, we have that

$$\int_{s+1-iM^3}^{s+1+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz = \int_{s+1-iM^3}^{s+T-iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz$$
$$+ \int_{s+T-iM^3}^{s+T+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz + \int_{s+T+iM^3}^{s+1+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz.$$

We are going to bound now:
(1) The right-hand edge.

$$\begin{aligned}
&\left| \int_{s+T-iM^3}^{s+T+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz \right| \\
&\qquad \leqslant \int_{s+T-iM^3}^{s+T+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{\mathrm{Re}(z-s)} \frac{1}{|z-s|}dz \\
&\qquad \leqslant 2M^3 \underbrace{\left(\frac{M+\frac{1}{2}}{n}\right)}_{<1}^{T} \frac{1}{T} \xrightarrow{T\to\infty} 0,
\end{aligned}$$

so, as we are considering n and M are fixed, given a positive δ, $\delta < \frac{8}{nM}$, then we can find T_1 so when $T > T_1$, then

$$\left| \int_{s+T-iM^3}^{s+T+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}dz \right| < \frac{8}{nM}.$$

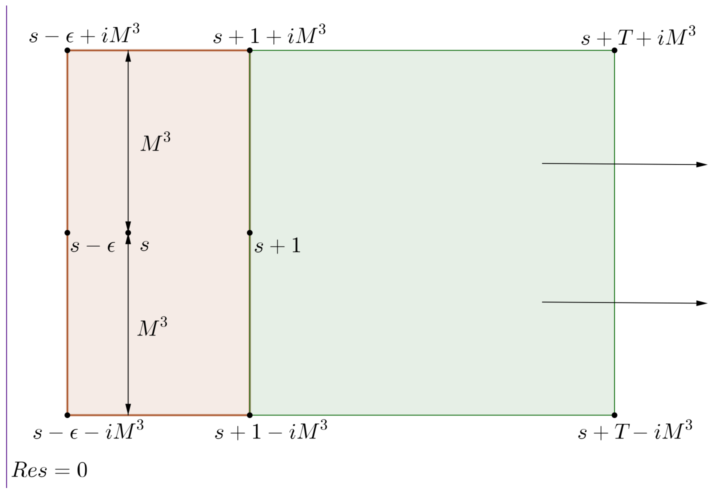

Figure 2.2 New integration contour

(2) The top and bottom edges.

$$\begin{aligned}
&\left|\int_{s+1-iM^3}^{s+T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\right| \\
&\leqslant \left|\int_0^T\left(\frac{M+\frac{1}{2}}{n}\right)^{1+u}\frac{1}{|1+u+iM^3|}du\right| \leqslant \left|\int_0^T\left(\frac{M+\frac{1}{2}}{n}\right)^{1+u}\frac{1}{M^3}du\right| \\
&= M^{-3}\left|\int_0^T\left(\frac{M+\frac{1}{2}}{n}\right)^{1+u}du\right| \leqslant M^{-3}\left[\frac{e^{\log\frac{M+\frac{1}{2}}{n}}}{\log\frac{M+\frac{1}{2}}{n}} - \underbrace{\frac{e^{(1+T)\log\frac{M+\frac{1}{2}}{n}}}{\log\frac{M+\frac{1}{2}}{n}}}_{\to 0 \text{ as } T\to\infty}\right],
\end{aligned}$$

so, again, as we are considering n and M to be fixed, we can choose $\delta < M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1}$, and find a T_0 such that, if $T > T_0$ then

$$\left|\int_{s+1-iM^3}^{s+T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\right| \leqslant 2M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1}.$$

The value of $\log \frac{M+\frac{1}{2}}{n}$ is smaller when $\frac{M+\frac{1}{2}}{n}$ is closer to one, that is, when $n = M+1$, so

$$\left|\log \frac{M+\frac{1}{2}}{n}\right|^{-1} \leqslant \left|\log \frac{M+\frac{1}{2}}{M+1}\right|^{-1} = \left(-\log\left(1-\frac{1}{2M+2}\right)\right)^{-1} \leqslant 2M+2$$

and then

$$\left|\int_{s+1-iM^3}^{s+T-iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s} dz\right| \leqslant 2M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)(2M+2) \leqslant \frac{8}{nM}.$$

Therefore, recalling that, as we have absolute convergence on $\operatorname{Re} s+1$, $|b_n| \leqslant ||f||_\infty$,

$$\sum_{n\geqslant M+1} \frac{|b_n|}{n^{\operatorname{Re} s}} \left|\int_{s+1-iM^3}^{s+1+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s} dz\right| \leqslant ||f||_\infty \frac{24}{M} \sum_{n\geqslant M+1} \frac{1}{n^{1+\varepsilon}}.$$

Define $K_3(\varepsilon) := 24\sum_{n\geqslant 1}\frac{1}{n^{1+\varepsilon}}$, and $K_4(\varepsilon)$ such that

$$\frac{1}{M^{1-\varepsilon}\log M} \leqslant K_4(\varepsilon) \quad \text{for every } M \in \mathbb{N},$$

then we get that

$$\sum_{n\geqslant M+1} \frac{|b_n|}{n^{\operatorname{Re} s}} \left|\int_{s+1-iM^3}^{s+1+iM^3} \left(\frac{M+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s} dz\right| \leqslant ||f||_\infty K_3(\varepsilon)K_4(\varepsilon)\frac{\log M}{M^\varepsilon}. \quad (2.10)$$

Second case: $n \leqslant M$.

Now we will apply again the Residue Theorem, but this time the integral over the rectangular contour shown picks a contribution of $2\pi i$ with

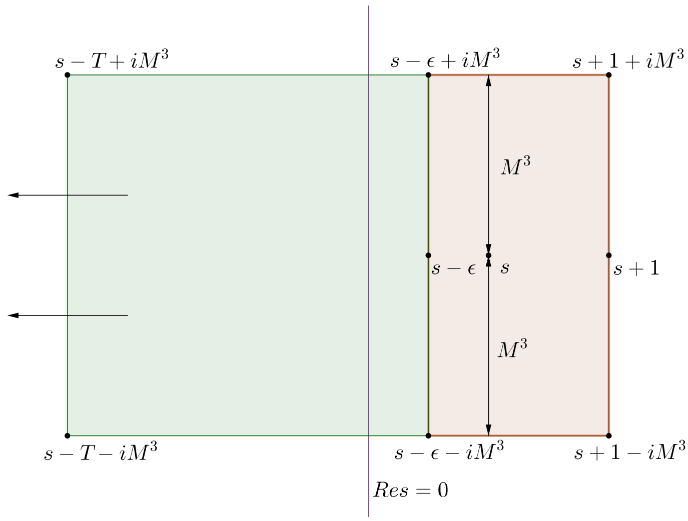

Figure 2.3 New integration contour

residue 1, so

$$\begin{aligned}&\int_{s+1-iM^3}^{s+1+iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz+\int_{s+1+iM^3}^{s-T+iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\\&+\int_{s-T+iM^3}^{s-T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz+\int_{s-T-iM^3}^{s+1-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\\&=2\pi i.\end{aligned}$$

We will bound three of these integrals as we did before:
(i) The left-hand edge.

$$\left|\int_{s-T+iM^3}^{s-T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\right|\leqslant 2M^{-3}\underbrace{\left(\frac{M+\frac{1}{2}}{n}\right)}_{>1}^{-T}\frac{1}{T}\xrightarrow{T\to\infty}0,$$

Then, it exists T_2 such that, if $T\geqslant T_2$, then

$$\left|\int_{s-T+iM^3}^{s-T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\right|\leqslant\frac{12}{nM}.$$

(ii) The top and bottom edges.

$$\begin{aligned}\left|\int_{s+1-iM^3}^{s-T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\right|&\leqslant\left|\int_{\operatorname{Re}s-T}^{1+\operatorname{Re}s}\left(\frac{M+\frac{1}{2}}{n}\right)^{u-\operatorname{Re}s}\frac{1}{M^3}du\right|\\&=M^{-3}\left|\frac{e^{\log\frac{M+\frac{1}{2}}{n}}}{\log\frac{M+\frac{1}{2}}{n}}-\underbrace{\frac{e^{-T\log\frac{M+\frac{1}{2}}{n}}}{\log\frac{M+\frac{1}{2}}{n}}}_{\to 0\text{ as }T\to\infty}\right|;\end{aligned}$$

and again, choosing the same $\delta<M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1}$, and the same T_0 as before, if $T>T_0$ then

$$\left|\int_{s+1-iM^3}^{s-T-iM^3}\left(\frac{M+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}dz\right|\leqslant 2M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1}.$$

Now, $\log\frac{M+\frac{1}{2}}{n}$ is smaller when $n=M$, and

$$\begin{aligned}2M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1}&\leqslant 2M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\left(\log(1+\frac{1}{2M})\right)^{-1}\\&\leqslant 2M^{-3}\left(\frac{M+\frac{1}{2}}{n}\right)\frac{1+\frac{1}{2M}}{\frac{1}{2M}}\leqslant\frac{12}{nM}.\end{aligned}$$

Therefore,

$$\left|\int_{s+1+iM^3}^{s-T+iM^3}g(z)dz+\int_{s-T+iM^3}^{s-T-iM^3}g(z)dz+\int_{s-T-iM^3}^{s+1-iM^3}g(z)dz\right|\leqslant\frac{36}{nM}$$

and then,

$$\left|\sum_{n=1}^{M} \frac{b_n}{n^s}\left(\int_{s+1+iM^3}^{s-T+iM^3} g(z)dz + \int_{s-T+iM^3}^{s-T-iM^3} g(z)dz + \int_{s-T-iM^3}^{s+1-iM^3} g(z)dz\right)\right|$$
$$\leqslant ||f||_\infty \frac{36}{M}\sum_{n=1}^{M}\frac{1}{n^{\mathrm{Re}\, s+1}} \leqslant ||f||_\infty \frac{36}{M}\sum_{n=1}^{M}\frac{1}{n^{\varepsilon+1}}.$$

Setting $K_5(\varepsilon) := 36\sum_{n\geqslant 1}\frac{1}{n^{\varepsilon+1}}$, and remembering the definition for $K_4(\varepsilon)$ we get that

$$\left|\int_{s+1+iM^3}^{s-T+iM^3} g(z)dz + \int_{s-T+iM^3}^{s-T-iM^3} g(z)dz + \int_{s-T-iM^3}^{s+1-iM^3} g(z)dz\right|$$
$$\leqslant ||f||_\infty K_5(\varepsilon)K_4(\varepsilon)\frac{\log M}{M^\varepsilon}. \quad (2.11)$$

Now, using (2.9), (2.10) and (2.11), we get the final conclusion:

$$
\begin{aligned}
&\left| f(s) - \sum_{n=1}^{M} \frac{b_n}{n^s} \right| \\
&\quad = \left| 2\pi i f(s) - 2\pi i \sum_{n=1}^{M} \frac{b_n}{n^s} \right| \frac{1}{2\pi} \\
&\quad = \frac{1}{2\pi} \left| 2\pi i f(s) - \left(\int_{s+1-iM^3}^{s+1+iM^3} g(z)dz + \int_{s+1+iM^3}^{s-T+iM^3} g(z)dz \right.\right. \\
&\qquad\qquad \left.\left. + \int_{s-T+iM^3}^{s-T-iM^3} g(z)dz + \int_{s-T-iM^3}^{s+1-iM^3} g(z)dz \right) \sum_{n=1}^{M} \frac{b_n}{n^s} \right| \\
&\quad \leqslant \frac{1}{2\pi} \left| 2\pi i f(s) - \sum_{n=1}^{M} \frac{b_n}{n^s} \int_{s+1-iM^3}^{s+1+iM^3} g(z)dz \right| \\
&\qquad + \frac{1}{2\pi} ||f||_\infty K_5(\varepsilon) K_6(\varepsilon) \frac{\log M}{M^\varepsilon} \\
&\quad = \frac{1}{2\pi} \left| 2\pi i f(s) - \sum_{n=1}^{\infty} \frac{b_n}{n^s} \int_{s+1-iM^3}^{s+1+iM^3} g(z)dz \right| \\
&\qquad + \frac{1}{2\pi} \left| \sum_{n=M+1}^{\infty} \frac{b_n}{n^s} \int_{s+1-iM^3}^{s+1+iM^3} g(z)dz \right| \\
&\qquad + \frac{1}{2\pi} ||f||_\infty K_5(\varepsilon) K_6(\varepsilon) \frac{\log M}{M^\varepsilon} \\
&\quad \leqslant \frac{1}{2\pi} ||f||_\infty \left(K_1(\varepsilon) + K_2(\varepsilon) + K_3(\varepsilon) K_4(\varepsilon) + K_5(\varepsilon) K_6(\varepsilon) \right) \\
&\quad = ||f||_\infty c_\varepsilon \frac{\log M}{M^\varepsilon}.
\end{aligned}
$$

□

As we explained above, this is a quantitative version of Bohr's original result. To enunciate this result, first we need to give the following definition.

Definition 2.21. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ a Dirichlet series, we define its abscissa of boundedness as

$$\sigma_b(D) = \inf\{\sigma \in \mathbb{C} : D(s) \text{ extends to a bounded function in } \mathbb{C}_\sigma\}.$$

Theorem 2.22 (Bohr's Theorem)**.** *Let $D(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ be a Dirichlet series. Then $\sigma_u(D) = \sigma_b(D)$.*

Remark 2.23. The proof of Theorem 2.22 can be given following the same lines of the proof of Theorem 2.20, although Bohr's original result only assumes absolute convergence in a suficiently remote half-plane, and the hypothesis of the limit function f being in $\mathcal{H}_\infty(\mathbb{C}_+)$ is necessary if you want to get the approximation error in terms of the norm of the function, $||f||_\infty$. Nevertheless, if our hypothesis is that D is a Dirichlet series that extends as a bounded holomorphic function to $\mathbb{C}_+$, then by Theorem 2.22 $D \in \mathcal{H}_\infty(\mathbb{C}_+)$, and then we can apply Theorem 2.20 to get the bound for the approximation error.

As we mentioned when we introduced the space $\mathcal{H}_\infty(\mathbb{C}_+)$, this space is not only a vector space but a Banach algebra, and this is a remarkable consequence of Bohr's result. Proving this will be the final aim of this section, but first we will need some auxiliary results. These results and their proofs can also be found in [12] (see Propositions 1.11 and 1.19), on in Chapter 4 of [28].

Proposition 2.24.

$$\sum_{n=1}^{N} |a_n|^2 = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n n^{it} \right| dt$$

Proof.

$$\begin{aligned}\lim_{R\to+\infty}\frac{1}{2R}\int_{-R}^{R}&\left|\sum_{n=1}^{N}a_n n^{it}\right|dt\\ &=\lim_{R\to+\infty}\frac{1}{2R}\int_{-R}^{R}\left(\sum_{n=1}^{N}a_n n^{it}\right)\left(\sum_{m=1}^{N}\bar{a}_m m^{-it}\right)dt\\ &=\sum_{m,n=1}^{N}a_n\bar{a}_m+\left(\lim_{R\to+\infty}\frac{1}{2R}\int_{-R}^{R}n^{it}m^{-it}dt\right)\\ &=\sum_{m,n=1}^{N}a_n\bar{a}_m\delta_{m,n}=\sum_{n=1}^{N}|a_n|^2.\end{aligned}$$

□

Proposition 2.25. *For every Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}_+)$, we have that*

$$\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{\frac{1}{2}}\leqslant\left|\left|\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right|\right|_\infty.$$

In particular, $|a_n|\leqslant\left|\left|\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right|\right|_\infty$.

Proof. We fix $\varepsilon>0$. By Bohr's Theorem 2.20,

$$\sup_{\operatorname{Re}s>\varepsilon}\left|f(s)-\sum_{n=1}^{M}\frac{a_n}{n^s}\right|\leqslant c_\varepsilon\frac{\log M}{M^\varepsilon}||f||_\infty,$$

so there is a M_0 such that $M\geqslant M_0$ implies

$$\sup_{\operatorname{Re}s>\varepsilon}\left|f(s)-\sum_{n=1}^{M}\frac{a_n}{n^s}\right|\leqslant\varepsilon.$$

Fix $\sigma>0$ such that $\operatorname{Re}\sigma>\varepsilon$, and then

$$\sup_{t\in\mathbb{R}}\left|\sum_{n=1}^{M}\frac{a_n}{n^{\sigma+it}}\right|\leqslant||f||_\infty+\varepsilon,$$

and applying Proposition 2.24, for $M \geqslant M_0$,

$$\left(\sum_{n=1}^{N}\left|\frac{a_n}{n^\sigma}\right|^2\right)^{\frac{1}{2}} = \left(\lim_{R\to+\infty}\frac{1}{2R}\int_{-R}^{R}\left|\sum_{n=1}^{N}\frac{a_n}{n^{\sigma+it}}\right|^2\right)^{\frac{1}{2}} \leqslant ||f||_\infty + \varepsilon.$$

Hence if σ and ε both tend to 0, the claimed inequality is proved. □

Theorem 2.26. *$\mathcal{H}_\infty(\mathbb{C}_+)$ is a Banach complex space.*

Proof. Since $H_\infty(\mathbb{C}_+)$ is a Banach space, we just have to show that $\mathcal{H}_\infty(\mathbb{C}_+)$ is closed in $H_\infty(\mathbb{C}_+)$. Take a sequence $\{D^{(k)}\}_k = \left\{\sum_{n=1}^{\infty}\frac{a_n^{(k)}}{n^s}\right\}_k$ in $\mathcal{H}_\infty(\mathbb{C}_+)$ and assume that it converges to f in $H_\infty(\mathbb{C}_+)$. It has to be shown that $f \in \mathcal{H}_\infty(\mathbb{C}_+)$. By Proposition 2.25, for every k, l, and all n,

$$|a_n^{(k)} - a_n^{(l)}| \leqslant ||D^{(k)} - D^{(l)}||_\infty.$$

Hence for each fixed n, the sequence $\{a_n^{(k)}\}_k$ is Cauchy and therefore converges to some a_n. Even more, for each $\varepsilon > 0$ there exists k_0 such that

$$|a_n^{(k)} - a_n| < \varepsilon \text{ for every } k \geqslant k_0 \text{ and every } n.$$

Define the Dirichlet series $D(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$. Our aim now is to prove that $f = D$ on some appropriate half-plane, so we can use Bohr's Theorem 2.20 to extend convergence to $\mathbb{C}_+$. To do that we again apply Proposition 2.25 which assures that for every k and n we have $|a_n^{(k)}| \leqslant ||D^{(k)}||_\infty$. Taking limits in n we see that the boundedness of $\{D^{(k)}\}_k$ in $\mathcal{H}_\infty(\mathbb{C}_+)$ implies the boundedness of $\{a_n\}$ in $\mathbb{C}$, hence $\sigma_a(D) \leqslant 1$. We finally prove that $f = D$ on $\mathbb{C}_1$. To do this we take some $s \in \mathbb{C}_1$ and $\varepsilon > 0$, then we can find a fixed $k \geqslant k_=$ such that

$$|f(s) - D^{(k)}(s)| < \varepsilon \quad \text{and} \quad |a_n^{(k)} - a_n| < \varepsilon \text{ for all } n.$$

Moreover, we can choose $N \in \mathbb{N}$ large enough such that

$$\left| D^{(k)}(s) - \sum_{n=1}^{N} a_n^{(k)} \frac{1}{n^s} \right| < \varepsilon \quad \text{and} \quad \left| D(s) - \sum_{n=1}^{N} a_n \frac{1}{n^s} \right| < \varepsilon.$$

Then

$$|f(s) - D(s)| \leqslant |f(s) - D^{(k)}(s)| + \left| D^{(k)}(s) - \sum_{n=1}^{N} a_n^{(k)} \frac{1}{n^s} \right|$$
$$+ \left| \sum_{n=1}^{N} a_n^{(k)} \frac{1}{n^s} - \sum_{n=1}^{N} a_n \frac{1}{n^s} \right| + \left| \sum_{n=1}^{N} a_n \frac{1}{n^s} - D(s) \right| \leqslant 2\varepsilon + \varepsilon \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}} + \varepsilon.$$

As $s \in \mathbb{C}_1$, $\sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}}$ is convergent and we can say that $f = D$ in $\mathbb{C}_1$, so applying Bohr's Theorem 2.20, $D \in \mathcal{H}_\infty(\mathbb{C}_+)$ and the proof is finished. □

Remark 2.27. As we advanced $\mathcal{H}_\infty(\mathbb{C}_+)$ is a Banach algebra: if we consider the Dirichlet series $D_1(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $D_2(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \in \mathcal{H}_\infty(\mathbb{C}_+)$, and say they converge to the analytic bounded functions f and g respectively on $\mathbb{C}_+$, then

$$D(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad \text{where} \quad c_n = \sum_{kj=n} a_k b_j$$

is the *Dirichlet product* of the series D_1 and D_2 and it converges in some half-plane to the product fg, which is bounded and analytic on $\mathbb{C}_+$, so it follows from Theorem 2.20 to the $D \in \mathcal{H}_\infty(\mathbb{C}_+)$.

Chapter 3

Convergence of multiple Dirichlet series

In this chapter we introduce multiple Dirichlet series and we study their convergence. Let us properly define what we call a multiple Dirichlet series.

Definition 3.1. An ordinary multiple (or k-multiple) Dirichlet series is a series of the form

$$\sum_{m_1,\ldots,m_k=1}^{\infty} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}},$$

where $\{a_{m_1,\ldots,m_k}\} \subset \mathbb{C}$ is the sequence of coefficents of the series, and $s_1, \ldots, s_k \in \mathbb{C}$ are complex variables.

In this chapter our main goals are to obtain a theorem of regular convergence for multiple Dirichlet series that replicates the structure of Theorem 2.3 and to obtain the sets of regular convergence of multiple Dirichlet series, where now the dimensional jump gives a completely different situation. To achieve the first goal we will use the results from Section 1.3, more precisely Corollary 1.47, to which we will need to add the following lemma.

Lemma 3.2. *For every* $0 \leqslant a < \frac{\pi}{2}$*, the* k*-multiple sequence*

$$\{m_1^{-s_1}, \dots, m_k^{-s_k}\}_{m_1,\dots,m_k=1}^{\infty}$$

is of uniform bounded variation on $\mathcal{S}_{a,0}^k = \{(s_1, \dots, s_k) : |\operatorname{Arg} s_i| \leqslant a\}$*.*

Proof. Fix some $0 \leqslant a < \frac{\pi}{2}$. Taking into account that $0 < \cos a < 1$ and Lemma 2.2,

$$\begin{aligned}
\sum_{m_{i_1}=n_{i_1}}^{p_{i_1}} \cdots \sum_{m_{i_j}=n_{i_j}}^{p_{i_j}} &| \Delta_{m_{i_1},\dots,m_{i_j}}(m_1^{-s_1}, \dots, m_k^{-s_k})| \\
&= \left| \frac{1}{m_{l_1}^{s_{l_1}} \cdots m_{l_{k-j}}^{s_{l_{k-j}}}} \right| \prod_{r=1}^{j} \sum_{m_{i_r}=n_{i_r}}^{p_{i_r}} \left| \frac{1}{m_{i_r}^{s_{i_r}}} - \frac{1}{(m+1)_{i_r}^{s_{i_r}}} \right| \\
&\leqslant \prod_{r=1}^{j} \sum_{m_{i_r}=1}^{\infty} \frac{|s_{i_r}|}{\sigma_{i_r}} \left(\frac{1}{m_{i_r}^{\sigma_{i_r}}} - \frac{1}{(m+1)_{i_r}^{\sigma_{i_r}}} \right) \\
&\leqslant \prod_{r=1}^{j} \frac{1}{\cos(\operatorname{Arg}(s_{i_r}))} \leqslant \frac{1}{\cos(a)^j} \leqslant \frac{1}{\cos(a)^k} \leqslant 1.
\end{aligned}$$

□

Now we can give the announced extension of Theorem 2.3 and its natural corollary.

Theorem 3.3. *Let* $D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1},\dots,m_k^{s_k}}$ *be a* k*-multiple Dirichlet series which converges regularly at* $(z_1, \dots, z_k)$*. Then it converges regularly and uniformly on the angular region* $\mathcal{S}_{a,z}^k = \{(s_1, \dots, s_k) : |Arg(s_i - z_i)| \leqslant a < \frac{\pi}{2}\}$*.*

Proof. Let $t_i = s_i - z_i$, then the k-multiple Dirichlet series

$$\sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{t_1}, \dots, m_k^{t_k}}$$

converges regularly at $(0,\dots,0)$, that is, $\sum_{m_1,\dots,m_k} a_{m_1,\dots,m_k}$ is regularly convergent. Using Lemma 3.2, $\{m_1^{-s_1},\dots,m_k^{-s_k}\}_{m_1,\dots,m_k=1}^{\infty}$ is of uniform bounded variation on $\mathcal{S}_{a,0}^k$, so $D(t_1,\dots,t_k)$ converges regularly and uniformly on $\mathcal{S}_{a,0}^k$, and therefore $D(s_1,\dots,s_k)$ converges regularly and uniformly on $\mathcal{S}_{a,z}^k$. □

Corollary 3.4. *Let* $D(s_1,\dots,s_k) = \sum_{m_1,\dots,m_k} \frac{b_{m_1,\dots,m_k}}{m_1^{s_1},\dots,m_k^{s_k}}$ *a* k*-multiple Dirichlet series which converges regularly at* $(z_1,\dots,z_k)$*. Then it converges regularly on the product of complex half-planes* $\mathbb{C}_{\operatorname{Re} z_1}\times\cdots\times\mathbb{C}_{\operatorname{Re} z_k}$.

We should point out here that Corollary 3.4 does not characterize the sets of regular convergence of multiple Dirichlet series: as multiple Dirichlet series have several complex variables, we cannot take infima to define abscissae of convergence unless we make some assumptions. This is illustrated in [25], where it is stated that *associated abscissae of regular convergence* can be defined, and they are related by a convex function. This related abscissae, which characterize the sets of regular convergence of double Dirichlet series, will be studied in Section 3.2.
Regarding the holomorphy of the multiple Dirichlet series, the same argument used in the one variable case applies: Theorem 3.3 and Weierstrass convergence Theorem are enough to assure that multiple Dirichlet series define holomorphic functions in the region where they are convergent, which gives rise to the study of spaces of multiple Dirichlet series. However, before we begin that study we should give some results that generalize the relationship (2.1) for multiple Dirichlet series. Their proofs can be deduced easily from the ones for Dirichlet series, but we give some indications of how to proceed.

3.1 Absolute, uniform and regular convergence

When studying the absolute convergence of k-multiple Dirichlet series we get the natural extension of Proposition 2.7, which as expected has the same form of Corollary 3.4.

Proposition 3.5. *If a k-multiple Dirichlet series converges absolutely in a point $(z_1, \dots, z_k)$, then it converges absolutely on $\mathbb{C}_{\operatorname{Re} z_1} \times \cdots \times \mathbb{C}_{\operatorname{Re} z_k}$.*

Proof. It is enough to note that, for $(s_1, \dots, s_k) \in \mathbb{C}_{\operatorname{Re} z_1} \times \cdots \times \mathbb{C}_{\operatorname{Re} z_k}$,

$$\sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} \frac{|a_{m_1,\dots,m_k}|}{m_1^{\operatorname{Re} s_1} \cdots m_k^{\operatorname{Re} s_k}} \leqslant \sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} \frac{|a_{m_1,\dots,m_k}|}{m_1^{\operatorname{Re} z_1} \cdots m_k^{\operatorname{Re} z_k}},$$

Indeed, this is enough because of Remark 1.15. □

Another relation between the regular and the absolute convergence of a k-multiple Dirichlet series is given by the following proposition, which extends Proposition 2.10 to the k-dimensional case.

Proposition 3.6. *Let $D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$ a k-multiple Dirichlet series, and $\{a_{m_1,\dots,m_k}\}$ a bounded k-multiple sequence. Then $D(s_1, \dots, s_k)$ converges absolutely (and therefore, regularly) on $\mathbb{C}_1^k$.*

Proof. This follows easily from the fact that $\sum_{m_1,\dots,m_k} \frac{1}{m_1^{s_1} \cdots m_k^{s_k}}$ converges absolutely if and only if $\operatorname{Re} s_j > 1$ for each $1 \leqslant j \leqslant k$. □

Corollary 3.7. *If a k-multiple Dirichlet series*

$$D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$$

converges regularly at a point $(z_1, \dots, z_k)$, then it converges absolutely on $\mathbb{C}_{\operatorname{Re} z_1+1} \times \cdots \times \mathbb{C}_{\operatorname{Re} z_k+1}$.

Proof. Making a simple change of variable

$$\sum_{m_1,\ldots,m_k} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}} = \sum_{m_1,\ldots,m_k} \frac{a_{m_1,\ldots,m_k}}{m_1^{z_1}\cdots m_k^{z_k}} \frac{1}{m_1^{s_1-z_1}\cdots m_k^{s_k-z_k}},$$

where $\{\frac{a_{m_1,\ldots,m_k}}{m_1^{z_1}\cdots m_k^{z_k}}\}_{m_1,\ldots,m_k=1}^{\infty}$ is bounded, given the regular convergence of the series. □

Trivially, if a k-multiple Dirichlet series $D(s_1,\ldots,s_k)$ converges absolutely on $\mathbb{C}_{\sigma_1}\times\cdots\times\mathbb{C}_{\sigma_k}$, then it converges regularly and uniformly on $\mathbb{C}_{\sigma_1}\times\cdots\times\mathbb{C}_{\sigma_k}$, but there is also a relation between the absolute and the uniform convergence of a k-multiple Dirichlet series that extends Proposition 2.12.

Proposition 3.8. *If a k-multiple Dirichlet series converges regularly and uniformly on $\mathbb{C}_{\sigma_1}\times\cdots\times\mathbb{C}_{\sigma_k}$, then it converges absolutely on $\mathbb{C}_{\sigma_1+\frac{1}{2}}\times\cdots\times\mathbb{C}_{\sigma_k+\frac{1}{2}}$.*

We will give the proof of the 2-dimensional case, which illustrates the differences in the proof between the one variable Dirichlet series case and the multiple Dirichlet series case.

Proof. By the Cauchy- Schwartz inequality,

$$\sum_{k,l=1}^{\infty} \frac{|a_{k,l}|}{k^{\sigma_1+\frac{1}{2}+\varepsilon} l^{\sigma_2+\frac{1}{2}+\varepsilon}} \leqslant \left(\sum_{k,l=1}^{\infty} \frac{|a_{k,l}|^2}{k^{2\sigma_1+\varepsilon} l^{2\sigma_2+\varepsilon}}\right)^{\frac{1}{2}} \left(\sum_{k,l=1}^{\infty} \frac{1}{k^{1+\varepsilon} l^{1+\varepsilon}}\right)^{\frac{1}{2}},$$

and the second series on the right-hand side is convergent, so we only need to prove the convergence of the first series of the right-hand side. As we have uniform convergence of the Dirichlet series on $\mathbb{C}_{\sigma_1}\times\mathbb{C}_{\sigma_2}$, then all the partial sums are bounded by a constant M, so

$$
\begin{aligned}
M^2 \geqslant & \left|\sum_{k_1=1}^{m}\sum_{l=1}^{n}\frac{a_{k,l}}{k^{\sigma_1+\varepsilon+i\tau_1}l^{\sigma_2+\varepsilon+i\tau_2}}\right|^2 = \sum_{k=1}^{m}\sum_{l=1}^{n}\frac{|a_{k,l}|^2}{k^{2\sigma_1+\varepsilon}l^{2\sigma_2+\varepsilon}} \\
& + 2\operatorname{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{kl}}}{i^{\sigma_1+\varepsilon+i\tau_1}j^{\sigma_2+\varepsilon+i\tau_2}k^{\sigma_1+\varepsilon-i\tau_1}l^{\sigma^2+\varepsilon-i\tau_2}} \\
& + 2\operatorname{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{il}\bar{a_{kj}}}{i^{\sigma_1+\varepsilon+i\tau_1}l^{\sigma_2+\varepsilon+i\tau_2}k^{\sigma_1+\varepsilon-i\tau_1}j^{\sigma^2+\varepsilon-i\tau_2}} \\
& + 2\operatorname{Re}\sum_{i=1}^{m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{il}}}{i^{\sigma_1+\varepsilon+i\tau_1}j^{\sigma_2+\varepsilon+i\tau_2}i^{\sigma_1+\varepsilon-i\tau_1}l^{\sigma^2+\varepsilon-i\tau_2}} \\
& + 2\operatorname{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{j=1}^{n}\frac{a_{ij}\bar{a_{kj}}}{i^{\sigma_1+\varepsilon+i\tau_1}j^{\sigma_2+\varepsilon+i\tau_2}k^{\sigma_1+\varepsilon-i\tau_1}j^{\sigma^2+\varepsilon-i\tau_2}} \\
& = \sum_{k=1}^{m}\sum_{l=1}^{n}\frac{|a_{k,l}|^2}{k^{2\sigma_1+\varepsilon}l^{2\sigma_2+\varepsilon}} \\
& + 2\operatorname{Re}\left[\sum_{1\leqslant i<k\leqslant m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{kl}}+a_{il}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}\left(\frac{i}{k}\right)^{i\tau_1}\left(\frac{j}{l}\right)^{i\tau_2}}\right] \\
& + 2\operatorname{Re}\left[\sum_{i=1}^{m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{il}}}{i^{2\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}\left(\frac{j}{l}\right)^{i\tau_2}}\right] \\
& + 2\operatorname{Re}\left[\sum_{1\leqslant i<k\leqslant m}\sum_{j=1}^{n}\frac{a_{ij}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}j^{2\sigma_2+\varepsilon}\left(\frac{i}{k}\right)^{i\tau_1}}\right].
\end{aligned}
$$

We take the average value by integrating with respect to τ_1, τ_2 to get

$$
\begin{aligned}
M^2 \geqslant & \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{|a_{k,l}|^2}{k^{2\sigma_1+\varepsilon}l^{2\sigma_2+\varepsilon}}d\tau_1 d\tau_2 \\
& + \frac{1}{4T^2}2\operatorname{Re}\int_{-T}^{T}\int_{-T}^{T}\sum_{1\leqslant i<k\leqslant m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{kl}}+a_{il}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}\left(\frac{i}{k}\right)^{i\tau_1}\left(\frac{j}{l}\right)^{i\tau_2}}d\tau_1 d\tau_2 \\
& + \frac{1}{4T^2}2\operatorname{Re}\int_{-T}^{T}\int_{-T}^{T}\sum_{i=1}^{m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{il}}}{i^{2\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}\left(\frac{j}{l}\right)^{i\tau_2}}d\tau_1 d\tau_2
\end{aligned}
$$

$$+\frac{1}{4T^2}2\,\mathrm{Re}\int_{-T}^{T}\int_{-T}^{T}\sum_{1\leqslant i<k\leqslant m}\sum_{j=1}^{n}\frac{a_{ij}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}j^{2\sigma_2+\varepsilon}(\frac{i}{k})^{i\tau_1}}d\tau_1 d\tau_2$$

$$=\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{|a_{k,l}|^2}{k^{2\sigma_1}l^{2\sigma_2}}$$

$$+2\,\mathrm{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{kl}}+a_{il}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}}\frac{1}{4T^2}\int_{-T}^{T}(\frac{i}{k})^{i\tau_1}d\tau_1\int_{-T}^{T}(\frac{j}{l})^{i\tau_2}d\tau_2$$

$$+2\,\mathrm{Re}\sum_{i=1}^{m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{il}}}{i^{2\sigma_1+\varepsilon}(jl)^{\sigma+\varepsilon 2}}\frac{1}{2T}\int_{-T}^{T}(\frac{j}{l})^{i\tau_2}d\tau_2$$

$$+2\,\mathrm{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{j=1}^{n}\frac{a_{ij}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}j^{2\sigma_2+\varepsilon}}\frac{1}{2T}\int_{-T}^{T}(\frac{i}{k})^{i\tau_1}d\tau_1.$$

Calculating the integrals, taking into account that the integers in every quotient are always different, we get that

$$M^2\geqslant\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{|a_{k,l}|^2}{k^{2\sigma_1+\varepsilon}l^{2\sigma_2+\varepsilon}}$$

$$+2\,\mathrm{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{kl}}+a_{il}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}}\frac{\sin(T\log(\frac{i}{k}))\sin(T\log(\frac{j}{l}))}{T^2\log(\frac{i}{k})\log(\frac{j}{l})}$$

$$+2\,\mathrm{Re}\sum_{i=1}^{m}\sum_{1\leqslant j<l\leqslant n}\frac{a_{ij}\bar{a_{il}}}{i^{2\sigma_1+\varepsilon}(jl)^{\sigma_2+\varepsilon}}\frac{\sin(T\log(\frac{j}{l}))}{T\log(\frac{j}{l})}$$

$$+2\,\mathrm{Re}\sum_{1\leqslant i<k\leqslant m}\sum_{j=1}^{n}\frac{a_{ij}\bar{a_{kj}}}{(ik)^{\sigma_1+\varepsilon}j^{2\sigma_2+\varepsilon}}\frac{\sin(T\log(\frac{i}{k}))}{T\log(\frac{i}{k})}.$$

It suffices to take limits when $T\to\infty$ to get that all partial sums

$$\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{|a_{k,l}|^2}{k^{2\sigma_1+\varepsilon}l^{2\sigma_2+\varepsilon}}$$

are bounded, and therefore the series is convergent (because all of its terms are positive) and the proof is complete. □

3.2 Convergence formulae

In Tetsuzô Kojima's last work [25], published one year after his death, the author performs an exhaustive study of the convergence sets of double general Dirichlet series. After introducing the notion of regular convergence, he proves a first theorem of regular convergence of double general Dirichlet series which contains Theorem 3.3 for the double ordinary case. Throughout the following pages he works to characterize the sets of regular convergence for those series via some convergence formulae, which extend his previous work published in [24] and reproduced here in Theorem 2.18. The interest of this work resides in the remarkable differences between the one variable and the two variables case, since the double case turns out to be far more complex. For instance, consider a double Dirichlet series which is defined as the product of two Dirichlet series, one in the variable s and the other one in the variable t. Since the convergence of each series is independent from the other one, the region of regular convergence of the double Dirichlet series is clearly the product of complef half-planes defined by the corresponding absicissae of convergence of the one variable series. However, these kind of sets, which naturally extend the one dimensional case, are not the only ones possible in the double case, as we will see with some other examples in the end of this section.

Unfortunately, Kojima's work probably was not noticed for a few years, given that other authors such as Leja and Adams independently obtained and published some results very realted to or partially included in the ones found in [25] (the details on such publications can be found in Adams' survey [1]). In this section we will reproduce here Kojima's characterization of sets of regular convergence for double general Dirichlet series, and we will add some new convergence formulae for the ordinary case which can be adapted for the general double case for some particular frequencies which grow at a pace very similar to the the pace at which the sequence $\{\log n\}_n$ grows.

Let us start by setting the necessary background for double general Dirichlet series. Theorem 3.3 can be also proven for the general case, but before we show how, let us introduce the formal definition of what we mean by a double general Dirichlet series.

Definition 3.9. A double general Dirichlet series is a series of the form

$$\sum_{m,n=1}^{\infty} a_{m,n} e^{-\lambda_m s - \mu_n t},$$

where $\{\lambda_m\}_m$ and $\{\mu_n\}$ are increasing sequences of real numbers diverging to $+\infty$ and s and t are complex variables.

To use the arguments of Theorem 3.3 in the case of double general Dirichlet series we would need a lemma that plays the role of Lemma 3.2 for the general case.

Lemma 3.10. *If $\{\lambda_m\}_m$ and $\{\mu_n\}$ are increasing sequences of real numbers diverging to $+\infty$, the double sequence $\{e^{-\lambda_m s - \mu_n t}\}_{m,n=1}^{\infty}$ is of uniform bounded variation on $\mathcal{S}^2_{a,0} = \{(s,t) : |\operatorname{Arg} s| \leqslant a, |\operatorname{Arg} t| \leqslant a\}$ for every $0 < a < \frac{\pi}{2}$.*

Proof. First note that, for $s \in \mathbb{C}$ such that $|\operatorname{Arg} s| \leqslant a$,

$$\begin{aligned}
|\Delta_m e^{-\lambda_m s}| = |e^{-\lambda_m s} - e^{\lambda_{m+1}}| &\leqslant \left| \int_{\lambda_m}^{\lambda_{m+1}} -s e^{-xs} dx \right| \\
&= \left| \int_{e^{\lambda_m}}^{e^{\lambda_{m+1}}} -s u^{-s-1} du \right| \leqslant |s| \int_{e^{\lambda_m}}^{e^{\lambda_{m+1}}} u^{-\operatorname{Re} s - 1} du \\
&= |s| \left[\frac{u^{-\operatorname{Re} s}}{-\operatorname{Re} s} \right]_{u=e^{\lambda_m}}^{u=e^{\lambda_{m+1}}} = \frac{|s|}{\operatorname{Re} s} (e^{-\lambda_m s} - e^{-\lambda_{m+1} s})
\end{aligned}$$

so taking into account that $0 < \cos a < 1$,

$$\sum_{m=p}^{q} |\Delta_m e^{-\lambda_m s}| \leqslant \sum_{m=p}^{q} \frac{|s|}{\operatorname{Re} s}(e^{-\lambda_m s} - e^{-\lambda_{m+1} s})$$
$$= \frac{|s|}{\operatorname{Re} s}(e^{-\lambda_p s} - e^{-\lambda_{q+1} s}) \leqslant \frac{|s|}{\operatorname{Re} s} = \frac{1}{\cos(\arg(s))} \leqslant 1.$$

Analogously $\sum_{m=p}^{q} |\Delta_n e^{-\mu_n s}| \leqslant 1$ and

$$\sum_{m=p}^{q} \sum_{n=p'}^{q'} \left|\Delta_{m,n} e^{-\lambda_m s - \mu_n t}\right| = \sum_{m=p}^{q} |\Delta_m e^{-\lambda_m s}| \sum_{n=p'}^{q'} |\Delta_n e^{-\mu_n s}| \leqslant 1.$$

□

Now we can enunciate a theorem for convergence of general double Dirichlet series, whose prove is basically the same than the one of Theorem 3.3, but using Lemma 3.10.

Theorem 3.11. *Let $D(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} e^{-\lambda_m s - \mu_n t}$ be a general double Dirichlet series which converges regularly at (z_1, z_2). Then it converges regularly and uniformly on the angular region $\mathcal{S}^2_{a,z} = \{(s_1, s_2) \in \mathbb{C}^2 : |Arg(s_i - z_i)| \leqslant a\}$ for every $0 \leqslant a < \frac{\pi}{2}$.*

As we said earlier, the jump from the theorem of regular convergence for double Dirichlet series to the characterization of their sets of regular convergence is not immediate, in contrast to the one variable case. We will reproduce here Kojima's work on this matter, providing shorter proofs for his main results.

Theorem 3.12. *Let $D(s,t) = \sum_{i,j=1}^{\infty} a_{i,j} e^{-\lambda_i s - \mu_j t}$ be a formal general double Dirichlet series, where $\{\lambda_i\}$ and $\{\mu_j\}$ are strictly increasing sequences divergent to $+\infty$ which we will suppose to be of positive terms. Write*

$$\varphi(\alpha) = \limsup_{x+y\to\infty} \frac{\log\left(e^{-y\alpha} \left|\sum_{\lfloor x\rfloor}^{x} \sum_{\lfloor y\rfloor}^{y} a_{i,j}\right|\right)}{x+y} \tag{3.1}$$

where we assume that x and y are positive, increasing and independent variables, and

$$\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j} = \sum\{a_{i,j} \,:\, \lfloor x\rfloor \leqslant \lambda_i \leqslant x, \lfloor y\rfloor \leqslant \mu_j \leqslant y\},$$

where $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to x. Then the double Dirichlet series converges regularly in the set

$$R(D) = \{(s,t) \in \mathbb{C}^2_+ \,:\, \textit{there exists } \alpha \in \mathbb{R} \textit{ such that}$$
$$\varphi(\alpha) \in \mathbb{R},\ \mathrm{Re}\, s > \varphi(\alpha),\ \mathrm{Re}\, t > \alpha + \varphi(\alpha)\}.$$

Remark 3.13. Note that $\lim_{x+y\to\infty} e^{\frac{\alpha}{x+y}(\lfloor y\rfloor - y)} = 1$ for any $\alpha \in \mathbb{R}$, which implies that

$$\varphi(\alpha) = \limsup_{x+y\to\infty} \frac{\log\left(e^{-\lfloor y\rfloor\alpha}\left|\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}\right|\right)}{x+y}.$$

Proof. By Theorem 3.11 it is enough to prove that D converges regularly in $R(D)\cap\mathbb{R}^2$. Suppose $(s,t) \in R(D)\cap\mathbb{R}^2$, that is, there exist $\alpha \in \mathbb{R}$ and $\delta > 0$ arbitrarily small such that $s > \varphi(\alpha) + \delta$ and $t > \alpha + \varphi(\alpha) + \delta$. Since $\varphi(\alpha) + \frac{\delta}{2} > \varphi(\alpha)$ and

$$\lim_{x+y\to\infty} \frac{\lfloor x\rfloor + \lfloor y\rfloor}{x+y} = \lim_{x+y\to\infty} \frac{\lfloor x\rfloor + \lfloor y\rfloor + 1}{x+y} = \lim_{x+y\to\infty} \frac{\lfloor x\rfloor + \lfloor y\rfloor + 2}{x+y} = 1,$$

then there exists $X_0 > 0$ such that $x + y > X_0$ implies

$$\left|\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}\right| < e^{(\varphi(\alpha)+\frac{\delta}{2})(\lfloor x\rfloor + \lfloor y\rfloor) + \lfloor y\rfloor\alpha}, \tag{3.2}$$

$$\left|\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}\right| < e^{(\varphi(\alpha)+\frac{\delta}{2})(\lfloor x\rfloor + \lfloor y\rfloor + 1) + \lfloor y\rfloor\alpha}, \tag{3.3}$$

$$\left|\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}\right| < e^{(\varphi(\alpha)+\frac{\delta}{2})(\lfloor x\rfloor+\lfloor y\rfloor+2)+\lfloor y\rfloor\alpha}. \tag{3.4}$$

Define the sets $B_k = \{\lambda_i : \lfloor\lambda_i\rfloor = k\}$ and $B'_l = \{\mu_j : \lfloor\mu_j\rfloor = l\}$ and, by ordering them, consider the following notation: $B_k = \{\lambda_{k,1} \ldots, \lambda_{k,r_k}\}$, $B'_l = \{\mu_{l,1}, \ldots, \mu_{l,r'_l}\}$, so that $\{\lambda_i\}_{i\in\mathbb{N}} = \bigcup_{k\in\mathbb{N}_0} B_k$, $\{\mu_j\}_{j\in\mathbb{N}} = \bigcup_{l\in\mathbb{N}_0} B'_l$. Extend that notation to the coefficients, where $a_{k,i;l,j}$ is the coefficient that goes with $e^{-\lambda_{k,i}s-\mu_{l,j}t}$. Suppose that $k,l \in \mathbb{N}_0$ are fixed and let $1 \leqslant m \leqslant M \leqslant r_k$, $1 \leqslant n \leqslant N \leqslant r'_l$. The regular convergence of D will follow from the following claim:

$$\left|\sum_{i=m}^{M}\sum_{j=n}^{N} a_{k,i;l,j}e^{-\lambda_{k,i}s-\mu_{l,j}t}\right| \leqslant 16(e^{-\frac{\delta}{2}})^{(k+l)}. \tag{3.5}$$

Indeed, given $\varepsilon > 0$ we should find $P_0 > 0$ such that $\max(P,P') > P_0$ implies

$$\left|\sum_{i=P}^{Q}\sum_{j=P'}^{Q'} a_{i,j}e^{-\lambda_i s-\mu_j t}\right| < \varepsilon, \quad \text{for all } Q, Q' \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty}(e^{-\frac{\delta}{2}})^k$ is a convergent series of positive terms, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty}(e^{-\frac{\delta}{2}})^k < \frac{\varepsilon}{16}(1-e^{-\frac{\delta}{2}}). \tag{3.6}$$

Let P_1 be the smallest integer such that index of $\lfloor\lambda_{P_1}\rfloor = k_0$ and P'_1 the smallest integer such that index of $\lfloor\mu_{P'_1}\rfloor = k_0$ and define $P_0 = \max(P_1, P'_1)$. Suppose that, for $P, P', Q, Q' \in \mathbb{N}$, $\lfloor\lambda_P\rfloor = k$, $\lfloor\mu_{P'}\rfloor = l$,

$\lfloor \lambda_Q \rfloor = u$ and $\lfloor \mu_{Q'} \rfloor = v$. Then,

$$\begin{aligned} \left| \sum_{i=P}^{Q} \sum_{j=P'}^{Q'} a_{i,j} e^{-\lambda_i s + \mu_j t} \right| &\leqslant \sum_{g=k}^{u} \sum_{h=l}^{v} \left| \sum_{i=m_g}^{M_g} \sum_{j=n_h}^{N_h} a_{g,i;h,j} e^{-\lambda_{g,i} s - \mu_{h,j} t} \right| \\ &\leqslant 16 \sum_{g=k}^{u} \sum_{h=l}^{v} (e^{-\frac{\delta}{2}})^{(g+h)} \\ &\leqslant 16 \left(\sum_{g=k}^{\infty} (e^{-\frac{\delta}{2}})^k \right) \left(\sum_{h=l}^{\infty} (e^{-\frac{\delta}{2}})^l \right), \end{aligned} \tag{3.7}$$

where $1 \leqslant m_g \leqslant M_g \leqslant r_g$ and $1 \leqslant n_h \leqslant N_h \leqslant r'_h$ for all $k \leqslant g \leqslant v$, $l \leqslant h \leqslant v$. If we suppose that $\max(P, P') \geqslant P_0 = \max(P_1, P'_1)$, one of the sums in the right hand side of (3.7) satisfies (3.6), whereas the other one is smaller that $\frac{1}{1-e^{-\frac{\delta}{2}}}$. Hence,

$$\left| \sum_{i=P}^{Q} \sum_{j=P'}^{Q'} a_{i,j} e^{-\lambda_i s + \mu_j t} \right| < 16 \frac{\varepsilon}{16} (1 - e^{-\frac{\delta}{2}}) \frac{1}{1 - e^{-\frac{\delta}{2}}} = \varepsilon,$$

so the double Dirichlet series converges regularly in (s, t).

