Gradient Flows in Random Walk Spaces

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Contents

Declaro que esta disertación titulada *Gradient Flows in Random Walk Spaces* y el trabajo presentado en ella son míos. Lo confirmo:

- Este trabajo se realizó total o principalmente mientras se cursaban los estudios para la obtención del título de Doctor en la Universitat de València.
- Cuando se han consultado las publicaciones de otras personas, siempre se ha indicado claramente.
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- Han sido reconocidas todas las fuentes de ayuda.

Pisa, julio de 2021

Marcos Solera Diana

Declaramos que esta disertación presentada por **Marcos Solera Diana** titulada *Gradient Flows in Random Walk Spaces* se ha realizado bajo nuestra supervisión en la Universitat de València. También indicamos que este trabajo corresponde al proyecto de tesis aprobado por esta institución y cumple todos los requisitos para obtener el título de Doctor en Matemáticas.

Burjassot, julio de 2021

J.M.Mazón y J.J.Toledo

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Introducció

El món digital ha comportat l'aparició de molts tipus de dades, de mida i complexitat creixents. De fet, els dispositius moderns ens permeten obtenir fàcilment imatges de major resolució, així com recopilar dades sobre cerques a la xarxa, anàlisis sanitàries, xarxes socials, sistemes d'informació geogràfica, etc. En conseqüència, l'estudi i el tractament d'aquests grans conjunts de dades té un gran interès i valor. En aquest sentit, els grafs ponderats proporcionen un espai de treball natural i flexible on representar les dades. En aquest context, un vèrtex representa una dada concreta i a cada aresta se li assigna un pes segons alguna mesura de "semblança" adequadament triada entre els vèrtexs corresponents. Històricament, les principals eines per a l'estudi de grafs provenien de la combinatòria i la teoria de grafs. No obstant això, després de la implementació de l'operador laplacià (discret) associat a un graf en el desenvolupament de l'agrupació espectral als anys setanta, la teoria d'equacions diferencials parcials en grafs ha obtingut resultats importants en aquest camp (vegeu, per exemple, [63], [115] i les seves referències). Això ha provocat un gran augment de la investigació de les equacions diferencials parcials en grafs. A més, l'interès s'ha vist reforçat per l'estudi de problemes en el processament d'imatges. En aquesta àrea de recerca, els píxels juguen el paper dels vèrtexs i els pesos estan associats a la "similitud" entre els píxels corresponents. La forma en què es defineixen aquests pesos depèn del problema que ens ocupa (vegeu, per exemple, [79] i [114]).

D'una altra banda, sigui $J : \mathbb{R}^N \to \mathbb{R}$ una funció no negativa, radialment simètrica i contínua amb $\int_{\mathbb{R}^N} J(z)dz = 1$. Equacions d'evolució no local de la forma

$$u_t(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)dy - u(x,t) \tag{0.1}$$

i les seves variacions, han sorgit de manera natural en diversos camps científics com a mitjà per modelar una àmplia gamma de processos de difusió. Per exemple, en biologia ([53], [131]), sistemes de partícules ([38]), models de coagulació ([84]), models anisotròpics no locals per a transicions de fase ([1], [2]), finances matemàtiques mitjançant una teoria de control òptima ([36], [104]), processament d'imatges ([91], [109]), etc. Un raonament intuïtiu que explica el grau d'aplicabilitat d'aquest model prové de pensar en $u(x,t)$ com la densitat d'una "població" en un punt x en el moment t i en $J(y-x)$ com a la distribució de probabilitats de passar de y a x en un "salt". Aleshores, $\int_{\mathbb{R}^N} J(y-x)u(y,t)dy$ és la taxa a la qual els "individus" arriben a x des de qualsevol altre lloc i $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$ és la velocitat a la qual surten de la ubicació x. Per tant, en absència de fonts externes o internes, ens conduirà a l'equació (0.1) com a model per a l'evolució de la densitat de població al llarg del temps. Es pot trobar un ampli estudi d'aquest problema a [18].

En els dos paràgrafs anteriors, hem avançat dos casos en què hi ha un gran interès en l'estudi d'equacions diferencials parcials en un entorn no local (o discret). L'anàlisi de la formulació peridinàmica de la mecànica contínua (vegeu [111] i [145]), així com l'estudi dels processos de salt de Markov i altres models no locals, han augmentat aquest interès. Les referències sobre tots els temes esmentats fins ara es donen al llarg de la tesi (vegeu també [48], [64], [77], [79], [87], [88], [92], [109], [114], [143], [155], [156], [161]).

L'objectiu d'aquesta tesi és unificar en un marc ampli l'estudi de molts dels problemes esmentats anteriorment. Per fer-ho, observem que hi ha una forta relació entre alguns d'aquests problemes i la teoria de la probabilitat, i és en aquest camp on trobem els espais adequats per desenvolupar aquest estudi unificador. Sigui $(X, \mathcal{B})$ un espai mesurable i $P : X \times \mathcal{B} \to [0,1]$ un nucli de probabilitat de transició a X (vegeu la secció 1.1). A continuació, es pot definir una funció de transició markoviana de la següent manera: per a qualsevol $x \in X$ i $B \in \mathcal{B}$,

$$P_t(x, B) := e^{-t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} P^n(x, B), \quad t \in \mathbb{R}_+,$$

on P^n denota el nucli de probabilitat de transició al pas n. La família d'operadors associada, $P_t f(x) := \int f(y) P_t(x, dy)$, satisfà

$$\frac{\partial}{\partial t} P_t f(x) = \int P_t f(y) P(x, dy) - P_t f(x).$$

A més, si considerem un procés de Markov $(X_t)_{t \geqslant 0}$ associat a la funció de transició markoviana $(P_t)_{t \geqslant 0}$, i si denotem per μ_t la distribució de X_t, llavors la família $(\mu_t)_{t \geqslant 0}$ també compleix una equació lineal de la forma

$$\frac{\partial}{\partial t} \mu_t = \int P(y, \cdot) \mu_t(dy) - \mu_t.$$

En aquest context, algunes eleccions específiques de l'espai mesurable $(X, \mathcal{B})$ i de P donaran lloc a alguns dels problemes anteriors. Per exemple, si $X = \mathbb{R}^N$ i $P(x, dy) = J(y - x)dy$, recuperarem l'equació (0.1). A més, prenent X com el conjunt de vèrtexs d'un graf ponderat i definint adequadament la funció de probabilitat de transició en termes de pesos (vegeu l'Exemple 0.38), també podem recuperar l'equació de la calor en grafs.

Les observacions anteriors suggereixen que els espais de passeig aleatori proporcionen el marc adequat per complir els nostres objectius d'unificar una àmplia varietat de models no locals. Aquests espais estan constituïts per un espai mesurable $(X, \mathcal{B})$ i un nucli de probabi–litat de transició P en X que codifica els salts d'un procés de Markov. Adoptarem la notació $m_x := P(x, \cdot) \in \mathcal{P}(X, \mathcal{B})$ per a cada $x \in X$ (aquí $\mathcal{P}(X, \mathcal{B})$ indica l'espai de les mesures de probabilitat en $(X, \mathcal{B})$). A més, requerirem una mena de propietat d'estabilitat per a aquests espais, és a dir, l'existència d'una mesura invariant ν (vegeu la Definició 0.7). Aleshores, direm que $[X, \mathcal{B}, m, \nu]$ és un espai de passeig aleatori. Degut a la generalitat d'aquests espais, els resultats que obtindrem tindran un gran ventall d'aplicabilitat a una àmplia gamma de problemes d'evolució sorgits en diversos camps científics. Malauradament, aquest marc no cobreix problemes relacionats amb el nucli fraccionari degut a la seva naturalesa singular.

Durant els darrers anys i tenint en compte l'objectiu esmentat, hem estudiat alguns fluxos gradient en el marc general d'un espai de passeig aleatori. En particular, hem estudiat el flux de la calor, el flux per la variació total i problemes d'evolució del tipus Leray-Lions amb diferents tipus de condicions de frontera no homogènies. Concretament, juntament amb l'existència i la unicitat de solucions a aquests problemes i el comportament asimptòtic de les seves solucions, s'han estudiat una àmplia varietat de propietats, així com els operadors de difusió no locals que hi participen. Els nostres resultats s'han publicat a [**123**], [**124**], [**125**], [**126**] i [**146**].

Guió de la tesi

A continuació descrivim breument el contingut de la tesi. Per començar, al capítol 1, introduïm el marc general d'un espai de passeig aleatori. A continuació, a la secció 1.1, la relacionem amb nocions clàssiques de la teoria de cadenes de Markov i proporcionem una llista de resultats que esperem que ajudin el lector a tenir una bona idea sobre les propietats que gaudeixen aquests espais. Després d'introduir una propietat d'estabilitat per a espais de passeig aleatoris, anomenada m-connexió, dediquem la secció 1.2 a explorar les característiques que gaudeix aquesta noció i la relacionem amb conceptes coneguts d'ergodicitat. A conti–nuació, proporcionem una llista d'exemples d'espais de passeig aleatori d'interès particular, com els que es van esmentar al començament de la introducció. La resta del capítol es dedica a introduir els homòlegs no locals de nocions clàssiques com les de gradient, divergència, límit, perímetre, curvatura mitjana i curvatura de Ricci, així com de l'operador de Laplace. En fer-ho, obtenim resultats que imiten els resultats clàssics en el cas local i, a més, obtenim més caracteritzacions de la m-connexió d'un espai de passeig aleatori. També dediquem un espai a trobar condicions suficients per a l'existència de desigualtats de tipus Poincaré i relacionem aquestes desigualtats tant amb la bretxa espectral (o "gap") de l'operador de Laplace com amb desigualtats isoperimètriques. Finalment, la secció 1.7 està dedicada a la curvatura d'Ollivier-Ricci i la seva relació amb la desigualtat de tipus Poincaré.

El capítol 2 se centra en l'estudi del flux de la calor en espais de passeig aleatori. En el nostre context, associat al passeig aleatori $m = (m_x)_{x \in X}$, l'*operador de Laplace* Δ_m es defineix com

$$\Delta_m f(x) := \int_X (f(y) - f(x)) dm_x(y).$$

Suposant que la mesura invariant ν compleix una condició de reversibilitat respecte al passeig aleatori (vegeu la Definició 1.15), l'operador $-\Delta_m$ genera en $L^2(X, \nu)$ el semigrup markovià $(e^{t\Delta_m})_{t \geqslant 0}$ (vegeu el Teorema 2.4) anomenat *flux de la calor* a l'espai de passeig aleatori. A més, som capaços de caracteritzar la velocitat infinita de propagació del flux de la calor en termes de la m-connectivitat de l'espai de passeig aleatori (vegeu el Teorema 2.9). Així mateix, a la secció 2.2, estudiem el comportament asimptòtic del semigrup $(e^{t\Delta_m})_{t \geqslant 0}$ i amb l'ajut d'una desigualtat de Poincaré obtenim taxes de convergència de $(e^{t\Delta_m})_{t \geqslant 0}$. En aquest sentit, demostrem que, si ν és una mesura de probabilitat i $[X, \mathcal{B}, m, \nu]$ satisfà una desigualtat de Poincaré, el flux de la calor convergeix a la mitjana de la dada inicial amb taxa exponencial.

A la secció 2.3 introduïm la condició de curvatura-dimensió de Bakry-Émery. Schmuckenschlager [**144**] va considerar per primera vegada l'ús de la condició de curvatura-dimensió de Bakry-Émery per obtenir una definició vàlida de curvatura de Ricci fitada a cadenes de Markov. A més, al 2010, Lin i Yau [**112**] van aplicar aquesta idea als grafs. Posteriorment, aquest concepte de curvatura en el marc discret s'ha utilitzat amb freqüència (vegeu [**107**] i les seves referències). Tingueu en compte que, per definir la condició de curvatura-dimensió de Bakry-Émery, heu de fer ús d'un *carré du champ* Γ (vegeu [**22**, Secció 1.4.2]). En el marc dels semigrups de difusió de Markov, per obtenir bones desigualtats a partir d'aquesta condició de curvatura-dimensió, és essencial que el generador A del semigrup compleixi la fórmula de la regla de la cadena:

$$A(\Phi(f)) = \Phi'(f)A(f) + \Phi''(f)\Gamma(f) \quad \text{per a } f \in D(A) \text{ i } \Phi : \mathbb{R} \to \mathbb{R} \text{ suau},$$

que caracteritza els operadors de difusió en el context continu (vegeu [**22**]). Malauradament, aquesta regla de la cadena no es compleix en un entorn discret, i aquesta és una de les principals dificultats que sorgeix quan es treballa amb aquesta condició de curvatura-dimensió en espais mètrics de passeig aleatori. Seguint la teoria desenvolupada a [**22**], estudiarem la condició de curvatura-dimensió de Bakry-Émery en espais de passeig aleatori reversibles i la seva relació amb la desigualtat de Poincaré.

Finalment, la secció 2.4 es dedica a l'estudi de les desigualtats de transport en relació amb la condició de curvatura-dimensió de Bakry-Émery i la curvatura d'Ollivier-Ricci. Arran dels treballs de Marton i Talagrand ([**118**], [**152**]) sobre les desigualtats de transport, que relacionen les distàncies de Wasserstein amb l'entropia i la informació, aquest tema de recerca ha tingut un gran desenvolupament (vegeu [**94**]). Una de les claus d'aquesta teoria va ser el descobriment el 1986 per Marton [**117**] del vincle entre les desigualtats de transport i la concentració de la mesura. Tingueu en compte que les desigualtats de concentració de la mesura es poden obtenir mitjançant altres desigualtats funcionals, com ara les desigualtats isoperimètriques i les desigualtats de Sobolev logarítmiques (vegeu el llibre de text de Ledoux [**110**]). En aquesta secció provem que, sota la positivitat de la condició de curvatura-dimensió de Bakry-Émery o de la curvatura d'Ollivier-Ricci, es satisfà una desigualtat de transport-informació (Teoremes 2.27 i 2.34). A més, demostrem que si es satisfà una desigualtat de transport-informació, es compleix també una desigualtat de transport-entropia

(Teorema 2.31) i que, en general, la implicació inversa no es compleix.

