Renormalization of Quantum Fields in Curved Spacetime

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VNIVERSITAT DE VALÈNCIA

Programa de Doctorat en Física

Departament de Física Teòrica

Doctoral Thesis

May 2021

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Abstract

Quantum fields in curved spacetime undergo fluctuations that produce non-vanishing vacuum expectation values of the stress-energy tensor, i.e., energy can be generated due to the gravitational field. The same happens for other type of background fields like gauge or scalars. This effect plays an important role in the early Universe, in astrophysical compact objects, and in strong electromagnetic phenomena.

However, the computation of the stress-energy tensor, among others, is a highly nontrivial issue. In particular, non-trivial divergences appear when computing expectation values of local observables. The objective of my thesis is to tackle this issue by studying regularization and renormalization mechanisms for quantum fields in curved spacetime, especially in Friedman-Robertson-Walker-Lemaitre spacetimes.

On the one hand, this will be done by extending adiabatic regularization to include interacting fields (scalar, gauge fields). On the other hand, running of the coupling constant by introducing a mass parameter will be computed for general curved spacetime and a subtraction scheme, that naturally incorporates decoupling for higher massive fields will be obtained. A particular application will be given in the context of the cosmological constant problem.

Declaration

I declare that the work presented in this thesis was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Program.

The contents of chapters two, three, four and five are based on the following papers. Credit copyright American Physical Society and Elsevier.

- *"Adiabatic regularization with a Yukawa interaction"*; A. del Rio, A. Ferreiro, J. Navarro-Salas, and F. Torrenti, *Phys. Rev. D* **95**, 105003 (2017).
- *"Pair creation in electric fields, anomalies, and renormalization of the electric current"*; A. Ferreiro and J. Navarro-Salas, *Phys. Rev. D* **97**, 125012 (2018).
- *"Adiabatic expansions for Dirac fields, renormalization, and anomalies"*; J. F. G. Barbero, A. Ferreiro, J. Navarro-Salas, and E. J. S. Villaseñor, *Phys. Rev. D* **98**, 025016 (2018).

- *"Role of gravity in the pair creation induced by electric fields"*; A. Ferreiro, J. Navarro-Salas, and S. Pla, *Phys. Rev. D* **98**, 045015 (2018).
- *"Running couplings from adiabatic regularization"*; A. Ferreiro and J. Navarro-Salas, *Phys. Lett. B* **792**, 81 (2019).
- *"Breaking of adiabatic invariance in the creation of particles by electromagnetic fields"*; P. Beltran-Palau, A. Ferreiro, J. Navarro-Salas, and S. Pla, *Phys. Rev. D* **100** 085014 (2019).
- *"R-summed form of adiabatic expansions in curved spacetime"*; A. Ferreiro, J. Navarro-Salas and S. Pla, *Phys. Rev. D* **101**, 105011 (2020).
- *"Running gravitational couplings, decoupling, and curved spacetime renormalization"*; A. Ferreiro and J. Navarro-Salas, *Phys. Rev. D* **102**, 045021 (2020).

The text presented here should be understood as a dissertation submitted to the University of Valencia as required to obtain the degree of Doctor of Philosophy in Physics.

Except where indicated by specific reference in the text, this is the candidate's own work, done in collaboration with, and/or with the assistance of, the candidate's supervisors and collaborators. Any views expressed in the thesis are those of the author.

Valencia, May 7, 2021
Antonio Ferreiro

Acknowledgements

Este viaje no habría sido posible sin el apoyo, tanto científico como emocional de muchas personas que me han acompañado a lo largo de estos cuatro años. En primer lugar, quería agradecer a mis compañeros Francisco Torrentí, Adrián Del Rio, Sergi Nadal y Pau Beltrán. En especial a Silvia Pla, sin los viajes y las aventuras entre congresos y charlas, esto no habría sido lo mismo. También agradezco a Fernando Barbero y Eduardo Villaseñor, de los cuales he aprendido mucho y deseo, con todos ellos, poder continuar nuestras colaboraciones y amistades en mis proyectos futuros.

Quiero dar las gracias también a mis amigas, Cynthia, Gemma y Arantxa por sacarme una sonrisa, tan necesario en los malos momentos de este trayecto. También a Bart y a Jorge, por su paciencia y su amistad. En especial, a Iván, que desde aquellas noches estudiando Cálculo II hasta hoy, siendo ambos doctores (¡Quién nos lo iba a decir!), hemos vivido experiencias impagables. Doy las gracias también a Javi Roque por enseñarme a ver la ciencia desde los ojos de un niño.

Quiero agradecer a mis padres Isabel y Carlos por su apoyo incondicional, su sacrificio y por haberme convertido en la persona

que soy hoy. Sin ellos, nada de esto hubiera sido posible. A mi padre, por enseñarme el amor por la ciencia, la curiosidad, la paciencia y a arriesgarme para conseguir mis metas. A mi madre, por haberme enseñado a tener pasión por mi trabajo, el valor del esfuerzo, la convivencia y el respeto por los demás. En definitiva, a ambos por darme unos valores firmes para mantener allá por donde pise.

Finalmente, doy las gracias a José Navarro Salas, *Pepe*, por ser mi guía durante este camino. Desde que realicé, allá por el 2014, mi Trabajo Fin de Grado, Pepe ha estado presente para formarme no sólo en el contexto de la física sino también como investigador autónomo. Puede decir, sin duda alguna, que Pepe es la personificación de lo que considero que debe ser un científico y de cómo se debe hacer buena ciencia. Por todo ello, quiero agradecerle una vez más por haberme dado la oportunidad de compartir con él esta experiencia.

Resumen de la Tesis

Motivación y Contexto

Una de las mayores riquezas de la física fundamental es su relativa nitidez a la hora de clasificar fenómenos físicos en sus correspondientes teorías físicas. Si queremos estudiar la interacción de una partícula cargada en un campo electromagnético, nadie discutiría que la mejor forma es hacer uso de la Electrodinámica Cuántica (QED). Lo mismo ocurre con el estudio de interacciones nucleares o desintegraciones de partículas . Por otro lado, si deseamos estudiar la evolución del Universo, el colapso de una estrella o la propagación de ondas gravitacionales, la Teoría de la Relatividad General sería la teoría ideal para describir dichos fenómenos. En gran aproximación, el mundo observable parece dividirse en las descripciones de dos teorías fundamentales: la Teoría Cuántica de Campos y el Modelo Estándar y la Teoría de la Relatividad General.

Ahora bien, a pesar del enorme éxito cosechado por ambas teorías en cuanto a explicaciones y predicciones de nuevos fenómenos, quedan importantes cuestiones sin resolver. En primer lugar, existen observaciones empíricas que no pueden ser explicadas por ninguna de las dos teorías: la *materia oscura*, la *masa de los neutrinos* o el origen de la distribución del Fondo Cósmico de Microondas. En segundo lugar, la búsqueda de una teoría de la gravedad cuántica o de una teoría más fundamental que unifique ambas teorías surge directamente de la necesidad de acoplar de forma consistente la materia, cuantizada, a la gravedad tal y como obliga la Teoría de la Relatividad General.

Ante esta última cuestión, varias propuestas han ido adquiriendo forma durante las últimas décadas (ver [89] para un visión general). Sin embargo, todas se enfrentan a obstáculos tanto teóricos como prácticos. En especial, su conexión con posibles predicciones observables hoy en día se ve obstaculizada debido a su complejo formalismo matemático. Una propuesta intermedia consiste en aprovechar la formulación de Teoría Cuántica de Campos en espacio plano y generalizarla a espacios curvos. En efecto, dado que el obstáculo cualitativo de una teoría más fundamental es la cuantización de la gravedad, se puede *aparcar* momentáneamente el problema usando uno de los enfoques más fructíferos de la física contemporánea: las teorías de campos efectivas. Estas consisten en asumir que una teoría, en este caso la teoría cuántica de campos en espacio curvo es una descripción válida a escalas de energía (o longitud) mucho menor (o mayor) que cierta escala, en este caso la masa de Planck (o longitud de Planck):

$$M_{\mathrm{P}} = G^{-1/2}\hbar^{1/2}c^{1/2} \approx 0.2 \times 10^{-5}\mathrm{g} \tag{1}$$

$$l_{\mathrm{P}} = G^{1/2}\hbar^{1/2}c^{-3/2} \approx 1.4 \times 10^{-33}\mathrm{cm}\ , \tag{2}$$

donde usando unidades de $c = \hbar = 1$ implica $M_{\mathrm{P}} \approx 10^{19}$ GeV (usaremos a partir de ahora $c = \hbar = 1$). Excepto casos particulares como el interior de un agujero negro o el origen del Universo en el modelo ΛCDM no existen demasiados fenómenos que alcancen este tipo de energías y por lo tanto es una buena suposición asumir que la Teoría Cuántica de Campos en Espacios Curvos (QFTCS) pueda describir la mayor parte de fenómenos detectables hoy en día en el Universo.

El estudio de un campo cuántico en presencia de un campo clásico externo ya fue considerado previamente en el caso de un campo electromagnético antes de dar paso a una teoría cuántica de la electrodinámica. Uno de los efectos de esta teoría semiclásica es la producción espontánea de partículas (p.ej. un par electrón-positrón) debido a un campo electromagnético clásico, conocido como mecanismo de (Sauter-Heisenberg-Euler-)Schwinger [97, 99]. Análogamente es esperable que un campo gravitatorio también produzca partículas. En efecto, uno de los primeros resultados en QFTCS fue, a partir del uso novedoso de transformaciones de Bogoliubov, la producción de partículas en universos en expansión a finales de los años 60 [78] y, más tarde, en el contexto de colapso gravitatorio y agujeros negros [57]. En los modelos inflacionarios durante los primeros instantes del Universo, este efecto estaría detrás de las anisotropías que se observan hoy en día en el Fondo Cósmico de Microondas. Así mismo, se espera también que haya sido determinante para la formación de materia (electrones, fotones, etc.) durante el período cósmico conocido como Recalentamiento (o Reheating).

Otra consecuencia de la cuantización de un campo es la estructura no trivial del estado de vacío que, entre otras, deja su huella como *polarización* del vacío y sobre la cual se asientan importantes efectos como el efecto Lamb y el momento magnético del electrón o el muón. En el caso de un campo gravitatorio, la estructura del estado de vacío estaría codificada en el tensor momento-energía no nulo. Cuantificar la densidad de energía del vacío ha sido hasta ahora una tarea bastante esquiva, pero es necesaria para entender correctamente el efecto de la presencia de campos cuánticos en espacio-tiempo curvo. En particular, la energía de vacío podría actuar como fuente en universos en expansión dando lugar a una posible contribución a la expansión acelerada del Universo [88].

Una de las magnitudes más importantes en QFTCS es el valor esperado de vacío del tensor energía-momento $\langle 0|T_{ab}|0\rangle$. Este contiene información tanto de la producción de partículas como los efectos de polarización del vacío. Además, siguiendo la Teoría de la Relatividad General, esta magnitud tiene que contribuir en la dinámica del espacio-tiempo, a través de la ecuación semiclásica de Einstein[1]

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G \langle 0|T_{ab}|0\rangle. \tag{3}$$

La construcción de este tipo de objetos no está exenta de problemas. Las infinitas, no equivalentes, formas de seleccionar el estado de vacío, los límites de la aproximación semiclásica y las soluciones generales de las ecuaciones de los campos son algunos de ellos

[1] A lo largo de la tesis seguiremos las covenciones de signos de [86].

(para un estudio más detallado nos referimos a las referencias [15, 52, 64, 86, 112]).

Un problema adicional está relacionado con las divergencias que aparecen al calcular magnitudes como $\langle 0|T_{ab}|0\rangle$. Las divergencias en teorías de campos en espacio plano son ya conocidas y diversos métodos han sido diseñados para superarlas [91, 98]. En general, estos métodos consisten en aislar las divergencias de las diversas magnitudes a través de la *regularización* para posteriormente definir consistentemente los términos del Lagrangiano de tal forma que se obtenga un Lagrangiano *renormalizado* que resulte en cantidades finitas. De manera esquemática podemos construir el tensor energía-momento renormalizado como

$$\langle 0|T_{ab}|0\rangle_{\text{ren}} = \langle 0|T_{ab}|0\rangle - T_{ab}^{\text{sub}}, \tag{4}$$

de tal forma que ambas cantidades del lado derecho cancelen sus divergencias dando como resultado una cantidad finita. En QFTCS también se han ido construyendo distintos métodos de regularización y renormalización [15, 86] dando como resultado distintos T_{ab}^{sub}. Sin embargo, existen ciertas restricciones sobre los términos de sustracción [112]. Deben ser compatibles con la conservación covariante del tensor energía-momento $\nabla^a \langle 0|T_{ab}|0\rangle_{\text{ren}} = 0$ y construirse de manera local y geométrica. Finalmente, cabría exigir tener solamente un número finito de términos de sustracción, imitando el criterio de renormalizabilidad usual. Una consecuencia de estas exigencias es que ciertos resultados especialmente significativos, como las anomalías conformes, resultan ser esencialmente independientes del método particular de renormalización.

Durante la tesis, presentaremos algunos métodos de regularización ampliamente usados en la teoría cuántica de espacios curvos como regularización dimensional o regularización adiabática para el caso particular de métricas de Friedman-Lemaitre-Robertson-Walker. Antes de continuar, es importante matizar ciertos términos usados en este contexto. Estrictamente hablando, nos referimos a un método de regularización a un método que introduzca un regulador ν que en un cierto límite, normalmente $\nu \to \infty$ hace divergir la magnitud calculada. De esta forma se pueden aislar las divergencias de tal forma que se pueda construir un método o *esquema* de sustracción. Regularización adiabática obtiene un esquema de substracción sin necesidad de un regulador. Otro método conocido que usaremos es el esquema de sustracción DeWitt-Schwinger. Estos métodos se han estudiado en detalle [15, 86] asegurando los requisitos antes explicados, demostrando la equivalencia entre uno y otro y obteniendo resultados explícitos en métricas específicas relevantes en astrofísica y cosmología.

La mayor parte de resultados de regularización adiabática se han desarrollado para campos escalares libres [86] y sólo recientemente para campos de Dirac [31, 72, 73]. Uno de los objetivos de esta tesis será extender estos resultados para incluir campos escalares y Dirac en interacción con otros campos escalares clásicos y electromagnéticos. Ambas interacciones aparecen en diversos escenarios físicos relevantes en cosmología. En efecto, la mayor parte de modelos apuntan a la existencia de al menos un campo escalar como fuente del período inflacionario al comienzo del Universo. Este sería responsable tanto de la expansión acelerada, de las anisotropías observadas en el fondo cósmico como de la producción de materia observable durante Reheating [8, 55, 56, 70, 71]. Otro campo escalar, experimentalmente verificado, es el campo de Higgs que tiene un papel importante en la transición Electrodébil en los primeros instantes del Universo.

Campos electromagnéticos también pueden generar pares de partículas cargadas a partir del mecanismo (Sauter-Heisenberg-Euler-)Schwinger [97, 99]. Para poder tener una señal de este efecto en el laboratorio, la opción más eficiente requeriría alcanzar un campo eléctrico crítico y una escala de intensidad [41] de

$$\mathcal{E}_c = \frac{m_e^2}{e} \approx 10^{16}\mathrm{V/cm} \qquad I_c = \frac{\mathcal{E}_c^2}{8\pi} \approx 4 \times 10^{29}\mathrm{W/cm}^2 \,. \tag{5}$$

Láseres tradicionales no alcanzan estas escalas, lo cual explica que esta producción de partículas no haya sido observada. Avances recientes [1–3, 16, 33, 38, 40, 60] sugieren la posibilidad de alcanzar este tipo de efecto en el Extreme Light Infrastructure (ELI) [41]. Otro laboratorio potencial puede venir del campo de la cosmología y la astrofísica. En efecto, estrellas de neutrones altamente magnetizadas [96] y producción de campos electromagnéticos durante el Universo temprano [66] podrían alcanzar este tipo de escalas. Esta última posee un gran interés puesto que es uno de los posibles orígenes de los recientes descubrimientos de campos magnéticos a escala cosmológica [42].

La regularización adiabática, a pesar de restringirse a métricas particulares, es muy eficiente para cálculos numéricos. Esto es esencial no sólo para poder cuantificar, en escenarios físicamente motivados, la producción de partículas y la polarización de vacío, sino también para calcular la respuesta de estos fenómenos en el campo clásico que los produce, proceso conocido como *backreaction*.

La segunda parte de la tesis se centra en estudiar la parte finita que sobrevive a la renormalización y en especial a la dependencia de las constantes de acople de una escala arbitraria μ. Regularización dimensional, ampliamente utilizada en *scattering* de partículas en física de altas energías, proporciona una arbitrariedad a la hora de seleccionar un esquema de substracción de contratérminos particular, codificado en un parámetro de dimensiones de energía μ.

La introducción de este parámetro es bastante práctica para calcular cierto tipo de magnitudes. Así por ejemplo se utiliza con frecuencia el esquema de *Minimal Subtraction (MS)* en lugar del esquema *on-shell*. En este último ni los acoples ni las magnitudes tienen una dependencia explícita en μ y se relaciona directamente con cantidades físicas (masa, carga, etc.). MS es muy útil en Cromodinámica Cuántica (QCD) donde partículas como los quarks no tienen estados asintóticos definidos y las condiciones de on-shell no son las más adecuadas [98]. Dado que los resultados físicos no deben depender del esquema de renormalización que se use (on-shell, MS, etc.), la invariancia en μ resulta en las ecuaciones del grupo de renormalización, que permite predecir el comportamiento de las teorías en ciertos límites de energía. En efecto, dos resultados clásicos de la Teoría Cuántica de Campos en espacio plano son las funciones beta de las cargas de QED y QCD:

$$\beta^{\mathrm{QED}} \equiv \frac{de}{\mathrm{d}\log\mu} = \frac{e^3}{12\pi^2}\ ; \quad \beta^{\mathrm{QCD}} \equiv \frac{dg}{\mathrm{d}\log\mu} = -\left(11 - \frac{2n_f}{3}\right)\frac{g^3}{16\pi^2} \quad (6)$$

donde n_f es el número de sabores, y e y g los acoples de interacción de ambas teorías. Estas expresiones permiten inferir el comportamiento a altas energías de ambas teorías: la primera aumenta su carga efectiva y se hace no perturbativa y la segunda se vuelve más débil, y tiene *libertad asintótica*.

Uno de los resultados más importantes de QFTCS es que es *renormalizable*, es decir, que se necesita un número finito de términos en el Lagrangiano para absorber las divergencias. En efecto, la ecuación de Einstein semiclásica (renormalizada) sería de la forma [15]

$$\kappa^2 G_{ab} + \Lambda g_{ab} + \alpha^{(1)} H_{ab} + \beta^{(2)} H_{ab} = -\langle 0|T_{ab}|0\rangle_{\mathrm{ren}}\,. \quad (7)$$

Regularización dimensional y MS se pueden aplicar para calcular $\langle 0|T_{ab}|0\rangle_{\mathrm{ren}}$ y de esa forma obtener una dependencia en μ tanto para $\langle 0|T_{ab}|0\rangle_{\mathrm{ren}}$ como para las distintas constantes κ, Λ, α y β, imitando el *running* análogo a los casos de QED y QCD. Sin embargo, la interpretación física no es tan sencilla y dedicaremos parte de la tesis a analizarlo con más detalle.

Otro aspecto importante en la visión moderna de las teorías cuánticas de campos está relacionado con entender las teorías accesibles hasta ahora como teorías efectivas, en un cierto límite de energías, de otras teorías más fundamentales. Un ejemplo de teoría efectiva es el Lagrangiano de Euler-Heisenberg

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^4}{360\pi^2 m^4}\left[\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right] + \ldots, \quad (8)$$

donde el efecto del campo del electrón está codificado en términos de orden superior del campo electromagnético y en potencias inversas de la masa del electrón. Esta teoría efectiva permite estudiar de una manera simple efectos como el scatterring de fotones.

Un resultado fundamental en este contexto es el desacoplamiento de campos masivos a bajas energías. Esto significa que las contribuciones de los campos cuánticos son despreciables cuando la escala de energía del sistema es mucho menor que la masa del campo. Así por ejemplo, no necesitamos saber física de la masa del top para estudiar el átomo de Hidrógeno.

Sabemos por el teorema de Appelquist-Carazzone [7] que un esquema de sustraccion dependiente de la masa en teoría de campos perturbativa produce desacoplamiento. En efecto, tanto las funciones beta como la magnitud calculada tendería a cero en el caso $m \to \infty$. Sin embargo, usando MS no se hace explícito este desacoplamiento [74]. En QED, por ejemplo, esto se resuelve cambiando al esquema de substracción de momentos (MOM). Una cuestión importante es recuperar desacoplamiento para el caso de QFTCS.

Finalmente, es bien conocido que la renormalización en espacios curvos produce resultados en tensión con los datos observacionales. En efecto, tanto un regulador tipo cutoff como MS producen contribuciones a la constante cosmológica muchos órdenes superiores al valor observado. Esta discrepancia se conoce como *problema de la constante cosmológica* [22,113]. Por lo tanto, es fundamental comprender correctamente la renormalización en QFTCS para entender la lógica de esta aparente o real discrepancia con las observaciones.

Resultados y Conclusiones

Hemos dividido los resultados obtenidos durante la tesis en dos partes: regularización adiabática en teorías con interacciones y renormalización en espacios curvos generales.

En primer lugar, se generalizó el método de regularización adiabática de campos escalares cuánticos para incluir interacciones con un campo eléctrico clásico. Para ello, se extendió el método usual de expansión tipo WKB [86] para incluir la interacción con el potencial $A(t)$ definido a partir de $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ [46]. Un punto crucial fue darse cuenta que para que el método de regularización fuese consistente con la conservación del tensor momento energía era necesario que el potencial eléctrico A fuese de orden adiabático uno, análogamente a $\dot{a}$ en el caso de gravedad [49]. Esto es un resultado destacado puesto que había sido obviado en la mayor parte de la literatura de regularización adiabática [11,59,69,105].

Se desarrolló tanto la regularización del tensor momento-energía como del vector de corriente eléctrica y se obtuvo la anomalía de traza esperada en este caso [46]. También se generalizó el método de regularización adiabática para incluir el parámetro de renormalización μ [47]. Un resultado determinante para que la regularización adiabática con μ fuese consistente es que en primer lugar no generase nuevas divergencias en comparación con el método usual de regularización adiabática [86], es decir, el caso $\mu = 0$ y que la diferencia entre dos posibles esquemas de sustracción, es decir, dos regularizaciones con parámetros μ_1 y μ_2 solo se diferenciasen en finitos términos covariantes,

$$\langle 0|T_{ab}|0\rangle_{\text{ren}}^{\mu_1} - \langle 0|T_{ab}|0\rangle_{\text{ren}}^{\mu_2} = a + bG_{ab} + c\,^{(1)}H_{ab} + dT_{ab}^{EM} \,. \tag{9}$$

Aplicando la invariancia con respecto a μ de las ecuación semiclásicas de Einstein (7), se puede obtener la dependencia de las constantes de acople con μ, codificadas en las funciones beta $\beta_O = \mu \frac{d}{d\mu} O$. Para el caso de las constantes dimensionales obtenemos

$$\beta_\Lambda = \frac{1}{16\pi^2} \frac{\mu^6}{m^2 + \mu^2} \quad \beta_\kappa = \frac{\xi - \frac{1}{6}}{4\pi^2} \frac{\mu^4}{m^2 + \mu^2}, \tag{10}$$

mientras que para las adimensionales

$$\beta_e = \frac{e^3}{48\pi^2} \frac{\mu^2}{\mu^2 + m^2} \quad \beta_\alpha = \frac{\left(\xi - \frac{1}{6}\right)^2}{8\pi^2} \frac{\mu^2}{\mu^2 + m^2} \,. \tag{11}$$

Para el último caso, recuperamos en el límite $\mu^2 \gg m^2$ los resultados estándar de la teoría perturbativa de campos, usando regularización dimensional y MS [101]. Sin embargo, un resultado destacado es que para los casos dimensionales sí obtenemos diferencias con MS

$$\beta_{\Lambda}^{MS} = \frac{m^4}{16\pi^2} \quad \beta_{\kappa}^{MS} = \frac{m^2}{4\pi^2}\left(\frac{1}{6} - \xi\right) . \tag{12}$$

Estos resultados fueron publicados en [46,47,49] y están descritos en más detalle en el capítulo 2 de esta tesis.

En segundo lugar, se extendió la regularización adiabática para campos de Dirac en dos dimensiones interactuando con un campo eléctrico clásico. Para ello, se generó la expansión adiabática adaptando los resultados obtenidos del campo de Dirac libre [31,72,73] para ser consistente con la introducción del campo electromagnético y los resultados del caso previo del campo escalar. Se regularizó tanto la corriente eléctrica como el tensor momento-energía y se obtuvo correctamente la anomalía quiral y de traza.

Se encontró una arbitrariedad a la hora de generar la expansión adiabática ya presente en el caso del campo libre [31,72,73]. Para solucionarlo, se propuso un método alternativo a [31,72,73] para contruir la expansión adiabática que no generaba ninguna ambigüedad.

A partir del resultado obtenido de la anomalía quiral en dos dimensiones, obtuvimos un resultado peculiar: en el caso de un campo sin masa de Dirac, la invariancia adiabática del número de partículas queda rota, generando una corriente eléctrica aún en el caso de un potencial eléctrico que evoluciona adiabáticamente. En efecto, un potencial eléctrico $A(t)$ producirá partículas por muy lenta que sea su variación con respecto al tiempo. Esto es una diferencia importante entre el comportamiento de un campo gravitatorio, que sí respecta invariancia adiabática, con respecto al campo eléctrico. Este resultado se extendió para el caso de cuatro dimensiones con el mismo resultado: en presencia de un campo electromagnético, para un campo de Dirac sin masa, la anomalía quiral rompe la invariancia adiabática, produciendo partículas. Estos resultados fueron publicados en [10, 12, 46] y están descritos en más detalle en el capítulo 3 de esta tesis.

Finalmente, la regularización adiabática del campo de Dirac libre en cuatro dimensiones se extendió también para incluir interacción con un campo escalar clásico de la forma $g_Y \bar{\psi}\psi\phi$. En primer lugar, se asignó al campo escalar el orden adiabático uno en consonancia con su dimensión. Se obtuvo correctamente la regularización del tensor momento-energía y la anomalía conforme. Se discutió este último resultado y su consistencia con otros métodos de regularización. Finalmente, se obtuvo la renormalización de la teoría a través de la introducción de contratérminos en el Lagrangiano inicial. Estos resultados fueron publicados en [29] y están descritos en más detalle en el capítulo 4 de esta tesis. Se obtuvo además el running para las constantes de acoplo usando la regularización adiabática con el parámetro μ descrito anteriormente, un resultado novedoso de esta tesis.

En la segunda parte de la tesis, se extendió el método de subtracción de DeWitt-Schwinger (DS) para incluir un parámetro de renormalización μ, análogo a la expansión adiabática. Para ello, en vez de centrarnos en el valor de expectación del vacío del tensor momento-energía $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$, usamos la acción efectiva S_{eff} a *one* *loop* que contiene toda la dinámica clásica más las correspondientes correcciones cuánticas. La expansión de DeWitt-Schwinger de la acción efectiva permite aislar los términos divergentes, análogamente al caso de regularización adiabática.

Incluyendo el parámetro μ de forma consistente en los términos de sustracción, confirmamos que efectivamente su introducción no genera nuevas divergencias. Además, la diferencia entre regularizar la contribución cuántica a la acción efectiva respecto de dos parámetros distintos μ_1 y μ_2 resulta en

$$\Gamma_{\rm ren}^{\mu_1} - \Gamma_{\rm ren}^{\mu_2} = a + bR + cR^2 + dC_{abcd}C^{abcd} + eF_{ab}F^{ab}. \quad (13)$$

Esta diferencia está formada por términos que tienen que estar presentes en la acción original para ser renormalizable. En el caso de una métrica FLRW recuperaríamos los resultados de regularización adiabática con la introducción de μ. En efecto, para las constantes de acople, usando la invariancia de la acción efectiva con respecto a μ obtenemos las siguiente funciones beta:

$$\beta_e = \frac{e^3}{48\pi^2}\frac{\mu^2}{\mu^2+m^2} \quad \beta_\alpha = \frac{\bar{\xi}^2}{8\pi^2}\frac{\mu^2}{\mu^2+m^2} \quad \beta_\gamma = \frac{-1}{960\pi^2}\frac{\mu^2}{\mu^2+m^2}$$
$$\beta_\Lambda = \frac{1}{16\pi^2}\frac{\mu^6}{m^2+\mu^2} \quad \beta_\kappa = \frac{\bar{\xi}}{4\pi^2}\frac{\mu^4}{m^2+\mu^2}. \quad (14)$$

Otro resultado destacado es que estas funciones beta son consistentes con el desacoplamiento de campos masivos a bajas energías. Sabemos por el teorema de Appelquist-Carazzone [7] que un esquema de regularización dependiente de la masa en teoría de campos perturbativa produce desacoplamiento. En efecto, tanto las funciones beta como la magnitud calculada tendería a cero en el caso $m \to \infty$. Un resultado conocido es que MS no tiene este desacoplamiento [74]. Nuestra versión extendida del método de DeWitt-Schwinger es por lo tanto un esquema compatible con los resultados teorema de Appelquist-Carazzone en espacio curvo. En efecto, observando los resultados (14) concluimos que todas las funciones beta, incluyendo las dimensionales, tienden a cero en el límite $m \to \infty$. Estos resultados fueron publicados en [48] y están descritos de forma más detallada en el capítulo 5.

Una de las aplicaciones de este resultado es estudiar las posibles contribuciones de campos cuánticos a la constante cosmológica. Es bien conocido que métodos de regulador tipo cutoff generan contribuciones con una discrepancia de 120 órdenes de magnitud [113] con respecto a la constante cosmológica observada, mientras que la discrepancia se reduce a 32 órdenes de magnitud si utilizamos Minimal Subtraction [75]. En la tesis, hemos argumentado que ambos esquemas no son métodos prácticos para renormalización en QFTCS. En primer lugar, ya es bien conocido que un regulador tipo cutoff que genera problemas en sí mismo (puesto que no respeta covariancia general [75] y no recupera resultados importantes en QFTCS [62]).

Minimal Subtraction sí es compatible con estos últimos requisitos pero, como ya se ha comentado, no es compatible con el desacople de campos masivos. En teoría perturbativa de campos esto se soluciona *integrating out* los campos masivos [74, 92] y construyendo distintas teorías que contienen solo campos ligeros (con respecto a la escala de validez de la teoría). Para constantes adimensionales como la carga eléctrica la diferencia entre las distintas teorías son correcciones logarítimas. Sin embargo, para la constante cosmológica esta diferencia es del orden de $\sim m^4 \log\left(\frac{\mu^2}{m^2}\right)$, generando la enorme discrepancia con el valor observado. Sin embargo, usando un esquema de sustracción que sí incluye desacople, como el método extendido de DeWitt-Schwinger, evita este problema dado que la corrección generada es del orden $\sim \frac{\mu^6}{m^2}$. Por lo tanto, esto indica que el problema de la constante cosmológica parece estar más relacionado con una generalización incorrecta de ciertas herramientas usadas en teoría cuántica de campos perturbativa que con una predicción *catastrófica* de la propia teoría de campos.