Let us prove (3.5) to conclude. With the notation

$$A_{k,i;l,j} = \sum_{g=1}^{i} \sum_{h=1}^{j} a_{k,g;l,h}, \qquad A_{k,0;l,j} = 0 = A_{k,i;l,0},$$

we have

$$\begin{aligned} &\sum_{i=m}^{M} \sum_{j=n}^{N} a_{k,i;l,j} e^{-\lambda_{k,i} s - \mu_{l,j} t} \\ &\quad = \sum_{i=m}^{M} \sum_{j=n}^{N} (A_{k,i;l,j} - A_{k,i-1;l,j} - A_{k,i;l,j-1} + A_{k,i-1;l,j-1}) e^{-\lambda_{k,i} s - \mu_{l,j} t} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=m}^{M}\sum_{j=n}^{N} A_{k,i;l,j} e^{-\lambda_{k,i}s-\mu_{l,j}t} - \sum_{i=m-1}^{M-1}\sum_{j=n}^{N} A_{k,i;l,j} e^{-\lambda_{k,i+1}s-\mu_{l,j}t} \\
&\quad - \sum_{i=m}^{M}\sum_{j=n-1}^{N-1} A_{k,i;l,j} e^{-\lambda_{k,i}s-\mu_{l,j+1}t} + \sum_{i=m-1}^{M-1}\sum_{j=n-1}^{N-1} A_{k,i;l,j} e^{-\lambda_{k,i+1}s-\mu_{l,j+1}t} \\
&= \sum_{i=m}^{M-1}\sum_{j=n}^{N-1} A_{k,i;l,j} (e^{-\lambda_{k,i}s} - e^{-\lambda_{k,i+1}s})(e^{-\mu_{l,j}t} - e^{-\mu_{l,j+1}t}) \\
&\quad + \sum_{i=m}^{M-1} (A_{k,i;l,N} e^{-\mu_{l,N}t} - A_{k,i;l,n-1} e^{-\mu_{l,n}t})(e^{-\lambda_{k,i}s} - e^{-\lambda_{k,i+1}s}) \\
&\quad + \sum_{j=n}^{N-1} (A_{k,M;l,j} e^{-\lambda_{k,M}s} - A_{k,m-1;l,j} e^{-\lambda_{k,m}s})(e^{-\mu_{l,j}t} - e^{-\mu_{l,j+1}t}) \\
&\quad + A_{k,M;l,N} e^{-\lambda_{k,M}s-\mu_{l,N}t} - A_{k,m-1;l,N} e^{-\lambda_{k,m}s-\mu_{l,N}t} \\
&\quad - A_{k,M;l,n-1} e^{-\lambda_{k,M}s-\mu_{l,n}t} + A_{k,m-1;l,n-1} e^{-\lambda_{k,m}s-\mu_{l,Nn}t}.
\end{aligned}
$$

We will study four different cases depending on the sign of $\varphi(\alpha)$ and $\alpha+\varphi(\alpha)$. First, let us suppose that $\varphi(\alpha) \geqslant 0$ and $\alpha+\varphi(\alpha) \geqslant 0$, so that both $s, t > 0$. Then, since $k \leqslant \lambda_{k,i} < k+1$ for all $i \in \{1, \ldots, r_k\}$,

$$
e^{-\lambda_{k,i}s} \leqslant e^{-ks} \quad \text{for} \quad 1 \leqslant i \leqslant r_k. \tag{3.8}
$$

Analogously,

$$
e^{-\mu_{l,j}t} \leqslant e^{-lt} \quad \text{for} \quad 1 \leqslant j \leqslant r'_l. \tag{3.9}
$$

Note that for any $1 \leqslant i \leqslant r_k$ and $1 \leqslant j \leqslant r'_l$ we can find $x, y > 0$ so that $A_{k,i;l,j} = \sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}$. Then, by choosing them large enough, (3.2) gives that, for k and l large enough and any $1 \leqslant i \leqslant r_k$, $1 \leqslant j \leqslant r'_l$,

$$
|A_{k,i;l,j}| < e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l)+l\alpha}. \tag{3.10}
$$

Using (3.8), (3.9) and (3.10),

$$
\begin{aligned}
&\left|\sum_{i=m}^{M}\sum_{j=n}^{N} a_{k,i;l,j} e^{-\lambda_{k,i}s-\mu_{l,j}t}\right| \\
&\quad \leqslant e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l)+l\alpha}\Bigg\{\sum_{i=m}^{M-1}\sum_{i=n}^{N-1}|e^{-\lambda_{k,i}s}-e^{-\lambda_{k,i+1}s}||e^{-\mu_{l,j}t}-e^{-\mu_{l,j+1}t}| \\
&\qquad + \left(\sum_{i=m}^{M-1}|e^{-\lambda_{k,i}s}-e^{-\lambda_{k,i+1}s}|\right)\left(e^{-\mu_{l,N}t}+e^{-\mu_{l,n}t}\right) \\
&\qquad + \left(e^{-\lambda_{k,M}s}+e^{-\lambda_{k,m}s}\right)\left(\sum_{i=n}^{N-1}|e^{-\mu_{l,j}t}-e^{-\mu_{l,j+1}t}|\right) \\
&\qquad + \left(e^{-\lambda_{k,M}s}+e^{-\lambda_{k,m}s}\right)\left(e^{-\mu_{l,N}t}+e^{-\mu_{l,n}t}\right)\Bigg\} \\
&\quad \leqslant e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l)+l\alpha}\left(\sum_{i=m}^{M-1}|e^{-\lambda_{k,i}s}-e^{-\lambda_{k,i+1}s}|+e^{-\lambda_{k,M}s}+e^{-\lambda_{k,m}s}\right) \\
&\qquad \left(\sum_{i=n}^{N-1}|e^{-\mu_{l,j}t}-e^{-\mu_{l,j+1}t}|+e^{-\mu_{l,N}t}+e^{-\mu_{l,n}t}\right) \\
&\quad \leqslant 4e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l)+l\alpha}\left(e^{-\lambda_{k,M}s}+e^{-\lambda_{k,m}s}\right)\left(e^{-\mu_{l,N}t}+e^{-\mu_{l,n}t}\right) \\
&\quad \leqslant 4e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l)+l\alpha}(2e^{-ks})(2e^{-lt}) \\
&\quad \leqslant 16e^{-k(s-(\varphi(\alpha)+\frac{\delta}{2}))}e^{-l(t-(\varphi(\alpha)+\alpha+\frac{\delta}{2}))} \leqslant 16(e^{-\frac{\delta}{2}})^{(k+l)}.
\end{aligned}
$$

If now $\varphi(\alpha) \geqslant 0$ but $\alpha + \varphi(\alpha) < 0$, we will only prove the claim for $(s,t) \in R(D) \cap \mathbb{R}^2$ in which $t < 0$, and by Theorem 3.11 the regular convergence of such points will imply the regular convergence of the Dirichlet series at the points $(s,t) \in R(D) \cap \mathbb{R}^2$ in which $s > \varphi(\alpha) > 0$ and $\alpha + \varphi(\alpha) < 0 \leqslant t$. By proceeding analogously to the previous case, we can find $x, y > 0$ large enough so that

$$|A_{k,i;l,j}| < e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l+1)+l\alpha}, \tag{3.11}$$

for k and l large enough and all $1 \leqslant i \leqslant r_k$, $1 \leqslant j \leqslant r'_l$. Producing the adequate modification of (3.9),

$$e^{-\mu_{l,j}t} \leqslant e^{-(l+1)t} \quad \text{for} \quad 1 \leqslant j \leqslant r'_l, \tag{3.12}$$

and using it together with (3.11) and (3.8) as before, the claim is easily proved. For the case $\varphi(\alpha) < 0$, $\alpha + \varphi(\alpha) > 0$, modify (3.8) to get

$$e^{-\lambda_{k,i}s} \leqslant e^{-(k+1)s} \quad \text{for} \quad 1 \leqslant i \leqslant r_k. \tag{3.13}$$

and use it with (3.11) and (3.9) to prove the (3.5) for $(s,t) \in R(D) \cap \mathbb{R}^2$ with $s < 0$. For the remaining case in which $\varphi(\alpha) < 0$ and $\alpha + \varphi(\alpha) < 0$, analogously from (3.4) we have

$$|A_{k,i;l,j}| < e^{(\varphi(\alpha)+\frac{\delta}{2})(k+l+2)+l\alpha}, \tag{3.14}$$

for k and l large enough and all $1 \leqslant i \leqslant r_k$, $1 \leqslant j \leqslant r'_l$, which can be used together with (3.13) and (3.12) to prove the (3.5) for $(s,t) \in R(D) \cap \mathbb{R}^2$ with $s, t < 0$. □

The following lemma gives a description of the set $\mathbb{C}^2 \setminus \overline{R(D)}$ which is intuitively clear. Its proof is quite elementary, nevertheless, for the sake of completeness we present a proof with full detail.

Lemma 3.14. *The following properties about the function φ defined in (3.1) are satisfied.*

(1) *If there exists $\alpha \in \mathbb{R}$ such that $\varphi(\alpha)$ is finite, then φ is defined in $\mathbb{R}$ and it is decreasing and continuous.*

(2) *Consider the function defined by $\psi(\alpha) = \alpha + \varphi(\alpha)$. If there exists $\alpha \in \mathbb{R}$ such that $\varphi(\alpha)$ is finite, then ψ is defined in $\mathbb{R}$ and it is increasing and continuous.*

(3) *The curve parametrized by* $\{(\varphi(\alpha), \psi(\alpha)) : \alpha \in \mathbb{R}\}$ *is decreasing and convex, so* $R(D)$ *is a convex set.*

(4) $\sup_{\alpha\in\mathbb{R}} \varphi(\alpha) = +\infty$ *and* $\sup_{\alpha\in\mathbb{R}} \psi(\alpha) = +\infty$.

(5) $\mathbb{C}^2 \setminus \overline{R(D)} = \{(s,t) \in \mathbb{C}^2$: *there exists* $\alpha \in \mathbb{R}$ *s.t.* $\mathrm{Re}\, s < \varphi(\alpha)$ *and* $\mathrm{Re}\, t < \psi(\alpha)\}$.

Proof. (1) Let us suppose that there exists $\alpha \in \mathbb{R}$ such that $\varphi(\alpha)$ is finite, and for $x, y > 0$ let $F_\alpha(x,y) = e^{-y\alpha} \sum_{\lfloor x\rfloor}^{x} \sum_{\lfloor y\rfloor}^{y} a_{i,j}$, and $G_\alpha(x,y) = \frac{\log|F_\alpha(x,y)|}{x+y}$. Suppose $\beta \geqslant \alpha$, so $e^{-y(\beta-\alpha)} \leqslant 1$ for all $y > 0$ and then $F_\beta(x,y) = F_\alpha(x,y)e^{-y(\beta-\alpha)} \leqslant F_\alpha(x,y)$ for any $x, y > 0$, so

$$\begin{aligned} G_\alpha(x,y) \geqslant G_\beta(x,y) &= \frac{\log(|F_\alpha(x,y)|e^{-y(\beta-\alpha)})}{x+y} \\ &= G_\alpha(x,y) - \frac{y}{x+y}(\beta-\alpha) \geqslant G_\alpha(x,y) - (\beta-\alpha) \end{aligned} \tag{3.15}$$

for all $x, y > 0$. On the one hand, since $\varphi(\alpha)$ is finite, given any $\varepsilon > 0$ there exists some $X(\alpha,\varepsilon) > 0$ such that $x + y > X(\alpha,\varepsilon)$ implies $G_\alpha(x,y) < \varphi(\alpha) + \varepsilon$. By the left-hand side inequality in (3.15), $x + y > X(\alpha,\varepsilon)$ implies $G_\beta(x,y) < \varphi(\alpha) + \varepsilon$, and therefore there exists $\varphi(\beta) = \limsup_{x+y\to\infty} G_\beta(x,y)$ and $\varphi(\beta) \leqslant \varphi(\alpha) + \varepsilon$ for $\beta \geqslant \alpha$. Since $\varepsilon > 0$ is arbitrary, $\varphi(\beta) \leqslant \varphi(\alpha)$ for $\beta \geqslant \alpha$, showing that φ is defined on $[\alpha, +\infty)$ and that it is decreasing there. On the other hand, using the right-hand side inequality in (3.15) we get that $0 \leqslant G_\alpha(x,y) - G_\beta(x,y) \leqslant \beta - \alpha$, and it is easy to see then that $0 \leqslant \varphi(\alpha) - \varphi(\beta) \leqslant \beta - \alpha$.

Assume now that $\beta \leqslant \alpha$. Exchaging the roles of the parameters in (3.15) we get

$$0 \leqslant G_\beta(x,y) - G_\alpha(x,y) \leqslant \alpha - \beta \quad \text{for} \quad \beta \leqslant \alpha \text{ and } x, y > 0.$$

Since $\varphi(\alpha)$ is finite, we get that $G_\beta(x,y) \leqslant \varphi(\alpha)+\varepsilon+\alpha-\beta$ for all $\beta \leqslant \alpha$ and $x+y > X(\alpha,\varepsilon)$, so there exists $\varphi(\beta) = \limsup_{x+y\to\infty} G_\beta(x,y)$ and $\varphi(\beta) \leqslant \varphi(\alpha)+\alpha-\beta$, giving $0 \leqslant \varphi(\beta)-\varphi(\alpha) \leqslant \alpha-\beta$ for all $\beta \leqslant \alpha$. This gives that φ is defined in $(-\infty,\alpha]$ and by the previous case it is also decreasing there. Moreover, we have proved that $|\varphi(\alpha)-\varphi(\beta)| \leqslant |\alpha-\beta|$ for all $\alpha,\beta \in \mathbb{R}$, so φ is a non-expansive and therefore continuous.
(2) Let

$$\begin{aligned}\tilde{G}_\alpha(x,y) &= \frac{\log\left(e^{x\alpha}\left|\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}\right|\right)}{x+y}\\ &= \alpha + \frac{\log\left(e^{-y\alpha}\left|\sum_{\lfloor x\rfloor}^{x}\sum_{\lfloor y\rfloor}^{y} a_{i,j}\right|\right)}{x+y} = \alpha + G_\alpha(x,y).\end{aligned}$$

Define $\psi(\alpha) = \limsup_{x+y\to\infty} \tilde{G}_\alpha(x,y) = \alpha+\varphi(\alpha)$. Clearly ψ is defined wherever φ is defined and, by proceeding analogously to the previous case, ψ is increasing and it is a non-expansive in $\mathbb{R}$, so it is continuous.
(3) Let $\alpha \leqslant \beta \leqslant \gamma$, it is easy to see that $(\gamma-\beta)G_\alpha(x,y)+(\beta-\alpha)G_\gamma(x,y)+(\alpha-\gamma)G_\beta(x,y) = 0$ for all $x,y>0$, so $(\gamma-\alpha)G_\beta(x,y) = (\gamma-\beta)G_\alpha(x,y)+(\beta-\alpha)G_\gamma(x,y)$, which implies $(\gamma-\alpha)\varphi(\beta) \leqslant (\gamma-\beta)\varphi(\alpha)+(\beta-\alpha)\varphi(\gamma)$. This last condition can be written as

$$\begin{vmatrix}\varphi(\alpha) & \psi(\alpha) & 1\\ \varphi(\beta) & \psi(\beta) & 1\\ \varphi(\gamma) & \psi(\gamma) & 1\end{vmatrix} = \begin{vmatrix}\varphi(\alpha) & \alpha & 1\\ \varphi(\beta) & \beta & 1\\ \varphi(\gamma) & \gamma & 1\end{vmatrix} \leqslant 0, \quad \text{hence} \quad \begin{vmatrix}\varphi(\gamma) & \psi(\gamma) & 1\\ \varphi(\beta) & \psi(\beta) & 1\\ \varphi(\alpha) & \psi(\alpha) & 1\end{vmatrix} \geqslant 0.$$

Recall the following characterization of the convexitiy of a curve by determinants. If $P_j = (x_j,y_j)$ for $j=1,2,3$ are three arbitrary points in the curve satisfying $x_1<x_2<x_3$, and the sign of the determinant

$$\Delta = \begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

is constant for any such choice of points, then the the convexity of the curve does not change, being the curve concave if $\Delta \leqslant 0$ and convex if $\Delta \geqslant 0$. This characterization can be proved easily, just note that the point P_3 is above the line passing through P_1 and P_2 if and only if $\Delta > 0$, it is below the line if and only if $\Delta < 0$ and it is on the line if and only if $\Delta = 0$. Recalling that φ is decreasing is enough to get that the curve parametrized by $\{(\varphi(\alpha), \psi(\alpha)) : \alpha \in I\}$ is convex, so $R(D)$ is a convex set. The fact that the curve is decreasing comes from the fact that φ being decreasing and ψ being increasing imply $(\varphi(\alpha) - \varphi(\beta))(\psi(\alpha) - \psi(\beta)) \leqslant 0$ for all $\alpha, \beta \in \mathbb{R}$.
(4) Since φ is decreasing and continuous,

$$L_1 := \sup_{\alpha \in \mathbb{R}} \varphi(\alpha) = \lim_{\alpha \to -\infty} \varphi(\alpha),$$

and since ψ is increasing and continuous,

$$L_2 := \sup_{\alpha \in \mathbb{R}} \psi(\alpha) = \lim_{\alpha \to +\infty} \psi(\alpha) = \lim_{\alpha \to +\infty} \alpha + \varphi(\alpha).$$

If $L_1 \in \mathbb{R}$, then $\lim_{\alpha \to -\infty} \alpha + \varphi(\alpha) = -\infty$, so the curve parametrized by $\{(\varphi(\alpha), \psi(\alpha)) : \alpha \in \mathbb{R}\}$ would decrease asymptotically to the left of the vertical line $x = L_1$, and for $\varphi(\alpha)$ close enough to L_1, the curve would have to be concave, which contradicts (3). In other words, assuming L_1 is finite and $L_2 = -\infty$ implies that the slope of the curve parametrized by $\{(\varphi(\alpha), \psi(\alpha)) : \alpha \in \mathbb{R}\}$ is negative and that it grows faster the closest its abscissa is to the value L_1, giving that the curve must be concave, contradicting (3). Therefore $\lim_{\alpha \to -\infty} \varphi(\alpha) = +\infty$. Analogously it can be proved that $L_2 = +\infty$.

(5) It is enough to prove that

$$\mathbb{R}^2 \setminus \overline{R(D)} = \{(s,t) \in \mathbb{R}^2 : \text{there exists } \alpha \in R \text{ such that } s < \varphi(\alpha) \text{ and } t < \psi(\alpha)\}.$$

One inclusion is clear, so we will suppose that $(s_0, t_0) \in \mathbb{R}^2 \setminus \overline{R(D)}$ but for all $\alpha \in \mathbb{R}$ either $s_0 \geqslant \varphi(\alpha)$ or $t_0 \geqslant \psi(\alpha)$. Since we know by (3) that $\sup_{\alpha\in\mathbb{R}} \varphi(\alpha) = +\infty$, it is not possible that for all $\alpha \in \mathbb{R}$ we have $s_0 \geqslant \varphi(\alpha)$. By the analogous argument it is not possible that for all $\beta \in \mathbb{R}$ we have $t_0 \geqslant \psi(\beta)$, so the sets $A = \{\alpha \in \mathbb{R} : s_0 \geqslant \varphi(\alpha)\}$ and $B = \{\beta \in \mathbb{R} : t_0 \geqslant \psi(\beta)\}$ are non-empty, and clearly $A \cup B = \mathbb{R}$. Note that if $\alpha \in A \cap B$ then $(s_0, t_0) \in \overline{R(D)}$, which contradicts the hypothesis, so $A \cap B = \emptyset$ where

$$A = \{\alpha \in \mathbb{R} : s_0 \geqslant \varphi(\alpha),\, t_0 < \psi(\alpha)\}$$

and

$$B = \{\beta \in \mathbb{R} : s_0 < \varphi(\beta),\, t_0 \geqslant \psi(\beta)\}.$$

From the monotonicity of either φ or ψ one gets that $\beta \leqslant \alpha$ for all $\beta \in B, \alpha \in A$, and since $A \cup B = \mathbb{R}$ we have that $\sup B = \inf A = \gamma$. It is a clear consequence of the continuity of both φ and ψ that $s_0 = \varphi(\gamma)$ and $t_0 = \psi(\gamma)$, which is a contradiction. Therefore the other inclusion is proved. □

Finally, we prove that $R(D)$ is the set of regular convergence of the double Dirichlet series D, understanding that the series cannot converge regularly outside $\overline{R(D)}$. This extends the natural behaviour of general Dirichlet series, in which nothing can be said about the convergence at the vertical line given by the abscissa.

Theorem 3.15. *Let $D(s,t) = \sum_{i,j=1}^{\infty} a_{i,j} e^{-\lambda_i s - \mu_j t}$ be a formal general double Dirichlet series, where $\{\lambda_i\}$ and $\{\mu_j\}$ are strictly increasing se-*

quences divergent to $+\infty$ *which we will suppose to be positive. If we take* φ *defined by* (3.1)*, then the double Dirichlet series* D *does not converge regularly in* $\mathbb{C}^2 \setminus \overline{R(D)}$*.*

Proof. Note that it is enough to prove the statement for $(s_0, t_0) \in \mathbb{R}^2 \setminus \overline{R(D)}$. Arguing by contradiciton, suppose that the D converges regularly on (s_0, t_0). On the one hand, by Lemma 3.14 there exists $\alpha \in \mathbb{R}$ such that $s_0 < \varphi(\alpha)$ and $t_0 < \psi(\alpha)$, so there exists $\delta > 0$ such that $s_0 + \delta < \varphi(\alpha)$ and $t_0 + \delta < \psi(\alpha)$. On the other hand, the regular convergence implies the boundedness of the partial double sums (this is a straightforward consequence of Definition 1.12) so there exists some $K > 0$ such that

$$|S(m,n)| := \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} e^{-\lambda_i s_0 - \mu_j t_0} \right| < K \quad \text{for every} \quad (m,n) \in \mathbb{N}^2.$$

Using the same notation form the proof of Theorem 3.12, for every $(k,l) \in \mathbb{N}_0^2$,

$$|S_{k,M;l,n}| := \left| \sum_{g=0}^{k} \sum_{h=0}^{l} \sum_{i=1}^{M_g} \sum_{j=1}^{N_h} a_{g,i;h,j} e^{-\lambda_{g,i} s_0 - \mu_{h,j} t_0} \right| < K,$$

where $M_g = r_g$ and $N_h = r'_h$ for $1 \leqslant g < k$, $1 \leqslant h < l$ and $1 \leqslant M_k \leqslant r_k$, $1 \leqslant N_l \leqslant r'_l$. Let $x, y > 0$ and $\alpha \in \mathbb{R}$, and let $\tilde{F}_\alpha(x,y) = e^{-\lfloor y \rfloor \alpha} \sum_{\lfloor x \rfloor}^{x} \sum_{\lfloor y \rfloor}^{y} a_{i,j} = e^{-l\alpha} \sum_1^M \sum_1^N a_{k,i;l,j}$ for some $k = \lfloor x \rfloor$, $l = \lfloor y \rfloor$, $1 \leqslant M \leqslant r_k$ and $1 \leqslant N \leqslant r'_l$. Define $\xi_{k,i} = e^{\lambda_{k,i} s_0}$ and $\eta_{l,j} = e^{\mu_{l,j} t_0 - l\alpha}$, and consider the notation $S_{k,0;l,n} = S_{k-1, r_{k-1}; l,n}$, $S_{k,m;l,0} = S_{k,m;l-1,r'_{l-1}}$. Then

$$\tilde{F}_\alpha(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} \left(S_{k,i;l,j} - S_{k,i-1;l,j} - S_{k,i;l,j-1} + S_{k,i-1;l,j-1} \right) \xi_{k,i} \eta_{l,j}$$

$$\begin{aligned}
&= \sum_{i=1}^{M}\sum_{j=1}^{N} S_{k,i;l,j}\xi_{k,i}\eta_{l,j} - \sum_{i=0}^{M-1}\sum_{j=1}^{N} S_{k,i;l,j}\xi_{k,i+1}\eta_{l,j} \\
&\quad - \sum_{i=1}^{M}\sum_{j=0}^{N-1} S_{k,i;l,j}\xi_{k,i}\eta_{l,j+1} + \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} S_{k,i;l,j}\xi_{k,i+1}\eta_{l,j+1} \\
&= \sum_{i=1}^{M-1}\sum_{j=1}^{N-1} S_{k,i;l,j}(\xi_{k,i} - \xi_{i,k+1})(\eta_{l,j} - \eta_{l,j+1}) \\
&\quad + \sum_{i=1}^{M-1} (\xi_{k,i} - \xi_{i,k+1})\left(S_{k,i;l,N}\eta_{l,N} - S_{k,i;l,0}\eta_{l,1}\right) \\
&\quad + \sum_{j=1}^{N-1} (\eta_{l,j} - \eta_{l,j+1})\left(S_{k,M;l,j}\xi_{k,M} - S_{k,0;l,j}\xi_{k,1}\right) \\
&\quad + S_{k,M;l,N}\xi_{k,M}\eta_{l,N} - S_{k,M;l,0}\xi_{k,M}\eta_{l,1} \\
&\quad - S_{k,0;l,N}\xi_{k,1}\eta_{l,N} + S_{k,0;l,0}\xi_{k,1}\eta_{l,1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\tilde{F}_\alpha(x,y)| &\leqslant K\left(\sum_{i=1}^{M} |\xi_{k,i} - \xi_{i,k+1}| + \xi_{k,1} + \xi_{k,M}\right) \\
&\qquad \left(\sum_{j=1}^{N} |\eta_{l,j} - \eta_{l,j+1}| + \eta_{l,1} + \eta_{l,N}\right) \\
&= K\left(\operatorname{sign}(s_0)(\xi_{k,M} - \xi_{k,1}) + \xi_{k,1} + \xi_{k,M}\right) \\
&\qquad \left(\operatorname{sign}(t_0)(\eta_{l,N} - \eta_{l,1}) + \eta_{l,1} + \eta_{l,N}\right) \\
&\leqslant 4K \max(\xi_{k,1}, \xi_{k,M}) \max(\eta_{l,1}, \eta_{l,N}).
\end{aligned}$$

Since $x - 1 < \lfloor x \rfloor = k \leqslant \lambda_{k,i} < x$ for all $1 \leqslant i \leqslant M$, $e^{\lambda_{k,i}s_0} < e^{xs_0} \leqslant e^{xs_0+|s_0|}$ if $s_0 \geqslant 0$ and $e^{\lambda_{k,i}} \leqslant e^{(x-1)s_0} = e^{xs_0+|s_0|}$ if $s_0 < 0$, so $\max(\xi_{k,1}, \xi_{k,M}) \leqslant e^{xs_0+|s_0|}$. Analogously, $\max(\eta_{l,1}, \eta_{l,N}) \leqslant e^{y(t_0-\alpha)+|\alpha|+|t_0|}$, so

$$|\tilde{F}_\alpha(x,y)| \leqslant 4Ke^{xs_0+|s_0|}e^{y(t_0-\alpha)+|\alpha|+|t_0|} = \tilde{K}e^{xs_0+y(t_0-\alpha)}.$$

Consequently, for any $x, y > 0$,

$$\frac{\log|\tilde{F}_\alpha(x,y)|}{x+y} \leqslant \frac{\tilde{K}}{x+y} + \frac{xs_0 + y(t_0-\alpha)}{x+y} \leqslant \frac{\tilde{K}}{x+y} + \varphi(\alpha) - \delta,$$

so by Remark 3.13

$$\begin{aligned}\varphi(\alpha) = \limsup_{x+y\to\infty} \frac{\log|\tilde{F}_\alpha(x,y)|}{x+y} &\leqslant \limsup_{x+y\to\infty} \frac{\tilde{K}}{x+y} + \varphi(\alpha) - \delta \\ &= \varphi(\alpha) - \delta < \varphi(\alpha),\end{aligned}$$

a contradiction. Hence, if D converges regularly at $(s_0, t_0) \in \mathbb{R}^2$ then $(s_0, t_0) \in \overline{R(D)}$. □

Theorems 3.12 and 3.15 show us that the sets of regular convergence for double Dirichlet series are given by decreasing convex curves, which allow more diversity of examples for such sets in comparison with the one variable case. Before we give examples of those sets, we give the new formulae for the sets of regular convergence of an ordinary double Dirichlet series which are inspired by Theorem 3.12 and extend the traditional formula for ordinary Dirichlet series (2.2) and (2.3). This formulae will have the advantage of being simpler to compute, giving the possibility of obtaining the examples we mentioned for sets of regular convergence that are not merely products of complex half-planes, which is the trivial example obtained for double Dirichlet series that can be written as the product of two Dirichlet series, one in each of the complex variables.

In the one variable case a distinction needed to be made between the cases of the abscissa being either positive or negative. This discernment can actually be stated equivalently by checking whether the Dirichlet series is convergent at the origin or not. This is the condition that can be properly translated into the double case, and it will produce two formulae, depending on whether the double Dirichlet series is regularly convergent at the origin of $\mathbb{C}^2$ (note that regular convergence at the origin is trivially equivalent to the regular convergence of the double sequence of coefficients). We will replicate the structure of the previous formulae to give the first formula, which can be used for double Dirichlet series that are not regularly convergent are the origin.

Theorem 3.16. *Let $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ be a formal double Dirichlet series which is not regularly convergent at the origin. If we write*

$$f_+(\alpha) = \limsup_{M+N\to\infty} \frac{\log\left(N^{-\alpha}|A_{M,N}|\right)}{\log(MN)} \tag{3.16}$$

where $A_{M,N} = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n}$, then the double Dirichlet series converges regularly in the set

$$R_+(D) = \{(s,t) \in \mathbb{C}^2 : \textit{there exists } \alpha \in \mathbb{R} \textit{ such that } f_+(\alpha) \in \mathbb{R},\ \operatorname{Re} s > f_+(\alpha),\ \operatorname{Re} t > \alpha + f_+(\alpha)\}.$$

Proof. Arguing as in Theorem 3.12, by Theorem 3.11, it is enough to prove that D converges regularly in $R_+(D) \cap \mathbb{R}^2$. Suppose $(s,t) \in R_+(D) \cap \mathbb{R}^2$, that is, there exist $\alpha \in \mathbb{R}$ and $\delta > 0$ arbitrarily small such that $s > f_+(\alpha) + \delta$ and $t > \alpha + f_+(\alpha) + \delta$. We want to check the condition of convergence in a restricted sense from Definition 1.12, so with the notation $\Delta m(s) = m^{-s} - (m+1)^{-s}$, $\Delta n(t) = n^{-t} - (n+1)^{-t}$,

$$\sum_{m=M+1}^{P} \sum_{n=N+1}^{Q} \frac{a_{m,n}}{m^s n^t}$$
$$= \sum_{m=M+1}^{P} \sum_{n=N+1}^{Q} (A_{m,n} - A_{m-1,n} - A_{m,n-1} + A_{m-1,n-1}) m^{-s} n^{-t}$$

$$
\begin{aligned}
&= \sum_{m=M+1}^{P} \sum_{n=N+1}^{Q} A_{m,n} m^{-s} n^{-t} - \sum_{m=M}^{P-1} \sum_{n=N+1}^{Q} A_{m,n} (m+1)^{-s} n^{-t} \\
&\quad - \sum_{m=M+1}^{P} \sum_{n=N}^{Q-1} A_{m,n} m^{-s} (n+1)^{-t} \\
&\quad + \sum_{m=M}^{P-1} \sum_{n=N}^{Q-1} A_{m,n} (m+1)^{-s} (n+1)^{-t} \\
&= \sum_{m=M+1}^{P-1} \sum_{n=N+1}^{Q-1} A_{m,n} \Delta m(s) \Delta n(t) \qquad (3.17) \\
&\quad + \sum_{m=M+1}^{P-1} (A_{m,Q} Q^{-t} - A_{m,N} (N+1)^{-t}) \Delta m(s) \\
&\quad + \sum_{n=N+1}^{Q-1} (A_{P,n} P^{-s} - A_{M,n} (M+1)^{-s}) \Delta n(t) \\
&\quad + A_{P,Q} P^{-s} Q^{-t} - A_{M,Q} (M+1)^{-s} Q^{-t} \\
&\quad - A_{P,N} P^{-s} (N+1)^{-t} + A_{M,N} (M+1)^{-s} (N+1)^{-t}
\end{aligned}
$$

To bound the terms appearing on the right-hand side of (3.17) we need to separate some cases. First suppose $s > f_+(\alpha) \geqslant 0$ and $t > \alpha + f_+(\alpha) \geqslant 0$. On the one hand,

$$
\begin{aligned}
|\Delta m(s)| = \left| \int_m^{m+1} \frac{s}{x^{s+1}} dx \right| \leqslant \int_m^{m+1} \frac{|s|}{x^{s+1}} \leqslant s \max_{m \leqslant x \leqslant m+1} \left(\frac{1}{x^{s+1}} \right) \\
= \frac{s}{m^{s+1}} = \frac{1}{m^{f_+(\alpha)+\frac{\delta}{2}}} \frac{s}{m^{1+\frac{\delta}{2}}},
\end{aligned} \qquad (3.18)
$$

and analogously,

$$
|\Delta n(t)| \leqslant \frac{1}{n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} \frac{t}{n^{1+\frac{\delta}{2}}}. \qquad (3.19)
$$

On the other hand, given $\delta > 0$ there exists some $K_0 \in \mathbb{N}$ such that $m + n > K_0$ implies $f_+(\alpha) + \dfrac{\delta}{2} > \dfrac{\log(n^{-\alpha}|A_{m,n}|)}{\log(mn)}$, so $|A_{m,n}| \leqslant$

$m^{f_+(\alpha)+\frac{\delta}{2}}n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}$. Then, taking $M, N \in \mathbb{N}$ such that $M+N > K_0$,

$$|A_{m,n}\Delta m(s)\Delta n(t)| \leqslant \frac{m^{f_+(\alpha)+\frac{\delta}{2}}n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{m^{f_+(\alpha)+\frac{\delta}{2}}n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{st}{(mn)^{1+\frac{\delta}{2}}} = \frac{st}{(mn)^{1+\frac{\delta}{2}}}. \tag{3.20}$$

We also get

$$|A_{m,l}\Delta m(s)l^{-t}| \leqslant \frac{m^{f_+(\alpha)+\frac{\delta}{2}}l^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{m^{f_+(\alpha)+\frac{\delta}{2}}l^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{s}{m^{1+\frac{\delta}{2}}}l^{-\frac{\delta}{2}} = \frac{s}{m^{1+\frac{\delta}{2}}}l^{-\frac{\delta}{2}}, \tag{3.21}$$

$$|A_{k,n}\Delta n(t)k^{-s}| \leqslant \frac{k^{f_+(\alpha)+\frac{\delta}{2}}n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{k^{f_+(\alpha)+\frac{\delta}{2}}n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{t}{n^{1+\frac{\delta}{2}}}k^{-\frac{\delta}{2}} = \frac{t}{n^{1+\frac{\delta}{2}}}k^{-\frac{\delta}{2}}, \tag{3.22}$$

and finally

$$\begin{aligned}
&|A_{P,Q}P^{-s}Q^{-t}| \\
&\quad\leqslant \frac{P^{f_+(\alpha)+\frac{\delta}{2}}Q^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{P^{f_+(\alpha)+\frac{\delta}{2}}Q^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}P^{-\frac{\delta}{2}}Q^{-\frac{\delta}{2}} \leqslant M^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}, \\
&|A_{M,Q}(M+1)^{-s}Q^{-t}| \\
&\quad\leqslant \frac{M^{f_+(\alpha)+\frac{\delta}{2}}Q^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{(M+1)^{f_+(\alpha)+\frac{\delta}{2}}Q^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}(M+1)^{-\frac{\delta}{2}}Q^{-\frac{\delta}{2}} \leqslant M^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}, \\
&|A_{P,N}P^{-s}(N+1)^{-t}| \\
&\quad\leqslant \frac{P^{f_+(\alpha)+\frac{\delta}{2}}N^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{P^{f_+(\alpha)+\frac{\delta}{2}}(N+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}P^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \leqslant M^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}, \\
&|A_{M,N}(M+1)^{-s}(N+1)^{-t}| \\
&\quad\leqslant \frac{M^{f_+(\alpha)+\frac{\delta}{2}}N^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{(M+1)^{f_+(\alpha)+\frac{\delta}{2}}(N+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}(M+1)^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \\
&\quad\leqslant M^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}.
\end{aligned} \tag{3.23}$$

Now let us study the case in which $f_+(\alpha) < 0$ but $\alpha + f_+(\alpha) > 0$. Given Theorem 3.11 it is enough to prove the regular convergence for the

$(s,t) \in R_+(D) \cap \mathbb{R}^2$ with $s<0$, so for $s<0$ we have

$$\begin{aligned}|\Delta m(s)| &\leqslant |s| \max_{m \leqslant x \leqslant m+1} \left(\frac{1}{x^{s+1}}\right) = |s| \max(m^{-(s+1)}, (m+1)^{-(s+1)}) \\ &\leqslant \max\left(\frac{1}{m^{f_+(\alpha)+\frac{\delta}{2}}} \frac{|s|}{m^{1+\frac{\delta}{2}}}, \frac{1}{(m+1)^{f_+(\alpha)+\frac{\delta}{2}}} \frac{|s|}{(m+1)^{1+\frac{\delta}{2}}}\right) \\ &\leqslant \frac{1}{(m+1)^{f_+(\alpha)+\frac{\delta}{2}}} \frac{|s|}{m^{1+\frac{\delta}{2}}}.\end{aligned} \tag{3.24}$$

For this case we need to adapt (3.20), (3.21) and (3.23). Since $\delta > 0$ may be chosen arbitrary small we will suppose that $f_+(\alpha) + \frac{\delta}{2} < 0$. Taking again $M, N \in \mathbb{N}$ such that $M + N > K_0$,

$$|A_{m,n}\Delta m(s)\Delta n(t)| \leqslant \frac{m^{f_+(\alpha)+\frac{\delta}{2}} n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{(m+1)^{f_+(\alpha)+\frac{\delta}{2}} n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} \frac{|s|t}{(mn)^{1+\frac{\delta}{2}}} \leqslant \frac{2^{-f_+(\alpha)}|s|t}{(mn)^{1+\frac{\delta}{2}}}. \tag{3.25}$$

and also

$$|A_{m,l}\Delta m(s) l^{-t}| \leqslant \frac{m^{f_+(\alpha)+\frac{\delta}{2}} l^{\alpha+f(\alpha)+\frac{\delta}{2}}}{(m+1)^{f_+(\alpha)+\frac{\delta}{2}} l^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} \frac{|s|}{m^{1+\frac{\delta}{2}}} l^{-\frac{\delta}{2}} = \frac{2^{-f_+(\alpha)}|s|}{m^{1+\frac{\delta}{2}}} l^{-\frac{\delta}{2}}. \tag{3.26}$$

Moreover,

$$\begin{aligned}&|A_{M,Q}(M+1)^{-s}Q^{-t}| \\ &\quad\leqslant \frac{M^{f_+(\alpha)+\frac{\delta}{2}} Q^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{(M+1)^{f_+(\alpha)+\frac{\delta}{2}} Q^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} (M+1)^{-\frac{\delta}{2}} Q^{-\frac{\delta}{2}} \\ &\quad\leqslant 2^{-f_+(\alpha)} (M+1)^{-\frac{\delta}{2}} Q^{-\frac{\delta}{2}} \leqslant 2^{-f_+(\alpha)} M^{-\frac{\delta}{2}} N^{-\frac{\delta}{2}},\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&|A_{M,N}(M+1)^{-s}(N+1)^{-t}| \\
&\qquad \leqslant \frac{M^{f_+(\alpha)+\frac{\delta}{2}}N^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{(M+1)^{f_+(\alpha)+\frac{\delta}{2}}(N+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}(M+1)^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \\
&\qquad \leqslant 2^{-f_+(\alpha)}(M+1)^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \leqslant 2^{-f_+(\alpha)}M^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}.
\end{aligned} \tag{3.28}$$

There is only one case left to study, since regular convergence at any point (s,t) with $f_+(\alpha) < s < 0$ and $\alpha + f_+(\alpha) < t < 0$ would imply, by Theorem 3.11, regular convergence at the origin. Then, for $s > f_+(\alpha) > 0$ and $\alpha + f_+(\alpha) < t < 0$ we need to adapt (3.20), (3.22) and (3.23). First,

$$\begin{aligned}
|\Delta n(t)| &\leqslant |t| \max_{n\leqslant x\leqslant n+1}\left(\frac{1}{x^{s+1}}\right) = |t|\max(n^{-(t+1)},(n+1)^{-(t+1)}) \\
&\leqslant \max\left(\frac{1}{n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{|t|}{n^{1+\frac{\delta}{2}}}, \frac{1}{(n+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{|t|}{(n+1)^{1+\frac{\delta}{2}}}\right) \\
&\leqslant \frac{1}{(n+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{|t|}{n^{1+\frac{\delta}{2}}}.
\end{aligned} \tag{3.29}$$

so for $M+N > K_0$,

$$\begin{aligned}
|A_{m,n}\Delta m(s)\Delta n(t)| &\leqslant \frac{m^{f_+(\alpha)+\frac{\delta}{2}}n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{m^{f_+(\alpha)+\frac{\delta}{2}}(n+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}\frac{s|t|}{(mn)^{1+\frac{\delta}{2}}} \\
&\leqslant \frac{2^{-(\alpha+f_+(\alpha))}s|t|}{(mn)^{1+\frac{\delta}{2}}},
\end{aligned} \tag{3.30}$$

and also

$$\begin{aligned} |A_{k,n}\Delta n(t)k^{-s}| &\leqslant \frac{k^{f_+(\alpha)+\frac{\delta}{2}} n^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{k^{f_+(\alpha)+\frac{\delta}{2}}(n+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} \frac{|t|}{n^{1+\frac{\delta}{2}}} k^{-\frac{\delta}{2}} \\ &= \frac{2^{-(\alpha+f_+(\alpha))}|t|}{n^{1+\frac{\delta}{2}}} k^{-\frac{\delta}{2}}. \end{aligned} \tag{3.31}$$

Moreover

$$\begin{aligned} &|A_{P,N}P^{-s}(N+1)^{-t}| \\ &\qquad \leqslant \frac{P^{f_+(\alpha)+\frac{\delta}{2}} N^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{P^{f_+(\alpha)+\frac{\delta}{2}}(N+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} P^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \\ &\qquad \leqslant 2^{-(\alpha+f(\alpha))} P^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \leqslant 2^{-(\alpha+f_+(\alpha))} M^{-\frac{\delta}{2}} N^{-\frac{\delta}{2}}, \\ &|A_{M,N}(M+1)^{-s}(N+1)^{-t}| \\ &\qquad \leqslant \frac{M^{f_+(\alpha)+\frac{\delta}{2}} N^{\alpha+f_+(\alpha)+\frac{\delta}{2}}}{(M+1)^{f_+(\alpha)+\frac{\delta}{2}}(N+1)^{\alpha+f_+(\alpha)+\frac{\delta}{2}}} (M+1)^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \\ &\qquad \leqslant 2^{-(\alpha+f_+(\alpha))}(M+1)^{-\frac{\delta}{2}}(N+1)^{-\frac{\delta}{2}} \leqslant 2^{-(\alpha+f_+(\alpha))} M^{-\frac{\delta}{2}} N^{-\frac{\delta}{2}}. \end{aligned} \tag{3.32}$$

Now we will combine all these cases to bound the terms appearing in the right-hand side of (3.17). Let $C = \max(1, 2^{-f_+(\alpha)}, 2^{-(\alpha+f_+(\alpha))})$, so combining (3.20), (3.25) and (3.30),

$$|A_{m,n}\Delta m(s)\Delta n(t)| \leqslant \frac{C|st|}{(mn)^{1+\frac{\delta}{2}}}, \tag{3.33}$$

and combining (3.21) with (3.26) and (3.22) with (3.31) we get, for $N \leqslant l$,

$$|A_{m,l}\Delta m(s)l^{-t}| \leqslant \frac{C|s|}{m^{1+\frac{\delta}{2}}} l^{-\frac{\delta}{2}} \leqslant \frac{C|s|}{m^{1+\frac{\delta}{2}}} N^{-\frac{\delta}{2}}, \tag{3.34}$$

and, for $M \leqslant k$,

$$|A_{k,n}\Delta n(t)k^{-s}| \leqslant \frac{C|t|}{n^{1+\frac{\delta}{2}}}k^{-\frac{\delta}{2}} \leqslant \frac{C|t|}{n^{1+\frac{\delta}{2}}}M^{-\frac{\delta}{2}}. \tag{3.35}$$

Finally, combining (3.23), (3.27), (3.28) and (3.32) we get

$$\begin{aligned} |A_{P,Q}P^{-s}Q^{-t}| &\leqslant CM^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}, \\ |A_{M,Q}(M+1)^{-s}Q^{-t}| &\leqslant CM^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}, \\ |A_{P,N}P^{-s}(N+1)^{-t}| &\leqslant CM^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}, \\ |A_{M,N}(M+1)^{-s}(N+1)^{-t}| &\leqslant CM^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}}. \end{aligned} \tag{3.36}$$

Now note that we can use (3.33) to bound the first term on the right hand side in (3.17), (3.34) for the second one, (3.35) for the third one and (3.36) for the last four terms, so we get

$$\begin{aligned} &\left| \sum_{m=M+1}^{P} \sum_{n=N+1}^{Q} \frac{a_{m,n}}{n^s n^t} \right| \\ &\qquad \leqslant \sum_{m=M+1}^{P-1} \sum_{n=N+1}^{Q-1} \frac{C|st|}{(mn)^{1+\frac{\delta}{2}}} + \sum_{m=M+1}^{P-1} 2\frac{C|s|}{m^{1+\frac{\delta}{2}}}N^{-\frac{\delta}{2}} \\ &\qquad + \sum_{n=N+1}^{Q-1} 2\frac{C|t|}{n^{1+\frac{\delta}{2}}}M^{-\frac{\delta}{2}} + 4CM^{-\frac{\delta}{2}}N^{-\frac{\delta}{2}} \\ &\qquad \leqslant C\left(\sum_{m=M+1}^{\infty} \frac{|s|}{m^{1+\frac{\delta}{2}}} + 2M^{-\frac{\delta}{2}}\right)\left(\sum_{n=N+1}^{\infty} \frac{|t|}{n^{1+\frac{\delta}{2}}} + 2N^{-\frac{\delta}{2}}\right). \end{aligned}$$

If $S_M(s) = \sum_{m=M+1}^{\infty} \frac{|s|}{m^{1+\frac{\delta}{2}}} + 2M^{-\frac{\delta}{2}}$, clearly $|S_m(s)| \leqslant |s|\zeta(1+\frac{\delta}{2}) + 2$ and, for any fixed s, $\lim_{M\to\infty} S_M(s) = 0$, so given $\varepsilon > 0$ there exists $K_1 \in \mathbb{N}$ such that $M > K_1$ implies $S_M(s) < \frac{\varepsilon}{C(|t|\zeta(1+\frac{\delta}{2})+2)}$. Choosing K_2 so $N > K_2$ implies $S_N(t) < \frac{\varepsilon}{C(|s|\zeta(1+\frac{\delta}{2})+2)}$ and $K = \max(K_0, K_1, K_2)$ is enough to get that $\max(M,N) > K$ implies

$$\left| \sum_{m=M+1}^{P} \sum_{n=N+1}^{Q} \frac{a_{m,n}}{n^s n^t} \right| \leqslant C S_M(s) S_N(t) < \varepsilon,$$

so the double Dirichlet series converges regularly in (s,t) according to Definition 1.12 and Theorem 1.14. □

With exactly the same proof one can state the analogous lemma of Lemma 3.14 for this case, where again the last point is the geometrical description needed for the necessitiy of the previous theorem, given below.

Lemma 3.17. *The following properties about the function f_+ defined in (3.16) are satisfied.*

(1) *If there exists $\alpha \in \mathbb{R}$ such that $f_+(\alpha)$ is finite, then f_+ is defined in $\mathbb{R}$ and it is decreasing and continuous.*

(2) *Consider the function defined by $g_+(\alpha) = \alpha + f_+(\alpha)$. If there exists $\alpha \in \mathbb{R}$ such that $f_+(\alpha)$ is finite, then g_+ is defined in $\mathbb{R}$ and it is increasing and continuous.*

(3) *The curve parametrized by $\{(f_+(\alpha), g_+(\alpha)) : \alpha \in \mathbb{R}\}$ is decreasing and concave, that is, $R_+(D)$ is a convex set.*

(4) $\sup_{\alpha\in\mathbb{R}} f_+(\alpha) = +\infty$ *and* $\sup_{\alpha\in\mathbb{R}} g_+(\alpha) = +\infty$.

(5) $\mathbb{C}^2 \setminus \overline{R_+(D)} = \{(s,t) \in \mathbb{C}^2 :$ *there exists $\alpha \in \mathbb{R}$ such that* $\operatorname{Re} s < f_+(\alpha)$ *and* $\operatorname{Re} t < g_+(\alpha)\}$.

Theorem 3.18. *Let $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ be a double Dirichlet series which is not regularly convergent at the origin. If we take f_+ defined by (3.16), then the double Dirichlet series D does not converge regularly in $\mathbb{C}^2 \setminus \overline{R_+(D)}$.*

The proof follows the same scheme than the proof of Theorem 3.15. For the sake of completeness it is given below.

Proof. Note that it is enough to prove the statement for $(s,t) \in \mathbb{R}^2 \setminus \overline{R_+(D)}$. Suppose that the D converges regularly on (s,t). On the one hand, by Lemma 3.17 there exists $\alpha \in \mathbb{R}$ such that $s_0 < f_+(\alpha)$ and $t_0 < \alpha + f_+(\alpha)$, so there exists $\delta > 0$ such that $s_0 + \delta < f_+(\alpha)$ and $t_0 + \delta < \alpha + f_+(\alpha)$. On the other hand, we use again that the regular convergence implies the boundedness of the partial double sums so there exists some $K > 0$ such that

$$|S_{M,N}| := \left| \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}}{m^s n^t} \right| < K \quad \text{for every} \quad (M,N) \in \mathbb{N}^2.$$

Using the notation $S_{m,0} = 0 = S_{0,n}$,

$$\begin{aligned}
A_{M,N} &= \sum_{m=1}^{M} \sum_{n=1}^{N} (S_{m,n} - S_{m-1,n} - S_{m,n-1} + S_{m-1,n-1}) m^s n^t \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N} S_{m,n} m^s n^t - \sum_{m=1}^{M-1} \sum_{n=1}^{N} S_{m,n} (m+1)^s n^t \\
&- \sum_{m=1}^{M} \sum_{n=1}^{N-1} S_{m,n} m^s (n+1)^t + \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} S_{m,n} (m+1)^s (n+1)^t \\
&= \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} S_{m,n} (m^s - (m+1)^s)(n^t - (n+1)^t) \\
&+ \sum_{m=1}^{M-1} S_{m,n} (m^s - (m+1)^s) N^t \\
&+ \sum_{n=1}^{N-1} S_{m,n} M^s (n^t - (n+1)^t) + S_{m,n} M^s N^t,
\end{aligned}$$

so, if $s, t \geqslant 0$,

$$\begin{aligned}\frac{1}{N^\alpha}|A_{M,N}| &\leqslant \frac{K}{N^\alpha}\left(\sum_{m=1}^{M-1}|m^s-(m+1)^s|+M^s\right)\\ &\qquad\left(\sum_{n=1}^{N-1}|n^t-(n+1)^t|+N^t\right)\\ &= \frac{K}{N^\alpha}(2M^s-1)(2N^t-1) \leqslant 4KM^sN^{t-\alpha}.\end{aligned}$$

Therefore, since $s < f_+(\alpha)-\delta$ and $t-\alpha < f_+(\alpha)-\delta$,

$$\begin{aligned}\frac{\log(N^{-\alpha}|A_{M,N}|)}{\log(MN)} &\leqslant \frac{\log 4K}{\log(MN)} + \frac{s\log M + (t-\alpha)\log N}{\log(MN)}\\ &< \frac{\log 4K}{\log(MN)} + f_+(\alpha) - \delta,\end{aligned}$$

so

$$\begin{aligned}f_+(\alpha) &= \limsup_{M+N\to\infty} \frac{\log(N^{-\alpha}|A_{M,N}|)}{\log(MN)}\\ &\leqslant \limsup_{M+N\to\infty} \frac{\log 4K}{\log(MN)} + f_+(\alpha)-\delta = f_+(\alpha)-\delta < f_+(\alpha),\end{aligned}$$

a contradiction. For the remaining cases, note that if $s<0$, $t \geqslant 0$ then

$$N^{-\alpha}|A_{M,N}| \leqslant 2KN^{t-\alpha}, \quad \text{so} \quad \frac{\log(N^{-\alpha}|A_{M,N}|)}{\log(MN)} < \frac{\log 2K}{\log(MN)} + f_+(\alpha) - \delta.$$

If $s \geqslant 0$, $t<0$ then, since $s < f_+(\alpha)-\delta = g_+(\alpha)-\alpha-\delta$,

$$M^{\alpha}|A_{M,N}| \leqslant 2KM^{s+\alpha}, \quad \text{so} \quad \frac{\log(M^{\alpha}|A_{M,N}|)}{\log(MN)} < \frac{\log 2K}{\log(MN)} + g_+(\alpha) - \delta,$$

so

$$\begin{aligned} g_+(\alpha) &= \limsup_{M+N\to\infty} \frac{\log(M^\alpha |A_{M,N}|)}{\log(MN)} \\ &\leqslant \limsup_{M+N\to\infty} \frac{\log 2K}{\log(MN)} + g_+(\alpha) - \delta = g_+(\alpha) - \delta < g_+(\alpha). \end{aligned}$$

Either way we get the same contradiction. Hence, if D converges regularly at $(s_0, t_0) \in \mathbb{R}^2$ then $(s_0, t_0) \in \overline{R_+(D)}$. □

If Theorem 3.16 gives a formula to use with double Dirichlet series that are not regularly convergent at the origin, the next result gives the formula for those who actually are. Since its proof follows basically the same ideas and structure than the proof of Theorem 3.16, we omit it.

Theorem 3.19. *Let $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ be a formal double Dirichlet series which is regularly convergent at the origin. If we write*

$$f_-(\alpha) = \limsup_{M+N\to\infty} \frac{\log\left(N^{-\alpha}|R_{M,N}|\right)}{\log(MN)} \tag{3.37}$$

where $R_{M,N} = \sum_{m=M}^{\infty} \sum_{n=N}^{\infty} a_{m,n}$, then the double Dirichlet series converges regularly in the set

$$R_-(D) = \{(s,t) \in \mathbb{C}^2 \;:\; \textit{there exists } \alpha \in \mathbb{R} \textit{ such that} \\ f_-(\alpha) \in \mathbb{R},\, \operatorname{Re} s > f_-(\alpha),\, \operatorname{Re} t > \alpha + f_-(\alpha)\},$$

and it does not converge regularly in $\mathbb{C}^2 \setminus \overline{R_-(D)}$.