Al capítol 3 estudiem el flux per la variació total. Des de la seva introducció en el treball seminal de Rudin, Osher i Fatemi com a mitjà per resoldre el problema d'eliminació del soroll o "denoising" ([**143**]), el flux per la variació total s'ha mantingut com una de les eines més populars en el processament d'imatges[1]. A més, l'ús de filtres veïnals per part de Buades, Coll i Morel a [**47**], que va ser proposat originalment per P. Yaroslavsky ([**161**]), ha conduït a una extensa literatura sobre models no locals en el processament d'imatges (vegeu per exemple, [**48**], [**92**], [**109**], [**114**] i les seves referències). En conseqüència, hi ha un gran interès a estudiar el flux per la variació total en el context no local. A més, una línia de recerca diferent tracta una imatge com un graf discret ponderat, on els píxels es prenen com a vèrtexs i la "similitud" entre píxels com a pesos [2]. Per tant, l'estudi de l'operador 1-Laplacià i el flux per la variació total en espais de passeig aleatori té un àmbit d'aplicació potencialment ampli.

Consegüentment, introduïm l'operador 1-Laplacià associat a un espai de passeig aleatori i n'obtenim diverses caracteritzacions (vegeu el Teorema 3.13). A continuació, procedim a demostrar l'existència i la unicitat de solucions del flux per la variació total en espais de passeig aleatori i a estudiar el seu comportament asimptòtic amb l'ajut d'algunes desigualtats de tipus Poincaré. Com a resultat del nostre estudi, generalitzem els resultats obtinguts a [**120**] i [**121**] per al cas particular de $\mathbb{R}^N$ amb un nucli no singular, així com alguns resultats en teoria de grafs.

A més, al capítol 3 introduïm els conceptes de conjunt de Cheeger i conjunt calibrable en espais de passeig aleatori i caracteritzem la calibrabilitat d'un conjunt mitjançant l'operador 1-Laplacià. A més, estudiem el problema del valor propi del 1-Laplacià i el relacionem amb el problema del tall òptim de Cheeger. Aquests resultats s'apliquen, en particular, als grafs ponderats connexos, i complementen els resultats donats a [**57**], [**58**], [**59**] i [**99**].

El capítol 4 està dedicat a l'estudi de la descomposició (BV, L^p), $p = 1$ i $p = 2$, de funcions en espais de passeig aleatori. Per a aquesta tasca estudiem el model de Rudin-Osher-Fatemi amb termes de fidelitat de tipus L^2 i L^1 en espais de passeig aleatori. Obtenim les equacions d'Euler-Lagrange d'aquests problemes de minimització i procedim a obtenir un ampli ventall de resultats sobre les propietats que gaudeixen els minimitzadors.

Com a motivació, recordem el problema clàssic de la restauració d'imatges. Donada una imatge sorollosa/danyada $f : \Omega \to \mathbb{R}$ on, per exemple, Ω és un rectangle a $\mathbb{R}^2$, l'objectiu és eliminar el soroll o la corrupció per tal d'obtenir la desitjada imatge "neta" $u : \Omega \to \mathbb{R}$, que està relacionada amb l'original per la següent equació quan n és el soroll addicional:

$$f = u + n.$$

Malauradament, el problema de recuperar u a partir de f està mal plantejat (vegeu [**12**]). Per solucionar aquest problema, Rudin, Osher i Fatemi (vegeu [**143**]) van proposar resoldre el següent problema de minimització a $BV(\Omega)$:

$$\text{(0.2)} \qquad \text{Minimitzar} \int_\Omega |Du| \quad \text{subjecte a} \quad \int_\Omega u = \int_\Omega f \quad \text{i} \quad \int_\Omega |u-f|^2 = \sigma^2.$$

La primera restricció correspon a la suposició que el soroll té mitjana zero i la segona a que la seva desviació estàndard és σ. El problema (0.2) està naturalment relacionat amb el següent problema sense restriccions (anomenat model ROF):

$$\text{(0.3)} \qquad \min\left\{ \int_\Omega |Du| + \frac{\lambda}{2}\|u-f\|_2^2 \, : \, u \in BV(\Omega) \right\},$$

[1] Des del punt de vista matemàtic, l'estudi del flux per la variació total en $\mathbb{R}^N$ es va establir a [**12**].

[2] La forma en què es defineixen aquests pesos depèn del problema, vegeu, per exemple, [**79**] i [**114**].

per a algun multiplicador de Lagrange $\lambda > 0$. Chambolle i Lions ([**54**]) van demostrar un resultat d'existència i unicitat per a (0.2), així com la relació entre (0.2) i (0.3). La constant λ a (0.3) juga el paper d'un "paràmetre d'escala". Si ajustem λ, podem seleccionar el nivell de detall desitjat a la imatge reconstruïda.

Seguint el model ROF obtenim la següent descomposició (BV, L^2) de f:

$$f = u_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] = \underset{(u,v)\in BV(\Omega)\times L^2(\Omega)}{\arg\min} \left\{ \int_\Omega |Du| + \frac{\lambda}{2}\|v\|_2^2 \ : \ f = u + v \right\}.$$

Un problema variacional alternatiu sorgeix quan el terme de fidelitat $\|f-u\|_2^2$ se substitueix pel terme de fidelitat $\|f-u\|_1$. Això va ser proposat per Alliney (vegeu [**3**] i [**4**]) en espais unidimensionals i fou estudiat extensament per Chan, Esedoglu i Nikolova (vegeu [**55**] i [**56**]):

$$f = u_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] \in \underset{(u,v)\in BV(\Omega)\times L^1(\Omega)}{\arg\min} \left\{ \int_\Omega |Du| + \lambda\|v\|_1 \ : \ f = u + v \right\}. \tag{0.4}$$

La descomposició (BV, L^1) resultant difereix de la descomposició (BV, L^2) en diversos aspectes importants que han atret una atenció considerable en els darrers anys (vegeu [**19**], [**71**], [**78**], [**92**], [**162**] i les seves referències). Assenyalem que la descomposició (BV, L^1) és invariant per contrast (vegeu [**55**]), a diferència de la descomposició (BV, L^2).

L'ús de filtres veïnals per Buades, Coll i Morel a [**47**], que va ser proposat originalment per P. Yaroslavsky [**161**], ha conduït a una extensa literatura de models no locals en el processament d'imatges (vegeu, per exemple, [**48**], [**49**], [**92**], [**109**], [**114**] i les seves referències). Aquest model ROF no local, en una versió simplificada, té la forma

$$\min \left\{ \int_{\Omega\times\Omega} J(x-y)|u(x)-u(y)|dxdy + \frac{\lambda}{2}\|u-f\|_2^2 \ : \ u \in L^2(\Omega) \right\}. \tag{0.5}$$

D'altra banda, una imatge es pot veure com un graf ponderat on es prenen els píxels com a vèrtexs i els pesos estan relacionats amb la similitud entre píxels. Depenent del problema, hi ha diferents maneres de definir els pesos; vegeu, per exemple, [**79**], [**101**], [**102**] i [**114**]. El model ROF en un graf ponderat $G = (V(G), E(G))$ té la forma següent:

$$\min \left\{ \frac{1}{2} \sum_{x\in V(G)} \sum_{y\in V(G)} |u(y)-u(x)|w_{xy} + \frac{\lambda}{2} \sum_{x\in V(G)} |u(x)-f(x)|^2 \sum_{y\sim x} w_{xy} \ : \ u \in L^2(G, \nu_G) \right\}. \tag{0.6}$$

Els problemes (0.5) i (0.6) són casos particulars del model ROF següent en un espai de passeig aleatori $[X, \mathcal{B}, m, \nu]$:

$$\min \left\{ \frac{1}{2} \int_X \int_X |u(y)-u(x)|dm_x(y)d\nu(x) + \frac{\lambda}{2} \int_X |u(x)-f(x)|^2 d\nu(x) \ : \ u \in L^2(X,\nu) \right\},$$

que és una de les motivacions d'aquest capítol i anomenem el model m-ROF. Un altre problema que ens interessa és la descomposició (BV, L^1) en un espai de passeig aleatori $[X, \mathcal{B}, m, \nu]$:

$$\min \left\{ \frac{1}{2} \int_X \int_X |u(y)-u(x)|dm_x(y)d\nu(x) + \lambda \int_X |u(x)-f(x)|d\nu(x) \ : \ u \in L^1(X,\nu) \right\},$$

que té com a cas particular la descomposició (BV, L^1) en grafs. Tingueu en compte que, en el cas local, és a dir, per al problema (0.4), el fet que existeixi un minimitzador per a cada dada en L^1 és una conseqüència del mètode directe del càlcul de variacions. Tot i això, en el nostre context, no tenim prou propietats de compacitat per aplicar aquest mètode. Per tant, la prova del fet que $M(f,\lambda) \neq \emptyset$ (vegeu la Definició 4.13) per cada $f \in L^1(X,\nu)$, es farà després de l'estudi del problema geomètric associat a la descomposició (BV, L^1) (que es tracta a la secció 4.2.1).

En resum, el nostre objectiu és estudiar la descomposició (BV, L^p), $p = 1, 2$, de funcions en espais de passeig aleatori, desenvolupant una teoria general que es pugui aplicar, en particular, a grafs discrets i models no locals.

Finalment, al capítol 5, estudiem problemes d'evolució de tipus p-Laplacià com el que es dona al següent model de referència:

$$u_t(t,x) = \int_{\Omega \cup \partial_m \Omega} |u(y) - u(x)|^{p-2}(u(y) - u(x))dm_x(y), \quad x \in \Omega,\ 0 < t < T,$$

amb condicions de frontera Neumann no homogènies, on $\Omega \in \mathcal{B}$ i $\partial_m \Omega$ és la m-frontera de Ω (vegeu la Definició 0.51). Aquest model de referència es pot considerar com l'equivalent no local al problema d'evolució clàssic:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u), & x \in U,\ 0 < t < T, \\ -|\nabla u|^{p-2}\nabla u \cdot \eta = \varphi, & x \in \partial U,\ 0 < t < T, \end{cases}$$

on U és un domini suau i fitat a $\mathbb{R}^n$, i η és el vector normal exterior a ∂U. De fet, el nostre estudi es desenvolupa amb una generalitat molt més gran que ens permet cobrir una àmplia varietat de problemes. Ara procedirem a donar més detalls.

Estudiem l'existència i unicitat de solucions "mild" i fortes de problemes de difusió no lineals i no locals de tipus p-Laplacià amb condicions de frontera no lineals. Els problemes es plantegen en un subconjunt W d'un espai de passeig aleatori reversible $[X, \mathcal{B}, m, \nu]$. La difusió no local pot ocórrer en W, en la frontera no local $\partial_m W$, o en tots dos conjunts alhora. Suposarem que W_m és m-connex (vegeu les Definicions 0.51 i 0.32) i que $\nu(W_m) < \infty$. Les formulacions dels problemes de difusió que estudiem són les següents:

$$(0.7) \qquad \begin{cases} v_t(t,x) - \operatorname{div}_m \mathbf{a}_p u(t,x) = f(t,x), & x \in W,\ 0 < t < T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0 < t < T, \\ -\mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p} u(t,x) \in \beta(u(t,x)), & x \in \partial_m W,\ 0 < t < T, \\ v(0,x) = v_0(x), & x \in W, \end{cases}$$

i, per a condicions de frontera dinàmiques i no lineals,

$$(0.8) \qquad \begin{cases} v_t(t,x) - \operatorname{div}_m \mathbf{a}_p u(t,x) = f(t,x), & x \in W,\ 0 < t < T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0 < t < T, \\ w_t(t,x) + \mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p} u(t,x) = g(t,x), & x \in \partial_m W,\ 0 < t < T, \\ w(t,x) \in \beta(u(t,x)), & x \in \partial_m W,\ 0 < t < T, \\ v(0,x) = v_0(x), & x \in W, \\ w(0,x) = w_0(x), & x \in \partial_m W, \end{cases}$$

on γ i β són grafs maximals monòtons a $\mathbb{R} \times \mathbb{R}$ amb $0 \in \gamma(0) \cap \beta(0)$, $\operatorname{div}_m \mathbf{a}_p$ és un operador de tipus Leray-Lions no local (el model del qual és un operador de difusió no local de tipus p-Laplacià) i $\mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p}$ és un operador de frontera no local de tipus Neumann (vegeu la subsecció 5.1.1 per obtenir més informació). De fet, resoldrem aquests problemes amb major generalitat, ja que no només els considerarem per a un conjunt W i la seua frontera no local $\partial_m W$, sinó per a qualssevol dos conjunts disjunts Ω_1 i $\Omega_2 \in \mathcal{B}$ tal que la seva unió sigui m-connexa i de mesura finita.

Aquests problemes es poden veure com els homòlegs no locals dels problemes de difusió locals governats per l'operador de difusió p-Laplacià (o un operador de Leray-Lions) on dues altres no-linealitats són induïdes per γ i β (vegeu, per exemple, [**13**] i [**33**] per als problemes locals). A [**18**], i les seves referències, es pot trobar una interpretació del procés de difusió no local implicat en aquest tipus de problemes.

Sobre les no linealitats (provocades per) γ i β no imposem cap altra hipòtesi a part de la natural (vegeu el treball de Ph. Bénilan, M. G. Crandall i P. Sacks [**33**]):

$$0 \in \gamma(0) \cap \beta(0)$$

i (per tal que es produeixi la difusió)

$$\nu(W)\Gamma^- + \nu(\partial_m W)\mathfrak{B}^- < \nu(W)\Gamma^+ + \nu(\partial_m W)\mathfrak{B}^+,$$

on

$$\Gamma^- := \inf\{\mathrm{Ran}(\gamma)\},\ \Gamma^+ := \sup\{\mathrm{Ran}(\gamma)\},\ \mathfrak{B}^- := \inf\{\mathrm{Ran}(\beta)\} \text{ i } \mathfrak{B}^+ := \sup\{\mathrm{Ran}(\beta)\}.$$

Per tant, treballem amb una classe bastant general de problemes de difusió no lineals i no locals amb condicions de frontera no lineals que, en particular, inclouen la condició de frontera Dirichlet homogènia o la condició de frontera Neumann.

Amb el nostre enfocament general podem cobrir directament: problemes d'obstacles, amb obstacles unilaterals o bilaterals (ja sigui en W, en $\partial_m W$ o en ambdós alhora); el homòleg no local de problemes de Stefan, que impliquen grafs monòtons com el graf invers de

$$\theta_S(r) := \begin{cases} r & \text{if } r < 0, \\ [0,\lambda] & \text{si } r = 0, \\ \lambda + r & \text{si } r > 0, \end{cases}$$

per $\lambda > 0$; problemes de difusió en medis porosos, on grafs monòtons com $p_s(r) = |r|^{s-1}r$, $s > 0$ hi participen; i problemes de tipus Hele-Shaw, que impliquen grafs com

$$H(r) := \begin{cases} 0 & \text{si } r < 0, \\ [0,1] & \text{si } r = 0, \\ 1 & \text{si } r > 0. \end{cases}$$

A més, si $\gamma(r) = 0$ en el primer problema, la dinàmica només té lloc a la frontera no local i obtenim el problema d'evolució d'un operador de Dirichlet a Neumann no local com a cas particular.