Comentarios Finales y Futuras Direcciones

Durante la tesis han aparecido diversas cuestiones que merecen especial atención y que no han podido ser estudiadas con más detalle.

En primer lugar, aunque se ha conseguido desarrollar por primera vez la regularizacióna adiabática para un campo escalar cargado en presencia de un campo electromagnético en espacio curvo, queda por generalizar el procedimiento para un campo de Dirac en espacio curvo[2]. Otra posible extensión de nuestros resultados sería determinar correctamente las magnitudes relevantes en la produc-

[2] El caso límite de Minkowski ha sido desarrollado recientemente en [14].

ción de partículas en el caso del Universo temprano y estudiar también los posibles efectos en el origen de los campos magnéticos cósmicos [11, 59, 69, 105].

En segundo lugar, una aplicación interesante asociada a la introducción del parámetro μ en la regularización adiabática es su potencial ventaja en la determinación de espectros de producción de partículas en ciertos escenarios físicos. Por ejemplo, para un campo de Dirac cuantizado interactuando con un campo escalar clásico y aproximadamente constante $\Phi \approx$ cte, se puede obtener el bilinear $\langle 0|\bar{\psi}\psi|0\rangle_{\rm ren}$ renormalizado usando regularización adiabática descrita en el capítulo cuatro. En el caso de la regularización estándar, con $\mu = 0$, obtendríamos $\langle 0|\bar{\psi}\psi|0\rangle_{\rm ren} \sim \Phi^3$ que puede ser arbitrariamente grande. Si en cambio elegimos $\mu = \Phi$ se obtiene $\langle 0|\bar{\psi}\psi|0\rangle_{\rm ren} \approx 0$. La elección de μ permite por lo tanto calibrar las ecuaciones semiclásicas. Esto también podría permitir extender los límites de las aproximaciones semiclásicas, estudiados recientemente [6, 93]. Una de las líneas de investigación que estamos tratando actualmente consiste en entender este tipo de regularización y calibración, y sus consecuencias en los espectros de densidad de energía de producción de campos escalares y de Dirac durante Preheating, donde no sólo existe un campo escalar constante sino que evoluciona acompañado de un cierto potencial.

Continuando en este contexto, otra propuesta es realizar cálculos y simulaciones numéricas de producción de fluctuaciones del campo de Dirac durante preheating, incluyendo backreaction, es decir, la respuesta de dicha producción tanto en el campo gravitatorio como en el campos escalar clásico. Tener en cuenta la regularización de los observables es especialmente importante en campos de Dirac puesto que no existe una distinción nítida entra modos infrarrojos y ultravioletas como en el caso escalar. Otra propuesta sería calcular magnitudes renormalizadas necesarias para la producción de ondas gravitacionales [24].

Finalmente, otra cuestión que surgió al estudiar expansiones de tipo DeWitt-Schwinger, como la expansión de Parker-Raval [84, 85, 88], es entender cómo influye en la evolución del Universo las posibles soluciones de vacío. En efecto, una de las plausibles explicaciones para la aceleración actual del Universo tiene su origen en los efectos del vacío cuántico de un campo escalar con una masa muy por debajo de las masas del Modelo Estándar. En este sentido, una posibilidad es estudiar extensiones de esta soluciones para incluir otro tipo de campos, masas, aproximaciones de la solución del vacío y posibles efectos añadidos como la dependencia en temperatura, o la interaccion con campos electromagnéticos y escalares. Por otro lado, sería interesante analizar el comportamiento de este tipo de soluciones a altas energías, es decir, en los primeros instantes del Universo y entender la influencia de ellas en las dinámicas de Inflación y Reheating.

Contents

Chapter 1.

Summary of the Thesis

1.1. Motivation

One of the best features of fundamental physics is its relative sharpness to classify physical phenomena into its corresponding theories. If we wish to study the interaction between a charged particle in an electromagnetic field, nobody would argue that the best option is to make use of quantum electrodynamics (QED). The same happens with the study of nuclear interactions or particle disintegration. On the other hand, if we want to study the evolution of the space-time of our Universe, the collapse of a star or the propagation of gravitational waves, the theory of General Relativity (GR) would be the ideal theory for describing these phenomena. With great approximation, the observable world seems until now to be divided into descriptions of two fundamental theories: Quantum Field Theory (QFT) together with the Standard Model (SM) and General Relativity (GR).

Nonetheless, in spite of the enormous success of both theories as to explaining and predicting new phenomena, several questions remain unsolved. First, several experimental observations cannot be explained by these two theories: *Dark Matter*, *neutrino masses* or the origin of the distribution of the Cosmic Microwave Background. Secondly, the search of a theory of quantum gravity or a more fundamental theory that unifies both theories arises of the need of coupling consistently (quantized) matter to gravity, as required by GR.

In view of this last question, several proposals have been shaped during the last decades (see [89] for a general view). However, they face both theoretical and practical obstacles. Specially, its connection to possible observable predictions is in the present day obstructed by its mathematical complexities. An intermediate proposal consist on taking advantage of QFT in flat spacetime and generalize it to curved spacetime. Indeed, since a qualitative obstacle in a more fundamental theory is the quantization of gravity, it can be set aside momentarily using one of the most successful approaches of modern physics: the effective field theory approach. This consist on assuming that any theory, in this case, Quantum Field Theory in Curved Space-time (QFTCS) is a valid description up to energy (or length) scales much smaller (or bigger) than a certain scale, in this case the Planck Mass (or Planck length):

$$M_P = G^{-1/2}\hbar^{1/2}c^{1/2} \approx 0.2 \times 10^{-5}\text{g}$$
$$l_P = G^{1/2}\hbar^{1/2}c^{-3/2} \approx 1.4 \times 10^{-33}\text{cm}\,, \qquad (1.1)$$

where using units of $c = \hbar = 1$ implies $M_P \approx 10^{19}$GeV (we will use from here on the units $c = \hbar = 1$) . Except in some particular cases such as the interior of a black hole or at the very beginning of the Universe in the ΛCDM model, there isn't many phenomena that reach this energy scales and therefore it seems a good assumption that QFTCS could describe most parts of the observational phenomena in the present.

The study of a quantum field in the presence of a classical external field was previously considered in the case of an electromagnetic field before the arise of QED. One of the effects of this semi-classical theory is the spontaneous particle production (e.g. electron-positron pair) due to the electromagnetic field, also known as (Sauter-Heisenberg-Euler-)Schwinger mechanism [97,99]. Analogously, it is expected that a gravitational field also produces particles. Indeed, one of the first results of QFTCS, through the novel use of Bogoliubov transformations, was the production of particles in the case of an expanding universe [78], and later on, in the context of gravitational collapse and black holes [57]. In the inflationary models during the first instants of our Universe, this effect would be the cause of the observed anisotropies of the Cosmic Microwave Background. In addition, it is expected to be vital for the formation of matter (electrons, photons, etc.) during a period also known as Reheating.

Another consequence of the field quantization is its non-trivial structure of the vacuum state, which, among others, can leave a print as vacuum polarization. Important effects such as the Lamb shift or the anomalous magnetic moment of the electron field rely on this. In the case of a gravitational field, the vacuum state structure is encoded in the non vanishing stress-energy tensor. Quantifying the energy density of the vacuum has been an elusive task, but necessary to correctly understand the effect of the presence of a quantum field in a curved spacetime. In particular, the energy of the vacuum could act as a a source in expanding universes giving rise to a possible contribution of the accelerated expansion of the Universe [88].

One of the most important quantities of QFTCS is the vacuum expectation value of the stress-energy tensor $\langle 0|T_{ab}|0\rangle$. This contains information of both the particle production and the vacuum polarization effects. Moreover, following General Relativity, this magnitude has to contribute to the dynamics of space-time, through the semi-classical Einstein equation[1]

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G\langle 0|T_{ab}|0\rangle. \tag{1.2}$$

The construction of this kind of object is not exempted of obstacles. The infinite and in-equivalent forms of selecting the vacuum state, the limits of the semi-classical approximation and the general solu-

[1]We will use in this thesis the sign conventions used in [86].

tions of the equations of fields are among them (for a more detailed description we refer to [15, 52, 64, 86, 112]).

An additional problem is related with the divergences that appear when computing magnitudes like $\langle 0|T_{ab}|0\rangle$. The divergences in quantum field theories in flat space are well known and several methods have been defined to overcome them [91, 98]. In general, these methods consist on isolating the divergences of the magnitude through regularization and afterwards defining consistently the counter-terms of the Lagrangian such that the renormalized Lagrangian results in finite quantities. In general, we define the renormalized stress-energy tensor as

$$\langle 0|T_{ab}|0\rangle_{\text{ren}} = \langle 0|T_{ab}|0\rangle - T_{ab}^{\text{sub}}, \tag{1.3}$$

in such a way that both quantities from the right hang side cancel their corresponding divergences giving as a result a finite quantity. In QFTCS, different regularization and renormalization techniques [15, 86] have also been constructed, resulting in different T^{ab}_{sub}. Nevertheless, there exists some restrictions on them [112]. Among others, the renormalization has to be compatible with the conservation of the stress energy tensor $\nabla^a\langle 0|T_{ab}|0\rangle_{\text{ren}} = 0$; and has to be constructed in both local and covariant way. Finally, we would require to have only a finite number of subtraction terms, imitating the usual renormalizability criteria. A consequence of these requirements is that certain relevant results, such as the conformal anomaly, must result essentially independent of the particular renormalization prescription.

During this thesis, we will present some well known regularization methods in QFTCS, e.g. dimensional regularization and adiabatic regularization for the particular case of Friedman-Lemaitre-Robertson-Walker space-times. Before continuing, it is important to clarify some terminology. Strictly speaking, we refer to a regularization method to some method that introduces a regulator ν, that at some limit $\nu \to$ makes the calculated magnitude divergent. That way, we can isolate the divergences in such a way that we can construct a subtraction scheme. Adiabatic regularization obtains a subtraction scheme without invoking a regulator (we could still apply dimensional regularization but it is not necessary). Another well-known method we will use is the DeWitt-Schwinger subtraction scheme (DS). These methods have been studied in detail [15,86] ensuring the above mentioned results, proving the equivalence between them and obtaining explicit results in some well motivated metrics for astrophysics and cosmology.

Most of the results of adiabatic regularization have been developed for free scalar fields [86] and only recently of Dirac fields [31,72,73]. One of the aims of this thesis will be to further extend these results to include scalar and Dirac fields interacting with classical scalar and electromagnetic fields. Both interactions appear in different physical scenarios relevant for cosmology. Indeed, most of the models point towards the existence of at least one scalar field as a source of the inflationary expansion at the beginning of the Universe. This would be responsible for the accelerated expansion, the anisotropies of the cosmic microwave background, and for the production of observable matter during Reheating [8,55,56,70,71].

Another scalar field, experimental verified, is the Higgs field which has a leading role in the electroweak transition during the first instants of the Universe.

Electromagnetic fields can also generate pairs of charged particles through the (Sauter-Heisenberg-Euler-)Schwinger mechanism [97, 99]. In order to obtain a signal of this effect in the laboratory the most efficient option would be to reach a critical electric field and a critical intensity of [41]

$$\mathcal{E}_c = \frac{m_e^2}{e} \approx 10^{16}\mathrm{V/cm} \qquad I_c = \frac{\mathcal{E}_c^2}{8\pi} \approx 4 \times 10^{29}\mathrm{W/cm}^2 \,. \tag{1.4}$$

Traditional lasers do not reach these scales, which is why this effect has not been observed yet. Recent advances [1–3, 16, 33, 38, 40, 60] suggest the possibility of reaching this type of effect in the Extreme Light Infrastructure (ELI) [39]. Another potential laboratory could come from cosmology and astrophysics. Indeed, highly magnetized neutron stars [96] and the production of electromagnetic fields during the early Universe [66] could reach the necessary scales. This last option has a big interest since it is one of the possible origins of the recent discoveries of magnetic fields at cosmological scales [42].

Adiabatic regularization, although restricting to a particular metric, is enormously efficient for numerical computations. This is essential not only to be able to quantify, in physical motivated scenarios, particle production and vacuum polarization, but also to compute the answer of these phenomena on the classical background field, also known as backreaction.

The second part of the thesis is focused in studying the finite part that survives the renormalization and in particular the dependence of the coupling constants with an arbitrary scale μ. Dimensional regularization, which has been extensively used in computing scattering amplitudes in particles physics at high energy, provides an arbitrariness when selecting a concrete subtraction scheme of counter-terms, which is codified in an energy dimension-full parameter μ.

The introduction of this parameter is very useful for computing certain kind of magnitudes. For example, a frequently used subtraction scheme is Minimal Subtraction (MS), as opposed to the on-shell scheme. In the latter, neither the couplings nor the magnitudes have a explicit μ dependence and is related directly to physical quantities. MS is very efficient in quantum chromodynamics (QCD) where fields as quarks do not have defined asymptotic states and the conditions imposed to on-shell scheme are not suitable [98]. Since physical results can not depend on the renormalization scheme, the invariance under μ results in the equation of the renormalization group, which allows to predict the behavior of the theories at some energy limit. Indeed, two classical results of quantum field theory in flat spacetime are the beta functions of the charges of QED and QCD

$$\beta^{\text{QED}} \equiv \frac{de}{\mathrm{d}\log\mu} = \frac{e^3}{12\pi^2}\ ;\ \beta^{\text{QCD}} \equiv \frac{dg}{\mathrm{d}\log\mu} = \left(\frac{2n_f}{3} - 11\right)\frac{g^3}{16\pi^2} \tag{1.5}$$

where n_f is the number of flavours and e and g the coupling of the interaction of both theories. These expressions allow us to infer the high energy behaviour of both theories: the first one increases its effective charge and becomes non perturbative while the second one becomes weaker, also known as asymptotic freedom.

One of the most important results of quantum field theory in curved spacetime is that is is renormalizable, i.e., only a finite number of terms in the Lagrangian are needed to absorb the possible divergences. Indeed, the renormalized semi-classical Einstein equation is of the form [15]

$$\kappa^2 Gab + \Lambda g_{ab} + \alpha^{(1)} H_{ab} + \beta^{(2)} H_{ab} = -\langle 0|T_{ab}|0\rangle_{\text{ren}} \, . \tag{1.6}$$

Dimensional Regularization and MS can be applied to compute $\langle 0|T_{ab}|0\rangle_{\text{ren}}$ such that we obtain a μ dependence both for $\langle 0|T_{ab}|0\rangle_{\text{ren}}$ and for the different couplings constants κ, Λ, α and β, obtaining an analog result to QED and QCD. Nevertheless, the physical interpretation is not straightforwards and we will dedicate part of the thesis to analyze this in more detail.

Another important aspect of the modern approach to quantum field theory is related to understand the today's accessible theories as effective theories, in an certain energy range of more fundamental theories. An example of a effective theory is the Euler-Heisenberg Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1 e^4}{360 \pi^2 m^4} \left[\left(F_{\mu\nu} F^{\mu\nu} \right)^2 + \frac{7}{4} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 \right] \tag{1.7}$$

where the effect of the electron field has been encoded in higher order terms of the electromagnetic field in a limit where this field is much smaller than the electron mass. This theory allows to study in a more simpler way effects like photon scattering.

A fundamental result in this context is the decoupling of massive fields at low energies. This means that contributions of quantum fields are negligible when the energy scale of the system is much smaller then the mass of the field. For example, we do not need to know the physics of the top mass to study in detail the Hydrogen atom.

We know from the Appelquist-Carazzone theorem [7] that a mass dependent subtraction scheme in perturbative quantum field theories is compatible with decoupling. Indeed, the beta functions go to zero in the limit $m \to \infty$. However, using MS this decoupling is not explicit [74]. In QED, for example, this is solved by changing to the Momentum subtraction scheme (MOM). An important question is to recover decoupling in the case of QFTCS.

Finally, it is well-known that renormalization in quantum field theory produces results in tension with observational data. Indeed, both a cutoff regulator and MS produces contributions to the cosmological constant many orders of magnitudes higher than the observed quantity. This discrepancy is known as the cosmological constant problem [22, 113]. It is fundamental in this sense to better understand renormalization of quantum field in curved spacetime to better understand the logic of this apparent or real discrepancy with observations

1.2. Results and Conclusions

We have divided the obtained results in this thesis in two parts: adiabatic regularization with interactions and renormalization in curved spacetimes.

First, we generalized adiabatic regularization for quantum scalar fields to include interaction with a classical electric field. For this, we have extended the usual WKB expansion [86] to include interactions with a potential $A(t)$ defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ [49]. A crucial point was to realize that in order to have a regularization methods consistent with the conservation of the stress-energy tensor it was necessary that the electric potential A to be of adiabatic order one, analog to $\dot{a}$ in case of gravity [49]. This was an outstanding result since it has been ignored in most of the literature in adiabatic regularization [11,59,69,105].

We develop both the regularization of the stress-energy tensor and the electric current and we obtained the expected trace anomaly for this case [46]. We also introduced an extension of adiabatic regularization to include the parameter μ [47]. An essential result was to check that adiabatic regularization with the parameter μ is consistent. This required that we did not generate new divergences in comparison to the standard adiabatic regularization [86] (the case of $\mu = 0$) and that the difference between two possible subtraction schemes, i.e., two regularizations with parameters μ_1 and μ_2 had a difference parametrized by covariant terms as

$$\langle 0|T_{ab}|0\rangle^{\mu_1}_{\rm ren} - \langle 0|T_{ab}|0\rangle^{\mu_2}_{\rm ren} = a + bG_{ab} + c^{(1)}H_{ab} + dT^{EM}_{ab} \ . \quad (1.8)$$

Applying the invariance with respect to μ in the semi-classical Einsteins equations (1.6), we can obtain the dependence of the coupling constants with respect to μ, encoded in the beta functions $\beta_O = \mu \frac{d}{d\mu} O$. For the case of the dimensionfull constants we obtain

$$\beta_\Lambda = \frac{1}{16\pi^2} \frac{\mu^6}{m^2 + \mu^2} \quad \beta_\kappa = \frac{\xi - \frac{1}{6}}{4\pi^2} \frac{\mu^4}{m^2 + \mu^2}, \tag{1.9}$$

while for the adimensional ones

$$\beta_e = \frac{e^3}{48\pi^2} \frac{\mu^2}{\mu^2 + m^2} \quad \beta_\alpha = \frac{\left(\xi - \frac{1}{6}\right)^2}{8\pi^2} \frac{\mu^2}{\mu^2 + m^2} \,. \tag{1.10}$$

For the last one, we recover in the limit $\mu^2 \gg m^2$ the standard results of perturbative quantum field theory, using dimensional regularization and MS [98, 101]. However, an intriguing result is that for the dimensional case we do obtain a difference with respect to MS

$$\beta_\Lambda^{MS} = \frac{m^4}{16\pi^2} \quad \beta_\kappa^{MS} = \frac{m^2}{4\pi^2} \left(\frac{1}{6} - \xi \right) \,. \tag{1.11}$$

These results were published in [46, 47, 49] and are described in more detail in chapter 2.

In second place, we developed adiabatic regularization for a Dirac field interacting with an electromagnetic field in two dimensions. First, we generated the adiabatic expansion adapting the results obtained for the free Dirac field [31, 72, 73] to be consistent with the introduction with the classical electric field and the previous results for the scalar field. We regularized both the electric current and the stress-energy tensor and we obtained the correct chiral and trace anomaly. There exists an arbitrariness when generating the adiabatic expansion which is present even in the free case [31,72,73].

We presented a possible solution to this last issue and it consists on an iterative method that generated an adiabatic expansion without any ambiguity. From the result obtained of the chiral anomaly in two dimensions, we obtained a peculiar result: in the case of a mass-less Dirac field, adiabatic invariance of the particle number is broken. Indeed, an electric potential A will produce particles independent of the slowness or adiabaticity of the time evolution. This is an important difference in comparison to the behavior of the gravitational field which do respect adiabatic invariance. This result was extended for the case of four dimensions with the same conclusion: in presence of an electromagnetic field, for a mass-less Dirac field, the chiral anomaly breaks the adiabatic invariance, producing particles. These result were published in [10,12,46] and are part of chapter 3 of the thesis.

Finally, adiabatic regularization for a free Dirac field [31,72,73] was extended to also include an interaction with a classical scalar field of the form $g_Y\bar{\psi}\psi\phi$. First, the scalar field was assigned with adiabatic order one, consistent with its dimension. We obtained the regularization of the stress-energy tensor and the conformal anomaly. This last result was discussed and the consistency with other methods were checked. Finally, we obtained the renormalization of the Lagrangian by the introduction of finite counter-terms.

These results were published in [29] and are part of chapter 4. We also obtained the running of the coupling constants using adiabatic regularization with a μ parameter, a novel result of this thesis.

In the second part of this thesis we extended the DS expansion to include a renormalization parameter μ, analog to the adiabatic expansion. For this, instead of focusing in the expectation value of the stress-energy tensor $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$, we can equivalently use the effective action S_{eff} which contains all the classical dynamics plus the correspondent quantum corrections at a given *loop* (corrections with increasing order of $\hbar$). The DeWitt-Schwinger expansion of the effective action allows to isolate the divergent terms, analogously to the adiabatic regularization case.

Introducing the μ parameter in a consistent way into the subtraction terms, we confirmed that it is indeed correct and no extra divergences wold be generated. Moreover, the difference between regularizing the quantum contribution to the effective action with two different parameter μ_1 and μ_2 resulted in

$$\Gamma^{\mu_1}_{\text{ren}} - \Gamma^{\mu_2}_{\text{ren}} = a + bR + cR^2 + dC_{abcd}C^{abcd} + eF_{ab}F^{ab}. \quad (1.12)$$

This difference is formed by terms that have to been present in order for the original action to be renormalizable at any loop. In the case of the FLRW metric we recover the results from adiabatic regularization with the introduction of μ. Indeed, for the coupling constant, using the invariance of the effective action with respect to

μ we obtain the following beta functions

$$\beta_e = \frac{e^3}{48\pi^2}\frac{\mu^2}{\mu^2+m^2} \quad \beta_\alpha = \frac{\bar{\xi}^2}{8\pi^2}\frac{\mu^2}{\mu^2+m^2} \quad \beta_\gamma = \frac{-1}{960\pi^2}\frac{\mu^2}{\mu^2+m^2}$$
$$\beta_\Lambda = \frac{1}{16\pi^2}\frac{\mu^6}{m^2+\mu^2} \quad \beta_\kappa = \frac{\bar{\xi}}{4\pi^2}\frac{\mu^4}{m^2+\mu^2}. \tag{1.13}$$

An outstanding result is that this beta functions are consistent with the decoupling of massive fields at low energy. We know, by the Appelquist-Carazzone theorem [7] that a mass dependent subtraction scheme in perturbative QFT is compatible with decoupling. Indeed, both the beta functions and the compute magnitude would ten to zero in the case of $m \to \infty$. A well-known result is that MS does not have this decoupling [74]. Our extended DeWitt-Schwinger subtraction scheme is therefore a scheme compatible with the Appelquist-Carazzone theorem in curved spacetime. Indeed, observing the results (1.13) we conclude that all the beta functions, including the dimensional ones, go to zero in the limit $m \to$. These results were published in [48] and are described in more detailed in chapter 5.

One of the applications of this result is to study possible contributions from quantum fields to the cosmological constant. It is well-known that cut-toff regulators generate contributions with a discrepancy of 120 orders of magnitude [113] with respect to the observed cosmological constant, while the discrepancy is reduced to 32 orders of magnitude [75] if we use Minimal Subtraction. In this thesis, we argue that both schemes are not practical methods for renormalization in QFTCS. First, it is well understood that a cutoff regulator generates problems in itself (since it does not respect general covariance [75] and does not recover important results in QFTCS [112]).

Minimal Subtraction is compatible with these last requirements but, as we have already stated, it is not compatible with the decoupling of massive fields. In perturbative quantum field theory this is solved by integrating out by hand the massive fields [74, 92] and building different theories that contain only light fields (with respect to the validity scale of the theory). For adimensional coupling constants, e.g. electric charge, this difference between theories has only logarithmic corrections. However, for the cosmological constant this difference is of the order of $\sim m^4 \log\left(\frac{\mu^2}{m^2}\right)$, generating the enormous discrepancy with the observed value. Nevertheless, using a subtraction scheme that is compatible with decoupling, such as the extended DeWitt-Schwinger scheme, we avoid this problem since the generated correction is of order $\sim \frac{\mu^6}{m^2}$. This indicates that the problem of the cosmological constant is more related to an incorrect generalization of specific tools in perturbative quantum field theory than a catastrophically prediction of field theories.

1.3. Methodology

The methods employed in this thesis are essentially mathematical computations of physical relevant observable, consult of bibliography, analyzing different theoretical descriptions of physical phenomena, extending mathematical tools of different theories and some numerical computations.

We have used mostly tools from modern areas of physics such as Quantum Field Theory, General Relativity, Electrodynamics and Cosmology. Also specific tools from mathematics such as differential equations, real and complex calculus, and functional analysis were used. Specific methods have already been commented along the introductory text.

Part I.

Adiabatic Regularization and Running Couplings in Interacting Theories

Introduction and Motivation

One of the first applications of quantum fields propagating in a curved space-time was the computation of the particle number expectation value of the vacuum in a FLRW metric [79–81]. In a statically bounded smooth expansion, the particle number can be computed by the expectation value of the number operator in terms of creation and annihilation operator of the late time Minkowski vacuum evaluated in early time vacuum. This, for general time dependence of the expansion parameter, gives rise to a non vanishing expectation value that can be interpreted as a particles being spontaneously produced by the vacuum. For example, the generation of inhomogeneities of the Cosmic Microwave Background can be interpreted as the non vanishing particle number in a configuration of two, late and early times, asymptotically Minkowski limit between a De Sitter expansion [54].

The particle number magnitude does not need to be regularized since it is finite at the late time Minkowski limit. However, for non-bounded expansions, like the current accelerated Universe, this description is no longer valid, and the particle number is indeed divergent. Furthermore, even in a bounded expansion, one would wish that the expectation value is finite at intermediate steps. Therefore, we require a regularization procedure to cure this divergences resulting in finite observables. In this context, adiabatic regularization was proposed [78] (see for a historical review [82]). In a FLRW spacetime, adiabatic regularization is introduced by computing the v.e.v. of the particle number in form a momentum-space integral $\langle n \rangle = \int dk^3 n_k$ such that n_k diverges as $k \to \infty$. Adiabatic regularization fixes uniquely the divergences of n_k such that they can be subtracted from the original n_k.

It was later generalize to regularize the v.e.v. of the stress-energy tensor [51, 53, 83] in such a way that locality and covariance of the renormalization were maintained. The v.e.v. of of the stress-energy tensor also carries UV divergences. In order to regularize expectation values, we introduce an asymptotic expansion of the mode function of the quantized fields, with increasingly higher number of time derivative of the scale factor, also called, *adiabatic order*. On dimensional grounds, an increasingly adiabatic order is equivalent of a decreasing momentum, such that a given order $n+1$ it will no longer be divergent. The adiabatic regularization prescription consist on subtracting from the original v. e.v. the n divergent adiabatic order terms. For the stress-energy tensor this means to subtract up to adiabatic order four,

$$\langle T_{ab} \rangle_{\text{ren}} = \int d^3k \langle T_{ab} \rangle_k - \langle T_{ab} \rangle_k^{(0-4)}.$$

These subtraction terms have been shown to be reabsorbed in the usual Einstein-Hilbert term with extra finite higher order terms [19], which agrees with the results of more general renormalization methods in curved spacetime. However, a standard result with renormalization in flat spacetime involves the running of the couplings. These were not computed until now for adiabatic regularization. It is important for a renormalization prescription to obtain well established result such as the running of the electric charge. Moreover, a correct interpretation of the running of the gravitational couplings allow to explore in more detail the possible effects of these running in the history of our Universe [102, 103].

Adiabatic regularization is a very efficient method for numerical purposes since it is relatively simple to incorporate numerical integration of the modes to a given accuracy. This is important since the computation of higher order contributions, i.e., possible backreaction effects make the calculations rather involved. Other regularization prescriptions such as point-splitting involve differentiation and longer computations, which makes it almost impossible for most physical interesting models, except for very specific cases such as the De-Sitter solution [20].

Most of the results for adiabatic regularization focused on a free scalar field in FLRW space-times, but recent generalization to quantized Dirac field have been proposed [31, 72, 73], and extra background fields have also been incorporated. In this part of the thesis we will extend adiabatic regularization to include an electromagnetic background for a scalar fields in four dimensions and a Dirac field in two dimensional spacetime. We will also incorporate adiabatic regularization with a classical background field with a Yukawa interaction. In the four dimensional cases we will make use of the extended adiabatic regularization that includes a μ parameter in order to obtain the running of the couplings.

In chapter 2 and 3 we will focus on adiabatic regularization with a time varying background gauge field [10, 12, 46, 47, 49]. Electric fields also generated new divergences in the computation of local quantities of quantum fields. We have extended adiabatic regularization to consistently tackle this issue by subtracting the correct divergent pieces of these quantities, while satisfying the conservation of the stress-energy tensor and the correct computations of the trace anomaly and the chiral anomaly in case of the Dirac field.

In chapter 2 we extend the usual WKB ansatz to generate the adiabatic expansion of the modes of the quantum scalar field to include the interaction with the potential A [46]. We discuss the assignation of adiabatic order one to the potential A in order to generate the expansion in consistence with the conservation of the renormalized stress-energy tensor [49]. We also introduce an arbitrary μ parameter in the expansion such that it does not generate any new divergences and correctly results in covariant finite contributions between two different parametrizations μ_1 and μ_2, that can be reabsorbed in the original Lagrangian [47].

We correctly regularize both the stress-energy tensor and the electric current, which are the two magnitudes that enter in the semiclassical Einstein-Maxwell equations. Furthermore we reproduce the conformal anomaly. Finally we use the μ invariance to generate the correct beta functions for QED and discuss the corresponding beta functions of the Newton and cosmological constant.

In chapter 3, we extended adiabatic regularization for the case of two dimensional Dirac field. We propose a novel procedure [10] to generate the adiabatic expansion for Dirac fields, since the standard WKB ansatz is no longer possible [72]. Moreover, this method avoids an unnecessary arbitrariness that appears other approaches [31,72,73]. We correctly regularize both the stress-energy tensor and the electric current and recover the conformal and chiral anomaly. We link the chiral anomaly in two dimensions with the breaking of the adiabatic invariance in the case of a slow varying electric fields that produce pairs of electron-positron type of particle [12].