These formaule are not only valid for the ordinary double case, since there are some frequencies $\{\lambda_m\}_m$ and $\{\mu_n\}_n$ for which the analogues of Theorems 3.16, 3.18 and 3.19 can be used in the same manner. These frequencies will have to grow at practically the same speed as the sequence $\{\log n\}_n$, but nevertheless they define Dirichlet series that are not properly ordinary Dirichlet series, so these formulae deserve a separate mention.

Theorem 3.20. *Let* $D(s,t)=\sum_{m,n=1}^{\infty} a_{m,n} e^{-\lambda_m s-\mu_n t}$ *be a formal general double Dirichlet series and suppose that the frequencies* $\{\lambda_m\}_m$ *and* $\{\mu_n\}_n$ *satisfy*

$$\limsup_{m\to\infty} \frac{\log m}{\lambda_m} \leqslant 1, \qquad \limsup_{n\to\infty} \frac{\log n}{\mu_n} \leqslant 1, \tag{3.38}$$

and that there exist some constants C_1 *and* C_2 *such that*

$$|e^{\lambda_{m+1}}-e^{\lambda_m}| \leqslant C_1, \qquad |e^{\mu_{n+1}}-e^{\mu_n}| \leqslant C_2.$$

Writing

$$f_-(\alpha)=\limsup_{M+N\to\infty} \frac{\log\left(e^{-\mu_N\alpha}|R_{M,N}|\right)}{\lambda_M+\mu_N},$$

$$f_+(\alpha)=\limsup_{M+N\to\infty} \frac{\log\left(e^{-\mu_N\alpha}|A_{M,N}|\right)}{\lambda_M+\mu_N},$$

where $R_{M,N}=\sum_{m=M}^{\infty}\sum_{n=N}^{\infty} a_{m,n}$ *and* $A_{M,N}=\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}$*, the double Dirichlet series converges regularly in the set*

$$R_-(D)=\{(s,t)\in\mathbb{C}^2 : \text{there exists } \alpha\in\mathbb{R} \text{ such that}$$
$$f_-(\alpha)\in\mathbb{R},\ \operatorname{Re} s>f_-(\alpha),\ \operatorname{Re} t>\alpha+f_-(\alpha)\}$$

and does not converge regularly in $\mathbb{C}^2\setminus\overline{R_-(D)}$ *if it is regularly convergent at the origin, and if the series does not converge regularly at the origin then it converges regularly in the set*

$$R_+(D)=\{(s,t)\in\mathbb{C}^2 : \text{there exists } \alpha\in\mathbb{R} \text{ such that}$$
$$f_+(\alpha)\in\mathbb{R},\ \operatorname{Re} s>f_+(\alpha),\ \operatorname{Re} t>\alpha+f_+(\alpha)\}$$

and does not converge regularly in $\mathbb{C}^2\setminus\overline{R_+(D)}$.

Proof. Let us suppose first that the general double Dirichlet series does not converge regularly at the origin. The proof follows along the same lines of the ordinary case, the main differences appearing in the proofs of the sufficiency for both formulae. One the one hand, the bound given in (3.18) should be replaced by

$$\begin{aligned}|e^{-\lambda_m s} - e^{\lambda_{m+1}}| &\leqslant \left| \int_{\lambda_m}^{\lambda_{m+1}} -se^{-xs}dx \right| = \left| \int_{e^{\lambda_m}}^{e^{\lambda_{m+1}}} -su^{-s-1}du \right| \\ &\leqslant |e^{\lambda_{m+1}} - e^{\lambda_m}||s| \max_{e^{\lambda_m} \leqslant u \leqslant e^{\lambda_{m+1}}} |u^{-s-1}| \\ &\leqslant C_1|s|e^{-\lambda_m(f_+(\alpha)+\frac{\delta}{2})}e^{-\lambda_m(1+\frac{\delta}{2})}\end{aligned}$$

if $s > f_+(\alpha) > 0$, and (3.24) should be replaced by

$$|e^{-\lambda_m s} - e^{\lambda_{m+1}}| \leqslant C_1|s|e^{-\lambda_{m+1}(f_+(\alpha)+\frac{\delta}{2})}e^{-\lambda_{m+1}(1+\frac{\delta}{2})}$$

for $f_+(\alpha) < s < 0$. Also (3.19) and (3.29) should be replaced by the analogous bounds, so the analogues of inequalities (3.20), (3.21), (3.22) and (3.23) can be obtained. For the case in which either $f_+(\alpha)$ or $\alpha + f_+(\alpha)$ is negative, one should use that

$$e^{\lambda_{m+1}-\lambda_m} \leqslant C_1+1 \quad \text{for all } n \in \mathbb{N}, \qquad e^{\mu_{n+1}-\mu_n} \leqslant C_2+1 \quad \text{for all } n \in \mathbb{N}.$$

Defining in this case $C = \max(1, (C_1+1)^{-f_+(\alpha)}, (C_2+1)^{-(\alpha+f_+(\alpha))})$ and using that both $\sum_{m=1}^{\infty} e^{-\lambda_m s}$ and $\sum_{n=1}^{\infty} e^{-\mu_n t}$ are convergent in $\mathbb{C}_1$ because of (3.38) and the first formula in (2.4), is enough to complete the proof for the sufficiency following the steps of the proof of Theorem 3.16. The remaining parts of the proof are obtained by reproducing the proofs of Lemma 3.17 (analogous to the proof of Lemma 3.14) and the proof of Theorem 3.19. The same should be done for the formula involving $f_-(\alpha)$ in the case in which the series is regularly convergent at the origin. □

Finally, we can obtain some information extra for the curve that defines the sets of regular convergence of double Dirichlet series and we can give some examples to illustrate the different possibilities for this curve. Let us asume that

$$l=\liminf_{M+N\to\infty}\frac{\log\left(|A_{M,N}|\right)}{\lambda_M+\mu_N}\leqslant\limsup_{M+N\to\infty}\frac{\log\left(|A_{M,N}|\right)}{\lambda_M+\mu_N}=f_+(0)=g_+(0).$$

Recall the following elementary property.

$$\limsup_{n\to\infty}a_n+\liminf_{n\to\infty}b_n\leqslant\limsup_{n\to\infty}a_n+b_n\leqslant\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n,\quad(3.39)$$

and consider $a_{M,N}=\frac{\log\left(|A_{M,N}|\right)}{\lambda_M+\mu_N}$ and $b_{M,N}=\frac{\alpha\mu_N}{\lambda_n+\mu_N}$. Note that, for $\alpha>0$,

$$\limsup_{M+N\to\infty}b_{M,N}=\alpha\quad\text{and}\quad\liminf_{M+N\to\infty}b_{M,N}=0.$$

Since $f_+(\alpha)=\limsup_{M+N\to\infty}(a_{M,N}-b_{M,N})$, using (3.39),

$$f_+(\alpha)\geqslant\liminf_{M+N\to\infty}a_{M,N}+\limsup_{M+N\to\infty}(-b_{M,N})=l-\limsup_{M+N\to\infty}b_{M,N}=l.$$

On the other hand, if $\alpha\leqslant 0$, $a_{M,N}-b_{M,N}\geqslant a_{m,n}\geqslant l$, so $f_+(\alpha)\geqslant l$ also when $\alpha\leqslant 0$. Moreover, since f_+ is decreasing, $f_+(\alpha)\leqslant f_+(0)$, so for every $\alpha>0$,

$$l\leqslant f_+(\alpha)\leqslant f_+(0).$$

Analogously one can get that that $l\leqslant g_+(\alpha)$ for every $\alpha\in\mathbb{R}$ and that $g_+(\alpha)\leqslant g_+(0)$ for every $\alpha<0$, due to g_+ being increasing. Therefore the curve $\{(f_+(\alpha),g_+(\alpha)):\alpha\in\mathbb{R}\}$ defined in the plane $\operatorname{Re}s\times\operatorname{Re}t$ is included in the set

$$[l,+\infty)^2\setminus(f_+(0),\infty)^2=[l,f_+(0)]\times[l,+\infty)\cup[l,+\infty)\times[l,f_+(0)],$$

where $(f_+(0), f_+(0))$ is the point of the curve where it is cut by the bisector. In the case in which

$$l = \lim_{M+N\to\infty} \frac{\log(|A_{M,N}|)}{\lambda_M + \mu_N} = f_+(0),$$

this condition turns out to be a necessary one for the curve to be the union of the vertical and the horizontal rays touching at the point $(f_+(0), f_+(0))$.

With these formulae at hand we can give examples for double Dirichlet series with sets of regular convergence defined, for instance, by oblique lines in the plane $\operatorname{Re} s \times \operatorname{Re} t$. If k is a natural number, define $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ with $a_{m,m^k} = 1$ and $a_{m,n} = 0$ otherwise. Then clearly D is not regularly convergent at the origin and

$$f_+(\alpha) = \limsup_{m\to\infty} \frac{\log m^{-k\alpha+1}}{\log m^{k+1}} = \frac{1-k\alpha}{k+1}, \quad \text{and} \quad g_+(\alpha) = \frac{1+\alpha}{k+1}.$$

Since $\frac{g_+(\beta)-g_+(\alpha)}{f_+(\beta)-f_+(\alpha)} = -\frac{1}{k}$, the functions f_+ and g_+ parametrize the line with slope $-\frac{1}{k}$ which passes through the point $(\frac{1}{k+1}, \frac{1}{k+1})$. If we define $\tilde{D}(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{n,m}}{m^s n^t}$,

$$g_+(\alpha) = \limsup_{m\to\infty} \frac{\log m^{\alpha+1}}{\log m^{k+1}} = \frac{\alpha+1}{k+1}, \quad \text{and} \quad f_+(\alpha) = \frac{1-k\alpha}{k+1},$$

so we obtain the line with slope k that goes which passes through the point $(\frac{1}{k+1}, \frac{1}{k+1})$.

Chapter 4

Bounded multiple Dirichlet series

4.1 The algebra $\mathcal{H}_\infty(\mathbb{C}_+^2)$

In the study of multiple Dirichlet series, the case of double series is usually completely analogous to the k-multiple case, and this is exactly what happens when studying the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$, $k \in \mathbb{N}$. However, introducing first the space of bounded double Dirichlet series will allow us to present the ideas of the theory in a simpler and clearer way, since the inductive techniques used in the k-dimensional case often force us to merge some theorems in one big statement where the intuition can sometimes be lost. With this intention, we begin by extending the definition of $\mathcal{H}_\infty(\mathbb{C}_+)$ to double Dirichlet series.

Definition 4.1. We denote by $\mathcal{H}_\infty(\mathbb{C}_+^2)$ the space of all Dirichlet series which are regularly convergent on $\mathbb{C}_+^2$ and define a bounded function there.

The aim now is to see that $\mathcal{H}_\infty(\mathbb{C}_+^2)$ is a Banach algebra when we endow it with the norm $||\cdot||_\infty$,

$$||D||_\infty = \sup_{\substack{\operatorname{Re} s>0 \\ \operatorname{Re} t>0}} \left| \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} \right|.$$

4.1.1 The vector-valued perspective

Remark 4.2. If a double Dirichlet series D is in $\mathcal{H}_\infty(\mathbb{C}_+^2)$, then every row and column subseries of the double Dirichlet series is convergent, because of the regular convergence, so:

1. For every $m \in \mathbb{N}$ and every $t \in \mathbb{C}_+$,

$$\sum_{n=1}^{\infty} \frac{a_{mn}}{m^s n^t} = \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} \text{ converges.}$$

2. For every $n \in \mathbb{N}$ and every $s \in \mathbb{C}_+$,

$$\sum_{m=1}^{\infty} \frac{a_{mn}}{m^s n^t} = \frac{1}{n^t} \sum_{m=1}^{\infty} \frac{a_{mn}}{m^s} \text{ converges.}$$

As regular convergence guarantees that the iterated sums exist and coincide with the sum, then for all $(s,t) \in \mathbb{C}_+^2$

$$\lim_{M\to\infty} \sum_{m=1}^{M} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} = D_t(s) = D(s,t)$$
$$= D_s(t) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^t} \sum_{m=1}^{\infty} \frac{a_{mn}}{m^s},$$

and trivially $||D_s||_\infty \leqslant ||D||_\infty$ and $||D_t||_\infty \leqslant ||D||_\infty$ for every $s,t \in \mathbb{C}_+$, so both of the functions D_s and D_t can be seen as Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}_+)$ when the variable that appears in the subindex is fixed in $\mathbb{C}_+$.

We denote the coefficients of these Dirichlet series by

$$\alpha_m(t) = \sum_{n=1}^{\infty} \frac{a_{mn}}{n^t}, \qquad \beta_n(s) = \sum_{m=1}^{\infty} \frac{a_{mn}}{m^s},$$

which can be seen as Dirichlet series themselves. Moreover, by using Proposition 2.25 again, we have that $|\alpha_m(t)| \leqslant ||D_t||_\infty \leqslant ||D||_\infty$ for every $t \in \mathbb{C}_+$, so $||\alpha_m||_\infty \leqslant ||D||_\infty$, and therefore $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+)$ for every $m \in \mathbb{N}$. Analogously $\beta_n \in \mathcal{H}_\infty(\mathbb{C}_+)$ with $||\beta_n||_\infty \leqslant ||D||_\infty$ for every $n \in \mathbb{N}$. The one-dimensional subseries of D are

$$\sum_{m=1}^{\infty} \frac{a_{m,n}}{m^s n^t} = \frac{1}{n^t}\alpha_m(t), \qquad \sum_{n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} = \frac{1}{m^s}\beta_n(s),$$

but the factors $\frac{1}{n^t}$ and $\frac{1}{m^s}$ do not affect the convergence of the respective series for the respective fixed values of t and s in $\mathbb{C}_+$, so from now on we will instead say that the one-dimensional subseries are α_m and β_n, and the same in the k-dimensional case.

The discussion above justifies that we can see $\mathcal{H}_\infty(\mathbb{C}_+^2)$ as the space of double Dirichlet series which are convergent on $\mathbb{C}_+^2$ and whose row and column subseries are in $\mathcal{H}_\infty(\mathbb{C}_+)$. From this perspective, we can think of double Dirichlet series as vector-valued Dirichlet series whose coefficients are Dirichlet series, where this coefficients may be seen as either the row subseries or the column subseries of the double Dirichlet series. Vector-valued Dirichlet series were studied extensively in [11], where the authors give a vector-valued version of the Theorem 2.20. This version is going to be a fundamental tool in the double case, as we will be able to argue inductively using the vector-valued perspective to go through the induction jump as we try to build an analogue of Theorem 2.20 for double Dirichlet series. The next theorem could be obtained directly from the arguments of the proof of Theorem 3.2 of

[13] and Theorem 2.20, but we give here a simple proof for the sake of completeness.

Theorem 4.3. *Let X be a Banach space. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series with coefficients $\{a_n\} \subset X$ and $\sigma > 0$. Suppose that D is convergent on $\mathbb{C}_\sigma$ to a function $f \in H_\infty(\mathbb{C}_+, X)$, such that $f(s) = D(s)$ in $\mathbb{C}_\sigma$. Then $D \in \mathcal{H}_\infty(\mathbb{C}_+, X)$ and, for every $\delta > 0$, D converges uniformly to f in $\mathbb{C}_\delta$. Furthermore, for each $\delta > 0$ there exists $c_\delta > 0$ only dependent on δ such that*

$$\sup_{\operatorname{Re} s > \delta} \left\| \sum_{n=1}^{N} \frac{a_n}{n^s} - f(s) \right\| \leqslant c_\delta \frac{\log N}{N^\delta} ||f||_\infty.$$

Proof. Let $x^* \in X^*$. Then, $x^* f : \mathbb{C}_+ \to \mathbb{C}$, $x^* f(s) = \sum_{n=1}^{\infty} \frac{x^*(a_n)}{n^s}$ for $s \in \mathbb{C}_+$, and by Theorem 2.20 for every positive δ there exists $c_\delta > 0$ such that

$$\sup_{\operatorname{Re} s > \delta} \left| \sum_{n=1}^{N} \frac{x^*(a_n)}{n^s} - x^* f(s) \right| \leqslant c_\delta \frac{\log N}{N^\epsilon} ||x^* f||_\infty, \text{ that is,}$$

$$\sup_{\operatorname{Re} s > \delta} \left| x^* \left(\sum_{n=1}^{N} \frac{a_n}{n^s} - f(s) \right) \right| \leqslant c_\delta \frac{\log N}{N^\epsilon} ||f||_\infty \text{ for all } x^* \in S_{X^*},$$

and therefore

$$\sup_{\operatorname{Re} s > \delta} \left\| \sum_{n=1}^{N} \frac{a_n}{n^s} - f(s) \right\|_X \leqslant c_\delta \frac{\log N}{N^\epsilon} ||f||_\infty \quad \text{for every} \quad s \in \mathbb{C}_\delta.$$

□

Corollary 4.4. *In Theorem 4.3 we can exchange the right-hand side of the inequality, $c_\varepsilon \frac{\log N}{N^\varepsilon} ||f||_\infty$ for $c_{\varepsilon-\delta} \frac{\log N}{N^{\varepsilon-\delta}} ||f||_{\mathbb{C}_\delta}$, for $0 < \delta < \varepsilon$.*

Proof. Let $\varepsilon > 0$. Suppose the conditions for Theorem 4.3 hold, and for $0 < \delta < \varepsilon$ define $g(t) := f(t + \delta)$, $g(t) := \sum_{n=1}^{\infty} \frac{a_n}{n^{t+\delta}}$. We can apply

Theorem 4.3 to g for $\varepsilon - \delta > 0$, and we get

$$\sup_{\mathrm{Re}\, t > \varepsilon - \delta} \left|\left| \sum_{n=1}^{N} \frac{a_n}{n^{t+\delta}} - g(t) \right|\right|_X \leqslant c_{\varepsilon-\delta} \frac{\log N}{N^{\varepsilon-\delta}} ||g||_\infty = c_{\varepsilon-\delta} \frac{\log N}{N^{\varepsilon-\delta}} ||f||_{\mathbb{C}_\delta}.$$

With the notation $u = t + \delta$, rewriting the previous equation for f,

$$\sup_{\mathrm{Re}\, u > \varepsilon} \left|\left| \sum_{n=1}^{N} \frac{a_n}{n^{u}} - f(u) \right|\right|_X \leqslant c_{\varepsilon-\delta} \frac{\log N}{N^{\varepsilon-\delta}} ||f||_{\mathbb{C}_\delta}.$$

□

Following the inductive intuition form Remark 4.2, we will work to formalize the identification between double Dirichlet series and vector-valued Dirichlet series, that is, we will establish an isometry between the spaces $\mathcal{H}_\infty(\mathbb{C}^2_+)$ and $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$. We do this, but before we need the following proposition.

Proposition 4.5. *If $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ with $D \in \mathcal{H}_\infty(\mathbb{C}^2_+)$, then for every $m, n \in \mathbb{N}$,*

$$|a_{m,n}| \leqslant ||\alpha_m||_\infty \leqslant ||D||_\infty, \qquad |a_{m,n}| \leqslant ||\beta_n||_\infty \leqslant ||D||_\infty.$$

Proof. If $D \in \mathcal{H}_\infty(\mathbb{C}^2_+)$ Remark 4.2 guarantees that $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+)$ and $||\alpha_m||_\infty \leqslant ||D||_\infty$ for every $m \in \mathbb{N}$, where the fact that $|a_{m,n}| \leqslant ||\alpha_m||_\infty$ for every $m, n \in \mathbb{N}$ is a direct application of Proposition 2.25. The proof for the second chain of inequalities is analogous. □

Proposition 4.6. *The mapping defined by*

$$\begin{array}{rcl} \Psi : \mathcal{H}_\infty(\mathbb{C}^2_+) & \to & \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+)) \\ D(s,t) = \displaystyle\sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} & \to & \displaystyle\sum_{m=1}^{\infty} \frac{1}{m^s} \left(\sum_{n=1}^{\infty} \frac{a_{m,n}}{n^t} \right) \end{array}$$

is an isometry into.

Proof. Consider

$$D(s,t)=\sum_{m,n=1}^{\infty}\frac{a_{m,n}}{m^s n^t},\qquad F(s)(t)=\sum_{m=1}^{\infty}\frac{1}{m^s}\left(\sum_{n=1}^{\infty}\frac{a_{m,n}}{n^t}\right)=\sum_{m=1}^{\infty}\frac{\alpha_m(t)}{m^s}.$$

Suppose $D\in\mathcal{H}_\infty(\mathbb{C}_+^2)$. Obviously $\Psi(D)=F$, so it remains to be seen that $F\in\mathcal{H}_\infty(\mathbb{C}_+,\mathcal{H}_\infty(\mathbb{C}_+))$. Remark 4.2 gives that $\alpha_m\in\mathcal{H}_\infty(\mathbb{C}_+)$ so the coefficients of F as a vector-valued Dirichlet series are in $\mathcal{H}_\infty(\mathbb{C}_+)$. Moreover D is regularly convergent, so by Proposition 1.16, for each $(s,t)\in\mathbb{C}_+^2$

$$F(s)(t)=\sum_{m=1}^{\infty}\frac{\alpha_m(t)}{m^s}=\sum_{m=1}^{\infty}\frac{1}{m^s}\left(\sum_{n=1}^{\infty}\frac{a_{m,n}}{n^t}\right)=\sum_{m,n=1}^{\infty}\frac{a_{m,n}}{m^s n^t}=D(s,t).$$

We only need to check that F is convergent on $\mathbb{C}_+$ as a vector-valued Dirichlet series, so we need to prove that $F(s)=\lim_{M\to\infty}\sum_{m=1}^{M}\frac{\alpha_m}{m^s}$ in $\mathcal{H}_\infty(\mathbb{C}_+)$ for all $s\in\mathbb{C}_+$. To do that, choose $\varepsilon>0$ and fix $t\in\mathbb{C}_+$. Recall the Dirichlet series

$$D_t(s)=\sum_{m=1}^{\infty}\frac{\alpha_m(t)}{m^s},$$

where $D_t\in\mathcal{H}_\infty(\mathbb{C}_+)$ by Remark 4.2. Applying Theorem 2.20, given $\delta>0$ there exists $c_\delta>0$ such that

$$\sup_{\operatorname{Re}s>\delta}\left|\sum_{m=1}^{M}\frac{\alpha_m(t)}{m^s}-D_t(s)\right|<c_\delta\frac{\log M}{M^\delta}||D_t||_\infty\leqslant c_\delta\frac{\log M}{M^\delta}||D||_\infty,$$

where the last inequality is given also by Remark 4.2. Then, since $D_t(s)=F(s)(t)$ for all $s,t\in\mathbb{C}_+$,

$$\sup_{\operatorname{Re}t>0}\sup_{\operatorname{Re}s>\delta}\left|\sum_{m=1}^{M}\frac{\alpha_m(t)}{m^s}-F(s)(t)\right|\leqslant c_\delta\frac{\log M}{M^\delta}||D||_\infty.$$

Choosing M_0 such that $M \geqslant M_0$ implies $c_\delta \frac{\log M}{M^\delta} ||D||_\infty < \varepsilon$, we have that, for $M \geqslant M_0$

$$\sup_{\operatorname{Re} s > \delta} \left|\left| \sum_{m=1}^{M} \frac{\alpha_m}{m^s} - F(s) \right|\right|_\infty \leqslant c_\delta \frac{\log M}{M^\delta} ||D||_\infty < \varepsilon,$$

and we get that F is convergent in $\mathbb{C}_+$. Moreover,

$$\begin{aligned} ||F||_\infty = \sup_{\operatorname{Re} s > 0} ||F(s)||_\infty &= \sup_{\operatorname{Re} s > 0} ||D_s||_\infty \\ &= \sup_{\operatorname{Re} s > 0} \left(\sup_{\operatorname{Re} t > 0} |D(s,t)| \right) = ||D||_\infty, \end{aligned}$$

so F is also bounded, and then it is in $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$. This also proves that Ψ is an isometry, and Ψ is into because both D and $F = \Psi(D)$ are characterized by the coefficients $a_{m,n}$. Indeed, if one has $\Psi(D_1) = \Psi(D_2)$ with

$$\Psi(D_j)(s)(t) = \sum_{m=1}^{\infty} \frac{1}{m^s} \left(\sum_{n=1}^{\infty} \frac{a_{m,n}^{(j)}}{n^t} \right), \quad j = 1, 2$$

then by Proposition 4.5

$$|a_{m,n}^{(1)} - a_{m,n}^{(2)}| \leqslant ||\Psi(D_1) - \Psi(D_2)||_\infty = 0,$$

so $D_1 = D_2$. □

It is a natural question to ask if the isometry from Proposition 4.6 is also onto, that is, if a bounded vector-valued Dirichlet series with coefficients in $\mathcal{H}_\infty(\mathbb{C}_+)$ can be seen as a double Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}_+^2)$. We give a positive answer to this question in Theorem 4.11. This will be the fundamental step into making the vector-valued perspective useful in the study of $\mathcal{H}_\infty(\mathbb{C}_+^2)$, as it will allow us to go back and forth between vector-valued Dirichlet series and double Dirichlet series.

The next results are the first steps of proving that $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ is isometrically included in $\mathcal{H}_\infty(\mathbb{C}_+^2)$. Following the definition of regular convergence, we will need to prove that every bounded vector-valued Dirichlet series produces a double Dirichlet series that converges in $\mathbb{C}_+$ and that can also be obtained by taking iterated infinite sums when the order of the sums is exchanged. The lemma below deals with the first of these aims, and it will later be key when we try to get a double case analogue for Theorem 2.20.

Lemma 4.7. *Let* $F \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$. *Write*

$$F(s)(t) = \sum_{m=1}^{\infty} \frac{1}{m^s} \left(\sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} \right),$$

with

$$\alpha_m(t) = \sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} \in \mathcal{H}_\infty(\mathbb{C}_+)$$

and $F(s) = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s}$. *Then for every* $\delta > 0$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{mn}}{m^s n^t} \text{ converges to } F(s)(t)$$

uniformly on $\mathbb{C}_\delta^2$.

Proof. Fix $\delta > 0$ and $\varepsilon > 0$. By Theorem 4.3 there exists c_δ such that

$$\sup_{\operatorname{Re} s > \delta} \left\| \sum_{m=1}^{M} \frac{\alpha_m}{m^s} - F(s) \right\|_\infty < c_\delta \frac{\log M}{M^\delta} ||F||_\infty.$$

Hence, if M_0 is such that $c_\delta \frac{\log M}{M^\delta} ||F||_\infty < \frac{\varepsilon}{2}$ for all $M \geqslant M_0$, then for every $M \geqslant M_0$

$$\sup_{\operatorname{Re} s > \delta} \left\| \sum_{m=1}^{M} \frac{\alpha_m}{m^s} - F(s) \right\|_\infty < \frac{\varepsilon}{2}. \tag{4.1}$$

Now fix $s \in \mathbb{C}_\delta$, and define

$$F_{M,s}(t) = \sum_{m=1}^{M} \frac{\alpha_m(t)}{m^s} = \sum_{n=1}^{\infty} \frac{1}{n^t} \left(\sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right).$$

As every $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+)$, $F_{M,s} \in \mathcal{H}_\infty(\mathbb{C}_+)$ and by (4.1), for every $M \geqslant M_0$ and every $s \in \mathbb{C}_\delta$,

$$||F_{M,s}(t)||_\infty \leqslant ||F(s)||_\infty + \frac{\varepsilon}{2} \leqslant ||F||_\infty + \frac{\varepsilon}{2}.$$

Again, by Theorem 2.20,

$$\begin{aligned} \sup_{\operatorname{Re} t > \delta} \left| \sum_{n=1}^{N} \frac{1}{n^t} \left(\sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right) - F_{M,s}(t) \right| &< c_\delta \frac{\log N}{N^\delta} ||F_{M,s}||_\infty \\ &\leqslant c_\delta \frac{\log N}{N^\delta} (||F||_\infty + \frac{\varepsilon}{2}). \end{aligned}$$

Choosing N_0 such that $N \geqslant N_0$ implies $c_\delta \frac{\log N}{N^\delta}(||F||_\infty + \frac{\varepsilon}{2}) < \frac{\varepsilon}{2}$, we have that, for every $s \in \mathbb{C}_\delta$ and for every $N \geqslant N_0$,

$$\sup_{\operatorname{Re} t > \delta} \left| \sum_{n=1}^{N} \frac{1}{n^t} \left(\sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right) - F_{M,s}(t) \right| < \frac{\varepsilon}{2},$$

and then,

$$\sup_{\substack{\operatorname{Re} s > \delta \\ \operatorname{Re} t > \delta}} \left| \sum_{n=1}^{N} \frac{1}{n^t} \left(\sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right) - F_{M,s}(t) \right| < \frac{\varepsilon}{2}. \tag{4.2}$$

Combining (4.1) and (4.2), for every $N \geqslant N_0$ and every $M \geqslant M_0$,

$$\sup_{\substack{\operatorname{Re} s > \delta \\ \operatorname{Re} t > \delta}} \left| \sum_{n=1}^{N} \frac{1}{n^t} \left(\sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right) - F(s)(t) \right| < \varepsilon.$$

□

The next lemma deals with the second of the aims mentioned above, but before we prove it we need to recall another useful result from [12] on how the coefficients of a Dirichlet series can be recovered from its limit function, as we will need that technique in the next proof.

Proposition 4.8 (Proposition 1.9 of [12])**.** *If $D(t) = \sum_{m=1}^{\infty} \frac{a_m}{m^t}$ is a Dirichlet series, and $\kappa > \sigma_a(D)$, then*

$$a_m = \lim_{R\to\infty} \frac{1}{2Ri} \int_{\kappa-iR}^{\kappa+iR} D(t)m^t dt.$$

Remark 4.9. If $F \in H_\infty(\mathbb{C}_+, H_\infty(\mathbb{C}_+))$, let $f : \mathbb{C}_+^2 \to \mathbb{C}_+$, $f(s,t) = F(s)(t)$. Then $f \in H_\infty(\mathbb{C}_+^2)$. Let us check this. The function f is trivially bounded on $\mathbb{C}_+^2$. To check that f is analytic and bounded on $\mathbb{C}_+^2$ we will use Hartog's theorem (see for instance [11, Theorem 15.7]) and show that it is separably analytic in each variable. For a fixed s, the function $F(s)$ is analytic because it is in $H_\infty(\mathbb{C}_+)$. If we fix $t \in \mathbb{C}_+$, then we are dealing with $\delta_t \circ F$ (where δ_t is the evaluation at t) which is analytic as it is the composition of two analytic functions.

Lemma 4.10. *If a double Dirichlet series D is absolutely convergent in $\mathbb{C}_{\sigma_1} \times \mathbb{C}_{\sigma_2}$ to a function $f \in H_\infty(\mathbb{C}_+^2)$, then both its row and column subseries α_m, β_n are in $\mathcal{H}_\infty(\mathbb{C}_+)$.*

Proof. Take $\sigma = \max(\sigma_1, \sigma_2)$ and write

$$D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}, \quad \alpha_m(t) = \sum_{n=1}^{\infty} \frac{a_{m,n}}{n^t}, \quad \beta_n(s) = \sum_{m=1}^{\infty} \frac{a_{m,n}}{m^s}.$$

Since absolute convergence implies regular convergence, the absolute convergence of D gives the absolute convergence of every subseries, so the column subseries β_n are absolutely convergent in $\mathbb{C}_\sigma$ for every $n \in \mathbb{N}$. We are now able to apply Lemma 1.16 to the double Dirichlet series D

and
$$D(s,t) = \sum_{n=1}^{\infty} \frac{1}{n^t} \left(\sum_{m=1}^{\infty} \frac{a_{m,n}}{m^s} \right) = \sum_{n=1}^{\infty} \frac{\beta_n(s)}{n^t}, \quad (s,t) \in \mathbb{C}^2_\sigma.$$

For a fixed $s \in \mathbb{C}_\sigma$ define formally the Dirichlet series $D_s(t) = \sum_{n=1}^{\infty} \frac{\beta_n(s)}{n^t}$, as well as $f_s(t) = f(s,t)$. Since D is absolutely convergent in $\mathbb{C}^2_\sigma$, $D_s(t)$ is absolutely convergent for $t \in \mathbb{C}_\sigma$ and $D_s(t) = D(s,t) = f(s,t) = f_s(t)$ for $t \in \mathbb{C}_\sigma$, where f_s is a bounded analytic function on $\mathbb{C}_+$ for the $s \in \mathbb{C}_\sigma$ we had previously fixed. Choosing $\delta > 0$ and applying Bohr's Lemma 2.20 we get that D_s converges uniformly to f_s on $\mathbb{C}_\delta$. As this can be done for every $\delta > 0$, D_s converges to f_s on $\mathbb{C}_+$ and then $D_s \in \mathcal{H}_\infty(\mathbb{C}_+)$. We have defined D_s because we want to get convergence of the columns subseries $\beta_n(s)$ in $\mathbb{C}_+$, and these are precisely the coefficients of D_s. With this aim, we will apply to D_s Proposition 4.8. We take $s \in \mathbb{C}_\sigma$ and $n \in \mathbb{N}$ both fixed and $\kappa = \sigma + 1 > \sigma \geqslant \sigma_a(D_s)$, and then

$$\beta_n(s) = \lim_{R\to\infty} \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} D_s(t) n^t dt = \lim_{R\to\infty} \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} f_s(t) n^t dt.$$

We define formally for $s \in \mathbb{C}_+$

$$\begin{aligned} l_n(s) &= \lim_{R\to\infty} \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} f_s(t) n^t dt \\ &= n^{\sigma+1} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(s, \sigma+1+i\tau) n^{i\tau} d\tau. \end{aligned}$$

We are going to show that $l_n(s)$ is well defined not only for $s \in \mathbb{C}_\sigma$ but for $s \in \mathbb{C}_+$, and that it coincides to the corresponding column subseries $\beta_n(s)$ in $\mathbb{C}_\sigma$. In other words, we are going to extend the convergence of $\beta_n(s)$ to $\mathbb{C}_+$ via the function $l_n(s)$. The first thing we have to do is to assure that the limit which defines l_n exists, that is, $\left\{ \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} f_s(t) n^t dt \right\}_{R \in \mathbb{R}_+}$ is a convergent net in $\mathbb{C}$, for the fixed $s \in \mathbb{C}_+$. We will check the Cauchy condition.

First, fix $t \in \mathbb{C}_\sigma$. If $s \in \mathbb{C}_\sigma$, by Proposition 1.16

$$D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{a_{m,n}}{n^t} = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s} = D_t(s),$$

and D_t is absolutely convergent to f_t on $\mathbb{C}_\sigma$, which is analytic and bounded on $\mathbb{C}_+$, so using Bohr's Lemma 2.20 for every $\delta > 0$ there exists $c_\delta > 0$ such that

$$\sup_{\operatorname{Re} s > \delta} \left| f_t(s) - \sum_{m=1}^{M} \frac{\alpha_m(t)}{m^s} \right| \leqslant c_\delta \frac{\log M}{M^\delta} ||f_t||_\infty \leqslant c_\delta \frac{\log M}{M^\delta} ||f||_\infty .$$

Therefore

$$\sup_{\operatorname{Re} t > \sigma} \sup_{\operatorname{Re} s > \delta} \left| f(s,t) - \sum_{m=1}^{M} \frac{\alpha_m(t)}{m^s} \right| \leqslant c_\delta \frac{\log M}{M^\delta} ||f||_\infty . \tag{4.3}$$

Now, taking M_0 such that $M \geqslant M_0$ implies $c_\delta \frac{\log M}{M^\delta} ||f||_\infty < \frac{\varepsilon}{3}$,

$$\begin{aligned}
&\left| \frac{1}{2U} \int_{-U}^{U} f_s(\sigma + 1 + i\tau) n^{i\tau} d\tau - \frac{1}{2R} \int_{-R}^{R} f_s(\sigma + 1 + i\tau) n^{i\tau} d\tau \right| \\
&\leqslant \left| \frac{1}{2U} \int_{-U}^{U} \left(f_s(\sigma + 1 + i\tau) - \sum_{m=1}^{M_0} \frac{\alpha_m(\sigma + 1 + i\tau)}{m^s} \right) n^{i\tau} d\tau \right| \\
&+ \left| \frac{1}{2U} \int_{-U}^{U} \sum_{m=1}^{M_0} \frac{\alpha_m(\sigma + 1 + i\tau)}{m^s} n^{i\tau} d\tau \right. \\
&\qquad\qquad \left. - \frac{1}{2R} \int_{-R}^{R} \sum_{m=1}^{M_0} \frac{\alpha_m(\sigma + 1 + i\tau)}{m^s} n^{i\tau} d\tau \right| \\
&+ \left| \frac{1}{2R} \int_{-R}^{R} \left(f_s(\sigma + 1 + i\tau) - \sum_{m=1}^{M_0} \frac{\alpha_m(\sigma + 1 + i\tau)}{m^s} \right) n^{i\tau} d\tau \right|
\end{aligned}$$

and then we can bound the first and third terms,

$$\begin{aligned}\left|\frac{1}{2U}\int_{-U}^{U}\left(f_s(\sigma+1+i\tau)-\sum_{m=1}^{M_0}\frac{\alpha_m(\sigma+1+i\tau)}{m^s}\right)n^{i\tau}d\tau\right| \\ \leqslant \frac{1}{2U}\int_{-U}^{U}\sup_{\operatorname{Re}t>\sigma}\sup_{\operatorname{Re}s>\delta}\left|f(s,t)-\sum_{m=1}^{M}\frac{\alpha_m(t)}{m^s}\right||n^{i\tau}|d\tau \\ \leqslant c_\delta\frac{\log M}{M^\delta}||f||_\infty<\frac{\varepsilon}{3}.\end{aligned}$$

To bound the remaining term, let us note that we can apply Proposition 4.8 to every $\alpha_m\in\mathcal{H}_\infty(\mathbb{C}_+)$, so for every $m\in\mathbb{N}$,

$$\begin{aligned}\lim_{R\to\infty}\frac{1}{2Ri}\int_{\sigma+1-iR}^{\sigma+1+iR}\alpha_m(t)n^t dt \\ =\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\alpha_m(\sigma+1+i\tau)n^{\sigma+1}n^{i\tau}d\tau=a_{m,n},\end{aligned}$$

and then for all $M\in\mathbb{N}$ and all $s\in\mathbb{C}_+$,

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\sum_{m=1}^{M}\frac{\alpha_m(\sigma+1+i\tau)}{m^s}n^{\sigma+1}n^{i\tau}d\tau=\sum_{m=1}^{M}\frac{a_{m,n}}{m^s}. \tag{4.4}$$

Moreover, given the previous $\varepsilon>0$ we can find some $R_m>0$ such that, if $U,R>R_m$,

$$\begin{aligned}\Bigg|\frac{1}{2U}\int_{-U}^{U}\alpha_m(\sigma+1+i\tau)n^{\sigma+1}n^{i\tau}d\tau \\ -\frac{1}{2R}\int_{-R}^{R}\alpha_m(\sigma+1+i\tau)n^{\sigma+1}n^{i\tau}d\tau\Bigg|<\frac{\varepsilon}{3M_0}\end{aligned}$$

Then, if $R_0 = \max_{1\leqslant m\leqslant M_0} R_m$, $U, R \geqslant R_0$ implies

$$\begin{aligned}\Bigg| \frac{1}{2U}\int_{-U}^{U}\sum_{m=1}^{M_0}\frac{\alpha_m(\sigma+1+i\tau)}{m^s}n^{i\tau}d\tau \\ - \frac{1}{2R}\int_{-R}^{R}\sum_{m=1}^{M_0}\frac{\alpha_m(\sigma+1+i\tau)}{m^s}n^{i\tau}d\tau\Bigg| \\ \leqslant \sum_{m=1}^{M_0}\frac{1}{m^{\operatorname{Re}s}}\Bigg|\frac{1}{2U}\int_{-U}^{U}\alpha_m(\sigma+1+i\tau)n^{i\tau}d\tau \\ - \frac{1}{2R}\int_{-R}^{R}\alpha_m(\sigma+1+i\tau)n^{i\tau}d\tau\Bigg| < \frac{\varepsilon}{3}.\end{aligned}$$

Using these three bounds together, if $s \in \mathbb{C}_\delta$,

$$\left|\frac{1}{2U}\int_{-U}^{U} f_s(\sigma+1+i\tau)n^{i\tau}d\tau - \frac{1}{2R}\int_{-R}^{R} f_s(\sigma+1+i\tau)n^{i\tau}d\tau\right| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Therefore $\left\{\frac{1}{2Ri}\int_{\sigma+1-iR}^{\sigma+1+iR} f_s(t)n^t dt\right\}_{R\in\mathbb{R}_+}$ is uniformly Cauchy on $(s,R) \in \mathbb{C}_\delta \times [0,+\infty[$ and then uniformly convergent on $\mathbb{C}_\delta \times [0,+\infty[$, so for every $s \in \mathbb{C}_+$ there exists

$$\begin{aligned}l_n(s) = \lim_{R\to\infty}\frac{1}{2Ri}\int_{\sigma+1-iR}^{\sigma+1+iR} f_s(t)n^t dt \\ = n^{\sigma+1}\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R} f_s(\sigma+1+i\tau)n^{i\tau}d\tau \in \mathbb{C}.\end{aligned}$$

Then we can apply Bohr's Lemma 2.20 to extend the convergence of $\beta_n(s)$ to $\mathbb{C}_+$ as $l_n(s)$ is the uniform limit on $\mathbb{C}_\delta \times [0,+\infty[$ of the analytic functions

$$g_n(s,R) = \frac{1}{2R}\int_{-R}^{R} f_s(\sigma+1+i\tau)n^{\sigma+1}n^{i\tau}d\tau$$

and it is bounded by $n^{\sigma+1}||f||_\infty$. Moreover, it can be shown that $l_n(s)$ actually coincides with the definition of $\beta_n(s)$ in as the column subseries. To do that fix $s \in \mathbb{C}_+$ and take $0 < \delta < \operatorname{Re} s$. Recall (4.3) and choose

M_0 such that $M \geqslant M_0$ implies $c_\delta \frac{\log M}{M^\delta} ||f||_\infty < \frac{\varepsilon}{3n^{\sigma+1}}$. Also by (4.4) one can find R_0 such that, if $R \geqslant R_0$

$$\left| \frac{1}{2R} \int_{-R}^{R} \sum_{m=1}^{M} \frac{\alpha_m(\sigma+1+i\tau)}{m^s} n^{\sigma+1} n^{i\tau} d\tau - \sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right| < \frac{\varepsilon}{3}.$$

Moreover, for the fixed $s \in \mathbb{C}_+$, there exists some $R_1 \geqslant R_0$ such that $R \geqslant R_1$ implies

$$\left| l_n(s) - \frac{1}{2R} \int_{-R}^{R} f_s(\sigma+1+i\tau) n^{\sigma+1} n^{i\tau} \right| < \frac{\varepsilon}{3}.$$

Therefore, for $R \geqslant R_1$, $M \geqslant M_0$,

$$\begin{aligned} &\left| l_n(s) \ - \sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right| \\ &\qquad \leqslant \left| l_n(s) - \frac{1}{2R} \int_{-R}^{R} f_s(\sigma+1+i\tau) n^{\sigma+1} n^{i\tau} d\tau \right| \\ &\qquad + \frac{1}{2R} \int_{-R}^{R} \sup_{\operatorname{Re} t > \sigma} \sup_{\operatorname{Re} s > \delta} \left| f(s,t) - \sum_{m=1}^{M} \frac{\alpha_m(t)}{m^s} \right| n^{\sigma+1} |n^{i\tau}| d\tau \\ &\qquad + \left| \frac{1}{2R} \int_{-R}^{R} \sum_{m=1}^{M} \frac{\alpha_m(\sigma+1+i\tau)}{m^s} n^{\sigma+1} n^{i\tau} d\tau - \sum_{m=1}^{M} \frac{a_{m,n}}{m^s} \right| < \varepsilon. \end{aligned}$$

This proves that $l_n(s) = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{a_{m,n}}{m^s}$ for every $s \in \mathbb{C}_+$. This can be done in an analogous way to get the convergence of the row subseries α_m. □

The previous lemma is the final piece we need to prove that the isometry i n Proposition 4.6 is also onto. We do this in the theorem below.

Theorem 4.11.

$$\begin{array}{rcl}\Psi : \mathcal{H}_\infty(\mathbb{C}_+^2) & \to & \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+)) \\ D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} & \to & \sum_{m=1}^{\infty} \frac{1}{m^s} \left(\sum_{n=1}^{\infty} \frac{a_{m,n}}{n^t} \right)\end{array}$$

then Ψ is a bijective isometry.

Proof. By Proposition 4.6, we only need to see that Ψ is onto. Take $F \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$,

$$F(s)(t) = \sum_{m=1}^{\infty} \frac{1}{m^s} \left(\sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} \right) = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s},$$

where $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+)$ for every $m \in \mathbb{N}$ and define the double Dirichlet series

$$D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}.$$

We need to show that $D \in \mathcal{H}_\infty(\mathbb{C}_+^2)$. Lemma 4.7 gives the convergence of the series D to the function $f \in H_\infty(\mathbb{C}_+^2)$ on $\mathbb{C}_+^2$, where $f(s,t) = F(s)(t)$. In addition to that, as $\alpha_m(t)$ are the row subseries of D, and they belong to $\mathcal{H}_\infty(\mathbb{C}_+)$, they are convergent on $\mathbb{C}_+$ and we just need to show convergence of the column subseries on $\mathbb{C}_+$. In the first place, applying Proposition 2.25 we get that, for every $m \in \mathbb{N}$,

$$|\alpha_m(t)| \leqslant ||F||_\infty \quad \text{for every } t \in \mathbb{C}_+, \quad \text{so} \quad ||\alpha_m||_\infty \leqslant ||F||_\infty,$$

and applying the same result to α_m,

$$|a_{m,n}| \leqslant ||\alpha_m||_\infty \leqslant ||F||_\infty \quad \text{for every } m, n \in \mathbb{N}.$$

Now, the coefficients are bounded and the series D converges absolutely on $\mathbb{C}_1^2$. Applying now Lemma 4.10, one gets the convergence of the column subseries β_n and therefore the double Dirichlet series $D(s,t)$ converges regularly to the bounded analytic function $f(s,t) = F(s)(t)$, and then it belongs to $\mathcal{H}_\infty(\mathbb{C}_+^2)$. □

The identification of the spaces $\mathcal{H}_\infty(\mathbb{C}_+^2)$ and $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$ given by the isometry of Theorem 4.11 allows us to finally give the extension of Bohr's original result, Theorem 2.22.

Corollary 4.12. *If a double Dirichlet series D is absolutely convergent in $\mathbb{C}_{\sigma_1} \times \mathbb{C}_{\sigma_2}$ to a function $f \in H_\infty(\mathbb{C}_+^2)$, then it converges uniformly to f in $\mathbb{C}_\delta^2$ and $D \in \mathcal{H}(\mathbb{C}_+^2)$.*

Proof. Define $\sigma = \max(\sigma_1, \sigma_2)$. By Lemma 4.10 the row and column subseries of D are in $\mathcal{H}_\infty(\mathbb{C}_+)$, so it only remains to prove the uniform convergence of D to the function f. To do that, define formally $F = \Psi(D)$, so F is a vector-valued Dirichlet series whose coefficients α_m are in $\mathcal{H}_\infty(\mathbb{C}_+)$, and as D converges absolutely on $\mathbb{C}_\sigma^2$, F converges absolutely on $\mathbb{C}_\sigma$ and there it coincides with f_t. As $f_t \in H_\infty(\mathbb{C}_+)$, using Theorem 4.3 we get that $F \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$. Then, as $F = \Psi(D)$ and Ψ is a bijective isometry, $D \in \mathcal{H}_\infty(\mathbb{C}_+^2)$. Lemma 4.7 gives the uniform convergence of the double Dirichlet series. □

4.1.2 $\mathcal{H}_\infty(\mathbb{C}_+^2)$ as a Banach algebra

With Corollary 4.12 we can extend Theorem 2.26 to show that $\mathcal{H}_\infty(\mathbb{C}_+^2)$ is a complete space.

Theorem 4.13. *$\mathcal{H}_\infty(\mathbb{C}_+^2)$ is a complex Banach space.*

Proof. Consider $\{D^{(k)}\}$ a Cauchy sequence of double Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}_+^2) \subset H_\infty(\mathbb{C}_+^2)$, which is complete, so there exists $f \in H_\infty(\mathbb{C}_+^2)$ such that $f = \lim_k D^{(k)}$ in $H_\infty(\mathbb{C}_+^2)$.

Now, for every $k \in \mathbb{N}$, $D^{(k)} \in \mathcal{H}_\infty(\mathbb{C}_+^2)$, so

$$D^{(k)}(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}^{(k)}}{m^s n^t} = \sum_{m=1}^{\infty} \frac{\alpha_m^{(k)}(t)}{m^s} = D_t^{(k)}(s)$$
$$= \sum_{n=1}^{\infty} \frac{\beta_n^{(k)}(s)}{n^t} = D_s^{(k)}(t),$$

where $\alpha_m^{(k)}(t), D_t^{(k)}, \beta_n^{(k)}, D_s^{(k)} \in \mathcal{H}_\infty(\mathbb{C}_+)$ for all $m,n \in \mathbb{N}$ and all $s,t \in \mathbb{C}_+$. By Proposition 2.25

$$|\alpha_m^{(k)}(t)| \leqslant ||D_t^{(k)}||_\infty \leqslant ||D^{(k)}||_\infty \quad \text{for every } t \in \mathbb{C}_+,$$

so

$$||\alpha_m^{(k_1)} - \alpha_m^{(k_2)}||_\infty \leqslant ||D^{(k_1)} - D^{(k_2)}||_\infty,$$

hence $\{\alpha_m^{(k)}\}_k$ is a Cauchy sequence in $\mathcal{H}_\infty(\mathbb{C}_+)$ and therefore, by Theorem 2.26, for each $m \in \mathbb{N}$ there exists $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+)$ such that $\alpha_m = \lim_k \alpha_m^{(k)}$ in $\mathcal{H}_\infty(\mathbb{C}_+)$. Moreover, again by Proposition 2.25,

$$|a_{m,n}^{(k_1)} - a_{m,n}^{(k_2)}| \leqslant ||\alpha_m^{(k_1)} - \alpha_m^{(k_2)}||_\infty \leqslant ||D^{(k_1)} - D^{(k_2)}||_\infty,$$

so there exists $a_{m,n} := \lim_k a_{m,n}^{(k)}$. If we write $\alpha_m(t) = \sum_{n=1}^\infty \frac{b_{m,n}}{n^t}$, using Proposition 2.25 one more time we get that

$$|b_{m,n} - a_{m,n}^{(k)}| \leqslant ||\alpha_m - \alpha_m^{(k)}||_\infty,$$

so

$$b_{m,n} = \lim_k a_{m,n}^{(k)} \quad \text{for every } m,n \in \mathbb{N}.$$

Analogously there exists $\beta_n \in \mathcal{H}_\infty(\mathbb{C}_+)$, $\beta_n = \lim_k \beta_n^{(k)}$ and $\beta_n(s) = \sum_{m=1}^\infty \frac{a_{m,n}}{m^s}$. Define

$$D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}.$$

We have to check that $D(s,t)$ is the limit of the sequence $\{D^{(k)}\}$ in $\mathcal{H}_\infty(\mathbb{C}_+^2)$, that is,

(i) For each $(s,t) \in \mathbb{C}_+^2$,

$$\lim_{(M,N)\to(\infty,\infty)} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}}{m^s n^t} = F(s,t).$$

(ii) For every $m \in \mathbb{N}$ $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+)$.

(iii) For every $n \in \mathbb{N}$ $\beta_n \in \mathcal{H}_\infty(\mathbb{C}_+)$.

The second and the third points have already been seen, so there is only the first one left to check. First let us see that $D(s,t)$ is not everywhere divergent. Consider $\{D^{(k)}\}$, which is a Cauchy sequence in $\mathcal{H}_\infty(\mathbb{C}_+^2)$ and then is bounded by some constant C. Using Proposition 2.25 again,

$$|a_{m,n}^{(k)}| \leqslant ||\alpha_m^{(k)}|| \leqslant ||D^{(k)}|| \leqslant C \quad \text{for all } m,n,k \in \mathbb{N}.$$

Then $|a_{m,n}| = \lim_k |a_{m,n}^{(k)}| \leqslant C$ for all $m,n \in \mathbb{N}$, so $D(s,t)$ converges absolutely on $\mathbb{C}_1^2$. Now we will see that $D(s,t) = f(s,t)$ on $\mathbb{C}_1^2$.