La motivació per a l'estudi d'aquests problemes de difusió no local de tipus p-Laplacià que fan ús d'operadors de frontera Neumann no locals la proporcionen els operadors de frontera Neumann no locals estudiats (per al cas lineal) a [**72**] i [**96**]. Tanmateix, a causa de la generalitat de les hipòtesis considerades en el nostre estudi, els resultats que obtenim condueixen a resultats nous d'existència i unicitat per a una àmplia gamma de problemes. Això és cert fins i tot quan es consideren els problemes en grafs discrets ponderats o a $\mathbb{R}^N$ amb un passeig aleatori induït per un nucli no singular, espais per als quals només s'han estudiat alguns casos particulars d'aquests problemes (després donem algunes referències). En aquests espais, i per a l'operador p-Laplacià no local, el problema (0.7) té les formulacions següents (vegeu l'Exemple 0.38 i la Definició 0.51, per a les definicions necessàries i notacions):

$$\begin{cases} v_t(t,x) = \dfrac{1}{d_x} \displaystyle\sum_{y \in V(G)} w_{x,y}|u(y)-u(x)|^{p-2}(u(y)-u(x)), & x \in W,\ 0<t<T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0<t<T, \\ \dfrac{1}{d_x} \displaystyle\sum_{y \in W_{m^G}} w_{x,y}|u(y)-u(x)|^{p-2}(u(y)-u(x)) \in \beta(u(t,x)), & x \in \partial_{m^G} W,\ 0<t<T, \\ u(x,0) = u_0(x), & x \in W, \end{cases}$$

per a grafs discrets ponderats i

$$\begin{cases} v_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(y-x)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x \in W,\ 0<t<T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0<t<T, \\ \displaystyle\int_{W_{m^J}} J(y-x)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy \in \beta(u(t,x)), & x \in \partial_{m^J} W,\ 0<t<T, \\ v(x,0) = v_0(x), & x \in W. \end{cases}$$

per al cas de $\mathbb{R}^N$ amb un passeig aleatori induït pel nucli no singular J. Hem detallat aquests problemes amb formulacions conegudes per mostrar fins a quin punt els problemes (0.7) i (0.8) cobreixen problemes específics no locals de gran interès.

La teoria de semigrups no lineals serà la base per a l'estudi de l'existència i unicitat de les solucions dels problemes anteriors. Aquest estudi es desenvoluparà a la secció 5.3, on demostrem, com a cas particular del Teorema 5.22, l'existència de solucions "mild" al problema (0.8) per a dades generals en L^1, i de solucions fortes si requerim condicions d'integrabilitat addicionals a les dades. A més, s'obté un principi de contracció i comparació. El mateix es fa per al problema (0.7) al Teorema 5.28. Vegeu [**23**], [**24**], [**28**], [**32**], [**43**], [**67**], [**68**] i [**69**] per obtenir més informació sobre aquesta teoria; es pot trobar un resum de la mateixa a l'apèndix d'aquesta tesi.

Per aplicar la teoria de semigrups no lineals, el nostre primer objectiu és demostrar l'existència i unicitat de les solucions del problema

$$(0.9) \qquad \begin{cases} \gamma(u(x)) - \mathrm{div}_m \mathbf{a}_p u(x) \ni \varphi(x), & x \in W, \\ \mathcal{N}_1^{\mathbf{a}_p} u(x) + \beta(u(x)) \ni \phi(x), & x \in \partial_m W, \end{cases}$$

per a grafs monòtons maximals γ i β. Això és l'homòleg no local de problemes (locals) el·líptics quasilineals amb condicions de frontera no lineals (vegeu [**14**] i [**33**] per a l'estudi general del cas local) i és un problema interessant en si mateix a conseqüència de la generalitat amb la qual treballem. Amb aquest objectiu, farem ús de les desigualtats de tipus Poincaré generalitzada introduïdes a la secció 0.6, però fins i tot amb aquestes desigualtats no podem obtenir arguments de compacitat com els que s'utilitzen a la teoria local o en problemes de difusió fraccionària. En conseqüència, hem d'aprofitar al màxim els arguments de fitació i monotonicitat per demostrar els nostres resultats, sent les idees principals una implementació de les que s'utilitzen a [**14**] i [**33**] (vegeu també [**16**] per a un cas molt particular). El mateix passa amb els problemes de difusió. L'estudi del problema (0.9) es desenvoluparà a la secció 5.1, on demostrem, per a un problema més general, l'existència de solucions (Teorema 5.15) i un principi de contracció i comparació (Teorema 5.14). Al final d'aquesta secció treballem amb un altre operador de frontera Neumann no local.

Per a problemes el·líptics lineals o quasilineals amb condicions de frontera, els obstacles compliquen l'existència de solucions. L'aparició d'aquesta dificultat s'entén millor quan es té en compte la continuïtat de la solució entre l'interior del domini i la frontera a través de la traça. De fet, per a un domini suau i fitat Ω a $\mathbb{R}^N$, γ amb domini fitat $[0,1]$ i $\beta(r) = 0$ per a tot r, no és possible trobar una solució feble de

$$\begin{cases} -\Delta u + \gamma(u) \ni \varphi & \text{a } \Omega, \\ \nabla u \cdot \eta = \phi & \text{a } \partial\Omega, \end{cases}$$

per a dades que satisfan $\varphi \leqslant 0$, $\phi \leqslant 0$ i $\phi \not\equiv 0$ (vegeu [**14**]). No obstant això, en el nostre context no local aquest tipus de continuïtat no se satisfà i l'estudi d'aquests problemes de difusió no local amb obstacles, per tant, difereix de l'estudi dels problemes de difusió local (vegeu [15] per a un estudi detallat d'aquests problemes locals). En particular, no necessitem imposar cap hipòtesi sobre les no linealitats γ i β a part de les naturals.

Hi ha una llarga llista de referències per als homòlegs el·líptics i parabòlics locals dels problemes que estudiem; vegeu, per exemple, [13], [14], [28], [29], [30], [33], [157], i les seves referències. Vegeu també [103], i les seves referències, per a un problema de Hele-Shaw amb condicions de frontera dinàmiques. Per a alguns problemes particulars no locals ens referim a [16], [17], [18], [34], [50], [60], [97], [106] i [125]. Per a problemes de difusió fraccionària vegeu, per exemple, [119], on es consideren les condicions de frontera Dirichlet i Neumann; a [39], [40], [65], [90] i [137], on s'estudien equacions fraccionàries en medis porosos, vegeu també l'estudi [158] de J. L. Vázquez i les seves referències; i a [153] i [154] per a problemes de difusió fraccionària per al problema de Stefan.

Introduction

The digital world has brought with it many different kinds of data of increasing size and complexity. Indeed, modern devices allow us to easily obtain images of higher resolution, as well as to collect data on internet searches, healthcare analytics, social networks, geographic information systems, business informatics, etc. Consequently, the study and treatment of these big data sets is of great interest and value. In this respect, weighted discrete graphs provide a natural and flexible workspace in which to represent the data. In this context, a vertex represents a data point and each edge is weighted according to an appropriately chosen measure of "similarity" between the corresponding vertices. Historically, the main tools for the study of graphs came from combinatorial graph theory. However, following the implementation of the graph Laplacian in the development of spectral clustering in the seventies, the theory of partial differential equations on graphs has obtained important results in this field (see, for example, [**63**], [**115**] and the references therein). This has prompted a big surge in the research of partial differential equations on graphs. Moreover, interest has been further bolstered by the study of problems in image processing. In this area of research, pixels are taken as the vertices and the "similarity" between pixels as the weights. The way in which these weights are defined depends on the problem at hand (see, for instance, [**79**] and [**114**]).

On another note, let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radially symmetric and continuous function with $\int_{\mathbb{R}^N} J(z)dz = 1$. Nonlocal evolution equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)dy - u(x,t), \tag{0.1}$$

and variations of it, have naturally arisen in various scientific fields as a means of modelling a wide range of diffusion processes. For example, in biology ([**53**], [**131**]), particle systems ([**38**]), coagulation models ([**84**]), nonlocal anisotropic models for phase transition ([**1**], [**2**]), mathematical finances using optimal control theory ([**36**], [**104**]), image processing ([**91**], [**109**]), etc. An intuitive reasoning for the wide applicability of this model comes from thinking of $u(x,t)$ as the density of a "population" at a point x at time t and of $J(y-x)$ as the probability distribution of moving from y to x in one "jump". Then, $\int_{\mathbb{R}^N} J(y-x)u(y,t)dy$ is the rate at which "individuals" are arriving at x from anywhere else and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x. Therefore, in the absence of external or internal sources, we are lead to equation (0.1) as a model for the evolution of the population density over time. An extensive study of this problem can be found in [**18**].

In the previous two paragraphs, we have brought forward two instances in which there is great interest in the study of partial differential equations in a nonlocal (or discrete) setting. Further interest has arisen following the analysis of the peridynamic formulation of the continuous mechanic (see [**111**] and [**145**]), the study of Markov jump processes and other nonlocal models. References on all of the topics mentioned thus far are given all along the thesis (see also [**48**], [**64**], [**77**], [**79**], [**87**], [**88**], [**92**], [**109**], [**114**], [**143**], [**155**], [**156**], [**161**]).

The aim of this thesis is to unify into a broad framework the study of many of the previously mentioned problems. In order to do so, we note that there is a strong relation between some of these problems and probability theory, and it is in this field in which we find the appropriate spaces in which to develop this unifying study. Let $(X, \mathcal{B})$ be a measurable space and $P : X \times \mathcal{B} \to [0,1]$ a transition probability kernel on X (see Section 1.1). Then, a Markovian transition function can be defined as follows: for any $x \in X$, $B \in \mathcal{B}$, let

$$P_t(x, B) := e^{-t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} P^n(x, B), \quad t \in \mathbb{R}_+,$$

where P^n denotes the n-step transition probability kernel. The associated family of operators, $P_t f(x) := \int f(y) P_t(x, dy)$, satisfy

$$\frac{\partial}{\partial t} P_t f(x) = \int P_t f(y) P(x, dy) - P_t f(x).$$

Moreover, if we consider a Markov process $(X_t)_{t \geqslant 0}$ associated with the Markovian transition function $(P_t)_{t \geqslant 0}$, and if we denote by μ_t the distribution of X_t, then the family $(\mu_t)_{t \geqslant 0}$ also satisfies a linear equation of the form

$$\frac{\partial}{\partial t} \mu_t = \int P(y, \cdot) \mu_t(dy) - \mu_t.$$

In this setting, particular choices of the measurable space $(X, \mathcal{B})$ and of P will lead to some of the previous problems. For example, if $X = \mathbb{R}^N$ and $P(x, dy) = J(y - x)dy$, we will recover equation (0.1). Moreover, taking X to be the set of vertices of a weighted discrete graph and appropriately defining the transition probability function in terms of the weights (see Example 1.38) we are also able to recover the heat equation on graphs.

The previous remarks suggest that the appropriate setting in which to fulfill our goals of unifying a wide variety of nonlocal models into broad framework is provided by random walk spaces. These spaces are constituted by a measurable space $(X, \mathcal{B})$ and a transition probability kernel P on X which encodes the jumps of a Markov process. We will adopt the notation $m_x := P(x, \cdot) \in \mathcal{P}(X, \mathcal{B})$ for each $x \in X$ (here $\mathcal{P}(X, \mathcal{B})$ denotes the space of probability measures on $(X, \mathcal{B})$). Additionally, we will require a kind of stability property to hold for these spaces, that is, the existence of an invariant measure ν (see Definition 1.7). Then, we will say that $[X, \mathcal{B}, m, \nu]$ is a random walk space. Owing to the generality of these spaces, the results that we obtain will have a wide range of applicability to a large spectrum of evolution problems arising in a variety of scientific fields. Unfortunately, this framework does not cover problems related with the fractional kernel due to its singular nature.

During the last years and with the aforementioned goal in mind, we have studied some gradient flows in the general framework of a random walk space. In particular, we have studied the heat flow, the total variation flow, and evolutions problems of Leray-Lions type with different types of nonhomogeneous boundary conditions. Specifically, together with the existence and uniqueness of solutions to these problems and the asymptotic behavior of its solutions, a wide variety of their properties have been studied, as well as the nonlocal diffusion operators involved in them. Our results have been published in [**123**], [**124**], [**125**], [**126**] and [**146**].

Structure of the work

Let us shortly describe the contents of the thesis. To start with, in Chapter 1, we introduce the general framework of a random walk space. Then, in Section 1.1, we relate it to classical notions in Markov chain theory and provide a list of results which we hope aid the reader in getting a good idea about the properties which these spaces enjoy. After introducing a stability property for random walk spaces, called m-connectedness, we devote Section 1.2 to exploring the characteristics enjoyed by this notion and we relate it to known concepts of ergodicity. We then provide a list of examples of random walk spaces of particular interest, as those that were mentioned at the beginning of the introduction. The rest of the chapter is dedicated to introducing the nonlocal counterparts of classical notions like those of gradient, divergence, boundary, perimeter, mean curvature and Ricci curvature, as well as of the Laplace operator. In doing so, we obtain results which mimic classic results in the local case and, moreover, obtain further characterizations of the m-connectedness of a random walk space. We also spend some time in finding sufficient conditions for Poincaré type inequalities to hold and relate them both to the gap of the Laplace operator and to isoperimetric inequalities. Finally, Section 1.7 is devoted to the Ollivier-Ricci curvature and its relation with the Poincaré type inequality.