In chapter 4, we extend adiabatic regularization for a Dirac field to include a Yukawa interaction. The advantage of the method proposed in [10] for two dimensional Dirac fields with an electric field is that it can also be used for this case. We consistently generate the adiabatic expansion and the regularization of the stress-energy tensor and the bi-linear $\langle \bar{\psi}\psi \rangle$ by requiring the classical scalar field Φ to be of adiabatic order one, analog to A and $\dot{a}$. We obtain the conformal anomaly and compare this result with other standard regularization methods in general curved spacetimes. We also add the μ parameter analog to the scalar case which supports the robustness of this arbitrariness since it again generate the correct finite and covariant quantities between two parametrizations. We briefly comment on the running of the gravitational couplings and the new couplings generated by the Yukawa interaction.

Chapter 2.

Adiabatic Regularization for a Scalar Field in an EM Background

Consider a massive charged scalar field and a classical electromagnetic background in a general curved spacetime. We start from the classical Einstein-Maxwell theory

$$S = \int d^4x \sqrt{-g} \left(-\Lambda + \frac{R}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + S_{\mathrm{M}} \tag{2.1}$$

coupled to a quantized charged scalar field described by the action

$$S_{\mathrm{M}} = \int d^4x \sqrt{-g} \left((D_\mu \phi)^\dagger D^\mu \phi - m^2 |\phi|^2 - \xi R |\phi|^2 \right) , \tag{2.2}$$

with $D_\mu = \nabla_\mu + iqA_\mu$. The scalar field obeys the Klein-Gordon field equation

$$(D_\mu D^\mu + m^2 + \xi R)\phi = 0 \,. \tag{2.3}$$

By variations of the action with respect to the metric we obtain the stress energy of the scalar field

$$\begin{aligned}
T_{\mu\nu} &:= \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = \left(\frac{1}{2} - \frac{3}{2}\xi\right) m^2 g_{\mu\nu}\left\{\phi, \phi^\dagger\right\} \\
&+ (1-2\xi)\left\{D_\mu\phi, D_\nu\phi^\dagger\right\} + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}g^{\rho\sigma}\left\{D_\rho\phi, D_\sigma\phi^\dagger\right\} \\
&-2\xi\left\{D_\mu D_\nu\phi, \phi^\dagger\right\} + \xi g_{\mu\nu}\left\{\Box\phi, \phi^\dagger\right\} \\
&-\xi\left[R_{\mu\nu} - \left(\frac{1}{2} - \frac{3}{2}\xi\right) R g_{\mu\nu}\right]\left\{\phi, \phi^\dagger\right\}
\end{aligned} \tag{2.4}$$

where the symbol $\{\}$ denotes the anti-commutator. The electric current is

$$j^\nu = iq\left[\phi^\dagger D^\nu\phi - (D^\nu\phi)^\dagger\phi\right] \,. \tag{2.5}$$

The semi-classical equations are obtained from the Einstein-Maxwell equations by replacing the classical source terms by its corresponding vacuum expectation values

$$\langle T_{\alpha\beta}\rangle_{\text{ren}} + T^{EM}_{\alpha\beta} = \frac{-G_{\alpha\beta}}{8\pi G} - \Lambda g_{\alpha\beta} \qquad \nabla_\alpha F^{\alpha\beta} = \langle j^\beta\rangle_{\text{ren}} \,, \tag{2.6}$$

where $T^{EM}_{\alpha\beta} = \left(\frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}g_{\alpha\beta} - F_{\alpha}^{\ \rho}F_{\beta\rho}\right)$. We include here only the expectation value of the quantized complex scalar field. It is a standard result that both $\langle T_{\alpha\beta}\rangle$ and $\langle j^{\beta}\rangle$ diverge for a general metric and EM field configuration. Therefore, we need to regularize and renormalize these bilinears in order to obtain the finite, physical semiclassical equations. It is also useful for future discussion the conservation of the left hand side of the Einsteins equation in (2.6)

$$\nabla_{\alpha}\langle T^{\alpha\beta}\rangle_{\rm ren} = -\nabla_{\alpha}T^{\alpha\beta}_{EM} = \langle j_{\alpha}\rangle_{\rm ren}F^{\alpha\beta} \tag{2.7}$$

where we have use the Maxwell equation from (2.6).

Assuming that the electric field is spatially homogeneous and the magnetic field is zero, we take the electric field in the direction of the x axis. For our purposes it is very convenient to choose a gauge such that only the x-component of the vector potential is nonvanishing: $A_{\mu} = (0, -A(t), 0, 0)$. Therefore, the field strength is given by $F_{0i} = (-\dot{A}(t), 0, 0)$. In a FLRW metric $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$ the Klein-Gordon equation (2.3) becomes

$$\ddot{\phi} - \frac{\vec{\nabla}^2}{a^2}\phi + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{2iqA}{a^2}\partial_x\phi + \frac{q^2A^2}{a^2}\phi + (m^2 + \xi R)\phi = 0 \tag{2.8}$$

with $\phi = \phi(\vec{x}, t)$. Here $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$. We can do a Fourier expansion with respect to space

$$\phi(x) = \frac{1}{\sqrt{2(2\pi a)^3}}\int d^3\vec{k}[A_{\vec{k}}e^{i\vec{k}\vec{x}}h_{\vec{k}}(t) + B^{\dagger}_{\vec{k}}e^{-i\vec{k}\vec{x}}h^{*}_{-\vec{k}}(t)]\,, \tag{2.9}$$

where $A^\dagger_{\vec{k}}, B^\dagger_{\vec{k}}$ and $A_{\vec{k}}, B_{\vec{k}}$ are the usual creation and annihilation operators. The normalization condition is

$$h_k^* \dot{h}_k - \dot{h}_k^* h_k = 2i \qquad \text{and} \qquad h_k \dot{h}_{-k} - \dot{h}_k h_{-k} = 0\,. \tag{2.10}$$

These ensure the standard commutation relations. The Klein-Gordon equation is then

$$\ddot{h}_{\vec{k}} + \left(a^{-2}\left(k_x - qA\right)^2 + a^{-2}k_\perp^2 + m^2 + \sigma\right) h_{\vec{k}} = 0 \tag{2.11}$$

where k_x and $k_\perp^2 = k_y^2 + k_z^2$ the 3-momentum parallel and perpendicular to the direction of the electric field respectively and $\sigma = (6\xi - 3/4)(\dot{a}/a)^2 + (6\xi - 3/2)\ddot{a}/a$. We can now construct physical observables for the scalar field. The two-point function

$$\langle 0|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)|0\rangle = \frac{1}{2(2\pi)^3 a^3}\int d^3k e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}|h_k(t)|^2. \tag{2.12}$$

For the scalar stress-energy tensor we define

$$\langle T_{\mu\nu}\rangle := \langle 0|T_{\mu\nu}|0\rangle = \frac{1}{(2\pi a)^3}\int d^3k \langle T_{\mu\nu}\rangle_k\,. \tag{2.13}$$

From (2.4) and the field expansion of (2.9) we have

$$\begin{aligned}\langle T_{00}\rangle_k = \quad & |\dot{h}_k|^2 + \left(m^2 + a^{-2}P^2\right)|h_k|^2 + \left(\frac{9}{4} - 12\xi\right)\left(\frac{\dot{a}}{a}\right)^2 |h_k|^2 \\ & \left(6\xi - \frac{3}{2}\right)\frac{\dot{a}}{a}(\dot{h}_k h_k^* + \dot{h}_k^* h_k)\end{aligned} \tag{2.14}$$

$$\langle T_{ii}\rangle_k a^{-2} = (4\xi - 1)\, P^2 a^{-2}|h_k|^2 + 2a^{-2}(k_i - qA\delta_{ix})^2|h_k|^2$$
$$+ (4\xi - 1)\, m^2|h_k|^2 - (4\xi - 1)\, |\dot{h}_k|^2 + \left(8\xi - \frac{3}{2}\right)\frac{\dot{a}}{a}\left(\dot{h}_k h_k^* + \dot{h}_k^* h_k\right)$$
$$+ \left(\left(\frac{9}{4} + 24\xi^2 - 17\xi\right)\left(\frac{\dot{a}}{a}\right)^2 + \left(24\xi^2 - 4\xi\right)\left(\frac{\ddot{a}}{a}\right)\right)|h_k|^2 \qquad (2.15)$$

where we have defined $P^2 = (k_x - qA)^2 + k_\perp^2$. The electric current is

$$\langle 0|\vec{j}|0\rangle = \frac{1}{(2\pi a)^3 a^2}\int d^3k\left(\vec{k} + q\vec{A}\right)|h_k|^2. \qquad (2.16)$$

In the ultraviolet, i.e., large k the modes behave as $|h_k| \sim k^{-1}$. As a consequence the stress-energy tensor can have quartic, quadratic and logarithmic divergences. The electric current on the other side has cubic, quadratic and logarithmic divergences. This defines the adiabatic terms we need to subtract: up to order four for the stress-energy tensor and up to adiabatic order three for the electric current.

2.1. Adiabatic Regularization

The adiabatic expansion for the scalar field modes is based on the usual WKB ansatz

$$h_{\vec{k}} = W_{\vec{k}}^{-1/2}(t)e^{-i\int^t W_{\vec{k}}(t')dt'}\ ,\ \ W_{\vec{k}}(t) = \omega^{(0)} + \omega^{(1)} + \omega^{(2)} + \cdots \qquad (2.17)$$

where $W_{\vec{k}}(t)$ is a real function. One can substitute the above ansatz into Eq. (2.11) and the Wronskian condition (2.10). We then get the equations (we drop the $\vec{k}$ index for simplicity)

$$W^2 = a^{-2}P^2 + m^2 + \sigma + W^{1/2}\frac{d^2}{dt^2}W^{-1/2} \, . \tag{2.18}$$

We have to solve order by order to obtain the different terms of the expansion. As usual [86], we will consider $a(t)$ of adiabatic order zero, $\dot{a}(t)$ of adiabatic order one, etc. However, to get an unique series expansion we have to assign also an adiabatic order to the vector potential function $A(t)$. We will choose $A(t)$ to be of adiabatic order 1. This assignment of adiabatic order 1 is consistent with the scaling dimension of the field $A(t)$, as it possesses the same dimensions as $\dot{a}$. The mass dimension of the scale factor $a(t)$ is zero, while that of $\dot{a}(t)$, or the field $A(t)$, is unity (We will reexamine this point in connection with the conservation of the stress-energy tensor requirement in 2.2.1). Therefore, $\dot{A}(t)$ will be of adiabatic order 2, $\ddot{A}(t)$ of order 3 and so on.

On the other hand as stated in [47] there is an arbitrariness in choosing the zeroth order of the expansion which can be parametrized by a parameter μ

$$\omega^{(0)} = \omega \equiv \sqrt{m^2 + \mu^2 + \frac{\vec{k}^2}{a^2}}. \tag{2.19}$$

In order to obtain the expansion we rewrite (2.18) as

$$W^2 = \omega^2 + \sigma - \mu^2 + W^{1/2}\frac{d^2}{dt^2}W^{-1/2} \tag{2.20}$$

and fix μ^2 to be of adiabatic order two while ω^2 is of adiabatic order zero. The next terms are iterative calculated and result in

$$\begin{aligned} \omega^{(1)} = & \quad -\frac{qAk_x}{a^2\omega} \\ \omega^{(2)} = & \quad -\frac{q^2A^2k_x^2}{2a^4\omega^3} + \frac{q^2A^2}{2a^2\omega} + \frac{3\xi\dot{a}^2}{a^2\omega} - \frac{3\dot{a}^2}{8a^2\omega} - \frac{\mu^2}{2\omega} \\ & \quad + \frac{3\xi\ddot{a}}{a\omega} - \frac{3\ddot{a}}{4a\omega} + \frac{3\dot{\omega}^2}{8\omega^3} - \frac{\ddot{\omega}}{4\omega^2} \end{aligned} \tag{2.21}$$

The same procedure can be repeated for all higher orders.

2.2. Regularization of the Stress-Energy Tensor and the Electric Current

Since all of the relevant results only involve $\langle T_{00}\rangle$ and $\langle T_i^i\rangle$ and for simplicity purposes, we will only compute adiabatic expansions of these terms, but the generalization for each component is straightforward. We start performing the adiabatic expansion for both components

$$\langle T_{00}\rangle = \frac{1}{(2\pi a)^3}\int d^3k \langle T_{00}\rangle_k \qquad \langle T_{00}\rangle_k = \langle T_{00}\rangle_k^{(0)} + \langle T_{00}\rangle_k^{(1)} + ...$$
$$\langle T^i_i\rangle = \frac{1}{(2\pi a)^3}\int d^3k \langle T^i_i\rangle_k \qquad \langle T^i_i\rangle_k = \langle T^i_i\rangle_k^{(0)} + \langle T^i_i\rangle_k^{(1)} + ... \qquad . \tag{2.22}$$

All the divergences are encapsulated in the first four adiabatic terms. This can be seen by dimensional grounds, since each adiabatic order increases the dimension. To obtain the expansion, we plug (2.17) up to adiabatic order four using the obtained $\omega^{(n)}$, into (2.15). For example, the result for the first two terms are:

$$\langle T_{00}\rangle_k^{(0)} = 2\omega, \quad \langle T_{00}\rangle_k^{(1)} = -\frac{2qAk_x}{a^2\omega}, \tag{2.23}$$
$$\langle T^i_i\rangle_k^{(0)} = \frac{2m^2}{\omega} - 2\omega, \quad \langle T^i_i\rangle_k^{(1)} = \frac{2qA(m^2+\mu^2)k_x}{a^2\omega^3} + \frac{2qAk_x}{a^2\omega}. \tag{2.24}$$

Note that in standard results form free fields in curved spacetime, odd terms vanishes [86], wheres in the case of an additional electromagnetic field this is no longer the case. Finally, for the electric current the only non vanishing component is

$$\langle j_x\rangle = \frac{1}{(2\pi a)^3 a^2}\int d^3k \langle j\rangle_k; \quad \langle j\rangle_k = (k_x - qA)\,|h_k(t)|^2. \tag{2.25}$$

The electric current has the divergences encoded only in the first three terms and therefore we repeat the procedure by only including (2.17) up to adiabatic order three, obtaining

$$\langle j\rangle_k^{(0)} = \frac{k_x}{a\omega}, \qquad \langle j\rangle_k^{(1)} = \frac{qAk_x^2}{a^3\omega^3} - \frac{qA}{a\omega}, \qquad \ldots \tag{2.26}$$

The regularized components are

$$\begin{aligned}
\langle T_{00}\rangle_{\rm ren} &= \frac{1}{(2\pi a)^3}\int d^3k\left[\langle T_{00}\rangle_k - \langle T_{00}\rangle_k^{(0-4)}\right]\\
\langle T_i^i\rangle_{\rm ren} &= \frac{1}{(2\pi a)^3}\int d^3k\left[\langle T_i^i\rangle_k - \langle T_i^i\rangle_k^{(0-4)}\right]\\
\langle j_x\rangle_{\rm ren} &= \frac{1}{(2\pi)^3 a^5}\int d^3k\left[(k_x - qA)\,|h_k(t)|^2 - \langle j\rangle_k^{(0-3)}\right]
\end{aligned} \tag{2.27}$$

where the super-index $^{0-n}$ denotes the sum of the first adiabatic terms up to order n. Note here that we have maintained the $\rangle\langle$ symbols for the subtraction terms, but they do not dependent on the actual vacuum state. In general we would define e.g.

$$\langle T_{00}\rangle_{\rm ren} \equiv \langle T_{00}\rangle - T_{00}^{\rm sub}; \qquad T_{00}^{\rm sub} \equiv \frac{1}{(2\pi a)^3}\int d^3k\langle T_{00}\rangle_k^{(0-4)}. \tag{2.28}$$

2.2.1. Conservation of the Stress-Energy Tensor and Adiabatic Order

We have required that the electric potential A is of the same adiabatic order as $\dot{a}$, i.e., of adiabatic order one. This may seem an arbitrariness but it is a mandatory prescription for a consistent regularization. To see this, let us assume for simplicity $\mu = 0$. We recall the leading order of the adiabatic expansion

$$\omega_k^{(0)} = \sqrt{k^2/a^2 + m^2} \equiv \omega \ , \tag{2.29}$$

here, since the potential is of adiabatic order one it does not appear in the leading order. Let us now assume instead that A is of adiabatic order zero such that the leading order is now

$$\omega_k^{(0)} = \sqrt{(k - qA)^2/a^2 + m^2} \equiv \bar{\omega} \ . \tag{2.30}$$

All the subtraction terms of both the stress-energy tensor and the electric current are going to be different. In principle, this could be possible since it is a standard result of QFTCS that two regularization prescriptions can differ [112]. However one of the conditions for any regularization is that the conservation of the stress-energy tensor has to hold, i.e.,

$$\nabla_\alpha \langle T^{\alpha\beta} \rangle_{\text{ren}} = -\nabla_\alpha T^{\alpha\beta}_{EM} = \langle j_\alpha \rangle_{\text{ren}} F^{\alpha\beta}. \tag{2.31}$$

Adiabatic regularization for free fields ensures that this holds for each adiabatic order [86] and accordingly, we wish that it still holds for interacting fields. Let us compute the 00 component of (2.31) for a FLRW spacetime and a time dependent electric field,

$$\partial_0 \langle T^{00} \rangle_{\text{ren}} + 3\frac{\dot{a}}{a} \langle T^{00} \rangle_{\text{ren}} + \dot{a} a \delta_{ij} \langle T^{ij} \rangle_{\text{ren}} = \frac{\dot{A}}{a^2} \langle j_x \rangle_{\text{ren}}. \tag{2.32}$$

One can check that this is valid for each adiabatic order by using the aforementioned prescription (2.29), i.e.

$$\partial_0 \langle T^{00} \rangle^{(n)} + 3\frac{\dot{a}}{a} \langle T^{00} \rangle^{(n)} + +\dot{a}a\delta_{ij} \langle T^{ij} \rangle^{(n)} - \frac{\dot{A}}{a^2} \langle j_x \rangle^{(n-1)} = 0 \quad (2.33)$$

for $n = 0, ..., 4$. Here we have defined

$$\langle T^{ab} \rangle^{(n)} := \frac{1}{(2\pi a)^3} \int d^3k \langle T^{ab} \rangle_k^{(n)} \quad \langle j_x \rangle^{(n)} := \frac{1}{(2\pi)^3 a^5} \int d^3k \langle j \rangle_k^{(n)}. \quad (2.34)$$

We have also taken into account that A is of adiabatic order one and therefore $\dot{A}$ of adiabatic order two. Let us now assume option (2.30), we compute the adiabatic subtractions for the stress-energy tensor and the electric current current and we find that the conservation of the stress-energy tensor takes now the form

$$\partial_0 \langle T^{00} \rangle^{(n)} + 3\frac{\dot{a}}{a} \langle T^{00} \rangle^{(n)} + +\dot{a}a\delta_{ij} \langle T^{ij} \rangle^{(n)} - \frac{\dot{A}}{a^2} \langle j_x \rangle^{(n)} = 0 \quad (2.35)$$

since now $\dot{A}$ is of adiabatic order one for $n = 0, ..., 3$. Since the electric current has only to be regularize to adiabatic order three by consistency with the adiabatic regularization prescription, for $n = 4$ this results in

$$\partial_0 \langle T^{00} \rangle^{(4)} + 3\frac{\dot{a}}{a} \langle T^{00} \rangle^{(4)} + +\dot{a}a\delta_{ij} \langle T^{ij} \rangle^{(4)} \neq 0 \quad (2.36)$$

and therefore the prescription fails to fulfill the requirement. In most of the literature [27, 67, 68], it is assumed this specific implementation of the adiabatic renormalization program without realizing an underlying inconsistency when gravity is turned on. This fact has been largely overlooked in the literature. However, one can check that if we turn off gravity, and only have an electromagnetic background field both schemes are equivalent and consistent. We will not address this result here and refer to [49] for further details.

In conclusion, the gauge field should enter at the next to leading order in the adiabatic expansion: $A(t)$ should be treated as a field of adiabatic order 1, in the same footing as $\dot{a}(t)$, as displayed in Table 2.1.

Field	Adiabatic order assignment
$a(t)$	0
$\dot{a}(t)$, $A(t)$	1
$\ddot{a}(t)$, $\dot{a}^2(t)$, $A^2(t)$, $\dot{a}(t)A(t)$	2
$\dddot{a}(t)$, $\ddot{a}(t)\dot{a}(t)$, $A^3(t)$, $\dot{a}(t)\dot{A}(t)$, ...	3
$\ddddot{a}(t)$, ...	4

Table 2.1.: We summarize the adiabatic order assignment for different numbers of derivatives for the metric and the gauge field.

2.2.2. Conformal Anomaly

Another nontrivial test for our proposal is to reproduce the trace anomaly for the quantized charged scalar field for $\xi = 1/6$ and $m = 0$ (and $\mu = 0$). To evaluate the trace anomaly in the adiabatic regularization method, we have to start with a massive field and take the mass-less limit at the end of the calculation. Moreover, for a massive charged field $T^{\mu}_{\mu} = 2m^2\phi\phi^{\dagger}$. However, this formal identification does not imply that $\langle T^{\mu}_{\mu}\rangle_{\rm ren} = 2m^2\langle\phi\phi^{\dagger}\rangle_{\rm ren}$. The divergences of the stress-energy tensor components have terms of fourth adiabatic order, while the divergences of $\langle\phi\phi^{\dagger}\rangle$ involve only terms untill second adiabatic order. Therefore, in order to evaluate the trace anomaly by using the above formal expression, the adiabatic subtractions for $\langle\phi\phi^{\dagger}\rangle$ should also include subtractions up to fourth adiabatic order. The same argument has been used to work out the trace anomaly of a real scalar field [86]. Therefore,

$$\langle T^{\mu}_{\mu}\rangle_{\rm ren} = \lim_{m\to 0} 2m^2(\langle\phi\phi^{\dagger}\rangle_{\rm ren} - \langle\phi\phi^{\dagger}\rangle^{(4)}) \ . \qquad (2.37)$$

The fourth-order subtraction term, which produces a nonzero finite contribution when the mass vanishes, is codified in $\langle\phi\phi^{\dagger}\rangle^{(4)}$. The piece $m^2\langle\phi\phi^{\dagger}\rangle_{ren}$ vanishes when $m^2 \to 0$. The remaining term produces the anomaly, which in terms of adiabatic expansion of the functions is

$$\langle\phi\phi^{\dagger}\rangle^{(n)} = \frac{2\pi}{2(2\pi a)^3}\int_0^{\infty}\int_{-\infty}^{\infty}|k_{\perp}|\left(W^{-1/2}\right)^{(4)} dk_x dk_{\perp} \ . \qquad (2.38)$$

After some computation, the trace anomaly is finally given by

$$\langle T^{\mu}_{\mu}\rangle_{\rm ren} = \frac{a^{(4)}}{240\pi^2 a} + \frac{\ddot{a}^2}{240\pi^2 a^2} + \frac{a^{(3)}\dot{a}}{80\pi^2 a^2} - \frac{\dot{a}^2\ddot{a}}{80\pi^2 a^3} - \frac{q^2\dot{A}^2}{48\pi^2 a^2} \ . \qquad (2.39)$$

This last term is in full agreement with the well-known trace anomaly for a background electromagnetic field in Minkowski spacetime [35]. The remaining terms reproduce the trace anomaly of the gravitational background with FLRW metric $ds^2 = dt^2 - a^2(t)d\vec{x}^2$. The result is twice the value obtained for a real scalar field [86]. In covariant form, we get

$$\langle T_\mu^\mu \rangle_{\rm ren} = \frac{1}{1440\pi^2}\left\{\Box R - \left(R^{\mu\nu}R_{\mu\nu} - \frac{R^2}{3}\right)\right\} + \frac{q^2}{96\pi^2}F_{\mu\nu}F^{\mu\nu}. \tag{2.40}$$

The ability to reproduce the conformal anomaly is a nontrivial test for our renormalization scheme. One can check that using the prescription of (2.30), the anomaly is not recovered, adding and additional obstacle for this prescription (see [46] for more detail).

2.3. Renormalization and Running Couplings

Until now, the results obtained assumed $\mu = 0$ consistent with the standard results of adiabatic regularization for a free field. The introduction of the mass scale μ leads to an inherent ambiguity in the adiabatic renormalization scheme, as also happens in dimensional regularization. It is natural to compare the renormalized current at two different scales: $\langle j^\beta \rangle_{\rm ren}(\mu) - \langle j^\beta \rangle_{\rm ren}(\mu_0) = \langle j^\beta \rangle^{(0-3)}(\mu_0) - \langle j^\beta \rangle^{(0-3)}(\mu)$. By using the above adiabatic expansion we find[1] (we rewrite the result in covariant terms)

$$\langle j^\beta \rangle_{\text{ren}}(\mu) - \langle j^\beta \rangle_{\text{ren}}(\mu_0) = -2\delta_q \nabla_\alpha F^{\alpha\beta}, \tag{2.41}$$

with $\delta_q = \frac{1}{6(4\pi)^2} \log\left(\frac{\mu^2+m^2}{\mu_0^2+m^2}\right)$. The semi-classical Maxwell equations should be independent of the μ, and therefore it is natural to include a dependence of μ in the coupling parameter, in this case the electric charge

$$\frac{1}{q^2(\mu)} \nabla_\alpha F^{\alpha\beta} = \langle j^\beta \rangle_{ren}(\mu) \; . \tag{2.42}$$

The independence of μ implies that we must also have

$$\frac{1}{q^2(\mu_0)} \nabla_\alpha F^{\alpha\beta} = \langle j^\beta \rangle_{ren}(\mu_0) \; . \tag{2.43}$$

Demanding now physical equivalence between between (2.42) and (2.43), and using (2.41), one obtains the running of the electric charge

$$\frac{1}{q^2(\mu)} - \frac{1}{q^2(\mu_0)} = -\frac{1}{48\pi^2} \log\left(\frac{\mu^2+m^2}{\mu_0^2+m^2}\right) , \tag{2.44}$$

in full agreement with the result obtained within perturbative scalar QED in Minkowski space in the limit $\mu^2 \gg m^2$ (using, for instance, dimensional regularization and the modified minimal subtraction scheme [104]). Note that, for getting the above result, there has been no need to assume a generic form for the electromagnetic

[1] For simplicity, we have reabsorbed here the charge dependence into the electromagnetic field term of the original Lagrangian.

background. It has been enough to use a background potential of the form $A_\mu = (0, -A(t), 0, 0)$. We also remark that (2.44) has been obtained without using any perturbative expansion in the coupling constant q.

Comparing now $\langle T_{\mu\nu}\rangle_{\text{ren}}(\mu)$ and $\langle T_{\mu\nu}\rangle_{\text{ren}}(\mu_0)$ we find (for simplicity we take here $\mu_0 = 0$)

$$\begin{aligned} &\langle T_{\alpha\beta}\rangle_{\text{ren}}(\mu) - \langle T_{\alpha\beta}\rangle_{\text{ren}}(m) = \\ &a g_{\mu\nu} + b G_{\mu\nu} + c\,{}^{(1)}H_{\mu\nu} + d\left(\frac{1}{4} g_{\alpha\beta} F_{\rho\sigma} F^{\rho\sigma} - F_\alpha{}^\rho F_{\beta\rho}\right) \end{aligned} \tag{2.45}$$

with

$$\begin{aligned} a &= \frac{-1}{(8\pi)^2}\frac{4}{3}(1-6\xi)\left(-\mu^2 + m^2 \log\left(\frac{\mu^2+m^2}{m^2}\right)\right) \\ b &= \frac{1}{(8\pi)^2}\frac{1}{9}(1-6\xi)^2 \log\left(\frac{\mu^2+m^2}{m^2}\right) \\ c &= \frac{1}{(8\pi)^2}\left(2m^2\mu^2 - \mu^4 + 2m^4 \log\left(\frac{\mu^2+m^2}{m^2}\right)\right) \\ d &= \frac{1}{3(4\pi)^2} \log\left(\frac{\mu^2+m^2}{m^2}\right). \end{aligned} \tag{2.46}$$

Here ${}^{(1)}H_{\mu\nu}$ is the conserved curvature tensor obtained by functionally differentiating the quadratic curvature Lagrangian R^2 with respect to the metric. The extra term $c\,{}^{(1)}H_{\mu\nu}$ implies the existence of a modification of general relativity due to quantum effects, as first pointed out in [109] for asymptotically flat spacetimes. Here there is no need to introduce the additional conserved tensor, $H^{(2)}_{\mu\nu}$, coming from the Lagrangian $R_{\mu\nu}R^{\mu\nu}$. This is because, in a FLRW spacetime, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ are not independent. As long as we treat the gravitational field as a classical background, no terms of higher order in the curvature are required.

At this point we should remark that expression (2.45) is compatible with the ambiguities in the quantization of the stress-energy tensor found in the algebraic approach to QFT in curved spacetime [61,63,112]. To be more precise, any two local and covariant procedures of renormalization of the stress-energy tensor should differ at most in a linear combination of conserved local terms: $\alpha g_{\mu\nu} + \beta G_{\mu\nu} + \gamma\, {}^{(1)}H_{\mu\nu} + \delta\, {}^{(2)}H_{\mu\nu}$. In a FLRW spacetime, $H^{(2)}_{\mu\nu}$ is proportional to $H^{(1)}_{\mu\nu}$, hence δ can be reabsorbed into γ. Moreover, since we have an additional external field (the electromagnetic background), the ambiguity should also include the electromagnetic stress-energy tensor. Therefore, given two prescriptions to renormalize the stress-energy tensor, denoted by $\langle T_{\alpha\beta}\rangle_{\rm ren}$ and $\langle \tilde{T}_{\alpha\beta}\rangle_{\rm ren}$, the difference for the expected stress-energy tensor is parametrized by (2.45) where the constant parameters a, b, c and d are not constrained within the axiomatic approach. We can identify $\langle \tilde{T}_{\alpha\beta}\rangle_{\rm ren}$ with the standard adiabatic prescription to renormalize the stress-energy tensor $\langle \tilde{T}_{\alpha\beta}\rangle_{\rm ren} \equiv \langle T_{\alpha\beta}\rangle_{\rm ren}(0)$, and $\langle T_{\alpha\beta}\rangle_{\rm ren}$ with our modified adiabatic prescription (parametrized by the mass scale μ): $\langle T_{\alpha\beta}\rangle_{\rm ren} \equiv \langle T_{\alpha\beta}\rangle_{\rm ren}(\mu)$. Therefore, the constant and finite parameters a, b, c and d naturally acquire a dependence on the scale μ as in (2.46). Furthermore, as we will see now, this implies a natural running for the gravitational coupling constants.