- As $f = \lim_k D^{(k)}$ in $H_\infty(\mathbb{C}_+^2)$, given $\varepsilon > 0$ there exists some $k_1 \in \mathbb{N}$ such that $k \geqslant k_1$ implies

$$\sup_{\substack{\operatorname{Re} s>0 \\ \operatorname{Re} t>0}} |f(s,t) - D^{(k)}(s,t)| < \varepsilon,$$

  that is, if $k \geqslant k_1$ then

$$|f(s,t) - D^{(k)}(s,t)| < \varepsilon \qquad \text{for every } (s,t) \in \mathbb{C}_+^2.$$

- Since $\{D^{(k)}\}$ is a Cauchy sequence in $\mathcal{H}_\infty(\mathbb{C}_+^2)$, using Proposition 2.25 we have that, given $\varepsilon > 0$ there exists $k_2 \in \mathbb{N}$ such that

$k, l \geqslant k_2$ implies

$$|a_{m,n}^{(k)} - a_{m,n}^{(l)}| \leqslant ||D^{(k)} - D^{(l)}||_\infty < \varepsilon,$$

and therefore, if $k \geqslant k_2$ then

$$\lim_{l \to \infty} |a_{m,n}^{(k)} - a_{m,n}^{(l)}| = |a_{m,n}^{(k)} - a_{m,n}| < \varepsilon.$$

- Taking $k_0 = \max\{k_1, k_2\}$ then for a fixed pair $(s,t) \in \mathbb{C}_1^2$ there are $M, N \in \mathbb{N}$ such that

  $$\left| \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}^{(k_0)}}{m^s n^t} - D^{(k_0)}(s,t) \right| < \varepsilon,$$

  and

  $$\left| \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}}{m^s n^t} - D(s,t) \right| < \varepsilon.$$

  Actually, as we have absolute convergence of $D^{(k_0)}$ and D, we could choose M and N to be independent of (s,t). However, this is not relevant as we are going to need to fix the point (s,t) to finish the argument.

Finally, for a fixed pair $(s,t) \in \mathbb{C}_1^2$,

$$\begin{aligned} |f(s,t) - D(s,t)| &\leqslant |f(s,t) - D^{(k_0)}(s,t)| + \left| D^{(k_0)}(s,t) - \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}^{(k_0)}}{m^s n^t} \right| \\ &\quad + \left| \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}^{(k_0)} - a_{m,n}}{m^s n^t} \right| + \left| \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{m,n}}{m^s n^t} - D(s,t) \right| \\ &< \varepsilon \left[3 + \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{m^{\operatorname{Re} s} n^{\operatorname{Re} t}} \right]. \end{aligned}$$

Since $(s,t) \in \mathbb{C}^2_1$ implies that

$$\zeta(\operatorname{Re} s, \operatorname{Re} t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^{\operatorname{Re} s} n^{\operatorname{Re} t}} = \zeta(\operatorname{Re} s)\zeta(\operatorname{Re} t) < \infty,$$

we get that $D(s,t) = f(s,t)$ on $\mathbb{C}^2_1$. Applying now Theorem 4.12, $D \in \mathcal{H}_\infty(\mathbb{C}^2_+)$ and it coincides with f on $\mathbb{C}^2_+$. This completes the proof. □

A consequence of the last argument from the proof of the previous theorem is the following, which tries to generalize the analogous useful consequence of Bohr's Theorem 2.20.

Definition 4.14. For a double Dirichlet series $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ we can define

$$\sigma_b(D) = \inf\{\sigma \in \mathbb{R} : \sup_{\operatorname{Re} s \geqslant \sigma, \operatorname{Re} t \geqslant \sigma} |D(s,t)| < \infty\},$$

and

$$\sigma_u(D) = \inf\{\sigma \in \mathbb{R} : \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{a_{m,n}}{m^s n^t} \text{ converges uniformly to } D(s,t) \text{ on } \mathbb{C}^2_\sigma\}.$$

Corollary 4.15. *If $D \in \mathcal{H}_\infty(\mathbb{C}^2_+)$, then $\sigma_b(D) = \sigma_u(D)$.*

To complete the extension of the simple case it only remains to see that $\mathcal{H}_\infty(\mathbb{C}^2_+)$ is a Banach algebra. However, in the double case the proof is a little longer, as it requires to go through the vector-valued Dirichlet series again.

Proposition 4.16. *If $D(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}$ where $\{b_m\}_m \subset X$, X a Banach space, and D defines a bounded holomorphic function on $\mathbb{C}_+$, then $||b_m||_X \leqslant ||D||_\infty$ for every $m \in \mathbb{N}$.*

Proof. If $x^* \in X^*$ with $||x^*|| \leqslant 1$ and if $s \in \mathbb{C}_+$ is fixed, the convergence of $\sum_{m=1}^{\infty} \frac{b_m}{m^s}$ altogether with the linearity and continuity of x^* give that $(x^*D)(s) = x^*\left(\sum_{m=1}^{\infty} \frac{b_m}{m^s}\right) = \sum_{m=1}^{\infty} \frac{x^*(b_m)}{m^s}$ is a scalar Dirichlet series convergent for that value of s, and as $||x^*D||_\infty \leqslant ||D||_\infty$, $x^*D \in \mathcal{H}_\infty(\mathbb{C}_+)$. Proposition 2.25 gives that $|x^*(b_m)| \leqslant ||x^*D||_\infty \leqslant ||D||_\infty$ for every $m \in \mathbb{N}$. Therefore $||b_m||_X \leqslant ||D||_\infty$. □

Theorem 4.17. *Let X be a Banach algebra, $D_1(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $D_2(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ where $D_1, D_2 \in H_\infty(\mathbb{C}_+, X)$. If $c_k = \sum_{nm=k} a_n b_m$ and $D(s) = \sum_{k=1}^{\infty} \frac{c_k}{n^s}$, then $D \in H_\infty(\mathbb{C}_+, X)$ and $D(s) = D_1(s)D_2(s)$ for every $s \in \mathbb{C}_+$.*

Proof. Using Proposition 4.16, $||a_n||_X \leqslant ||D_1||_\infty$, and that implies $\sigma_a(D_1) \leqslant 1$. Analogously $\sigma_a(D_2) \leqslant 1$, and then, if $\sigma = \operatorname{Re} s > 1$ we can define

$$\sum_{k=1}^{\infty} \frac{\tilde{c}_k}{k^\sigma} := \left(\sum_{n=1}^{\infty} \frac{||a_n||}{n^\sigma}\right)\left(\sum_{m=1}^{\infty} \frac{||b_m||}{m^\sigma}\right), \quad \tilde{c}_k = \sum_{nm=k} ||a_n||\,||b_m||.$$

It is obvious that $||c_k|| \leqslant \tilde{c}_k$, so if $\sigma > 1$ the series $\sum_{k=1}^{\infty} \frac{||c_k||}{k^\sigma}$ is convergent and $D_1(s)D_2(s) = \sum_{k=1}^{\infty} \frac{c_k}{k^s}$ converges absolutely on $\mathbb{C}_1$. Now we can apply Theorem 4.3 to $D_1(s)D_2(s) \in H_\infty(\mathbb{C}_+, X)$ and we get that $\sum_{k=1}^{\infty} \frac{c_k}{k^s}$ converges in $\mathbb{C}_+$ and it coincides with $D_1(s)D_2(s)$. □

Theorem 4.18. *If $D_1, D_2 \in \mathcal{H}_\infty(\mathbb{C}_+^2)$, then $D_1D_2 \in \mathcal{H}_\infty(\mathbb{C}_+^2)$.*

Proof. If $D_1, D_2 \in \mathcal{H}_\infty(\mathbb{C}_+^2)$, $D_1(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ and $D_2(s,t) = \sum_{m,n=1}^{\infty} \frac{b_{m,n}}{m^s n^t}$, consider

$$\Psi(D_1)(s)(t) = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s}, \qquad \Psi(D_2)(s,t) = \sum_{m=1}^{\infty} \frac{\gamma_m(t)}{m^s},$$

where $\Psi(D_1), \Psi(D_2) \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$. Then by Theorem 4.17,

$$\Psi(D_1)(s)(t)\Psi(D_2)(s)(t) = \sum_{p=1}^{\infty} \left(\sum_{mk=p} \alpha_m(t)\gamma_k(t) \right) \frac{1}{p^s},$$

with $\Psi(D_1)(s)\Psi(D_2)(s) \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$, but then by Remark 2.27

$$\alpha_m(t)\gamma_k(t) = \left(\sum_{n=1}^{\infty} \frac{a_{m,n}}{n^t} \right) \left(\sum_{n=1}^{\infty} \frac{b_{m,n}}{n^t} \right) = \sum_{q=1}^{\infty} \left(\sum_{nl=q} a_{m,n} b_{k,l} \right) \frac{1}{q^t},$$

$\alpha_m \gamma_k \in \mathcal{H}_\infty(\mathbb{C}_+)$, so by Theorem 4.11

$$\begin{aligned} D_1(s,t)D_2(s,t) &= \sum_{p=1}^{\infty} \left(\sum_{mk=p} \sum_{q=1}^{\infty} \left(\sum_{nl=q} a_{m,n} b_{k,l} \right) \frac{1}{q^t} \right) \frac{1}{p^s} \\ &= \sum_{p=1}^{\infty} \left(\sum_{q=1}^{\infty} \underbrace{\sum_{mk=p} \sum_{nl=q}}_{\text{finite sums}} a_{m,n} b_{k,l} \frac{1}{q^t} \right) \frac{1}{p^s} = \\ & \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left(\sum_{mk=p} \sum_{nl=q} a_{m,n} b_{k,l} \right) \frac{1}{p^s q^t}, \end{aligned}$$

and $D_1(s,t)D_2(s,t) \in \mathcal{H}_\infty(\mathbb{C}_+^2)$. □

4.2 The algebras $\mathcal{H}_\infty(\mathbb{C}_+^k)$.

In this section we study the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$, $k \in \mathbb{N}$, following the structure of the previous section, with the aim of proving that, for any $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is a Banach algebra. First let us define properly these spaces.

Definition 4.19. For every $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is the space of all k-multiple Dirichlet series that are regularly convergent on $\mathbb{C}_+^k$ to a bounded holomorphic function $f \in H_\infty(\mathbb{C}_+^k)$.

The aim of this section is to see that:

- The space $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is complete for every $k \in \mathbb{N}$.
- The space $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is an algebra for every $k \in \mathbb{N}$.

For the completeness of the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ awe need the k-dimensional extension of Theorem 2.20. However, the inductive argument that we are going to develop requires combining the extension of Theorems 2.20, 4.7 and 4.10 in a single statement. Before we give this next result, given $k \in \mathbb{N}$ and $\delta > 0$, consider the constant c_δ from Theorem 2.20 and define

$$C_\delta(\mathcal{M}) = \sum_{j=1}^{k} c_\delta^j \left[\sum_{\substack{\mathcal{A} \in \mathcal{P}(\mathcal{M}) \\ |\mathcal{A}|=j}} \prod_{N \in \mathcal{A}} \frac{\log N}{N^\delta} \right],$$

where $\mathcal{M} = \{M_1, \dots, M_k\} \subset \mathbb{N}$ and $\mathcal{P}(\mathcal{M})$ denotes the family of subsets of $\mathcal{M}$. This new constant, which only depends on δ and on the indexes $M_1, \dots, M_k$, plays an important role in the theorem below and it verifies that $\lim_{\substack{M_j \to \infty \\ 1 \leqslant j \leqslant k}} C_\delta\{M_1, \dots, M_k\} = 0$. Indeed,

$$\begin{aligned} C_\delta(\mathcal{M}) = \sum_{j=1}^{k} c_\delta^j \left[\sum_{\substack{\mathcal{A} \in \mathcal{P}(\mathcal{M}) \\ |\mathcal{A}|=j}} \prod_{N \in \mathcal{A}} \frac{\log N}{N^\delta} \right] &\leqslant \sum_{j=1}^{k} c_\delta^j \binom{k}{j} \left(\max_{1 \leqslant l \leqslant k} \frac{\log M_l}{M_l^\delta} \right)^j \\ &= \left(1 + c_\delta \max_{1 \leqslant l \leqslant k} \frac{\log M_l}{M_l^\delta} \right)^k - 1 \xrightarrow[1 \leqslant j \leqslant k]{M_j \to \infty} 0. \end{aligned}$$

Theorem 4.20. *Consider a multiple Dirichlet series*

$$D(s_1, \dots, s_k) = \sum_{m_1, \dots, m_k = 1}^{\infty} \frac{a_{m_1, \dots, m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$$

which converges regularly on $\mathbb{C}_\sigma^k$ to a function $f \in H_\infty(\mathbb{C}_+^k)$ for some positive σ. Then $D \in \mathcal{H}_\infty(\mathbb{C}_+^k)$ and, for every positive δ, D converges uniformly to f on $\mathbb{C}_\delta^k$. Furthermore,

(1) All of the $(k-1)$-dimensional subseries of D belong to $\mathcal{H}_\infty(\mathbb{C}_+^{k-1})$.

(2) For every $\delta > 0$

$$\sup_{\substack{\operatorname{Re} s_j > \delta \\ 1 \leqslant j \leqslant k}} \left| \sum_{m_1=1}^{M_1} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}} - f(s_1, \dots, s_k) \right| \leqslant C_\delta(\{M_1, \dots, M_k\}) ||f||_\infty.$$

Proof. We will prove this result by induction on k. The case $k = 1$ is just Theorem 2.20. Therefore, we will assume that the result is true for $k-1$ and we will prove it for k.

The $(k-1)$-dimensional subseries of D are obtained when we fix one of the summation indexes $m_1, \dots, m_k$ to one given value of $\mathbb{N}$. For instance, if $(s_1, \dots, s_k) \in \mathbb{C}_\sigma^k$,

$$D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}} = \sum_{m_1=1}^{\infty} \frac{\alpha_{m_1}(s_2, \dots, s_k)}{m_1^{s_1}},$$

where $\alpha_{m_1}(s_2, \dots, s_k) = \sum_{m_2,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_2^{s_2} \dots m_k^{s_k}}$ and the equality is justified by the regular convergence on $\mathbb{C}_\sigma^k$. We want to show that $\alpha_{m_1} \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$ for any given value of $m_1 \in \mathbb{N}$. To simplify the notation we write $m = m_1$, $\alpha_m = \alpha_{m_1}$, $s = s_1$, $t = (s_2, \dots, s_k)$. Then

$$D(s;t) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}} = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s} = D_t(s),$$

where, for every fixed $t \in \mathbb{C}_\sigma^{k-1}$, D_t is a Dirichlet series which converges absolutely to the function $f_t \in H_\infty(\mathbb{C}_+^{k-1})$, $f_t(s) = f(s;t)$. By Theorem

2.20, $D_t \in \mathcal{H}_\infty(\mathbb{C}_+)$ for every fixed $t \in \mathbb{C}_\sigma^{k-1}$. Now, fix $m \in \mathbb{N}$ and consider α_m which, by Proposition 4.8, satisfies

$$\alpha_m(t) = \lim_{R\to\infty} \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} D_t(s) m^s ds = \lim_{R\to\infty} \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} f_t(s) m^s ds.$$

Take $t \in \mathbb{C}_+^{k-1}$ and define formally

$$\begin{aligned} h_m(t) = \lim_{R\to\infty} \frac{1}{2Ri} \int_{\sigma+1-iR}^{\sigma+1+iR} f_t(s) m^s ds \\ = m^{\sigma+1} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(\sigma+1+iu;t) m^{iu} du. \end{aligned}$$

The idea is to show that h_m is well-defined for all $t \in \mathbb{C}_+^{k-1}$ and then that it extends α_m to $\mathbb{C}_+^{k-1}$. Thus, we need to show that

$$\left\{ \frac{1}{2R} \int_{-R}^{R} f(\sigma+1+iu;t) m^{iu} du \right\}_{R\in\mathbb{R}_+}$$

is a Cauchy net . Actually we are going to see that it is uniformly Cauchy with respect to t. Let $\varepsilon > 0$ and $t \in \mathbb{C}_+^{k-1}$, $t = (s_2, \dots, s_k)$. On the one hand, for a fixed $s \in \mathbb{C}_\sigma$ consider

$$\omega(s) = \sum_{m=1}^{\infty} \frac{a_{m,m_2,\dots,m_k}}{m^s} \quad \text{and} \quad D_s(t) = \sum_{m_2,\dots,m_k}^{\infty} \frac{\omega(s)}{m_2^{s_2} \cdots m_k^{s_k}},$$

where $\omega(s)$ is a convergent series as it is a one-dimensional subseries of D, and D_s is a $(k-1)$-multiple Dirichlet series which is regularly convergent on $\mathbb{C}_\sigma^{k-1}$ to the function g_s, with $g_s(t) = f(s;t)$ and $g_s \in H_\infty(\mathbb{C}_+^{k-1})$. For any positive δ the induction hypothesis gives that

$$\sup_{\substack{\operatorname{Re} s_j>\delta\\ 2\leqslant j\leqslant k}}\left|\sum_{m_2=1}^{M_2}\cdots\sum_{m_k=1}^{M_k}\frac{\omega(s)}{m_2^{s_2}\cdots m_k^{s_k}}-g_s(t)\right|\leqslant C_\delta(\{M_2,\dots,M_k\})||g_s||_\infty$$
$$\leqslant C_\delta(\{M_2,\dots,M_k\})||f||_\infty.$$

Therefore

$$\sup_{\operatorname{Re} s>\sigma}\sup_{\substack{\operatorname{Re} s_j>\delta\\ 2\leqslant j\leqslant k}}\left|\sum_{m_2=1}^{M_2}\cdots\sum_{m_k=1}^{M_k}\frac{\omega(s)}{m_2^{s_2}\cdots m_k^{s_k}}-f(s;t)\right|$$
$$\leqslant C_\delta(\{M_2,\dots,M_k\})||f||_\infty.$$

As $\lim_{M_2,\dots,M_k\to\infty} C_\delta(\{M_2,\dots,M_k\})=0$ we can choose $M_0\in\mathbb{N}$ such that $M_2,\dots,M_k\geqslant M_0$ implies $C_\delta(\{M_2,\dots,M_k\})||f||_\infty<\frac{\varepsilon}{3}$ and then

$$\sup_{\operatorname{Re} s>\sigma}\sup_{\substack{\operatorname{Re} s_j>\delta\\ 2\leqslant j\leqslant k}}\left|\sum_{m_2=1}^{M_2}\cdots\sum_{m_k=1}^{M_k}\frac{\omega(s)}{m_2^{s_2}\cdots m_k^{s_k}}-f(s;t)\right|<\frac{\varepsilon}{3}. \tag{4.5}$$

On the other hand, ω is a Dirichlet series which converges absolutely on $\mathbb{C}_\sigma$, so by Proposition 4.8

$$a_{m,m_2,\dots,m_k}=m^{\sigma+1}\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\omega(\sigma+1+iu)m^{iu}du.$$

Thus, for a fixed $t=(s_2,\dots,s_k)\in\mathbb{C}_+^{k-1}$,

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\sum_{m_2=1}^{M_0}\cdots\sum_{m_k=1}^{M_0}\frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}}m^{iu}du$$
$$=\frac{1}{m^{\sigma+1}}\sum_{m_2=1}^{M_0}\cdots\sum_{m_k=1}^{M_0}\frac{a_{m,m_2,\dots,m_k}}{m_2^{s_2}\cdots m_k^{s_k}}$$

so there exists $R_0 > 0$ such that $R_1, R_2 > R_0$ implies

$$\left| \frac{1}{2R_1} \int_{-R_1}^{R_1} \sum_{m_2=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}} m^{iu} du \right.$$
$$\left. - \frac{1}{2R_2} \int_{-R_2}^{R_2} \sum_{m_2=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}} m^{iu} du \right| < \frac{\varepsilon}{3}. \quad (4.6)$$

Using (4.5) and (4.6), with $R_1, R_2 > R_0$,

$$\left| \frac{1}{2R_1} \int_{-R_1}^{R_1} f(\sigma+1+iu;t) m^{iu} du - \frac{1}{2R_2} \int_{-R_2}^{R_2} f(\sigma+1+iu;t) m^{iu} du \right|$$
$$\leqslant \left| \frac{1}{2R_1} \int_{-R_1}^{R_1} \left(f(\sigma+1+iu;t) m^{iu} \right.\right.$$
$$\left.\left. - \sum_{m_2=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}} \right) m^{iu} du \right|$$
$$+ \left| \frac{1}{2R_1} \int_{-R_1}^{R_1} \sum_{m_2=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}} m^{iu} du \right.$$
$$\left. - \frac{1}{2R_2} \int_{-R_2}^{R_2} \sum_{m_2=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}} m^{iu} du \right|$$
$$+ \left| \frac{1}{2R_2} \int_{-R_2}^{R_2} \left(f(\sigma+1+iu;t) m^{iu} \right.\right.$$
$$\left.\left. - \sum_{m_2=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{\omega(\sigma+1+iu)}{m_2^{s_2}\cdots m_k^{s_k}} \right) m^{iu} du \right| < 3\frac{\varepsilon}{3} = \varepsilon.$$

This proves that h_m is well-defined in $\mathbb{C}_+^{k-1}$. As the choice of R_0 is independent of $t = (s_2, \ldots, s_k) \in \mathbb{C}_+^{k-1}$, h_m is the uniform limit on R of the functions

$$H_m(R,t) = \frac{1}{2R} \int_{-R}^{R} f(\sigma+1+iu;t) m^{\sigma+1+iu} du$$

which are holomorphic on $\mathbb{C}_+^{k-1}$ for every $R>0$, so h_m is also holomorphic and for all $r\in\mathbb{R}_+$, $t\in\mathbb{C}_+^{k-1}$,

$$|H_m(R,t)| \leqslant \frac{1}{2R}\int_{-R}^{R} ||f||_\infty m^{\sigma+1} du = ||f||_\infty m^{\sigma+1},$$

hence $||h_m|| \leqslant ||f||_\infty m^{\sigma+1}$ and therefore $h_m \in H_\infty(\mathbb{C}_+^{k-1})$. Using the induction hypothesis, α_m is a $(k-1)$-multiple Dirichlet series which converges regularly on $\mathbb{C}_\sigma^{k-1}$ and it coincides there with $h_m \in H_\infty(\mathbb{C}_+^{k-1})$, so $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$.
To sum up, we have proved that every $(k-1)$-dimensional subseries of D obtained by fixing the first summation index to any given positive integer belongs to $\mathcal{H}_\infty(\mathbb{C}_+^{k-1})$. The proof for the other $(k-1)$-dimensional subseries of D, obtained by fixing one of the other summation indexes instead of the first one, is analogous. This completes the proof of (1).
We set now to prove (2). We continue using the simplified notation we stated at the beginning of the proof. The idea is to use Theorem 4.3 with $\sum_{m=1}^\infty \frac{\alpha_m}{m^s}$ now that we have that $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$ for every $m\in\mathbb{N}$, and then to use the induction hypothesis. To do that, define $G:\mathbb{C}_+ \to H_\infty(\mathbb{C}_+^{k-1})$ by $G(s)(t) = f(s;t)$. We need to check that $G(s) \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$ for every $s\in\mathbb{C}_+$ so we can apply Theorem 4.3. If $(s;t)\in\mathbb{C}_\sigma^k$, then $G(s)(t) = \sum_{m=1}^\infty \frac{\alpha_m(t)}{m^s} = D_s(t)$, where D_s is a $(k-1)$-multiple Dirichlet series that converges regularly in $\mathbb{C}_\sigma^{k-1}$ to $f_s \in H_\infty(\mathbb{C}_+^{k-1})$, so by the induction hypothesis $D_s \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$ and therefore, as $D_s = G(s)$, we obtain $G(s)\in\mathcal{H}_\infty(\mathbb{C}_+^{k-1})$. Now, by Theorem 4.3

$$\sup_{\operatorname{Re} s>\delta} \left|\left| \sum_{m=1}^{M} \frac{\alpha_m}{m^s} - G(s) \right|\right|_\infty \leqslant c_\delta ||G||_\infty \frac{\log M}{M^\delta},$$

and then, since $||G||_\infty = ||f||_\infty$,

$$\sup_{\operatorname{Re} s > \delta} \sup_{\substack{\operatorname{Re} s_j > 0 \\ 2 \leqslant j \leqslant k}} \left| \sum_{m=1}^{M} \frac{\alpha_m(t)}{m^s} - f(s;t) \right| \leqslant c_\delta ||f||_\infty \frac{\log M}{M^\delta}. \tag{4.7}$$

Fix $s \in \mathbb{C}_\delta$ and define $D_{M,s} = \sum_{m=1}^{M} \frac{\alpha_m}{m^s}$. As $D_{M,s}$ is a linear combination of $\alpha_m \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$, then $D_{M,s} \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$. Note that

$$\begin{aligned} ||D_{M,s}||_\infty &\leqslant ||D_{M,s} - G(s)||_\infty + ||G(s)||_\infty \\ &\leqslant c_\delta ||f||_\infty \frac{\log M}{M^\delta} + ||f||_\infty = ||f||_\infty \left(c_\delta \frac{\log M}{M^\delta} + 1 \right). \end{aligned}$$

Moreover,

$$D_{M,s}(t) = \sum_{m=1}^{M} \frac{\alpha_m(t)}{m^s} = \sum_{m_2,\dots,m_k}^{\infty} \frac{1}{m_2^{s_2} \cdots m_k^{s_k}} \left(\sum_{m=1}^{M} \frac{a_{m,m_2,\dots,m_k}}{m^s} \right).$$

Now, by the induction hypothesis, if $\mathcal{M} = \{M_2, \dots, M_k\}$,

$$\begin{aligned}
&\sup_{\substack{\operatorname{Re} s_j>0\\ 2\leqslant j\leqslant k}} \left| \sum_{m_2=1}^{M_2} \cdots \sum_{m_k=1}^{M_k} \frac{1}{m_2^{s_2}\cdots m_k^{s_k}} \left(\sum_{m=1}^{M} \frac{a_{m,m_2,\dots,m_k}}{m^s} \right) - D_{M,s}(t) \right| \\
&\leqslant C_\delta(\{M_2, \dots, M_k\}) ||D_{M,s}||_\infty \\
&\leqslant ||f||_\infty \left(c_\delta \frac{\log M}{M^\delta} + 1 \right) \sum_{j=1}^{k-1} c_\delta^j \left[\sum_{\substack{\mathcal{A}\in\mathcal{P}(\mathcal{M})\\ |\mathcal{A}|=j}} \prod_{N\in\mathcal{A}} \frac{\log N}{N^\delta} \right] \\
&= ||f||_\infty \sum_{j=1}^{k-1} c_\delta^{j+1} \left[\sum_{\substack{\mathcal{A}\in\mathcal{P}(\mathcal{M})\\ |\mathcal{A}|=j}} \frac{\log M}{M^\delta} \prod_{N\in\mathcal{A}} \frac{\log N}{N^\delta} \right] \\
&\qquad + ||f||_\infty \sum_{j=1}^{k-1} c_\delta^j \left[\sum_{\substack{\mathcal{A}\in\mathcal{P}(\mathcal{M})\\ |\mathcal{A}|=j}} \prod_{N\in\mathcal{A}} \frac{\log N}{N^\delta} \right]
\end{aligned} \tag{4.8}$$

Therefore, if $(s;t) \in \mathbb{C}_\delta^k$, using (4.7) and (4.8)

$$\begin{aligned}
&\left| \sum_{m=1}^{M} \sum_{m_2=1}^{M_2} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m,m_2,\dots,m_k}}{m^s m_2^{s_2}\cdots m_k^{s_k}} - f(s;t) \right| \\
&\leqslant \left| \sum_{m=1}^{M} \sum_{m_2=1}^{M_2} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m,m_2,\dots,m_k}}{m^s m_2^{s_2}\cdots m_k^{s_k}} - D_{M,s}(t) \right| + |D_{M,s}(t) - f(s;t)| \\
&\leqslant ||f||_\infty \sum_{j=1}^{k-1} c_\delta^j \left[\sum_{\substack{\mathcal{A}\in\mathcal{P}(\mathcal{M})\\ |\mathcal{A}|=j}} \prod_{N\in\mathcal{A}} \frac{\log N}{N^\delta} \right] \\
&+ ||f||_\infty \sum_{j=1}^{k-1} c_\delta^{j+1} \left[\sum_{\substack{\mathcal{A}\in\mathcal{P}(\mathcal{M})\\ |\mathcal{A}|=j}} \frac{\log M}{M^\delta} \prod_{N\in\mathcal{A}} \frac{\log N}{N^\delta} \right] + c_\delta \frac{\log M}{M^\delta} \\
&= C_\delta(\{M, M_2, \dots, M_k\}) ||f||_\infty.
\end{aligned}$$

As $\lim_{M,M_2,\dots,M_k\to\infty} C_\delta(\{M,M_2,\dots,M_k\})=0$,

$$\lim_{M,M_2,\dots,M_k\to\infty} \sup_{\substack{\operatorname{Re} s_j>\delta \\ 1\leqslant j\leqslant k}} \left| \sum_{m=1}^{M}\sum_{m_2=1}^{M_2}\cdots\sum_{m_k=1}^{M_k} \frac{a_{m,m_2,\dots,m_k}}{m^s m_2^{s_2}\cdots m_k^{s_k}} - f(s;t)\right| = 0$$

and D converges uniformly to f on $\mathbb{C}^k_\delta$. It only remains to check that $D\in\mathcal{H}_\infty(\mathbb{C}^k_+)$. The uniform convergence of D in $\mathbb{C}^k_\delta$ for every positive δ gives the convergence of D to f in $\mathbb{C}^k_+$. Moreover, by (1) all of $(k-1)$-dimensional subseries of D are in $\mathcal{H}_\infty(\mathbb{C}^{k-1}_+)$ so they are regularly convergent on $\mathbb{C}^{k-1}_+$ and then all of their j-dimensional subseries are convergent, for $1\leqslant j\leqslant k$. All of those subseries account for all the j-dimensional subseries of D, for $1\leqslant j\leqslant k$, so D is regularly convergent on $\mathbb{C}^k_+$. Finally, as f is bounded, $||D||_\infty=||f||_\infty$ and $D\in\mathcal{H}_\infty(\mathbb{C}^k_+)$. □

Theorem 4.20 will allow us, as in the double case, to prove that $\mathcal{H}_\infty(\mathbb{C}^k_+)$ is a Banach space, for every $k\in\mathbb{N}$. Before we do that we give the extension of Proposition 2.25, as we will need it to prove the completeness of $\mathcal{H}_\infty(\mathbb{C}^k_+)$.

Proposition 4.21. *For every Dirichlet series in $D\in\mathcal{H}_\infty(\mathbb{C}^k_+)$, we have*

$$|a_{m_1,\dots,m_k}|\leqslant \left|\left|\sum_{m_{j_k}=1}^{\infty}\frac{a_{m_1,\dots,m_k}}{m_{j_k}^{s_{j_k}}}\right|\right|_\infty \leqslant\cdots\leqslant ||\alpha_{m_{j_1}}||_\infty\leqslant ||D||_\infty$$

for any order $m_{j_1},\dots,m_{j_k}$ of the indexes $m_1,\dots,m_k$.

Proof. By induction on k, the result from Proposition 2.25 corresponds to the case $k=1$. Suppose the result is true for any Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}^{k-1}_+)$, then if $D\in\mathcal{H}_\infty(\mathbb{C}^k_+)$,

$$D(s_1,\dots,s_k)=\sum_{m_{j_1}=1}^{\infty}\frac{\alpha_{m_{j_1}}(s_1,\dots,s_{j_1-1},s_{j_1+1},\dots,s_k)}{m_{j_1}^{s_{j_1}}},$$

for any choice m_{j_1} among the indexes $m_1, \ldots, m_k$. Then, on the one hand, fix an arbitrary $t_{j_1} = (s_1, \ldots, s_{j_1-1}, s_{j_1+1}, \ldots, s_k) \in \mathbb{C}_+^{k-1}$ and consider

$$D_{t_{j_1}}(s_{j_1}) = \sum_{m_{j_1}=1}^{\infty} \frac{\alpha_{m_{j_1}}(t_{j_1})}{m_{j_1}^{s_{j_1}}} = D(s_1, \ldots, s_k)$$

which is a Dirichlet series in $\mathcal{H}_\infty(\mathbb{C}_+)$, so for every $t_{j_1} \in \mathbb{C}_+^{k-1}$,

$$|\alpha_{m_{j_1}}(t_{j_1})| \leqslant ||D_{t_{j_1}}||_\infty = \sup_{\operatorname{Re} s_{j_1} > 0} |D_{t_{j_1}}(s_{j_1})| \leqslant ||D||_\infty.$$

Therefore $||\alpha_{m_{j_1}}||_\infty \leqslant ||D||_\infty$. On the other hand, using the induction hypothesis we get that

$$|a_{m_1,\ldots,m_k}| \leqslant \left\| \sum_{m_{j_k}=1}^{\infty} \frac{a_{m_1,\ldots,m_k}}{m_{j_k}^{s_{j_k}}} \right\|_\infty \leqslant \cdots \leqslant ||\alpha_{m_{j_1}}||_\infty.$$

Connecting this chain of inequalities to the previous inequality, we get the desired result. □

Remark 4.22. Proposition 4.21 guarantees that k-multiple Dirichlet series with a point of regular convergence are characterized by their multiple sequence of coefficients. Indeed, if the multiple Dirichlet series series

$$D(s_1, \ldots, s_k) = \sum_{m_1,\ldots,m_k=1}^{\infty} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$$

converges regularly at a point $(z_1, \ldots, z_k)$ with $\sigma_j = \operatorname{Re} z_j$, $1 \leqslant j \leqslant k$, then using Proposition 3.6 and Proposition 4.21

$$\left| \frac{a_{m_1,\ldots,m_k}}{m_1^{\sigma_1+1+\delta} \cdots m_k^{\sigma_k+1+\delta}} \right| \leqslant ||D||_\infty.$$

Therefore if the series D is identically zero, every coefficient is zero.

Theorem 4.23. *For every $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is a Banach space.*

Proof. We prove this result inductively. The case $k = 1$ is Theorem 2.26. For the general case, assume the result is true for $k-1$ and take $\{D^{(j)}\}_{j=1}^{\infty} \subset \mathcal{H}_\infty(\mathbb{C}_+^k)$ a Cauchy sequence,

$$D^{(j)}(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a^{(j)}_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}.$$

As $\{D^{(j)}\}_{j=1}^{\infty}$ is a sequence of bounded functions which is Cauchy with the supremum norm, it converges uniformly to some $f \in H_\infty(\mathbb{C}_+)$. Our aim now is to show that f can be represented by a Dirichlet series D in $\mathbb{C}_\sigma^k$ for some positive σ to apply Theorem 4.20.

For every $j \in \mathbb{N}$, Theorem 4.20 gives that $\alpha^{(j)}_{m_l} \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$ for $1 \leqslant l \leqslant k$. Moreover, Proposition 4.21 gives that $||\alpha^{(j_1)}_{m_l} - \alpha^{(j_2)}_{m_l}||_\infty \leqslant ||D^{(j_1)} - D^{(j_2)}||_\infty$ for $1 \leqslant l \leqslant k$, so $\{\alpha^{(j)}_{m_l}\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}_\infty(\mathbb{C}_+^{k-1})$, which, by our induction hypothesis, is complete, so for each $1 \leqslant l \leqslant k$ there exists $\lim_{j\to\infty} \alpha^{j}_{m_l} = \alpha_{m_l} \in \mathcal{H}_\infty^{k-1}(\mathbb{C}_+)$. Applying Proposition 4.21 again we get that

$$|a^{(j_1)}_{m_1,\dots m_k} - a^{(j_2)}_{m_1,\dots m_k}| \leqslant ||\alpha^{(j_1)}_{m_l} - \alpha^{(j_2)}_{m_l}||_\infty \leqslant ||D^{(j_1)} - D^{(j_2)}||_\infty,$$

so for every $m_1, \dots, m_k \in \mathbb{N}$ there exists $a_{m_1,\dots,m_k} = \lim_{j\to\infty} a^{(j)}_{m_1,\dots m_k}$ and therefore the coefficients of α_{m_l} are $\{a_{m_1,\dots,m_k}\}$. Define the (formal) k-multiple Dirichlet series $D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$. As $\{D^{(j)}\}_{j=1}^{\infty}$ is a Cauchy sequence it is bounded, so there exists a positive C such that

$$|a^{(j)}_{m_1,\dots m_k}| \leqslant ||D^{(j)}||_\infty \leqslant C \quad \text{for every} \quad j, m_1, \dots, m_k \in \mathbb{N},$$

and therefore $|a_{m_1,\dots m_k}| \leqslant C$ for every $m_1, \dots, m_k \in \mathbb{N}$. Then by Proposition 3.6 the multiple Dirichlet series D converges absolutely on $\mathbb{C}_1^k$. Let us check that D and f coincide on $\mathbb{C}_\sigma^k$ for some positive σ. Given $\varepsilon > 0$, take $\delta > 0$ and $(s_1, \dots, s_k) \in \mathbb{C}^k_{1+\delta}$. There exists $j_0 \in \mathbb{N}$ such that $j \leqslant j_0$ implies

$$|f(s_1, \dots, s_k) - D^{(j_0)}(s_1, \dots, s_k)| < \varepsilon \quad \text{and} \quad ||D^{(j)} - D^{(j_0)}||_\infty < \frac{\varepsilon}{2}.$$

Then for all $m_1, \dots, m_k \in \mathbb{N}$

$$|a^{(j)}_{m_1,\dots,m_k} - a^{(j_0)}_{m_1,\dots,m_k}| \leqslant ||D^{(j)} - D^{(j_0)}||_\infty < \frac{\varepsilon}{2}$$

so

$$\lim_{j\to\infty} |a^{(j)}_{m_1,\dots,m_k} - a^{(j_0)}_{m_1,\dots,m_k}| = |a_{m_1,\dots,m_k} - a^{(j_0)}_{m_1,\dots,m_k}| \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Also, we can find $M_0 \in \mathbb{N}$ such that

$$\left| D^{(j_0)}(s_1, \dots, s_k) - \sum_{m_1=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{a^{(j_0)}_{m_1,\dots,m_k}}{m_1^{s_1}, \dots, m_k^{s_k}} \right| < \varepsilon$$

and

$$\left| D(s_1, \dots, s_k) - \sum_{m_1=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}, \dots, m_k^{s_k}} \right| < \varepsilon.$$

Then

$$\begin{aligned}
&|f(s_1, \dots, s_k) - D(s_1, \dots, s_k)| \\
&\quad\leqslant |f(s_1, \dots, s_k) - D^{(j_0)}(s_1, \dots, s_k)| \\
&\qquad + \left| D^{(j_0)}(s_1, \dots, s_k) - \sum_{m_1=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{a^{(j_0)}_{m_1,\dots,m_k}}{m_1^{s_1}, \dots, m_k^{s_k}} \right| \\
&\qquad + \sum_{m_1=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{|a^{(j_0)}_{m_1,\dots,m_k} - a_{m_1,\dots,m_k}|}{m_1^{\operatorname{Re} s_1} \cdots m_k^{\operatorname{Re} s_k}} \\
&\qquad + \left| \sum_{m_1=1}^{M_0} \cdots \sum_{m_k=1}^{M_0} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}, \dots, m_k^{s_k}} - D(s_1, \dots, s_k) \right|
\end{aligned}$$

$$< 2\varepsilon + \varepsilon \sum_{m_1,\dots,m_k=1}^{\infty} \frac{1}{m_1^{1+\delta} \cdots m_k^{1+\delta}} + \varepsilon$$
$$= \varepsilon \left(3 + \sum_{m_1,\dots,m_k=1}^{\infty} \frac{1}{m_1^{1+\delta} \cdots m_k^{1+\delta}} \right) = \varepsilon(3 + \zeta(1+\delta)^k),$$

so f and D coincide on $\mathbb{C}^k_{1+\delta}$ and by Theorem 4.20 $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$. □

Remark 4.24. This proof of Theorem 4.23 includes the case $k = 1$, understanding properly that in that case $k - 1 = 0$.

The next step will be now to establish the analogous isometry to that of Theorem 4.11. We do this in the next theorem.

Theorem 4.25. *The map* $\Psi : \mathcal{H}_\infty(\mathbb{C}^k_+) \to \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}^{k-1}_+))$ *defined by* $\Psi(D) = \tilde{D}$ *where*

$$D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$$

and

$$\tilde{D}(s_1)(s_2, \dots, s_k) = \sum_{m_1=1}^{\infty} \frac{1}{m_1^{s_1}} \left(\sum_{m_2,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_2^{s_2} \cdots m_k^{s_k}} \right)$$

is a bijective isometry.

Proof. Ψ is an isometry into. First, let us check that Ψ is well-defined. Suppose $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$, then by Theorem 4.20 $\alpha_{m_1} \in \mathcal{H}_\infty(\mathbb{C}^{k-1}_+)$ for every $m_1 \in \mathbb{N}$ where

$$\alpha_{m_1}(s_1, \dots, s_k) = \sum_{m_2,\dots,m_k}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_2^{s_2} \cdots m_k^{s_k}},$$

$$D(s_1, \dots, s_k) = \sum_{m_1}^{\infty} \frac{\alpha_{m_1}(s_2, \dots, s_k)}{m_1^{s_1}}.$$

Therefore $\tilde{D}$ is a formal vector-valued Dirichlet series with coefficients in $\mathcal{H}_\infty(\mathbb{C}^{k-1}_+)$. As D is regularly convergent in $\mathbb{C}^k_+$, the iterated sums converge and coincide with the sum of the series for any given order for the indexes. Choosing m_1 as the last index and computing the sum for the other indexes first we obtain $D(s_1, \dots, s_k) = \tilde{D}(s_1)(s_2, \dots, s_k)$, so $\tilde{D}$ is convergent in $\mathbb{C}_+$ and defines a bounded holomorphic function there. Then $\tilde{D} \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$ and Ψ is well-defined. To see that it is into it is sufficient to note that k-multiple Dirichlet series are characterized by its k-multiple sequence of coefficients, which is the same in $\mathcal{H}_\infty(\mathbb{C}_+^k)$ and $\mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$. Moreover,

$$||\tilde{D}||_\infty = \sup_{s_1 \in \mathbb{C}_+} ||D(s_1)||_\infty = \sup_{s_1 \in \mathbb{C}_+} \cdots \sup_{s_k \in \mathbb{C}_+} |\tilde{D}(s_1)(s_2, \dots, s_k)|$$
$$= \sup_{\substack{s_j \in \mathbb{C}_+ \\ 1 \leqslant j \leqslant k}} |D(s_1, \dots, s_k)| = ||D||_\infty.$$

<u>Ψ is onto.</u> Given $\tilde{D}(s_1) = \sum_{m_1}^\infty \frac{\tilde{\alpha}_{m_1}}{m_1^{s_1}}$ with $\tilde{D} \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$ and $\tilde{\alpha}_{m_1} \in \mathcal{H}_\infty^{k-1}(\mathbb{C}_+)$ for every $m_1 \in \mathbb{N}$,

$$\tilde{D}(s_1)(s_2, \dots, s_k) = \sum_{m_1=1}^\infty \frac{1}{m_1^{s_1}} \left(\sum_{m_2, \dots, m_k = 1}^\infty \frac{a_{m_1, \dots, m_k}}{m_2^{s_2} \cdots m_k^{s_k}} \right),$$

define formally

$$D(s_1, \dots, s_k) = \sum_{m_1, \dots, m_k = 1}^\infty \frac{a_{m_1, \dots, m_k}}{m_1^{s_1} \cdots m_k^{s_k}}.$$

Using Proposition 4.21,

$$|a_{m_1, \dots, m_k}| \leqslant ||\tilde{\alpha}_{m_1}||_\infty \text{ for all } m_1, \dots, m_k \in \mathbb{N},$$

and by Proposition 4.16

$$||\tilde{\alpha}_{m_1}||_\infty \leqslant ||\tilde{D}||_\infty, \text{ for all } m_1 \in \mathbb{N},$$

so the coefficients of D are bounded and D is absolutely convergent in $\mathbb{C}_1^k$. As $\tilde{D} \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$, the function $f : \mathbb{C}_+^k \to \mathbb{C}$ with $f(s_1, \dots, s_k) = \tilde{D}(s_1)(s_2, \dots, s_k)$ is holomorphic and as D and f coincide on $\mathbb{C}_1^k$, by Theorem 4.20 $D \in \mathcal{H}_\infty(\mathbb{C}_+^k)$. □

The final result of this section is that, for each $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is a Banach algebra. We are going to rely on the isometry form Theorem 4.25 and on Theorem 4.17 to do so.

Theorem 4.26. *For every $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is a Banach algebra. Furthermore, if $D_1, D_2 \in \mathcal{H}_\infty(\mathbb{C}_+^k)$,*

$$D_1(s_1, \dots, s_k) = \sum_{m_1, \dots, m_k = 1}^{\infty} \frac{a_{m_1, \dots, m_k}}{m_1^{s_1} \cdots m_k^{s_k}},$$

$$D_2(s_1, \dots, s_k) = \sum_{n_1, \dots, n_k = 1}^{\infty} \frac{b_{n_1, \dots, n_k}}{n_1^{s_1} \cdots n_k^{s_k}},$$

then

$$(D_1 D_2)(s_1, \dots, s_k) = \sum_{l_1, \dots, l_k}^{\infty} \frac{c_{l_1, \dots, l_k}}{l_1^{s_1} \cdots l_k^{s_k}}$$

where $c_{l_1, \dots, l_k} = \sum_{\substack{m_j n_j = l_j \\ 1 \leqslant j \leqslant k}} a_{m_1, \dots, m_k} b_{n_1, \dots, n_k}$.

Proof. By induction on k, the case $k = 1$ is given in Remark 2.27. Now suppose it is true for $k - 1$ and consider $D_1, D_2 \in \mathcal{H}_\infty(\mathbb{C}_+^k)$. By Theorem 4.25, $\Psi(D_1) = \tilde{D}_1, \Psi(D_2) = \tilde{D}_2 \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$. Then by Theorem 4.17 $\tilde{D}_1 \tilde{D}_2 \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$, so $\Psi^{-1}(\tilde{D}_1 \tilde{D}_2) \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+^{k-1}))$, so it remains to check that $\Psi^{-1}(\tilde{D}_1 \tilde{D}_2)$ actually coincides with the expression for $D_1 D_2$ of our statement. By Theorem 4.17, if $\tilde{D}_1(s_1) = \sum_{m_1=1}^{\infty} \frac{\alpha_{m_1}}{m_1^{s_1}}$, $\tilde{D}_2(s_1) = \sum_{n_1=1}^{\infty} \frac{\beta_{n_1}}{n_1^{s_1}}$,

$$\tilde{D}_1(s_1) \tilde{D}_2(s_1) = \sum_{l_1=1}^{\infty} \frac{1}{m_1^{s_1}} \sum_{m_1 n_1 = l_1} \alpha_{m_1} \beta_{n_1}$$

and by the induction hypothesis applied to $\alpha_{m_1},\ \beta_{m_1} \in \mathcal{H}_\infty(\mathbb{C}_+^{k-1})$,

$$\alpha_{m_1}\beta_{m_1}(s_2,\dots,s_k) = \sum_{l_2,\dots,l_k=1}^{\infty} \frac{1}{l_2^{s_2}\cdots l_k^{s_k}} \left(\sum_{\substack{m_j n_j = l_j \\ 2\leqslant j\leqslant k}} a_{m_1,\dots,m_k} b_{n_1,\dots,n_k} \right).$$

Using that $D_1D_2 \in \mathcal{H}_\infty(\mathbb{C}_+^k)$ and that regular convergence allows to compute the sum in any order of the indexes,

$$\begin{aligned}(D_1D_2)(s_1,\dots,s_k)& \\ &= \sum_{l_1=1}^{\infty} \frac{1}{m_1^{s_1}} \sum_{m_1n_1=l_1} \sum_{l_2,\dots,l_k=1}^{\infty} \frac{1}{l_2^{s_2}\cdots l_k^{s_k}} \sum_{\substack{m_j n_j = l_j \\ 2\leqslant j\leqslant k}} a_{m_1,\dots,m_k} b_{n_1,\dots,n_k} \\ &= \sum_{l_1,\dots,l_k=1}^{\infty} \frac{1}{l_1^{s_1}\cdots l_k^{s_k}} \sum_{\substack{m_j n_j = l_j \\ 1\leqslant j\leqslant k}} a_{m_1,\dots,m_k} b_{n_1,\dots,n_k}.\end{aligned}$$

□

4.3 The spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ and $\mathcal{A}(\mathbb{C}_+^k)$ and their isometries.

The aim of this section is twofold. First, to prove that the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$, for $k \in \mathbb{N}$, are all isometrically isomorphic independently form their dimension. Second, we introduce a new family of spaces $\mathcal{A}(\mathbb{C}_+^k)$, and we show that they are also isomorphic independently from $k \in \mathbb{N}$. The spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ have been defined and studied in the previous section, so we define and characterize now the spaces $\mathcal{A}(\mathbb{C}_+^k)$.

Definition 4.27. We denote by $\mathcal{A}(\mathbb{C}_+^k)$ the set of all k-multiple Dirichlet series which are regularly convergent on $\mathbb{C}_+^k$ and define uniformly continuous functions on $\mathbb{C}_+^k$.

Recently in [3], the space $\mathcal{A}(\mathbb{C}_+)$ has been proven to be isometrically isomorphic to $\mathcal{A}_u(B_{c_0})$, the algebra of holomorphic functions on the unit ball of c_0 which are the uniform limit of a sequence of polynomials. Following the lines of this work, we give the following characterization.

Theorem 4.28. *For every $k \in \mathbb{N}$, $\mathcal{A}(\mathbb{C}^k_+)$ is a closed subspace of $\mathcal{H}_\infty(\mathbb{C}^k_+)$. Moreover, $D \in \mathcal{A}(\mathbb{C}^k_+)$ if and only if it is the uniform limit of a multiple sequence of Dirichlet polynomials.*

Proof. First, let us see that $\mathcal{A}(\mathbb{C}^k_+)$ is a subspace of $\mathcal{H}_\infty(\mathbb{C}^k_+)$. Let $D \in \mathcal{A}(\mathbb{C}^k_+)$, $D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}$. We can find $\delta > 0$ such that $|D(s_1, \dots, s_k) - D(s'_1, \dots, s'_k)| < 1$ if $|s_j - s'_j| < \delta$ for every $1 \leqslant j \leqslant k$. Take $n \in \mathbb{N}$ with $\frac{1}{n} < \delta$. Since D is regularly convergent on $\mathbb{C}^k_+$, it is absolutely convergent on $\mathbb{C}^k_1$. Thus, if $\operatorname{Re} s_j \geqslant 1 + \frac{1}{n}$ for every $1 \leqslant j \leqslant k$,

$$
\begin{aligned}
|D(s_1, \dots, s_k)| &= \left| \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}} \right| \\
&\leqslant \sum_{m_1,\dots,m_k=1}^{\infty} \frac{|a_{m_1,\dots,m_k}|}{(m_1\cdots m_k)^{1+\frac{1}{n}}} := M < \infty.
\end{aligned}
$$

Suppose now that $0 < \operatorname{Re} s_1 < 1 + \frac{1}{n}$, $\operatorname{Re} s_j \geqslant 1 + \frac{1}{n}$ for every $2 \leqslant j \leqslant k$,

$$
\begin{aligned}
|D(s_1, \dots, s_k)| \leqslant \sum_{\nu=0}^{n} &|D(s_1 + \frac{\nu}{n}, \dots, s_k) - D(s_1 + \frac{\nu+1}{n}, \dots, s_k)| \\
&+ |D(s_1 + \frac{n+1}{n}, \dots, s_k)| \leqslant n + 1 + M.
\end{aligned}
$$

If now $0 < \operatorname{Re} s_j < 1 + \frac{1}{n}$ for $j = 1, 2$ and $\operatorname{Re} s_j \geqslant 1 + \frac{1}{n}$ for every $3 \leqslant j \leqslant k$,

$$|D(s_1, \dots, s_k)| \leqslant \sum_{\nu=0}^{n} |D(s_1, s_2 + \frac{\nu}{n}, \dots, s_k) - D(s_1, s_2 + \frac{\nu+1}{n}, \dots, s_k)|$$
$$+ |D(s_1, s_2 + \frac{n+1}{n}, \dots, s_k)| \leqslant (n+1) + (n+1+M) = 2(n+1) + M.$$

By finite induction we can assure that $|D(s_1, \dots, s_k)| \leqslant k(n+1) + M$ for every $(s_1, \dots, s_k) \in \mathbb{C}^k_+$, so $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$.
It only remains to check that $\mathcal{A}(\mathbb{C}^k_+)$ is closed in $\mathcal{H}_\infty(\mathbb{C}^k_+)$, but this is a consequence of the fact that the uniform limit of uniformly continuous functions is also uniformly continuous.