Chapter 2 focuses on the study of the heat flow in random walk spaces. In our context, associated with the random walk $m = (m_x)_{x\in X}$, the *Laplace operator* Δ_m is defined as

$$\Delta_m f(x) := \int_X (f(y) - f(x))dm_x(y).$$

Assuming that the invariant measure ν satisfies a reversibility condition with respect to the random walk (see Definition 1.15), the operator $-\Delta_m$ generates in $L^2(X,\nu)$ a Markovian semigroup $(e^{t\Delta_m})_{t\geqslant 0}$ (see Theorem 2.4) called the *heat flow* in the random walk space. Moreover, we are able to characterise the infinite speed of propagation of the heat flow in terms of the m-connectedness of the random walk space (see Theorem 2.9). In addition, in Section 2.2, we study the asymptotic behaviour of the semigroup $(e^{t\Delta_m})_{t\geqslant 0}$ and with the help of a Poincaré inequality we obtain rates of convergence of $(e^{t\Delta_m})_{t\geqslant 0}$ as $t \to \infty$. In Section 2.3 we introduce the Bakry-Émery curvature-dimension condition and study its relation to the Poincaré inequality. Lastly, Section 2.4 is devoted to the study of transport inequalities in relation with the Bakry-Émery curvature-dimension condition and the Ollivier-Ricci curvature.

In Chapter 3 we study the total variation flow. For this purpose, we introduce the 1-Laplacian operator associated with a random walk space and obtain various characterizations of it (see Theorem 3.13). We then proceed to prove existence and uniqueness of solutions of the total variation flow in random walk spaces and to study its asymptotic behaviour with the help of some Poincaré type inequalities. As a result of our study, we generalize results obtained in [**120**] and [**121**] for the particular case of $\mathbb{R}^N$ with a nonsingular kernel as well as some results in graph theory.

Moreover, in Chapter 3 we introduce the concepts of Cheeger and calibrable sets in random walk spaces and characterise the calibrability of a set by using the 1-Laplacian operator. Furthermore, we study the eigenvalue problem of the 1-Laplacian and relate it to the optimal Cheeger cut problem. These results apply, in particular, to locally finite weighted connected discrete graphs, complementing the results given in [**57**], [**58**], [**59**] and [**99**].

Chapter 4 is dedicated to the study of the (BV, L^p)-decomposition, $p = 1$ and $p = 2$, of functions in random walk spaces. This is done by studying the Rudin-Osher-Fatemi model both with L^2 and with L^1 fidelity terms. We obtain the Euler-Lagrange equations of these minimization problems and proceed to obtain a wide range of results on the properties enjoyed by the minimizers.

Finally, in Chapter 5, we study p-Laplacian type evolution problems like the one given in the following reference model:

$$u_t(t,x) = \int_{\Omega\cup\partial_m\Omega} |u(y) - u(x)|^{p-2}(u(y) - u(x))dm_x(y), \quad x \in \Omega,\ 0 < t < T,$$

with nonhomogeneous Neumann boundary conditions, where $\Omega \in \mathcal{B}$ and $\partial_m\Omega$ is the m-boundary of Ω (see Definition 1.51). This reference model can be regarded as the nonlocal counterpart to the classical evolution problem

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u), & x \in U,\ 0 < t < T, \\ -|\nabla u|^{p-2}\nabla u \cdot \eta = \varphi, & x \in \partial U,\ 0 < t < T, \end{cases}$$

where U is a bounded smooth domain in $\mathbb{R}^n$, and η is the outer normal vector to ∂U. In fact, our study develops with a far greater generality which allows us to cover a wide variety of problems such as: obstacle problems, the nonlocal counterpart of Stefan like problems, diffusion problems in porous media, Hele-Shaw type problems, and the evolution problem for a nonlocal Dirichlet-to-Neumann operator.

Passeigs aleatoris

Aquest capítol és una traducció al valencià del capítol 1.

El personatge principal dels espais marc sobre els quals es desenvolupa la tesi és el passeig aleatori.

Definition 0.1. Sigui $(X, \mathcal{B})$ un espai mesurable on la σ-àlgebra $\mathcal{B}$ està comptablement generada. Un passeig aleatori a $(X, \mathcal{B})$ és una família de mesures de probabilitat $(m_x)_{x \in X}$ en $\mathcal{B}$ tal que $x \mapsto m_x(B)$ és una funció mesurable de X per cada $B \in \mathcal{B}$ fixat.

La notació i la terminologia escollides en aquesta definició provenen de [**134**]. Com s'assenyala en eixe article de Y. Ollivier, els geometres poden pensar en m_x com un substitut de la noció de una bola centrada a x, mentre que, en termes probabilístics, podem pensar en aquestes mesures de probabilitat com les generadores d'una cadena de Markov. En efecte, en aquest darrer cas, la probabilitat de transició de x a y en n passos és

$$dm_x^{*n}(y) := \int_{z \in X} dm_z(y) dm_x^{*(n-1)}(z), \quad n \geqslant 1$$

i $m_x^{*0} = \delta_x$, la mesura de dirac a x. A la següent secció aprofundirem en aquesta darrera perspectiva que serà la principal al llarg del nostre treball, la primera perspectiva tindrà un paper important a la secció 0.7. Per tant, prenem un descans momentani de la construcció del que serà el nostre espai marc per tal de recordar alguns resultats de la teoria clàssica de les cadenes de Markov que creiem que ajudaran a proporcionar motivació.

0.1. Cadenes de Markov

En aquesta secció ens submergim en la terminologia probabilística que allotja el nostre treball. Això serà especialment útil per a lectors amb antecedents probabilístics, ja que servirà per aclarir on cau exactament el nostre treball en aquest camp. A més, recordem resultats ben coneguts per proporcionar més informació sobre la naturalesa dels passeigs aleatoris. Amb aquest objectiu, comencem donant la definició d'una cadena de Markov en temps discret i homogènia en el temps, i la propietat de Markov que satisfà. Els resultats d'aquesta secció es poden trobar a [**76**], [**100**] i [**128**].

Definition 0.2. Sigui $(\Omega, \mathcal{F}, \mathcal{P})$ un espai de probabilitat i $(X, \mathcal{B})$ un espai mesurable on la σ-àlgebra $\mathcal{B}$ està comptablement generada. Una cadena de Markov en temps discret i homogènia en el temps és una successió de variables aleatòries definides en Ω i amb valors en X, $\{X_n : n = 0, 1, 2, \ldots\}$, tal que

$$\mathcal{P}(X_{n+1} \in B \mid X_0, X_1, \ldots, X_n) = \mathcal{P}(X_{n+1} \in B \mid X_n) \quad \forall B \in \mathcal{B},\ n = 0, 1, \ldots \tag{0.1}$$

La identitat (0.1) s'anomena propietat de Markov.

La propietat de Markov indica que el futur del procés és independent del passat donat el seu valor actual, de manera que, intuïtivament, podem dir que no té memòria. Es pot trobar una gran varietat d'exemples de cadenes de Markov a [**76**], [**100**] o citeMeyn.

En aquesta tesi no treballarem mai directament amb les variables aleatòries, sinó que utilitzarem un enfocament diferent, però equivalent, de les cadenes de Markov. Per cada $x \in X$ i $B \in \mathcal{B}$, sigui

$$P(x, B) := \mathcal{P}(X_{n+1} \in B | X_n = x). \tag{0.2}$$

Això defineix un nucli estocàstic en X, el que significa que

- $P(x, \cdot)$ és una mesura de probabilitat en $\mathcal{B}$ per a qualsevol $x \in X$, i
- $P(\cdot, B)$ és una funció mesurable en X per a qualsevol $B \in \mathcal{B}$.

El nucli estocàstic P també es coneix com a nucli de probabilitat de transició. La "homogeneïtat en el temps" de la cadena Markov es refereix al fet que P, tal com es defineix a (0.2), és independent de n. Aquest nucli estocàstic és el que anteriorment hem anomenat passeig aleatori, de manera que, en la nostra terminologia, $m_x(B) = P(x, B)$. De la mateixa manera, denotem per m_x^{*n} el nucli de probabilitat de transició en n passos $P^n(x, B) := \mathcal{P}^n(X_{n+1} \in B \,|\, X_0 = x)$, $x \in X$, $B \in \mathcal{B}$ que, com abans, es pot definir recursivament mitjançant la següent equació

$$P^n(x, B) = \int_X P^{n-1}(y, B) P(x, dy) = \int_X P(y, B) P^{n-1}(x, dy)$$

per a $B \in \mathcal{B}$, $x \in X$ i $n = 1, 2, \ldots$, amb $P^0(x, \cdot) = \delta_x(\cdot)$. Tingueu en compte que, com es pot demostrar fàcilment per inducció,

$$dm_x^{*(n+k)}(y) = \int_{z \in X} dm_x^{*k}(z) dm_z^{*n}(y)$$

per a tot n, $k \in \mathbb{N}$.

Aleshores es pot demostrar que, de fet, començant amb un passeig aleatori (o nucli estocàstic) a X, podem construir una cadena de Markov (X_n) de manera que els nuclis de probabilitat de transició coincideixin amb el passeig aleatori donat (vegeu [**128**, Teorema 3.4.1]). En particular, està demostrat que per a una distribució de probabilitat inicial donada μ a $\mathcal{B}$ es pot construir la mesura de probabilitat P_μ a $\mathcal{F}$ de manera que $P_\mu(X_0 \in B) = \mu(B)$ per a $B \in \mathcal{B}$ i, a més, per a tot $n = 0, 1, \ldots$, $x \in X$ i $B \in \mathcal{B}$,

$$P_\mu(X_{n+1} \in B \,|\, X_n = x) = m_x(B).$$

Quan μ és la mesura de Dirac a $x \in X$, denotem P_μ per P_x. De la mateixa manera, els operadors esperança corresponents es denoten E_μ i E_x, respectivament.

Ara definirem alguns conceptes generals d'estabilitat per a les cadenes de Markov que s'utilitzaran durant tot aquest treball. Aquestes nocions d'alguna manera oferiran una visió del comportament a llarg termini del procés a mesura que evoluciona amb el temps. Amb aquest objectiu introduïm els conceptes següents.

Definition 0.3.
(i) Sigui $A \in \mathcal{B}$, el temps d'ocupació η_A és el nombre de visites de la cadena de Markov a A després del temps zero [3]:

$$\eta_A := \sum_{n=1}^{\infty} \chi_{\{X_n \in A\}}.$$

(ii) Per a qualsevol conjunt $A \in \mathcal{B}$, la variable aleatòria

$$\tau_A := \min\{n \geqslant 1 \,:\, X_n \in A\},$$

s'anomena el temps de primera tornada a A. Assumim que $\inf \varnothing = \infty$.

El primer i menys restrictiu concepte d'estabilitat és el de φ-irreductibilitat, on φ és una mesura en $\mathcal{B}$. Amb aquest concepte exigim que, independentment del punt de partida, siguem capaços d'assolir qualsevol conjunt "important" en un nombre finit de salts. Els conjunts "importants" s'entendran com aquells que tenen una mesura positiva respecte a la

[3]Denotem la característica d'un subconjunt A d'un conjunt X per χ_A, i.e., $\chi_A : X \to \{0, 1\}$ està definida per

$$\chi_A(x) := \begin{cases} 1 & \text{si } x \in A, \\ 0 & \text{si } x \notin A. \end{cases}$$

mesura φ. També podem entendre que estem demanant que la cadena Markov no estigui realment constituïda per dues cadenes. Definim

$$\begin{aligned} U(x, A) &:= \sum_{n=1}^{\infty} m_x^{*n}(A) \\ &= E_x[\eta_A]. \end{aligned}$$

DEFINITION 0.4. Sigui φ una mesura en $\mathcal{B}$. Un passeig aleatori m és φ-irreductible si, per a tot $x \in X$,

$$\varphi(A) > 0 \Rightarrow U(x, A) > 0.$$

Alternativament (vegeu [**128**, Proposició 4.2.1]), també podem entendre aquest concepte utilitzant τ_A. Llavors, m és φ-irreductible si, per a qualsevol $x \in X$,

$$\varphi(A) > 0 \Rightarrow P_x(\tau_A < \infty) > 0,$$

és a dir, a partir de qualsevol punt $x \in X$ tenim una probabilitat positiva d'arribar a qualsevol conjunt de mesures positives en temps finit.

En lloc de prendre alguna mesura possiblement arbitrària per definir la irreductibilitat del passeig aleatori, podem prendre la *mesura d'irreducibilitat maximal* que defineix l'abast de la cadena de manera més completa. Això es fa a través de la següent proposició. Sigui

$$K_{a_{\frac{1}{2}}}(x, A) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} m_x^{*n}(x, A), \quad x \in X, A \in \mathcal{B}(X);$$

aquest nucli defineix per a qualsevol $x \in X$ una mesura de probabilitat equivalent[4] a $m_x^{*0}(\cdot) + U(x, \cdot)$ (que pot ser infinit per a molts conjunts).

PROPOSITION 0.5. *(*[**128**, Proposition 4.2.2]*) Donat un passeig aleatori φ-irreductible m en $(X, \mathcal{B})$, existeix una mesura de probabilitat ψ a $\mathcal{B}$ tal que*
(i) m és ψ-irreductible;
(ii) per a qualsevol altra mesura φ', el passeig aleatori m és φ'-irreductible si, i només si, $\varphi' \ll \psi$;
(iii) si $\psi(A) = 0$, llavors $\psi(\{y : P_y(\tau_A < \infty) > 0\}) = 0$;
(iv) la mesura de probabilitat ψ és equivalent a

$$\psi'(A) := \int_X K_{a_{\frac{1}{2}}}(y, A) d\varphi'(y),$$

per a qualsevol mesura de irreductibilitat finita φ'.

Una mesura que compleix les condicions de la proposició anterior s'anomena mesura d'irreductibilitat maximal. Per comoditat de notació, direm que m és ψ-irreductible si és φ-irreductible per a alguna mesura φ i la mesura ψ és una mesura d'irreductibilitat maximal.

Es pot obtenir una noció d'estabilitat més forta demanant, no només que $U(x, A) > 0$, sinó que $U(x, A) = \infty$ per a qualsevol $x \in X$ i cada conjunt mesurable A amb $\varphi(A) > 0$. Com a alternativa, podem reforçar el requisit que hi hagi una probabilitat positiva d'arribar a tots els conjunts de mesura φ-positiva independentment d'on comencem i, en canvi, exigir que, de fet, això acabi succeint. Aquests enfocaments condueixen als diversos conceptes de recurrència.

DEFINITION 0.6.
(i) Sigui $A \in \mathcal{B}$. El conjunt A s'anomena recurrent si $U(x, A) = E_x[\eta_A] = \infty$ per a tot $x \in A$.
(ii) La cadena de Markov és recurrent si és ψ-irreductible i $U(x, A) = E_x[\eta_A] = \infty$ per a tot $x \in X$ i tot $A \in \mathcal{B}(X)$ amb $\psi(A) > 0$.