The semi-classical Maxwell-Einstein equations are given by (2.42) together with

$$\langle T_{\alpha\beta}\rangle_{\rm ren}(\mu) + T^{EM}_{\alpha\beta}(\mu) = \frac{-G_{\alpha\beta}}{8\pi G(\mu)} - \Lambda(\mu) g_{\alpha\beta} - \alpha^1(\mu) H^{(1)}_{\alpha\beta}\ , \tag{2.47}$$

with $T^{EM}_{\alpha\beta} = \frac{1}{q^2(\mu)}\left(\frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}g_{\alpha\beta} - F_{\alpha}^{\ \rho}F_{\beta\rho}\right)$. The coupling Λ is related to the cosmological constant Λ_c by the relation $\Lambda = \Lambda_c/(8\pi G)$. Enforcing that the above equations be independent of the scale μ, we obtain, using the above results for a and b, the running of the Newton gravitational constant G and Λ. The running of q can also be obtained, and coincides with the result (2.44), derived directly from the renormalization of the electric current. We can encode the running of all the couplings in the beta functions $\beta_O = \mu\frac{d}{d\mu}O$. For the case of the dimensionfull constants we obtain

$$\beta_\Lambda = \frac{1}{16\pi^2}\frac{\mu^6}{m^2+\mu^2} \quad \beta_\kappa = \frac{\xi-\frac{1}{6}}{4\pi^2}\frac{\mu^4}{m^2+\mu^2} \tag{2.48}$$

where $\kappa = (8\pi G)^{-1}$, while for the adimensional ones

$$\beta_q = \frac{q^3}{48\pi^2}\frac{\mu^2}{\mu^2+m^2} \quad \beta_\alpha = \frac{\left(\xi-\frac{1}{6}\right)^2}{8\pi^2}\frac{\mu^2}{\mu^2+m^2}\ . \tag{2.49}$$

For the last one, we recover in the limit $\mu^2 \gg m^2$ the standard results of perturbative quantum field theory, using dimensional regularization and MS [86]. However, an intriguing result is that for the dimensional case we do obtain a difference with respect to

MS [101]

$$\beta_{\Lambda}^{MS} = \frac{m^4}{16\pi^2} \quad \beta_{\kappa}^{MS} = \frac{m^2}{4\pi^2}\left(\frac{1}{6} - \xi\right) . \tag{2.50}$$

We will see in chapter 5 how this difference is determinant for the correct decoupling of the gravitational couplings in the limit where $m^2 \gg \mu^2$.

Chapter 3.

Adiabatic regularization for a 2-D Dirac field in an EM Background

We consider two-dimensional spinor QED in an expanding spacetime described by the metric $ds^2 = dt^2 - a^2(t)dx^2$. The classical action is given by

$$\mathcal{S} = \int dx^2 \sqrt{-g} \left(-\frac{1}{4q} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi \right) , \tag{3.1}$$

and the corresponding Dirac equation reads

$$(i\underline{\gamma}^{\mu} \nabla_{\mu} - m)\psi = 0, \tag{3.2}$$

where $\nabla_{\mu} \equiv \partial_{\mu} - \Gamma_{\mu} - iqA_{\mu}$ and Γ_{μ} is the spin connection. $\underline{\gamma}^{\mu}(x)$ are the spacetime-dependent Dirac matrices satisfying the anticommutation relations $\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}$. These gamma matrices are related with the Minkowskian ones by $\underline{\gamma}^0(t) = \gamma^0$ and $\underline{\gamma}^1(t) = \gamma^1/a(t)$, and the components of the spin connections are $\Gamma_0 = 0$ and $\Gamma_1 = (\dot{a}/2)\gamma_0\gamma_1$. Therefore, $\underline{\gamma}^\mu \Gamma_\mu = -\frac{\dot{a}}{2a}\gamma_0$ and we fix a gauge for the potential as $A_\mu = (0, -A(t))$. The Dirac equation (3.2) becomes

$$\left(i\gamma^0\partial_0 + \frac{i}{2}\frac{\dot{a}}{a}\gamma^0 + \left(\frac{i}{a}\partial_1 + \frac{A_1}{a}\right)\gamma^1 - m\right)\psi = 0. \tag{3.3}$$

From now on we will use the Weyl representation (with $\gamma^5 \equiv \gamma^0\gamma^1$)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

Expanding the field in momentum modes $\psi(t,x) = \sum_k \psi_k(t)e^{ikx}$, (3.3) is converted into

$$\left(\partial_0 + \frac{\dot{a}}{2a} + \frac{i}{a}(k+A)\gamma^5 + im\gamma^0\right)\psi_k = 0\,. \tag{3.5}$$

We can construct two independent spinor solutions

$$u_k(t,x) = \frac{e^{ikx}}{\sqrt{2\pi a}}\begin{pmatrix} h_k^I(t) \\ -h_k^{II}(t) \end{pmatrix} \quad v_k(t,x) = \frac{e^{-ikx}}{\sqrt{2\pi a}}\begin{pmatrix} h_{-k}^{II*}(t) \\ h_{-k}^{I*}(t) \end{pmatrix}, \tag{3.6}$$

where $h_k^I(t)$ and $h_k^{II}(t)$ are appropriate solutions of the equations

$$i\partial_t \begin{pmatrix} h^I \\ h^{II} \end{pmatrix} = \begin{pmatrix} \frac{-k-A}{a} & -m \\ -m & \frac{k+A}{a} \end{pmatrix}\begin{pmatrix} h^I \\ h^{II} \end{pmatrix}. \tag{3.7}$$

The normalization condition $|h^I|^2 + |h^{II}|^2 = 1$ leads to the usual Dirac scalar products

$$(u_k, u_{k'}) = \delta(k - k'); \quad (v_k, v_{k'}) = \delta(k - k'); \quad (u_k, v_{k'}) = 0\,. \quad (3.8)$$

These conditions guarante the anticommutation relations for the creation and annihilation operators B_k and D_k, defined by the expansion of the Dirac field operator in terms of the above spinors

$$\psi(t, x) = \int dk \left[B_k u_k(t, x) + D_k^\dagger v_k(t, x) \right]\,. \quad (3.9)$$

The usual equal-time anticommutation relation holds

$$\{\psi_\alpha(t, x), \psi_\beta^\dagger(t, y)\} = \delta(x - y)\delta_{\alpha\beta}\,. \quad (3.10)$$

The stress-energy tensor is given by

$$T^m_{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{V_{\nu a}}{\det V} \frac{\delta S_m}{\delta V^\mu_a} = \frac{i}{2} \left[\bar{\psi} \underline{\gamma}_{(\mu} \nabla_{\nu)} \psi - (\nabla_{(\mu} \bar{\psi}) \underline{\gamma}_{\nu)} \psi \right]\,, \quad (3.11)$$

which for a FLRW spacetime, introducing the expansion (3.9) with the modes (3.8) can be simplified in

$$\langle T_{00} \rangle = \frac{1}{2\pi a} \int_0^\infty dk \rho_k(t)\,, \quad \rho_k(t) \equiv i \left(h_k^I \frac{\partial h_k^{I*}}{\partial t} + h_k^{II} \frac{\partial h_k^{II*}}{\partial t} \right)\,, \quad (3.12)$$

and

$$\langle T_{ii}\rangle = \frac{1}{2\pi}\int_0^\infty dk p_k(t)\ , \quad p_k(t) \equiv (k+qA)\left(|h_k^I|^2 - |h_k^{II}|^2\right)\ . \tag{3.13}$$

The electric current is

$$\langle j^x\rangle = \frac{1}{2\pi a^2}\int_{-\infty}^\infty dk\left(|h_k^{II}|^2 - |h_k^I|^2\right)\ . \tag{3.14}$$

Both magnitudes $\langle T_{ab}\rangle$ and $\langle j^a\rangle$ carry divergences. In particular the stress-energy tensor has quartic and logarithmic divergences and therefore need to be subtracted up to adiabatic order two whereas the electric current only need to be subtracted up to adiabatic order one.

3.1. Adiabatic Regularization

As mentioned in the previous section the Dirac equation in terms of the modes can be expressed as[1]

$$i\partial_t\begin{pmatrix} h^I \\ h^{II}\end{pmatrix} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & -\alpha(t)\end{pmatrix}\begin{pmatrix} h^I \\ h^{II}\end{pmatrix}\ , \tag{3.15}$$

[1] We will present here the regularization procedure for a more general configuration as (3.15), since the same mechanism will be performed in the next chapter but with different coefficients α and β.

which is a Schrödinger type equation

$$i\partial_t \boldsymbol{h} = \boldsymbol{H}(t)\boldsymbol{h} \tag{3.16}$$

with

$$\boldsymbol{H}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & -\alpha(t) \end{pmatrix} . \tag{3.17}$$

The adiabatic expansion consists on an iterative procedure to transform (3.15) in such a way that we get consistent approximations for its solutions at the desired adiabatic order. The three basic features of the proposed procedure are [10]:

1. At each step we introduce a change of variables defined by an unitary transformation. This guarantees that the normalization condition $|h^I|^2 + |h^{II}|^2 = 1$ is automatically preserved.

2. At each step it will be evident how to truncate the resulting equations to reach the desired order in the adiabatic approximation.

3. The truncated equations involve a diagonal time-dependent Hamiltonian and, hence, can be trivially solved.

We first diagonalize the matrix Hamiltonian

$$\boldsymbol{H}(t) = \boldsymbol{U}_0(t)\boldsymbol{D}_0(t)\boldsymbol{U}_0^\dagger(t) \tag{3.18}$$

with

$$\boldsymbol{D}_0(t) = \begin{pmatrix} \omega_0(t) & 0 \\ \\ 0 & -\omega_0(t) \end{pmatrix},$$

$$\boldsymbol{U}_0(t) = \begin{pmatrix} \sqrt{\frac{\omega_0(t)+\alpha(t)}{2\omega_0(t)}} & \sigma_\beta\sqrt{\frac{\omega_0(t)-\alpha(t)}{2\omega_0(t)}} \\ \\ \sigma_\beta\sqrt{\frac{\omega_0(t)-\alpha(t)}{2\omega_0(t)}} & -\sqrt{\frac{\omega_0(t)+\alpha(t)}{2\omega_0(t)}} \end{pmatrix}, \qquad (3.19)$$

where $\omega_0(t) := \sqrt{\alpha^2(t) + \beta^2(t)}$ and σ_β denotes the sign of $\beta(t)$.

We introduce a change of variables $\boldsymbol{h}_0(t) := \boldsymbol{U}_0^\dagger(t)\boldsymbol{h}(t)$. They satisfy the new Schrödinger equation

$$i\partial_t \boldsymbol{h}_0 = \boldsymbol{H}_0(t)\boldsymbol{h}_0\,, \qquad (3.20)$$

where $\boldsymbol{H}_0(t) := \boldsymbol{D}_0(t) - i\boldsymbol{U}_0^\dagger(t)\partial_t\boldsymbol{U}_0(t)$ has the explicit form

$$\boldsymbol{H}_0 = \begin{pmatrix} \omega_0 & i\sigma_\beta\dfrac{\omega_0\dot{\alpha} - \alpha\dot{\omega}_0}{2\omega_0\sqrt{\omega_0^2 - \alpha^2}} \\ \\ -i\sigma_\beta\dfrac{\omega_0\dot{\alpha} - \alpha\dot{\omega}_0}{2\omega_0\sqrt{\omega_0^2 - \alpha^2}} & -\omega_0 \end{pmatrix}.$$

Here we have used dots to represent time derivatives and lightened the notation by not writing the explicit time dependence.

The key observation at this point is to realize that the lowest adiabatic order of the off-diagonal terms of $\boldsymbol{H}_0$ is one unit higher than that of the corresponding ones in $\boldsymbol{H}$. If we repeat now the previous procedure (diagonalization of the Hamiltonian and "unitary change of variables") this same behavior will occur at each iteration order.

Once the non-diagonal elements of the Hamiltonian surpass a certain adiabatic order n we will discard them. By doing this the resulting Schrödinger equation can then trivially solved (because the corresponding Hamiltonian is diagonal) and, by undoing the sequence of changes of variables arrive at an approximate solution to (3.15).

If at a certain iteration order $j \geq 0$ we have $\boldsymbol{h}_j$ obtained from the Schrödinger equation associated with the Hamiltonian $\boldsymbol{H}_j$ the objects in the $j+1$ step are given by

$$\boldsymbol{h}_{j+1} = \boldsymbol{U}^{\dagger}_{j+1}\boldsymbol{h}_j \quad \boldsymbol{H}_{j+1} = \qquad \boldsymbol{D}_{j+1} - i\boldsymbol{U}^{\dagger}_{j+1}\partial_t\boldsymbol{U}_{j+1}\,, \tag{3.21}$$

with the diagonal matrix $\boldsymbol{D}_{j+1}$ and the unitary matrix $\boldsymbol{U}_{j+1}$ obtained by diagonalizing $\boldsymbol{H}_j$:

$$\boldsymbol{H}_j = \boldsymbol{U}_{j+1}\boldsymbol{D}_{j+1}\boldsymbol{U}^{\dagger}_{j+1}\,. \tag{3.22}$$

Notice that $\boldsymbol{h}_{j+1}$ satisfies the Schrödinger equation

$$i\partial_t\boldsymbol{h}_{j+1} = \boldsymbol{H}_{j+1}\boldsymbol{h}_{j+1}\,. \tag{3.23}$$

The explicit expressions for the $\boldsymbol{H}_j$, $\boldsymbol{U}_j$ and $\boldsymbol{D}_j$ are

$$\boldsymbol{H}_j = \begin{pmatrix} \omega_j & S_{j-1}\frac{\omega_{j-1}\dot{\omega}_j-\omega_j\dot{\omega}_{j-1}}{2\omega_j\sqrt{\omega_j^2-\omega_{j-1}^2}} \\ S^*_{j-1}\frac{\omega_{j-1}\dot{\omega}_j-\omega_j\dot{\omega}_{j-1}}{2\omega_j\sqrt{\omega_j^2-\omega_{j-1}^2}} & -\omega_j \end{pmatrix}$$

$$\boldsymbol{U}_j = \begin{pmatrix} \sqrt{\frac{\omega_j+\omega_{j-1}}{2\omega_j}} & iS_{j-1}\sqrt{\frac{\omega_j-\omega_{j-1}}{2\omega_j}} \\ iS_{j-1}\sqrt{\frac{\omega_j-\omega_{j-1}}{2\omega_j}} & (-1)^{j+1}\sqrt{\frac{\omega_j+\omega_{j-1}}{2\omega_j}} \end{pmatrix},$$

$$\boldsymbol{D}_j = \begin{pmatrix} \omega_j & 0 \\ 0 & -\omega_j \end{pmatrix},$$

where the positive frequencies ω_j satisfy the recurrence

$$\omega_{j+1}^2 = \omega_j^2 + \frac{(\omega_{j-1}\dot{\omega}_j - \omega_j\dot{\omega}_{j-1})^2}{(2\omega_j)^2(\omega_j^2 - \omega_{j-1}^2)}\,, \tag{3.24}$$

with initial data

$\omega_0^2 = \alpha^2 + \beta^2\,, \quad \omega_1^2 = \omega_0^2 + \frac{(\omega_0\dot{\alpha}-\dot{\omega}_0\alpha)^2}{4\omega_0^2\beta^2}$. The S_j coefficients are given by

$$S_j = \begin{cases} -is_{j-1}\,, & j \text{ even} \\ s_{j-1}\,, & j \text{ odd} \end{cases} \tag{3.25}$$

where $s_j = s_{j-1}\text{sign}(\omega_{j-1}\dot{\omega}_j - \dot{\omega}_{j-1}\omega_j)$ for the case of $j \geq 1$, and $s_0 = \sigma_\beta \text{sign}(\alpha\dot{\omega}_0 - \omega_0\dot{\alpha})$.

Several comments are in order now. First it is important to notice that the lowest adiabatic weight of the non-diagonal terms of the Hamiltonian $\boldsymbol{H}_j$ is larger than j (although even higher orders may be present). This fact suggests a terminating criterion to obtain an approximate solution valid at adiabatic order n: replace the Hamiltonian $\boldsymbol{H}_n$ by its diagonal part $\boldsymbol{D}_n$ and approximate $\boldsymbol{h}_n$ by $\tilde{\boldsymbol{h}}_n$ satisfying the Schrödinger equation$\partial_t \tilde{\boldsymbol{h}}_n = \boldsymbol{D}_n \tilde{\boldsymbol{h}}_n$ This way we get

$$\begin{aligned}\tilde{\boldsymbol{h}}_n(t) &= \widetilde{\boldsymbol{U}}_n(t,t_0)\mathfrak{h}(t_0)\\ &:= \begin{pmatrix} \exp(-i\int_{t_0}^t \omega_n) & 0 \\ 0 & \exp(i\int_{t_0}^t \omega_n) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix},\end{aligned}$$

where our choice of initial data selects positive frequencies.

The final form for the approximate solution $\boldsymbol{h}(t|n)$ of (3.15) can be obtained by undoing the unitary transformations introduced above

$$\boldsymbol{h}(t) \sim \boldsymbol{h}(t|n) := \boldsymbol{U}_0(t)\boldsymbol{U}_1(t)\cdots\boldsymbol{U}_n(t)\widetilde{\boldsymbol{U}}_n(t,t_0)\mathfrak{h}(t_0)\,. \tag{3.26}$$

From this last expression it is straightforward to obtain the adiabatic expansion of $\boldsymbol{h}$ to order n.

For the 1+1 dimensional Dirac field we only need to compute up to adiabatic order two in order to subtract the divergences of (3.12), (3.13) and (3.30). We compute $\boldsymbol{h}(t|2)$ from (3.26). After a straightforward computation we find now

$$h^I \sim \sqrt{\frac{\omega - k/a}{2\omega}}\left(1 + \phi^{(1)} + \phi^{(2)}\right) \exp\left(-i \int \omega_2\right) \qquad (3.27)$$

where $\omega^2 = m^2 + k^2/a^2$ and

$$\begin{aligned}
\phi^{(1)} &= -\frac{Aq}{2a\omega} - \frac{Akq}{2a^2\omega^2} + \frac{im^2\dot{a}}{4a\omega^3} - \frac{i\dot{a}}{4a\omega} - \frac{ik\dot{a}}{4a^2\omega^2} \\
\phi^{(2)} &= \frac{A^2kq^2}{2a^3\omega^3} - \frac{5A^2m^2q^2}{8a^2\omega^4} + \frac{A^2q^2}{2a^2\omega^2} - \frac{7iAkm^2q\dot{a}}{8a^3\omega^5} + \frac{iAkq\dot{a}}{2a^3\omega^3} \\
&- \frac{3iAm^2q\dot{a}}{4a^2\omega^4} + \frac{iAq\dot{a}}{2a^2\omega^2} + \frac{iq\dot{A}}{4a\omega^2} + \frac{ikq\dot{A}}{4a^2\omega^3} - \frac{k\ddot{a}}{8a^2\omega^3} + \frac{m^2\ddot{a}}{8a\omega^4} \\
&- \frac{\ddot{a}}{8a\omega^2} + \frac{3km^2\dot{a}^2}{8a^3\omega^5} - \frac{k\dot{a}^2}{8a^3\omega^3} - \frac{11m^4\dot{a}^2}{32a^2\omega^6} + \frac{15m^2\dot{a}^2}{32a^2\omega^4} - \frac{\dot{a}^2}{8a^2\omega^2} \\
\omega_2 &= -\frac{A^3km^2q^3}{2a^4\omega^5} + \frac{35\dot{a}^2A^2m^6q^2}{16a^4\omega^9} - \frac{45\dot{a}^2A^2m^4q^2}{16a^4\omega^7} - \frac{5A^4m^4q^4}{8a^4\omega^7} \\
&+ \frac{3\dot{a}^2A^2m^2q^2}{4a^4\omega^5} + \frac{A^2m^2q^2}{2a^2\omega^3} + \frac{A^4m^2q^4}{2a^4\omega^5} + \frac{5\dot{a}^2Akm^4q}{8a^4\omega^7} - \frac{3\dot{a}^2Akm^2q}{8a^4\omega^5} \\
&- \frac{\dot{a}\dot{A}km^2q}{4a^3\omega^5} + \frac{Akq}{a^2\omega} - \frac{5\dot{a}A\dot{A}m^4q^2}{4a^3\omega^7} + \frac{\dot{A}^2m^2q^2}{8a^2\omega^5} + \frac{\dot{a}A\dot{A}m^2q^2}{a^3\omega^5} \\
&+ \frac{3\dot{a}^2m^6\ddot{a}}{16a^3\omega^9} - \frac{m^4\ddot{a}^2}{32a^2\omega^7} - \frac{\dot{a}^2m^4\ddot{a}}{4a^3\omega^7} + \frac{m^2\ddot{a}^2}{32a^2\omega^5} + \frac{\dot{a}^2m^2\ddot{a}}{16a^3\omega^5} - \frac{37\dot{a}^4m^8}{128a^4\omega^{11}} \\
&+ \frac{31\dot{a}^4m^6}{64a^4\omega^9} - \frac{29\dot{a}^4m^4}{128a^4\omega^7} - \frac{\dot{a}^2m^4}{8a^2\omega^5} + \frac{\dot{a}^4m^2}{32a^4\omega^5} + \frac{\dot{a}^2m^2}{8a^2\omega^3} + \omega. \qquad (3.28)
\end{aligned}$$

The asymptotic expansion for h^{II} can be obtained from that of h^I by substituting $a(t)$ for $-a(t)$ and introducing a global minus sign [46].

3.2. Regularization of the Stress-Energy Tensor and Electric current

The adiabatic subtractions for the stress-energy tensor are calculated by plugging h^I and h^{II} from the last section into (3.12) and (3.13). The regularized tensor is then

$$\langle T_{00}\rangle_{\rm ren} = \frac{1}{2\pi a}\int_{-\infty}^{\infty} dk\left(\rho_k + \omega + \frac{kqA}{\omega} + \frac{m^2q^2A^2}{2a^2\omega^3} - \frac{k^2m^2\dot{a}^2}{8a^4\omega^5}\right)$$
$$\langle T_{11}\rangle_{\rm ren} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk p_k + \frac{k^2}{a\omega} + \frac{km^2qA}{a\omega^3} + \frac{kqA}{a\omega} - \frac{m^4\ddot{a}}{4\omega^5} + \frac{m^2\ddot{a}}{4\omega^3}$$
$$+\frac{5m^6\dot{a}^2}{8a\omega^7} - \frac{3m^4\dot{a}^2}{4a\omega^5} + \frac{m^2\dot{a}^2}{8a\omega^3} + \frac{3m^4q^2A^2}{2a\omega^5} - \frac{m^2q^2A^2}{2a\omega^3}\,. \tag{3.29}$$

For the electric current we obtain

$$\langle j^x\rangle_{\rm ren} = \frac{q}{2\pi a^2}\int_{-\infty}^{\infty} dk\left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k}{a\omega} - \frac{qm^2}{a\omega^3}A\right). \tag{3.30}$$

One can check that the conservation of the stress-energy holds as in the 4D scalar field, $\nabla_\mu\langle T^{\mu\nu}\rangle_{\rm ren} + \nabla_\mu T^{\mu\nu}_{elec} = 0$, with $T^{elec}_{\mu\nu} = \frac{1}{2}E^2 g_{\mu\nu}$. Again, if we were to choose A of adiabatic order zero, the conservation would fail [49].

To account for the trace anomaly, the trace of the energy momentum tensor can be written as: $T^\mu_\mu = m\bar{\psi}\psi$. After renormalization we have a residual contribution when the mass goes to zero

$$\langle T^{\mu}_{\mu} \rangle_{\rm ren} = \lim_{m\to 0} -m\langle \bar{\psi}\psi \rangle^{(2)} = \lim_{m\to 0} \frac{-m}{2\pi a} \int_{-\infty}^{\infty} dk \left(h^{I*}h^{II} + h^{II*}h^{I} \right)^{(2)} . \tag{3.31}$$

By using the adiabatic expansion from (3.27) and (3.28) and integrating we can write:

$$\langle T^{\mu}_{\mu} \rangle_{ren} = -\frac{\ddot{a}}{12\pi a} = -\frac{R}{24\pi} , \tag{3.32}$$

where in the last step we have used the expression of the two-dimensional scalar curvature in the terms of the expansion factor. The result agrees with the value of the trace anomaly for a Dirac spinor in two dimensions [37], which in turn coincides with the trace anomaly of a real scalar field [15, 28, 36].

3.3. Chiral Anomaly

To test the self-consistency of the above adiabatic expansion we are also going to show how the chiral anomaly is obtained from it. We will consider the axial current

$$j^{\mu}_{A} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi , \tag{3.33}$$

which is conserved in the massless limit. To evaluate the expectation value $\langle \nabla_{\mu} j^{\mu}_{A} \rangle$ we will reintroduce the mass and evaluate the right-hand-side of

$$\langle \nabla_\mu j^\mu_A \rangle = 2im\langle \bar{\psi}\gamma^5\psi \rangle, \tag{3.34}$$

in the limit $m \to 0$. Since the formal expression for $\langle \nabla_\mu j^\mu_A \rangle$ has divergences till second adiabatic order we need to perform subtractions in $\langle \bar{\psi}\gamma^5\psi \rangle$ up to second adiabatic order. Therefore,

$$\langle \nabla_\mu j^\mu_A \rangle_{ren} = -\lim_{m \to 0} 2im\langle \bar{\psi}\gamma^5\psi \rangle^{(2)} . \tag{3.35}$$

By writing $\langle \nabla_\mu j^\mu_A \rangle$ in terms of $\{h^I, h^{II}\}$

$$\langle \bar{\psi}\gamma^5\psi \rangle = \frac{1}{2\pi a}\int_{-\infty}^{\infty} dk(h^{II*}h^I - h^{I*}h^{II}) , \tag{3.36}$$

and using our adiabatic series expansion, we arrive at

$$\langle \bar{\psi}\gamma^5\psi \rangle^{(2)} = \frac{iq\dot{A}}{2\pi am} . \tag{3.37}$$

This result leads immediately to the axial anomaly in two dimensions

$$\langle \nabla_\mu j^\mu_A \rangle_{ren} = \frac{q\dot{A}}{a\pi} = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} , \tag{3.38}$$

where $\epsilon^{01} = |g|^{-1/2} = a^{-1}$. This result reproduces exactly the chiral anomaly for spinor QED_2 [86]. For a massive field we obtain $\langle \nabla_\mu j^\mu_A \rangle_{\rm ren} = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + 2im\langle \bar{\psi}\gamma^5\psi \rangle_{\rm ren}$.

3.4. Breaking of Adiabatic invariance

The obtained chiral anomaly obtained in the last section can be interpreted as follows. We use the fact that in 2-D the axial current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is related by duality to the electric current $\langle j_\mu \rangle_{\rm ren} = q\epsilon_{\mu\nu}\langle j_5^\nu \rangle_{\rm ren}$. Hence, the result for the axial anomaly (3.38) implies

$$\langle j^x \rangle_{ren} = \frac{q^2(A(-\infty) - A(+\infty))}{\pi}, \tag{3.39}$$

where we have integrated the differential equation. The result (3.39) implies that for a massless field a net electric current will be generated by an electric potential independently of the time evolution history of the potential. This means that if we consider a slowly varying electric potential, i.e., and adiabatically slow evolution, there will be still particle production, which is a counterexample of the adiabatic invariance for a time dependent field as the gravitational field.

In order to see this in more detail, we will analyze a particular electric field configuration that exemplifies the adiabatic evolution of the field. We first start with a brief introduction of the adiabatic invariance in a gravitational time dependent field and then proceed with the electric field analogue. The former will be introduced for the scalar field case since it only has pedagogical purposes. The same scalar case can also be performed for the electric case [12], with a similar result as the Dirac field case. We omit this computation here and focus only in the Minkowski spacetime limit $a = 1$.

3.4.1. A brief orientation: Adiabatic invariance in FLRW

The adiabatic invariance of the particle number operator in an expanding universe can be easily illustrated with a simple two-dimensional example borrowed from [15]. This example, although well-know, will serve to better clarify the main idea of the next sections. Consider the following metric

$$ds^2 = dt^2 - a^2(t)dx^2 = C(\eta)(d\eta^2 - dx^2)\,, \tag{3.40}$$

where $d\eta = a^{-1}(t)dt$ and the conformal scale factor is given by the function $C(\eta) = 1 + B(1 + \tanh \rho\eta)$, with B a positive constant. This represents a smooth expansion bounded by asymptotically static and flat spacetime regions. The expansion factor has smoothly shifted from $a_{in} \equiv a(-\infty) = 1$ to $a_{out} \equiv a(+\infty) = \sqrt{1+2B}$.

In the remote past the normalized modes of a scalar field are assumed to behave as the positive frequency modes in Minkowski space

$$\frac{1}{\sqrt{2(2\pi)\omega_{in}}} e^{ikx} e^{-i\omega_{in}t}\,, \tag{3.41}$$

with $\omega_{in} = \sqrt{k^2 + m^2}$. As time evolves these modes behave, in the remote future, as a mixture of positive and negative frequency modes of the form

$$\frac{\alpha_k}{\sqrt{2(2\pi)\omega_{out}}} e^{ikx} e^{-i\omega_{out}t} + \frac{\beta_k}{\sqrt{2(2\pi)\omega_{out}}} e^{ikx} e^{+i\omega_{out}t}\,, \tag{3.42}$$

with $\omega_{out} = \sqrt{(\frac{k}{a_{out}})^2 + m^2}$. α_k and β_k are the so-called Bogoliubov coefficients. The annihilation operators for physical particles at late times a_k are related to the annihilation and creation operators at early times (A_k and $A_k^\dagger$) by the relations

$$a_k = \alpha_k A_k + \beta_k^* A_{-k}^\dagger \,. \tag{3.43}$$

The average density number of created particles n_k, with momentum k, is given by

$$n_k = |\beta_k|^2 = \frac{\sinh^2(\pi \frac{\omega_-}{\rho})}{\sinh(\pi \frac{\omega_{in}}{\rho}) \sinh(\pi \frac{a_{out}\omega_{out}}{\rho})} \,, \tag{3.44}$$

where $\omega_- = \frac{1}{2}(a_{out}\omega_{out} - \omega_{in})$. It is very easy to check that in the adiabatic limit, that is, for an extremely slow expansion $\rho \to 0$, the density number of created particles goes to $n_k \sim e^{-2\pi\omega_{in}/\rho} \to 0$. This shows the fact that the particle number is an adiabatic invariant. This behavior of the particle number observable is generic, and it can be extended to isotropically expanding universes in four dimensions, irrespective of the value of the mass [79–81].