Let us prove now the characterization of functions in $\mathcal{A}(\mathbb{C}^k_+)$. For the necessary condition let us suppose that $D : \mathbb{C}^k_+ \to \mathbb{C}$ is the uniform limit of a multiple sequence $\{P_{n_1,\dots,n_k}\}$ of multiple Dirichlet polynomials,

$$P_{n_1,\dots,n_k}(s_1, \dots, s_k) = \sum_{m_1=1}^{M_{n_1}} \cdots \sum_{m_k=1}^{M_{n_k}} \frac{b_{m_1,\dots,m_k,n_1,\dots,n_k}}{m_1^{s_1} \cdots m_k^{s_k}}.$$

Since $\mathcal{H}_\infty(\mathbb{C}^k_+)$ is a Banach space and $P_{n_1,\dots,n_k} \in \mathcal{H}_\infty(\mathbb{C}^k_+)$ for every $n_1, \dots, n_k \in \mathbb{N}$, then $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$. Thus, D can be pointwise represented by a multiple Dirichlet series and it is uniformly continuous because it is the uniform limit of uniformly continuous functions. For the sufficient condition, let us suppose now that

$$D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$$

pointwise on $\mathbb{C}^k_+$ with D uniformly continuous on $\mathbb{C}^k_+$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $(s_1, \dots, s_k), (s_1', \dots, s_k') \in \mathbb{C}^k_+$ with $|s_j - s_j'| < \delta$ for every $1 \leqslant j \leqslant k$ then $|D(s_1, \dots, s_k) - D(s_1', \dots, s_k')| < \frac{\varepsilon}{2}$. Since D is bounded on $\mathbb{C}^k_+$, its multiple Dirichlet series is uniformly convergent in

$\mathbb{C}_\sigma^k$ for every $\sigma > 0$. Take $\sigma = \frac{\delta}{2}$ and define

$$Q_{M_1,\dots,M_k}(s_1,\dots,s_k) = \sum_{m_1=1}^{M_1} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m_1,\dots,m_k}}{m_1^{\frac{\delta}{2}} \cdots m_k^{\frac{\delta}{2}}} \frac{1}{m_1^{s_1} \cdots m_k^{s_k}}$$

and $g(s_1,\dots,s_k) = D(s_1 + \frac{\delta}{2},\dots,s_k + \frac{\delta}{2})$. Then $Q_{M_1,\dots,M_k}$ converges uniformly to g on $\mathbb{C}_+^k$. Moreover, as D is uniformly continuous on $\mathbb{C}_+^k$, $||D - g||_\infty < \frac{\varepsilon}{2}$. Thus

$$||D - Q_{M_1,\dots,M_k}||_\infty \leqslant ||D - g||_\infty + ||g - Q_{M_1,\dots,M_k}||_\infty,$$

so it is enough to take $M_1,\dots,M_k \in \mathbb{N}$ large enough to get the conclusion. □

Both of the isometries we want to prove are essentially a consequence of Bohr's fundamental Lemma (see [12, Theorem 3.2]), which establishes a relationship between Dirichlet polynomials and power series of functions of finitely many variables. For the sake of completeness we give this lemma, but before we introduce some notation. Given $m \in \mathbb{N}$ and considering $\mathfrak{p} = \{p_n\}$ the sequence of prime numbers, we write the decomposition of m using multiindexes $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ as follows,

$$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = \mathfrak{p}^\alpha, \quad \text{where} \quad \alpha = (\alpha_1,\dots,\alpha_r,0,\dots).$$

Moreover, given $M \in \mathbb{N}$, we define $\pi(M)$ as the number of prime numbers which are smaller than or equal to M.

Lemma 4.29 (Bohr's fundamental Lemma)**.** *With the notation introduced above*

$$\sup_{\mathrm{Re}\, s > 0} \left| \sum_{m=1}^{M} \frac{a_m}{m^s} \right| = \sup_{\omega \in \mathbb{D}^{\pi(M)}} \left| \sum_{1 \leqslant \mathfrak{p}^\alpha \leqslant M} a_{\mathfrak{p}^\alpha} z^\alpha \right|.$$

Lemma 4.29 can be extended to the k-multiple case in the following way.

Lemma 4.30. *Extending the notation of Lemma 4.29 to the k-multiple case,*

$$\sup_{\substack{\operatorname{Re} s_j>0 \\ 1\leqslant j\leqslant k}} \left| \sum_{m_1=1}^{M_1} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}} \right| = \sup_{\substack{z_j\in\mathbb{D}^{\pi(M_j)} \\ 1\leqslant j\leqslant k}} \left| \sum_{1\leqslant \mathfrak{p}^{\alpha_1}\leqslant M_1} \cdots \sum_{1\leqslant \mathfrak{p}^{\alpha_k}\leqslant M_k} a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1}\cdots z_k^{\alpha_k} \right|.$$

Proof. Suppose $M_1,\dots,M_k\in\mathbb{N}$ are fixed, $\pi(M_j)$ is the number of prime numbers which are less than or equal to M_j for $1\leqslant j\leqslant k$. For each $1\leqslant j\leqslant k$ take $1\leqslant m_j\leqslant M_j$ and $\alpha_j\in\mathbb{N}_0^{(\mathbb{N})}$ such that $m_j=\mathfrak{p}^{\alpha_j}$. Fix $(s_1,\dots,s_k)\in\mathbb{C}^k_+$, then

$$\frac{1}{m_j^{s_j}} = \frac{1}{(\mathfrak{p}^{\alpha_j})^{s_j}} = \frac{1}{(\mathfrak{p}^{s_j})^{\alpha_j}}, \quad \text{with} \quad \alpha_j=(\alpha_{j_1},\dots,\alpha_{j_{\pi(M_j)}},0,\dots).$$

Define $\Lambda_j=\{\alpha_j\in\mathbb{N}_0^{(\mathbb{N})} : 1\leqslant \mathfrak{p}^{\alpha_j}\leqslant M_j\}$ and $\Lambda=\Lambda_1\times\cdots\times\Lambda_k$. Then

$$\sum_{m_1=1}^{M_1}\cdots\sum_{m_k=1}^{M_k}\frac{a_{m_1,\dots,m_k}}{m_1^{s_1},\dots,m_k^{s_k}} = \sum_{(\alpha_1,\dots,\alpha_k)\in\Lambda} a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}} \frac{1}{(\mathfrak{p}^{s_1})^{\alpha_1}\cdots(\mathfrak{p}^{s_k})^{\alpha_k}}.$$

Consider now the sum $\sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1}\cdots z_k^{\alpha_k}$. It defines an analytic function on $\mathbb{D}^{\pi(M_1)}\times\cdots\times\mathbb{D}^{\pi(M_k)}$ and clearly

$$\begin{aligned}
&\sup_{\substack{\operatorname{Re} s_j>0\\ 1\leqslant j\leqslant k}} \left| \sum_{m_1=1}^{M_1}\cdots\sum_{m_k=1}^{M_k} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1},\ldots,m_k^{s_k}} \right| \\
&\quad = \sup_{\substack{\operatorname{Re} s_j>0\\ 1\leqslant j\leqslant k}} \left| \sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}} \frac{1}{(\mathfrak{p}^{s_1})^{\alpha_1}\cdots(\mathfrak{p}^{s_k})^{\alpha_k}} \right| \\
&\quad \leqslant \sup_{\substack{z_j\in\mathbb{D}^{\pi(M_j)}\\ 1\leqslant j\leqslant k}} \left| \sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1}\cdots z_k^{\alpha_k} \right|,
\end{aligned}$$

where the converse inequality also holds because of the fact that $\{\frac{1}{r\mathfrak{p}^{i\tau}} : \tau\in\mathbb{R}\}$ is dense $r\mathbb{T}$ for every $0<r<1$, a consequence of the well-known result of Kronecker (see for example [21, Lemma 2.2]). □

Remark 4.31. Note that the proof we have given for Lemma 4.30 is also valid for Lemma 4.29.

Now we give a lemma that serves as a version of Montel's Theorem (see [12, Theorem 2.17] for instance), for example) for the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$, which is an extension of Lemma 5.2 of [4].

Lemma 4.32. *Let $k\in\mathbb{N}$ and suppose $\{D_n\}$ is a bounded sequence in $\mathcal{H}_\infty(\mathbb{C}_+^k)$. Then there exists a function $D\in\mathcal{H}_\infty(\mathbb{C}_+^k)$ and a subsequence $\{D_{n_d}\}$ that converges uniformly to D on $\mathbb{C}_\delta^k$ for every $\delta>0$.*

Proof. We can assume that $||D_n||_\infty\leqslant 1$ for every $n\in\mathbb{N}$. Using Montel's Theorem [12] there exists a function $f\in H_\infty(\mathbb{C}_+^k)$ and a subsequence of $\{D_n\}$, which we will denote by $\{D_n\}$ to keep the notation simpler, which is uniformly convergent to f on the compact subsets of $\mathbb{C}_+^k$. Hence we have that $||f||_\infty\leqslant\limsup_{n\to\infty}||D_n||_\infty\leqslant 1$. Consider a bijection $\Phi:\mathbb{N}\to\mathbb{N}^k$ and write $m_j=\Phi_j(m)$ for every $m\in\mathbb{N}$ and $1\leqslant j\leqslant k$.

If $D_n(s_1,\dots,s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{n,m_1,\dots,m_k}}{m_1^{s_1},\dots,m_k^{s_k}}$, then for $\mathbf{s} = (s_1,\dots,s_k) \in \mathbb{C}^k_1$ we have that

$$D_n(s_1,\dots,s_k) = \sum_{m=1}^{\infty} \frac{a_{n,\Phi_1(m),\dots,\Phi_k(m)}}{\Phi_1(m)^{s_1}\cdots\Phi_k(m)^{s_k}} = \sum_{m=1}^{\infty} \frac{a_{n,\Phi(m)}}{\Phi(m)^{\mathbf{s}}}.$$

Using Proposition 4.21, $|a_{n,\Phi(1)}| \leqslant ||D_n||_\infty \leqslant 1$ for every $n \in \mathbb{N}$, so there exists a subsequence $\{a_{n_l,\Phi(1)}\}$ which converges to some $a_{\Phi(1)} \in \mathbb{C}$. Analogously we can find a subsequence $\{a_{n_{l_h},\Phi(2)}\}$ which converges to some $a_{\Phi(2)} \in \mathbb{C}$. Using a diagonal argument there exists a subsequence $\{a_{n_d,\Phi(m)}\}$ that converges to some $a_{\Phi(m)}$ for every $m \in \mathbb{N}$. Define now $D(s_1,\dots,s_k) = \sum_{m=1}^{\infty} \frac{a_{\Phi(m)}}{\Phi(m)^{\mathbf{s}}}$. Since $|a_{\Phi(m)}| \leqslant \lim_{n\to\infty} |a_{n,\Phi(1)}| \leqslant 1$, the series D is absolutely convergent on $\mathbb{C}^k_1$ and if $\mathbf{s} = (s_1,\dots,s_k) \in \mathbb{C}^k_1$ then $D(s_1,\dots,s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}$. To get that $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$ it will be enough to see that $f = D$ in $\mathbb{C}^k_1$ and then use Theorem 4.20. Let $\varepsilon > 0$, take $\delta > 0$ and $(s_1,\dots,s_k) \in \mathbb{C}^k_{1+\delta}$. The series $\sum_{m_1,\dots,m_k=1}^{\infty} \frac{1}{m_1^{1+\delta}\cdots m_k^{1+\delta}}$ is absolutely convergent so we can choose $M_0 \in \mathbb{N}$ such that

$$\sum_{m_1,\dots,m_k=1}^{\infty} \left|\frac{1}{m_1^{s_1}\cdots m_k^{s_k}}\right| - \sum_{m_1=1}^{M_0}\cdots\sum_{m_k=1}^{M_0} \left|\frac{1}{m_1^{s_1}\cdots m_k^{s_k}}\right| < \frac{\varepsilon}{6}.$$

Take $d \in \mathbb{N}$ large enough so that $|a_{n_d,m_1,\dots,m_k} - a_{m_1,\dots,m_k}| \leqslant \frac{\varepsilon}{3M_0^k}$ for every $1 \leqslant m_j \leqslant M_0$, $1 \leqslant j \leqslant k$, and also $|D_{n_d}(s_1,\dots,s_k) - f(s_1,\dots,s_k)| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}
&|f(s_1,\dots,s_k) - D(s_1,\dots,s_k)| \\
&\quad \leqslant |f(s_1,\dots,s_k) - D_n(s_1,\dots,s_k)| + |D_n(s_1,\dots,s_k) - D(s_1,\dots,s_k)| \\
&\quad \leqslant |f(s_1,\dots,s_k) - D_n(s_1,\dots,s_k)| + \sum_{\substack{m_j\leqslant M_0\\ 1\leqslant j\leqslant k}} \frac{|a_{n,m_1,\dots,m_k} - a_{m_1,\dots,m_k}|}{m_1^{s_1},\dots,m_k^{s_k}}
\end{aligned}$$

$$+2\left(\sum_{m_1,\dots,m_k=1}^{\infty}\left|\frac{1}{m_1^{s_1}\cdots m_k^{s_k}}\right|-\sum_{m_1=1}^{M_0}\cdots\sum_{m_k=1}^{M_0}\left|\frac{1}{m_1^{s_1}\cdots m_k^{s_k}}\right|\right)<3\frac{\varepsilon}{3}=\varepsilon.$$

It only remains to see that $\{D_n\}$ converges uniformly to D on $\mathbb{C}_\delta^k$ for every $\delta > 0$. Let $\varepsilon > 0$ and take $C_\delta\{M_1,\dots,M_k\}$ as in Theorem 4.20, which only depends on δ and M_j, $1 \leqslant j \leqslant k$. Choose $M_0 \in \mathbb{N}$ such that $C_\delta\{M_0,\dots,M_0\} < \frac{\varepsilon}{4}$. Since for every $d \in \mathbb{N}$ we have that D, $D_{n_d} \in \mathcal{H}_\infty(\mathbb{C}_+^k)$ and $||D||_\infty = ||f||_\infty = 1 = ||D_{n_d}||_\infty$, by Theorem 4.20

$$\sup_{\substack{\operatorname{Re} s_j>\delta\\ 1\leqslant j\leqslant k}}\left|\sum_{m_1=1}^{M_0}\cdots\sum_{m_k=1}^{M_0}\frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}-D(s_1,\dots,s_k)\right| \leqslant C_\delta(\{M_0,\dots,M_0\}),$$

and

$$\sup_{\substack{\operatorname{Re} s_j>\delta\\ 1\leqslant j\leqslant k}}\left|\sum_{m_1=1}^{M_0}\cdots\sum_{m_k=1}^{M_0}\frac{a_{n_d,m_1,\dots,m_k}}{m_1^{s_1},\dots,m_k^{s_k}}-D_{n_d}(s_1,\dots,s_k)\right| \leqslant C_\delta(\{M_0,\dots,M_0\}).$$

Then, choosing $d \in \mathbb{N}$ large enough so that $|a_{n_d,m_1,\dots,m_k} - a_{m_1,\dots,m_k}| \leqslant \frac{\varepsilon}{2M_0^k}$, for every $1 \leqslant m_j \leqslant M_0$, $1 \leqslant j \leqslant k$,

$$\begin{aligned}&|D_{n_d}(s_1,\dots,s_k)-D(s_1,\dots,s_k)|\\ &\quad\leqslant 2C_\delta(\{M_0,\dots,M_0\})+\sum_{m_1=1}^{M_0}\cdots\sum_{m_k=1}^{M_0}\frac{|a_{n_d,m_1,\dots,m_k}-a_{m_1,\dots,m_k}|}{m_1^{\delta}\cdots m_k^{\delta}}\\ &\quad< 2\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Therefore $\{D_n\}$ converges uniformly to D on $\mathbb{C}_\delta$ for any $\delta > 0$. $\square$

Now we are finally ready to state the main theorem of this section which is the extension to the k-multiple case of [21, Lemma 2.3]. Our proof extends a somewhat simplified version of the proof that appears in [12, Theorem 3.8] for one variable. We begin by proving the extension of [3, Theorem 2.5] and the we build over that.

Theorem 4.33. *The spaces $\mathcal{H}_\infty(\mathbb{C}^k_+)$, $k \in \mathbb{N}$, are all ismoetrically isomorphic to $H_\infty(B_{c_0})$ and the spaces $\mathcal{A}(\mathbb{C}^k_+)$ are all isometrically isomorphic to $\mathcal{A}_u(B_{c_0})$.*

Proof. Since, for every $k \in \mathbb{N}$, the spaces c_0^k and c_0 are isometrically isomorphic, the spaces $H_\infty(B_{c_0^k})$, for $k \in \mathbb{N}$, are all isometrically isomorphic to $H_\infty(B_{c_0})$. Hence it is enough to show that, for any given $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}^k_+)$ is isometrically isomorphic to the corresponding $H_\infty(B_{c_0^k})$, and the same argument holds for the case of the spaces $\mathcal{A}(\mathbb{C}^k_+)$ and $\mathcal{A}_u(B_{c_0^k})$. Lemma 4.30 establishes a bijective isometry between the spaces

$$\operatorname{span}\left\{\frac{1}{m_1^{s_1}\cdots m_k^{s_k}} : m_1,\dots,m_k \in \mathbb{N}\right\}$$

and

$$\operatorname{span}\left\{z_1^{\alpha_1}\cdots z_k^{\alpha_k} : \alpha_1,\dots,\alpha_k \in \mathbb{N}_0^{(\mathbb{N})}\right\},$$

which extends to a surjective isometry between the respective closures of these spaces. Following the arguments of the proof of [3, Theorem 2.5],

$$\overline{\operatorname{span}\left\{z_1^{\alpha_1}\cdots z_k^{\alpha_k} : \alpha_1,\dots,\alpha_k \in \mathbb{N}_0^{(\mathbb{N})}\right\}}^{||\cdot||_{B_{c_0^k}}} = \mathcal{A}_u(B_{c_0^k}),$$

and Theorem 4.28 gives that the closure of the space of Dirichlet monomials with the norm $||\cdot||_\infty$ is exactly $\mathcal{A}(\mathbb{C}^k_+)$. Let $\mathcal{B}_u : \mathcal{A}_u(B_{c_0^k}) \to \mathcal{A}(\mathbb{C}^k_+)$ be the isometry between these spaces, take $f \in \mathcal{A}_u(B_{c_0^k})$ and $D \in \mathcal{A}(\mathbb{C}^k_+)$

such that $\mathcal{B}_u(f) = D$. On the one hand, if $D \in \mathcal{A}(\mathbb{C}_+^k)$ is of the form

$$D(s_1, \ldots, s_k) = \sum_{m_1,\ldots,m_k=1}^{\infty} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}},$$

then by Theorem 4.28 $D \in \mathcal{H}_\infty(\mathbb{C}_+^k)$ and it is the uniform limit of a multiple sequence of k-multiple Dirichlet polynomials

$$P_{n_1,\ldots,n_k}(s_1, \ldots, s_k) = \sum_{m_1=1}^{M_{n_1}} \cdots \sum_{m_k=1}^{M_{n_k}} \frac{b_{m_1,\ldots,m_k,n_1,\ldots,n_k}}{m_1^{s_1}\cdots m_k^{s_k}}.$$

By using Remark 4.22, for every $m_1, \ldots, m_k \in \mathbb{N}$,

$$|b_{m_1,\ldots,m_k,n_1,\ldots,n_k} - a_{m_1,\ldots,m_k}| \leqslant ||P_{n_1,\ldots,n_k} - D||_\infty \xrightarrow[1\leqslant j\leqslant k]{n_j\to\infty} 0.$$

Now, for each $n_1, \ldots, n_k \in \mathbb{N}$ consider $f_{n_1,\ldots,n_k}$ the polynomial over $B_{c_0^k}$ such that $\mathcal{B}_u(f_{n_1,\ldots,n_k}) = P_{n_1,\ldots,n_k}$. Then $f_{n_1,\ldots,n_k} \in H_\infty(B_{c_0^k})$ and, since $\mathcal{B}_u$ is an isometry,

$$||f_{n_1,\ldots,n_k} - f||_\infty = ||P_{n_1,\ldots,n_k} - D||_\infty \xrightarrow[1\leqslant j\leqslant k]{n_j\to\infty} 0,$$

so by the Cauchy integral formula, for $0 < r < 1$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{N}_0^{(\mathbb{N})}$,

$$|c_{\alpha_1,\ldots,\alpha_k}(f_{n_1,\ldots,n_k}) - c_{\alpha_1,\ldots,\alpha_k}(f)| \leqslant \frac{1}{r^{|\alpha|}}||f_{n_1,\ldots,n_k} - f||_\infty.$$

Therefore

$$\begin{aligned} c_{\alpha_1,\ldots,\alpha_k}(f) &= \lim_{n_1,\ldots,n_k\to\infty} c_{\alpha_1,\ldots,\alpha_k}(f_{n_1,\ldots,n_k}) \\ &= \lim_{n_1,\ldots,n_k\to\infty} b_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k},n_1,\ldots,n_k} = a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}}. \end{aligned}$$

We define now the extension of $\mathcal{B}_u$,

$$\begin{aligned} \mathcal{B} : H_\infty(B_{c_0^k}) &\longrightarrow \quad \mathcal{H}_\infty(\mathbb{C}^k_+) \\ \sum_{\substack{(\alpha_1,\dots,\alpha_k) \\ \alpha_j \in \mathbb{N}_0^{(\mathbb{N})},\, 1\leqslant j \leqslant k}} c_{\alpha_1,\dots,\alpha_k} z_1^{\alpha_1}\cdots z_k^{\alpha_k} &\longrightarrow \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}}{m_1^{s_1},\dots,m_k^{s_k}}. \end{aligned}$$

Step 1. $\mathcal{B}$ is well-defined, linear and an injective contraction.
Let $f \in H_\infty(B_{c_0^k})$, and for $0 < r < 1$ consider

$$f_r(z) = f(rz) = \sum_{\substack{(\alpha_1,\dots,\alpha_k) \\ \alpha_j \in \mathbb{N}_0^{(\mathbb{N})},\, 1\leqslant j \leqslant k}} r^{|\alpha|} c_{\alpha_1,\dots,\alpha_k} z_1^{\alpha_1}\cdots z_k^{\alpha_k}.$$

For every $0 < r < 1$, $rB_{c_0^k}$ is contained in a compact subset of $B_{c_0^k}$, so $f_r \in \mathcal{A}_u(B_{c_0^k})$ with $||f_r||_\infty \leqslant ||f||_\infty = \sup_{0<r<1} ||f_r||_\infty$. Then $D_r = \mathcal{B}_u(f_r) \in \mathcal{A}(\mathbb{C}^k_+)$ for every $0 < r < 1$, with

$$D_r(s_1,\dots,s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}(r)}{m_1^{s_1},\dots,m_k^{s_k}} = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{|r|^{\alpha} c_{\alpha_1,\dots,\alpha_k}(f)}{m_1^{s_1},\dots,m_k^{s_k}},$$

and $||D_r||_\infty = ||f_r||_\infty \leqslant ||f||_\infty$. By using Lemma 4.32 there exists a function $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$ and a subnet $\{D_{r_l}\}$ which is uniformly convergent to D on $\mathbb{C}^k_\delta$ for every $\delta > 0$, and if $D(s_1,\dots,s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}$ then

$$\begin{aligned} |a_{m_1,\dots,m_k} - c_{\alpha_1,\dots,\alpha_k}(f)| &= \lim_{r_l \to 1} |a_{m_1,\dots,m_k} - |r_l|^{\alpha} c_{\alpha 1,\dots,\alpha_k}(f)| \\ &\leqslant \lim_{r_l\to 1} ||D - D_{r_l}||_\infty \\ &= \lim_{r_l \to 1} \sup_{\mathbb{C}^k_\delta} |D(s_1,\dots,s_k) - D_{r_l}(s_1,\dots,s_k)| = 0. \end{aligned}$$

Therefore $D = \mathcal{B}(f)$ and

$$||B(f)||_\infty = ||D||_\infty \leqslant \limsup_{r_l \to 1} ||D_{r_l}||_\infty = \limsup_{r_l \to 1} ||f_r||_\infty \leqslant ||f||_\infty.$$

Step 2. $\mathcal{B}$ is surjective and it is an isometry.
Let $D \in \mathcal{H}_\infty(\mathbb{C}_+^k)$, $D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{m_1,\dots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}}$ and $\sigma > 0$. Define

$$D_\sigma(s_1, \dots, s_k) = D(\sigma s_1, \dots, \sigma s_k) = \sum_{m_1,\dots,m_k} \frac{a_{m_1,\dots,m_k}}{m_1^\sigma, \dots, m_k^\sigma} \frac{1}{m_1^{s_1}, \dots, m_k^{s_k}},$$

$D_\sigma \in \mathcal{A}(\mathbb{C}_+^k)$ for every $\sigma > 0$ and

$$||D_\sigma||_\infty = \sup_{\substack{\operatorname{Re} s_j > \sigma \\ 1 \leqslant j \leqslant k}} |D(s_1, \dots, s_k)| \leqslant ||D||_\infty.$$

Consider now $f_\sigma = \mathcal{B}_u^{-1}(D_\sigma) \in \mathcal{A}_u(B_{c_0^k})$ for each $\sigma > 0$, whose coefficients are $c_{(\alpha_1,\dots,\alpha_k)}(f_\sigma) = \dfrac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}}{\mathfrak{p}^{\sigma\alpha_1} \cdots \mathfrak{p}^{\sigma\alpha_k}}$. Since $||f_\sigma||_\infty = ||D_\sigma|| \leqslant ||D||_\infty$, we can use Montel's Theorem (see for example [12, Theorem 2.17]) to find a subnet $\{f_{\sigma_l}\}$ which converges on the compact-open topology to a function $f \in H_\infty(B_{c_0^k})$ satisfying that $||f||_\infty \leqslant \limsup_{\sigma_l \to 0} ||D_{\sigma_l}||_\infty \leqslant ||D||_\infty$. As in the case of $\mathcal{A}_u(B_{c_0^k})$, for every $\alpha_1, \dots \alpha_k \in \mathbb{N}_0^{(\mathbb{N})}$

$$c_{\alpha_1,\dots,\alpha_k}(f) = \lim_{\sigma_l \to 0} c_{\alpha_1,\dots,\alpha_k}(f_{\sigma_l}) = \lim_{\sigma_l \to 0} \frac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}}{\mathfrak{p}^{\sigma_l\alpha_1} \cdots \mathfrak{p}^{\sigma_l\alpha_k}} = a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}},$$

so $\mathcal{B}(f) = D$ with $||f||_\infty \leqslant ||D||_\infty = ||B(f)||_\infty$. □

Remark 4.34. The second part of Theorem 4.33 actually appears as [12, Theorem 3.8] for $k = 1$. In this version the authors also prove that, if $\mathcal{B}(f) = D$, then $D(s) = f(\frac{1}{\mathfrak{p}^s})$ for $s \in \mathbb{C}_+$. Moreover, the proof of [12, Theorem 3.8] does not need Lemma 4.32. We now give a proof of Theorem 4.33 which is based on the proof [12, Theorem 3.8], proving that

$$\begin{aligned} D(s_1,\dots,s_k) &= \lim_{j\to\infty} B(f_{n_j}) \\ &= \lim_{j\to\infty} f_{n_j}(\{\frac{1}{p_\nu^{s_1}}\}_{\nu=1}^{n_j},\dots,\{\frac{1}{p_\nu^{s_k}}\}_{\nu=1}^{n_j}) = f(\frac{1}{\mathfrak{p}^{s_1}},\dots,\frac{1}{\mathfrak{p}^{s_k}}). \end{aligned}$$

Then we will get Lemma 4.32 as a consequence, proving that are equivalent, since one can always get one of the results as a consequence of the other.

Proof of Theorem 4.33 following Theorem 3.8 of [12]. Taking the first step regarding the isometry $\mathcal{B}_u$ from the previous proof, we define

$$\begin{aligned} \mathcal{B} : H_\infty(B_{c_0^k}) &\longrightarrow \quad \mathcal{H}_\infty(\mathbb{C}^k_+) \\ \sum_{\substack{(\alpha_1,\dots,\alpha_k)\\ \alpha_j\in\mathbb{N}_0^{(\mathbb{N})},\,1\leqslant j\leqslant k}} c_{\alpha_1,\dots,\alpha_k} z_1^{\alpha_1}\cdots z_k^{\alpha_k} &\longrightarrow \sum_{m_1,\dots,m_k=1}^{\infty} \frac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}}{m_1^{s_1},\dots,m_k^{s_k}}. \end{aligned}$$

Step 1. $\mathcal{B}$ is well-defined and it is an injective contraction.
First, fix $M_1,\dots,M_k\in\mathbb{N}$, let $M=\sum_{j=1}^k M_j$ and consider $f\in H_\infty(\mathbb{D}^M)$. Define $I_{M_j}=\{\mathfrak{p}^\alpha:\alpha\in\mathbb{N}_0^{M_j}\}$, the set of positive integers whose factors are the first M_j prime numbers. Take $(s_1,\dots,s_k)\in\mathbb{C}^k_+$ and define $z_j=(p_1^{-s_j},\dots,p_{M_j}^{-s_j})\in\mathbb{D}^{M_j}$, so $(z_1,\dots,z_k)\in\mathbb{D}^M$. Since f is holomorphic on $\mathbb{D}^M$ the series $\sum_{\gamma\in\mathbb{N}_0^M} c_\gamma(f)u^\gamma$ is sumable in $\mathbb{D}^M$ and therefore absolutely convergent. Particularly if $u=(z_1,\dots,z_k)\in\mathbb{D}^M$ and $\Lambda=\prod_{j=1}^k\mathbb{N}_0^{M_j}$,

$$\sum_{(\alpha_1,\dots,\alpha_k)\in\Lambda} |c_{(\alpha_1,\dots,\alpha_k)}(f) z_1^{\alpha_1}\cdots z_k^{\alpha_k}| < \infty.$$

Define

$$a_{m_1,\dots,m_k} = \begin{cases} c_{\alpha_1,\dots,\alpha_k}(f), & \text{if } m_j=\mathfrak{p}^{\alpha_j}, 1\leqslant j\leqslant k,\\ 0, & \text{otherwise.}\end{cases}$$

Define formally

$$D(s_1,\dots,s_k)=\sum_{m_1,\dots,m_k=1}^{\infty}\frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}.$$

Then

$$\begin{aligned}D(s_1,\dots,s_k)&=\sum_{m_1,\dots,m_k=1}^{\infty}\frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}\\&=\sum_{(\alpha_1,\dots,\alpha_k)\in\Lambda}c_{(\alpha_1,\dots,\alpha_k)}z_1^{\alpha_1}\cdots z_k^{\alpha_k}=f(z_1,\dots,z_k)\end{aligned}$$

for every $(s_1,\dots,s_k)\in\mathbb{C}_+^k$, so D is regularly convergent in $\mathbb{C}_+^k$. Moreover, as f is bounded, D is bounded and $\mathcal{B}$ is well-defined in the finite dimensional case with

$$\begin{aligned}||\mathcal{B}(f)||_\infty&=\sup_{\substack{\operatorname{Re}s_j>0\\1\leqslant j\leqslant k}}\left|\sum_{(m_1,\dots,m_k)\in I}\frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}\right|\\&\leqslant\sup_{(z_1,\dots,z_k)\in\mathbb{D}^M}|f(z_1,\dots,z_k)|=||f||_\infty,\end{aligned}$$

so in this case $\mathcal{B}$ is a contraction, and it is injective because remark 4.22 guarantees that each multiple Dirichlet series is characterized by its coefficients.

Now we take $f\in H_\infty(B_{c_0^k})$. We want that $D=\mathcal{B}(f)\in\mathcal{H}_\infty(\mathbb{C}_+^k)$. By the Cauchy Integral Formula, $|a_{m_1,\dots,m_k}|=|c_{(\alpha_1,\dots,\alpha_k)}(f)|\leqslant||f||_\infty$, so by Proposition 3.6 the multiple Dirichlet series

$$D(s_1,\dots,s_k)=\sum_{m_1,\dots,m_k=1}^{\infty}\frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}$$

converges absolutely for $(s_1,\dots,s_k)\in\mathbb{C}_1^k$. Consider $f_n\in H_\infty(\mathbb{D}^{nk})$ the restiriction of f to $\prod_{j=1}^k\mathbb{D}^n$ and $D_n=B(f_n)$. By what has been done in the finite dimensional case,

$$||B(f_n)||_\infty \leqslant ||f_n||_\infty \leqslant ||f||_\infty, \quad \text{so} \quad \sup_{n\in\mathbb{N}} ||B(f_n)||_\infty \leqslant ||f||_\infty.$$

Therefore $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H_\infty(\mathbb{C}^k_+)$. As $\mathbb{C}^k$ is a separable normed space, we can use Montel's Theorem [12, Theorem 2.17] to obtain a subsequence $\{\mathcal{B}(f_{n_j})\}_{j=1}^\infty$ which is uniformly convergent on compat sets to a holomorphic function $g : \mathbb{C}^k_+ \to \mathbb{C}$. As $||\mathcal{B}(f_{n_j})||_\infty \leqslant ||f||_\infty$ for every $j \in \mathbb{N}$, $||g||_\infty \leqslant ||f||_\infty$ and $g \in H_\infty(\mathbb{C}^k_+)$. On the other hand, $\mathcal{B}(f_{n_j})$ also converges to D, so D and g coincide on $\mathbb{C}^k_1$. Applying Theorem 4.20, the series D coincides with g on $\mathbb{C}^k_+$ and therefore belongs to $\mathcal{H}_\infty(\mathbb{C}^k_+)$, with $||D||_\infty \leqslant ||f||_\infty$ and

$$\begin{aligned} D(s_1, \dots, s_k) &= \lim_{j\to\infty} \mathcal{B}(f_{n_j}) \\ &= \lim_{j\to\infty} f_{n_j}(\{\frac{1}{p_\nu^{s_1}}\}_{\nu=1}^{n_j}, \dots, \{\frac{1}{p_\nu^{s_k}}\}_{\nu=1}^{n_j}) = f(\frac{1}{\mathfrak{p}^{s_1}}, \dots, \frac{1}{\mathfrak{p}^{s_k}}). \end{aligned}$$

Step 2. $\mathcal{B}$ is surjective.

Consider $D \in \mathcal{H}_\infty(\mathbb{C}^k_+)$, $D(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^\infty \frac{a_{m_1,\dots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}}$. If D is uniformly convergent on $\mathbb{C}^k_+$ then, using Theorem 4.28, $D \in \mathcal{A}(\mathbb{C}^k_+)$ and by what has been stated at the beginning oh the proof there exists some $h \in \mathcal{A}_u(B^k_{c_0}) \subset H_\infty(B_{c_0^k})$ such that $\mathcal{B}(h) = D$. In the general case we do not have uniform convergence of D in $\mathbb{C}^k_+$, but by Theorem 4.20 we can assure the uniform convergence on $\mathbb{C}^k_\delta$ for every $\delta > 0$. Thus, define

$$D_\delta(s_1, \dots, s_k) = \sum_{m_1,\dots,m_k=1}^\infty \frac{a_{m_1,\dots,m_k}}{m_1^\delta \cdots m_k^\delta} \frac{1}{m_1^{s_1} \cdots m_k^{s_k}},$$

$$D_{\delta,M_1,\dots,M_k}(s_1, \dots, s_k) = \sum_{m_1=1}^{M_1} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m_1,\dots,m_k}}{m_1^\delta \cdots m_k^\delta} \frac{1}{m_1^{s_1} \cdots m_k^{s_k}}.$$

Now $D_{\delta,M_1,\dots,M_k}$ converge uniformly to D_δ on $\mathbb{C}_+^k$ so there exists $h_\delta \in H_\infty(B_{c_0^k})$ such that $\mathcal{B}(h_\delta) = D_\delta$, with $||h_\delta||_\infty = ||D_\delta||_\infty$ and

$$c_{(\alpha_1,\dots,\alpha_k)}(h) = \frac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}}{(\mathfrak{p}^{\alpha_1})^\delta \cdots (\mathfrak{p}^{\alpha_k})^\delta}.$$

As

$$||D_\delta||_\infty = \sup_{\substack{\operatorname{Re} s_j > 0 \\ 1 \leqslant j \leqslant k}} |D_\delta(s_1,\dots,s_k)| = \sup_{\substack{\operatorname{Re} s_j > \delta \\ 1 \leqslant j \leqslant k}} |D(s_1,\dots,s_k)| \leqslant ||D||_\infty,$$

$||h_\delta||_\infty \leqslant ||D||_\infty$ for every positive δ. Given $M_1,\dots,M_k \in \mathbb{N}$, let $h_{\delta,M_1,\dots,M_k}$ be the restriction of h_δ to $\prod_{j=1}^k \mathbb{D}^{M_j}$. The function $h_{\delta,M_1,\dots,M_k}$ is holomorphic on $\prod_{j=1}^k \mathbb{D}^{M_j}$ and its monomial series expansion is

$$\sum_{(\alpha_1,\dots,\alpha_k)\in\Lambda} \left| \frac{a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}}}{(\mathfrak{p}^{\alpha_1})^\delta \cdots (\mathfrak{p}^{\alpha_k})^\delta} z_1^{\alpha_1} \cdots z_k^{\alpha_k} \right| < \infty$$

for every $(z_1,\dots,z_k) \in \prod_{j=1}^k \mathbb{D}^{M_j}$ and every $\delta > 0$. Define

$$K_\delta^{(j)} = \left\{ \left(\frac{\omega_\nu}{p_\nu} \right)_{\nu=1}^{M_j} : \omega = (\omega_\nu)_{\nu=1}^{M_j} \in \mathbb{D}^{M_j} \right\}, \quad K_\delta = \prod_{j=1}^k K_\delta^{(j)}$$

clearly $\prod_{j=1}^k \mathbb{D}^{M_j} = \cup_{\delta>0} K_\delta$. Let us see that the series

$$\sum_{(\alpha_1,\dots,\alpha_k)\in\Lambda} a_{\mathfrak{p}^{\alpha_1},\dots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1} \cdots z_k^{\alpha_k}$$

defines a holomorphic function, say $F_{M_1,\dots,M_k}$, on $\prod_{j=1}^k \mathbb{D}^{M_j}$. If a point $(z_1,\dots,z_k) \in \prod_{j=1}^k \mathbb{D}^{M_j}$, we can find a positive δ such that $(z_1,\dots,z_k) \in K_\delta$. Then there exists $\omega^{(j)} \in \mathbb{D}^{M_j}$ such that $z_j = (\frac{\omega_1^{(j)}}{p_1^\delta}, \dots, \frac{\omega_{M_j}^{(j)}}{p_{M_j}^\delta})$, for every $1 \leqslant j \leqslant k$: Hence

$$\begin{aligned}&\sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} |a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1}\cdots z_k^{\alpha_k}| \\ &\qquad = \sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} \left| \frac{a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}}}{(\mathfrak{p}^{\alpha_1})^\delta,\ldots,(\mathfrak{p}^{\alpha_k})^\delta} (\omega^{(1)})^{\alpha_1}\cdots(\omega^{(k)})^{\alpha_k}\right| < \infty,\end{aligned}$$

and also

$$\begin{aligned}&\sup_{(z_1,\ldots,z_k)\in K_\delta} |F_{M_1,\ldots,M_k}(z_1,\ldots,z_k)| \\ &= \sup_{\substack{(z_1,\ldots,z_k)\\ \in\prod_{j=1}^k \mathbb{D}^{M_j}}} \sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} \left| \frac{a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}}}{(\mathfrak{p}^{\alpha_1})^\delta,\ldots,(\mathfrak{p}^{\alpha_k})^\delta} z_1^{\alpha_1}\cdots z_k^{\alpha_k}\right| \\ &= \sup_{\substack{(z_1,\ldots,z_k)\\ \in\prod_{j=1}^k \mathbb{D}^{M_j}}} |h_{\delta,M_1,\ldots,M_k}(z_1,\ldots,z_k)| = ||h_\delta||_\infty \leqslant ||D||_\infty.\end{aligned}$$

Therefore

$$\sup_{M_1,\ldots,M_k\in\mathbb{N}} \sup_{\substack{(z_1,\ldots,z_k)\\ \in\prod_{j=1}^k \mathbb{D}^{M_j}}} \sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} |a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1}\cdots z_k^{\alpha_k}| \leqslant ||D||_\infty,$$

so by Hilbert's criterion for Reinhardt domains (see [12, Theorem 15.26]) there exists $F \in H_\infty(B_{c_0^k})$ such that $c_{(\alpha_1,\ldots,\alpha_k)}(F) = a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}}$ for every $(\alpha_1,\ldots,\alpha_k) \in \prod_{j=1}^k \mathbb{N}_0^{(\mathbb{N})}$ and

$$||F||_\infty = \sup_{M_1,\ldots,M_k\in\mathbb{N}} \sup_{\substack{(z_1,\ldots,z_k)\\ \in\prod_{j=1}^k \mathbb{D}^{M_j}}} \sum_{(\alpha_1,\ldots,\alpha_k)\in\Lambda} |a_{\mathfrak{p}^{\alpha_1},\ldots,\mathfrak{p}^{\alpha_k}} z_1^{\alpha_1}\cdots z_k^{\alpha_k}| \leqslant ||D||_\infty,$$

so $\mathcal{B}(F) = D$ and $||F||_\infty \leqslant ||\mathcal{B}(F)||_\infty$. Thus, finally, $\mathcal{B}$ is a bijective isometry. □

Proof of Lemma 4.32. Suppose $\{D_n\}_n \subset \mathcal{H}_\infty(\mathbb{C}_+^k)$ is a bounded sequence. For every $n \in \mathbb{N}$ consider $g_n = \mathcal{B}^{-1}(D_n) \in H_\infty(B_{c_0^k})$ and apply Montel's Theorem [12, Theorem 2.17] to the sequence $\{g_n\}$ to find a subsequence $\{g_{n_l}\}$ which converges uniformly on the compact subsets of B_{c_0} to some $g \in H_\infty(B_{c_0^k})$. Define $D = \mathcal{B}(g)$, and then for every $\delta > 0$,

$$\sup_{\substack{\operatorname{Re} s_j \geqslant \delta \\ 1 \leqslant j \leqslant k}} |D_{n_l}(s_1, \ldots, s_k) - D(s_1, \ldots, s_k)| =$$

$$\sup_{\substack{\operatorname{Re} s_j \geqslant \delta \\ 1 \leqslant j \leqslant k}} |(g_{n_l} - g)\left(\frac{1}{\mathfrak{p}^{s_1}}, \ldots, \frac{1}{\mathfrak{p}^{s_k}}\right)| \leqslant \sup_{(x_1,\ldots,x_k) \in K_\delta^k} |(g_{n_l} - g)(x_1, \ldots, x_k)|,$$

where $K_\sigma = \{x = (x_n)_n \in B_{c_0} : |x_n| \leqslant \frac{1}{p_n^\sigma}, \forall n \in \mathbb{N}\}$, which is a compact subset of $B_{c_0^k}$. Since $\lim_{l\to\infty} \sup_{(x_1,\ldots,x_k)\in K_\delta} |(g_{n_l} - g)(x_1, \ldots, x_k)| = 0$, $\{D_{n_k}\}$ converges uniformly to D on $\mathbb{C}_\delta^k$ for any $\delta > 0$. □

Remark 4.35. As it has been proved in Theorem 4.33, the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ are all isometrically isomorphic independently form the dimension $k \in \mathbb{N}$. However the actual isometry that transforms Dirichlet monomials into Dirichelt monomials is lost in translation. We give an example here to clarify how this isometry works.

Take for example c_0 and c_0^2 and $\psi : c_0^2 \to c_0$ and isometric isomorphism (a particular example could be $\psi(\{x_i\}_i, \{y_l\}_l) = \{z_h\}_h$, where $z_h = x_{2i}$ if h is even and $z_h = y_{2l-1}$ if h is odd). If $\frac{a_{m,n}}{m^s n^t}$ is a double Dirichlet monomial, think of m and n as belonging to two different copies of $\mathbb{N}$, one per index of the double Dirichlet series the monomial belongs to. Now, take α and β the multi-indices such that $m = \mathfrak{p}^\alpha$ and $n = \mathfrak{p}^\beta$. Since $\alpha, \beta \in c_{00} \subset c_0$, there extists $\gamma := \psi(\alpha, \beta)$, and then we can take $j = \mathfrak{p}^\gamma \in \mathbb{N}$. Now define $b_j = a_{m,n}$. This is where the methodology of Theorem 4.33 makes sense of this apparently arbitrary definition, as from the point of view of functions in infinetly many variables you are just defining $d_\gamma = d_{\psi(\alpha,\beta)} := c_{\alpha,\beta}$ and then through the relation between natural numbers and the multi-indices from their decomposition in primer factors you complete this definition with

$$b_j := d_\gamma = d_{\psi(\alpha,\beta)} = c_{\alpha,\beta} = a_{m,n}.$$

Finally, the map $\Psi : \mathcal{H}_\infty(\mathbb{C}_+^2) \to \mathcal{H}_\infty(\mathbb{C}_+)$ that ψ induces satisfies $\Psi\left(\frac{a_{m,n}}{m^s n^t}\right) = \frac{b_j}{j^u}$. What is proved in Theorem 4.33 using infinite dimensional complex analysis is that Ψ can be extended form monomials to infinite sums as an isometric isomorphism.

Chapter 5

Composition operators on spaces of multiple Dirichlet series

We dedicate this chapter to the study of composition operators on the algebras $\mathcal{H}_\infty(\mathbb{C}_+^k)$. Composition operators of spaces of Dirichlet series were introduced in [17] where the authors gave sufficient and necessary conditions for a symbol ϕ to define a composition operator C_ϕ of the space $\mathcal{H}^2$ of Dirichlet series whose coefficients are in l^2. This result was later extended by Bayart in [5] and [4] to the spaces of Dirichlet series $\mathcal{H}^p$ that he defined, including the case $p = \infty$, which we denote by $\mathcal{H}_\infty(\mathbb{C}_+)$. We will start by studying this case in Section 5.1, and we will extend those results to the spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ in Section 5.2. Finally, we also study in this chapter superposition operators on Hardy spaces of Dirichlet series, in Section 5.3.

5.1 Composition operators on $\mathcal{H}_\infty(\mathbb{C}_+)$

The first theorem that characterized composition operators of spaces of Dirichlet series appeared in [17], and we recall it below.

Theorem 5.1 ([17], Theorem B). *An analytic function $\phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ defines a bounded composition operator $C_\phi : \mathcal{H}^2 \to \mathcal{H}^2$ if and only if*

(a) it is of the form $\phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}$, where $\mathcal{D}$ is the set of Dirichlet series which converge in some remote half-plane.

(b) ϕ has an analyitic extension to $\mathbb{C}_+$, also denoted by ϕ, such that

(i) $\phi(\mathbb{C}_+) \subset \mathbb{C}_+$ if $0 < c_0$, and

(ii) $\phi(\mathbb{C}_+) \subset \mathbb{C}_{\frac{1}{2}}$ if $c_0 = 0$.

The proof of this theorem is actually developed throughout [17], but we are specially interested in Theorem A of [17], as the methods used to prove this theorem are also used when studying composition operators of spaces like $\mathcal{H}_\infty(\mathbb{C}_+^k)$.

Theorem 5.2 ([17], Theorem A). *An analytic function $\phi : \mathbb{C}_\theta \to \mathbb{C}_{\frac{1}{2}}$ generates a composition operator $C_\phi : \mathcal{H}^2 \to \mathcal{D}$ if and only if it is of the form $\phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}$, where $\mathcal{D}$ is the set of Dirichlet series which converge on some sufficiently remote half-plane.*

Here it should be understood that φ is a Dirichlet series which converges in some remote half-plane and has an analytic continuation $\tilde{\varphi}$ defined in $\mathbb{C}_\theta$. We will split the proof of Theorem 5.2 in order to separate the results we are going to use independently below.