[4]Dues mesures μ i ν són equivalents si $\mu \ll \nu$ i $\nu \ll \mu$, és a dir, si coincideixen en quins conjunts tenen mesura zero.

(iii) Sigui $A \in \mathcal{B}$. El conjunt A s'anomena Harris recurrent si

$$P_x[\eta_A = \infty] = 1$$

per a tot $x \in A$.
(iv) La cadena de Markov és Harris recurrent si és ψ-irreductible i tot $A \in \mathcal{B}(X)$ amb $\psi(A) > 0$ és Harris recurrent.

Es dedueix que qualsevol conjunt Harris recurrent és recurrent. De fet, per a la recurrència necessitem que el nombre esperat de visites sigui infinit, mentre que la Harris recurrència implica que el nombre de visites és infinit gairebé segur. En particular, per [**128**, Teorema 9.0.1] tenim que una cadena recurrent difereix per un conjunt de ψ-null d'una cadena Harris recurrent.

A més, es demostra que hi ha una dicotomia en el sentit que les cadenes irreductibles de Markov no poden ser "parcialment estables", o bé posseeixen aquestes propietats d'estabilitat de manera uniforme en x o la cadena és inestable de manera ben definida (però no entrarem en això, vegeu [**128**] per obtenir més informació).

Una altra propietat d'estabilitat que utilitzarem ve donada per l'existència d'una mesura invariant. Es tracta d'una mesura que proporciona una distribució tal que, si la cadena comença distribuïda d'aquesta manera, queda distribuïda així en tot moment. A més, aquestes mesures resulten ser les que defineixen el comportament a llarg termini de la cadena.

Definition 0.7. Un mesura σ-finita ν en $\mathcal{B}$ és invariant amb respecte a el passeig aleatori m si

$$\nu(A) := \int_X m_x(A) d\nu(x) \quad \text{per a tot } A \in \mathcal{B}.$$

Per descomptat, si una mesura invariant és finita, es pot normalitzar a una mesura de probabilitat (estacionària). Per tant, al llarg d'aquesta tesi, sempre que requerim la finitud de la mesura invariant, considerarem directament una mesura de probabilitat invariant.

Remark 0.8. Tingueu en compte que, si m és un passeig aleatori a $(X, \mathcal{B})$, llavors m^{*n} també és un passeig aleatori a $(X, \mathcal{B})$ per a qualsevol $n \in \mathbb{N}$. A més, si ν és invariant respecte a m, ν és invariant respecte a m^{*n} per a qualsevol $n \in \mathbb{N}$.

Unint la irreductibilitat i l'existència d'una mesura invariant obtenim els següents resultats.

Proposition 0.9. *(*[**128**, Proposition 10.0.1]*) Si el passeig aleatori m és recurrent, admet una mesura invariant única (sense tenir en compte la multiplicació per constants).*

Proposition 0.10. *(*[**128**, Proposition 10.1.1]*) Si el passeig aleatori m és ψ-irreductible i admet una mesura de probabilitat invariable, llavors és recurrent; així, en particular, la mesura de probabilitat invariant és única.*

A més, donem el següent teorema relacionant les mesures d'irreducibilitat invariants i les maximals per a cadenes recurrents.

Theorem 0.11. *(*[**128**, Theorem 10.4.9]*) Si el passeig aleatori m és recurrent (i, per tant, ψ-irreductible), la única (sense tenir en compte la multiplicació per constants) mesura invariant ν respecte a m és equivalent a ψ (per tant, ν és una mesura maximal d'irreductibilitat).*

Un altre concepte ben conegut és el de la mesura ergòdica.

Definition 0.12. Es diu que un conjunt $B \in \mathcal{B}$ és invariant (o absorbent o estocàsticament tancat) (respecte a m) si $m_x(B) = 1$ per a qualsevol $x \in B$.

Es diu que una mesura de probabilitat invariant ν és ergòdica (respecte a m) si $\nu(B) = 0$ o $\nu(B) = 1$ per a qualsevol conjunt invariant $B \in \mathcal{B}$.

Es pot trobar un estudi profund de la teoria ergòdica aplicada a les cadenes de Markov a [**76**, Capítol 5]. Allà podem trobar la construcció d'un sistema dinàmic associat a una cadena de Markov de manera que la noció anterior d'ergodicitat sigui equivalent a la noció clàssica d'ergodicitat per a aquest sistema dinàmic ([**76**, Teorema 5.2.11]). El resultat següent garanteix que la unicitat de la mesura invariant implica la seva ergodicitat.

PROPOSITION 0.13. *(*[**100**, Proposition 2.4.3]*) Sigui m un passeig aleatori en X. Si m té una mesura de probabilitat invariant única ν, ν és ergòdica.*

Que juntament amb la Proposició 0.10 i el Teorema 0.11 implica el següent.

COROLLARY 0.14. *Si m és un passeig aleatori ψ-irreductible que admet una mesura de probabilitat invariant, la mesura de probabilitat invariant és única, ergòdica i equivalent a ψ.*

Finalment, l'última propietat que introduïm és l'existència d'una mesura reversible respecte a la cadena de Markov. Aquesta condició de reversibilitat en una mesura és més forta que la condició d'invariancia. Primer definim el producte tensorial d'una mesura σ-finita i d'un nucli estocàstic.

DEFINITION 0.15. Si ν és una mesura σ-finita a $(X, \mathcal{B})$ i N és un nucli estocàstic a $(X, \mathcal{B})$, definim el producte tensorial de ν i N, denotat per $\nu \otimes N$, que és una mesura de $(X \times X, \mathcal{B} \times \mathcal{B})$, per

$$\nu \otimes N(A \times B) = \int_A N(x, B) d\nu(x), \ (A, B) \in \mathcal{B} \times \mathcal{B}.$$

Utilitzant la nostra notació m per al passeig aleatori, designem el producte tensorial de ν i m per $\nu \otimes m_x$.

Una mesura σ-finita ν a $\mathcal{B}$ és reversible respecte al passeig aleatori m si la mesura $\nu \otimes m_x$ a $\mathcal{B} \times \mathcal{B}$ és simètrica, i.e., per a tot $(A, B) \in \mathcal{B} \times \mathcal{B}$,

$$\nu \otimes m_x(A \times B) = \nu \otimes m_x(B \times A).$$

Equivalentment, ν és reversible respecte a m si, per a totes les funcions mesurables i fitades f definides a $(X \times X, \mathcal{B} \times \mathcal{B})$,

$$\int_X \int_X f(x, y) dm_x(y) d\nu(x) = \int_X \int_X f(y, x) dm_x(y) d\nu(x).$$

Tingueu en compte que si ν és reversible respecte a m, llavors és invariant respecte a m (vegeu [**76**, Proposition 1.5.2]).

Associat a una cadena de Markov podem definir el següent operador que tindrà un paper molt important en molts dels nostres desenvolupaments.

DEFINITION 0.16. Si ν és una mesura invariant respecte a m, definim l'operador lineal M_m a $L^1(X, \nu)$ de la manera següent

$$M_m f(x) := \int_X f(y) dm_x(y), \quad f \in L^1(X, \nu).$$

M_m s'anomena *operador mitjana* a $[X, \mathcal{B}, m]$ (vegeu, per exemple, [**134**]).

Tingueu en compte que, si $f \in L^1(X, \nu)$ llavors, usant la invariància de ν respecte a m,

$$\int_X \int_X |f(y)| dm_x(y) d\nu(x) = \int_X |f(x)| d\nu(x) < \infty,$$

així $f \in L^1(X, m_x)$ per a ν-quasi tot $x \in X$, per tant, M_m està ben definit de $L^1(X, \nu)$ en si mateix.

REMARK 0.17. Sigui ν una mesura invariant respecte a m. Se segueix que

$$\|M_m f\|_{L^1(X,\nu)} \leqslant \|f\|_{L^1(X,\nu)} \quad \forall f \in L^1(X, \nu),$$

de manera que M_m és una contracció a $L^1(X, \nu)$. De fet, com que $M_m f \geqslant 0$ si $f \geqslant 0$, tenim que M_m és una contracció positiva a $L^1(X, \nu)$.

A més, per la desigualtat de Jensen, tenim que, per a $f \in L^1(X,\nu) \cap L^2(X,\nu)$,

$$\begin{aligned}\|M_m f\|^2_{L^2(X,\nu)} &= \int_X \left(\int_X f(y) dm_x(y) \right)^2 d\nu(x) \\ &\leqslant \int_X \int_X f^2(y) dm_x(y) d\nu(x) \\ &= \int_X f^2(x) d\nu(x) = \|f\|^2_{L^2(X,\nu)}.\end{aligned}$$

Per tant, M_m és un operador lineal a $L^2(X,\nu)$ amb domini

$$D(M_m) = L^1(X,\nu) \cap L^2(X,\nu).$$

En conseqüència, si ν és una mesura de probabilitat, M_m és un operador lineal i fitat de $L^2(X,\nu)$ en si mateix que satisfà $\|M_m\|_{\mathcal{B}(L^2(X,\nu),L^2(X,\nu))} \leqslant 1$.

Tingueu en compte que, fent ús d'aquest operador, tenim que $B \in \mathcal{B}$ és invariant respecte a m (Definició 0.12) si, i només si, $M_m \chi_B \geqslant \chi_B$. Podem debilitar lleugerament aquesta noció de la següent manera.

Definition 0.18. Diem que $B \in \mathcal{B}$ és ν-invariant (respecte a m) si $M_m \chi_B = \chi_B$ ν-gairebé pertot.

De la mateixa manera, definim la noció d'una funció harmònica (o ν-invariant).

Definition 0.19. Es diu que una funció $f \in L^1(X,\nu)$ és harmònica (respecte a m) si $M_m f = f$ ν-gairebé pertot.

Per tant, podem recordar un resultat clàssic que caracteritza l'ergodicitat de ν (vegeu, per exemple, [**100**, Lemma 5.3.2]).

Proposition 0.20. *Sigui ν una mesura de probabilitat invariant. Aleshores ν és ergòdica si, i només si, cada funció harmònica és una constant ν-gairebé pertot.*

0.1.1. φ-Essential Irreducibility. Es pot prendre una direcció diferent per definir la irreductibilitat d'un passeig aleatori (vegeu [**133**] o [**142**, Definition 4.4]).

Definition 0.21. Sigui φ una mesura a $\mathcal{B}$. Un passeig aleatori m és φ-essencialment irreductible si, per a φ-quasi tot $x \in X$,

$$\varphi(A) > 0 \Rightarrow U(x,A) > 0.$$

Tot i que aquest canvi pot semblar petit, en realitat condueix a una classe més àmplia i més salvatge de models "irreductibles". No obstant això, hi ha un bon resultat de dicotomia:

Proposition 0.22. *(*[**133**, Proposition 2]*) Sigui ν una mesura invariant respecte al passeig aleatori m de manera que m és ν-essencialment irreductible, llavors només es pot produir un dels dos casos següents:*
(i) existeix $X_1 \in \mathcal{B}$ tal que $\nu(X \backslash X_1) = 0$, X_1 és invariant respecte a m i

$$\nu \ll U(x,.) \quad \textit{per a tot } x \in X_1$$

i.e., la restricció de la cadena de Markov a X_1 és ν-irreductible;
(ii) existeix $X_2 \in \mathcal{B}$ tal que $\nu(X \backslash X_2) = 0$, X_2 és invariant respecte a m i

$$\nu \perp U(x,.) \quad \textit{per a tot } x \in X_2.$$

Com que la majoria d'exemples d'aplicació dels nostres resultats entraran en la primera categoria d'aquest teorema, seran aplicables els resultats anteriors d'aquesta secció. No obstant això, alguns exemples extrems entraran en la segona categoria, un cas que la literatura clàssica no sol cobrir (vegeu [**142**, Capítol 4] per a una discussió d'alguns dels resultats amb l'ús d'aquesta forma debilitada). Per tant, ara procedirem a desenvolupar alguns dels resultats que hem donat per a cadenes de Markov φ-irreductibles però per a cadenes de Markov

φ-essencialment irreductibles (suposant, a més, que φ és una mesura invariant). En aquest punt, tornem al treball de la tesi i recuperem la nostra terminologia en què les cadenes de Markov es denominen passeigs aleatoris i la noció d'irreductibilitat essencial de φ s'anomenarà m-connectivitat.

0.2. Espais de passeig aleatori

Seguim desenvolupant els espais en què treballarem.

Definition 0.23. Sigui $(X, \mathcal{B})$ un espai mesurable on la σ-àlgebra $\mathcal{B}$ està comptablement generada. Sigui m un passeig aleatori a $(X, \mathcal{B})$ i ν una mesura invariant respecte a m. L'espai mesurable juntament amb m i ν s'anomena espai de passeig aleatori i es denota per $[X, \mathcal{B}, m, \nu]$.

Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Si (X, d) és un espai mètric polonès (espai topològic completament metritzable i separable), $\mathcal{B}$ és el seu Borel σ-àlgebra i ν és una mesura de Radon (és a dir, ν és interiorment regular [5] i localment finita [6]), llavors diem que $[X, \mathcal{B}, m, \nu]$ és un espai mètric de passeig aleatori i el denotem per $[X, d, m, \nu]$. A més, tal com es fa a [**134**], quan calgui, també assumirem que cada mesura m_x té primer moment finit, és a dir, per a alguns (per tant, qualsevol) $z \in X$ i per a qualsevol $x \in X$ es té $\int_X d(z,y) dm_x(y) < +\infty$.

Definition 0.24. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Diem que $[X, \mathcal{B}, m, \nu]$ és m-connex si, per a tot $D \in \mathcal{B}$ amb $\nu(D) > 0$ i ν-quasi tot $x \in X$,

$$\sum_{n=1}^{\infty} m_x^{*n}(D) > 0,$$

i.e., m és ν-essencialment irreductible.

Tingueu en compte que, en aquesta definició, exigim que el passeig aleatori sigui ν-essencialment irreductible amb el requisit addicional que ν sigui en realitat una mesura invariant (com es va fer a la Proposició 0.22) . Tanmateix, la mesura d'irreductibilitat i la mesura invariant solen introduir-se per separat, tal com es veu a la secció anterior. Tot i això, aquesta definició proporciona una noció unificadora més senzilla, l'elecció de la qual es justifica a més a més amb el Teorema 0.11. Observeu que, tal com es va esmentar breument a la secció anterior, el concepte fonamental és que es pot arribar a totes les parts de l'espai després d'un cert nombre de salts, independentment del punt de partida (excepte, com a màxim, d'un ν-conjunt nul de punts).