3.4.2. Adiabatic Invariance in an Electric Field

In order to study the adiabatic limit for the electric pair production, we need to consider a bounded potential $A(t)$. Note that $A(t)$ will play a somewhat similar role to the conformal factor $C(\eta)$ for the expanding spacetime. To this end we choose for convenience a Sauter-type electric pulse [97] of the form

$$E(t) = -\frac{\rho A_0}{2} \cosh^{-2}(\rho t) \qquad (3.45)$$

which can be described, in the Coulomb gauge, by the potential ($E(t) = -\dot{A}(t)$)

$$A(t) = \frac{1}{2} A_0 \left(\tanh(\rho t) + 1\right) \ . \qquad (3.46)$$

This potential is bounded both at early and late times. The adiabatic limit is an extremely slow evolution of the potential, obtained when $\rho \to 0$. We have to remark that the adiabatic limit is not the limit of a vanishing electric field. If the electric field had support in a bounded period of time, there would not be production of particles when $E(t) \to 0$. But the adiabatic limit is a more subtle limit, in which the electric field varies very slowly. Although $E \to 0$ when $\rho \to 0$, the width of the pulse is also very large maintaining constant and non-vanishing the integral

$$\int_{-\infty}^{+\infty} E_{\rho_1}(t) dt = \int_{-\infty}^{+\infty} E_{\rho_2}(t) dt = \text{ constant} = -q A_0 \ . \qquad (3.47)$$

With this input the mode equations (3.7) can be solved exactly in terms of hypergeometric functions

$$h_k^{I/II}(t) = \pm\sqrt{\frac{\omega_{\text{in}} \mp k}{2\omega_{\text{in}}}} z^{-i\frac{\omega_{\text{in}}}{2\rho}} (1-z)^{i\frac{\omega_{\text{out}}}{2\rho}} \times F\left(i\frac{2\omega_- \pm qA_0}{2\rho}, 1 + i\frac{2\omega_- \mp qA_0}{2\rho}, 1 - i\frac{\omega_{\text{in}}}{\rho}; z\right) \tag{3.48}$$

where $z = \frac{A(t)}{A_0}$, $\omega_{\text{in}} = \sqrt{k^2+m^2}$, $\omega_{\text{out}} = \sqrt{(k+qA_0)^2+m^2}$ and $\omega_\pm = \frac{1}{2}(\omega_{\text{out}} \pm \omega_{\text{in}})$. We have fixed the initial condition in order to recover the positive frequency solution for a free field at early times $t \to -\infty$

$$h_k^{I/II}(t) \sim \pm\sqrt{\frac{\omega_{in} \mp k}{2\omega_{in}}} e^{-i\omega_{in} t} \,. \tag{3.49}$$

At late times $t \to +\infty$ the modes can be written as

$$h_k^{I/II}(t) \sim \pm\sqrt{\frac{\omega_{out} \mp (k+qA_0)}{2\omega_{out}}} \alpha_k e^{-i\omega_{out} t} + \sqrt{\frac{\omega_{out} \pm (k+qA_0)}{2\omega_{out}}} \beta_k e^{i\omega_{out} t} \,. \tag{3.50}$$

α_k and β_k are the Bogoliubov coefficients satisfying the relation $|\alpha_k|^2 + |\beta_k|^2 = 1$. These coefficients relate the early time creation and annihilation operators B_k, D_k with the late time operators b_k, d_k

as follows

$$b_k = \alpha_k B_k + \beta_k^* D_{-k}^\dagger \tag{3.51}$$
$$d_k = \alpha_{-k} D_k - \beta_{-k}^* B_{-k}^\dagger \, . \tag{3.52}$$

The density of created quanta is given by

$$N_k = \langle 0|b_k^\dagger b_k|0\rangle + \langle 0|d_k^\dagger d_k|0\rangle \equiv n_k + \bar{n}_k, \tag{3.53}$$

where $n_k = |\beta_k|^2$ and $\bar{n}_k = |\beta_{-k}|^2$. Therefore, the particle number is also

$$\langle N\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, N_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(|\beta_k|^2 + |\beta_{-k}|^2\right) \, . \tag{3.54}$$

The matching of (3.48) with (3.50) at late times determines the Bogoliubov coefficients. For the beta coefficients we get

$$\beta_k = \sqrt{\frac{\omega_{out}}{\omega_{in}}\frac{\omega_{in}-k}{\omega_{out}+k+qA_0}}\frac{\Gamma(1-i\frac{\omega_{in}}{\rho})\Gamma(-i\frac{\omega_{out}}{\rho})}{\Gamma(1+i\frac{\omega_- + qA_0/2}{\rho})\Gamma(1+i\frac{\omega_- - qA_0/2}{\rho})} \, . \tag{3.55}$$

Therefore,

$$|\beta_k|^2 = \frac{\omega_{in}-k}{\omega_{out}+k+qA_0}\frac{2\omega_- + qA_0}{2\omega_- - qA_0}\frac{\cosh(2\pi\frac{\omega_-}{\rho}) - \cosh(\pi\frac{qA_0}{\rho})}{2\sinh(\pi\frac{\omega_{in}}{\rho})\sinh(\pi\frac{\omega_{out}}{\rho})} \, . \tag{3.56}$$

The number of particles decreases as $\rho \to 0$ and increases as $m \to 0$. For fermions, the relation $|\alpha_k|^2 + |\beta_k|^2 = 1$ implies that $|\beta_k|^2 \leq 1$

for any value of k, according to Pauli's exclusion principle. In the massless case, irrespective of the value of ρ, one obtains

$$\lim_{m\to 0} |\beta_k|^2 = 1 \tag{3.57}$$

for $k \in (0, qA_0)$, and hence

$$N_k = \begin{cases} 0 & \text{for } k \notin (-qA_0, qA_0) \\ 1 & \text{for } k \in (-qA_0, qA_0) \quad . \end{cases} \tag{3.58}$$

The total density of created quanta is

$$\langle N \rangle = \frac{1}{2\pi} \int_{-|qA_0|}^{|qA_0|} dk\, N_k = \frac{|qA_0|}{\pi}. \tag{3.59}$$

This implies that the particle number is not and adiabatic invariant for the massless case. The same result (3.59) occurs for the scalar field in 1+1 D, but in this case the result was obtained by performing the adiabatic limit $\rho \to 0$. Here, we did not have to do this since the result is independent of the Dirac field history between the two asymptotic limits [12].

For massive fermions and in the limit $\rho \to 0$, expression (3.56) behaves essentially as

$$|\beta_k|^2 \sim e^{-\frac{\pi}{\rho}\delta} , \tag{3.60}$$

where $\delta = 2\omega_+ - |qA_0|$. For $m \neq 0$, the former has a minimum at $k = -\frac{qA_0}{2}$, with value $\delta_{min} = \sqrt{(qA_0)^2 + 4m^2} - |qA_0| > 0$. Hence,

$\delta > 0$ and $|\beta_k|^2 \to 0$. Therefore we can conclude that the particle number is an adiabatic invariant only for massive Dirac fields.

Using the renormalization method described in the last section for a Dirac field interacting with an homogeneous time-dependent electric field, the vacuum expectation value of the electric current $j^\mu = -q\bar{\psi}\gamma^\mu\psi$ is given by

$$\langle j^x \rangle_{\text{ren}} = \frac{q}{2\pi} \int dk \left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k}{\omega} - \frac{qm^2}{\omega^3} A \right) . \tag{3.61}$$

To study the explicit dependence of the electric current $\langle j^x \rangle$ with the mass, we can compute their time derivative

$$\partial_t \langle j^x \rangle_{\text{ren}} = \frac{2qm}{\pi} \left(\int \mathcal{I} \left(h_k^{II} h_k^{I*} \right) dk \right) - \frac{q^2}{\pi} \dot{A}. \tag{3.62}$$

It is immediate to see that in the massless limit the first term vanishes, and the equation below can be easily integrated. With $A(-\infty) = 0$ as initial condition one obtains

$$\langle j^x \rangle_{\text{ren}} = -\frac{q^2 A(t)}{\pi} . \tag{3.63}$$

The same result can be obtained by analyzing the behaviour of the Bogoliubov coefficients from (3.50) in the case of $m \to 0$. In conclusion, in the special case of mass-less Dirac fields, where the axial symmetry is broken trough the chiral anomaly the adiabatic invariance is broken, i.e., there is a net production of electric current (or particles) in an infinite slow varying electric potential. A similar result was obtained for the four dimensional case [12].

Chapter 4.

Adiabatic regularization of Dirac Fields in a Scalar field background

We consider the theory defined by the action functional of the form $S = S[g_{\mu\nu}, \Phi, \psi, \nabla\psi]$, where ψ represents a Dirac field, Φ is a scalar field, and $g_{\mu\nu}$ stands for the spacetime metric. We decompose the action as $S = S_g + S_m$, where S_m is the matter sector

$$S_m = \int d^4x\sqrt{-g}\ \left\{\frac{i}{2}[\bar{\psi}\underline{\gamma}^{\mu}\nabla_{\mu}\psi - (\nabla_{\mu}\bar{\psi})\underline{\gamma}^{\mu}\psi)] - m\bar{\psi}\psi - g_Y\Phi\bar{\psi}\psi\right\}\ , \tag{4.1}$$

and S_g is the gravity-scalar sector. Here, $\underline{\gamma}^{\mu}(x)$ are the spacetime-dependent Dirac matrices satisfying the anti-commutation relations $\{\underline{\gamma}^{\mu}, \underline{\gamma}^{\nu}\} = 2g^{\mu\nu}$, related to the usual Minkowski ones by the vierbein field $V^a_{\mu}(x)$, defined through $g_{\mu\nu}(x) = V^a_{\mu}(x)V^b_{\nu}(x)\eta_{ab}$. On the other hand, $\nabla_\mu \equiv \partial_\mu - \Gamma_\mu$ is the covariant derivative associated to the spin connection Γ_μ; m is the mass of the Dirac field; and g_Y is the dimensionless coupling constant of the Yukawa interaction. In (4.1), both the metric $g_{\mu\nu}(x)$ and the scalar field $\Phi(x)$ are regarded as classical external fields. The Dirac spinor $\psi(x)$ will be our quantized field, living in a curved spacetime and possessing a Yukawa coupling to the classical field Φ. The Dirac equation is

$$(i\underline{\gamma}^\mu \nabla_\mu - m - g_Y \Phi)\psi = 0 \,, \tag{4.2}$$

and the stress-energy tensor is given by [15]

$$T^m_{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{i}{2} \left[\bar{\psi} \underline{\gamma}_{(\mu} \nabla_{\nu)} \psi - (\nabla_{(\mu} \bar{\psi}) \underline{\gamma}_{\nu)} \psi \right] \,. \tag{4.3}$$

The complete theory, including the gravity-scalar sector in the action, can be described by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{16\pi G} + \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \right\} + S_m \,, \tag{4.4}$$

where S_m is the action for the matter sector given in (4.1). We will reconsider the form of the action in Section 4.4, in view of the counterterms required to cancel the UV divergences of the quantized Dirac field. However, let us work for the moment with the action (4.4). The Einstein equations are then

$$G^{\mu\nu} + 8\pi G(\nabla^\mu \Phi \nabla^\nu \Phi - \frac{1}{2} g^{\mu\nu} \nabla^\rho \Phi \nabla_\rho \Phi + g^{\mu\nu} V(\Phi)) = -8\pi G T^{\mu\nu}_m \,, \tag{4.5}$$

and the equation for the scalar field is

$$\Box\Phi + \frac{\partial V}{\partial\Phi} = -g_Y\bar{\psi}\psi\,. \tag{4.6}$$

The semiclassical equations are obtained from (4.5) and (4.6) by replacing $T_m^{\mu\nu}$ and $\bar{\psi}\psi$ by the corresponding (renormalized) vacuum expectation values $\langle T_m^{\mu\nu}\rangle_{\rm ren}$ and $\langle\bar{\psi}\psi\rangle_{\rm ren}$,

$$G^{\mu\nu} + \kappa(\nabla^\mu\Phi\nabla^\nu\Phi - \frac{1}{2}g^{\mu\nu}\nabla^\rho\Phi\nabla_\rho\Phi + g^{\mu\nu}V(\Phi)) = -\kappa\langle T_m^{\mu\nu}\rangle_{\rm ren} \quad , \tag{4.7}$$

$$\Box\Phi + \frac{\partial V}{\partial\Phi} = -g_Y\langle\bar{\psi}\psi\rangle_{\rm ren} \quad . \tag{4.8}$$

with $\kappa = 8\pi G$. In a spatially flat FLRW spacetime, the time-dependent gamma matrices are related with the Minkowskian ones by $\underline{\gamma}^0(t) = \gamma^0$ and $\underline{\gamma}^i(t) = \gamma^i/a(t)$, and the components of the spin-connections are $\Gamma_0 = 0$ and $\Gamma_i = (\dot{a}/2)\gamma_0\gamma_i$. The Dirac equation with the Yukawa interaction $i\underline{\gamma}^\mu\nabla_\mu\psi - m\psi = g_Y\Phi\psi$, taking Φ as a homogenous scalar field $\Phi = \Phi(t)$, is then

$$\left(\partial_0 + \frac{3}{2}\frac{\dot{a}}{a} + \frac{1}{a}\gamma^0\vec{\gamma}\vec{\nabla} + i(m + s(t))\gamma^0\right)\psi = 0\,, \tag{4.9}$$

where we have defined $s(t) \equiv g_Y\Phi(t)$. If we expand the field ψ as $\psi = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\psi_{\vec{k}}(t)e^{i\vec{k}\vec{x}}$, and we substitute it into (4.9), we obtain the following differential equation for $\psi_{\vec{k}}$:

$$\left(\partial_t + \frac{3\dot{a}}{2a} + i\gamma^0\vec{\gamma}\frac{\vec{k}}{a} + i\gamma^0(m + s(t))\right)\psi_{\vec{k}} = 0 \,. \tag{4.10}$$

In order to solve this equation, it is convenient to write the Dirac field in terms of two-component spinors of the generic form

$$\psi_{\vec{k},\lambda}(t) = \frac{1}{a^{3/2}(t)}\begin{pmatrix} h_k^I(t)\xi_\lambda(\vec{k}) \\ h_k^{II}(t)\frac{\vec{\sigma}\vec{k}}{k}\xi_\lambda(\vec{k}) \end{pmatrix} \,, \tag{4.11}$$

where ξ_λ with $\lambda = \pm 1$ are two constant orthonormal two-spinors ($\xi_\lambda^\dagger \xi_{\lambda'} = \delta_{\lambda,\lambda'}$), eigenvectors of the helicity operator $\frac{\vec{\sigma}\vec{k}}{2k}\xi_\lambda = \frac{\lambda}{2}\xi_\lambda$. The time-dependent functions h_k^I and h_k^{II} satisfy the first-order coupled equations

$$h_k^{II} = \frac{ia}{k}\left(\frac{\partial h_k^I}{\partial t} + i(m+s)h_k^I\right) \,, \quad h_k^I = \frac{ia}{k}\left(\frac{\partial h_k^{II}}{\partial t} - i(m+s)h_k^{II}\right) \,. \tag{4.12}$$

Given a particular solution $\{h_k^I(t), h_k^{II}(t)\}$ to equations (4.12), one can construct the modes

$$u_{\vec{k},\lambda}(t) = \frac{e^{i\vec{k}\vec{x}}}{\sqrt{(2\pi)^3 a^3(t)}}\begin{pmatrix} h_k^I(t)\xi_\lambda(\vec{k}) \\ h_k^{II}(t)\frac{\vec{\sigma}\vec{k}}{k}\xi_\lambda(\vec{k}) \end{pmatrix} \,. \tag{4.13}$$

Equation (4.13) will be a solution of positive-frequency type in the adiabatic regime. A solution of negative-frequency type can be obtained by applying a charge conjugate transformation $C\psi =$

$-i\gamma^2\psi^*$ (we follow here the convention in [91])

$$v_{\vec{k},\lambda}(t) = Cu_{\vec{k},\lambda}(t) = \frac{e^{-i\vec{k}\vec{x}}}{\sqrt{(2\pi)^3 a^3(t)}} \begin{pmatrix} h_k^{II*}(t)\xi_{-\lambda}(\vec{k}) \\ h_k^{I*}(t)\frac{\vec{\sigma}\vec{k}}{k}\xi_{-\lambda}(\vec{k}) \end{pmatrix} . \quad (4.14)$$

The Dirac inner product is defined as $(\psi_1, \psi_2) = \int d^3x a^3 \psi_1^\dagger \psi_2$. The normalization condition for the above four-spinors, $(u_{\vec{k}\lambda}, v_{\vec{k}'\lambda'}) = 0$, $(u_{\vec{k}\lambda}, u_{\vec{k}'\lambda'}) = (v_{\vec{k}\lambda}, v_{\vec{k}'\lambda'}) = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k} - \vec{k}')$, reduces to

$$|h_k^I|^2 + |h_k^{II}|^2 = 1 . \quad (4.15)$$

Since the Dirac scalar product is preserved by the cosmological evolution, the normalization condition (4.15) holds at any time. This ensures also the standard anticommutation relations for the creation and annihilation operators ($\{B_{\vec{k},\lambda}, B^\dagger_{\vec{k}',\lambda'}\} = \delta^3(\vec{k} - \vec{k}')\delta_{\lambda\lambda'}$, $\{B_{\vec{k},\lambda}, B_{\vec{k}',\lambda'}\} = 0$, and similarly for the $D_{\vec{k},\lambda}$, $D^\dagger_{\vec{k}',\lambda'}$ operators), defined by the Fourier expansion of the Dirac field operator

$$\psi(x) = \int d^3\vec{k} \sum_\lambda \left[B_{\vec{k}\lambda} u_{\vec{k}\lambda}(x) + D^\dagger_{\vec{k}\lambda} v_{\vec{k}\lambda}(x) \right] . \quad (4.16)$$

In order to perform the adiabatic expansion, t is useful to write the modes equation of motion (4.12) as

$$i\partial_t \begin{pmatrix} h^I \\ h^{II} \end{pmatrix} = \begin{pmatrix} m + s(t) & a^{-1}k \\ a^{-1}k & -m - s(t) \end{pmatrix} \begin{pmatrix} h^I \\ h^{II} \end{pmatrix} . \quad (4.17)$$

We define the vacuum state $|0\rangle$ as $B_{\vec{k},\lambda}|0\rangle \equiv D_{\vec{k},\lambda}|0\rangle \equiv 0$, and denote any expectation value on this vacuum as e.g. $\langle T_{\mu\nu}\rangle \equiv \langle 0|T_{\mu\nu}|0\rangle$. In the quantum theory, the vacuum expectation values of the stress-energy tensor (4.3) take the form

$$\langle 0|T_{00}|0\rangle = \frac{1}{2\pi^2 a^3}\int_0^\infty dk k^2 \rho_k\,, \quad \rho_k \equiv 2i\left(h_k^I \frac{\partial h_k^{I*}}{\partial t} + h_k^{II}\frac{\partial h_k^{II*}}{\partial t}\right) \tag{4.18}$$

and

$$\langle T_{ii}\rangle = \frac{1}{2\pi^2 a}\int_0^\infty dk k^2 p_k\,, \quad p_k \equiv -\frac{2k}{3a}(h_k^I h_k^{II*} + h_k^{I*}h_k^{II})\,. \tag{4.19}$$

whereas the two point function is

$$\langle 0|\bar{\psi}\psi|0\rangle = \frac{1}{2\pi^2 a^3}\int_0^\infty dk k^2 \left(|h_k^I|^2 - |h_k^{II}|^2\right)\,. \tag{4.20}$$

4.1. Adiabatic Regularization

In order to regularize the aforementioned bilinears, we need to perform the adiabatic expansion of the modes, identically as in the 2D of the last chapter. We will not repeat the complete procedure since from (3.15) it is easy to see that we can use the result (3.26) with $\alpha = m + s$ and $\beta = a^{-1}k$. The difference between the four and two dimensional case is that for the former we need to compute up to adiabatic order four in order to fully regularize the stress-energy tensor.

Additionally, we will introduce here again the μ parameter presented in chapter two and in [47] to further extend the regularization method for the for Dirac fields and for Yukawa interactions. In order to this we can rewrite the equation of motions (4.21) as

$$i\partial_t \begin{pmatrix} h^I \\ h^{II} \end{pmatrix} = \begin{pmatrix} m+\mu+\tilde{s}(t) & a^{-1}k \\ a^{-1}k & -m-\mu-\tilde{s}(t) \end{pmatrix} \begin{pmatrix} h^I \\ h^{II} \end{pmatrix} . \tag{4.21}$$

where $\tilde{s}(t) = s(t) - \mu$. It can be seen that the difference between this and the standard procedure [10,29] is a shift $s(t) \mapsto \tilde{s}(t)$ and $m \mapsto m+\mu$.

The product $\boldsymbol{U}_0(t)\boldsymbol{U}_1(t)\cdots\boldsymbol{U}_4(t)$ can be exactly computed in principle, however we only need its adiabatic expansion to fourth order. After a long but conceptually direct computation we get

$$h^I \sim \sqrt{\frac{\omega+m}{2\omega}}\Big(1+(\omega-m)\sum_{n=1}^{4}\phi^{(n)}\Big)\exp\Big(-i\int\omega_4\Big) \tag{4.22}$$

where $\omega^2 = (m+\mu)^2 + k^2/a^2$ and

$$\begin{aligned} \phi^{(1)} &= -\frac{im\dot{a}}{4a\omega^3} - \frac{i\mu\dot{a}}{4a\omega^3} + \frac{\tilde{s}}{2\omega^2} \\ \phi^{(2)} &= -\frac{(m+\mu)\ddot{a}}{8a\omega^4} + \frac{11(m+\mu)^3\dot{a}^2}{32a^2\omega^6} - \frac{(m+\mu)^2\dot{a}^2}{32a^2\omega^5} - \frac{(m+\mu)\dot{a}^2}{8a^2\omega^4} \\ &+ \frac{7i(m+\mu)^2\tilde{s}\dot{a}}{8a\omega^5} + \frac{i(m+\mu)\tilde{s}\dot{a}}{8a\omega^4} - \frac{i\tilde{s}\dot{a}}{4a\omega^3} - \frac{5(m+\mu)\tilde{s}^2}{8\omega^4} - \frac{\tilde{s}^2}{8\omega^3} - \frac{i\dot{\tilde{s}}}{4\omega^3} \end{aligned} \tag{4.23}$$

for the first two terms. The third and fourth adiabatic order terms can be computed in the same way and can be found in [10]. The asymptotic expansion for h^{II} can be easily obtained from that of h^{I} by performing the exchange $m \mapsto -m$ and $s \mapsto -s$.

4.2. Regularization of the Stress-Energy Tensor

We start by performing the adiabatic expansion of the energy density in momentum space (4.18)

$$\rho_k = \rho_k^{(0)} + \rho_k^{(1)} + \rho_k^{(2)} + \rho_k^{(3)} + \rho_k^{(4)} + \dots\,, \tag{4.24}$$

where $\rho_k^{(n)}$ is of nth adiabatic order. Applying the mode expansion of (4.22) in (4.18) we can iteratively obtain each terms of the expansion (4.24)

$$\begin{aligned}
\rho_k^{(0)} &= -2\omega\,,\\
\rho_k^{(1)} &= -\frac{2(m+\mu)(s+\mu)}{\omega}\,,\\
\rho_k^{(2)} &= -\frac{\dot{a}^2(m+\mu)^4}{4a^2\omega^5} + \frac{\dot{a}^2(m+\mu)^2}{4a^2\omega^3} + \frac{(m+\mu)^2\,(s+\mu)^2}{\omega^3} - \frac{(s+\mu)^2}{\omega}\,.
\end{aligned} \tag{4.25}$$

The rest of the terms can be computed in the same way and can be found in [29]. We note that if we turn off the Yukawa coupling (and $\mu = 0$), we recover the results obtained in [31]. The

Yukawa interaction produces new contributions and, in particular, we have now non-zero terms at first and third adiabatic orders. Note here that in the UV limit, $\rho_k^{(0)} \sim k$, $(\rho_k^{(1)} + \rho_k^{(2)}) \sim k^{-1}$, and $(\rho_k^{(3)} + \rho_k^{(4)}) \sim k^{-3}$. This indicates that subtracting the zeroth-order term will cancel the natural quartic divergence of the stress-energy tensor, subtracting up to second order will cancel also the quadratic divergence, and subtracting up to fourth order will cancel the logarithmic divergence. Therefore, defining the adiabatic subtraction terms as

$$T_{00}^{\rm sub} \equiv \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 (\rho_k^{(0)} + \rho_k^{(1)} + \rho_k^{(2)} + \rho_k^{(3)} + \rho_k^{(4)}) \tag{4.26}$$

and the renormalized 00-component of the stress-energy tensor is

$$\langle T_{00} \rangle_{\rm ren} \equiv \langle T_{00} \rangle - T_{00}^{\rm sub} = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 (\rho_k - \rho_k^{(0-4)}) \ . \tag{4.27}$$

where$\rho_k^{(0-4)} = \rho_k^{(0)} + \rho_k^{(1)} + \rho_k^{(2)} + \rho_k^{(3)} + \rho_k^{(4)}$This integral is, by construction, finite. The method proceeds in the same way for the pressure. The renormalized ii-component of the stress-energy tensor is given by

$$\langle T_{ii} \rangle_{\rm ren} \equiv \langle T_{ii} \rangle - T_{ii}^{\rm sub} = \frac{1}{2\pi^2 a} \int_0^\infty dk k^2 (p_k - p_k^{(0-4)}) \ , \tag{4.28}$$

where $p_k^{(0-4)} \equiv p_k^{(0)} + p_k^{(1)} + p_k^{(2)} + p_k^{(3)} + p_k^{(4)}$, and

$$T_{ii}^{\rm sub} \equiv \frac{1}{2\pi^2 a} \int_0^\infty dk k^2 p_k^{(0-4)} \ . \tag{4.29}$$

The corresponding adiabatic terms for the pressure are

$$
\begin{aligned}
p_k^{(0)} &= -\frac{2\omega}{3} + \frac{2(m+\mu)^2}{3\omega}, \\
p_k^{(1)} &= \frac{2(m+\mu)\,(g\Phi+\mu)}{3\omega} - \frac{2(m+\mu)^3\,(g\Phi+\mu)}{3\omega^3}, \\
p_k^{(2)} &= -\frac{5\dot{a}^2(m+\mu)^6}{12a^2\omega^7} + \frac{\dot{a}^2(m+\mu)^4}{2a^2\omega^5} - \frac{\dot{a}^2(m+\mu)^2}{12a^2\omega^3} + \frac{(s+\mu)^2}{3\omega} \\
&- \frac{4(m+\mu)^2\,(s+\mu)^2}{3\omega^3} + \frac{(m+\mu)^4\ddot{a}}{6a\omega^5} - \frac{(m+\mu)^2\ddot{a}}{6a\omega^3} + \frac{(m+\mu)^4\,(s+\mu)^2}{\omega^5}.
\end{aligned}
\tag{4.30}
$$

As before, we see that in the UV limit, $p_k^{(0)} \sim k$, $(p_k^{(1)} + p_k^{(2)}) \sim k^{-1}$, and $(p_k^{(3)} + p_k^{(4)}) \sim k^{-3}$. Subtracting the zeroth-order term eliminates the quartic divergence, subtracting up to second order removes the quadratic divergence, and subtracting up to fourth order removes the logarithmic divergence.

Finally, we are also interested in computing the renormalized expectation value $\langle\bar{\psi}\psi\rangle_{\rm ren}$. The formal (unrenormalized) expression for this quantity is

$$
\langle\bar{\psi}\psi\rangle = \frac{-1}{\pi^2a^3}\int_0^\infty dk k^2 \langle\bar{\psi}\psi\rangle_k\,, \quad \langle\bar{\psi}\psi\rangle_k \equiv |h_k^I|^2 - |h_k^{II}|^2\,. \tag{4.31}
$$

We define the corresponding terms in the adiabatic expansion as $\langle\bar{\psi}\psi\rangle_k = \langle\bar{\psi}\psi\rangle_k^{(0)} + \langle\bar{\psi}\psi\rangle_k^{(1)} + \langle\bar{\psi}\psi\rangle_k^{(2)} + \langle\bar{\psi}\psi\rangle_k^{(3)} + \dots$. Due to the Yukawa interaction, ultraviolet divergences arrive till the third adiabatic order. In general, we have

$$\langle\bar{\psi}\psi\rangle_k^{(n)} = \frac{\omega+m}{2\omega}\left(|F|^2\right)^{(n)} - \frac{\omega-m}{2\omega}\left(|G|^2\right)^{(n)} . \tag{4.32}$$

From here, we obtain

$$\begin{aligned}
\langle\bar{\psi}\psi\rangle_k^{(0)} &= \frac{\mu+m}{\omega} \\
\langle\bar{\psi}\psi\rangle_k^{(1)} &= \frac{s-\mu}{\omega} - \frac{(\mu+m)^2(s-\mu)}{\omega^3} \\
\langle\bar{\psi}\psi\rangle_k^{(2)} &= -\frac{5\dot{a}^2(\mu+m)^5}{8a^2\omega^7} + \frac{7\dot{a}^2(\mu+m)^3}{8a^2\omega^5} - \frac{\dot{a}^2(\mu+m)}{4a^2\omega^3} + \frac{(\mu+m)^3\ddot{a}}{4a\omega^5} \\
&- \frac{(\mu+m)\ddot{a}}{4a\omega^3} + \frac{3(\mu+m)^3(s-\mu)^2}{2\omega^5} - \frac{3(\mu+m)(s-\mu)^2}{2\omega^3} \quad .
\end{aligned} \tag{4.33}$$

In this case, we observe that in the UV limit we have $(\langle\bar{\psi}\psi\rangle_k^{(0)} + \langle\bar{\psi}\psi\rangle_k^{(1)}) \sim k^{-1}$, and $(\langle\bar{\psi}\psi\rangle_k^{(2)} + \langle\bar{\psi}\psi\rangle_k^{(3)}) \sim k^{-3})$. Subtracting up to first order eliminates the quadratic divergence, and up to third order removes the logarithmic one. The renormalized quantity is then

$$\langle\bar{\psi}\psi\rangle_{\rm ren} = \frac{-1}{\pi^2 a^3}\int_0^\infty dk k^2 \left(\langle\bar{\psi}\psi\rangle_k - \langle\bar{\psi}\psi\rangle_k^{(0-3)}\right). \tag{4.34}$$

4.3. Conformal Anomaly

In the massless limit the classical action of the theory enjoys invariance under the conformal transformations

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \,, \quad \Phi(x) \to \Omega^{-1}(x)\Phi(x) \,, \tag{4.35}$$

with

$$\psi(x) \to \Omega^{-3/2}(x)\psi(x) \,, \quad \bar{\psi}(x) \to \Omega^{-3/2}(x)\bar{\psi}(x). \tag{4.36}$$

Variation of the action yields the identity

$$g^{\mu\nu} T^m_{\mu\nu} + \Phi \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta \Phi} = 0 \,, \tag{4.37}$$

which, in our case, turns out to be $g^{\mu\nu} T_{\mu\nu} - g_Y \Phi \bar{\psi}\psi = 0$. At the quantum level the theory will lose its conformal invariance as a consequence of renormalization and generates an anomaly

$$g^{\mu\nu} \langle T^m_{\mu\nu} \rangle_{\rm ren} - g_Y \Phi \langle \bar{\psi}\psi \rangle_{\rm ren} = C_f \neq 0 \,. \tag{4.38}$$

C_f is independent of the quantum state and depends only on local quantities of the external fields.