Lemma 5.3. *Let m be a positive integer, and let $f(s) = \sum_{n=m}^{\infty} \frac{a_n}{n^s}$ be a Dirichet series from the class $\mathcal{D}$, starting from the index m. Then $m^s f(s) \to a_m$ uniformly as $\operatorname{Re} s \to \infty$.*

Proof. Define $g(s) = m^s f(s) = a_m + \sum_{n=m+1}^{\infty} a_n \left(\frac{m}{n}\right)^s$. Note that the series that defines $g(s)$ converges if and only if the series that defines $f(s)$ converges. Now, take $\sigma > \sigma_a(f)$ and $\operatorname{Re} s \geqslant \sigma$, then

$$\sum_{n=m+1}^{\infty} |a_n| \left(\frac{m}{n}\right)^{\operatorname{Re} s} = m^{\operatorname{Re} s} \sum_{n=m+1}^{\infty} \frac{|a_n|}{n^{\operatorname{Re} s}} \leqslant m^{\operatorname{Re} s} \sum_{n=m+1}^{\infty} \frac{|a_n|}{n^{\sigma}} < \infty,$$

so the series that defines $g(s)$ converges absolutely for $\operatorname{Re} s \geqslant \sigma$ and then the sequence $\left(|a_n| \frac{m^\sigma}{n^\sigma}\right)_{n=m+1}^{\infty}$ is convergent to zero and therefore bounded, say by K. On the one hand, as $g(s)$ converges absolutely on $\mathbb{C}_{\sigma_a(f)}$, it converges uniformly on $\mathbb{C}_{\sigma_a(f)}$, and given $\varepsilon > 0$ there exists a certain $M_0 \in \mathbb{N}$ such that $M \geqslant M_0$ implies

$$\left| \sum_{n=m+1}^{\infty} a_n \left(\frac{m}{n}\right)^s - \sum_{n=m+1}^{M} a_n \left(\frac{m}{n}\right)^s \right| < \frac{\varepsilon}{2} \quad \text{for every } s \in \mathbb{C}_{\sigma_a(f)}.$$

On the other hand, $\left(\frac{m}{m+1}\right)^s \to 0$ uniformly when $\operatorname{Re} s \to 0$, so for the previous $M_0 \in \mathbb{N}$, given $\varepsilon > 0$ we can find R_0 such that $\operatorname{Re} s > R_0$ implies $\left(\frac{m}{m+1}\right)^{\operatorname{Re} s} < \frac{\varepsilon}{2KM_0}$. Then, for $s > R_0 + \sigma = R_1$,

$$\begin{aligned} \left| \sum_{n=m+1}^{M_0} a_n \left(\frac{m}{n}\right)^s \right| &\leqslant \sum_{n=m+1}^{M_0} |a_n| \left(\frac{m}{n}\right)^\sigma \left(\frac{m}{n}\right)^{\operatorname{Re} s} \\ &\leqslant K \sum_{n=m+1}^{M_0} \left(\frac{m}{n}\right)^{\operatorname{Re} s - \sigma} \leqslant KM_0 \left(\frac{m}{m+1}\right)^{\operatorname{Re} s - \sigma} < \frac{\varepsilon}{2}. \end{aligned}$$

Finally, for $s \in \overline{\mathbb{C}}_\sigma$

$$\begin{aligned} \left| \sum_{n=m+1}^{\infty} a_n \left(\frac{m}{n}\right)^s \right| &\leqslant \left| \sum_{n=m+1}^{\infty} a_n \left(\frac{m}{n}\right)^s - \sum_{n=m+1}^{M_0} a_n \left(\frac{m}{n}\right)^s \right| \\ &\quad + \left| \sum_{n=m+1}^{M_0} a_n \left(\frac{m}{n}\right)^s \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Lemma 5.4. *Let* $\{f_n\}_n$ *be a sequence of continuous functions* $f_n : \mathbb{R} \to \mathbb{C}$ *such that for all* $n \in \mathbb{N}$ *there exists*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(t)dt = a_n \in \mathbb{C}.$$

If $\sum_{n=1}^{\infty} f_n$ *is uniformly convergent in* $\mathbb{R}$ *then there exists*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{n=1}^{\infty} f_n(t)dt = \sum_{n=1}^{\infty} a_n \in \mathbb{C}.$$

Proof. Given $T > 0$, $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $m \geqslant n \geqslant n_0$ implies $|\sum_{k=n}^{m} f_k(t)| < \frac{\varepsilon}{4}$ for all $t \in \mathbb{R}$, so $|\sum_{k=n}^{\infty} f_k(t)| \leqslant \frac{\varepsilon}{4}$ for all $t \in \mathbb{R}$. Then, for a fixed $n \geqslant n_0$ define

$$\begin{aligned} \Delta_n &= \left| \frac{1}{2T} \int_{-T}^{T} \sum_{k=1}^{\infty} f_k(t)dt - \sum_{k=1}^{n} a_k \right| \\ &\leqslant \left| \sum_{k=1}^{n} \frac{1}{2T} \int_{-T}^{T} f_k(t)dt - \sum_{k=1}^{n} a_k \right| + \left| \frac{1}{2T} \int_{-T}^{T} \sum_{k=n+1}^{\infty} f_k(t)dt \right|. \end{aligned}$$

If $k \leqslant n$ then there exists $T_k > 0$ such that $T \geqslant T_k$ implies

$$\left| \frac{1}{2T} \int_{-T}^{T} f_k(t)dt - a_k \right| < \frac{\varepsilon}{4n},$$

so if $T \geqslant \max_{1\leqslant k\leqslant n} T_k$ then

$$\Delta_n \leqslant \sum_{k=1}^{n} \frac{\varepsilon}{4n} + \frac{1}{2T} \int_{-T}^{T} \frac{\varepsilon}{4} dt = \frac{\varepsilon}{2}.$$

Moreover, for $m \geqslant n \geqslant n_0$

$$\left| \sum_{k=n}^{m} a_k \right| = \left| \sum_{k=n}^{m} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_k(t)dt \right| \leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{k=n}^{m} f_k(t)dt \right| \leqslant \frac{\varepsilon}{4},$$

so $\sum_{k=1}^{\infty} a_k$ converges and

$$\left|\frac{1}{2T}\int_{-T}^{T}\sum_{k=1}^{\infty} f_k(t)dt - \sum_{k=1}^{\infty} a_k\right| \leqslant \Delta_n + \left|\sum_{k=n+1}^{\infty} a_k\right| \leqslant \frac{3\varepsilon}{4} < \varepsilon,$$

so there exists

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{n=1}^{\infty} f_n(t)dt = \sum_{n=1}^{\infty} a_n \in \mathbb{C}.$$

□

Lemma 5.5. *Let G and H be multiplicative semigroups contained in $\mathbb{Q}_+$ and let $\varphi : \mathbb{C}_\sigma \to \mathbb{C}$ be a function that can be represented as an absolutely convergent Dirichlet series over G and also over H, that is, for all $s \in \mathbb{C}_\sigma$, $\varphi(s) = \sum_{g\in G} b_g g^{-s} = \sum_{h\in H} d_h h^{-s}$. Then $b_g = 0$ for all $g \in G \setminus H$ and $d_h = 0$ for all $h \in H \setminus G$, that is, $\varphi(s) = \sum_{g\in G\cap H} b_g g^{-s}$.*

Proof. First, if $G = \{g_n\}_n \in \mathbb{N}$ and $m \in \mathbb{N}$, using Lemma 5.4

$$\begin{aligned}\lim_{T\to\infty}&\frac{1}{2T}\int_{-T}^{T}\varphi(\sigma+1+it)g_m^{\sigma+1+it}dt\\ &= \sum_{n=1}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} b_{g_n} g_n^{-(\sigma+1+it)} g_m^{\sigma+1+it}dt\\ &= b_{g_n}\sum_{n=1}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\frac{g_m}{g_n}\right)^{\sigma+1+it} dt = b_{g_m},\end{aligned}$$

because $\frac{g_m}{g(n)} \in \mathbb{Q}\setminus\{1\}$ for $m \neq n$. Then, since $g_m \in G\setminus H$ implies $\frac{g_m}{h_n} \in \mathbb{Q}\setminus\{1\}$, for all $g_m \in G\setminus H$ we have that

$$\begin{aligned}b_{g_m} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(\sigma+1+it)g_m^{\sigma+1+it}dt\\ &= d_{h_n}\sum_{n=1}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\frac{g_m}{h_n}\right)^{\sigma+1+it} dt = 0.\end{aligned}$$

□

Lemma 5.6. *Suppose that $\phi : \mathbb{C}_\theta \to \mathbb{C}_\vartheta$ is an analytic function such that $k^{-\phi} \in \mathcal{D}$ for every $k \in \mathbb{N}$. Then there exists some $\sigma \geqslant \theta$ such that $\phi(s) = c_0 s + \varphi(s)$, where $c_0 \in \mathbb{N}_0$ and φ is a Dirichlet series which converges absolutely in $\mathbb{C}_\sigma$.*

Proof. By hypothesis, there exists some $\sigma_k > 0$ such that

$$k^{-\phi(s)} = \sum_{n=N(k)}^{\infty} \frac{a_n^{(k)}}{n^s} \qquad \text{for every } s \in \mathbb{C}_{\sigma_k},$$

where $N(k) \in \mathbb{N}$ is the index of the first non-zero coefficient of the series. Applying Lemma 5.3 we arrive at $N(k)^s k^{-\phi(s)} \to a_{N(k)}^{(k)}$ uniformly as $\operatorname{Re} s \to \infty$, that is, $e^{s \log N(k) - \phi(s) \log k} \to a_{N(k)}^{(k)}$ uniformly as $\operatorname{Re} s \to \infty$, where log stands for the natural logarithm. Define $F_k, G_k : \mathbb{C}_{\sigma_k} \to \mathbb{C}$, $F_k(s) = N(k)^s k^{-\phi(s)}$ and $G_k(s) = s \log N(k) - \phi(s) \log k$. We have that $F_k(s) = e^{G_k(s)}$ and we want to find a branch of the complex logarithm $\log_\alpha$ such that

$$\lim_{\operatorname{Re} s \to \infty} G_k(s) = \lim_{\operatorname{Re} s \to \infty} \log_\alpha F_k(s) + 2\pi i q_k, \quad \text{for some } q_k \in \mathbb{Z}.$$

First, as $a_{N(k)}^{(k)} \neq 0$, there exists $R_0 > 0$ such that $\operatorname{Re} s > R_0$ implies $|F_k(s) - a_{N(k)}^{(k)}| < |a_{N(k)}^{(k)}|$, and therefore $F_k(s) \neq 0$ for every $s \in \mathbb{C}_{R_0}$. Now write $a_{N(k)}^{(k)} = re^{i\alpha}$, and consider the branch of the complex logarithm $\log_\alpha$. We have already stated that $F_k(\mathbb{C}_{R_0}) = B(a_{N(k)}^{(k)}, r)$, so $\log_\alpha F_k$ is well defined and analytic on $\mathbb{C}_{R_0}$. Choose $R_0' \geqslant R_0$ such that $\operatorname{Re} s > R_0'$ implies $|\log_\alpha F_k(s) - \log_\alpha a_{N(k)}^{(k)}| < 1$. As $e^{G_k(s)} = F_k(s) = e^{\log_\alpha F_k(s)}$ for every $s \in \mathbb{C}_{R_0'}$, we can find $q_k(s) \in \mathbb{Z}$ such that $\log_\alpha F_k(s) + 2\pi i q_k(s) = G_k(s)$ for every $s \in \mathbb{C}_{R_0'}$. Now, $\log_\alpha F_k(\mathbb{C}_{R_0'}) \subset B(\log_\alpha a_{N(k)}^{(k)}, 1)$, so $G_k(\mathbb{C}_{R_0'}) \subset B(\log_\alpha a_{N(k)}^{(k)}, 1) + 2\pi i \mathbb{Z}$, but G_k being analytic gives that $G_k(\mathbb{C}_{R_0'})$ is connected, and therefore the function $q_k(s)$ must be constant, that is, there is a unique $q_k \in \mathbb{Z}$ such that $\log_\alpha F_k(s) + 2\pi i q_k = G_k(s)$

for every $s \in \mathbb{C}_{R_0'}$. Thus,

$$\lim_{\operatorname{Re} s \to \infty} s \log N(k) - \phi(s) \log k = \log_\theta a^{(k)}_{N(k)} + 2\pi i q_k,$$

and therefore

$$\lim_{\operatorname{Re} s \to \infty} \frac{\phi(s)}{s} = \frac{\log N(k)}{k} =: c_0.$$

Note that c_0 is independent of k as it is ϕ. Moreover, as $c_0 = \dfrac{\log N(k)}{\log k}$, $k^{c_0} = N(k) \in \mathbb{N}$ for every $k \in \mathbb{N}$. This means that $c_0 \in \mathbb{N}_0$, as it is given by Lemma 3.2 of [17].

Define now $\varphi(s) = \phi(s) - c_0 s$ for $s \in \mathbb{C}_\theta$. We want to see that $\varphi \in \mathcal{D}$, that is, that there exists some $\sigma > 0$ such that φ can be represented as a converging Dirichlet series on $\mathbb{C}_\sigma$. Consider $k^{-\phi}$ and $s \in \mathbb{C}_\theta$, then

$$\begin{aligned} k^{-\varphi(s)} = k^{-\phi(s)} k^{c_0 s} &= \sum_{n=k^{c_0}}^{\infty} a_n^{(k)} \left(\frac{k^{c_0}}{n}\right)^s \\ &= a^{(k)}_{N(k)} + h_k(s) = a^{(k)}_{N(k)} \left(1 + \frac{h_k(s)}{a^{(k)}_{N(k)}}\right), \end{aligned}$$

where

$$h_k(s) = \sum_{n=1}^{\infty} a^{(k)}_{N(k)+n} \left(1 + \frac{n}{k^{c_0}}\right)^{-s}.$$

Since

$$k^{-\varphi(s)} = k^{-\phi(s)} k^{c_0 s} = N(k)^s k^{-\phi(s)} \longrightarrow a^{(k)}_{N(k)+n}$$

uniformly as $\operatorname{Re} s \to \infty$, then $h_k(s) \longrightarrow 0$ uniformly as $\operatorname{Re} s \to \infty$, so we can find R_k such that $R \geqslant R_k$ implies that the series which defines $h_k(s)$ is absolutely convergent in $\mathbb{C}_{R_k}$ and also that $|h_k(s)| < |a^{(k)}_{N(k)+n}|$, so $\left|\dfrac{h_k(s)}{a^{(k)}_{N(k)}}\right| < 1$. This allows to take the same branch of the complex logarithm, $\log_\alpha$, and proceed as before, so there exists $q_k' \in \mathbb{Z}$ such that, for all $s \in \mathbb{C}_{R_k}$,

$$-\varphi(s)\log k = \log_\alpha a^k_{N(k)+n} + \log_\alpha\left(1 + \frac{h_k(s)}{a^{(k)}_{N(k)}}\right) + 2\pi i q'_k.$$

Expanding $\log(1+z)$ in a Taylor series around $z_0 = 0$ with $z = \frac{h_k(s)}{a^{(k)}_{N(k)}}$ we get, for all $s \in \mathbb{C}_{R_k}$,

$$-\varphi(s)\log k = \log_\alpha a^k_{N(k)+n} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{h_k(s)}{a^{(k)}_{N(k)}}^n + 2\pi i q'_k.$$

The absolute convergence of both the Taylor series of $\log(1+z)$ and the series defining $h_k(s)$ allows us to open the brackets in the expression $h_k(s)^n$ and to rearrange the terms for $s \in \mathbb{C}_{R_k}$, so we can write

$$\varphi(s) = \sum_{q=0}^{\infty}\sum_{n_1,\ldots,n_q}^{\infty}\beta_{n_1,\ldots,n_q}\left(1+\frac{n_1}{k^{c_0}}\right)^{-s}\cdots\left(1+\frac{n_q}{k^{c_0}}\right)^{-s}$$

which converges absolutely in $\mathbb{C}_{R_k}$. This means that, given a $k \in \mathbb{N}$, we can see φ as a Dirichlet series over the multiplicative semigroup generated by $\{1+\frac{j}{k^{c_0}}\}_{j\in\mathbb{N}}$ which we will denote by $\mathcal{G}(k^{c_0})$. Now, $\bigcap_{k\in\mathbb{N}}\mathcal{G}(k^{c_0}) = \mathbb{N}$, because $\mathcal{G}(2^{c_0}) \cap \mathcal{G}(3^{c_0}) = \mathbb{N}$. Therefore, by Lemma 5.5, φ is a Dirichlet series over $\mathbb{N}$ which is absolutely convergent in $\mathbb{C}_\sigma$, where $\sigma = \max(R_2, R_3)$.

□

Proof of Theorem 5.2. *Necessity.* Suppose that $f \circ \phi \in \mathcal{D}$ for every $f \in \mathcal{H}^2$. For any $k \in \mathbb{N}$ consider $k^{-\phi(s)}$, which belongs to $\mathcal{D}$ because $k^{-s} \in \mathcal{H}^2$. Then, applying Lemma 5.6 is enough.

<u>*Sufficiency*</u> Suppose $\phi : \mathbb{C}_\theta \to \mathbb{C}_{\frac{1}{2}}$ is an analytic mapping and that there exists some $\sigma \geqslant \max(\theta, \frac{1}{2})$ such that

$$\phi(s) = c_0 s + \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad \text{for } s \in \mathbb{C}_\sigma,$$

with $c_0 \in \mathbb{N}_0$ and the series $\varphi(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$ absolutely convergent in $\mathbb{C}_\sigma$. If $k \in \mathbb{N}$ and $\operatorname{Re} s > \sigma$,

$$k^{-\phi(s)} = k^{-c_0 s - c_1} e^{-\log k \sum_{n=2}^{\infty} \frac{c_n}{n^s}} = k^{-c_0 s - c_1} \prod_{n=2}^{\infty} e^{-\log k \frac{c_n}{n^s}}.$$

Now, if $f \in \mathcal{H}^2$, $f(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}$, using the equalities above and expanding the exponential function as a power series,

$$\sum_{k=1}^{\infty} \frac{a_k}{k^{\phi(s)}} = \sum_{k=1}^{\infty} a_k k^{-c_0 s - c_1} \prod_{n=2}^{\infty} \left(1 + \sum_{r=1}^{\infty} \frac{(-c_n \log k)^r}{r!} \frac{1}{(n^r)^s}\right).$$

We claim that we can rearrange the terms in the latter sum to write it as a Dirichlet series wherever it is absolutely convergent. The idea behind this rearrangement is that we are trying to determine the Nth coefficient of the Dirichlet series that will result from rearranging the terms of the product above, which is multiplying an infinite number of infinite series. To do that, we are going to arrange them in the same coefficient if they accompany the same $\frac{1}{N^s}$. Indeed, if $N \in \mathbb{N}$, $N > 1$, choose $n_1, \dots, n_d \in \mathbb{N}$, $n_j > 1$ for every $1 \leqslant j \leqslant d$, and $r_1, \dots, r_d \in \mathbb{N}$ such that $N = n_1^{r_1} \cdots n_d^{r_d}$ (Note that this is always possible as $n_1 = N$, $r_1 = 1$ is always an admissible choice). Now define

$$b_{n_1^{r_1}, \dots, n_d^{r_d}} = \prod_{j=1}^{d} \frac{(-c_{n_j} \log k)^{r_j}}{r_j!}.$$

Then for every possible factorization of N we have to consider the product of the coefficients of its factors, and then take the sum of this products over all the possible admissible factorizations. Thus, define

$$B_{N,k} = \sum\{b_{n_1^{r_1},\dots,n_d^{r_d}} : N = n_1^{r_1}\cdots n_d^{r_d}\}$$

where the n_j and the r_j satisfy the conditions that have been set before. Define also $B_{1,k} = 1$. Thus, since $\operatorname{Re} s > \sigma \geqslant \frac{1}{2}$ gives the absolute convergence of both the Dirichlet series φ and $\sum_{k=1}^{\infty} \frac{a_k}{k^s}$,

$$\sum_{k=1}^{\infty} \frac{a_k}{k^{\phi(s)}} = \sum_{k=1}^{\infty} a_k k^{-c_0 s - c_1} \sum_{N=1}^{\infty} \frac{B_{N,k}}{N^s} = \sum_{k=1}^{\infty}\sum_{N=1}^{\infty} \frac{a_k B_{N,k}}{(k^{c_0} N)^s},$$

where the double series on the right-hand side can be again rearranged into a Dirichlet series which will be absolutely convergent in $\mathbb{C}_\sigma$.
At this moment it only remains to be seen that

$$\sum_{k=1}^{\infty} \frac{a_k}{k^{\phi(s)}} = \sum_{k=1}^{\infty} a_k k^{-c_0 s - c_1} \prod_{n=2}^{\infty} \left(1 + \sum_{r=1}^{\infty} \frac{(-c_n \log k)^r}{r!} \frac{1}{(n^r)^s}\right).$$

is absolutely convergent on a certain half-plane. Formally,

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty} a_k k^{-c_0 s - c_1} \prod_{n=2}^{\infty} \left(1 + \sum_{r=1}^{\infty} \frac{(-c_n \log k)^r}{r!} \frac{1}{(n^r)^s}\right)\right| \\
&\qquad \leqslant \sum_{k=1}^{\infty} |a_k| k^{-c_0 \operatorname{Re} s - \operatorname{Re} c_1} \prod_{n=2}^{\infty} \left(1 + \sum_{r=1}^{\infty} \frac{(|c_n| \log k)^r}{r!} \frac{1}{(n^r)^{\operatorname{Re} s}}\right) \\
&\qquad = \sum_{k=1}^{\infty} |a_k| k^{-c_0 \operatorname{Re} s - \operatorname{Re} c_1} \prod_{n=2}^{\infty} k^{\frac{|c_n|}{n^{\operatorname{Re} s}}} = \sum_{k=1}^{\infty} |a_k| k^{-c_0 \operatorname{Re} s - \operatorname{Re} c_1 + \sum_{n=2}^{\infty} \frac{|c_n|}{n^{\operatorname{Re} s}}},
\end{aligned}$$

and it is well known that $\sum_{n=2}^{\infty} \frac{|c_n|}{n^{\operatorname{Re} s}}$ tends to uniformly to zero as $\operatorname{Re} s \to \infty$, so given $\varepsilon > 0$ one can find R_0 such that $\operatorname{Re} s \geqslant R_0$ implies

$\sum_{n=2}^{\infty} \frac{|c_n|}{n^{\operatorname{Re} s}} < \varepsilon$. Now, if $c_0 \neq 0$,

$$\sum_{k=1}^{\infty} |a_k| k^{-c_0 \operatorname{Re} s - \operatorname{Re} c_1 + \sum_{n=2}^{\infty} \frac{|c_n|}{n^{\operatorname{Re} s}}} \leqslant \sum_{k=1}^{\infty} \frac{|a_k| k^{\varepsilon}}{(k_0^c)^{\operatorname{Re} s} k^{\operatorname{Re} c_1}} \leqslant \sum_{k=1}^{\infty} \frac{|a_k|}{k^{\operatorname{Re} s + \operatorname{Re} c_1 - \varepsilon}},$$

where the last series on the right-hand side is convergent if $\operatorname{Re} s > \max(R_0, \frac{1}{2} - \operatorname{Re} c_1 + \varepsilon)$.
In the case that $c_0 = 0$, we claim that $\operatorname{Re} c_1 > \frac{1}{2}$. Then

$$\sum_{k=1}^{\infty} |a_k| k^{-c_0 \operatorname{Re} s - \operatorname{Re} c_1 + \sum_{n=2}^{\infty} \frac{|c_n|}{n^{\operatorname{Re} s}}} \leqslant \sum_{k=1}^{\infty} \frac{|a_k|}{k^{\operatorname{Re} c_1 - \varepsilon}},$$

so it is enough to choose ε such that $\varepsilon < \operatorname{Re} c_1 - \frac{1}{2}$ to get the convergence of the series on the right-hand side.
Proof of the claim. If $c_0 = 0$, $\phi = \varphi$, so $\varphi : \mathbb{C}_\theta \to \mathbb{C}_{\frac{1}{2}}$. If φ is constant, trivially $\operatorname{Re} c_1 > \frac{1}{2}$. If it is not constant, then there exists a minimum $N \in \mathbb{N}$, $N > 1$, such that $c_N \neq 0$. First suppose that $\varphi(s) = c_1 + c_N N^{-s}$, and write $s = x + iy$ and $c_N = |c_N| e^{i\alpha}$. Then

$$\operatorname{Re} \varphi(s) = \operatorname{Re} c_1 + \operatorname{Re} c_N N^s = \operatorname{Re} c_1 + |c_N| e^{-x \log N} \cos(\alpha - y \log N).$$

Take $x_0 > \theta$, $y_0 = \frac{\alpha - \pi}{\log N}$, then $\cos(\alpha - y_0 \log N) = -1$, so

$$\frac{1}{2} < \operatorname{Re} \varphi(s_0) = \operatorname{Re} c_1 - |c_N| e^{-x_0 \log N} < \operatorname{Re} c_1.$$

Now, if $\varphi(s) = c_1 + c_N N^s + \sum_{m=N+1}^{\infty} \frac{c_m}{m^s}$, define $G(s) = c_N N^s$ and $H(s) = \sum_{m=N+1}^{\infty} \frac{c_m}{m^s}$. Clearly $\lim_{\operatorname{Re} s \to \infty} \frac{H(s)}{G(s)} = 0$, so there exists a certain R_0 such that $\operatorname{Re} s > R_0$ implies $H(s) < \frac{1}{2} G(s)$. By the previous case we can find $s_0 = x_0 + iy_0 \in \mathbb{C}_+$, with x_0 as large as needed, such that $\operatorname{Re} G(s_0) = -|G(s_0)|$. Therefore,

$$\frac{1}{2} < \operatorname{Re} \varphi(s_0) = \operatorname{Re} c_1 + \operatorname{Re} G(s_0) + \operatorname{Re} H(s_0) \\ < \operatorname{Re} c_1 - |G(s_0)| + \frac{1}{2}|G(s_0)| \leqslant \operatorname{Re} c_1,$$

which proves the claim in the general case. □

The hypothesis of Theorem 5.2 can be weakened as the condition of analiticity of the symbol can be dropped. We show this in the following lemma.

Lemma 5.7. *Let $\theta \geqslant 0$ and suppose that $\phi : \mathbb{C}_\theta \to \mathbb{C}$ is a function such that $p^{-\phi}, q^{-\phi} \in \mathcal{H}^\infty(\mathbb{C}_\sigma)$, where p and q are different prime numbers and $\sigma \geqslant \theta$. Then ϕ is analytic in $\mathbb{C}_\sigma$.*

Remark 5.8. If p and q are different prime numbers such that $p^{-\phi}, q^{-\phi} \in \mathcal{D}$, there exist $\sigma_1, \sigma_2 \geqslant \theta$ such that $p^{-\phi} \in \mathcal{H}^\infty(\mathbb{C}_{\sigma_1})$ and $q^{-\phi} \in \mathcal{H}^\infty(\mathbb{C}_{\sigma_2})$, so if $\sigma = \max(\sigma_1, \sigma_2)$ then $p^{-\phi}, q^{-\phi} \in \mathcal{H}^\infty(\mathbb{C}_\sigma)$.

Proof. By hypothesis, $D(s) = \frac{1}{p^{\phi(s)}} \in \mathcal{H}^\infty(\mathbb{C}_\sigma)$ with $|D(s)| > 0$ for every $s \in \mathbb{C}_\sigma$, so there exists $L : \mathbb{C}_\sigma \to \mathbb{C}$ a holomorphic logarithm of D, that is, $D(s) = e^{L(s)}$ for every $s \in \mathbb{C}_\sigma$. As we have $e^{L(s)} = D(s) = e^{-(\log p)\phi(s)}$, there exists a function $k : \mathbb{C}_\sigma \to \mathbb{Z}$ such that $-(\log p)\phi(s) = L(s) + 2k(s)\pi i$. Analogously for $\tilde{D}(s) = \frac{1}{q^{\phi(s)}}$, there exists $\tilde{L} : \mathbb{C}_\sigma \to \mathbb{C}$ a holomorphic logarithm of $\tilde{D}$ and a function $\tilde{k} : \mathbb{C}_\sigma \to \mathbb{Z}$ such that $-(\log q)\phi(s) = \tilde{L}(s) + 2\tilde{k}(s)\pi i$. Therefore, for every $s \in \mathbb{C}_\sigma$,

$$\frac{1}{-\log p}(L(s) + 2k(s)\pi i) = \frac{1}{-\log q}(\tilde{L}(s) + 2\tilde{k}(s)\pi i),$$

so, if $h : \mathbb{C}_\sigma \to \mathbb{C}$ is defined as

$$h(s) = \frac{1}{2\pi i}\left(\frac{L(s)}{\log p} - \frac{\tilde{L}(s)}{\log q}\right) = \frac{\tilde{k}(s)}{\log p} - \frac{k(s)}{\log q} \in \frac{1}{\log p}\mathbb{Z} + \frac{1}{\log q}\mathbb{Z},$$

then $h(\mathbb{C}_\sigma)$ is a countable set and by the open mapping property, h is a constant function. Now suppose $\tilde{k}$ is not constant and take $s_1, s_2 \in \mathbb{C}_\sigma$ such that $\tilde{k}(s_1) \neq \tilde{k}(s_2)$. Since $h(s_1) = h(s_2)$,

$$\frac{\tilde{k}(s_1)}{\log q} - \frac{k(s_1)}{\log p} = \frac{\tilde{k}(s_2)}{\log q} - \frac{k(s_2)}{\log p},$$

so

$$\frac{\log q}{\log p} = \frac{k(s_1) - k(s_2)}{\tilde{k}(s_1) - \tilde{k}(s_2)} \in \mathbb{Q},$$

a contradiction. Therefore, $\tilde{k}$ is constant and $\phi(s) = \frac{1}{-\log q}(\tilde{L}(s) + 2\tilde{k}\pi i)$ for all $s \in \mathbb{C}_\sigma$, so ϕ is analytic in $\mathbb{C}_\sigma$.

□

Before we get into the case of $\mathcal{H}_\infty(\mathbb{C}_+)$, we are going to need some lemmas that will play the role of a Maximum Modulus Principle for Dirichlet series. The first one of them extends Lemma 4.30. Given $D(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$, $D \in \mathcal{H}_\infty(\mathbb{C}_+)$, recall that for $t \in \mathbb{R}$

$$\overline{\lim}\, |D(it)| = \inf_{r>0} \sup_{s \in \mathbb{D}^+(n^{it}, r)} |D(s)|,$$

where $\mathbb{D}^+(n^{it}, r) = \{s \in \mathbb{C} : |s - n^{it}| < r, \operatorname{Re} s > 0\}$.

Lemma 5.9. *For $D(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$, $D \in \mathcal{H}_\infty(\mathbb{C}_+)$, we have that*

$$\sup_{\operatorname{Re} s > 0} \left| \sum_{n=1}^\infty \frac{a_n}{n^s} \right| = \sup_{t \in \mathbb{R}} \overline{\lim}\, |D(it)|.$$

Proof. Let us write

$$A = \sup_{t \in \mathbb{R}} \overline{\lim}\, |D(it)| \quad \text{and} \quad B = \sup_{\operatorname{Re} s > 0} \left| \sum_{n=1}^\infty \frac{a_n}{n^s} \right|.$$

By definition $A \leqslant B$. For the converse inequality let us fix $\varepsilon > 0$ and consider the function

$$g_\varepsilon(s) := e^{-\varepsilon\sqrt{s}} \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{defined for } \operatorname{Re} s > 0,$$

where $\sqrt{s}$ denotes the principal square root of s. Then g_ε is a holomorphic function on $\mathbb{C}_+$. Taking now $s = re^{i\alpha} \in \mathbb{C}_+$, we have

$$|g_\varepsilon(s)| = e^{-\varepsilon \operatorname{Re}\sqrt{s}} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| = e^{-\varepsilon\sqrt{r}\cos\frac{\alpha}{2}} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leqslant Be^{-\varepsilon\sqrt{r}\cos\frac{\pi}{4}},$$

and this tends to 0 as $r \to \infty$. Hence there exists $R > 0$ such that $|g_\varepsilon(re^{i\alpha})| \leqslant A$ for $r \geqslant R$. Taking now $\Delta = \{s \in \mathbb{C}_+ : |s| < R\}$, we have that, for $|t| < R$,

$$|\overline{\lim}\, g_\varepsilon(it)| = e^{-\varepsilon \operatorname{Re}\sqrt{it}}\, \overline{\lim}\, |D(it)| \leqslant Ae^{-\varepsilon\sqrt{t}\cos\frac{\pi}{4}} \leqslant A.$$

Since ∞ is not accessible from the open set Δ, by the maximum modulus principle for subharmonic functions (Theorem 1 of [15]), $|g_\varepsilon(s)| \leqslant A$ for all $s \in \Delta$. This, altogether with $|g_\varepsilon(re^{i\alpha})| \leqslant A$ for $r \geqslant R$, gives that $|g_\varepsilon(s)| \leqslant A$ for every $s \in \mathbb{C}_+$. Letting $\varepsilon \to 0$ we get the desired conclusion. □

Corollary 5.10. *For $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $D \in \mathcal{H}_\infty(\mathbb{C}_+)$ with a continuous extension to $\overline{\mathbb{C}_+}$,*

$$\sup_{\operatorname{Re} s>0} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| = \sup_{t\in\mathbb{R}} |D(it)|.$$

Corollary 5.11. *Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series such that $\sigma_b(D) < \infty$ and consider $\sigma > \sigma_b(D)$. Then*

$$\sup_{\operatorname{Re} s>\sigma} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| = \sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+it}} \right|.$$

Theorem 5.12. *Let* $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be a non constant Dirichlet series in* $\mathcal{H}_\infty(\mathbb{C}_+)$*. We have that, for every* $0 < \sigma < \eta$,

$$\sup_{\operatorname{Re} s > \sigma} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| > \sup_{\operatorname{Re} s > \eta} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right|$$

Proof. Define $D_\sigma(s) = D(s+\sigma) = \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \frac{1}{n^s}$ and $D_\eta(s) = D(s+\eta) = \sum_{n=1}^{\infty} \frac{a_n}{n^\eta} \frac{1}{n^s}$. Clearly D_σ and D_η belong to $\mathcal{H}_\infty(\mathbb{C}_+)$, and

$$\sup_{\operatorname{Re} s > \sigma} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| = \sup_{\operatorname{Re} s > 0} |D_\sigma(s)| = ||D_\sigma||_\infty,$$

$$\sup_{\operatorname{Re} s > \eta} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| = \sup_{\operatorname{Re} s > 0} |D_\eta(s)| = ||D_\eta||_\infty.$$

Moreover, if $f_\sigma, f_\eta : B_{c_0} \to \mathbb{C}$ are the unique bounded holomorphic functions on the open unit ball of c_0 such that their monomial coefficients are $c_\alpha(f_\sigma) = \frac{a_{\mathfrak{p}^\alpha}}{\mathfrak{p}^{\alpha\sigma}}$ and $c_\alpha(f_\eta) = \frac{a_{\mathfrak{p}^\alpha}}{\mathfrak{p}^{\alpha\eta}}$ respectively, we have by Theorem 4.33 that

$$||D_\sigma||_\infty = ||f_\sigma||_{B_{c_0}}, \quad \text{and} \quad ||D_\eta||_\infty = ||f_\eta||_{B_{c_0}}.$$

Clearly

$$f_\eta(z) = f(\frac{1}{\mathfrak{p}^\eta} z) = f_\sigma(\frac{\mathfrak{p}^\sigma}{\mathfrak{p}^\eta} z) = f_\sigma(\frac{z_1}{2^{\eta-\sigma}}, \frac{z_2}{3^{\eta-\sigma}}, \dots, \frac{z_k}{p_k^{\eta-\sigma}}, \dots),$$

for every $z = (z_1, z_2, \dots) \in B_{c_{00}}$ and, by density, for every $z \in B_{c_0}$. Hence

$$||f_\eta||_{B_{c_0}} = ||f_\sigma||_{\frac{1}{\mathfrak{p}^{\eta-\sigma}} B_{c_0}} \leqslant ||f_\sigma||_{\frac{1}{2^{\eta-\sigma}} B_{c_0}}.$$

If we denote by $\tilde{f}_\eta$ and $\tilde{f}_\sigma$ the Aron-Berner extensions of f_η and f_σ respectively, we have the Davie and Gamelin result (see [10])

$$||\tilde{f}_\eta||_{B_{l_\infty}} = ||f_\eta||_{B_{c_0}} \leqslant ||f_\sigma||_{\frac{1}{2^{\eta-\sigma}}B_{c_0}} = ||\tilde{f}_\sigma||_{\frac{1}{2^{\eta-\sigma}}B_{l_\infty}}.$$

As $\tilde{f}_\sigma$ is $\omega(l_\infty, l_1)-$continuous on $\frac{1}{2^{\eta-\sigma}}\bar{B}_{l_\infty}$, which is $\omega(l_\infty, l_1)-$compact, $|\tilde{f}_\sigma|$ has a maximum in that compact set, i. e. there exists $z_0 \in B_{l_\infty}$ with $||z_0|| \leqslant \frac{1}{2^{\eta-\sigma}}$ and $||\tilde{f}_\sigma||_{\frac{1}{2^{\eta-\sigma}}B_{l_\infty}} = |\tilde{f}_\sigma(z_0)|$. Now we have two possibilities. Either the holomorphic mapping $\xi(\lambda) : 2^{\eta-\sigma}\mathbb{D} \to \mathbb{C}$ defined by $\xi(\lambda) = f_\sigma(\lambda z_0)$ is constant or not. If $\xi(\lambda)$ is constant this implies that $|\tilde{f}_\sigma|$ has a maximum on $\frac{1}{2^{\eta-\sigma}}B_{l_\infty}$ at 0, which is an interior point. Hence, by the Maximum Modulus Theorem applied to the restriction of that function to any complex line crossing zero, $\tilde{f}_\sigma$ is constant on the whole B_{l_∞}, thus f_σ is constant on B_{c_0}. This implies that D_σ is constant, which clearly implies that D is a constant function. This is a contradiction with our hypothesis. Hence ξ is not constant, and by the Maximum Modulus Theorem, there exists λ_0, $1 < \lambda_0 < 2^{\eta-\sigma}$, such that $|\tilde{f}_\sigma(\lambda_0 z_0)| > |\tilde{f}_\sigma(z_0)|$. In that case

$$\sup_{\operatorname{Re} s>\sigma}\left|\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right| = ||D_\sigma||_\infty = ||\tilde{f}_\sigma||_{B_{l_\infty}} \geqslant \tilde{f}_\sigma(\lambda_0 z_0) >$$
$$||\tilde{f}_\sigma||_{\frac{1}{2^{\eta-\sigma}}B_{l_\infty}} \geqslant ||\tilde{f}_\eta||_{B_{l_\infty}} = ||D_\eta||_\infty = \sup_{\operatorname{Re} s>\eta}\left|\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right|,$$

and the conclusion follows. □

We add another proof of this last result which may be more elementary.

Proof. First, by Corollary 5.11 and since D is not constant, $|a_1| < \sup_{\operatorname{Re} s>\sigma}|D(s)| = \sup_{\operatorname{Re} s=\sigma}|D(s)|$. Indeed, this is obviously true when $a_1 = 0$ and if $a_1 > 0$ and D is not constant there exists $N > 1$ such that $a_n \neq 0$, so by Proposition 2.25, $|a_1| < \sqrt{|a_1|^2 + \frac{|a_N|^2}{n^{2\sigma}}} \leqslant$

$\sup_{\operatorname{Re} s>\sigma}|D(s)|$. Consider $\varepsilon = \frac{1}{2}(\sup_{\operatorname{Re} s>\sigma}|D(s)| - |a_1|)$. Since by Lemma 5.3 $\lim_{\operatorname{Re} s\to\infty} D(s) = a_1$, there exists $\gamma > \sigma$ such that $\operatorname{Re} s > \gamma$ implies $|D(s) - a_1| < \varepsilon$, and therefore $\sup_{\operatorname{Re} s>\gamma}|D(s)| \leqslant |a_1| + \varepsilon < \sup_{\operatorname{Re} s>\sigma}|D(s)|$. If $0 < \theta < 1$ be such that $\eta = (1-\theta)\sigma + \theta\gamma$, Hadamard's three lines theorem gives that

$$\sup_{\operatorname{Re} s=\eta}|D(s)| = \sup_{\operatorname{Re} s>\sigma}|D(s)|^{(1-\theta)} \sup_{\operatorname{Re} s>\gamma}|D(s)|^{\theta} < \sup_{\operatorname{Re} s>\sigma}|D(s)|.$$

□

Theorem 5.12 can be used in the context of composition operators, as it gives information on the range of a Dirichlet series that is part of the symbol of a composition operator.

Proposition 5.13 ([17], Proposition 4.2). *Suppose $\phi : \mathbb{C}_\theta \to \mathbb{C}_\vartheta$ is an holomorphic mapping of the form $\phi(s) = c_0 s + \varphi(s)$ for some $c_0 \in \mathbb{N}_0$ and $\varphi(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \in \mathcal{D}$. Then if φ is constant, that constant value lies in the closed half-plane $\overline{\mathbb{C}}_{\vartheta - c_0\theta}$; if the function φ is non-constant then it extends to a holomorphic mapping $\varphi : \mathbb{C}_\theta \to \mathbb{C}_{\vartheta - c_0\theta}$. Moreover, for every $\theta' \in \mathbb{R}$ with $\theta' > \theta$, if φ is non-constant then φ maps $\mathbb{C}_{\theta'}$ to $\mathbb{C}_{\vartheta' - c_0\theta}$ for some $\vartheta' = \vartheta'(\theta') > \vartheta$.*

Proof. We shall reproduce the proof of [17] and also show a new proof of this result. First, by assumption, $\operatorname{Re}\phi(s) = c_0 \operatorname{Re} s + \operatorname{Re}\varphi(s) > \vartheta$ for $s \in \mathbb{C}_\theta$, so that $\operatorname{Re}\varphi(s) > \vartheta - c_0 \operatorname{Re} s$ for $s \in \mathbb{C}_\theta$. As $\varphi \in \mathcal{D}$, there exist $\sigma > 0$ and $M > 0$ such that

$$-\operatorname{Re}\varphi(s) \leqslant |\operatorname{Re}\varphi(s)| \leqslant |\varphi(s)| \leqslant M \quad \text{for } \operatorname{Re} s > \sigma.$$

On the other hand, if $\theta < \operatorname{Re} s \leqslant \sigma$, $-\operatorname{Re}\varphi(s) < c_0 \operatorname{Re} s - \vartheta < c_0\sigma - \vartheta$. Consider $2^{-\varphi}$, which by Lemma 5.1 is a Dirichlet series since $\varphi \in \mathcal{D}$ and $\frac{1}{2^s} \in \mathcal{H}^2$. As $|2^{-\varphi(s)}| = 2^{-\operatorname{Re}\varphi(s)} \leqslant 2^{\max(M, c_0\sigma - \vartheta)}$, $2^{-\varphi}$ is bounded throughout $\mathbb{C}_\theta$. Define $f_\varepsilon(s) = 2^{-\varphi(s+\varepsilon)}$, then f_ε is holomorphic on

$\mathbb{C}_\theta$ and continuous on $\overline{\mathbb{C}}_\theta$, so by the Maximum Modulus Theorem, for $s \in \mathbb{C}_{\theta+\varepsilon}$,

$$|2^{-\varphi(s)}| \leqslant \sup_{\operatorname{Re} s=\theta+\varepsilon} 2^{-\operatorname{Re}\varphi(s)} \leqslant \sup_{\operatorname{Re} s=\theta+\varepsilon} 2^{\vartheta-c_0 \operatorname{Re} s} = 2^{\vartheta-c_0(\theta+\varepsilon)}. \tag{5.1}$$

Taking $\varepsilon \to 0$ gives the conclusion, where the strict inequalitiy is obtained when φ is nonconstant as a consequence of the open mapping property of holomorphic mappings. The argument regarding (5.1) can be substituted by Corollary 5.11. For the second part of the theorem, let $\theta' > \theta$ and suppose that φ is nonconstant. Applying Theorem 5.12 to $D(s) = 2^{-\varphi(s-\theta)}$ we get that the function $M_D(t) = \sup_{s\in\mathbb{C}_t} |D(s)|$ is strictly decreasing on $\mathbb{C}_\theta$, but this can also be proved without the need of calling this theorem. By Theorem 12.8 of [30] the function $M_D(t)$ is decreasing and logarithmically convex, and it cannot be constant since $\lim_{\operatorname{Re} s\to\infty} \varphi(s) = c_1$ and therefore the image under φ of a sufficiently remote half-plane is a perturbed disk centred at c_1, so that the function $2^{-\varphi}$ there assumes values larger in modulus that $2^{-\operatorname{Re} c_1}$. Because $\log M_D$ is convex, M_D has to be strictly decreasing, as it is obviously already decreasing in $\mathbb{C}_\theta$ and cannot be constant in any closed interval as that would contradict either the assumption that φ is nonconstant or that $\log M_D$ is convex. In fact, if M_D was constant in the interval $[a, b]$, the function $\log M_D$ would be constant there. Moreover, as φ is not constant, we could choose $x > b$ such that $\log M_D(x) < \log M_D(y) = C$ for every $y \in [a, b]$. Defining $G : [0, 1] \to [\frac{a+b}{2}, x]$ with $G(\alpha) = (1-\alpha)(\frac{a+b}{2}) + \alpha x$, then there exists α_0 such that $G(\alpha_0) = b$, $\log M_D(G(\alpha_0)) = C$. But then

$$\alpha_0 \log M_D(\frac{a+b}{2}) + (1-\alpha_0)\log M_D(x) < \alpha_0 C + (1-\alpha_0)C = C,$$

so $\log M_D$ could not be convex. Now, using that M_D is strictly decreasing, for every $\theta' > \theta$ we have

$$\sup_{\operatorname{Re} s > \theta'} |2^{-\varphi(s)}| = \sup_{\operatorname{Re} s > \theta'} 2^{-\operatorname{Re}\varphi(s)} = M_D(\theta' - \theta) < M_D(0) = 2^{c_0\theta - \vartheta},$$

so there exists some $\vartheta' > \vartheta$ such that $\operatorname{Re}\varphi(s) > \vartheta' - c_0\theta$. □

Corollary 5.14. *Suppose $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ is an holomorphic mapping of the form $\phi(s) = c_0 s + \varphi(s)$ for some $c_0 \in \mathbb{N}_0$ and such that φ can be represented as a converging Dirichlet series in $\mathbb{C}_\sigma$ for some $\sigma > 0$. Then if φ is constant, that constant is non-negative, and if the function φ is non-constant then $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. Moreover, if φ is non-constant then for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\varphi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\delta$.*

Corollary 5.15. *Suppose $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ is an holomorphic mapping of the form $\phi(s) = c_0 s + \varphi(s)$ for some $c_0 \in \mathbb{N}_0$ and such that φ can be represented as a converging Dirichlet series in $\mathbb{C}_\sigma$ for some $\sigma > 0$. Then for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\delta$.*

Proof. If ϕ is constant, $\phi(s) = c_1$ for all $s \in \mathbb{C}_+$, then that constant is positive and if $\delta = \frac{c_1}{2}$, $\phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\delta$ for every $\varepsilon > 0$. Suppose now ϕ is not constant. If φ is constant then $\phi(s) = c_0 s + c_1$, so $c_0 \geqslant 1$ and $\phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\varepsilon$ for every $\varepsilon > 0$. Otherwise, φ is not constant and the result is a direct application of Corollary 5.14. □

Corollary 5.15 was the last result we needed before stating the theorem that characterizes composition operators on $\mathcal{H}_\infty(\mathbb{C}_+)$, which was given by Bayart in [4].

Theorem 5.16. *An analytic function $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ generates a composition operator $C_\phi : \mathcal{H}_\infty(\mathbb{C}_+) \to \mathcal{H}_\infty(\mathbb{C}_+)$ if and only if it is of the form $\phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N}_0$ and such that φ can be represented as a converging Dirichlet series in $\mathbb{C}_\sigma$ for some $\sigma > 0$.*

Proof. The necessity follows form Lemma 5.6 and the fact that $\frac{1}{k^s} \in \mathcal{H}_\infty(\mathbb{C}_+)$ for all $k \in \mathbb{N}$. For the sufficiency, take $f \in \mathcal{H}_\infty(\mathbb{C}_+)$, $f(s) = \sum_{k=1}^\infty \frac{a_k}{k^s}$; $\phi : \mathbb{C}_+ \to \mathbb{C}_+$, $\phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}$; and fix $\varepsilon > 0$. Consider $f_n(s) = \sum_{k=1}^n \frac{a_k}{k^s}$, and by Theorem 5.2 $f_n \circ \phi$ is a Dirichlet series convergent in some half-plane. Moreover, by Corollary 5.15, given $\varepsilon > 0$ there exists some $\delta > 0$ such that $\phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\delta$ and therefore the sequence $\{f_n \circ \phi\}_n$ is uniformly bounded on $\mathbb{C}_\varepsilon$, say by C, as there it converges uniformly to $f \circ \phi$. Write $(f_n \circ \phi)(s) = \sum_{k=1}^\infty \frac{b_k^n}{k^s}$. For a fixed k, the sequence $\{b_k^n\}_n$ is bounded by Ck^ε, so it converges to some $b_k \in \mathbb{C}$, $|b_k| \leqslant Ck^\sigma$ for all $\sigma > \varepsilon$. Defining $F(s) = \sum_{k=1}^\infty \frac{b_k}{k^s}$, since $|b_k| \leqslant Ck^{2\varepsilon}$, $F(s)$ converges absolutely in $\mathbb{C}_{1+2\varepsilon}$, and it is clear that in that half-plane the sequence $\{f_n \circ \phi\}_n$ converges to F. It is enough to note that $||f \circ \phi||_\infty \leqslant ||f||_\infty < \infty$ to apply Theorem 2.20 and get that $f \circ \phi$ actually coincides with F and that it is in $\mathcal{H}_\infty(\mathbb{C}_+)$. □

Remark 5.17. The statemaent of Theorem 5.16 can be strengthened using Theorem 3.1 of [29], which we will reproduce below for the sake of completeness. The strengthened version of Theorem 5.16 will also be stated below as a corollary.

Theorem 5.18 (Theorem 3.1 of [29])**.** *Suppose that φ is analytic with no zeros in $\mathbb{C}_+$ and that the harmonic conjugate of* $\log|\varphi|$ *is bounded in $\mathbb{C}_+$. If φ can be represented as a convergent Dirichlet series in some half-plane $\mathbb{C}_{\sigma_0}$, then this Dirichlet series converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$.*

Proof. Observe first that the Dirichlet series φ will be uniformly bounded in every half-plane $\mathbb{C}_\theta$ when $\theta > \sigma_0 + 1$. We fix such an abscissa θ and choose $0 < \alpha < 1$ small enough. Then the function φ^α is analytic in $\mathbb{C}_+$ and has the property that $|\varphi(s)|^\alpha$ is controlled by $\mathrm{Re}[\varphi(s)]^\alpha$ times a constant that only depends on α. Indeed, $\varphi^\alpha = |\varphi|^\alpha e^{i\alpha \operatorname{Arg}\varphi}$, and the harmonic conjugate of $\log|\varphi|$ is $\operatorname{Arg}\varphi$, so $|\operatorname{Arg}\varphi| \leqslant k$. Take α such that

$\alpha k < \frac{\pi}{4}$, so that $\left|\frac{\operatorname{Im}\varphi\alpha}{\operatorname{Re}\varphi^\alpha}\right| = |\tan(\alpha \operatorname{Arg}\varphi)| \leqslant 1$, and then $|\varphi|^\alpha \leqslant \sqrt{2}\operatorname{Re}\varphi^\alpha$. Given any $s = \sigma + it \in \mathbb{C}_\varepsilon$, $\varepsilon > 0$, we can now apply Harnack's inequality to the positive harmonic function $u := \operatorname{Re}\varphi^\alpha$ at the points $\sigma + it$ and $\vartheta + it$. Indeed, we will prove that $v(\sigma) \leqslant C_{\varepsilon,\theta} v(\theta)$ for every $\varepsilon < \sigma < \theta$ and every v a harmonic function on $\mathbb{C}_+$. First, consider the map $\psi_\theta : \mathbb{C}_+ \to \mathbb{D}$ defined by $\psi_\theta(s) = \frac{s-\theta}{s+\theta}$, which satisfies $-1 < \psi_\theta(\varepsilon) < \psi_\theta(\sigma) < \psi_\theta(\theta) = 0$ for every $\varepsilon < \sigma < \theta$. Consider $\delta = \frac{1+\psi_\theta(\varepsilon)}{2}$, and the closed disc $\overline{B(0,\delta)}$, so any harmonic function F defined on $\mathbb{D}$ is continuous in $\overline{B(0,\delta)}$ and harmonic in its interior. If $\varphi_\theta(\varepsilon) < x < 0$, that is, $\varepsilon < \sigma < \theta$ with $x = \psi_\theta(\sigma)$, $v = F \circ \psi_\theta$, the using Harnack's inequality we get

$$
\begin{aligned}
v(\sigma) = F(\psi_\theta(\sigma)) = F(x) &\leqslant \frac{\delta + |x|}{\delta - |x|} F(0) \leqslant \frac{2}{\frac{1+\psi_\theta(\varepsilon)}{2}} F(0) \\
&\leqslant \frac{4}{1+\psi_\theta(\varepsilon)} F(0) = \frac{4}{1+\psi_\theta(\varepsilon)} F(\psi_\theta(\theta)) = \frac{4}{1+\psi_\theta(\varepsilon)} v(\theta).
\end{aligned}
$$

Applying this inequality to any positive harmonic function defined on $\mathbb{C}_+$ as $v_\tau(\sigma) = v(\sigma + i\tau)$ for a fixed $\tau \in \mathbb{R}$ gives $v(\sigma + i\tau) \leqslant C_{\varepsilon,\theta} v(\theta + i\tau)$ if $0 < \sigma \leqslant \theta$. This implies that φ^α and hence φ is uniformly bounded in $\mathbb{C}_\varepsilon$. By Theorem 2.22, it follows that the Dirichlet series representing φ converges uniformly in every half-plane $\mathbb{C}_\varepsilon$. □

Corollary 5.19. *An analytic function $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ generates a composition operator $C_\phi : \mathcal{H}_\infty(\mathbb{C}_+) \to \mathcal{H}_\infty(\mathbb{C}_+)$ if and only if it is of the form $\phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N}_0$ and φ a Dirichlet series which converges uniformly in $\mathbb{C}_\varepsilon$ for all $\varepsilon > 0$.*

Proof. By Theorem 5.16, if ϕ defines a composition operator C_ϕ : $\mathcal{H}_\infty(\mathbb{C}_+) \to \mathcal{H}_\infty(\mathbb{C}_+)$ then $\phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N}_0$ and such that φ can be represented as a converging Dirichlet series in $\mathbb{C}_\sigma$ for some $\sigma > 0$. Applying Corollary 5.14, $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$, so φ is analytic with no zeros in $\mathbb{C}_+$ and $\operatorname{Log}\varphi(s) = \log|\varphi(s)| + i \operatorname{Arg}\varphi(s)$, where log denotes the principal branch of the logarithm and Arg is the principal argument.

Since $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$, $\operatorname{Arg} \varphi(s) \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and therefore is bounded. Now we can apply Theorem 5.18 to get that the Dirichlet series φ converges uniformly in each half-plane $\mathbb{C}_\varepsilon$ for each $\varepsilon > 0$. $\square$

Remark 5.20. The hypothesis of ϕ being analytic can also be removed form the statement of Theorem 5.16 since it can be removed from the statement of Lemma 5.6. The proof of this fact is given in Lemma 5.7.

Before we get into the multiple case, we can see that there is a connection between composition operators of $\mathcal{H}_\infty(\mathbb{C}_+)$ and composition operators of $H_\infty(B_{c_0})$, which is of course given by the isometric isomorphism between these two spaces, the Bohr transform.

Proposition 5.21. *Let ϕ be a holomorphic function which defines a composition operator $C_\phi : \mathcal{H}_\infty(\mathbb{C}_+) \to \mathcal{H}_\infty(\mathbb{C}_+)$. Then C_ϕ induces a composition operator $C_\psi : H_\infty(B_{c_0}) \to H_\infty(B_{c_0})$.*

Proof. Recall the bijective isometry $\mathcal{B} : H_\infty(B_{c_0}) \to \mathcal{H}_\infty(\mathbb{C}_+)$ from Theorem 4.33, with $k = 1$. Clearly $T = \mathcal{B}^{-1} \circ C_\phi \circ \mathcal{B}$ is an operator, $T : H_\infty(B_{c_0}) \to H_\infty(B_{c_0})$, but we will see that it is actually a composition operator. Consider $\mathfrak{p} = \{p_j\}_{j \in \mathbb{N}}$ the sequence of prime numbers and define $\phi_j = C_\phi(\frac{1}{p_j^s}) = \frac{1}{p_j^{\phi(s)}}$. Since $\frac{1}{p_j^s} \in \mathcal{H}_\infty(\mathbb{C}_+)$, $\phi_j \in \mathcal{H}_\infty(\mathbb{C}_+)$ for every $j \in \mathbb{N}$, so $\mathcal{B}^{-1}(\phi_j) \in H_\infty(B_{c_0})$ for every $j \in \mathbb{N}$. Define formally $\Phi = (\phi_1, \ldots, \phi_j, \ldots)$ and $\psi = (\mathcal{B}^{-1}(\phi_1), \ldots, \mathcal{B}^{-1}(\phi_j), \ldots)$, with the notation $\mathcal{B}^{-1}\Phi = \psi$. Now, consider a polynomial $f(z) = \sum_{\alpha \in \Lambda} c_\alpha z^\alpha$ with Λ finite.