Recordarem ara com es va introduir originalment aquesta noció a [**123**]. Això ens servirà per introduir una notació que utilitzarem en alguns resultats.

Definition 0.25. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Donat un conjunt ν-mesurable D, definim

$$N_D^m := \{x \in X \,:\, m_x^{*n}(D) = 0, \ \forall n \in \mathbb{N}\}.$$

Per a $n \in \mathbb{N}$, també definim

$$H_{D,n}^m := \{x \in X \,:\, m_x^{*n}(D) > 0\},$$

i

$$H_D^m := \bigcup_{n \in \mathbb{N}} H_{D,n}^m = \left\{ x \in X \,:\, \sum_{n=1}^{\infty} m_x^{*n}(D) > 0 \right\}.$$

[5] ν és interiorment regular si, per a qualsevol conjunt obert U, $\nu(U)$ és el màxim de $\nu(K)$ en tots els subconjunts compactes K de U.

[6] ν és localment finita si cada punt de X té un entorn U per al qual $\nu(U)$ és finit

Amb aquesta notació tenim que $[X, \mathcal{B}, m, \nu]$ és m-connex si, i només si, $\nu(N_D^m) = 0$ per a tot $D \in \mathcal{B}$ tal que $\nu(D) > 0$. Tingueu en compte que N_D^m i H_D^m són disjunts i

$$X = N_D^m \cup H_D^m.$$

També cal tenir en compte que N_D^m, $H_{D,n}^m$ i H_D^m pertanyen a $\mathcal{B}$. Al següent resultat veiem que N_D^m és invariant i H_D^m és ν-invariant (recordeu les Definicions 0.12 i 0.18).

Proposition 0.26. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $D \in \mathcal{B}$. Si $N_D^m \neq \varnothing$, aleshores:*
(i)

$$m_x^{*n}(H_D^m) = 0 \text{ (per tant } m_x^{*n}(N_D^m) = 1) \quad \text{per a tot } x \in N_D^m \text{ i } n \in \mathbb{N},$$

i.e., N_D^m és invariant respecte a m.
(ii)

$$m_x^{*n}(H_D^m) = 1 \text{ (per tant } m_x^{*n}(N_D^m) = 0) \quad \text{per a } \nu\text{-quasi tot } x \in H_D^m, \text{ i per a tot } n \in \mathbb{N}.$$

i.e., H_D^m és ν-invariant respecte a m.

Conseqüentment, per a tot $x \in N_D^m$ i ν-quasi tot $y \in H_D^m$ tenim que $m_x \perp m_y$, i.e. m_x i m_y són mútuament singulars[7].

Proof. *(i)*: Suposem que $m_x^{*k}(H_D^m) > 0$ per a algun $x \in N_D^m$ i $k \in \mathbb{N}$, aleshores, com $H_D^m = \cup_n H_{D,n}^m$ existeix $n \in \mathbb{N}$ tal que $m_x^{*k}(H_{D,n}^m) > 0$ però en aquest cas tenim

$$m_x^{*(n+k)}(D) = \int_{z \in X} m_z^{*n}(D) dm_x^{*k}(z) \geqslant \int_{z \in H_{D,n}^m} m_z^{*n}(D) dm_x^{*k}(z) > 0$$

atès que $m_z^{*n}(D) > 0$ per a tot $z \in H_{D,n}^m$, i això contradiu que $x \in N_D^m$. La segona afirmació de *(i)* és llavors immediata.

(ii): Fixeu $n \in \mathbb{N}$. Utilitzant la invariància de ν respecte a m^{*n} i l'afirmació *(ii)* tenim que

$$\begin{aligned} \nu(N_D^m) &= \int_X m_x^{*n}(N_D^m) d\nu(x) = \int_{H_D^m} m_x^{*n}(N_D^m) d\nu(x) + \int_{N_D^m} d\nu(x) \\ &= \int_{H_D^m} m_x^{*n}(N_D^m) d\nu(x) + \nu(N_D^m). \end{aligned}$$

Conseqüentment, $m_x^{*n}(N_D^m) = 0$ per a ν-quasi tot $x \in H_D^m$.

Aleshores, es segueix la primera afirmació de *(ii)*. □

Aquest resultat exemplifica com un passeig aleatori m que no és m-connex en realitat es compon de dos (o més) passeigs aleatoris separats, un amb salts a H_D^m i l'altre a N_D^m. A més, podem restringir la mesura invariant a qualsevol d'aquests subconjunts per tal d'obtenir mesures invariants per als passeigs aleatoris restringits tal com es veu al següent resultat.

Proposition 0.27. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $D \in \mathcal{B}$. Per a tot $n \in \mathbb{N}$ i $A \in \mathcal{B}$,*

$$\nu(A \cap H_D^m) = \int_{H_D^m} m_x^{*n}(A) d\nu(x),$$

i

$$\nu(A \cap N_D^m) = \int_{N_D^m} m_x^{*n}(A) d\nu(x).$$

[7]Dues mesures positives μ i ν són mútuament singulars si hi ha dos conjunts disjunts A i B a $\mathcal{B}$ la unió dels quals és X de manera que μ és zero en tots els subconjunts mesurables de B mentre que ν és zero en tots els subconjunts mesurables de A.

Proof. Per la invariància de ν respecte a m^{*n} i la Proposició 0.26 tenim que, per a qualsevol $A \in \mathcal{B}$,

$$\nu(A \cap H_D^m) = \int_X m_x^{*n}(A \cap H_D^m) d\nu(x) = \int_{H_D^m} m_x^{*n}(A \cap H_D^m) d\nu(x) = \int_{H_D^m} m_x^{*n}(A) d\nu(x).$$

De la mateixa manera, un demostra l'altra afirmació. □

En el resultat següent veiem que, donat un espai de passeig aleatori $[X, \mathcal{B}, m, \nu]$, si comencem a ν-casi qualsevol punt x en un conjunt $D \in \mathcal{B}$ de ν-mesura positiva, hi ha una probabilitat positiva que eventualment tornem a D. En els termes de la secció anterior, tenim que $P_x(\tau_D < \infty) > 0$ per a ν-quasi tot $x \in D$.

Corollary 0.28. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Per a tot $D \in \mathcal{B}$, tenim que*

$$\nu(D \cap N_D^m) = 0.$$

En conseqüència, si $\nu(D) > 0$, llavors $D \subset H_D^m$ excepte, com a molt, a un conjunt ν-nul; per tant, per a ν-quasi tot $x \in D$ existeix $n \in \mathbb{N}$ tal que $m_x^{*n}(D) > 0$.

Proof. Se segueix de la Proposició 0.27 que

$$\nu(D \cap N_D^m) = \int_{N_D^m} m_x^{*n}(D) d\nu(x) = 0.$$ □

Finalment, donarem un altre enfocament per definir un espai de passeig aleatori m-connex. Aquest enfocament requereix la noció de m-interacció entre conjunts i és molt útil per proporcionar intuïció no només per al concepte de m-connexió, sinó també per a la condició de reversibilitat d'una mesura.

Definition 0.29. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i A, $B \in \mathcal{B}$. Definim la *m-interacció* entre A i B com a

$$L_m(A, B) := \int_A \int_B dm_x(y) d\nu(x) = \int_A m_x(B) d\nu(x).$$

Tingueu en compte que, sempre que $L_m(A, B) < +\infty$, si ν és reversible respecte a m, tenim que

$$L_m(A, B) = L_m(B, A).$$

Una possible interpretació d'aquesta noció és la següent: per a una població que es distribueix originalment segons ν i que es mou segons la llei proporcionada per la caminada aleatòria m, $L_m(A, B)$ mesura quants les persones passen de A a B en un sol salt. Aleshores, si ν és reversible respecte a m, $L_m(A, B)$ també és igual a la quantitat d'individus que passen de B a A en un salt.

Per tal de facilitar la notació fem la següent definició.

Definition 0.30. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Diem que $[X, \mathcal{B}, m, \nu]$ és un espai de passeig aleatori reversible si ν és reversible respecte a m. A més, si $[X, d, m, \nu]$ és un espai mètric de passeig aleatori i ν és reversible respecte a m, direm que $[X, d, m, \nu]$ és un espai mètric de passeig aleatori reversible.

El resultat següent proporciona una caracterització de la m-connexió en termes de la m-interacció entre conjunts.

Proposition 0.31. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Les següents afirmacions són equivalents:*

(i) $[X, \mathcal{B}, m, \nu]$ és m-connex.

(ii) Si $A, B \in \mathcal{B}$ satisfan $A \cup B = X$ i $L_m(A, B) = 0$, aleshores $\nu(A) = 0$ o $\nu(B) = 0$.

(iii) Si $A \in \mathcal{B}$ és un conjunt ν-invariant, aleshores $\nu(A) = 0$ o $\nu(X \backslash A) = 0$.

PROOF. *(i)* ⇒ *(ii)*: Suposem que $[X, \mathcal{B}, m, \nu]$ és m-connex i siguin A, B igual que a l'enunciat *(ii)*. Si

$$0 = L_m(A, B) = \int_A \int_B dm_x(y) d\nu(x),$$

aleshores existeix $N_1 \in \mathcal{B}$, $\nu(N_1) = 0$, tal que

$$m_x(B) = 0 \ \text{ per a tot } x \in A \backslash N_1.$$

Ara, com que ν és invariant respecte a m,

$$0 = \nu(N_1) = \int_X m_x(N_1) d\nu(x),$$

i, en conseqüència, existeix $N_2 \in \mathcal{B}$, $\nu(N_2) = 0$, tal que

$$m_x(N_1) = 0 \quad \forall x \in X \backslash N_2.$$

Per tant, per a $x \in A \backslash (N_1 \cup N_2)$,

$$\begin{aligned} m_x^{*2}(B) &= \int_X \chi_B(y) dm_x^{*2}(y) = \int_X \left(\int_X \chi_B(y) dm_z(y) \right) dm_x(z) \\ &= \int_X m_z(B) dm_x(z) = \int_A m_z(B) dm_x(z) + \underbrace{\int_B m_z(B) dm_x(z)}_{=0, \text{ atès que } x \in A \backslash N_1.} \\ &= \underbrace{\int_{A \backslash N_1} m_z(B) dm_x(z)}_{=0, \text{ ja que } z \in A \backslash N_1.} + \underbrace{\int_{N_1} m_z(B) dm_x(z)}_{=0, \text{ perquè } x \in A \backslash N_2.} = 0 \end{aligned}$$

Treballant com anteriorment, trobem $N_3 \in \mathcal{B}$, $\nu(N_3) = 0$, de manera que

$$m_x(N_1 \cup N_2) = 0 \quad \forall x \in X \backslash N_3.$$

Llavors, per a $x \in A \backslash (N_1 \cup N_2 \cup N_3)$, tenim que

$$\begin{aligned} m_x^{*3}(B) &= \int_X \chi_B(y) dm_x^{*3}(y) = \int_X \left(\int_X \chi_B(y) dm_z^{*2}(y) \right) dm_x(z) \\ &= \int_X m_z^{*2}(B) dm_x(z) \leqslant \int_A m_z^{*2}(B) dm_x(z) + \underbrace{\int_B m_z^{*2}(B) dm_x(z)}_{=0, \text{ ja que } x \in A \backslash (N_1 \cup N_2).} \\ &\leqslant \underbrace{\int_{A \backslash (N_1 \cup N_2)} m_z^{*2}(B) dm_x(z)}_{=0, \text{ atès que } z \in A \backslash (N_1 \cup N_2).} + \underbrace{\int_{N_1 \cup N_2} m_z^{*2}(B) dm_x(z)}_{=0, \text{ perquè } x \in A \backslash N_3.} = 0. \end{aligned}$$

Inductivament, aconseguim que

$$m_x^{*n}(B) = 0 \quad \text{per a } \nu\text{-quasi tot } x \in A \text{ i tot } n \in \mathbb{N}.$$

Conseqüentment,

$$A \subset N_B^m$$

excepte, com a molt, a un conjunt ν-nul, per tant $\nu(B) > 0$ implica que $\nu(A) = 0$.

(ii) ⇒ *(iii)*: Tingueu en compte que, si A és ν-invariant, llavors $L_m(A, X \backslash A) = 0$.

(iii) ⇒ *(i)*: Sigui $D \in \mathcal{B}$ amb $\nu(D) > 0$. Aleshores, per la Proposició 0.26, tenim que H_D^m és ν-invariant però, pel Corol·lari 0.28, $\nu(H_D^m) \geqslant \nu(D) > 0$ i, per tant, $\nu(N_D^m) = 0$. □

Observeu que aquest resultat també justifica l'elecció de la terminologia utilitzada, ja que la caracterització de la m-connexió donada recorda d'alguna manera la definició d'un espai topològic connex.

Utilitzem també aquest moment per introduir la noció de m-connexió per a un subconjunt d'un espai de passeig aleatori reversible.

Definition 0.32. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i $\Omega \in \mathcal{B}$ amb $\nu(\Omega) > 0$. Sigui $\mathcal{B}_\Omega$ la següent σ-àlgebra

$$\mathcal{B}_\Omega := \{B \in \mathcal{B} \,:\, B \subset \Omega\}.$$

Diem que Ω és *m-connex (respecte a ν)* si $L_m(A, B) > 0$ per a qualsevol parell de conjunts A, $B \in \mathcal{B}_\Omega$ que no siguin ν-nuls tal que $A \cup B = \Omega$.

Si un espai de passeig aleatori $[X, \mathcal{B}, m, \nu]$ no és m-connex, llavors podem obtenir descomposicions no trivials de X com la següent.

Definition 0.33. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i $\Omega \in \mathcal{B}$ amb $0 < \nu(\Omega) < \nu(X)$. Suposem que $\Omega_1, \Omega_2 \in \mathcal{B}$ satisfan: $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, $\nu(\Omega_1) > 0$, $\nu(\Omega_2) > 0$ i $L_m(\Omega_1, \Omega_2) = 0$. Aleshores, escriurem $\Omega = \Omega_1 \sqcup_m \Omega_2$.

Ara som capaços de caracteritzar la m-connexió d'un espai de passeig aleatori en termes de l'ergodicitat de la mesura invariant (recorda el Corol·lari 0.14).