To calculate the conformal anomaly in the adiabatic regularization method, we have to start with a massive field (and $\mu = 0$) and take the massless limit at the end of the calculation. Therefore,

$$C_f = g^{\mu\nu} \langle T^m_{\mu\nu} \rangle_{\rm ren} - g_Y \Phi \langle \bar{\psi}\psi \rangle_{\rm ren} = \lim_{m \to 0} m (\langle \bar{\psi}\psi \rangle_{\rm ren} - \langle \bar{\psi}\psi \rangle^{(4)}) \ . \tag{4.39}$$

Since the divergences of the stress-energy tensor have terms of fourth adiabatic order, the adiabatic subtractions for $\langle \bar{\psi}\psi \rangle$ should also include them. The fourth order subtraction term, which produces a non-zero finite contribution when $m \to 0$, is codified in $\langle \bar{\psi}\psi \rangle^{(4)}$. The term $m \langle \bar{\psi}\psi \rangle_{ren}$ vanishes when $m \to 0$. The remaining piece produces the anomaly [recall (4.31)-(4.32)]. Applying the adiabatic expansion computed in Section 4.2 and doing the integrals we obtain (see [29] for more details)

$$\begin{aligned} C_f = \quad & \frac{a^{(4)}}{80\pi^2 a} + \frac{s^2 \ddot{a}}{8\pi^2 a} + \frac{\ddot{a}^2}{80\pi^2 a} + \frac{3 s \dot{s} \dot{a}}{4\pi^2 a} + \frac{s^2 \dot{a}^2}{8\pi^2 a^2} \\ & + \frac{3 \dot{a} a^{(3)}}{80\pi^2 a^2} - \frac{\dot{a}^2 \ddot{a}}{60\pi^2 a^3} + \frac{s \ddot{s}}{4\pi^2} + \frac{\dot{s}^2}{8\pi^2} + \frac{s^4}{8\pi^2} \ . \end{aligned} \tag{4.40}$$

Since C_f is an scalar, we must be able to rewrite the above result as a linear combination of covariant scalar terms made out of the metric, the Riemann tensor, covariant derivatives, and the external scalar field Φ. Our result is

$$\begin{aligned} C_f = \quad & \frac{1}{2880\pi^2} \left[-11 \left(R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R^2 \right) + 6 \Box R \right] \\ & + \frac{g_Y^2}{8\pi^2} \left[\nabla^\mu \Phi \nabla_\mu \Phi + 2 \Phi \Box \Phi + \frac{1}{6} \Phi^2 R + g_Y^2 \Phi^4 \right] . \end{aligned} \tag{4.41}$$

In the absence of Yukawa interaction ($h = 0$, $g_Y = 0$) we reproduce the well-known trace anomaly for spin-1/2 fields (restricted to our FLRW spacetime) [15]. We recall that the trace anomaly is generically given for a conformal free field of spin $0, 1/2$ or 1 in terms of three coefficients

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle_{ren} = aC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + bG + c\Box R \,, \tag{4.42}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is proportional to the Euler density. The coefficients a and b are independent of the renormalization scheme and are given by [36,58]

$$\begin{aligned} a &= \frac{1}{120(4\pi)^2}(N_s + 6N_f + 12N_v) \,, \\ b &= \frac{-1}{360(4\pi)^2}(N_s + 11N_f + 62N_v) \,, \end{aligned} \tag{4.43}$$

where N_s is the number of real scalar fields, N_f is the number of Dirac fields, and N_v is the number of vector fields. Our results with $g_Y = 0$ fit the values in (4.43). [We note that in the FLRW spacetime of adiabatic regularization the Weyl tensor vanishes identically]. In contrast, the coefficient c depends in general on the particular renormalization scheme [111]. A local counterterm proportional to R^2 in the action can modify the coefficient c. For instance, for vector fields the point-splitting and the dimensional regularization method predict different values for c.

When the Yukawa interaction is added, the general form of the conformal anomaly is

$$g^{\mu\nu}\left\langle T^{m}_{\mu\nu}\right\rangle_{\rm ren}+\Phi\frac{1}{\sqrt{-g}}\langle\frac{\delta S_m}{\delta\Phi}\rangle_{\rm ren}=aC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+bG+c\Box R$$
$$+d\,g_Y^2\nabla^\mu\Phi\nabla_\mu\Phi+e\,g_Y^2\Phi\Box\Phi+f g_Y^2\Phi^2R+g\,g_Y^4\Phi^4\,. \qquad (4.44)$$

Now, the coefficients f and g are independent of the renormalization scheme but d and e are not since the finite Lagrangian counterterms required by the renormalizability of the Yukawa interaction can be modified by

$$\frac{\delta Z}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi-\frac{\delta\xi_2}{2}R\Phi^2-\frac{\delta\lambda_4}{4!}\Phi^4\,, \qquad (4.45)$$

which might alter the values of the coefficients d and e, but not the coefficients f and g. Note that, due to classical conformal invariance, one should consider only those counterterms having dimensionless coupling parameters. Therefore, our results for the f and g coefficients are (including a quantized scalar field, see [29])

$$f=\frac{1}{3(4\pi)^2}N_f\,,\quad g=\frac{-1}{3(4\pi)^2}\left(\frac{3}{2}N_s-6N_f\right)\,. \qquad (4.46)$$

The same result can be obtained via heat-kernel in general curved spacetime [29].

4.4. Renormalization and Running of the Couplings

The ultraviolet divergent terms of the adiabatic subtractions can be univocally related to particular counterterms in a Lagrangian including the background gravity-scalar sector. By writing

$$L = L_m - (\Lambda + \delta\Lambda) + \frac{1}{16\pi}\left(G^{-1} + \delta G^{-1}\right) R \frac{1}{2}(1+\delta Z)\, g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \\ - \sum_{i=1}^{4} \frac{\lambda_i + \delta\lambda_i}{i!}\Phi^i - (\xi_1 + \delta\xi_1)\, R\Phi - \frac{1}{2}(\xi_2 + \delta\xi_2)\, R\Phi^2 \tag{4.47}$$

the semi-classical Einstein Equation is

$$\frac{G^{\mu\nu}}{8\pi G_B} + \Lambda_B g^{\mu\nu} + (1+\delta Z)(\nabla^\mu\Phi\nabla^\nu\Phi - \frac{1}{2}g^{\mu\nu}\nabla^\rho\Phi\nabla_\rho\Phi) \\ + g^{\mu\nu}\sum_{i=1}^{4}\frac{(\lambda_{iB})}{i!}\Phi^i - 2\sum_{i=1}^{2}\frac{\xi_{iB}}{i!}(G^{\mu\nu}\Phi^i - g^{\mu\nu}\Box\Phi^i + \nabla^\mu\nabla^\nu\Phi^i) = -\langle T_m^{\mu\nu}\rangle \tag{4.48}$$

where we have defined the bare constants as $\alpha_B \equiv \alpha + \delta\alpha$. In order to fix the counterterms $\delta\alpha$ we can use dimensional regularization to integrate the divergent adiabatic terms of the stress-energy tensor (see [29] for more details).

For example, for the zeroth adiabatic order, we have

$$-\frac{1}{2\pi^2 a^3}\int_0^\infty dk k^{n-2}\rho_k^{(0)} \approx \frac{(m+\mu)^4}{8\pi^2}\frac{1}{n-4} \tag{4.49}$$

$$-\frac{1}{2\pi^2 a^3}\int_0^\infty dk k^{n-2} a^2 p_k^{(0)} \approx -\frac{(m+\mu)^4 a^2}{8\pi^2}\frac{1}{n-4}\,. \quad (4.50)$$

This can be done for all the subtraction terms [29]. The complete divergent expressions can be written covariantly as

$$\begin{aligned}
\langle T_{\mu\nu}\rangle_{Ad}^{(0)} &\approx \frac{(m+\mu)^4}{8\pi^2(n-4)} g_{\mu\nu},\\
\langle T_{\mu\nu}\rangle_{Ad}^{(1)} &\approx \frac{(g_Y\Phi-\mu)(m+\mu)^3}{2\pi^2(n-4)} g_{\mu\nu},\\
\langle T_{\mu\nu}\rangle_{Ad}^{(2)} &\approx \frac{3(g_Y\Phi-\mu)^2(m+\mu)^2}{4\pi^2(n-4)} g_{\mu\nu} - \frac{(m+\mu)^2}{24\pi^2(n-4)} G_{\mu\nu},\\
\langle T_{\mu\nu}\rangle_{Ad}^{(3)} &\approx -\frac{m+\mu}{12\pi^2(n-4)}\big[G_{\mu\nu}(g_Y\Phi-\mu) - \Box\Phi g_{\mu\nu} + g_Y\nabla_\mu\nabla_\nu\Phi\\
&-6(g_Y\Phi-\mu)^3 g_{\mu\nu}\big],\\
\langle T_{\mu\nu}\rangle_{Ad}^{(4)} &\approx \frac{-1}{24\pi^2(n-4)}\Big[G_{\mu\nu}(g_Y\Phi-\mu)^2 - g_{\mu\nu}\Box(g_Y\Phi-\mu)^2 + \nabla_\mu\nabla_\nu\Phi^2\\
&-6g_Y^2(\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\Phi\nabla^\rho\Phi) - 3(g_Y\Phi-\mu)^4 g_{\mu\nu}\Big], \quad (4.51)
\end{aligned}$$

and can be consistently removed by the renormalization parameters

$$\begin{aligned}
&\delta\Lambda = -\frac{m^4}{8\pi^2(n-4)}\,, \quad \delta G^{-1} = \frac{m^2}{3\pi(n-4)}\,, \quad \delta Z = -\frac{g_Y^2}{4\pi^2(n-4)}\\
&\delta\lambda_1 = -\frac{m^3 g_Y}{2\pi^2(n-4)}\,, \quad \delta\lambda_2 = -\frac{3m^2 g_Y^2}{2\pi^2(n-4)}\,, \quad \delta\lambda_3 = -\frac{3mg_Y^3}{\pi^2(n-4)}\\
&\delta\lambda_4 = -\frac{3g_Y^4}{\pi^2(n-4)} \quad \delta\xi_1 = -\frac{mg_Y}{24\pi^2(n-4)}\,, \quad \delta\xi_2 = -\frac{g_Y^2}{24\pi^2(n-4)}\,.
\end{aligned} \quad (4.52)$$

Note that the μ dependence has disappeared. This is a consequence of the fact that any regularization scheme defines the same divergences. We remark that the set of needed counterterms are all possible counterterms having couplings with non-negative mass dimension, up to Newton's coupling constant. This is also in agreement with the results in perturbative QFT in flat spacetime.

The renormalizability of the Yukawa interaction $g_Y \varphi \bar{\psi} \psi$ of a quantized massive scalar field φ with a massive quantized Dirac field ψ requires to add terms of the form $\frac{\lambda Z_\lambda}{4!} \varphi^4$, $\frac{\kappa Z_\kappa}{3!} \varphi^3$, and also a term linear in φ [104]. The presence of a curved background would require to add the terms $\xi_1 R \varphi$ and $\xi_2 R \varphi^2$. We note that a term of the form $\xi_2 R \varphi^2$ is required by renormalization for a purely quantized scalar field φ if a self-interaction term of the form $\frac{\lambda}{4!} \varphi^4$ appears in the bare Lagrangian [17,21]. Here we have found that the Yukawa interaction demands the presence of the renormalized terms $\xi_1 R \varphi$ and $\xi_2 R \varphi^2$ (as well as the terms $\lambda_i \varphi^i$), even if they are not present in the bare Lagrangian. Similar counterterms have been identified in the approach in Ref. [9].

Finally, let us briefly comment about the μ dependence. Following the approach of chapter two and [47] we can compute the difference between the renormalized stress-energy tensor with two different parametrizations μ_1 and μ_2. Note that we can not use directly (4.51) since these expressions only contain the pole term and not the finite parts that survive when $n \to 4$. In any case we can compute the difference

$$\langle T_{ab}\rangle_{\text{ren}}(\mu) - \langle T_{ab}\rangle_{\text{ren}}(m) =$$
$$a_1 g_{ab} + a_2 G_{ab} + b_1(\nabla^{\mu}\Phi\nabla^{\nu}\Phi - \frac{1}{2}g^{\mu\nu}\nabla^{\rho}\Phi\nabla_{\rho}\Phi)g^{\mu\nu}$$
$$+\sum_{i=1}^{4}\frac{c_i}{i!}\Phi^i - 2\sum_{i=1}^{2}\frac{d_i}{i!}(G^{\mu\nu}\Phi^i - g^{\mu\nu}\Box\Phi^i + \nabla^{\mu}\nabla^{\nu}\Phi^i) \tag{4.53}$$

Here we have again chosen $\mu_1 = \mu$ and $\mu_2 = 0$ for simplicity. Following the same procedure as in chapter two, i.e, forcing the invariance of the semiclassical equation (4.48) with (4.53) we can obtain the beta function for each coupling. For the gravitational couplings we found

$$\beta_{\Lambda} = \frac{-1}{8\pi^2}\frac{\mu^5}{m+\mu} \quad \beta_{\kappa} = \frac{-1}{8\pi^2}\frac{\mu^3}{m+\mu} \tag{4.54}$$

where $\kappa = (8\pi G)^{-1}$, while for the Yukawa coupling and the mass contribution of the classical scalar field λ_2 we find

$$\beta_g = \frac{-g_Y^3}{8\pi^2}\frac{\mu}{\mu+m} \quad \beta_{\lambda 2} = \frac{g_Y^2}{2\pi^2}\frac{\mu}{\mu+m}\left(\mu^2 - M^2\right) . \tag{4.55}$$

A possible application of this renormalization has been compute in [44]. See also [45] for a similar computation in general curved spacetime.

Part II.

Renormalization, Running Couplings and Decoupling in Curved Spacetime

Introduction and Motivation

In the first part of this thesis, we have focused on adiabatic regularization to overcome the divergences that appear when computing vacuum expectation values of relevant magnitudes, e.g. the stress energy-tensor or the electric current, in a FLRW spacetime. The main advantage of adiabatic regularization is its numerical efficiency both for quantifying the energy density of the quantum fields and the possible backreaction to the expansion of the universe.

However, usually we wish to obtain robust results that are valid for general curved spacetime, in the spirit of general covariance. A very common and useful approach in this case is to use the path integral formalism, where we define the propagator $G(x, x')$ of a given quantum field and compute the quantum contributions through the stress-energy tensor of the effective action.

Again, divergences appear when computing the propagator or the effective action and a renormalization mechanism is needed. In this case, it is very useful to perform the (DeWitt-Schwinger) proper-time expansion (DS) [99, 109]. This is the equivalent of the adiabatic expansion of the modes h_k explained in part one.

In chapter 5, we will present several subtraction schemes built from the DS expansion to construct finite magnitudes. We will compute the subtraction terms for the effective action since it is more transparent to obtain the running of the coupling constants encoded in its corresponding beta functions. For pedagogical purposes, we will include a charged scalar field and a classical electromagnetic background. We will present two examples to see how the regularization and subtraction mechanism works: the R-summed form of the DeWitt-Schwinger expansion, also known as Parker-Raval approximation, and the constant electromagnetic field. We will also describe Minimal Subtraction scheme (MS) and consequently obtain the running of the coupling constants.

There is an equivalence between adiabatic expansion in FLRW spacetimes, also known as Parker-Fulling (PF) expansion and the DeWitt-Schwinger expansion [13, 30]. As a consequence a natural question is whether it is possible to introduce an arbitrary mass parameter μ in the later, equivalently to PF. We will see that this is possible and leads to what we defined as extended DeWitt-Schwinger subtraction scheme. We will introduce this scheme and compute its corresponding beta function.

A relevant result of renormalization is the expected decoupling of higher massive particles, as enforced by the Appelquist-Carazzone theorem [7]. This means that particles with mass higher than the relevant physical energy scale should not contribute to any computed observable. This ensures that for low energy physics we do not need to know about the related very high energy physics, hence supporting the effective field theory framework. The minimal subtraction (MS) scheme in dimensional regularization [106, 107] is a very efficient method to evaluate the behavior of the running couplings. However, MS does not fulfill the decoupling theorem and one needs to resort to a mass-dependent scheme to capture the low energy behavior of the beta function. This is the case in perturbative quantum field theory in flat spacetime. However, the same results in curved spacetime when using minimal subtraction in dimensional regularization [17, 86, 101]. We will see that the beta functions of the extended DeWitt-Schwinger subtraction scheme do decouple, i.e. they vanish in the limit $m \to \infty$.

Another interesting feature we can extract from the renormalization techniques and the proper-time expansion is the effective field theory approach. This consist on designing a theory that is valid up to some scale and that generally is more convenient to use for computations of physical motivated scenarios. We have explained that QFTCS is in itself a effective field theory since we have lay aside the quantum properties of the gravitational fields by assuming it can be neglected below some energy scale M_p. However, computing the effective action in general curved spacetime is generally not possible, except in some particular cases where some special symmetry is present.

A possible effective field theory that can be constructed is an effective action that encodes the information of some quantum field with mass m in curved spacetime where it happens that the mass m is greater as any possible construction of the gravitational field tensor, i.e., $m^2 \gg R$, $m^4 \gg R_{ab}R^{ab}$, etc[1]. In chapter 6, we will use the proper-time DeWitt-Schwinger expansion to build this effective theory. The physical motivation behind this is to analyze the possible radiate corrections to the cosmological constant. Indeed, the observation of the cosmological constant today is performed at a scale where $m^2 \gg \mathcal{R}$ is valid for all the massive Standard Model particles. We will see that the traditionally known as *cosmological constant problem* arises in the context of effective field theory and the tools that we have develop so far will be useful to perform a critical analysis of the cosmological constant problem.

[1]We define this condition as $m^2 \gg \mathcal{R}$.

Chapter 5.

Extended DeWitt-Schwinger Subtraction Scheme

Let us assume a scalar field ϕ on a general smooth four-dimensional spacetime

$$S = \int d^4x\sqrt{-g}\left[-\Lambda + \frac{R}{16\pi G} + \frac{1}{2}\left(\nabla_\mu\phi\nabla^\mu\phi - m^2\phi^2 - \xi R\phi^2\right)\right] \quad (5.1)$$

with the associated Klein-Gordon equation

$$\left(\Box_x + m^2 + \xi R\right)\phi = 0\,. \quad (5.2)$$

From the matter section of the action (6.11) we can construct the functional integral

$$Z[J] = \langle out, 0|0, in\rangle = \int \mathcal{D}[\phi]\exp\left(iS_M + i\int d^n x J(x)\phi(x)\right)\,. \quad (5.3)$$

The functional allows to obtain the n point function. In particular for the two point function we can define the Feynman propagator

$$iG_F(x,y) := \langle 0|T(\phi(x)\phi(y))|0\rangle = \left(\frac{\delta^2 \log Z}{\delta J(x)\delta J(y)}\right)_{J=0}, \qquad (5.4)$$

which follows the equation

$$\left(\Box_x + m^2 + \xi R\right) G_F(x,x') = -(-g)^{-\frac{1}{2}}\delta^4(x-x')\,. \qquad (5.5)$$

To implement the renormalization program it is very useful to construct an adiabatic expansion of $G(x,x')$ in terms of the number of derivatives of the background metric. Since we are interested in the coincident limit of the propagator G_F we can introduce the Riemann normal coordinates y^μ for the point x with origin in the point x', and we expand consequently

$$\begin{aligned} g_{\mu\nu} = \quad & \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma \\ & +[\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\nu\delta}]y^\alpha y^\beta y^\gamma y^\delta + ... \end{aligned} \qquad (5.6)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor, and the coefficients are evaluated at $y=0$. Defining $\mathcal{G}_{\mathrm{F}}(x,x') = (-g(x))^{1/2}G_F(x,x')$, we can write

$$\mathcal{G}_{\mathrm{F}}(x,x') = (2\pi)^{-n}\int d^n k e^{-iky}\mathcal{G}_{\mathrm{F}}(k) \qquad (5.7)$$

with $ky = \delta^{\alpha\beta}k_\alpha y_\beta$. Now we can use equation (5.5) with (5.6), and introducing (5.7) we can iteratively obtain the adiabatic expansion (see [15, 21] for more detail). The result up to adiabatic four (four derivatives of the metric) is

$$\mathcal{G}_{\mathrm{F}}(k) \approx$$
$$(k^2-m^2)^{-1} - (\frac{1}{2}-\xi)R(k^2-m^2)^{-2} + \frac{i}{2}(\frac{1}{6}-\xi)R_{;\alpha}\partial^{\alpha}(k^2-m^2)^{-2}$$
$$-\frac{1}{3}a_{\alpha\beta}\partial^{\alpha}\partial^{\beta}(k^2-m^2)^{-2} + \left[(\frac{1}{6}-\xi)^2R^2+\frac{2}{3}a^{\lambda}_{\lambda}\right](k^2-m^2)^{-3} \tag{5.8}$$

where $\partial_\alpha = \partial/\partial k^\alpha$, and

$$a_{\alpha\beta} = \frac{1}{2}(\xi-\frac{1}{6})R_{;\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}R^{\lambda}_{\alpha\beta;\lambda} - \frac{1}{30}R^{\lambda}_{\alpha}R_{\lambda\beta}$$
$$+\frac{1}{60}R^{\kappa\lambda}_{\alpha\beta}R_{\kappa\lambda} + \frac{1}{60}R^{\lambda\mu\kappa}_{\alpha}R_{\lambda\mu\kappa\beta}. \tag{5.9}$$

Substituting (5.8) in (5.7) we obtain

$$\mathcal{G}_{\mathrm{F}}(k) \approx \int d^nk\frac{e^{-iky}}{(2\pi)^n}$$
$$\left[a_0(x,x') + a_1(x,x')\left(\frac{-\partial}{\partial m^2}\right) + a_2(x,x')\left(\frac{\partial}{\partial m^2}\right)^2\right](k^2-m^2)^{-1} \tag{5.10}$$

with $a_0(x,x') = 1$ and

$$a_1(x,x') = (\frac{1}{6}-\xi)R - \frac{1}{2}(\frac{1}{6}-\xi)R_{;\alpha}y^{\alpha} - \frac{1}{3}a_{\alpha\beta}y^{\alpha}y^{\beta} \tag{5.11}$$
$$a_2(x,x') = \frac{1}{2}(\frac{1}{6}-\xi)^2R^2 + \frac{1}{3}a^{\lambda}_{\lambda}. \tag{5.12}$$

Here the geometric terms are valuated at x'. If we now introduce the proper time s representation $K^{-1} = -i\int_0^\infty ds e^{isK}$ in (5.10) and integrate $d^n k$ we finally obtain, using (5.7)

$$G_{\rm F}(x,x') = -\frac{i[-g(x)]^{-\frac{1}{2}}}{(4\pi)^{n/2}}\int_0^\infty ids(is)^{-n/2}$$
$$e^{-im^2 s+(\sigma/2is)}\left(1+(is)a_1(x,x')+(is)^2 a_2(x,x')+...\right) \tag{5.13}$$

where $\sigma(x,x') = \frac{1}{2}y_\alpha y^\alpha$.

We recall here that the coefficient a_j is of adiabatic order $2j$. In four spacetime dimensions, and for arbitrary ξ, the first two terms in (5.7) are divergent in the UV limit, namely, when $s \to 0$ and $\sigma = 0$. For instance, the first two leading terms in the adiabatic expansion are, after performing the ds integral,

$${}^{(2)}G_F(x,x') = -i\frac{|g(x)|^{-1/4}}{(4\pi)^2}\left[\frac{m}{\sqrt{-2\sigma}}K_1(m\sqrt{-2\sigma})+\frac{a_1}{2}K_0(m\sqrt{-2\sigma})\right] \tag{5.14}$$

where K are the modified Bessel functions of second kind. The factor $|g(x)|^{-1/4}$ in the above expression is evaluated in Riemann normal coordinates with origin at x' [30]. Higher-order terms do not involve any UV divergences for the two-point function. However, the fourth adiabatic order term, a_2, is necessary to tame the logarithmic divergences of the stress-energy tensor and the effective action [25,26,32] (see also Refs. [15,86]).

For a general coordinate space, an equivalent result was obtained [32]

$$G^{\mathrm{DS}}(x,x') = -i\int_0^\infty ds e^{-im^2 s}\langle x,s|x',0\rangle \tag{5.15}$$

with

$$\langle x,s|x',0\rangle = \frac{i\Delta^{\frac{1}{2}}(x,x')}{(4\pi)^2}(is)^{-n/2}e^{\frac{\sigma}{2is}}$$
$$\left(1 + a_1(x,x')(is) + a_2(x,x')(is)^2 + \cdots\right) \tag{5.16}$$

where $\Delta(x,x')$ is the Van Vleck-Morette determinant

$$\Delta(x,x') = -\det\left[\partial_\mu\partial_\nu\sigma(x,x')\right][g(x)g(x')]^{-1/2}\,. \tag{5.17}$$

In normal coordinates Δ reduces to $[-g(x)]^{-\frac{1}{2}}$ and both results are equivalent. The stress-energy tensor can be obtained by a metric variation

$$\frac{2}{(-g)^{\frac{1}{2}}}\frac{\delta Z[0]}{\delta g^{\mu\nu}} = i\langle \mathrm{o}ut,0|T_{\mu\nu}|0,\mathrm{i}n\rangle\,. \tag{5.18}$$

On the other hand, it is useful for the discussion of renormalization to define the effective action W as $Z[0]=e^{iW}$, such that

$$W = -i\log\langle \mathrm{o}ut,0|0,\mathrm{i}n\rangle \tag{5.19}$$

and from (5.18) we find

$$\frac{2}{(-g)^{\frac{1}{2}}}\frac{\delta W}{\delta g^{\mu\nu}} = \frac{\langle out,0|T_{\mu\nu}|0,in\rangle}{\langle out,0|0,in\rangle}. \tag{5.20}$$

Since the DeWitt-Schwinger expansion is performed for the Feynman propagator, it is useful to write the effective action in terms of this magnitude. For this, we can obtain a relation between the propagator and the effective action as (see [15])

$$W = -i\log Z[0] = -\frac{1}{2}itr\log\left(-G_F\right). \tag{5.21}$$

Here G_F is interpreted as an operator acting on the space of vector $|x\rangle$, normalized by

$$\langle x|x'\rangle = \delta^n(x-x')[-g(x)]^{-1/2} \tag{5.22}$$

such that $G_F(x,x') = \langle x|G_f|x\rangle$. After some manipulations, and using the DeWitt-Schwinger proper time integral $G_F = -K^{-1} = -i\int_0^\infty \langle x|e^{-iKs}|x\rangle ds$ we can finally write

$$W = \frac{i}{2}\int_{m^2}^{\infty} dm^2 \int d^n x[-g(x)]^{1/2} \lim_{x'\to x} G_{\rm F}(x,x'). \tag{5.23}$$

It will be useful for further discussion to define the effective action as

$$W = \frac{1}{2}\int d^n x[-g(x)]^{1/2}\int_0^\infty (is)^{-1}e^{-ims}\lim_{x'\to x}\langle x,s|x',0\rangle ds \tag{5.24}$$

where we have used results (5.23) and (5.15).

5.1. Equivalence with Parker-Fulling adiabatic expansion

Both the adiabatic (proper time) DeWitt-Schwinger expansion (DS) and the adiabatic expansion for FLRW space-times, also known as Parker-Fulling (PF) adiabatic expansion are very similar in its construction. Indeed, the different orders of the expansion are defined in terms of adiabatic order, which for the gravitational field if based on the derivatives of the metric (or expansion parameter a in case of PF expansion). An important result, is the robustness of the equivalence between the two methods. We follow here the description of [30] and [50]. We will compute the first terms of both adiabatic expansions of the propagator G_F to see the equivalence.

In order to compute both the DS and PF expansion, let us assume a spatially flat metric of the form $ds^2 = dt^2 - a^2(t)d\vec{x}^2$. The scalar field satisfies the equation

$$(\Box + m^2 + \xi R)\phi = 0\,, \tag{5.25}$$

where $R = 6(\dot{a}^2/a^2 + \ddot{a}/a)$. The quantized field is expanded in Fourier modes as

$$\phi(x) = \frac{1}{\sqrt{2(2\pi a^3)}} \int d^3\vec{k}[A_{\vec{k}} f_{\vec{k}}(x) + A^{\dagger}_{\vec{k}} f^{*}_{\vec{k}}(x)]\,, \tag{5.26}$$

where $f_{\vec{k}}(x) = e^{i\vec{k}\vec{x}} h_k(t)$ and $A^{\dagger}_{\vec{k}}$ and $A_{\vec{k}}$ are the usual creation and annihilation operators. Substituting (5.26) into (5.25) we find $\ddot{h}_k +$

$\left[\omega^2 + \sigma\right] h_k = 0$, where $\sigma = (6\xi - \frac{3}{4})(\frac{\dot{a}^2}{a^2}) + (6\xi - \frac{3}{2})(\frac{\ddot{a}}{a})$ and $\omega = \sqrt{\frac{k^2}{a^2} + m^2}$. The adiabatic expansion for the scalar field modes is based on the usual WKB ansatz [15,86]

$$h_k(t) = \frac{1}{\sqrt{W_k(t)}} e^{-i\int^t W_k(t')dt'} , \quad W_k(t) = \omega^{(0)} + \omega^{(1)} + \cdots , \tag{5.27}$$

where the adiabatic order is based on the number of derivatives of the expansion factor $a(t)$. The function $W_k(t)$ obeys the differential equation

$$W_k^2 = \omega^2 + \sigma + \frac{3}{4}\frac{\dot{W}_k^2}{W_k^2} - \frac{1}{2}\frac{\ddot{W}_k}{W_k} \ . \tag{5.28}$$

If we now fix the leading term as $\omega^{(0)} = \omega$, one can substitute the ansatz into Eq. (5.28), and solve order by order to obtain recursively the different terms of the expansion:

$$\begin{aligned}
\omega^{(1)} &= \omega^{(3)} = 0 \\
\omega^{(2)} &= \frac{1}{2\omega^3}\left\{\sigma\omega^2 + \frac{3}{4}\dot{\omega}^2 - \frac{1}{2}\omega\ddot{\omega}\right\} \\
\omega^{(4)} &= \frac{1}{2\omega^3}\left\{2\sigma\omega\omega^{(2)} - 5\omega^2(\omega^{(2)})^2 + \frac{3}{2}\dot{\omega}\dot{\omega}^{(2)} - \frac{1}{2}(\omega\ddot{\omega}^{(2)} + \omega^{(2)}\ddot{\omega})\right\}.
\end{aligned} \tag{5.29}$$

From the mode expansion, we can expand any observable at any fixed adiabatic order. For the two-point function at the coincident limit $G(x,x) \sim \int dkk^2 W_k^{-1}$, we have

$$^{(2n)}G_{PF}(x,x) = \frac{1}{4\pi^2 a^3}\int_0^\infty dkk^2 \left\{\omega^{-1} + (W^{-1})^{(2)} + ... + (W^{-1})^{(2n)}\right\}, \tag{5.30}$$

where the first terms are

$$(W^{-1})^{(2)} = \frac{m^2\dot{a}^2}{2a^2\omega^5} + \frac{m^2\ddot{a}}{4a\omega^5} - \frac{5m^4\dot{a}^2}{8a^2\omega^7} - \frac{\bar{\xi}R}{2\omega^3} \tag{5.31}$$

$$(W^{-1})^{(4)} = -\frac{\omega^{(4)}}{\omega^2} + \frac{\left(\omega^{(2)}\right)^2}{\omega^3}. \tag{5.32}$$

Just as the DS expansion, only the first two terms in (5.30) are divergent, in such a way that it serves to isolate all the ultraviolet divergences of the propagator. More precisely, we have

$$^{(2)}G_{\rm PF}(x,x) = \frac{R}{288\pi^2} + \frac{1}{4\pi^2 a^3}\int_0^\infty dkk^2 \left[\frac{1}{\omega} - \frac{\bar{\xi}R}{2\omega^3}\right]. \tag{5.33}$$

After subtracting the divergences, one gets a finite result. Even though we have written (5.28) in a compact form, we can further expand this expression and obtain an analytic expression for $\omega^{(2n)}$ in terms of the lower adiabatic orders (see for instance Ref. [30]).