Then

$$\begin{aligned} T(f)(z) &= \mathcal{B}^{-1}\left(C_\phi\left(\mathcal{B}\left(\sum_{\alpha\in\Lambda} c_\alpha z^\alpha\right)\right)\right) = \mathcal{B}^{-1}\left(C_\phi\left(\sum_{\alpha\in\Lambda}\frac{c_\alpha}{(\mathfrak{p}^\alpha)^s}\right)\right) \\ &= \mathcal{B}^{-1}\left(\sum_{\alpha\in\Lambda}\frac{c_\alpha}{(\mathfrak{p}^{\phi(s)})^\alpha}\right) = \mathcal{B}^{-1}\left(\sum_{\alpha\in\Lambda} c_\alpha \Phi(s)^\alpha\right) \\ &= \sum_{\alpha\in\Lambda} c_\alpha(\mathcal{B}^{-1}\Phi)(z)^\alpha = \sum_{\alpha\in\Lambda} c_\alpha\psi(z)^\alpha = C_\psi(f)(z) \end{aligned} \tag{5.2}$$

Therefore, T coincides with the composition operator C_ψ on finite polynomials, so we will see that they actually are the same operator. First we need to see that C_ψ is well defined, that is, that ψ is a holomorphic function with $\psi(B_{c_0}) \subset B_{c_0}$. It is clear that ψ is holomorphic by definition, and the fact that it satisfies $\psi(B_{c_0}) \subset B_{c_0}$ is actually a consequence of Montel's Theorem. First, as $\phi(\mathbb{C}_+) \subset \mathbb{C}_+$, $\operatorname{Re}\phi(s) > 0$ for every $s \in \mathbb{C}_+$ and then $|\phi_j(s)| = \frac{1}{p_j^{\operatorname{Re}\phi(s)}} < 1$ for every $s \in \mathbb{C}_+$, so the sequence $\{\phi_j\}_j$ is bounded. Let $F_j = \mathcal{B}^{-1}(\phi_j) \in H_\infty(B_{c_0})$, $||F_j||_\infty = ||\phi_j||_\infty \leqslant 1$, so by Montel's Theorem there exists a subsequence $\{F_{j_l}\}$ which is uniformly convergent on compact sets to a function $F \in H_\infty(B_{c_0})$. Now, using that $F_j(\frac{1}{\mathfrak{p}^s}) = \phi_j(s)$ for every $j \in \mathbb{N}$ and every $s \in \mathbb{C}_+$, and the fact that for every previously fixed $s \in \mathbb{C}_+$ we have that $\lim_{j\to\infty}\phi_j(s) = 0$,

$$F(\frac{1}{\mathfrak{p}^s}) = \lim_{l\to\infty} F_{j_l}(\frac{1}{\mathfrak{p}^s}) = \lim_{l\to\infty}\phi_{j_l}(s) = 0 \quad \text{for every } s \in \mathbb{C}_+.$$

This implies that $F(z) = 0$ for every $z \in B_{c_0}$, as the set $\{\frac{1}{\mathfrak{p}^{i\tau}} : \tau \in \mathbb{R}\}$ is dense in $\mathbb{T}^\infty$, so $\{\frac{1}{\mathfrak{p}^{\sigma+i\tau}} : \tau \in \mathbb{R}\}$ is dense in $r\mathbb{T}^\infty$, where $r = ||\frac{1}{\mathfrak{p}^{\sigma+i\tau}}||_\infty = \frac{1}{2^\sigma}$. Then, as the sequence $\{F_j\}$ has a subsequence which is convergent uniformly to 0 on compact sets, it has to converge pointwise to zero, giving that $\psi(B_{c_0}) \subset B_{c_0}$. Indeed, if $\{F_j(z_0)\}$ were not convergent to 0 for some $z_0 \in B_{c_0}$, one could find a compact subset K containing z_0 and a subsequence $\{F_{j_r}\}$ such that $|F_{j_r}(z)| > \varepsilon$ for every $r \in \mathbb{N}$ and every $z \in K$. Choosing $s \in \mathbb{C}_+$ such that $\frac{1}{\mathfrak{p}^s} \in K$, since $\lim_{r\to\infty} \phi_{j_r}(s) = 0$ then there would exist some $r_0 \in \mathbb{N}$ such that $r \geqslant r_0$ implies $|\phi_{j_r}(s)| < \varepsilon$, but $|\phi_{j_r}(s)| = |F_{j_r}(\frac{1}{\mathfrak{p}^s})| > \varepsilon$ for every $r \in \mathbb{N}$, which is a contradiction.

To see that T and C_ψ are the same operator, it is sufficient to find a topology in which the finite polynomials on B_{c_0} are dense in $H_\infty(B_{c_0})$ and such that makes T and C_ψ continuous, so we can extend (5.2) by continuity. Knowing that $\mathcal{B}(f)(s) = f(\frac{1}{\mathfrak{p}^s})$ for every $f \in H_\infty(B_{c_0})$ and every $s \in \mathbb{C}_+$, we define $G : \mathbb{C}_+ \to B_{c_0}$ as $G(s) = \frac{1}{\mathfrak{p}^s}$ and consider τ_σ the topology of the uniform convergence on the half-planes $\overline{\mathbb{C}_\sigma}$ for $\mathcal{H}_\infty(\mathbb{C}_+)$ and for $H_\infty(B_{c_0})$ we consider τ_{K_σ} the topology of the uniform convergence on the compact subsets of B_{c_0} of the form $K_\sigma = \overline{G(\overline{\mathbb{C}_\sigma})} = \overline{\{\frac{1}{\mathfrak{p}^s} : \operatorname{Re} s \geqslant \sigma\}}$. We should note that these topologies define metrizable spaces since we can take $\sigma = \frac{1}{n}$, $n \in \mathbb{N}$, and we get the same topologies. First, since for every $\sigma > 0$ there exists some $0 < r < 1$ such that $\sup_{\operatorname{Re} s \geqslant \sigma} ||\frac{1}{\mathfrak{p}^s}||_\infty \leqslant r$, then any $f \in H_\infty(B_{c_0})$ has a uniformly convergent Taylor series on K_σ, that is, $f(z) = \sum_{m=0}^\infty P_m(f)(z)$ uniformly for every $z \in K_\sigma$. Moreover, as the set of finite polynomials on B_{c_0} is dense in the space of homogeneous polynomials on B_{c_0} with the topology of $||\cdot||_\infty$ (see the proof of Theorem 2.5 of [3] for more details), then we can extend (5.2) to homogeneous polynomials. Since the topology of $||\cdot||_\infty$ is finer than topology τ_{K_σ}, if we prove that T and C_ψ are continuous with the topology τ_{K_σ}, then we will be able to extend (5.2) to $H_\infty(B_{c_0})$ to get that $T = C_\psi$ as operators of $H_\infty(B_{c_0})$.

To see that T is continuous, we just have to check that C_ϕ is continuous for the topology τ_σ and that $\mathcal{B}$ defines a homeomorphism with the resepctive topologies τ_{K_σ} and τ_σ. To prove that C_ϕ is continuous we have to use, for the first time in the proof, the characterization of the composition operators of $\mathcal{H}_\infty(\mathbb{C}_+)$, Theorem 5.16, so we can apply Corollary 5.15 to ϕ to get that for every $\sigma > 0$ there exists some $\delta(\sigma) > 0$

such that $\phi(\mathbb{C}_\sigma) \subset \mathbb{C}_{\delta(\sigma)}$ (this differs form the case of $\mathcal{H}^p$, where the density of the polynomials in the space gives the result directly, without the need to call the characterization of the form of the symbol for the composition operators). Now, let $\{D_n\} \subset \mathcal{H}_\infty(\mathbb{C}_+)$ be a sequence converging to $D \in \mathcal{H}_\infty(\mathbb{C}_+)$ with τ_σ. As

$$||C_\phi(D_n) - C_\phi(D)||_{\overline{\mathbb{C}_\sigma}} = ||D_n \circ \phi - D \circ \phi||_{\overline{\mathbb{C}_\sigma}} \leqslant ||D_n - D||_{\overline{\mathbb{C}_{\delta(\sigma)}}},$$

C_ϕ is continuous. Now, to see that $\mathcal{B}$ is a homeomorphism with the respective topologies, suppose that $\{f_n\} \subset H_\infty(B_{c_0})$ is a sequence converging to $f \in H_\infty(B_{c_0})$ with τ_{K_σ}. Then, using the continuity of f_n and f,

$$\begin{aligned} ||\mathcal{B}(f_n) - \mathcal{B}(f)||_{\overline{\mathbb{C}_\sigma}} &= \sup_{\operatorname{Re} s \geqslant \sigma} |\mathcal{B}(f_n)(s) - \mathcal{B}(f)(s)| \\ &= \sup_{\operatorname{Re} s \geqslant \sigma} |f_n(\frac{1}{\mathfrak{p}^s}) - f(\frac{1}{\mathfrak{p}^s})| = \sup_{z \in K_\sigma} |f_n(z) - f(z)| = ||f_n - f||_{K_\sigma}, \end{aligned}$$

so clearly $\mathcal{B} : (H_\infty(B_{c_0}), \tau_{K_\sigma}) \to (\mathcal{H}_\infty(\mathbb{C}_+), \tau_\sigma)$ is a homeomorphism between these topological spaces, giving that T is continuous with the topology τ_{K_σ}.

Now, to prove that C_ψ is continuous with the topology τ_{K_σ}, recall that $\psi = (\mathcal{B}^{-1}(\phi_1), \ldots, \mathcal{B}^{-1}(\phi_j), \ldots)$. Hence, if $\sigma > 0$, take $z \in G(\overline{\mathbb{C}_\sigma})$, that is, $z = \frac{1}{\mathfrak{p}^s}$ for some $s \in \mathbb{C}_+$, and then $\mathcal{B}^{-1}(\phi_j)(z) = \phi_j(s) = \frac{1}{p_j^{\phi(s)}}$, so $\mathcal{B}^{-1}(\phi_j)(G(\overline{\mathbb{C}_\sigma})) \subset \{\frac{1}{p_j^{\phi(s)}} : \operatorname{Re} s \geqslant \sigma\} \subset \{\frac{1}{p_j^s} : \operatorname{Re} s \geqslant \delta(\sigma)\}$ and therefore $\psi(K_\sigma) \subset \overline{\{\frac{1}{\mathfrak{p}^s} : \operatorname{Re} s \geqslant \delta(\sigma)\}} = K_{\delta(\sigma)}$. Then, if $\{f_n\} \subset H_\infty(B_{c_0})$ is a sequence converging to $f \in H_\infty(B_{c_0})$ with τ_{K_σ},

$$||C_\psi(f_n) - C_\psi(f)||_{K_\sigma} = ||f_n \circ \psi - f \circ \psi||_{K_\sigma} =\leqslant ||f_n - f||_{K_{\delta(\sigma)}},$$

which gives the continuity of C_ψ with the topology τ_{K_σ}.

Finally, as every function in $H_\infty(B_{c_0})$ is the limit of a series of finite polynomials with the topology τ_{K_σ}, and by (5.2) T and C_ψ coincide on finite polynomials, then $T = C_\psi$. $\square$

This relation we have just showed only works one way. In the one-dimensional case, if you take $F : B_{c_0} \to B_{c_0}$ defined by $F(z) = (z_1, 0, \dots)$, then it defines a composition operator C_F on $H_\infty(B_{c_0})$. Suppose there exists some $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ such that $C_F = \mathcal{B}^{-1} \circ C_\phi \circ \mathcal{B}$, then $\mathcal{B}^{-1}(\frac{1}{3^{\phi(s)}}) = 0$ and hence $\frac{1}{3^{\phi(s)}} = 0$, which is a contradiction.

5.2 Composition operators on $\mathcal{H}_\infty(\mathbb{C}_+^2)$

We dedicate this section to obtaining a characterization for the composition operators of $\mathcal{H}_\infty(\mathbb{C}_+^2)$, which we state below.

Theorem 5.22. *Consider a function $\phi : \mathbb{C}_+^2 \to \mathbb{C}_+^2$, $\phi = (\phi_1, \phi_2)$. The operator $C_\phi : \mathcal{H}_\infty(\mathbb{C}_+^2) \to \mathcal{H}_\infty(\mathbb{C}_+^2)$, $C_\phi(f) = f \circ \phi$, is a composition operator if and only if, for $j = 1, 2$ and $(s,t) \in \mathbb{C}_+^2$ we have*

$$\phi_j(s,t) = c_0^{(j)} s + d_0^{(j)} t + \varphi_j(s,t)\,, \tag{5.3}$$

where

$$c_0^{(j)}, d_0^{(j)} \in \mathbb{N}_0$$

and

$$\varphi_j : \mathbb{C}_+^2 \to \mathbb{C}$$
$$\varphi_j(s,t) = \sum_{m,n=1}^{\infty} \frac{d_{m,n}^{(j)}}{m^s n^t}$$

is a double Dirichlet series that converges regularly and uniformly in $\mathbb{C}_\varepsilon^2$ for every $\varepsilon > 0$.

The proof of this theorem will be divided into two parts. The proof that symbols satisfying (5.3) leads to composition operators on $\mathcal{H}_\infty(\mathbb{C}^2_+)$ will follow the arguments of the one-variable case, with extra difficulties since we are working with double Dirichlet series and we have to be very careful with the convergence of the series. The proof of the converse part will require really new arguments, which were not necessary in the one-dimensional case (see in particular the proof of the forthcoming Theorem 5.33).

5.2.1 The sufficient condition

We can now start our work towards the characterization of composition operators on $\mathcal{H}_\infty(\mathbb{C}^2_+)$ stated in Theorem 5.22. We do this in two steps. First we see that the symbols ϕ as in the statement of the theorem indeed define composition operators (see Theorem 5.29). Once we have this we show that these are in fact the only symbols defining a composition operator on $\mathcal{H}_\infty(\mathbb{C}^2_+)$ (this follows from Theorem 5.33). We begin by showing that a symbol as in (5.3) defines a composition operator on the space of double Dirichlet polynomials (finite series).

Lemma 5.23. *Let $\phi : \mathbb{C}^2_+ \to \mathbb{C}^2_+$ be an analytic function and suppose there exists some $\sigma > 0$ such that $\phi_j(s,t) = c_j s + d_j t + \varphi_j(s,t)$ for $j = 1,2$ and $(s,t) \in \mathbb{C}^2_\sigma$, where $\varphi_j(s,t) = \sum_{m,n=1}^{\infty} \frac{b^{(j)}_{m,n}}{m^s n^t}$ converges absolutely in $\mathbb{C}^2_\sigma$ and $c_j, d_j \in \mathbb{N}_0$, $j = 1,2$. Then, if D is a double Dirichlet polynomial, $D \circ \phi$ is a double Dirichlet series in $\mathcal{H}^\infty(\mathbb{C}^2_+)$.*

Proof. We are going to see that $k^{-\varphi_1(s,t)}$ can be written as a double Dirichlet series in $\mathbb{C}^2_\sigma$. Using the expansion of the exponential function, if $(s,t) \in \mathbb{C}^2_\sigma$,

$$\begin{aligned} k^{-\varphi_1(s,t)} &= e^{-\log k \sum_{m,n=1}^{\infty} \frac{b^{(1)}_{m,n}}{m^s n^t}} \\ &= \prod_{m,n=1}^{\infty} e^{-\log k \frac{b^{(1)}_{m,n}}{m^s n^t}} \\ &= k^{-b^{(1)}_{1,1}} \prod_{\substack{m,n=1,\\(m,n)\neq(1,1)}}^{\infty} \left(1+\sum_{r=1}^{\infty} \frac{(-\log k\, b^{(1)}_{m,n})^r}{r!} \frac{1}{(m^r)^s} \frac{1}{(n^r)^t}\right). \end{aligned}$$

Let us see how can we rearrange this expression for $k^{-\varphi_1(s,t)}$ into a double Dirichlet series. For each $(M,N) \in \mathbb{N}^2$, because of the absolute convergence, we may define $A_{k,M,N}$ in the following way:

- If $(M,N) = (1,1)$, then $A_{k,1,1} = 1$.
- If $M \neq 1$ and $N = 1$, consider all possible factorizations of M as $M = m_1^{r_1} \cdots m_d^{r_d}$ where $m_1, \ldots, m_d \in \mathbb{N} \setminus \{1\}$ are all different, $r_1, \ldots, r_d \in \mathbb{N}$ (there is at least one such factorization by setting $m_1 = M$ and $r_1 = 1$). Now define

$$A_{k,M,1} = \sum_{m_1^{r_1} \cdots m_d^{r_d} = M} \left[\prod_{j=1}^{d} \frac{(-\log k\; b^{(1)}_{m_j,1})^{r_j}}{r_j!}\right].$$

- If $M = 1$, $N \neq 1$, proceeding analogously to the previous case, define

$$A_{k,1,N} = \sum_{n_1^{r_1} \cdots n_d^{r_d} = N} \left[\prod_{j=1}^{d} \frac{(-\log k\; b^{(1)}_{1,n_j})^{r_j}}{r_j!}\right].$$

- If both $M, N \neq 1$, combining the two previous cases, we define

$$A_{k,M,N} = \sum_{\substack{m_1^{r_1} \cdots m_d^{r_d} = M\\ n_1^{r_1} \cdots n_d^{r_d} = N}} \left[\prod_{j=1}^{d} \frac{(-\log k\; b^{(1)}_{m_j,n_j})^{r_j}}{r_j!}\right].$$

Then for any $(s,t) \in \mathbb{C}^2_\sigma$,

$$k^{-\varphi_1(s,t)} = k^{-b^{(1)}_{1,1}} \sum_{M,N=1}^{\infty} \frac{A_{k,M,N}}{M^s N^t}.$$

With the same idea one gets

$$l^{-\varphi_2(s,t)} = l^{-b^{(2)}_{1,1}} \sum_{M,N=1}^{\infty} \frac{B_{l,M,N}}{M^s N^t}.$$

As these two double Dirichlet series are absolutely convergent in $\mathbb{C}^2_\sigma$, they can be multiplied to obtain

$$k^{-\varphi_1(s,t)} l^{-\varphi_2(s,t)} = k^{-b^{(1)}_{1,1}} l^{-b^{(2)}_{1,1}} \sum_{M,N=1}^{\infty} \frac{C_{k,l,M,N}}{M^s N^t}.$$

Finally, let $D(s,t) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{k,l}}{k^s l^t}$. Then, for $(s,t) \in \mathbb{C}^2_\sigma$,

$$\begin{aligned}(D \circ \phi)(s,t) &= \sum_{k=1}^{K} \sum_{l=1}^{L} a_{k,l}\, k^{-c_1 s - d_1 t - b^{(1)}_{1,1}}\, l^{-c_2 s - d_2 t - b^{(2)}_{1,1}} \sum_{M,N=1}^{\infty} \frac{C_{k,l,M,N}}{M^s N^t} \\ &= \sum_{M,N=1}^{\infty} \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{k^{-b^{(1)}_{1,1}} l^{-b^{(2)}_{1,1}} C_{k,l,M,N}}{(k^{c_1} l^{c_2} M)^s (k^{d_1} l^{d_2} N)^t}\end{aligned}$$

which can be rearranged into a double Dirichlet series which still is absolutely convergent on $\mathbb{C}^2_\sigma$. Moreover, as $\|D \circ \phi\|_\infty \leqslant \|D\|_\infty < \infty$, Corollary 4.12 guarantees that $D \circ \phi \in \mathcal{H}^\infty(\mathbb{C}^2_+)$. □ □

Following the same scheme as in the one-dimensional case, some results concerning the range of the symbols of the composition operators are needed.

Remark 5.24. Suppose ϕ is an analytic function as in (5.3), where the Dirichlet series φ_j converge regularly on $\mathbb{C}^2_\sigma$ for some $\sigma > 0$. Then the function $\tilde{\varphi}_j(s,t) = \phi(s,t) - c^{(j)}_0 s - d^{(j)}_0 t$, defined on $\mathbb{C}^2_+$, is clearly analytic and coincides with φ_j on $\mathbb{C}^2_\sigma$. In other words, $\tilde{\varphi}_j$ is an analytic extension of φ_j to $\mathbb{C}^2_+$. For the sake of clarity in the notation we will write φ_j also for the extension, identifying the Dirichlet series with the extension. This is, for example, how the statement of Lemma 5.25 should be understood. On the other hand, if we suppose that each φ_j converges regularly on $\mathbb{C}^2_+$, then they define analytic functions. Therefore if ϕ is as in (5.3), then by Hartogs' theorem, it is analytic. This is the case, for example, in Lemma 5.28 and Theorem 5.29.

Lemma 5.25. *Suppose $\phi : \mathbb{C}^2_+ \to \mathbb{C}^2_+$ is an analytic function such that $\phi_j(s,t) = c_j s + d_j t + \varphi_j(s,t)$ for $j = 1, 2$, where $\varphi_j(s,t) = \sum_{m,n=1}^{\infty} \frac{b^{(j)}_{m,n}}{m^s n^t}$ converges in $\mathbb{C}^2_+$ and $c_j, d_j \in \mathbb{N}_0$, $j = 1, 2$. Then, $\operatorname{Re}\varphi_j(s,t) \geqslant 0$ for all $(s,t) \in \mathbb{C}^2_+$, $j = 1, 2$.*

Proof. Fix $s_0 \in \mathbb{C}_+$ and consider $\phi_j(s_0,t) = c_j s_0 + d_j t + \varphi_j(s_0,t) = d_j t + (c_j s_0 + \varphi_j(s_0,t))$. Using the first part of [17, Proposition 4.2], $\operatorname{Re}(c_j s_0 + \varphi_j(s_0,t)) \geqslant 0$ for all $t \in \mathbb{C}_+$, but also for all $s_0 \in \mathbb{C}_+$. Fixing now $t_0 \in \mathbb{C}_+$ and using again [17, Proposition 4.2], $\operatorname{Re}\varphi_j(s_0,t_0) \geqslant 0$ for all $(s_0,t_0) \in \mathbb{C}_+$. □ □

Remark 5.26. If $f : \mathbb{C}^2_+ \to \overline{\mathbb{C}_+}$ is a holomorphic function such that $\operatorname{Re} f(s_0,t_0) = 0$ for some $(s_0,t_0) \in \mathbb{C}^2_+$, then in fact $\operatorname{Re} f(s,t) = 0$ for all $(s,t) \in \mathbb{C}^2_+$. Indeed, if we define $f_{t_0} : \mathbb{C}_+ \to \overline{\mathbb{C}_+}$ as $f_{t_0}(s) = f(s,t_0)$ and suppose that f_{t_0} is not constant, by the open mapping property $f_{t_0}(\mathbb{C}_+)$ is an open set, which contradicts the fact that $f(s_0,t_0) = i\tau_0$ for some $\tau_0 \in \mathbb{R}$. Therefore, it is constant and $f_{t_0}(s) = i\tau_0$ for all $s \in \mathbb{C}_+$. Proceeding in the same way, defining for each $s \in \mathbb{C}_+$ a function $f_s : \mathbb{C}_+ \to \overline{\mathbb{C}_+}$ by $f_s(t) = f(s,t)$ we conclude that $f(s,t) = f_s(t) = i\tau_0$ for all $t \in \mathbb{C}_+$. This gives $\operatorname{Re} f \equiv 0$ in $\mathbb{C}^2_+$.
This allows to strengthen Lemma 5.25 to say that either $\operatorname{Re}\varphi_j(s,t) > 0$ for all $(s,t) \in \mathbb{C}^2_+$, or $\operatorname{Re}\varphi_j(s,t)$ is constant and equal to zero.

We aim now at an analogue of Corollary 5.15 for double Dirichlet series. A fundamental tool is the extension of Theorem 5.12, fow whcihc we give two proofs, as in the one-dimensional case.

Lemma 5.27. *Let* $D(s,t) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t}$ *be a non-constant double Dirichlet series in* $\mathcal{H}^\infty(\mathbb{C}_+^2)$*, and* $0 < \sigma_1 < \eta_1$*,* $0 < \sigma_2 < \eta_2$*. Then*

$$\sup_{\substack{\operatorname{Re} s > \sigma_1 \\ \operatorname{Re} t > \sigma_2}} \left| \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} \right| > \sup_{\substack{\operatorname{Re} s > \eta_1 \\ \operatorname{Re} t > \eta_2}} \left| \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} \right| .$$

Proof. Define $D_\sigma(s,t) = D(s+\sigma_1, t+\sigma_2) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^{\sigma_1} n^{\sigma_2}} \frac{1}{m^s n^t}$ and $D_\eta(s,t) = D(s+\eta_1, t+\eta_2) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^{\eta_1} n^{\eta_2}} \frac{1}{m^s n^t}$. Clearly D_σ and D_η belong to $\mathcal{H}_\infty(\mathbb{C}_+^2)$, and

$$\sup_{\substack{\operatorname{Re} s > \sigma_1 \\ \operatorname{Re} t > \sigma_2}} \left| \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} \right| = \sup_{\substack{\operatorname{Re} s > 0 \\ \operatorname{Re} t > 0}} |D_\sigma(s,t)| = ||D_\sigma||_\infty,$$

$$\sup_{\substack{\operatorname{Re} s > \eta_1 \\ \operatorname{Re} t > \eta_2}} \left| \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{m^s n^t} \right| = \sup_{\substack{\operatorname{Re} s > 0 \\ \operatorname{Re} t > 0}} |D_\eta(s,t)| = ||D_\eta||_\infty.$$

Moreover, if $f_\sigma, f_\eta : B_{c_0 \times c_0} \to \mathbb{C}$ are the unique bounded holomorphic functions on the open unit ball of $c_0 \times c_0$ such that their monomial coefficients are $c_{(\alpha,\beta)}(f_\sigma) = \frac{a_{\mathfrak{p}^\alpha, \mathfrak{p}^\beta}}{\mathfrak{p}^{\alpha\sigma_1} \mathfrak{p}^{\beta\sigma_2}}$ and $c_{(\alpha,\beta)}(f_\eta) = \frac{a_{\mathfrak{p}^\alpha, \mathfrak{p}^\beta}}{\mathfrak{p}^{\alpha\eta_1} \mathfrak{p}^{\alpha\eta_2}}$ respectively, we have by Theorem 4.11 that

$$||D_\sigma||_\infty = ||f_\sigma||_{B_{c_0 \times c_0}}, \quad \text{and} \quad ||D_\eta||_\infty = ||f_\eta||_{B_{c_0 \times c_0}}.$$

If f is the unique bounded holomorphic function such that $c_{(\alpha,\beta)}(f) = a_{\mathfrak{p}^\alpha,\mathfrak{p}^\beta}$, then

$$\begin{aligned} f_\eta(z_1,z_2) &= f(\frac{1}{\mathfrak{p}^{\eta_1}}z_1,(\frac{1}{\mathfrak{p}^{\eta_2}}z_2) = f_\sigma(\frac{\mathfrak{p}^{\sigma_1}}{\mathfrak{p}^{\eta_1}}z_1,\frac{\mathfrak{p}^{\sigma_1}}{\mathfrak{p}^{\eta_2}}z_2) \\ &= f_\sigma(\frac{z_1^{(1)}}{2^{\eta-\sigma}},\frac{z_1^{(2)}}{3^{\eta-\sigma}},\ldots,\frac{z_1^{(k)}}{p_k^{\eta-\sigma}},\ldots;\frac{z_2^{(1)}}{2^{\eta-\sigma}},\frac{z_2^{(2)}}{3^{\eta-\sigma}},\ldots,\frac{z_2^{(k)}}{p_k^{\eta-\sigma}},\ldots), \end{aligned}$$

for every $(z_1,z_2) \in B_{c_{00}\times c_{00}}$ with $z_j = (z_j^{(1)},z_j^{(2)},\ldots)$, $j=1,2$, and, by density, for every $(z_1,z_2) \in B_{c_0}$ with $z_j = (z_j^{(1)},z_j^{(2)},\ldots)$, $j=1,2$. Let $r = \min(\frac{1}{2^{\eta_1-\sigma_1}},\frac{1}{2^{\eta_2-\sigma_2}})$. Hence

$$||f_\eta||_{B_{c_0\times c_0}} \leqslant ||f_\sigma||_{rB_{c_0\times c_0}}.$$

If we denote by $\tilde{f}_\eta$ and $\tilde{f}_\sigma$ the Aron-Berner extensions of f_η and f_σ respectively, we have by the Davie and Gamelin result (see [10]) that

$$||\tilde{f}_\eta||_{B_{l_\infty}} = ||f_\eta||_{B_{c_0\times c_0}} \leqslant ||f_\sigma||_{rB_{c_0\times c_0}} = ||\tilde{f}_\sigma||_{rB_{l_\infty\times l_\infty}}.$$

As $\tilde{f}_\sigma$ is $\omega(l_\infty,l_1)-$continuous on $r\bar{B}_{l_\infty\times l_\infty}$, which is $\omega(l_\infty,l_1)-$compact, $|\tilde{f}_\sigma|$ has a maximum in that compact set, i. e. there exists $(z_1^*,z_2^*) \in B_{l_\infty\times l_\infty}$ with $||(z_1^*,z_2^*)||_\infty \leqslant r$ and $||\tilde{f}_\sigma||_{rB_{l_\infty\times l_\infty}} = |\tilde{f}_\sigma(z_1^*,z_2^*)|$. Now we have two possibilities. Either the holomorphic mapping $\xi : \frac{1}{r}\mathbb{D} \to \mathbb{C}$ defined by $\xi(\lambda) = f_\sigma(\lambda z_1^*,\lambda z_2^*)$ is constant or not. If ξ is constant this implies that $|\tilde{f}_\sigma|$ has a maximum on $rB_{l_\infty\times l_\infty}$ at 0, which is an interior point. Hence, by the Maximum Modulus Theorem applied to the restriction of that function to any complex line (each of them crossing zero), $\tilde{f}_\sigma$ is constant on the whole $B_{l_\infty\times\infty}$, thus f_σ is constant on $B_{c_0\times c_0}$. This implies that D_σ is constant, which clearly implies that D is a constant function. This is a contradiction with our hypothesis. If ξ is not constant, by the Maximum Modulus Theorem, there exists λ^*,

$1 < |\lambda^*| < r$, such that $|\tilde{f}_\sigma(\lambda^* z_1^*, \lambda^* z_2^*)| > |\tilde{f}_\sigma(z_1^*, z_2^*)|$. In that case

$$||D_\sigma||_\infty = ||\tilde{f}_\sigma||_{rB_{l_\infty \times l_\infty}} \geqslant |\tilde{f}_\sigma(\lambda^* z_1^*, \lambda^* z_2^*)| \\ > |\tilde{f}_\sigma(z_1^*, z_2^*)| = ||\tilde{f}_\sigma||_{rB_{l_\infty \times l_\infty}} \geqslant ||\tilde{f}_\eta||_{B_{l_\infty}} = ||D_\eta||_\infty,$$

and the conclusion follows. □

Proof. We first recall that for all Dirichlet series h belonging to $\mathcal{H}_\infty(\mathbb{C}_+)$ and for all $\sigma > 0$, by Corollary 5.11 $\sup_{\mathrm{Re}(s)=\sigma} |h(s)| = \sup_{\mathrm{Re}(s)>\sigma} |h(s)|$. This yields

$$\begin{aligned} \sup_{\substack{\mathrm{Re}(s)>\alpha_1 \\ \mathrm{Re}(t)>\alpha_2}} |D(s,t)| &= \sup_{\mathrm{Re}(t)>\alpha_2} \sup_{\mathrm{Re}(s)>\alpha_1} |D(s,t)| \\ &= \sup_{\mathrm{Re}(t)>\alpha_2} \sup_{\mathrm{Re}(s)=\alpha_1} |D(s,t)| \\ &= \sup_{\mathrm{Re}(s)=\alpha_1} \sup_{\mathrm{Re}(t)>\alpha_2} |D(s,t)| \\ &= \sup_{\substack{\mathrm{Re}(s)=\alpha_1 \\ \mathrm{Re}(t)=\alpha_2}} |D(s,t)|. \end{aligned}$$

Observe also that, since D is not constant,

$$|a_{1,1}| < \sup_{\substack{\mathrm{Re}(s)>\sigma_1 \\ \mathrm{Re}(t)>\sigma_2}} |D(s,t)|.$$

Again, this follows easily from the corresponding result in the one-dimensional case. Indeed, let $f(t) = \sum_{n=1}^{+\infty} a_{m,1} n^{-t}$ and $D_t(s) = D(s,t)$, so that $f(t)$ is the constant term of the Dirichlet series D_t. If f is constant, then there exists t' with $\mathrm{Re}(t') > \sigma_2$ such that $D_{t'}$ is not constant (otherwise D itself would be constant). We then write

$$|a_{1,1}| = f(2\sigma_2) < \sup_{\mathrm{Re}(s)>\sigma_1} |D_{t'}(s)| \leqslant \sup_{\substack{\mathrm{Re}(s)>\sigma_1 \\ \mathrm{Re}(t)>\sigma_2}} |D(s,t)|.$$

On the contrary, if f is not constant, we write

$$|a_{1,1}| < \sup_{\mathrm{Re}(t)>\sigma_2} |f(t)| \leqslant \sup_{\substack{\mathrm{Re}(s)>\sigma_1 \\ \mathrm{Re}(t)>\sigma_2}} |D(s,t)|,$$

where the first inequality is for example a consequence of Proposition 2.25, the argument being the same one used in the proof of Theorem 5.12. Let $\theta_1, \theta_2 \in (0,1)$ be such that $\eta_1 = (1-\theta_1)\sigma_1 + \theta_1\gamma$, $\eta_2 = (1-\theta_2)\sigma_2 + \theta_2\gamma$. Two successive applications of Hadamard's three lines theorem lead to

$$\begin{aligned} \sup_{\substack{\mathrm{Re}(s)=\eta_1 \\ \mathrm{Re}(t)=\eta_2}} |D(s,t)| &\leqslant \sup_{\substack{\mathrm{Re}(s)=\sigma_1 \\ \mathrm{Re}(t)=\sigma_2}} |D(s,t)|^{(1-\theta_1)(1-\theta_2)} \times \sup_{\substack{\mathrm{Re}(s)=\sigma_1 \\ \mathrm{Re}(t)=\gamma}} |D(s,t)|^{(1-\theta_1)\theta_2} \\ &\quad \times \sup_{\substack{\mathrm{Re}(s)=\gamma \\ \mathrm{Re}(t)=\sigma_2}} |D(s,t)|^{\theta_1(1-\theta_2)} \times \sup_{\substack{\mathrm{Re}(s)=\gamma \\ \mathrm{Re}(t)=\gamma}} |D(s,t)|^{\theta_1\theta_2} \\ &< \sup_{\substack{\mathrm{Re}(s)=\sigma_1 \\ \mathrm{Re}(s)=\sigma_2}} |D(s,t)|. \end{aligned}$$

□

We use now this result to obtain the extension of Corollary 5.15.

Lemma 5.28. *Suppose $\phi : \mathbb{C}_+^2 \to \mathbb{C}_+^2$ is a function such that $\phi_j(s,t) = c_j s + d_j t + \varphi_j(s,t)$ for $j = 1,2$, where $\varphi_j(s,t) = \sum_{m,n=1}^{\infty} \frac{b_{m,n}^{(j)}}{m^s n^t}$ converges regularly in $\mathbb{C}_+^2$ and $c_j, d_j \in \mathbb{N}_0$ for $j = 1,2$. Then, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\phi_j(\mathbb{C}_\varepsilon^2) \subset \mathbb{C}_\delta$ for all $j = 1,2$.*

Proof. Suppose $c_1 \neq 0$, that is $c_1 \in \mathbb{N}$. By Lemma 5.25, $\mathrm{Re}\,\phi_1(s,t) = c_1 \mathrm{Re}\, s + d_1 \mathrm{Re}\, t + \mathrm{Re}\,\varphi_1(s,t) > \varepsilon$ for $(s,t) \in \mathbb{C}_\varepsilon^2$, so for all $\varepsilon > 0$, $\phi_1(\mathbb{C}_\varepsilon^2) \subset \mathbb{C}_\varepsilon$. The same argument applies in the case $d_1 \neq 0$. Now, if $c_1 = 0 = d_1$, then $\phi_1(s,t) = \varphi_1(s,t)$ for all $(s,t) \in \mathbb{C}_+^2$, so $\varphi_1 : \mathbb{C}_+^2 \to \mathbb{C}_+^2$ and $\mathrm{Re}\,\varphi_1(s,t) > 0$ for all $(s,t) \in \mathbb{C}_+^2$. Now, if φ_1 is constant, then that constant has positive real part and the lemma is trivially satisfied. Otherwise we can apply Lemma 5.27 to $D(s,t) = 2^{-\varphi_1(s,t)}$, which by

Lemma 5.23 is a non-constant double Dirichlet series. Therefore, given $\varepsilon > 0$,

$$\sup_{(s,t)\in\mathbb{C}^2_\varepsilon} |2^{-\varphi_1(s,t)}| = \sup_{(s,t)\in\mathbb{C}^2_\varepsilon} 2^{-\operatorname{Re}\varphi_1(s,t)}$$
$$< \sup_{(s,t)\in\mathbb{C}^2_+} 2^{-\operatorname{Re}\varphi_1(s,t)} = \sup_{(s,t)\in\mathbb{C}^2_+} |2^{-\varphi_1(s,t)}|,$$

which implies that there exists some $\delta > 0$ such that

$$\inf_{(s,t)\in\mathbb{C}^2_\varepsilon} \operatorname{Re}\varphi_1(s,t) > \delta > \inf_{(s,t)\in\mathbb{C}^2_+} \operatorname{Re}\varphi_1(s,t) \geqslant 0,$$

that is, $\varphi_1(\mathbb{C}^2_\varepsilon) \subset \mathbb{C}_\delta$. □

Now we are ready to prove that the symbols we are dealing with actually define composition operators, which is done in the following theorem.

Theorem 5.29. *Let $\phi = (\phi_1, \phi_2) : \mathbb{C}^2_+ \to \mathbb{C}^2_+$ be a function such that $\phi_j(s,t) = c_j s + d_j t + \varphi_j(s,t)$ for $j = 1,2$ and $(s,t) \in \mathbb{C}^2_+$, where $\varphi_j(s,t) = \sum_{m,n=1}^\infty \frac{b^{(j)}_{m,n}}{m^s n^t}$ converges regularly in $\mathbb{C}^2_+$ and $c_j, d_j \in \mathbb{N}_0$ for $j = 1,2$. Then ϕ generates a composition operator $C_\phi : \mathcal{H}^\infty(\mathbb{C}^2_+) \to \mathcal{H}^\infty(\mathbb{C}^2_+)$.*

Proof. Take some $D = \sum_{k,l=1}^\infty \frac{a_{k,l}}{k^s l^t} \in \mathcal{H}^\infty(\mathbb{C}^2_+)$ and let us see that $D \circ \phi \in \mathcal{H}_\infty(\mathbb{C}^2_+)$. For each $(m,n) \in \mathbb{N}^2$ we denote by $D_{m,n}(s,t) = \sum_{k,l=1}^m \sum_{l=1}^n \frac{a_{k,l}}{k^s l^t}$ the partial sum of D and using Lemma 5.23 (note that ϕ is analytic and by Corollary 3.7 each φ_j converges absolutely on $\mathbb{C}^2_1$) we get that $D_{m,n} \circ \phi$ is a double series in $\mathcal{H}_\infty(\mathbb{C}^2_+)$, which we denote by $\sum_{k,l=1}^\infty \frac{c^{(m,n)}_{k,l}}{k^s l^t}$.
Now, by Lemma 5.28, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(\mathbb{C}^2_\varepsilon) \subset \mathbb{C}^2_\delta$. As a consequence of Corollary 4.12, the partial sums $D_{m,n}$ are uniformly convergent on $\mathbb{C}^2_\delta$ to D. Therefore the sequence $\{D_{m,n} \circ \phi\}_{m,n}$ is uniformly bounded on $\mathbb{C}^2_\varepsilon$, say by C. This implies that the horizontal translates

defined by $\sum_{k,l=1}^{\infty} \frac{c_{k,l}^{(m,n)}}{k^{s+\varepsilon} l^{t+\varepsilon}}$ are in $\mathcal{H}_\infty(\mathbb{C}_+^2)$ for every $m, n \in \mathbb{N}$. Applying now Proposition 4.5, which controls the coefficients of a series in $\mathcal{H}_\infty(\mathbb{C}_+^2)$ by its norm, we get that for every $k, l, m, n \in \mathbb{N}$,

$$\frac{|c_{k,l}^{(m,n)}|}{k^\varepsilon l^\varepsilon} \leqslant \sup_{(s,t)\in\mathbb{C}_\varepsilon^2} |D_{m,n}(\phi(s,t))| \leqslant \sup_{(s,t)\in\mathbb{C}_\delta^2} |D_{m,n}(s,t)| \leqslant C.$$

Therefore for fixed k and l we have

$$|c_{k,l}^{(m_1,n_1)} - c_{k,l}^{(m_2,n_2)}| \leqslant k^\varepsilon l^\varepsilon \sup_{(s,t)\in\mathbb{C}_\delta^2} |D_{m_1,n_1}(s,t) - D_{m_2,n_2}(s,t)|,$$

so the double sequence $\{c_{k,l}^{(m,n)}\}_{m,n}$ converges to some $c_{k,l} \in \mathbb{C}$ satisfying $|c_{k,l}| \leqslant C k^\sigma l^\sigma$ for all $\sigma > \varepsilon$. Define now $F(s,t) = \sum_{k,l=1}^{\infty} \frac{c_{k,l}}{k^s l^t}$. Since $|c_{k,l}| \leqslant C(kl)^{2\varepsilon}$, $F(s,t)$ converges absolutely in $\mathbb{C}_{1+\varepsilon}^2$, and there the double sequence $\{D_{m,n} \circ \phi\}_{m,n}$ clearly converges absolutely to F. It is enough to note that $\|D \circ \phi\|_\infty \leqslant \|D\|_\infty < \infty$ to apply Corollary 4.12 and get that $D \circ \phi$ actually coincides with F and that it is in $\mathcal{H}^\infty(\mathbb{C}_+^2)$.
□

5.2.2 The necessary condition

Theorem 5.29 gives the sufficient condition for the characterization of the composition operators of $\mathcal{H}_\infty(\mathbb{C}_+^2)$ in Theorem 5.22. To prove the necessity we use the vector-valued perspective introduced in Remark 4.2 to deal with double Dirichlet series (formalized in Lemma 5.30).

Lemma 5.30. *Let $\phi = (\phi_1, \phi_2) : \mathbb{C}_+^2 \to \mathbb{C}_+^2$ be inducing a composition operator $C_\phi : \mathcal{H}_\infty(\mathbb{C}_+^2) \to \mathcal{H}_\infty(\mathbb{C}_+^2)$. For each fixed $t \in \mathbb{C}_+$ and $j = 1, 2$, consider $\phi_{j,t} : \mathbb{C}_+ \to \mathbb{C}_+$ given by $\phi_{j,t}(s) = \phi_j(s,t)$. Then $\phi_{j,t}$ defines a composition operator of $\mathcal{H}_\infty(\mathbb{C}_+)$.*

Proof. We just deal with the case $j=1$, the other one being analogous. Take $D(s)=\sum_{m=1}^\infty \frac{a_m}{m^s}\in\mathcal{H}_\infty(\mathbb{C}_+)$, and define $b_{m,1}=a_m$, $b_{m,n}=0$ for $n\geqslant 2$, and $\tilde{D}(s,t)=\sum_{m,n=1}^\infty \frac{b_{m,n}}{m^s n^t}$. Clearly $\tilde{D}(s,t)=D(s)$ for every $(s,t)\in\mathbb{C}_+^2$, and $\tilde{D}\in\mathcal{H}_\infty(\mathbb{C}_+^2)$, so $\tilde{D}\circ\phi\in\mathcal{H}_\infty(\mathbb{C}_+^2)$. Now,

$$(\tilde{D}\circ\phi)(s,t)=\sum_{m,n=1}^\infty \frac{b_{m,n}}{m^{\phi_1(s,t)}n^{\phi_2(s,t)}}=\sum_{m=1}^\infty \frac{a_m}{m^{\phi_1(s,t)}}.$$

Then, for a fixed $t\in\mathbb{C}_+$,

$$(D\circ\phi_{1,t})(s)=\sum_{m=1}^\infty \frac{a_m}{m^{\phi_{1,t}(s)}}=\sum_{m=1}^\infty \frac{a_m}{m^{\phi_1(s,t)}}=(\tilde{D}\circ\phi)(s,t),$$

so $D\circ\phi\in\mathcal{H}_\infty(\mathbb{C}_+)$. $\square$ $\square$

We still need a further lemma before we give the main step towards the necessity in Theorem 5.22.

Lemma 5.31. *Consider $\varphi_2(s)=\sum_n \frac{a_n}{n^s}$ and $\varphi_3(s)=\sum_m \frac{b_m}{m^s}$ two Dirichlet series that converge absolutely in $\mathbb{C}_\sigma$ and let φ be any function defined on $\mathbb{C}_\sigma$. If there exists $c_0\in\mathbb{N}$ such that*

$$\sum_{n=1}^\infty a_n\left(\frac{2^{c_0}}{n}\right)^s=2^{c_0 s}\varphi_2(s)=\varphi(s)=3^{c_0 s}\varphi_3(s)=\sum_{m=1}^\infty b_m\left(\frac{3^{c_0}}{m}\right)^s$$

for all $s\in\mathbb{C}_\sigma$, then φ is also a Dirichlet series that converges absolutely in $\mathbb{C}_\sigma$.

Proof. Let $j \in \mathbb{N}$ such that it is not a multiple of 2^{c_0}. Then, using [12, Proposition 1.9])

$$\begin{aligned}
a_j &= \lim_{T\to\infty} \frac{1}{2T} \int_{\sigma+1-iT}^{\sigma+1+iT} 2^{c_0 s} \varphi_2(s) \left(\frac{j}{2^{c_0}}\right)^s ds \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{\sigma+1-iT}^{\sigma+1+iT} 3^{c_0 s} \varphi_3(s) \left(\frac{j}{2^{c_0}}\right)^s ds \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{\sigma+1-iT}^{\sigma+1+iT} \sum_{m=1}^{\infty} b_m \left(\frac{3^{c_0}}{m}\right)^s \left(\frac{j}{2^{c_0}}\right)^s ds \\
&= \sum_{m=1}^{\infty} b_m \lim_{T\to\infty} \frac{1}{2T} \int_{\sigma+1-iT}^{\sigma+1+iT} \left(\frac{3^{c_0}}{2^{c_0}} \frac{j}{m}\right)^s ds = 0,
\end{aligned}$$

because $\dfrac{3^{c_0} j}{2^{c_0} m}$ is not an integer. Hence, all the coefficients of φ_2 corresponding to non-multiples of 2^{c_0} are null, so

$$\varphi(s) = \sum_{n=1}^{\infty} a_n \left(\frac{2^{c_0}}{n}\right)^s = \sum_{j=1}^{\infty} a_{j2^{c_0}} \left(\frac{2^{c_0}}{j2^{c_0}}\right)^s = \sum_{j=1}^{\infty} \frac{a_{j2^{c_0}}}{j^s}.$$

Therefore φ is a Dirichlet series which converges absolutely in $\mathbb{C}_\sigma$. □ □

Remark 5.32. Although we have given an explicit proof, Lemma 5.31 is actually a direct consequence of the more general Lemma 5.5.

Theorem 5.33. *Let $\phi = (\phi_1, \phi_2) : \mathbb{C}_+^2 \to \mathbb{C}_+^2$ be inducing a composition operator $C_\phi : \mathcal{H}_\infty(\mathbb{C}_+^2) \to \mathcal{H}_\infty(\mathbb{C}_+^2)$. Then there exists some $\sigma > 0$ such that, for $j = 1, 2$, $\phi_j(s,t) = c_0^{(j)} s + d_0^{(j)} t + \varphi_j(s,t)$ for $(s,t) \in \mathbb{C}_\sigma^2$, where $c_0^{(j)}, d_0^{(j)} \in \mathbb{N}_0$ and $\varphi_j(s,t) = \sum_{m,n=1}^{\infty} \frac{b_{m,n}^{(j)}}{m^s n^t}$ is a double Dirichlet series that converges absolutely in $\mathbb{C}_\sigma^2$.*

Proof. The proof works for $j = 1$ or $j = 2$ so, to keep the notation simpler, we will drop the subscript and consider $\phi(s,t) : \mathbb{C}_+^2 \to \mathbb{C}_+$. On the one hand, by hypothesis $D_k(s,t) := k^{-\phi(s,t)} \in \mathcal{H}_\infty(\mathbb{C}_+^2)$ for every

$k \in \mathbb{N}$. Using the regular convergence of D_k, if $t \in \mathbb{C}_+$ is fixed,

$$k^{-\phi(s,t)} = \sum_{m,n=1}^{\infty} \frac{a_{m,n}^{(k)}}{m^s n^t} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{a_{m,n}^{(k)}}{n^t} = \sum_{m=1}^{\infty} \frac{\alpha_m^{(k)}(t)}{m^s} = D_{k,t}(s). \tag{5.4}$$

On the other hand, for $t \in \mathbb{C}_+$ still fixed, Lemma 5.30 gives that ϕ_t defines a composition operator of $\mathcal{H}_\infty(\mathbb{C}_+)$, so by Theorem 5.19 $\phi_t(s) = c_0(t)s + \varphi_t(s)$, where $c_0(t) \in \mathbb{N}_0$ and $\varphi_t(s) = \sum_{m=1}^{\infty} \frac{c_m(t)}{m^s}$ is a Dirichlet series that converges regularly and uniformly in the half-plane $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$.

We first show that $t \mapsto c_0(t)$ is constant on some half-plane $\overline{\mathbb{C}_{\sigma_0}}$. Using the argument in [17, page 316] (also in the proof of Lemma 5.6),

$$k^{c_0(t)} = \inf\Big(\{m \in \mathbb{N} \ : \ \alpha_m^{(k)}(t) \neq 0\}\Big).$$

This means that the series in (5.4) actually runs up for $m \geqslant k^{c_0}$ and

$$c_0(t) = \inf\Big(\{p \in \mathbb{N} \ : \ \alpha_{k^p}^{(k)}(t) \neq 0\}\Big).$$

Define

$$c_0 = \inf\Big(\{p \in \mathbb{N} \ : \ \alpha_{k^p}^{(k)} \text{ is not identically zero}\}\Big).$$

Let $a_{k^{c_0},N}^{(k)}$ be the first non-zero coefficient of $\alpha_{k^{c_0}}^{(k)}$. By Lemma 5.3 we can find $\sigma_0 > 0$ such that

$$\sup_{t \in \overline{\mathbb{C}_{\sigma_0}}} |N^t \alpha_{k^{c_0}}^{(k)}(t) - a_{k^{c_0},N}^{(k)}| \leqslant \frac{|a_{k^{c_0},N}^{(k)}|}{2}.$$

Hence $\alpha_{k^{c_0}}^{(k)}$ has no zeros in the half-plane $\overline{\mathbb{C}}_{\sigma_0}$ and therefore $c_0(t) = c_0$ for every $t \in \overline{\mathbb{C}}_{\sigma_0}$.

Since for each t the Dirichlet series φ_t converges uniformly on every half-plane strictly contained in $\mathbb{C}_+$, Proposition 2.12 implies that $\varphi_t(s)$ is absolutely convergent for every s with $\operatorname{Re} s > 1/2$. Take, then, $(s,t) \in \mathbb{C}_{\frac{1}{2}} \times \mathbb{C}_{\sigma_0}$. Following Theorem 5.2 and proceeding as in Lemma 5.23 (using the Taylor expansion of the exponential) we arrive at

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{\alpha_m^{(k)}(t)}{m^s} &= k^{-\phi(s,t)} \\ &= k^{-c_0 s} k^{-c_1(t)} \prod_{m=2}^{\infty} \left(1 + \sum_{j=1}^{\infty} \frac{(-1)^{-j} (c_m(t))^j (\log k)^j}{j!(m^j)^s} \right) .\end{aligned} \tag{5.5}$$

Expanding the product at the right-hand side yields a series whose terms we can rearrange (because all the involved series are absolutely convergent), into a Dirichlet series the coefficients of which we denote by $d_l^{(k)}(t)$. Hence

$$\sum_{m=1}^{\infty} \frac{\alpha_m^{(k)}(t)}{m^s} = \frac{k^{-c_1(t)}}{k^{c_0 s}} \left(1 + \sum_{l=2}^{\infty} \frac{d_l^{(k)}(t)}{l^s} \right) = \frac{k^{-c_1(t)}}{k^{c_0 s}} + \sum_{l=2}^{\infty} \frac{k^{-c_1(t)} d_l^{(k)}(t)}{(l k^{c_0})^s}. \tag{5.6}$$

So, we have arrived at an equality between Dirichlet series that converge absolutely on some half-plane and we may identify coefficients (recall that we already saw that the series in the left-hand side in fact starts at k^{c_0}). To begin with, $k^{-c_1(t)}$ is the coefficient corresponding to the term $(k^{c_0})^s$, so $k^{-c_1(t)} = \alpha_{k^{c_0}}^{(k)}(t)$ for all $t \in \mathbb{C}_{\sigma_0}$, and $\alpha_{k^{c_0}}^{(k)} \in \mathcal{H}_\infty(\mathbb{C}_+)$. Since this holds for every k, we can apply Lemma 5.6 to get that $c_1(t)$ is holomorphic in $\mathbb{C}_{\sigma_0}$ and that there exists some $\sigma_1 \geqslant \sigma_0$ such that $c_1(t) = d_0 t + \sum_{n=1}^{\infty} \frac{b_{1,n}}{n^t}$ for every $t \in \mathbb{C}_{\sigma_1}$, with $\sum_{n=1}^{\infty} \frac{b_{1,n}}{n^t}$ absolutely convergent in $\mathbb{C}_{\sigma_1}$.