Theorem 0.34. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i suposem que ν és una mesura de probabilitat. Aleshores,*

$$[X, \mathcal{B}, m, \nu] \text{ és } m\text{-connex} \;\Leftrightarrow\; \nu \text{ és ergòdic respecte a } m.$$

Proof. ($\Rightarrow$). Suposem que $B \in \mathcal{B}$ és invariant. Aleshores, B és ν-invariant; per tant, per la Proposició 0.31, tenim que $\nu(B) = 0$ o $\nu(B) = 1$.
($\Leftarrow$). Sigui $D \in \mathcal{B}$ amb $\nu(D) > 0$. Per la Proposició 0.26, tenim que N_D^m és invariant respecte a m. Llavors, com que ν és ergòdic, tenim que $\nu(N_D^m) = 0$ o $\nu(N_D^m) = 1$. Ara, com que $\nu(D) > 0$, pel Corol·lari 0.28, tenim que $\nu(N_D^m) = 0$ i, en conseqüència, $[X, \mathcal{B}, m, \nu]$ és m-connex. □

Finalment, donem una condició suficient per a la φ-irreductibilitat d'un passeig aleatori. Per això necessitem la següent definició (vegeu, per exemple, [**100**, Secció 7.2]).

Definition 0.35. Sigui $[X, d, m, \nu]$ un espai mètric de passeig aleatori reversible. Diem que $[X, d, m, \nu]$ té la *propietat forta de Feller a $x_0 \in X$* si

$$m_{x_0}(A) = \lim_{n\to+\infty} m_{x_n}(A) \quad \text{per a tot conjunt de Borel} \quad A \subset X$$

sempre que $x_n \to x_0$ en (X, d) a mesura que $n \to +\infty$.
Diem que $[X, d, m, \nu]$ té la *propietat forta de Feller* si té la propietat forta de Feller a tot punt de X.

Proposition 0.36. *Sigui $[X, d, m, \nu]$ un espai mètric de passeig aleatori reversible tal que $\operatorname{supp}\nu = X$. Suposem a més que $[X, d, m, \nu]$ té la propietat forta de Feller i que (X, d) és connex. Aleshores, m és ν-irreductible (i, per tant, m-connex).*

Proof. Recordem que la "setwise" convergència d'una successió de mesures de probabilitat equival a la convergència de les integrals de funcions mesurables i fitades. Per tant, atès que $[X, d, m, \nu]$ té la propietat forta de Feller i

$$m_x^{*k}(A) = \int_X m_y^{*(k-1)}(A)\,dm_x(y), \quad x \in X, A \in \mathcal{B},$$

$[X, d, m^{*k}, \nu]$ també té la propietat forta de Feller per a qualsevol $k \in \mathbb{N}$.
Sigui $D \in \mathcal{B}$ amb $\nu(D) > 0$. Vegem primer que H_D^m és obert o, equivalentment, que N_D^m és tancat. Si $(x_n)_{n\geqslant 1} \subset N_D^m$ és una successió tal que $\lim_{n\to\infty} x_n = x \in X$, llavors

$$m_x^{*k}(D) = \lim_{n\to\infty} m_{x_n}^{*k}(D) = 0$$

per a qualsevol $k \in \mathbb{N}$ i, per tant, $x \in N_D^m$.
Ara bé, H_D^m també és tancat. De fet, si $m_x(H_D^m) < 1$ per a algun $x \in H_D^m$, com que $[X, d, m, \nu]$ té la propietat forta de Feller, existeix $r > 0$ tal que $m_y(H_D^m) < 1$ per a tot

$y \in B_r(x) \subset H_D^m$. Per tant, per la Proposició 0.26, $\nu(B_r(x)) = 0$, cosa que està en contradicció amb $\operatorname{supp} \nu = X$. Per tant,

$$m_x(H_D^m) = 1 \Leftrightarrow x \in H_D^m.$$

Conseqüentment, donada una successió $(x_n)_{n\geqslant 1} \subset H_D^m$ tal que $\lim_{n\to\infty} x_n = x \in X$, tenim que

$$m_x(H_D^m) = \lim_{n\to\infty} m_{x_n}(H_D^m) = 1$$

i, per tant, $x \in H_D^m$. Aleshores, H_D^m és tancat i, així doncs, com que X és connex, obtenim que $X = H_D^m$ el que implica que $N_D^m = \emptyset$. □

Tingueu en compte que aquest resultat proporciona una relació entre la connectivitat topològica i la m-connexió d'un espai mètric de passeig aleatori.

0.3. Exemples

Example 0.37. Sigui $(\mathbb{R}^N, d, \mathcal{L}^N)$ un espai mètric mesurable, on d és la distància euclidiana i $\mathcal{L}^N$ és la mesura de Lebesgue. Per simplificar, escriurem dx en lloc de $d\mathcal{L}^N(x)$. Sigui $J : \mathbb{R}^N \to [0, +\infty[$ una funció mesurable, no negativa i radialment simètrica tal que $\int_{\mathbb{R}^N} J(x)dx = 1$. Sigui m^J el següent passeig aleatori en $(\mathbb{R}^N, d)$:

$$m_x^J(A) := \int_A J(x-y)dy \quad \text{per a tot } x \in \mathbb{R}^N \text{ i tot conjunt de Borel } A \subset \mathbb{R}^N.$$

Llavors, aplicant el teorema de Fubini, és fàcil veure que la mesura de Lebesgue $\mathcal{L}^N$ és reversible respecte a m^J.

Una interpretació similar a la que es dóna a la secció 0.2 per a la m-interacció entre conjunts, es pot donar per a m^J. En aquest cas, si a $\mathbb{R}^N$ considerem una població tal que cada individu que comença a la ubicació x salta a la ubicació y segons la distribució de probabilitats $J(x-y)$, aleshores, per a un conjunt de Borel A en $\mathbb{R}^N$, $m_x^J(A)$ mesura la proporció d'individus que van començar a x i van arribar a A després d'un salt.

Example 0.38. Considera un graf ponderat i localment finit $G = (V(G), E(G))$, on $V(G)$ és el conjunt de vèrtexs, $E(G)$ és el conjunt de arestes i cada aresta $(x, y) \in E(G)$ (escriurem $x \sim y$ si $(x, y) \in E(G)$) té assignat un pes positiu $w_{xy} = w_{yx}$.Suposem a més que $w_{xy} = 0$ si $(x, y) \notin E(G)$.

Una successió finita $\{x_k\}_{k=0}^n$ de vèrtexs del graf s'anomena *camí* si $x_k \sim x_{k+1}$ per a tot $k = 0, 1, ..., n-1$. La *longitud* d'un camí $\{x_k\}_{k=0}^n$ es defineix com el nombre n de arestes del camí. Amb aquesta terminologia, es diu que $G = (V(G), E(G))$ és *connex* si, per a qualssevol vèrtexs $x, y \in V$, hi ha un camí que connecta x amb y, és a dir, un camí $\{x_k {}_{k=0}^n$ tal que $x_0 = x$ i $x_n = y$. Finalment, si $G = (V(G), E(G))$ és connex, la *mètrica del graf* $d_G(x, y)$ entre dos vèrtexs diferents x, y es defineix com al mínim de les longituds dels camins que connecten x amb y. Tingueu en compte que aquesta mètrica és independent dels pesos.

Per a $x \in V(G)$ definim el pes de x com a

$$d_x := \sum_{y\sim x} w_{xy} = \sum_{y\in V(G)} w_{xy},$$

i l'entorn de x com a $N_G(x) := \{y \in V(G) : x \sim y\}$. Observeu que, per la definició de un graf localment finit, els conjunts $N_G(x)$ són finits. Quan tots els pesos són iguals a 1, d_x coincideix amb el grau del vèrtex x en un graf, és a dir, amb el nombre d'arestes que contenen x.

Per a qualsevol $x \in V(G)$ definim la següent mesura de probabilitat:

$$m_x^G := \frac{1}{d_x} \sum_{y\sim x} w_{xy}\, \delta_y.$$

No és difícil veure que la mesura ν_G definida per

$$\nu_G(A) := \sum_{x\in A} d_x, \quad A \subset V(G),$$

és una mesura reversible respecte a aquest passeig aleatori. Aleshores, $[V(G), \mathcal{B}, m^G, \nu_G]$ és un espai de passeig aleatori reversible ($\mathcal{B}$ és el σ-àlgebra de tots els subconjunts de $V(G)$) i $[V(G), d_G, m^G, \nu_G]$ és un espai mètric de passeig aleatori reversible.

Proposition 0.39. *Sigui $[V(G), d_G, m^G, \nu_G]$ l'espai de passeig aleatori associat a un graf ponderat, localment finit i connex $G = (V(G), E(G))$. Llavors, m^G és ν_G-irreductible.*

Proof. Agafeu $D \subset V(G)$ amb $\nu_G(D) > 0$ i vegem que $N_D^{m^G} = \varnothing$. Suposem que existeix $y \in N_D^{m^G}$, això implica que

$$(m^G)_y^{*n}(D) = 0 \quad \forall\, n \in \mathbb{N}. \tag{0.3}$$

Ara bé, donat $x \in D$, existeix un camí $\{x, z_1, z_2, \ldots, z_{k-1}, y\}$ ($x \sim z_1 \sim z_2 \sim \cdots \sim z_{k-1} \sim y$) de longitud k connectant x amb y i, per tant,

$$(m^G)_y^{*k}(\{x\}) \geqslant \frac{w_{yz_{k-1}} w_{z_{k-1}z_{k-2}} \cdots w_{z_2z_1} w_{z_1x}}{d_y d_{z_{k-1}} d_{z_{k-2}} \cdots d_{z_2} d_{z_1}} > 0;$$

cosa que està en contradicció amb (0.3). □

A la teoria de l'aprenentatge automàtic ([**87**], [**88**]), un exemple de graf discret ponderat pot venir donat per un núvol de punts a $\mathbb{R}^N$, $V = \{x_1, \ldots, x_n\}$, amb pesos w_{x_i,x_j} donats per

$$w_{x_i,x_j} := \eta(|x_i - x_j|), \quad 1 \leqslant i, j \leqslant n,$$

on el nucli $\eta : [0, \infty) \to [0, \infty)$ és un perfil radial que satisfà:

(i) $\eta(0) > 0$ i η és contínua a 0,
(ii) η és no-decreixent,
(iii) la integral $\int_0^\infty \eta(r) r^N dr$ és finita.

Example 0.40. Sigui $K : X \times X \to \mathbb{R}$ un nucli de Markov en un espai comptable X, i.e.,

$$K(x, y) \geqslant 0 \quad \forall x, y \in X, \qquad \sum_{y \in X} K(x, y) = 1 \quad \forall x \in X.$$

Llavors, si

$$m_x^K(A) := \sum_{y \in A} K(x, y), \quad x \in X,\ A \subset X$$

i $\mathcal{B}$ és la σ-àlgebra de tots els subconjunts de X, m^K és un passseig aleatori en $(X, \mathcal{B})$.

Recordem que, en la terminologia de la teoria de les cadenes de Markov discretes, una mesura π en X que satisfà

$$\sum_{x \in X} \pi(x) = 1 \quad \text{and} \quad \pi(y) = \sum_{x \in X} \pi(x) K(x, y) \quad \forall y \in X,$$

s'anomena mesura de probabilitat estacionària (o estat estacionari) a X. Per descomptat, π és una mesura de probabilitat estacionària si, i només si, π és una mesura de probabilitat invariant respecte a m^K. En conseqüència, si π és una mesura de probabilitat estacionària a X, llavors $[X, \mathcal{B}, m^K, \pi]$ és un espai de passeig aleatori.

A més, es diu que una mesura de probabilitat estacionària π és reversible respecte a K si es compleix la següent equació:

$$K(x, y)\pi(x) = K(y, x)\pi(y) \ \text{ per a tot } x, y \in X.$$

Aquesta condició es equivalent a

$$dm_x^K(y) d\pi(x) = dm_y^K(x) d\pi(y) \ \text{ per a tot } x, y \in X.$$

Tingueu en compte que, donat un graf ponderat i localment finit $G = (V(G), E(G))$ com a l'exemple 0.38, hi ha una definició natural d'una cadena de Markov als vèrtexs. De fet, definiu el nucli de Markov $K_G : V(G) \times V(G) \to \mathbb{R}$ per

$$K_G(x, y) := \frac{1}{d_x} w_{xy}.$$

Aleshores, m^G i m^{K_G} defineixen el mateix passeig aleatori. Si $\nu_G(V(G))$ és finit, l'única mesura de probabilitat reversible respecte a m^G ve donada per

$$\pi_G(x) := \frac{1}{\nu_G(V(G))} \sum_{z \in V(G)} w_{xz}.$$

Example 0.41. A partir d'un espai mètric de mesura (X, d, μ) podem obtenir un passeig aleatori, l'anomenat *passeig aleatori de pas ϵ associat a μ*, de la següent manera. Suposem que les boles de X tenen una mesura finita i que $\mathrm{Supp}(\mu) = X$. Donat $\epsilon > 0$, el passeig aleatori de pas ϵ en X, començant a $x \in X$, consisteix a saltar aleatòriament a la bola de radi ϵ al voltant de x, amb probabilitat proporcional a μ; és a dir,

$$m_x^{\mu,\epsilon} := \frac{\mu \llcorner B(x,\epsilon)}{\mu(B(x,\epsilon))}$$

on $\mu \llcorner B(x,\epsilon)$ indica la restricció de μ a $B(x,\varepsilon)$ (o, més exactament, a $\mathcal{B}_{B(x,\varepsilon)}$, on $\mathcal{B}$ és la σ-àlgebra de Borel associada a (X,d).).

Si $\mu(B(x,\varepsilon)) = \mu(B(y,\varepsilon))$ per a tot x, $y \in X$, aleshores μ és una mesura reversible respecte a $m^{\mu,\epsilon}$ i, per tant, $[X, d, m^{\mu,\epsilon}, \mu]$ és un espai mètric de passeig aleatori reversible.