To compare both adiabatic expansions, we have to restrict the DeWitt-Schwinger expansion of the Feynman propagator to the (spatially flat) FLRW universe considered above. Moreover, it is natural to compare the expansion of the two-point function $G_F(x,x')$

at the coincident limit $x = x'$. The comparison is highly nontrivial since in the DS formalism the coincidence limit is defined in terms of the geodesic distance with $\sigma \to 0$. We follow the analysis in Ref. [30].

From (5.14), the zeroth-order contribution ${}^{(0)}G_{DS}(x,x)$ can be re-expressed as [here $x \equiv (t, \vec{x})$ and $x' \equiv (t, \vec{x}')$]

$$\lim_{x \to x'} \frac{|g(x)|^{-1/4} m}{(2\pi)^2 \sqrt{-2\sigma}} K_1(m\sqrt{-2\sigma}) = \frac{R}{288\pi^2} + \lim_{\Delta\vec{x} \to 0} \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \frac{\sin(k|\Delta\vec{x}|)}{k|\Delta\vec{x}|} \frac{1}{\omega} , \quad (5.34)$$

where we have used

$$\frac{1}{-2\sigma} = \frac{1}{a^2 \Delta\vec{x}^2} - \frac{\dot{a}^2}{12 a^2} + O(\Delta\vec{x}^2) , \quad (5.35)$$

and

$$|g(x)|^{-1/4} = 1 - \left[2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right] \frac{\sigma}{6} + O(\sigma^{3/2}) . \quad (5.36)$$

Similarly, the second-order adiabatic contribution to ${}^{(2)}G_{DS}(x,x)$ is found to be

$$\lim_{x \to x'} \frac{|g(x)|^{-1/4}}{4\pi^2} \frac{a_1(x,x')}{2} K_0(m\sqrt{-2\sigma}) = \lim_{\Delta\vec{x} \to 0} \frac{-1}{4\pi^2 a^3} \int_0^\infty dk k^2 \frac{\sin(k|\Delta\vec{x}|)}{k|\Delta\vec{x}|} \frac{\bar{\xi} R}{2\omega^3} . \quad (5.37)$$

Therefore, taking into account (5.33), one can write

$$
\begin{aligned}
&{}^{(2)}G_{\rm PF}(x,x) = i^{(2)}G_{DS}(x,x) = \\
&\frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \left[\frac{1}{(\frac{k^2}{a^2}+m^2)^{1/2}} - \frac{\bar{\xi}R(x)}{2(\frac{k^2}{a^2}+m^2)^{3/2}}\right] + \frac{R(x)}{288\pi^2} \quad .
\end{aligned}
\tag{5.38}
$$

A detailed analysis can be found in Ref. [30]. It was explicitly checked (up to and including the sixth adiabatic order) that the Parker-Fulling expansion of the two-point function $G_{PF}(x,x)$ coincides with the corresponding DeWitt-Schwinger expansion of the two-point function at coincidence $G_{DS}(x,x)$, that is,

$$
\begin{aligned}
&{}^{(6)}G_{PF}(x,x) = i^{(6)}G_{DS}(x,x) = \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \\
&\left[\frac{1}{(\frac{k^2}{a^2}+m^2)^{1/2}} - \frac{\bar{\xi}R(x)}{2(\frac{k^2}{a^2}+m^2)^{3/2}}\right] + \frac{R(x)}{288\pi^2} + \frac{a_2(x)}{16\pi^2 m^2} + \frac{a_3(x)}{16\pi^2 m^4} \, .
\end{aligned}
\tag{5.39}
$$

This provides enough evidence for the equivalence at any adiabatic order,

$$
{}^{(2n)}G_{\rm PF}(x,x) = i^{(2n)}G_{DS}(x,x) \, . \tag{5.40}
$$

A similar result can be obtained for more involved asymptotic expansions, such as the R-summed form of the propagator found first by Parker and Toms [87](see also [65]) Here the equivalence is given as ${}^{(2n)}\bar{G}_{PF}(x,x) - {}^{(2n)}G_{PF}(x,x)$ for $n \geq 2$ is given by

$$ {}^{(2n)}\bar{G}_{PF}(x,x) - {}^{(2n)}G_{\mathrm{PF}}(x,x) = i\Big({}^{(2n)}\bar{G}_{\mathrm{DS}}(x,x) - {}^{(2n)}G_{\mathrm{DS}}(x,x)\Big) = \frac{1}{(4\pi)^2}\Big[M^2\log\left(\frac{M^2}{m^2}\right) - \bar{\xi}R + \sum_{j=2}^{n}(j-2)!\Big(\frac{\bar{a}_j}{M^{2j-2}} - \frac{a_j}{m^{2j-2}}\Big)\Big], \tag{5.41} $$

where the coefficients of the new expansion $\bar{a}_i$ are given by

$$ \bar{a}_n = \sum_{k=0}^{n} a_{n-k}\frac{\left(\xi - \frac{1}{6}\right)^k R^k}{k!} \ . \tag{5.42} $$

This was proven in [50] and demonstrate the robustness of the equivalence between the two methods.

5.2. DeWitt-Schwinger Subtraction Scheme

In this section we want to construct the DeWitt-Schwinger subtraction scheme, including regularization and renormalization, of the quantum charged scalar field coupled to an electromagnetic field in curved spacetime

$$ S_{\mathrm{M}} = \int d^4x\sqrt{-g}\left((D_\mu\phi)^\dagger D^\mu\phi - m^2|\phi|^2 - \xi R|\phi|^2\right) \ , \tag{5.43} $$

with $D_\mu = \nabla_\mu + iA_\mu$[1]. Here, the complete action is

$$S = \int d^4x\sqrt{-g}\left(-\Lambda + \frac{R}{16\pi G} - \frac{1}{4q^2}F_{\mu\nu}F^{\mu\nu}\right) + S_{\rm M}\,. \qquad (5.44)$$

The inclusion of an electromagnetic field can be carried out analog to the free field case (see [15, 86]). The only difference is an extra factor two due to the complex field and the DeWitt-Schwinger coefficients. We have then the following effective action

$$W = \int d^n x[-g(x)]^{1/2}\int_0^\infty (is)^{-1}e^{-ims}\lim_{x'\to x}\langle x, s|x', 0\rangle ds \qquad (5.45)$$

$$\begin{aligned}\langle x, s|x', 0\rangle &= \frac{i\Delta^{\frac{1}{2}}(x, x')}{(4\pi)^2}(is)^{-n/2}e^{\frac{\sigma}{2is}}\\ &\left(1 + a_1(x, x')(is) + a_2(x, x')(is)^2 + \cdots\right)\end{aligned} \qquad (5.46)$$

The divergences at the coincident limit are encoded in the first three coefficients,

$$\begin{aligned}a_0(x) &= 1\,, \qquad a_1(x) = -\left(\xi - \frac{1}{6}\right)R\\ a_2(x) &= \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R\\ &+ \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 - \frac{1}{12}F^{\mu\nu}F_{\mu\nu}.\end{aligned} \qquad (5.47)$$

[1] We have absorbed the electric charge in the electromagnetic field for simplicity, but the same calculations can be done without this.

We define the DeWitt-Schwinger subtraction scheme such that it subtracts the first three coefficients of the original divergent effective action. In terms of the effective Lagrangian at one loop $L^{(1)}$ defined as $W = \int dx^n \sqrt{-g} L^{(1)}$ this results in

$$L^{(1)}_{\text{ren}} = \int_0^\infty \frac{ds}{is} e^{-im^2 s} \left(\langle x, s | x, 0 \rangle - \frac{i}{(4\pi)^2} \sum_{j=0}^{2} a_j(x) (is)^{j-2} \right), \quad (5.48)$$

where the renormalized effective Lagrangian is constructed from the divergent initial Lagrangian $L^{(1)}$ and the subtraction terms. In general, the overall integral in the proper-time parameter s is finite and well-defined. There is no need to introduce any auxiliary regularization. However, each individual term generates ultraviolet divergences at $s = 0$ since this corresponds to $\sigma(x, x') \to 0$. One could also wish to manage these partial divergent integrals by introducing an accessory regularization procedure (i.e. a lower cut-off $s_0 > 0$ for the integral in ds or dimensional regularization) to make all single divergent terms well defined before performing the complete subtraction. The final result is the same, irrespective of the additional mass scale introduced in by the auxiliary regularization method. This is why the subtraction procedure acts also as a regularization method.

We will show how to perform this regularizations for two well-known examples: the effective action for a quantized scalar field in a constant electromagnetic background and the Parker-Raval solution for a quantized scalar field in curved spacetime.

5.2.1. Scalar Field in a Constant Electromagnetic Background

If we assume a constant electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu + a_\mu$ an explicit expression for the kernel and for the effective action can be obtained. We refer to the complete calculation to [86]. The exact one-loop effective Lagrangian is

$$L^{(1)} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s^3}\exp\left(-m^2 s\right)\frac{(s)^2\mathcal{G}}{\Im\mathfrak{m}[\cosh s\psi]} \tag{5.49}$$

where $\psi = \sqrt{2}\,(\mathcal{F}+i\mathcal{G})^{\frac{1}{2}}$, $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\mathcal{G} = \frac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}$. Here we have rotated the contour of the integration from the positive real axis to lie along the negative imaginary axis, by replacing $s \to -is$. If we introduce in (5.48) expression (5.49), taking into account the factor two of complex scalar fields, and the coefficients (5.47) for an electromagnetic field in flat spacetime, we obtain

$$L^{(1)}_{\text{ren}} = \lim_{K\to 0}\frac{1}{16\pi^2}\left[\int_K^\infty \frac{ds}{s^3}e^{-im^2 s}\frac{(s)^2\mathcal{G}}{\Im\mathfrak{m}[\cosh s\psi]} - \left(1-\frac{1}{3}s^2\mathcal{F}\right)\right] \tag{5.50}$$

where we have introduced an ultraviolet cutoff (remember that the UV divergence comes from $s \to 0$). This expression can not be directly integrated but some special cases can be calculated. For example, in the case where the electromagnetic field is weak in comparison with the mass, we can expand (5.49) such that

$$L_{\text{ren}}^{(1)} = \lim_{K\to 0} \frac{1}{16\pi^2} \int_K^\infty \frac{ds}{s^3} e^{-im^2 s} \left(\frac{s^4}{90} \left(\mathcal{G}^2 + \frac{1}{7}\mathcal{F}^2 \right) + ... \right) \quad (5.51)$$

obtaining the finite result

$$L_{\text{ren}}^{(1)} = \frac{1}{1440\pi^2 m^4} (7\mathcal{F}^2 + \mathcal{G}^2) + \mathcal{O}(m^{-6}) \, . \quad (5.52)$$

The same result could have been obtained without invoking the cutoff. This result is the scalar analog to the traditional Euler-Heisenberg Lagrangian, which is nothing but the quantum effects of scalar QED in the low field regime, where the quantum field has been integrated out and all the corrections can be accounted in terms of higher order electromagnetic field terms with an expansion in terms of the inverse mass of the quantum field.

5.2.2. Parker-Raval effective action

The Parker-Raval effective action is an expansion of the DeWitt-Schwinger type for a scalar field in curved spacetime but including an exponential factor of the type $e^{-is\left(\xi-\frac{1}{6}\right)R}$, which is a resumation of all the Ricci scalar dependence of the coefficients. This result has major physical consequences to account for the effective dynamics of the Universe and the observed cosmological acceleration. By integrating out the quantum fluctuations of an ultra-low-mass scalar field the effective gravitational dynamics provides negative pressure to suddenly accelerate the Universe, without the need of an underlying cosmological constant [84,85,88](see also Ref. [23]). This approach can also alleviate [34] the increasing H0 tension of the standard cosmological model.

The effective Lagrangian has the form

$$L^{(1)} = \frac{2i}{2(4\pi)^2} \sum_{j=0}^{2} \tilde{a}_j(x) \int_0^\infty e^{-is(m^2+(\xi-\frac{1}{6})R)} (is)^{j-3} ds \ . \tag{5.53}$$

where the modified coefficients $\tilde{a}_i$ are related to the initial a_i from (5.47) by $\tilde{a}_0 = a_0$, $\tilde{a}_1 = a_1 + (\xi - \frac{1}{6})R$ and $\tilde{a}_2 = a_2 - a_1(\xi - \frac{1}{6})R + \frac{1}{2}(\xi - \frac{1}{6})^2 R^2$. Dimensional regularization can be used to convert the divergent effective Lagrangian into a finite term. In this case we extend (5.53) to n dimensions

$$\begin{aligned} L^{(1)} &= \frac{i}{(4\pi)^{n/2}} \sum_{j=0}^{2} \tilde{a}_j(x) \int_0^\infty e^{-isM^2} (is)^{j-n/2-1} ds \\ &= \frac{i}{(4\pi)^{n/2}} \sum_{j=0}^{2} \tilde{a}_j(x) (M^2)^{n/2-j} \Gamma\left(j - \frac{n}{2}\right) \ . \end{aligned} \tag{5.54}$$

where we have defined $M^2 = m^2 + (\xi - \frac{1}{6})R$. In order to retain the units of $L^{(1)}$ as (lenght)$^{-4}$ even when $n \neq 4$, we introduce an arbitrary mass scale μ such that

$$L^{(1)} = \left(\frac{M^2}{\mu^2}\right)^{\frac{n}{2}-2} \frac{i}{(4\pi)^{n/2}} \sum_{j=0}^{2} \tilde{a}_j(x) (M^2)^{2-j} \Gamma\left(j - \frac{n}{2}\right) \ . \tag{5.55}$$

We can expand the first three divergent terms around the limit $n \to 4$ using

$$\Gamma\left(-\frac{n}{2}\right) = \frac{4}{n(n-2)}\left(\frac{2}{4-n} - \gamma + \frac{3}{2}\right) + \mathcal{O}(n-4)$$
$$\Gamma\left(1-\frac{n}{2}\right) = \frac{2}{(2-n)}\left(\frac{2}{4-n} - \gamma + 1\right) + \mathcal{O}(n-4)$$
$$\Gamma\left(2-\frac{n}{2}\right) = \frac{2}{4-n} - \gamma + \mathcal{O}(n-4) \tag{5.56}$$

and using $(m/\mu)^{n-4} = 1 + \frac{1}{2}(n-4)\log \mu^2/m^2 + ...$ we find

$$L^{(1)} \approx \frac{1}{(4\pi)^{\frac{n}{2}}} \sum_{j=0}^{\frac{n}{2}} \left[\frac{2}{n-4} + \psi\left(\frac{n}{2} - j + 1\right) + \log\left(\frac{\mu^2}{M^2}\right)\right] \frac{\left(-M^2\right)^{2-j}}{(\frac{n}{2}-j)!}\tilde{a}_j. \tag{5.57}$$

The same process can be performed for the subtraction terms of (5.48) and we obtain

$$L_{\text{sub}} \approx \frac{1}{(4\pi)^{\frac{n}{2}}} \sum_{j=0}^{\frac{n}{2}} \left[\frac{2}{n-4} + \psi\left(\frac{n}{2} - j + 1\right) + \log\left(\frac{\mu^2}{m^2}\right)\right] \frac{\left(-m^2\right)^{2-j}}{(\frac{n}{2}-j)!}a_j. \tag{5.58}$$

After performing the subtraction of (5.48) we obtain

$$L^{(1)}_{\text{ren}} = \frac{1}{64\pi^2}\left[2m^2\bar{\xi}R + 3\bar{\xi}^2R^2 - (2m^4 + 4m^2\bar{\xi}R + 4a_2)\log\left(\frac{m^2+\bar{\xi}R}{m^2}\right)\right] \tag{5.59}$$

with $\bar{\xi} = \xi - \frac{1}{6}$.

Finally, it is interesting to realize that the coefficient a_2 can include non-gravitational interactions. In the case of the complex scalar field we have a contribution to the effective action of the form $\sim F_{\mu\nu}F^{\mu\nu} \log\left(\frac{m^2+\bar{\xi}R}{m^2}\right)$. This can be interpreted as a non-perturbative gravitational dependence of the effective electric charge. A similar effect has been recently studied [95, 108] but further deeper understanding need to be carried out.

5.3. Minimal Subtraction Scheme

Let us consider again the proper time DeWitt-Schwinger expansion of the effective Lagrangian of the complex scalar field

$$L^{(1)} = \frac{i}{(4\pi)^2} \sum_{j=0}^{\infty} a_j(x) \int_0^{\infty} e^{-ism^2} (is)^{j-3} ds \, . \tag{5.60}$$

Introducing now dimensional regularization in the same form as before we obtain

$$\begin{aligned} L^{(1)} \approx & \frac{1}{(4\pi)^{\frac{n}{2}}} \sum_{j=0}^{\frac{n}{2}} \left[\frac{2}{n-4} + \psi\left(\frac{n}{2} - j + 1\right) + \log\left(\frac{\mu^2}{m^2}\right)\right] \frac{\left(-m^2\right)^{2-j}}{(\frac{n}{2}-j)!} a_j \\ & + \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_{n=3}^{\infty} \frac{(n-3)!}{m^{2n}} a_n \end{aligned} \quad . \tag{5.61}$$

Now, we have to give a prescription of which part will be subtracted. There are infinite one parameter family of performing this subtraction [18]. If we subtract all the terms of (5.61) except $L_{\rm fin}$, as in the on-shell regularization when computing the S-matrix in flat spacetime quantum field theory [91] we would do the DeWitt-Schwinger subtraction scheme explained in the section 5.2.

However, it can also useful (see Appendix A for a discussion in QED), to perform Minimal Subtraction or Modified Minimal Subtraction, which consists on maintaining the logarithmic contribution plus some extra constant term. In order to recover well-known results from running of the coupling constants in flat spacetime, let us subtract both the divergent part and the ψ term from (5.61), i.e.,

$$L^{(1)}_{\rm ren} \approx \frac{1}{(4\pi)^2}\left[\frac{m^4}{2} - m^2 a_1 + a_2\right]\log\left(\frac{\mu^2}{m^2}\right) + \left(\frac{m^2}{4\pi}\right)^2 \sum_{n=3}^{\infty}\frac{(n-3)!}{m^{2n}}a_n \tag{5.62}$$

after performing the limit $n \to 4$. There are two important issues here. First, the last summation term in (5.62) is only valid in the adiabatic limit and will in general have a different form. Secondly, the arbitrariness of subtracting $L_{\rm div}$, or $L_{\rm div} + L_{\rm log}$, or $L_{\rm div} + L_{\rm log} +$ constant... is parametrized by the μ scale and is in complete agreement with the standard result of quantum field theory in curved spacetime that any two renormalization prescription can differ only in local terms as included in coefficients a_0, a_1 and a_2. The complete effective action, together with the classical gravitational and electromagnetic term, is

$$S^{(1)} = \int dx^4 \sqrt{-g} \left[-\Lambda + \frac{1}{16\pi G} R + \frac{1}{4q^2} F_{\mu\nu} F^{\mu\nu} + L_{\text{eff}} \right] + S_{\text{HG}}, \tag{5.63}$$

where renormalization requires that the original classical Lagrangian be modified by the addition of higher derivative terms of the form $S_{\text{HG}} = \int dx^4 \sqrt{-g} \alpha_1 C^2 + \alpha_2 R^2$, where α_1 and α_2 are dimensionless coupling constants. Here $C^2 \equiv R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2$ is the square of the Weyl tensor.

Since we have introduced an arbitrary μ dependence in the renormalized Lagrangian L_{ren}, we need to ensure the μ-independence of the effective action, by introducing a dependence of μ into each of the coupling constants that absorb the contributions of L_{div}, namely Λ, G, q, α_1 and α_2. Demanding that the total effective Lagrangian, including the classical part, be μ-independent leads to the following beta-functions (see e.g. [101])

$$\begin{aligned} &\beta_{\Lambda}^{MS} = \frac{m^4}{16\pi^2} \quad \beta_{\kappa^{-1}}^{MS} = -\frac{m^2 \bar{\xi}}{4\pi^2} \quad \beta_{q}^{MS} = \frac{q^3}{48\pi^2} \\ &\beta_{\alpha_1}^{MS} = -\frac{1}{960\pi^2} \quad \beta_{\alpha_2}^{MS} = -\frac{1}{16\pi^2} \bar{\xi}^2 \quad , \end{aligned} \tag{5.64}$$

where $\kappa = 8\pi G$. The beta functions are defined as $\beta_O = \mu \frac{d}{d\mu} O$.

There are two interesting points to make. First, we recover the same beta function for scalar QED [98] in the Minimal Subtraction scheme in perturbative QFT in flat spacetime, $\beta^{MS} = \frac{q^3}{48\pi^2}$. However, in the limit $\mu^2 \ll m^2$, the beta function do not decouple, analog to the perturbative case in QED in flat spacetime (Appendix A). Moreover, for the cosmological constant and Newton constant the beta functions actually diverge, making them very sensitive to the higher massive fields. We will see in the next chapter that this result is the cause of one of the possible formulations of the cosmological constant problem.

5.4. Extended DeWitt-Schwinger μ-Subtraction

As we have seen, the equivalence between DeWitt-Schwinger adiabatic expansion and adiabatic Parker-Fulling expansion is very robust. Since we have proven that there is an inherent ambiguity in the definition of the subtraction terms in adiabatic regularization, it seems appropriate to analyze if there is an equivalent arbitrariness in the DeWitt-Schwinger subtraction scheme explained in the last section. The subtraction terms from (5.48) are of the form

$$L_{\text{sub}} = \frac{i}{2(4\pi)^2} \sum_{j=0}^{2} a_j(x) \int_0^\infty e^{-ism^2} (is)^{j-3} ds \, . \tag{5.65}$$

In the same way as the zeroth adiabatic order was defined as $\omega^{(0)} = \sqrt{a^{-2}k^2 + m^2 + \mu^2}$, we can propose the ansatz

$$L^{(0)}_{\text{Sub}} = \frac{2i}{2(4\pi)^2} \bar{a}_0 \int_0^\infty \frac{ds}{(is)^3} e^{-is\left(m^2+\mu^2\right)} \, . \tag{5.66}$$

An important improvement of this ansatz is the avoidance of the infrared divergence that usually appears in the mass-less limit, since in that limit we would have to add an arbitrary parameter either way (see [15]). In order to subtract all the divergences of the effective Lagrangian we need again to subtract the first three coefficients, such that

$$L^{\mu}_{\rm sub} := \frac{2i}{2(4\pi)^2} \sum_{j=0}^{2} \bar{a}_j(x) \int_0^\infty e^{-is\left(m^2+\mu^2\right)} (is)^{j-3} ds \, . \tag{5.67}$$

The new coefficients need to be modified with respect to (5.65) in order to not generate new divergences. The new coefficients are

$$\bar{a}_0 = a_0 \quad \bar{a}_1 = a_1 + \mu^2 \quad \bar{a}_2 = a_2 + a_1\mu^2 + \frac{1}{2}\mu^4 \, . \tag{5.68}$$

We can see from the coefficients that in this case μ^2 is of adiabatic order two, equivalently to the adiabatic Parker-Fulling expansion. In the same way as the Minimal Subtraction in dimensional regularization we have the renormalized effective Lagrangian expressed as $L^{(1)}_{\rm ren} = L^{(1)} - L_{\rm sub}(\mu)$. In order to obtain the beta functions of the coupling constant under this renormalization scheme, we ensure the invariance of the total effective action

$$S^{(1)} = \int d^4x \sqrt{-g} \left[-\Lambda + \frac{1}{16\pi G} R + \frac{1}{4q^2} F_{\mu\nu} F^{\mu\nu} + \alpha_1 C^2 + \alpha_2 R^2 + L_{\rm ren} \right] \tag{5.69}$$

under μ, i.e. $\mu\frac{d}{d\mu}S^{(1)} = 0$. This invariance imposes a running into the coupling constants of the background Lagrangian due to the μ dependence of the renormalized effective Lagrangian. It is useful to compute

$$\begin{aligned}\mu\frac{d}{d\mu}L_{\text{ren}} &= \frac{-i}{(4\pi)^2}\int_0^\infty e^{-is\left(m^2+\mu^2\right)}\left(2\mu^2 a_2 + 2\mu^4 a_1 + \mu^6\right) ds \\ &= \frac{-1}{8\pi^2(m^2+\mu^2)}\left(-\mu^2 a_2 + \mu^4 a_1 + \mu^6\right) . \end{aligned} \tag{5.70}$$

From this result, and the invariance of $S^{(1)}$, we obtain the beta functions

$$\begin{aligned}&\beta_1 = -\frac{1}{960\pi^2}\frac{\mu^2}{m^2+\mu^2} \quad \beta_2 = -\frac{\left(\xi-\frac{1}{6}\right)^2}{16\pi^2}\frac{\mu^2}{m^2+\mu^2} \\ &\beta_\Lambda = \frac{1}{16\pi^2}\frac{\mu^6}{m^2+\mu^2} \quad \beta_{\kappa^{-1}} = \frac{\xi-\frac{1}{6}}{4\pi^2}\frac{\mu^4}{m^2+\mu^2} \quad \beta_q = \frac{q^3}{48\pi^2}\frac{\mu^2}{m^2+\mu^2} \quad .\end{aligned} \tag{5.71}$$

It is interesting to note that we can recover the beta functions of minimal subtraction for the dimensionless couplings, including the electric charge, in the limit of $\mu^2 \gg m^2$. However, the dimensional constant differ in this limit. The main result is that this subtraction scheme generate beta functions that decouple in the limit $\mu^2 \ll m^2$, such that $\beta_a \to 0$ for all couplings, including the dimension-full ones.

In conclusion, the extended DeWitt-Schwinger subtraction scheme compatible with the results of the Appelquist-Carazzone theorem [7] for perturbative QFT in flat spacetime. This is an important difference with respect to Minimal Subtraction. In the next chapter we will explore this result in the context of the cosmological constant problem.

Chapter 6.

Heavy Fields, Decoupling and the Cosmological Constant Problem

"It isn't that they can't see the solution. It is that they can't see the problem."
— G. K. Chesterton, Scandal of Father Brown

As we have seen, contributions to the dynamics of the background fields, e.g. the gravitational field, from quantum fields can be computed through the effective action

$$S^{(1)} = \int d^4x \sqrt{-g} \left[-\Lambda + \frac{1}{16\pi G} R + \alpha_1 C^2 + \alpha_2 R^2 + L_{\text{ren}} \right] + S_M \,, \tag{6.1}$$

where S_M are the possible classical matter fields. We can derive from this action the semi-classical Einstein Equation

$$\frac{1}{8\pi G}G_{\mu\nu} + \Lambda g_{\mu\nu} + \beta_1^{(1)} H_{\mu\nu} + \beta_2^{(2)} H_{\mu\nu} = \langle T_{\mu\nu} \rangle_{\text{ren}} + T^M_{\mu\nu}. \tag{6.2}$$

The Standard Model of Cosmology, ΛCDM, assumes that $T^M_{\mu\nu}$ is mostly constituted by non-relativistic matter, and a non vanishing Λ. The contributions of higher order terms of the gravitational field are usually neglected, and so are quantum corrections. Introducing the FLRW metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) \tag{6.3}$$

where $a(t)$ is the expansion parameter and k the curvature, into the Einstein equation we find the Friedman equations

$$H^2(t) \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{6.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \,. \tag{6.5}$$

The energy density and pressure of the matter components are usually classified in two: relativistic fields (radiation) where $\rho = 3p$ and non-relativistic fields (matter) with $p = 0$. It is useful to define the energy density of each component in terms of the critical energy density $\rho_c = \frac{3H_0^2}{8\pi G}$

$$\Omega_M = \frac{\rho_M}{\rho_c} \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \quad \Omega_k = -\frac{k}{a_0^2 H_0^2} \,. \tag{6.6}$$

Cosmological observations from different sources: type IA supernova, Barionic Acoustic Oscillations and the Cosmic Microwave Background allow us to restrict the possible values of these contributions as shown in figure 6.1.

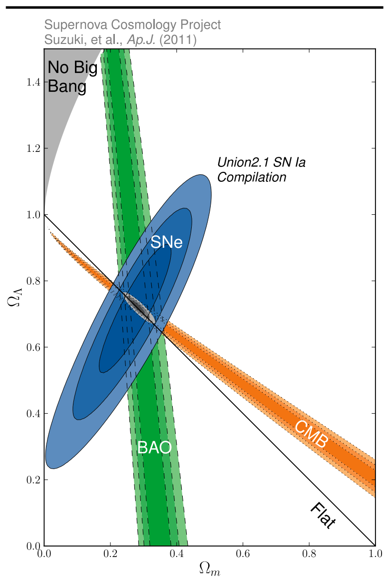

Figure 6.1.: Observational constraints from ype IA supernova, Barionic Acoustic Oscillations and the Cosmic Microwave Background for Ω_M and Ω_Λ. Here, $\Omega_k \approx 0$ as suggested by data from the Cosmic Microwave Background [43].