What we want to do now is to push further this idea, comparing coefficients in (5.6) in a systematic way to end up showing that every $c_m(t)$ can be written as a Dirichlet series absolutely convergent in $\mathbb{C}_{\sigma_1}$.

We do this by induction on $m \geqslant 2$ and start with the case $m = 2$. We take some $(s,t) \in \mathbb{C}_{\frac{1}{2}} \times \mathbb{C}_{\sigma_1}$ and note that the term corresponding to $l = 2$ in (5.6) is obtained by multiplying the term $m = 2$ and $j = 1$ (this carries $c_2(t)$) and 1's in (5.5). In this way we have

$$\sum_{m=1}^{\infty} \frac{\alpha_m^{(k)}(t)}{m^s} = \frac{k^{-c_1(t)}}{k^{c_0 s}} + \frac{-\log k\, c_2(t) k^{-c_1(t)}}{(2k_0^c)^s} + \sum_{l=3}^{\infty} \frac{k^{-c_1(t)} d_l^{(k)}(t)}{(lk^{c_0})^s}.$$

Identifying again coefficients we get $\alpha_{2k^{c_0}}^{(k)}(t) = -\log k\, c_2(t) k^{-c_1(t)}$, so

$$c_2(t) = \frac{-1}{\log k} k^{c_1(t)} \alpha_{2k^{c_0}}^{(k)} = k^{d_0 t} \frac{-1}{\log k} k^{-(-\sum_{n=1}^{\infty} \frac{b_{1,n}}{n^t})} \alpha_{2k^{c_0}}^{(k)} = k^{d_0 t} \psi_k(t)\,.$$

We need to see now that $\psi_k(t)$ is a Dirichlet series that converges absolutely in $\mathbb{C}_{\sigma_1}$. Note first that $\alpha_{2k^{c_0}}^{(k)}$ belongs to $\mathcal{H}_\infty(\mathbb{C}_+)$. On the other hand, we have just seen that $-\sum_{n=1}^{\infty} b_{1,n} n^{-t}$ is an absolutely convergent series in $\mathbb{C}_{\sigma_1}$, and a careful inspection of the proof of the sufficiency of Theorem 5.2 shows that $k^{\sum_{n=1}^{\infty} \frac{b_{1,n}}{n^t}}$ is an absolutely convergent Dirichlet series in $\mathbb{C}_{\sigma_1}$. This gives the claim. Letting now $k = 2, 3$ we have $2^{c_0 t}\psi_2(t) = c_2(t) = 3^{c_0 t}\psi_3(t)$ for every $t \in \mathbb{C}_{\sigma_1}$. Since $c_2(t)$ is analytic, Lemma 5.31 gives that $c_2(t) = \sum_{n=1}^{\infty} \frac{b_{2,n}}{n^t}$ is a Dirichlet series that converges absolutely in $\mathbb{C}_{\sigma_1}$. This completes the proof of the fact for $m = 2$.

Suppose now that $c_m(t)$ is analytic for every $2 \leqslant m \leqslant m_0$. We want to use again (5.6), comparing the coefficients of the term corresponding to $l = m_0$. Note that we get this factor by multiplying the term $m = m_0$ and $j = 1$ with all 1's (this brings $c_{m_0}(t)$) and the product of terms involving divisors of m_0 (this brings other $c_m(t)$'s, that we group in a term D_{m_0}). Let us be more precise. Starting from (5.5) we get

$$\sum_{m=1}^{\infty} \frac{\alpha_m^{(k)}(t)}{m^s} = \frac{-\log k\, c_{m_0}(t) k^{-c_1(t)} + D_{m_0(t)}}{(m_0 k^{c_0})^s} + \sum_{l \neq m_0}^{\infty} \frac{k^{-c_1(t)} d_l^{(k)}(t)}{(lk^{c_0})^s},$$

where D_{m_0} is given by

$$D_{m_0}(t) = \sum_{q>1} \sum_{m_1^{r_1}\cdots m_q^{r_q}=m_0} \prod_{h=1}^{q} \frac{\big(-\log k\, c_{m_h}(t)\big)^{r_h}}{r_h!}.$$

Since $m_1^{r_1}\cdots m_q^{r_q} = m_0$ for $q > 1$ implies that $m_h < m_0$ for all $1 \leqslant h \leqslant q$ we have D_{m_0} is a finite sum of finite products of Dirichlet series which by the induction hypothesis are absolutely convergent in $\mathbb{C}_{\sigma_1}$. Hence D_{m_0} is a Dirichlet series that converges absolutely on $\mathbb{C}_{\sigma_1}$. Then

$$\begin{aligned} c_{m_0}(t) &= \frac{-k^{-c_1(t)}}{\log k}(\alpha^{(k)}_{m_0 k^{c_0}} - D_m(t)) \\ &= k^{c_0 t}\frac{-1}{\log k}k^{(\sum_{n=1}^{\infty}\frac{b_{1,n}}{n^t})}(\alpha^{(k)}_{m_0 k^{c_0}} - D_{m_0}(t)) = k^{c_0 t}\psi_k^{(m_0)}(t), \end{aligned}$$

where, with the same argument as above, $\psi_k^{(m_0)}$ is again an absolutely convergent Dirichlet series on $\mathbb{C}_{\sigma_1}$. Once again by application of Lemma 5.31, $c_{m_0}(t) = \sum_{n=1}^{\infty}\frac{b_{m_0,n}}{n^t}$ is a Dirichlet series that converges absolutely on $\mathbb{C}_{\sigma_1}$.

Finally, for $(s,t) \in \mathbb{C}_{\frac{1}{2}} \times \mathbb{C}_{\sigma_1}$,

$$\phi(s,t) = c_0 s + d_0 t + \sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{\infty}\frac{b_{m,n}}{n^t} = c_0 s + d_0 t + \sum_{m,n=1}^{\infty}\frac{b_{m,n}}{m^s n^t},$$

where the last equality holds because the sums converge absolutely in $\mathbb{C}_{\frac{1}{2}} \times \mathbb{C}_{\sigma_1}$. □ □

Once we have established the form of the symbols of composition operators we may proceed as in [29] to strengthen the conditions on the symbol in terms of its uniform convergence in $\mathbb{C}^2_{\varepsilon}$ for every $\varepsilon > 0$, giving the proof of our main result.

Proof. of Theorem 5.22. To begin with, if ϕ_j is as in (5.3) and φ_j converges uniformly and regularly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$, then it converges regularly on $\mathbb{C}_+$ and Theorem 5.29 gives that C_ϕ defines a composition operator on $\mathcal{H}_\infty(\mathbb{C}_+^2)$.
For the necessary condition, Theorem 5.33 gives that each φ_j converges absolutely on $\mathbb{C}_\sigma^2$ for some $\sigma > 0$. We adapt the arguments of [29, Section 3] (also reproduced in the proof of Theorem 5.18) to see that in fact they converge uniformly on $\mathbb{C}_\varepsilon^2$ (we do it only for $j = 1$). Note first that Lemma 5.30, Lemma 5.7 and Hartogs' theorem give that ϕ is analytic. Then, by Lemma 5.25 (recall also Remark 5.24), we get $\varphi_1(\mathbb{C}_+^2) \subset \overline{\mathbb{C}_+}$, that is, $|\operatorname{Arg}\varphi_1| \leqslant \frac{\pi}{2}$ and $|\operatorname{Arg}\varphi_1^{1/2}| \leqslant \frac{\pi}{4}$, from which $\left|\frac{\operatorname{Im}\varphi_1^{1/2}}{\operatorname{Re}\varphi_1^{1/2}}\right| = |\tan(\operatorname{Arg}\varphi_1^{1/2})| \leqslant 1$ follows.

We consider now the function $u(s,t) = \operatorname{Re}(\varphi_1(s,t)^{1/2})$. Since φ_1 converges absolutely on $\mathbb{C}_\sigma^2$, then φ_1 is uniformly bounded there. Now it is easy to see that $|\operatorname{Re}\varphi_1^{1/2}| \leqslant |\operatorname{Re}\varphi_1|^{1/2} \leqslant |\varphi_1|^{1/2} \leqslant \sqrt{2}|\operatorname{Re}\varphi_1^{1/2}|$, where the last inequality is clear because $|\varphi_1^{1/2}| = \sqrt{\operatorname{Re}\varphi_1 + \operatorname{Im}\varphi_1} \leqslant \sqrt{2\operatorname{Re}\varphi_1}$, and then u is uniformly bounded on $\overline{\mathbb{C}_\theta^2}$, say by K. Our aim now is to see that, in fact, u is uniformly bounded on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$. By Remark 5.26 either φ_1 is identically zero or $\operatorname{Re}\varphi_1(s,t) > 0$ for every $(s,t) \in \mathbb{C}_+^2$. If it is identically zero, then so also is $\varphi_1^{1/2}$ and therefore u. If this is not the case, then (since $|\operatorname{Arg}\varphi_1^{1/2}| \leqslant \frac{\pi}{4}$) u is strictly positive.

If u is identically zero then the claim is trivially satisfied. We may then assume that u is positive and take $(s_0,t_0) = (\sigma_1 + i\tau_1, \sigma_2 + i\tau_2) \in \mathbb{C}_\varepsilon^2$. We know from [29, Section 3] that if v is a positive harmonic function defined on some $\mathbb{C}_{\sigma_0}$, then

$$v(\theta_1 + i\tau) \leqslant \frac{\theta_2}{\theta_1} v(\theta_2 + i\tau) \tag{5.7}$$

for every $\sigma_0 < \theta_1 \leqslant \theta_2$. Suppose that $\varepsilon < \sigma_1 < \sigma$ and consider $u_{t_0}(s) = u(s,t_0) = \operatorname{Re}(\varphi_{1,t_0}(s)^{1/2})$. Suppose now that $\varepsilon < \sigma_1 < \sigma$ then, since u_{t_0}

is a positive harmonic function (5.7) gives

$$u(s_0,t_0) = u_{t_0}(\sigma_1 + i\tau_1) \leqslant \frac{\sigma}{\sigma_1} u_t(\sigma + i\tau_1) \leqslant \frac{\sigma}{\varepsilon} u(\sigma + i\tau_1, \sigma_2 + i\tau_2)\,. \quad (5.8)$$

We distinguish two cases for t_0. First, if $\sigma \leqslant \sigma_2$ we immediately obtain (recall that u is uniformly bounded on $\overline{\mathbb{C}^2_\sigma}$ by K)

$$u(s_0,t_0) \leqslant \frac{\sigma}{\varepsilon} K\,.$$

On the other hand, if $\varepsilon < \sigma_2 < \sigma$, we can consider $\tilde{s} = \theta + i\tau_1 \in \overline{\mathbb{C}_\theta}$ and $t \mapsto u_{\tilde{s}}(t)$, which is again a positive harmonic function. Starting with (5.8) and using (5.7) again (this time for $u_{\tilde{s}}$) we get

$$u(s_0,t_0) \leqslant \frac{\sigma}{\varepsilon} u_{\tilde{s}}(\sigma_2 + i\tau_2) \leqslant \frac{\sigma^2}{\varepsilon^2} u_{\tilde{s}}(\sigma + i\tau_2) \leqslant \frac{\sigma^2}{\varepsilon^2} K.$$

The only case left to check is that in which $\varepsilon < \sigma_2 < \sigma$ and $\sigma_1 \geqslant \sigma$, but it is completely analogous to the one above, so we get $u(s,t) \leqslant \frac{\theta^2}{\varepsilon^2} K$ for every $(s,t) \in \mathbb{C}_\varepsilon$. Hence, $\varphi_1^{1/2}$, and therefore φ_1, is uniformly bounded in $\mathbb{C}_\varepsilon$ and consequently uniformly convergent. □

The characterization of compact composition operators on $\mathcal{H}_\infty(\mathbb{C}_+)$ given in [5, Theorem 18] still works for double Dirichlet series.

Theorem 5.34. *Let ϕ define a continuous composition operator $C_\phi : \mathcal{H}^\infty(\mathbb{C}^2_+) \to \mathcal{H}^\infty(\mathbb{C}^2_+)$. Then C_ϕ is compact if and only if $\phi(\mathbb{C}^2_+) \subset \mathbb{C}^2_\delta$ for some $\delta > 0$.*

Proof. First, suppose that there exists some $\delta > 0$ such that $\phi(\mathbb{C}^2_+) \subset \mathbb{C}^2_\delta$ and consider $\{D_n\}$ a bounded sequence in $\mathcal{H}_\infty(\mathbb{C}^2_+)$. By Lemma 4.32, there exists some subsequence $\{D_{n_k}\}$ in $\mathcal{H}_\infty(\mathbb{C}^2_+)$ and $D \in \mathcal{H}_\infty(\mathbb{C}^2_+)$ such that D_{n_k} converges uniformly to D on $\mathbb{C}^2_\delta$, and therefore $D_{n_k} \circ \phi$ converges to $D \circ \phi$ uniformly on $\mathbb{C}^2_+$. Hence, C_ϕ is compact. On the other hand, if C_ϕ is compact, choose the sequence $D_m(u,v) = m^{-u}$ in $\mathcal{H}_\infty(\mathbb{C}^2_+)$. Since

C_ϕ is compact there exists some subsequence D_{m_k} in $\mathcal{H}_\infty(\mathbb{C}^2_+)$ and some $\tilde{D} \in \mathcal{H}_\infty(\mathbb{C}^2_+)$ such that $\lim_{k\to\infty} \|D_{m_k} \circ \phi - \tilde{D}\|_\infty = 0$. Fix $(s,t) \in \mathbb{C}^2_+$, then $\tilde{D}(s,t) = \lim_{k\to\infty} m_k^{-\phi_1(s,t)} = 0$ since $\operatorname{Re}\phi_1(s,t) > 0$. Therefore

$$0 = \lim_{k\to\infty} \|D_{m_k} \circ \phi - \tilde{D}\|_\infty = \lim_{k\to\infty} \|D_{m_k} \circ \phi\|_\infty = \lim_{k\to\infty} m_k^{-\inf_{(s,t)\in\mathbb{C}^2_+} \operatorname{Re}\phi_1(s,t)},$$

so necessarily $\inf_{(s,t)\in\mathbb{C}^2_+} \operatorname{Re}\phi_1(s,t) > 0$ and there exists some $\delta_1 > 0$ such that $\phi_1(\mathbb{C}^2_+) \subset \mathbb{C}_{\delta_1}$. Applying the same idea to the sequence of functions n^{-v} in $\mathcal{H}_\infty(\mathbb{C}^2_+)$, one gets that there exists some $\delta_2 > 0$ such that $\phi_2(\mathbb{C}^2_+) \subset \mathbb{C}_{\delta_2}$. Taking $\delta = \min(\delta_1, \delta_2)$ we have $\phi(\mathbb{C}^2_+) \subset \mathbb{C}^2_\delta$. □

5.2.3 Composition operators on the spaces $\mathcal{H}_\infty(\mathbb{C}^2_+)$ and $H^\infty(B_{c_0^2})$

We finish this section by relating composition operators of $\mathcal{H}_\infty(\mathbb{C}^2_+)$ and composition operators on the corresponding space of holomorphic functions. This result extends the corresponding result for $\mathcal{H}^p$ and $H^p(\mathbb{T}^\infty)$, $1 \leqslant p < +\infty$ obtained in [5]. In that case, that is, for finite values of p, Dirichlet polynomials are dense in $\mathcal{H}^p$ and this was a key point in the proof done in [5], so the case $p = \infty$ requires a new proof.

Proposition 5.35. *Let ϕ define a continuous composition operator $C_\phi : \mathcal{H}^\infty(\mathbb{C}^2_+) \to \mathcal{H}^\infty(\mathbb{C}^2_+)$. Then C_ϕ induces a composition operator $C_\psi : H^\infty(B_{c_0^2}) \to H^\infty(B_{c_0^2})$.*

Proof. Let us recall how the bijective isometry $\mathcal{B} : H^\infty(B_{c_0^2}) \to \mathcal{H}_\infty(\mathbb{C}^2_+)$ from Theorem 4.33 (now with $k = 2$) is defined. For each $f \in H^\infty(B_{c_0^2})$ with coefficients $c_{\alpha,\beta}(f)$ (that can be computed through the Cauchy integral formula) we have

$$\mathcal{B}(f) = \sum_{\alpha,\beta} \frac{c_{\alpha,\beta}(f)}{(p^\alpha)^s (p^\beta)^t}.$$

Clearly $T = \mathcal{B}^{-1} \circ C_\phi \circ \mathcal{B}$ is an operator, $T : H^\infty(B_{c_0^2}) \to H^\infty(B_{c_0^2})$, and our aim is to see that it is actually a composition operator. For all $j \in \mathbb{N}$, define $D_1^{(j)} = C_\phi(\frac{1}{p_j^s}) = p_j^{-\phi_1(s,t)} \in \mathcal{H}_\infty(\mathbb{C}_+^2)$ and $D_2^{(j)} = C_\phi(\frac{1}{p_j^t}) = p_j^{-\phi_2(s,t)} \in \mathcal{H}_\infty(\mathbb{C}_+^2)$ and $D^{(j)} = (D_1^{(j)}, D_2^{(j)})$. Note that $\mathcal{B}^{-1}(D_1^{(j)}), \mathcal{B}^{-1}(D_2^{(j)}) \in H^\infty(B_{c_0^2})$ for every $j \in \mathbb{N}$. Define formally $\Phi = (\Phi_1, \Phi_2) = ((D_1^{(j)})_j, (D_2^{(j)})_j)$ and $\psi = (\psi_1, \psi_2) = ((\mathcal{B}^{-1} D_1^{(j)})_j, (\mathcal{B}^{-1} D_2^{(j)})_j)$. Then, if we consider a polynomial $f(z,\omega) = \sum_{\alpha,\beta\in\Lambda} c_{\alpha,\beta} z^\alpha \omega^\beta$ with Λ finite and we denote by $\mathfrak{p}$ the sequence of prime numbers, we get

$$\begin{aligned}
T(f)(z,\omega) &= \mathcal{B}^{-1}\left(C_\phi\left(\mathcal{B}\left(\sum_{\alpha,\beta\in\Lambda} c_{\alpha,\beta} z^\alpha \omega^\beta\right)\right)\right) \\
&= \mathcal{B}^{-1}\left(C_\phi\left(\sum_{\alpha,\beta\in\Lambda} \frac{c_{\alpha,\beta}}{(\mathfrak{p}^\alpha)^s (\mathfrak{p}^\beta)^t}\right)\right) \\
&= \mathcal{B}^{-1}\left(\sum_{\alpha,\beta\in\Lambda} \frac{c_{\alpha,\beta}}{(\mathfrak{p}^{\phi_1(s,t)})^\alpha (\mathfrak{p}^{\phi_2(s,t)})^\beta}\right) \\
&= \mathcal{B}^{-1}\left(\sum_{\alpha,\beta\in\Lambda} c_{\alpha,\beta} \Phi_1(s,t)^\alpha \Phi_2(s,t)^\beta\right) \\
&= \sum_{\alpha,\beta\in\Lambda} c_{\alpha,\beta} (\mathcal{B}^{-1}\Phi_1)(z,\omega)^\alpha (\mathcal{B}^{-1}\Phi_2)(z,\omega)^\beta \\
&= \sum_{\alpha,\beta\in\Lambda} c_{\alpha,\beta} \psi_1(z,\omega)^\alpha \psi_2(z,\omega)^\beta.
\end{aligned} \tag{5.9}$$

Therefore, T coincides with the composition operator C_ψ on finite polynomials, and we will see that they actually are the same operator. First we need to see that C_ψ is well-defined, namely that ψ is a holomorphic function with $\psi(B_{c_0^2}) \subset B_{c_0^2}$. The holomorphy of ψ follows from its definition. Let us define $F_1^{(j)} = \mathcal{B}^{-1}(D_1^{(j)})$ so that $\psi_1(z,w) = (F_1^{(j)}(z,w))$. Since $\phi(\mathbb{C}_+^2) \subset \mathbb{C}_+^2$, $|D_1^{(j)}(s,t)| = p_j^{-\operatorname{Re}\phi_1(s,t)} < 1$ for every $(s,t) \in \mathbb{C}_+^2$, so that $\|F_1^{(j)}\|_\infty = \|D_1^{(j)}\|_\infty \leqslant 1$. Assume by contradiction that there exists some $(z_0, w_0) \in B_{c_0^2}$ such that $\psi_1(z_0, w_0)$ does not belong to c_0.

Then there exists an increasing sequence of integers (j_r) and $\varepsilon > 0$ such that, for all $r \geqslant 1$, $|F_1^{(j_r)}(z_0, w_0)| \geqslant \varepsilon$. By Montel's theorem, we may extract from $(F_1^{(j_r)})$ a sequence, that we will still denote $(F_1^{(j_r)})$, converging uniformly on compact subsets of $B_{c_0^2}$ to some $F \in H^\infty(B_{c_0^2})$. Set $D = \mathcal{B}F$, so that $(D_1^{(j_r)})$ converges uniformly to D on each product of half-planes $\mathbb{C}_\varepsilon^2$. Now, for $(s,t) \in \mathbb{C}_+$,

$$D_1^{(j_r)}(s,t) = p_{j_r}^{-\phi_1(s,t)} \xrightarrow{r \to +\infty} 0$$

since $\operatorname{Re}\phi_1(s,t) > 0$. Thus D hence F are identically zero. But this contradicts $|F_1^{(j_r)}(z_0, w_0)| \geqslant \varepsilon$ for all $r \geqslant 1$. Finally, this yields that $\psi_1(B_{c_0^2}) \subset c_0$.

To see that T and C_ψ are the same operator we will define a topology on $H^\infty(B_{c_0^2})$ so that the finite polynomials on $B_{c_0^2}$ are dense in $H^\infty(B_{c_0^2})$ and such that T and C_ψ are continuous, with the aim of extending (5.9) by continuity. Define $G : \mathbb{C}_+ \to B_{c_0}$ by $G(s) = \frac{1}{\mathfrak{p}^s}$ and consider $\tilde{\tau}$ the topology of uniform convergence on the product of half-planes $\overline{\mathbb{C}_{\sigma_1}} \times \overline{\mathbb{C}_{\sigma_2}}$ for $\mathcal{H}_\infty(\mathbb{C}_+^2)$ and for $H^\infty(B_{c_0^2})$ we consider τ the topology of the uniform convergence on the compact subsets of B_{c_0} of the form $K_{\sigma_1,\sigma_2} = \overline{\left\{\left(\frac{1}{\mathfrak{p}^s}, \frac{1}{\mathfrak{p}^t}\right) : \operatorname{Re} s \geqslant \sigma_1, \operatorname{Re} t \geqslant \sigma_2\right\}} = \overline{G(\overline{\mathbb{C}_{\sigma_1}})} \times \overline{G(\overline{\mathbb{C}_{\sigma_2}})} = K_{\sigma_1} \times K_{\sigma_2}$. It should be noted that these topologies define metrizable spaces since we can take $\sigma_1 = \frac{1}{n}$, $\sigma_2 = \frac{1}{m}$, $n, m \in \mathbb{N}$, and we get the same topologies. First, since for every $\sigma_1, \sigma_2 > 0$ there exists some $0 < r < 1$ such that $\sup_{\substack{\operatorname{Re} s \geqslant \sigma_1 \\ \operatorname{Re} t \geqslant \sigma_2}} \left\|\left(\frac{1}{\mathfrak{p}^s}, \frac{1}{\mathfrak{p}^t}\right)\right\|_\infty \leqslant r$, then any $f \in H^\infty(B_{c_0^2})$ has a uniformly convergent Taylor series on K_{σ_1,σ_2}. Moreover, by adapting the arguments from the proof of [3, Theorem 2.5], the set of finite polynomials on $B_{c_0^2}$ is dense in the space of homogeneous polynomials on $B_{c_0^2}$ with the topology induced by $\|\cdot\|_\infty$. Therefore we can extend (5.9) to homogeneous polynomials. Again, since the topology of $\|\cdot\|_\infty$ is finer than topology τ, if we prove that T and C_ψ are continuous with the topology τ, then we will be able to extend (5.9) to $H^\infty(B_{c_0^2})$ to get that $T = C_\psi$ as operators of $H^\infty(B_{c_0^2})$.

To see that T is continuous, we just have to check that C_ϕ is continuous for the topology $\tilde{\tau}$ and that $\mathcal{B}$ defines a homeomorphism with the respective topologies τ and $\tilde{\tau}$. To prove that C_ϕ is continuous we have to apply Lemma 5.28 to ϕ to get that for every $\sigma_1, \sigma_2 > 0$ there exists some $\delta(\sigma_1), \delta(\sigma_2) > 0$ such that $\phi(\mathbb{C}_{\sigma_1} \times \mathbb{C}_{\sigma_2}) \subset \mathbb{C}_{\delta(\sigma_1)} \times \mathbb{C}_{\delta(\sigma_2)}$. Now, let $\{D_n\}_n \subset \mathcal{H}_\infty(\mathbb{C}_+^2)$ be a sequence convergent to $D \in \mathcal{H}_\infty(\mathbb{C}_+^2)$ with $\tilde{\tau}$. As

$$\begin{aligned}\|C_\phi(D_n) - C_\phi(D)\|_{\mathbb{C}_{\sigma_1}\times\mathbb{C}_{\sigma_2}} &= \|D_n \circ \phi - D \circ \phi\|_{\mathbb{C}_{\sigma_1}\times\mathbb{C}_{\sigma_2}} \\ &\leqslant \|D_n - D\|_{\mathbb{C}_{\delta(\sigma_1)}\times\mathbb{C}_{\delta(\sigma_2)}},\end{aligned}$$

C_ϕ is continuous. Now, to see that $\mathcal{B}$ is a homeomorphism with the respective topologies, suppose that $\{f_n\}_n \subset H^\infty(B_{c_0^2})$ is a sequence convergent to $f \in H^\infty(B_{c_0^2})$ with τ. Then, using the continuity of f_n and f,

$$\begin{aligned}\|\mathcal{B}(f_n) - \mathcal{B}(f)\|_{\mathbb{C}_{\sigma_1}\times\mathbb{C}_{\sigma_2}} &= \sup_{\substack{\operatorname{Re} s\geqslant\sigma_1\\ \operatorname{Re} t\geqslant\sigma_2}} |\mathcal{B}(f_n)(s,t) - \mathcal{B}(f)(s,t)| \\ &= \sup_{\substack{\operatorname{Re} s\geqslant\sigma_1\\ \operatorname{Re} t\geqslant\sigma_2}} \left| f_n\left(\frac{1}{\mathfrak{p}^s}, \frac{1}{\mathfrak{p}^t}\right) - f\left(\frac{1}{\mathfrak{p}^s}, \frac{1}{\mathfrak{p}^t}\right)\right| \\ &= \sup_{(z,\omega)\in K_{\sigma_1,\sigma_2}} |f_n(z,\omega) - f(z,\omega)| \\ &= \|f_n - f\|_{K_{\sigma_1,\sigma_2}},\end{aligned}$$

so clearly $\mathcal{B} : (H^\infty(B_{c_0^2}), \tau) \to (\mathcal{H}_\infty(\mathbb{C}_+^2), \tilde{\tau})$ is a homeomorphism between thsee topological spaces , so that T is continuous with the topology τ.

It remains to prove that C_ψ is continuous with the topology τ. Let us recall that $\psi = ((\mathcal{B}^{-1}(D_1^{(j)}))_j, (\mathcal{B}^{-1}(D_2^{(j)}))_j)$. Hence, if $\sigma_1, \sigma_2 > 0$, take $(z,\omega) \in G(\overline{\mathbb{C}_{\sigma_1}}) \times G(\overline{\mathbb{C}_{\sigma_2}})$, that is, $(z,\omega) = (\frac{1}{\mathfrak{p}^s}, \frac{1}{\mathfrak{p}^t})$ for some $(s,t) \in$

$\overline{\mathbb{C}_{\sigma_1}} \times \overline{\mathbb{C}_{\sigma_2}}$, and then $\mathcal{B}^{-1}(D_1^{(j)})(z,\omega) = D_1^{(j)}(s,t) = \frac{1}{p_j^{\phi_1(s,t)}}$, so

$$\mathcal{B}^{-1}(D_1^{(j)})(G(\overline{\mathbb{C}_{\sigma_1}}) \times G(\overline{\mathbb{C}_{\sigma_2}})) \subset \left\{ \frac{1}{p_j^{\phi_1(s,t)}} : \operatorname{Re} s \geqslant \sigma_1, \operatorname{Re} t \geqslant \sigma_2 \right\}$$
$$\subset \left\{ \frac{1}{p_j^s} : \operatorname{Re} s \geqslant \delta(\sigma_1) \right\},$$

and therefore

$$\psi_1(K_{\sigma_1,\sigma_2}) \subset \overline{\left\{ \frac{1}{\mathfrak{p}^s} : \operatorname{Re} s \geqslant \delta(\sigma_1) \right\}} = K_{\delta(\sigma_1)}.$$

Analogously, $\psi_2(K_{\sigma_1,\sigma_2}) \subset K_{\delta(\sigma_2)}$, so $\psi(K_{\sigma_1,\sigma_2}) \subset K_{\delta(\sigma_1)} \times K_{\delta(\sigma_2)} = K_{\delta(\sigma_1),\delta(\sigma_2)}$. Then, if $\{f_n\}_n \subset H^\infty(B_{c_0^2})$ is a sequence convergent to $f \in H^\infty(B_{c_0^2})$ with τ,

$$\|C_\psi(f_n) - C_\psi(f)\|_{K_{\sigma_1,\sigma_2}} = \|f_n \circ \psi - f \circ \psi\|_{K_{\sigma_1,\sigma_2}} \leqslant \|f_n - f\|_{K_{\delta(\sigma_1),\delta(\sigma_2)}},$$

which gives the continuity of C_ψ with the topology τ.

Finally, as every function in $H^\infty(B_{c_0^2})$ is the limit of a series of finite polynomials with the topology τ, and by (5.9), T and C_ψ coincide on finite polynomials, $T = C_\psi$. □

5.3 Superposition operators

Superposition operators are not, at least generally, directly related to composition operators in any way but the fact that they are defined through composition. This is the main reason for which we include this section, which deals with the superposition operators of Hardy spaces of Dirichlet series, in this chapter. In order to characterize such operators for said spaces, let us first recall the characterization done in [9] for the superposition operators of Hardy spaces on the disk, $H^p(\mathbb{D})$. Let us prepare the work by proving a series of lemmas.

Lemma 5.36. *Given a non-negative α and an entire function φ, there exists a positive number R such that $|\varphi(\omega)| \leqslant M|\omega|^\alpha$ whenever $|\omega| > R$ if and only if φ is a polynomial of degree at most $[\alpha]$, the greatest integer smaller than or equal to α.*

Proof. Consider $r > 0$ and $C(\omega, r) := \{\omega + re^{i\theta} : \theta \in [0, 2\pi]\}$. Suppose φ is a polynomial of degree N, then $\lim_{|z|\to\infty} \frac{\varphi(z)}{z^N} = a_N \in \mathbb{C} \setminus \{0\}$, so there exists some $R_0 > 1$ such that $|z| \geqslant R_0$ implies $\left|\frac{\varphi(z)}{z^N} - a_n\right| < 1$ and therefore $|\varphi(z)| \leqslant 1 + |a_N z^N| \leqslant (1 + |a_N|)|z^N|$.

On the other hand, if φ is an entire function, by the Cauchy integral formula we get that

$$|a_n(\varphi)| = \left|\frac{1}{2\pi i}\int_{C(0,r)} \frac{\varphi(z)}{z^{n+1}}dz\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi(re^{i\theta})|}{|r^{n+1}e^{i(n+1)\theta}|}d\theta$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{Mr^\alpha}{r^{n+1}}d\theta = Mr^{\alpha-(n+1)}.$$

Thus, taking $n > [\alpha]$, $n + 1 > \alpha$ and therefore $|a_n(\varphi)| \leqslant \frac{M}{r^{n+1-\alpha}} \xrightarrow{r\to\infty} 0$, so φ is a polynomial of degree at most $[\alpha]$. ☐

Lemma 5.37. *Let $\lambda \in \mathbb{T}$. A function φ is a polynomial of degree N if and only if $\psi(\omega) = \varphi(\lambda\omega)$ is a polynomial of degree N.*

Proof. Consider $\lambda \in \mathbb{T}$ and $\varphi(z) = \sum_{n=1}^N a_n z^n$. Write $\lambda = e^{i\theta}$, $\omega = |\omega|e^{i\alpha}$ and $b_n = a_n e^{in\theta}$, then

$$\psi(\omega) = \varphi(|\omega|e^{i(\theta+\alpha)}) = \sum_{n=1}^N a_n(|\omega|e^{i(\theta+\alpha)})^n = \sum_{n=1}^N a_n|\omega|^n e^{in(\theta+\alpha)}$$
$$= \sum_{n=1}^N (a_n e^{in\theta})\omega^n = \sum_{n=1}^N b_n\omega^n.$$

☐

Lemma 5.38. *A function φ is a polynomial of degree N if and only if $\psi(\omega) = \overline{\varphi(\overline{\omega})}$ is a polynomial of degree N.*

Proof. Write $\omega = |\omega|e^{i\alpha}$ and $a_n = |a_n|e^{i\theta_n}$. Then

$$\psi(\omega) = \sum_{n=1}^{N} \overline{a_n \overline{\omega}^n} = \sum_{n=1}^{N} \overline{|a_n||\omega|e^{i(n\alpha-\theta_n)}} = \sum_{n=1}^{N} |a_n||\omega|e^{i(\theta_n - n\alpha)} = \sum_{n=1}^{N} \overline{a_n}\omega^n.$$

□

Lemma 5.39. *If $c > 0$ is such that $cp < 1$ then $(1+z)^{-c} \in H^p(\mathbb{D})$.*

Proof. Define $I_\alpha(r) = \int_{-\pi}^{\pi} \frac{1}{|1-re^{it}|^\alpha} dt$ as a function defined for $0 \leqslant r < 1$. Consider the triangle defined by the points r, re^{it} and 1, which has an obtuse angle in r. Therefore, since $|1-e^{it}| = 2\left|\sin\left(\frac{t}{2}\right)\right| \geqslant \frac{|t|}{\pi}$ for $t \in [-\pi, \pi]$,

$$\begin{aligned} |1-re^{it}| > \max\{r|1-e^{it}|, 1-r\} \leqslant \frac{1}{2}\left(\frac{1}{2}|1-re^{it}| + 1 - r\right) \\ = \frac{1}{2}\left(\left|\sin\left(\frac{t}{2}\right)\right| + 1 - r\right) \geqslant \frac{1}{2}\left(\frac{1}{\pi}|t| + 1 - r\right) \geqslant \frac{1}{2\pi}(|t| + 1 + r). \end{aligned}$$

Moreover,

$$|1-re^{it}| \leqslant |1-e^{it}| + |e^{it} - re^{it}| = 2\left|\sin\left(\frac{t}{2}\right)\right| + 1 - r \leqslant |t| + 1 + r,$$

so, for $\alpha < 1$, $I_\alpha(r)$ is comparable to

$$\int_{-\pi}^{\pi} \frac{1}{(|t|+1-r)^\alpha} dt = \frac{2}{1-\alpha}\left[(1-r+\pi)^{1-\alpha} - (1-r)^{1-\alpha}\right].$$

Thus, $I_{cp}(r) < (2\pi)^\alpha \frac{2}{1-\alpha}\left[(1-r+\pi)^{1-\alpha} - (1-r)^{1-\alpha}\right]$ which tends to $\frac{2^{\alpha+1}\pi^\alpha}{1-\alpha}\left[\pi^{1-\alpha} - 1\right]$ when $\to 1$, so $(1+z)^{-c} \in H^p(\mathbb{D})$. □

The proof of the next lemma can be found in [14].

Lemma 5.40. *If a function* $f(z) \in H^p(\mathbb{D})$ $(0 < p < \infty)$, *then*

$$|f(z)| \leqslant 2^{\frac{1}{p}} ||f||_p (1 - |z|)^{-\frac{1}{p}}$$

We are ready now to reproduce the proof of the characterization of superporsition operators of Hardy spaces $H^p(\mathbb{D})$ given in [9].

Theorem 5.41. *If* $S_\varphi(H^p(\mathbb{D})) \subset H^q(\mathbb{D})$ *where* $S_\varphi(f) = \varphi \circ f$ *and* $p > 1$, *then* φ *is a polynomial of degree* $N \leqslant \left[\frac{p}{q}\right]$.

Proof. Choose $\varepsilon > 0$ such that $\varepsilon < \left[\frac{p}{q}\right] + 1 - \frac{p}{q}$, implying left $\left[\frac{p+\varepsilon}{q}\right] = \left[\frac{p}{q}\right]$. Define $u_\varepsilon(z) = \left(\frac{1}{1-z} - \frac{1}{2}\right)^{\frac{1}{p+\varepsilon}}$. Now $u_\varepsilon \in H^p(\mathbb{D})$ because of Lemma 5.39 and that, for every $z \in \overline{\mathbb{D}}$,

$$|u_\varepsilon(z)| = \frac{1}{2} \left|\frac{1+z}{1-z}\right|^{\frac{p}{p+\varepsilon}} \leqslant \frac{1}{|1-z|}^{\frac{p}{p+\varepsilon}}.$$

By hypothesis $f \circ u_\varepsilon \in H^q(\mathbb{D})$ so by Lemma 5.40 there exists some $c_0 > 0$ such that $|f(u_\varepsilon(z))| \leqslant c_0(1 - |z|)^{-\frac{1}{q}}$. Write $\omega_1 = u_\varepsilon(z)$, so $z = \frac{\omega_1^{p+\varepsilon} - \frac{1}{2}}{\omega_1^{p+\varepsilon} + \frac{1}{2}}$. Therefore

$$|f(\omega_1)| \leqslant c_0 \left(1 - \left|\frac{\omega_1^{p+\varepsilon} - \frac{1}{2}}{\omega_1^{p+\varepsilon} + \frac{1}{2}}\right|\right)^{-\frac{1}{q}} = \frac{c_0 |\omega_1^{p+\varepsilon} + \frac{1}{2}|^{\frac{1}{q}}}{(|\omega_1^{p+\varepsilon} + \frac{1}{2}| - |\omega_1^{p+\varepsilon} - \frac{1}{2}|)^{\frac{1}{q}}}.$$

Consider the set $\mathbb{C} \setminus B(0, \frac{1}{2})$ covered by a finite amount of angular sectors of the form $S_n := \{z \in \mathbb{C} : \frac{-\pi+2\pi n}{4(p+\varepsilon)} \leqslant \text{Arg}(z) \leqslant \frac{\pi+2\pi n}{4(p+\varepsilon)}\}_{n=0}^N$, where $N = [4(p+\varepsilon)]$. Define the function $g : \mathbb{C} \setminus B(0, \frac{1}{2}) \to \mathbb{C}$ as $g(z) = |z + \frac{1}{2}| - |z - \frac{1}{2}|$, which satisfies $|g(z)| \leqslant \left|z + \frac{1}{2} - z + \frac{1}{2}\right| = 1$ for every $z \in \mathbb{C} \setminus B(0, \frac{1}{2})$. Moreover, if $z = a + bi \in \mathbb{C} \setminus B(0, \frac{1}{2})$ and

$|\operatorname{Arg}(z)| \leqslant \frac{\pi}{4}$, that is, $b^2 \leqslant a^2$ and $a \geqslant \frac{1}{4}$, we have that

$$\begin{aligned}
\frac{1}{|g(z)|} &= \frac{1}{\sqrt{(a+\frac{1}{2})^2+b^2}-\sqrt{(a-\frac{1}{2})^2+b^2}} \\
&= \frac{\sqrt{(a+\frac{1}{2})^2+b^2}+\sqrt{(a-\frac{1}{2})^2+b^2}}{2a} \\
&\leqslant \frac{\sqrt{(a+\frac{1}{2})^2+a^2}+\sqrt{(a-\frac{1}{2})^2+a^2}}{2a} \\
&= \frac{\sqrt{(\sqrt{2}a+\frac{1}{2\sqrt{2}})^2+\frac{1}{8}}+\sqrt{(\sqrt{2}a-\frac{1}{2\sqrt{2}})^2+\frac{1}{8}}}{2a} \\
&\leqslant \frac{\sqrt{(\sqrt{2}a+\frac{1}{2\sqrt{2}})^2}+\sqrt{\frac{1}{8}}+\sqrt{(\sqrt{2}a-\frac{1}{2\sqrt{2}})^2}+\sqrt{\frac{1}{8}}}{2a} \\
&= \sqrt{2}+\frac{1}{2\sqrt{2}a} \leqslant 2\sqrt{2}.
\end{aligned}$$

Therefore for $\omega_1 \in \mathbb{C} \setminus B(0,\frac{1}{2}) \cap S_1$,

$$\begin{aligned}
|f(\omega_1)| &\leqslant c_0 2\sqrt{2}|\omega_1^{p+\varepsilon}+\frac{1}{2}|^{\frac{1}{q}} \leqslant c_0 2\sqrt{2}(|\omega_1|^{\frac{p+\varepsilon}{q}}+2^{-\frac{1}{q}}) \\
&\leqslant c_0 2\sqrt{2}|\omega_1|^{\frac{p+\varepsilon}{q}}(1+|\omega_1|^{-\frac{p+\varepsilon}{q}}2^{-\frac{1}{q}}) \leqslant c_0 2^{\frac{p+\varepsilon-1}{q}+\frac{3}{2}}|\omega_1|^{\frac{p+\varepsilon}{q}}
\end{aligned}$$

Now for $\omega_n = e^{\frac{2\pi n i}{4(p+\varepsilon)}}\omega_1$, $z = \frac{\omega_1^{p+\varepsilon}-\frac{1}{2}}{\omega_1^{p+\varepsilon}+\frac{1}{2}} = \frac{\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}-\frac{1}{2}}{\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}+\frac{1}{2}}$, and if $\omega_n \in \mathbb{C} \setminus B(0,\frac{1}{2}) \cap S_n$,

$$\begin{aligned}
|f(\omega_n)| &\leqslant c_0\left(1-\left|\frac{\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}-\frac{1}{2}}{\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}+\frac{1}{2}}\right|\right)^{-\frac{1}{q}} \\
&= \frac{c_0|\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}+\frac{1}{2}|^{\frac{1}{q}}}{(|\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}+\frac{1}{2}|-|\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}-\frac{1}{2}|)^{\frac{1}{q}}} \\
&\leqslant c_0 2^{\frac{3}{2}}|\omega_n^{p+\varepsilon}e^{-\frac{\pi n i}{2}}+\frac{1}{2}|^{\frac{1}{q}} \leqslant c_0 2^{\frac{p+\varepsilon-1}{q}+\frac{3}{2}}|\omega_n|^{\frac{p+\varepsilon}{q}}.
\end{aligned}$$

Since we have covered $\mathbb{C} \setminus B(0, \frac{1}{2})$ with the sectors S_n, if $|\omega| \geqslant \frac{1}{2}$ then $|f(\omega)| \leqslant c_0 2^{\frac{p+\varepsilon-1}{q}+\frac{3}{2}} |\omega|^{\frac{p+\varepsilon}{q}}$, so f is a polynomial of degree at most $\left[\frac{p+\varepsilon}{q}\right] = \left[\frac{p}{q}\right]$. □

Corollary 5.42. *An entire function φ defines a superposition operator $S_\varphi : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ if and only if φ is a polynomial of degree at most $\left[\frac{p}{q}\right]$.*

Proof. The necessary condition is Theorem 5.41, while the sufficient one is a consequence of the trivial fact that $f \in H^p(\mathbb{D})$ and $n \leqslant \frac{p}{q}$ imply $f^n \in H^q(\mathbb{D})$. □

Remark 5.43. As a matter of fact, the assumption of φ to be entire can be dropped. If $\varphi : \mathbb{C} \to \mathbb{C}$ induces a superposition operator S_φ mapping $H^p(\mathbb{D})$ into $H^q(\mathbb{D})$, then, since $f(z) = rz$ belongs to $H^p(\mathbb{D})$ for all $r > 0$, the function $z \mapsto \varphi(rz)$ is analytic in $\mathbb{D}$ for all $r > 0$, hence φ is entire.

We can extend this characterization to the spaces $H^p(\mathbb{T})$ in the following way.

Theorem 5.44. *An entire function φ defines a superposition operator $S_\varphi : H^p(\mathbb{T}) \to H^q(\mathbb{T})$ if and only if φ is a polynomial of degree at most $\left[\frac{p}{q}\right]$.*

Proof. First, if φ is a polynomial of degree at most $\left[\frac{p}{q}\right]$ then it defines a superposition operator on $\tilde{S}_\varphi : H^p(\mathbb{D}) \to H^q(\mathbb{D})$. Now, if $f \in H^p(\mathbb{T})$ then f is defined by the boundary values of a function $\tilde{f} \in H^p(\mathbb{D})$, and given the continuity of φ, $\varphi \circ f$ is defined by the boundary values of $\varphi \circ \tilde{f} \in H^q(\mathbb{D})$, so $\varphi \circ f \in H^q(\mathbb{T})$. On the other hand, if an entire function φ defines a superposition operator $S_\varphi : H^p(\mathbb{T}) \to H^q(\mathbb{T})$ and $f \in H^p(\mathbb{T})$, then f is defined by the boundary values of $\tilde{f} \in H^p(\mathbb{D})$. Since φ is continuous, the boundary limits of $\varphi \circ \tilde{f}$ are well defined wherever the boundary limits of $\tilde{f}$ are well defined, and coincide with $\varphi \circ f \in H^q(\mathbb{T})$, so *varphi* $\circ \tilde{f} \in H^q(\mathbb{D})$ and φ therefore defines a composition operator $\tilde{S}_\varphi : H^p(\mathbb{D}) \to H^q(\mathbb{D})$, implying that it is a polynomial of degree at most $\left[\frac{p}{q}\right]$.

Moreover, $\varphi \circ f \in H^q(\mathbb{T})$, so it is defined by the boundary values of a certain function $g \in H^q(\mathbb{D})$. □

Recall that the Bohr transform induces an isometric isomorphism from the Hardy space $H^p(\mathbb{T}^\infty)$ onto $\mathcal{H}^p$. The subspace of $\mathcal{H}^p$ consisting of Dirichlet series of the form $\sum_{k=1}^{\infty} a_{2^k} \frac{1}{(2^k)^s}$ is isometrically isomoporphic to $H^p(\mathbb{T}) \hookrightarrow H^p(\mathbb{T}^\infty)$. Let us recall also that $H^p(\mathbb{T})$ is isometrically isomorphic to $H^p(\mathbb{D})$. We are going to see that the superposition operators on $\mathcal{H}^p$ are in fact the same ones as on $H^p(\mathbb{D})$.

Theorem 5.45. *A function $\varphi : \mathbb{C} \to \mathbb{C}$ defines a superposition operator $S_\varphi : \mathcal{H}^p \to \mathcal{H}^q$ if and only if φ is a polynomial of degree at most $\lfloor \frac{p}{q} \rfloor$.*

Proof. We first see that if φ is a polynomial of degree $N \leqslant \lfloor \frac{p}{q} \rfloor$ then S_φ defines a superposition operator. Take first $\varphi_k(w) = w^k$. By Young's inequality we have $a^n \leqslant \frac{n}{p} a^p$ whenever $a > 0$ and $\frac{n}{p} < 1$. Then, since $\frac{kq}{p} \leqslant 1$ for all $1 \leqslant k \leqslant N$, given a Dirichlet polynomial P, we get

$$\begin{aligned}
\|\varphi_k(P)\|_q^q &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi_k(P(it))|^q dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|(P(it))^k\right|^q dt \\
&\leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{kq}{p} |P(it)|^p dt \leqslant \|P\|_p^p,
\end{aligned}$$

so $\varphi_k(P) \in \mathcal{H}^q$. Now, if $\varphi(w) = \sum_{k=0}^{N} b_k w^k$, then $\varphi(P) = \sum_{k=0}^{N} b_k \varphi_k(P) \in \mathcal{H}^q$ for every Dirichlet polynomial $P \in \mathcal{H}^p$. Using that Dirichlet polynomials are dense in $\mathcal{H}^p$ and $\mathcal{H}^q$ with the respective norms is enough to get the desired result.

On the other hand, suppose now that φ defines a superposition operator $S_\varphi : \mathcal{H}^p \to \mathcal{H}^q$. First of all, since $s \mapsto \varphi\left(\frac{\sqrt{2}R}{2^{-s}}\right)$ is holomorphic in $\mathbb{C}_{1/2}$, then taking two different branches of the complex logarithm we get that φ is holomorphic in $B(0,R) \setminus \{0\}$ for every $R > 0$. Moreover, $s \mapsto \varphi\left(\frac{\sqrt{2}R}{2^{-s}}\right)$ is an absolutely convergent Dirichlet series, and therefore is bounded in $\mathbb{C}_\sigma$ for every $\sigma > 1$. Since 0 is an isolated singularity of φ we get that φ is entire. Now let $f \in H^p(\mathbb{D})$, $\mathcal{B}(f) \in \mathcal{H}^p$ and $\varphi \circ \mathcal{B}(f) \in \mathcal{H}^q$. Since the Taylor series of φ converges absolutely on $\mathbb{C}$, the Taylor series of f converges absolutely on $\mathbb{D}$ and the Dirichlet series $\mathcal{B}(f)(s)$ converges absolutely in $\mathbb{C}_\sigma$ for any $\sigma > 1$ then there exists $\{a_n\}_n$ such that $(\varphi \circ \mathcal{B}(f))(s) = \sum_{n=0}^\infty \frac{a_n}{(2^n)^s}$ for every $s \in \mathbb{C}_\sigma$, $\sigma > 1$, and also $(\varphi \circ f)(z) = \sum_{n=0}^\infty a_n z^n$ for every $z \in \mathbb{D}$. Hence $\varphi \circ f = \mathcal{B}^{-1}(\varphi \circ \mathcal{B}(f)) \in H^q(\mathbb{D})$, thus φ defines a superposition operator $S_\varphi : H^p(\mathbb{D}) \to H^q(\mathbb{D})$ and consequently it is a polynomial of degree at most $\lfloor \frac{p}{q} \rfloor$. □ □

Remark 5.46. It is interesting to note here the differences between the spaces $\mathcal{H}^p$ and $\mathcal{H}_\infty(\mathbb{C}_+)$ regarding superposition operators. While only polynomials of a certain degree will define superposition operators $S_\varphi : \mathcal{H}^p \to \mathcal{H}^q$, the fact that $\mathcal{H}_\infty(\mathbb{C}_+)$ is an algebra gives trivially that any polynomial defines a superposition operator on $\mathcal{H}_\infty(\mathbb{C}_+)$. This leads easily to the fact that any entire function defines a superposition operator. Indeed, since entire functions are uniformly approximated by their Taylor series on all compact sets, in particular on the image of any function of $\mathcal{H}^\infty(\mathbb{C}_+)$, if $\varphi(z) = \sum_{n=1}^\infty a_n z^n$ and $f \in \mathcal{H}_\infty(\mathbb{C}_+)$, then $S_\varphi(f)$ is the uniform limit of $S_{\varphi_N}(f) \in \mathcal{H}_\infty(\mathbb{C}_+)$ where $\varphi_N = \sum_{n=1}^N a_n z^n$. Using that $\mathcal{H}_\infty(\mathbb{C}_+)$ is complete is enough to get that $S_\varphi : \mathcal{H}_\infty(\mathbb{C}_+) \to \mathcal{H}_\infty(\mathbb{C}_+)$ is a superposition operator.

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