As a consequence we know that apart from the matter content of the Universe we need a non-vanishing cosmological constant of

$$\rho_\Lambda = \Omega_\Lambda \rho_c \approx 10^{-47} \mathrm{GeV}^4 \,. \tag{6.7}$$

The discovery of the accelerated expansion of the Universe [90, 94], and the non-vanishing cosmological constant can be interpreted in two ways. We can assume that the cosmological constant is part of the classical gravitational background field, which value is determined by observation. This is the approach we have assumed so far. Another approach is to assume that there is a source, called dark energy, which has an equation of state $\rho = -p$ and that, at current times, mimics a cosmological constant term, i.e., $T_{ab} \approx \Lambda g_{ab}$. This is similar to the possible origin of inflationary expansion due to a scalar field with a potential. However, even if a dark energy proposal ends to be correct, there is no reason to assume that a cosmological constant term should be zero, and both contributions would be involved in the explanation of the current observations.

The question that arises is how does the quantum correction of, e.g. the Standard Model fields, contribute to the cosmological constant today. For this task it is interesting to analyze the physical scales present at this type of computation. Here, we will only study the case of the free massive fields [1]. In this case, the range of masses (see fig 6.2) would go from the electron mass $m_e \simeq 0.511$ MeV to the quark top mass $m_t \simeq 173.1$ GeV. On the other hand, the gravitational field can be parametrized by the Hubble constant today $H_0 \simeq 3.7 \times 10^{-41}$ GeV and the cosmological constant $\Lambda \simeq 10^{-47}\text{GeV}^4$. It is straightforward to realize that there is a big separation between the two scales. This will simplify enormously the computation of the effective action, and equivalently,

[1] A possible direction of research would be to include mass-less fields such as the photon field, interactions and possible beyond Standard fields as massive neutrinos.

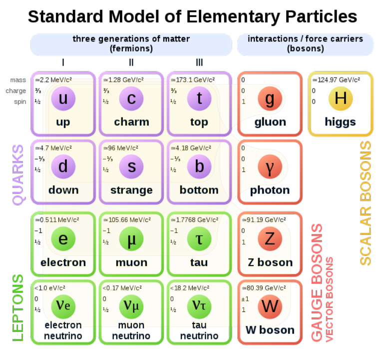

Figure 6.2.: Standard Model fields with its corresponding masses.

the stress-energy tensor; since we can construct an effective field theory where we would have $\mathcal{R} \ll m^2 \to \infty$ for each mass of the Standard Model. Here we have defined $\mathcal{R}$ as any possible terms constructed from the gravitational field, e.g. R^2 or $R_{ab}R^{ab}$.

When computing the possible corrections of the vacuum polarization of massive field to the cosmological constant, one encounters naturally the well-known cosmological constant problem [22,75,113]. Briefly, the formulation of the problem is as follows. If we compute the zero point energy density of a quantum field with mass m a divergent quantity results. Using a cuttoff M yields a contribution of $\sim M^4$ and using dimensional regularization and MS yields $\sim m^4$. In both cases the contribution is of many orders of magnitude higher than the observed quantity, such that the problem can be regarded as a regularization-independent result. In the following sections, we will analyze this argument in more detail by considering both cases and comparing these result with the already mentioned extended DeWitt-Schwinger subtraction scheme.

6.1. The Cosmological Constant Problem I

Let us assume a scalar field with mass m. We will work in a flat spacetime as in the original work [113]. The vacuum expectation value of the stress-energy tensor can be split in the non vanishing components of the energy density $\langle \rho \rangle$ and pressure $\langle p \rangle$ (see [75] for more details of the computations). In terms of the momentum integral we have

$$\langle \rho \rangle = \frac{1}{3(2\pi)^3} \int d^3k \sqrt{k^2 + m^2} \quad \langle p \rangle = \frac{1}{6(2\pi)^3} \int d^3k \frac{k^2}{\sqrt{k^2 + m^2}}. \tag{6.8}$$

Both integrals are divergent and a regularization method need to be imposed[2]. A natural option is to introduce a cutoff regulator M in the upper limit such that, for the case of $M \gg m$ we have

$$\langle \rho \rangle = \frac{M^4}{16\pi^2} \left(1 + \frac{m^2}{M^2} + ... \right) \qquad \langle p \rangle = \frac{1}{3} \frac{M^4}{16\pi^2} \left(1 - \frac{m^2}{M^2} + ... \right). \tag{6.9}$$

The regulator can be regarded as an intermediate step to finally obtain a finite renormalized result. However, from a effective field

[2]Note that we can use adiabatic regularization in this integrals giving a vanishing result for both terms.

theory point of view we can consider the cutoff M to be some physical limit of the theory. For example, we believe that at the Planck scale, the QFTCS in general is no longer valid. Choosing $M = M_{\rm P}$ this would mean that from (6.9) $\langle\rho\rangle \simeq \frac{M_{\rm Pl}^4}{16\pi^2} = 2\times 10^{71}{\rm GeV}^4$. Of course this is a "catastrophically" prediction, since we have seen that the observational value is $\Lambda_{\rm obs} \simeq 10^{-47}{\rm GeV}^4$ [75,113].

There are several comments to make at this point. First, results from (6.9) cannot be regarded as a physical prediction of the energy density since the use of this kind of regulator leads to inconsistent results. Just by analyzing (6.9) we find that $\langle\rho\rangle \neq -\langle p\rangle$, which implies that general covariance does not hold [4,76]. This is of no surprise since we know that this kind of regulator leads to undesirable breaking of symmetries, even in perturbative QED in flat spacetime [5]. It is also worth mentioning that quantum field theory in itself cannot be regarded as just a collection of low energy degrees of freedom, in this case $|k| \leq M$ if we wish to recover important results as the Casimir effect or the well-known anomalies [62]. This makes it rather complicated and obscure to use the techniques of perturbative flat spacetime in more general, non perturbative regimes such as QFTCS.

Second, even if we are able to construct a cut-off like regularization that is maintains covariancy of the results (see e.g. [100] donoghue) the use of physical cut-offs in field theories, even in flat spacetime, is a non trivial issue [74].

We know that the fields of the Standard Model in curved spacetime are renormalizable in the sense that we can always construct a theory with defined counter-terms that reabsorb the divergences of the different quantities. Therefore, we actually do not need to make use of a cut-off regulator at all and make us of a specific subtraction scheme, e.g. MS (or $\bar{\text{MS}}$ or μ scheme) in dimensional regularization. This can be done as in the last chapter by upgrading the above integrals to a d dimensional spacetime and then subtracting the pole terms. The final result is for μ scheme [75]

$$\langle\rho\rangle_{\text{ren}}^{\text{MS}} = \frac{m^4}{64\pi^2}\log\left(\frac{m^2}{\mu^2}\right) \qquad \langle p\rangle_{\text{ren}}^{\text{MS}} = -\frac{m^4}{64\pi^2}\log\left(\frac{m^2}{\mu^2}\right), \quad (6.10)$$

where we obtain now the correct equation of state $\langle\rho\rangle_{\text{ren}}^{\mu} = -\langle p\rangle_{\text{ren}}^{\mu}$. Here we would face a contribution of order m^4, which for the Masses of the Standard Model, e.g. $m_e \simeq 0.511\text{MeV}$, is still a very magnitudes far away from the observed quantity. One can still argue, that since a renormalizable theory can reabsorb the contributions into the couplings of the theory, there is no problem in doing this for the vacuum contribution. We could tune the couplings to describe the physics at the current (low) energy scales and have a perfectly valid description.

Nevertheless, we could go a step further and not only have a low energy description, i.e., an effective field theory description of the current observations but also connect this low energy theory with possible high energy formulations[3] , we would face again a cosmological constant problem[4] [22].

[3] We could take a more humble approach, by not considering the possibility of the existence of this kind of "bridge", (see [74]).

[4] A similar description of this formulation of the cosmological constant problem can be found in [77].

In order to properly understand this formulation let us first review a possible effective field theory that describes the massive Standard model fields in curved spacetime in the case where we observe the cosmological constant, i.e., $\mathcal{R} \ll m^2 \to \infty$. We consider here a quantized scalar field for simplicity but the same can be developed for Dirac and Gauge fields.

6.2. Effective Field Theory for a Scalar Field

Let us consider a quantum scalar field in curved spacetime

$$S = \int d^4x \sqrt{-g} \left(-\Lambda + \frac{R}{16\pi G} + \frac{1}{2} \left((\nabla_\mu \phi) \nabla^\mu \phi + m^2 \phi^2 + \xi R \phi^2 \right) \right). \tag{6.11}$$

The effective field theory is constructed by requiring that $\mathcal{R} \ll m^2$. A good approximation consists on taking the DeWitt-Schwinger proper time expansion for the effective Lagrangian as

$$L^{(1)} \approx \frac{i}{2(4\pi)^2} \sum_{j=0}^{\infty} a_j(x,x) \int_0^\infty \frac{ds}{(is)^3} e^{ism^2} (is)^j \tag{6.12}$$

where the coefficients a_j only depend on gravitational magnitudes, the first three being

$$a_0(x) = 1\,, \quad a_1(x) = -\bar{\xi}R$$
$$a_2(x) = \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\bar{\xi}^2 R^2 \quad . \tag{6.13}$$

We can perform dimensional regularization as in 5.3 obtaining

$$L^{(1)} \approx \frac{1}{(4\pi)^{\frac{n}{2}}}\sum_{j=0}^{\frac{n}{2}}\left[\frac{2}{n-4} + \psi\left(\frac{n}{2} - j + 1\right) + \log\left(\frac{\mu^2}{m^2}\right)\right] + \frac{\left(-m^2\right)^{2-j}}{(\frac{n}{2}-j)!}a_j$$
$$L_{\text{fin}}, \tag{6.14}$$

where we have defined the finite contribution as

$$L_{\text{fin}} = \left(\frac{m^2}{4\pi}\right)^{d/2}\sum_{n=3}^{\infty}\frac{(n-3)!}{m^{2n}}a_n. \tag{6.15}$$

Note that the infinite finite terms from (6.15) is an expansion of inverse powers of m^2 such that it is indeed an effective field theory. We have now to take care of the divergent part. The standard subtraction scheme for effective field theory maintains the logarithmic term (MS, $\bar{\text{MS}}$, μ-subtraction, etc.). Let us consider for simplicity μ-subtraction scheme, where we only maintain the logarithmic term.

In this case the renormalized effective Lagrangian takes the form

$$L^{\mu}_{\text{ren}} \approx \frac{-1}{32\pi^2}\left[\frac{m^4}{2} - m^2\tilde{a}_1 + \tilde{a}_2\right]\log\left(\frac{m^2}{\mu^2}\right) + \left(\frac{m^2}{4\pi}\right)^2 \sum_{n=3}^{\infty} \frac{(n-3)!}{m^{2n}} a_n. \tag{6.16}$$

The complete effective action results in

$$S^{(1)} = \int dx^4 \sqrt{-g}\left[-\Lambda + \frac{1}{16\pi G}R + \alpha_1 C^2 + \alpha_2 R^2 + L^{\mu}_{\text{ren}}\right], \tag{6.17}$$

where all the coupling constants have its corresponding μ dependence. Note that we recover the results obtained in (6.10), i.e., the cosmological constant Λ receives a contribution from L^{μ}_{ren} such that the observable quantity is $\Lambda_{\text{obs}} = \Lambda + \frac{m^4}{64\pi^2}\log\left(\frac{m^2}{\mu^2}\right) \simeq 10^{-47}\text{GeV}^4$. There is no apparent problem at this stage.

The cosmological constant problem as outlined in [22] arises when we wish to connect the effective field theory with the exact effective action (that is valid at high energy scales) in the limit of low energy.

6.3. The Cosmological Constant Problem II

Let us assume we have n light scalar fields $\mathcal{L}_i$ with mass m_i and couplings ξ_i and a heavy scalar field $\mathcal{H}$ with mass M and coupling

ξ_h. In the complete exact theory[5] we have the following effective action

$$S^{(1)} = \int dx^4 \sqrt{-g} \Bigg[-\Lambda_H(\mu) + \frac{1}{16\pi G_H(\mu)} R + \alpha_{1H}(\mu) C^2 + \alpha_{2H}(\mu) R^2 + \sum_{\mathcal{L}_i, \mathcal{H}} L^{\mu}_{\text{ren}} \Bigg] \quad . \tag{6.18}$$

The running of the coupling constants are determined by

$$\beta^{MS}_{\Lambda} = \sum_{\mathcal{L}_i} \frac{m_i^4}{32\pi^2} + \frac{M^4}{32\pi^2}, \qquad \beta^{MS}_{\kappa} = \sum_{\mathcal{L}_i} -\frac{m_i^2 \bar{\xi}_i}{8\pi^2} - \frac{M^2 \bar{\xi}_h}{8\pi^2},$$
$$\beta^{MS}_{\alpha_1} = \sum_{\mathcal{L}_i, \mathcal{H}} -\frac{1}{1920\pi^2}, \qquad \beta^{MS}_{\alpha_2} = \sum_{\mathcal{L}_i} -\frac{1}{32\pi^2} \bar{\xi}_i^2 - \frac{1}{32\pi^2} \bar{\xi}_h^2 \quad . \tag{6.19}$$

Here $\bar{\xi} = \xi - \frac{1}{6}$. In the low energy regime where $\mathcal{R} \ll M^2$ holds we have (using the results of (6.16))

$$S^{(1)} = \int dx^4 \sqrt{-g} \Bigg[-\Lambda_L(\mu) + \frac{1}{16\pi G_L(\mu)} R + \alpha_{1L}(\mu) C^2 + \alpha_{2L}(\mu) R^2 + \left(\frac{M^2}{4\pi}\right)^2 \sum_{n=3}^{\infty} \frac{(n-3)!}{M^{2n}} a_n + \sum_{\mathcal{L}_i} L^{\mu}_{\text{ren}} \Bigg]. \tag{6.20}$$

[5]Here we mean a theory that is valid at higher energies but below the Planck Mass, where QFTCS is meant to fail.

Here we have absorbed the problematic contribution into the effective coupling constant in the low energy regime. This is a result from integrating out the heavy massive field (see Appendix A) that is typically done in MS, since the beta functions (6.19) do not decouple[6].

We can automatically see where the problem is. The effective contribution to the cosmological constant differ between the two theories as

$$\Lambda_H^{MS}(\mu) = \Lambda_L^{MS}(\mu) + \frac{M^4}{64\pi^2}\log\left(\frac{\mu^2}{M^2}\right). \tag{6.21}$$

This is the cosmological constant problem as outlined in [22]. The problem has to do with the fact that we have good arguments to think that the cosmological constant is not only small but remains small at higher energies, at least at energies higher than for example the mass of the electron $m_e \simeq 0.511\text{MeV}$. Here the difference between both couplings just for the electron mass is $\Lambda_H(\mu) - \Lambda_L \simeq 10^{-16}\text{GeV}^4$ in comparison with the $\Lambda_L = \Lambda_{\text{obs}} \simeq 10^{-47}\text{GeV}^4$. The cosmological constant problem arises when we try to connect the low energy effective field theory with its more fundamental theory valid at high energy.

However, it is important to take into account that we have used a subtraction scheme that actually does not decouple and integrated out the corresponding heavy field by redefining the coupling constants of each theory (see Appendix A for the QED analog). This

[6]The beta functions of the coupling constants of (6.20) would be the same as (6.19) but without the contribution of the massive field $\mathcal{H}$.

is an artificial construct in order to safe Minimal Subtraction for it advantages in higher order loop calculations in scattering amplitudes, but there is no requirement of QFTCS that this subtraction scheme has any preference in comparison to others. Indeed, we can perform the extended DeWitt-Schwinger subtraction scheme[7] presented in 5.4 we would obtain

$$\Lambda_H^{DS}(\mu) = \Lambda_L^{DS}(\mu) + \frac{1}{128\pi^2}\mu^4 - \frac{M^2}{64\pi^2}\mu^2 + \frac{M^4}{64\pi^2}\log\left(\frac{\mu^2+M^2}{M^2}\right). \tag{6.22}$$

Of course, integrating out in this case is not needed since the beta functions in itself decouple, analog to the Momentum Subtraction scheme in perturbative QED (see Appendix A).

If we assume that there are no lighter massive fields than the electron mass (omitting possible beyond Standard Model fields), then the difference between the theory at an energy higher and lower than the mass of the electron $M = m_e$ gives now $\Lambda_H \simeq \Lambda_L + \frac{\mu^6}{192\pi^2 m_e^2}$. Taking now the low energy constant to be the observable magnitude $\Lambda_L = \Lambda_{\text{obs}} \simeq 10^{-47}\text{GeV}^4$ and e.g. $\mu \sim H_0 \simeq 3.7 \times 10^{-41}\text{GeV}$ we obtain $\Lambda_H - \Lambda_L \simeq 10^{-248}\text{GeV}^4$. This implies that the cosmological constant problem points towards the fact that Minimal Subtraction (or its modifications) may not be suitable for constructing effective field theories in curved spacetime, but it does not mean at all that we cannot construct such a theory and that it cannot be consistent with the current observational data.

[7]Note that even using the un-extended DeWitt-Schwinger subtraction scheme would not generate this problem since there is no explicit μ dependence.

In conclusion, if one takes the cuttoff approach to identify possible predictions one arrives at a catastrophically result of 122 orders of magnitudes. But the method by itself is well known to have troubles [112] even in gauge theories in flat spacetime [5,74]. Moreover, in a renormalizable theory as in QFTCS, cut-off regulator are actually not needed. The other possibility is to use Minimal Subtraction that does not carry these problems but we get again a similar catastrophe of more than 20 orders of magnitude. But as we have seen the problem comes from the construction of effective field theories and the fact that this scheme does not decouple even in flat spacetime so we have to integrate out by hand the fields. This is possible for example for QED, where the shift between the charge of the heavy theory and the light theory is just logarithmic (see Appendix A). However, this integrating out seems not to be possible in the case of the cosmological constant so we have to make use of other schemes, e.g. extended DeWitt-Schwinger subtraction schemes. There is no issue in doing that since Minimal Subtraction is just more optimal for computations in flat spacetime but has no preference as a scheme in any case in QFTCS.

6.4. Decoupling and Sensitivity of the Cosmological Constant

We have seen that we can built a effective field theory compatible with the observed cosmological constant. It is interesting to analyze if the cosmological constant can be sensible to quantum corrections and which conditions must hold. In order to do this, let us assume that we have a scalar field coupled to another classical scalar field with Klein Gordon equation

$$\left(\Box + m^2 + h^2\Phi^2 + \xi R\right)\phi = 0\,. \tag{6.23}$$

We can do the same computation of the DeWitt-Schwinger proper time expansion and regularize using the extended DeWitt-Schwinger subtraction scheme. (see [45] for the extended DeWitt-Schwinger subtraction scheme of the Yukawa model). Assuming that the classical external field Φ (and its derivative) is also smaller than the mass $h^2\Phi^2 \ll m^2$ we can use the expansion 6.2 and obtain

$$S^{(1)} = \int dx^4 \sqrt{-g}\Bigg[-\Lambda + \frac{1}{16\pi G}R + \alpha_1 C^2 + \alpha_2 R^2 + \left(\frac{m^2}{4\pi}\right)^2 \sum_{n=3}^{\infty} \frac{(n-3)!}{m^{2n}} a_n\Bigg].$$

Here we have omit the explicit μ dependence of the coupling constants. Using the DeWitt-Schwinger coefficient a_3 from [86, 110] which gives the first correction to the classical action in the effective field theory, such that

$$S^{(1)} \approx \int dx^4 \sqrt{-g} \Bigg[-\Lambda + \frac{R}{16\pi G} + \alpha_1 C^2 + \alpha_2 R^2 \\ - \frac{1}{m^2 12(4\pi)^2} \left(h^6 \Phi^6 + \frac{1}{2} h^4 \Phi^4 R + h^2 \Phi^2 \mathcal{G} \right) + \frac{\mathcal{R}^{(3)}_{\text{Grav}}}{m^2} \Bigg], \tag{6.24}$$

where $\mathcal{G} = \frac{1}{5}\Box R + \frac{1}{12}R^2 + \frac{1}{30}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1}{30}R_{\mu\nu}R^{\mu\nu}$ and $\mathcal{R}^{(3)}_{\text{Grav}}$ is sum of six dimensional operators constructed only my gravitational tensor. In this case, for an almost constant scalar field $\Phi \sim T$ we have a contribution to the cosmological constant as

$$\Lambda_{\text{eff}} = \Lambda + \frac{h^6 T^6}{192\pi^2 m^2}. \tag{6.25}$$

We can compare the contribution of (6.25) to the observed quantity $\Lambda \simeq 10^{-47}\text{GeV}^4$ for difference masses with respect to the field T (see figure 6.3). For the electron mass $m_e \simeq 0.511\text{MeV}$ we find that a induced cosmological constant is of the order of the observed Λ at a scale of $hT \simeq 3\text{eV}$, which is still far from the mass scale but enough for the cosmological constant to be sensible to this effect.

In conclusion, at the energy scales where we usually observe the cosmological constant, we can use the effective field theory description of e.g. (**??**), where here Λ would be the observed value $\Lambda \simeq 10^{-47}\text{GeV}^4$. The quantum contributions appear as higher order corrections, in alliance with the rules of the effective field theory approach. The corrections could be important in situations

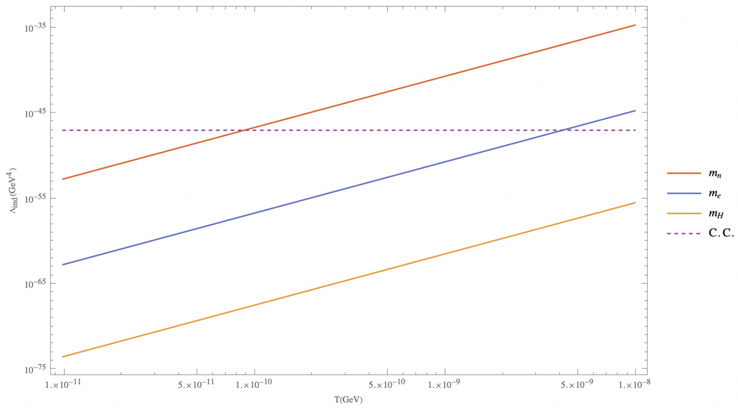

Figure 6.3.: We plot the induced cosmological constant term $\Lambda_{\text{ind}} \equiv \frac{h^6 T^6}{192\pi^2 m^2}$ for different masses: $m_n = 5$ eV,$m_e = 0.511$ MeV and $m_H = 125$ MeV and compare it with the observational cosmological constant $\Lambda_{\text{obs}} \simeq 10^{-47}\text{GeV}^4$. We take here $h = 1$.

where these extra contributions (e.g. (6.25)) approach the observed quantity.

Appendix A.

Vacuum polarization in perturbative QED

A.1. Renormalization and Subtraction Schemes

Let us assume a Dirac field and an electromagnetic field in QED

$$L = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{A.1}$$

The contribution of the one-loop vacuum polarization can be encoded in (see [98] for more details)

$$\Pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{-x(1-x)q^2+m^2}{\mu^2}\right) - \frac{e^2}{6\pi^2\epsilon} - \delta_{\text{sub}}, \tag{A.2}$$

where q is the external momentum of the photon propagator. Here δ_{sub} contains the divergent terms what we will subtract by reabsorbing it in the coupling constants of the Lagrangian (A.1), in this case into the electric charge e. The first possible prescription is the on-shell subtraction, which has the advantage of maintaining the physical quantities. In this case we have

$$\delta_{\text{sub}} = -\frac{e^2}{6\pi^2\epsilon} + \frac{e^2}{12\pi^2}\log\left(\frac{m^2}{\mu^2}\right). \tag{A.3}$$

The final quantity is then

$$\Pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{-x(1-x)q^2+m^2}{m^2}\right). \tag{A.4}$$

From this quantum contribution we can obtain the corrected Coulomb potential in momentum space [98]

$$V(\vec{q}) = \frac{e^2}{\vec{q}^2\left(1-\Pi(\vec{q}^2)\right)}. \tag{A.5}$$

The large momentum expansion $q^2 \gg m^2$ leads to

$$V^{\text{High}}(\vec{q}) = \frac{e^2}{\vec{q}^2\left(1-\frac{e^2}{12\pi^2}\log\left(\frac{\vec{q}^2}{e^{5/3}m^2}\right)\right)} \tag{A.6}$$

while the low energy limit $q^2 \ll m^2$ implies

$$V^{\text{low}}(\vec{q}) = \frac{e^2}{\vec{q}^2\left(1+\frac{e^2}{12\pi^2}\frac{\vec{q}^2}{30m^2}\right)}, \tag{A.7}$$

which in position space implies the well known Lamb-Shift [91]. Here, the electric charge $e \approx 0.302$ and the mass $m \approx 0.510\text{MeV}$ are the physical magnitudes we obtain from low energy experiments. As we have explained, we could also decide to subtract not (A.3) but

$$\delta_{\text{sub}} = -\frac{e^2}{6\pi^2\epsilon}. \tag{A.8}$$

In this case the corrected Coulomb potential has the form

$$V(\vec{q}) = \frac{e^2_{\text{MS}}}{\vec{q}^2\left(1 - \Pi_{\text{MS}}(\vec{q}^2)\right)} \tag{A.9}$$

with

$$\Pi_{MS}(q^2) = \frac{e^2_{MS}}{2\pi^2}\int_0^1 dxx(1-x)\log\left(\frac{-x(1-x)q^2 + m^2}{\mu^2}\right). \tag{A.10}$$

This is the case of Minimal Subtraction scheme in dimensional regularization. The contribution (A.10) must be independent of the scaling μ and therefore must include a variation in μ such that for a change from two different scales the charge is

$$e^2_{\text{MS}}(\mu) = \frac{e^2_{\text{MS}}(\mu_0)}{1 - \frac{e^2_{\text{MS}}(\mu_0)}{12\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)} \tag{A.11}$$

which compensates the change of scale of (A.10). This scaling dependence of the electric charge gives us the usual beta function $\beta_q = \frac{q^3_{\text{MS}}}{12\pi^2}$. We obtain again for both the high and the low energy limit

$$V_{\text{MS}}^{\text{High}}(\vec{q}) = \frac{e_{\text{MS}}^2(\mu)}{\vec{q}^2\left(1 - \frac{e_{\text{MS}}^2(\mu)}{12\pi^2}\log\left(\frac{\vec{q}^2}{\mu^2}\right)\right)} \tag{A.12}$$

$$V_{\text{MS}}^{\text{low}}(\vec{q}) = \frac{e_{\text{MS}^2}}{\vec{q}^2\left(1 + \frac{e_{\text{MS}}^2(\mu)}{12\pi^2}\left(\frac{\vec{q}^2}{30m^2} - \frac{1}{6}\log\left(\frac{m^2}{\mu^2}\right)\right)\right)}\,. \tag{A.13}$$

It is interesting to note that in the high energy limit, choosing the μ parameter to be $\mu = |q|$ we obtain

$$V_{\text{MS}}^{high}(\vec{q}) = \frac{e_{\text{MS}}^2(|q|)}{\vec{q}^2}, \tag{A.14}$$

which mimics the classical Coulomb potential but with a electric charge that *runs* with the momentum q. This gives us a well-founded equivalence between the μ parameter of dimensional regularization and the only relevant scale at high energy, in this case the momentum q.

The situation is dramatically different in the low energy limit since the assignation of $\mu = |q|$ does only introduce a problematic logarithmic term. Indeed, an important result for further discussion is that Minimal Subtraction do not decouple massive fields in the low energy regime [74]. Decoupling is a very relevant property for constructing effective fields theories as we will see in chapter 6. The physical motivation behind decoupling has to do with the fact that we believe, and experiments supports this idea, that high energy scales do not affect the low energy description. An equivalent explanation can be seen from the beta function $\beta_q = \frac{q_{\mathrm{MS}}^3}{12\pi^2}$ which is independent from the mass and therefore contributes equal to the charge at any physical scale.

A possible way of overcoming this problem is to change the subtraction scheme and use a more physical scheme, in this case the momentum subtraction scheme. In this scheme we subtract the value of the magnitude in (A.2) at $p^2 = -\mu_M^2$ obtaining

$$\Pi_{\mathrm{MOM}}(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dxx(1-x)\log\left(\frac{-x(1-x)q^2+m^2}{x(1-x)\mu_M^2+m^2}\right) \tag{A.15}$$

In the limit of $q^2 \ll m^2$ we obtain

$$\Pi_{\mathrm{MOM}}(q^2) = \frac{e^2}{2\pi^2}\left(\frac{p^2+\mu_M^2}{30m^2}+\mathcal{O}\left(\frac{p^4}{m^4}\right)\right) \tag{A.16}$$

such that the massive field does indeed decouple. This can also be seen by the corresponding beta function

$$\beta_{\mathrm{MOM}} = \frac{e^3}{2\pi^2}\int_0^1 dxx(1-x)\log\left(\frac{x(1-x)\mu_M^2}{x(1-x)\mu_M^2+m^2}\right) \tag{A.17}$$

which for $m^2 \ll \mu_M^2$ is $\beta_{\mathrm{MOM}} \approx \beta_{\mathrm{MS}} = \frac{e^3}{12\pi^2}$ but for $m^2 \gg mu_M^2$ is $\beta_{\mathrm{MOM}} \approx \frac{e^3}{60\pi^2}\frac{\mu_M^2}{m^2} \to 0$. The decoupling of the quantum contributions from massive fields in the low energy regime can be encapsulated in the Appelquist-Carazonne theorem [7] which ensures that mass dependent subtraction scheme, as MOM, decouple in this limit in perturbative QFT.

A.2. Effective Field Theory in QED

We have already seen that one of the flaws of Minimal Subtraction scheme (or its modifications) is related to the fact that it is not compatible with decoupling and we must use a more physical, mass dependent scheme such as Momentum subtraction scheme. However, MS is very efficient for loop calculations in gauge theories and therefore it is desirable to main this scheme [74,92]. The solution to this is to *integrate out* the massive fields by hand such that at the low energy regime there is no explicit radiative corrections of the massive field. The vacuum polarization contribution for the heavy field (A.10) in the limit of $q^2 \ll m^2$ is

$$\Pi(q^2) = \frac{e_H^2(\mu)}{2\pi^2} \int_0^1 x(1-x) \log\left(\frac{m^2}{\mu^2}\right) + \frac{q^2}{30m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right) \quad (A.18)$$

where we have dropped the MS index for simplicity and e_H is the running coupling of the high energy theory that includes all the fields, also the heavy field. The effective Lagrangian can encode this information by adding a correction to the original Lagrangian of the form [74]

$$L = \frac{e_L^2(\mu)}{240\pi^2 m^2} \partial_a F_{\mu\nu} \partial^a F^{\mu\nu}. \quad (A.19)$$

Apart from this extra higher order operator term we have a shift in the running coupling constant between the both theories

$$\frac{1}{e_L^2(\mu)} = \frac{1}{e_H^2(\mu)} - \frac{1}{12\pi^2} \log\left(\frac{m^2}{\mu^2}\right), \quad (A.20)$$

where both couplings have different beta functions, since e_H includes the running of the heavy field and e_L does not. This is called to integrate out the heavy degrees of freedom, which results in a theory that includes the quantum contribution of the heavy field in terms of higher order corrections and shifts in the couplings.

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