Example 0.42. Donat un espai de passeig aleatori $[X, \mathcal{B}, m, \nu]$ i $\Omega \in \mathcal{B}$ amb $\nu(\Omega) > 0$, sigui

$$m_x^{\Omega}(A) := \int_A dm_x(y) + \left(\int_{X \setminus \Omega} dm_x(y) \right) \delta_x(A) \quad \text{per a tot } A \in \mathcal{B}_\Omega \text{ i } x \in \Omega.$$

Llavors, m^Ω és un passeig aleatori en $(\Omega, \mathcal{B}_\Omega)$ i és fàcil veure que $\nu \llcorner \Omega$ és invariant respecte a m^Ω. Aleshores, $[\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega]$ és un espai de passeig aleatori. A més, si ν és reversible respecte a m, $\nu \llcorner \Omega$ és reversible respecte a m^Ω. Per descomptat, si ν és una mesura de probabilitat, podem normalitzar $\nu \llcorner \Omega$ per obtenir l'espai de passeig aleatori

$$\left[\Omega, \mathcal{B}_\Omega, m^\Omega, \frac{1}{\nu(\Omega)} \nu \llcorner \Omega \right].$$

Observeu que, si $[X, d, m, \nu]$ és un espai mètric de passeig aleatori i Ω és tancat, llavors $[\Omega, d, m^\Omega, \nu \llcorner \Omega]$ també és un espai mètric de passeig aleatori, on hem abusat de la notació i denotat per d la restricció de d a Ω.

En particular, en el context de l'exemple 0.37, si Ω és un subconjunt tancat i fitat de $\mathbb{R}^N$, obtenim l'espai mètric de passeig aleatori $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N \llcorner \Omega]$ on $m^{J,\Omega} := (m^J)^\Omega$; això és,

$$m_x^{J,\Omega}(A) := \int_A J(x-y)dy + \left(\int_{\mathbb{R}^n \setminus \Omega} J(x-z)dz \right) d\delta_x$$

per a tot conjunt de Borel $A \subset \Omega$ i $x \in \Omega$.

Utilitzant aquest darrer exemple podem caracteritzar els conjunts m-connexos de la següent manera (recordeu la Definició 0.32).

Proposition 0.43. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $\Omega \in \mathcal{B}$ amb $\nu(\Omega) > 0$. Aleshores,*

$$\Omega \text{ és } m\text{-connex} \Leftrightarrow [\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega] \text{ és } m^\Omega\text{-connex.}$$

Proof. Siguin A, $B \in \mathcal{B}_\Omega$ conjunts disjunts. Llavors, per a tot $x \in A$,

$$m_x^\Omega(B) = m_x(B) + m_x(X \setminus \Omega)\delta_x(B) = m_x(B)$$

i, per tant, $L_{m^\Omega}(A,B) = L_m(A,B)$. En conseqüència, el resultat se segueix per la Proposició 0.31. □

0.4. El gradient, la divergència i l'operador de Laplace no locals

Introduïm les nocions homòlogues no locals d'alguns conceptes clàssics.

Definition 0.44. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Donada una funció $u : X \to \mathbb{R}$ definim el seu gradient no-local $\nabla u : X \times X \to \mathbb{R}$ per

$$\nabla u(x, y) := u(y) - u(x) \quad \forall\, x, y \in X.$$

Altrament, donat $\mathbf{z} : X \times X \to \mathbb{R}$, la seua *m-divergència* $\mathrm{div}_m \mathbf{z} : X \to \mathbb{R}$ es defineix com

$$(\mathrm{div}_m \mathbf{z})(x) := \frac{1}{2} \int_X (\mathbf{z}(x, y) - \mathbf{z}(y, x)) dm_x(y).$$

Definim l'operador de Laplace (no local) de la següent manera (recordem la definició de l'operador mitjana donada a la Definició 0.16).

Definition 0.45. Sigui$[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori, definim *l'operador de Laplace* (o *Laplacià*) de $L^1(X, \nu)$ en si mateix com $\Delta_m := M_m - I$, i.e.,

$$\Delta_m u(x) = \int_X u(y) dm_x(y) - u(x) = \int_X (u(y) - u(x)) dm_x(y), \quad u \in L^1(X, \nu).$$

L'operador de Laplace també s'anomena operador drift (vegeu [**128**, Chapter 8]). Tingueu en compte que

$$\Delta_m f(x) = \mathrm{div}_m(\nabla f)(x).$$

Remark 0.46. Tenim que $\|\Delta_m f\|_1 \leqslant \|f\|_1$ i

$$\int_X \Delta_m f(x) d\nu(x) = 0 \quad \forall\, f \in L^1(X, \nu).$$

Com a la Observació 0.17, obtenim que Δ_m és un operador lineal a $L^2(X, \nu)$ amb domini

$$D(\Delta_m) = L^1(X, \nu) \cap L^2(X, \nu).$$

A més, si ν és una mesura de probabilitat, Δ_m és un operador lineal i fitat en $L^2(X, \nu)$ que satisfà $\|\Delta_m\| \leqslant 2$.

En el cas de l'espai de passeig aleatori associat a un graf ponderat i localment finit G (tal com es defineix a l'Exemple 0.38), l'operador de Laplace coincideix amb el Laplacià del graf estudiat per molts autors (vegeu, per exemple, [**26**], [**27**], [**75**] o [**105**]).

Proposition 0.47. *(Fórmula d'integració per parts) Sigui* $[X, \mathcal{B}, m, \nu]$ *un espai de passeig aleatori reversible. Aleshores,*

$$\int_X f(x) \Delta_m g(x) d\nu(x) = -\frac{1}{2} \int_{X \times X} \nabla f(x, y) \nabla g(x, y) d(\nu \otimes m_x)(x, y)$$

per a $f, g \in L^1(X, \nu) \cap L^2(X, \nu)$.

Proof. Atès que, per la reversibilitat de ν respecte a m,

$$\int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) = \int_X \int_X f(y)(g(x) - g(y)) dm_x(y) d\nu(x)$$

obtenim que

$$\int_X f(x)\Delta_m g(x)d\nu(x) = \int_X \int_X f(x)(g(y) - g(x))dm_x(y)d\nu(x)$$
$$= \frac{1}{2}\int_X \int_X f(x)(g(y) - g(x))dm_x(y)d\nu(x) + \frac{1}{2}\int_X \int_X f(x)(g(y) - g(x))dm_x(y)d\nu(x)$$
$$= \frac{1}{2}\int_X \int_X f(x)(g(y) - g(x))dm_x(y)d\nu(x) + \frac{1}{2}\int_X \int_X f(y)(g(x) - g(y))dm_x(y)d\nu(x)$$
$$= -\frac{1}{2}\int_{X\times X} \nabla f(x,y)\nabla g(x,y)d(\nu \otimes m_x)(x,y). \qquad \square$$

De fet, podem demostrar, de la mateixa manera, el següent resultat més general que serà útil al capítol 5.

Lemma 0.48. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i sigui $q \geqslant 1$. Si $Q \subset X \times X$ és un conjunt simètric (i.e., $(x,y) \in Q \Leftrightarrow (y,x) \in Q$) i $\Psi : Q \to \mathbb{R}$ és una funció antisimètrica $\nu \otimes m_x$-gairebé pertot (i.e., $\Psi(x,y) = -\Psi(y,x)$ per a $\nu \otimes m_x$-quasi tot $(x,y) \in Q$) amb $\Psi \in L^q(Q, \nu \otimes m_x)$ i $u \in L^{q'}(X,\nu)$, llavors*

$$\int_Q \Psi(x,y)u(x)d(\nu \otimes m_x)(x,y) = -\frac{1}{2}\int_Q \Psi(x,y)(u(y) - u(x))d(\nu \otimes m_x)(x,y).$$

En particular, si $\Psi \in L^1(Q, \nu \otimes m_x)$,

$$\int_Q \Psi(x,y)d(\nu \otimes m_x)(x,y) = 0.$$

Ara podem caracteritzar la m-connexió d'un espai de passeig aleatori en termes de l'ergodicitat de l'operador de Laplace. Després de Bakry, Gentil i Ledoux cite BGL, donem la següent definició.

Definition 0.49. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Diem que Δ_m és *ergodic* si, per a $u \in D(\Delta_m)$,

$$\Delta_m u = 0 \ \nu\text{-gairebé pertot} \Rightarrow u \text{ és igual a una constant } \nu\text{-gairebé pertot}$$

(sent aquesta constant 0 si ν no és finita), és a dir, totes les funcions harmòniques en $D(\Delta_m)$ (recordeu la Definició 0.19) són una constant ν -gairebé pertot.

Theorem 0.50. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i suposem que ν és una mesura de probabilitat. Aleshores,*

$$[X, \mathcal{B}, m, \nu] \text{ és } m\text{-connex} \Leftrightarrow \Delta_m \text{ és ergòdic}$$

Proof. ($\Leftarrow$) Sigui $D \in \mathcal{B}$ amb $\nu(D) > 0$ i recordeu que, pel Corol·lari 0.28, $\nu(H_D^m) \geqslant \nu(D) > 0$. Considereu la funció

$$u(x) := \chi_{H_D^m}(x), \quad x \in X,$$

i tingueu en compte que, com que ν és finita, $u \in L^1(X,\nu)$. Ara bé, com que, per la Proposició 0.26, H_D^m és ν-invariant, tenim que $\Delta_m u = 0$ ν-gairebé pertot. Per tant, per l'ergodicitat de Δ_m i recordant que $\nu(H_D^m) > 0$, obtenim que $u = \chi_{H_D^m} = 1$ ν-gairebé pertot i, així doncs, $\nu(N_D^m) = 0$.

($\Rightarrow$) Suposeu que $[X, \mathcal{B}, m, \nu]$ és m-connex i sigui $u \in L^1(X,\nu)$ de manera que u no és una constant ν-gairebé pertot, vegem que $\Delta_m u$ no és igual a 0 ν-gairebé pertot.

Existeix $U \in \mathcal{B}$ amb $0 < \nu(U) < 1$ tal que $u(x) < u(y)$ per a tot $x \in U$ i $y \in X \setminus U$. Aleshores, com que $L_m(U, X\setminus U) > 0$,

$$\mathcal{H}_m(u) = \int_X \int_X \nabla u(x,y)^2 dm_x(y)d\nu(x) \geqslant \int_U \int_{X\setminus U} \nabla u(x,y)^2 dm_x(y)d\nu(x) > 0$$

però $\mathcal{H}_m(u) = -\int_X u(x)\Delta_m u(x)d\nu(x)$ i, en conseqüència, $\Delta_m u$ no és igual a 0 ν-gairebé pertot. □

Aquest resultat, juntament amb el Teorema 0.34 mostra que els dos conceptes d'ergodicitat, el de la mesura invariant i el de l'operador de Laplace, són equivalents. Aquesta relació recupera la Proposició 0.20.

0.5. La frontera, el perímetre i la curvatura mitjana no locals

En aquesta secció introduïm les nocions homòlogues no locals dels conceptes de frontera, perímetre i curvatura mitjana.

La següent noció de frontera no local tindrà el paper de la frontera clàssica quan considerem les equacions homòlogues no locals d'equacions clàssiques al capítol 5, és a dir, s'imposaran condicions de frontera sobre aquest conjunt.

Definition 0.51. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $\Omega \in \mathcal{B}$. Definim la *m-frontera de* Ω com

$$\partial_m \Omega := \{x \in X \backslash \Omega \, : \, m_x(\Omega) > 0\}$$

i la seua *m-clausura* com

$$\Omega_m := \Omega \cup \partial_m \Omega.$$

Al capítol 3 s'utilitzarà àmpliament la següent noció de perímetre no local.

Definition 0.52. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $E \in \mathcal{B}$. El m-perímetre de E està definit per

$$P_m(E) := L_m(E, X\backslash E) = \int_E \int_{X\backslash E} dm_x(y) d\nu(x).$$

Pel que fa a la interpretació donada per a la m-interacció entre conjunts (a sota de la Definició 0.29), aquesta noció de perímetre es pot interpretar com a una mesura del flux total d'individus que creuen la "frontera" (en un sentit molt feble) d'un conjunt en un salt.

Lemma 0.53. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $E \in \mathcal{B}$ amb $\nu(E) < \infty$. Aleshores,*

$$P_m(E) = \nu(E) - \int_E \int_E dm_x(y) d\nu(x). \tag{0.4}$$

A més a més, $P_m(E) = P_m(X\backslash E)$ i

$$P_m(E) = \frac{1}{2}\int_X \int_X |\chi_E(y) - \chi_E(x)| dm_x(y) d\nu(x) = \frac{1}{2}\int_X \int_X |\nabla \chi_E(x,y)| dm_x(y) d\nu(x).$$

Proof. L'equació (0.4) es prova de manera senzilla. Ara bé,

$$P_m(E) - P_m(X\backslash E) = \int_E \int_{X\backslash E} dm_x(y)d\nu(x) - \int_{X\backslash E}\int_E dm_x(y)d\nu(x)$$

$$= \int_E \int_X dm_x(y)d\nu(x) - \int_E\int_E dm_x(y)d\nu(x) - \left(\int_X\int_E dm_x(y)d\nu(x) - \int_E\int_E dm_x(y)d\nu(x)\right)$$

$$= \int_E d\nu(x) - \int_X\int_X \chi_E(y) dm_x(y)d\nu(x)$$

$$= \nu(E) - \int_X \chi_E(x)d\nu(x)$$

$$= \nu(E) - \nu(E) = 0.$$

Per a la darrera afirmació, tingueu en compte que

$$\int_X\int_X |\chi_E(y) - \chi_E(x)| dm_x(y)d\nu(x) = P_m(E) + P_m(X\backslash E) = 2P_m(E).$$ □

La noció de m-perímetre es pot localitzar en un subconjunt de la següent manera.

DEFINITION 0.54. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $\Omega \in \mathcal{B}$ amb $\nu(\Omega) < \infty$. Aleshores, per a $E \in \mathcal{B}$, definim

$$P_m(E, \Omega) := L_m(E \cap \Omega, X \backslash E) + L_m(E \backslash \Omega, \Omega \backslash E).$$

Observeu que

$$L_m(E, X \backslash E) = L_m(E \cap \Omega, X \backslash E) + L_m(E \backslash \Omega, \Omega \backslash E) + L_m(E \backslash \Omega, X \backslash (E \cup \Omega))$$

i, en conseqüència, tenim que

$$P_m(E, \Omega) = \int_E \int_{X \backslash E} dm_x(y) d\nu(x) - \int_{E \backslash \Omega} \int_{X \backslash (E \cup \Omega)} dm_x(y) d\nu(x),$$

quan ambdues integrals són finites.

EXAMPLE 0.55. Sigui $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ l'espai mètric de passeig aleatori donat a l'Exemple 0.37. Aleshores,

$$P_{m^J}(E) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\chi_E(y) - \chi_E(x)| J(x - y) dy dx,$$

que coincideix amb el concepte de J-perímetre introduït a [**120**]. A més,

$$P_{m^{J,\Omega}}(E) = \frac{1}{2} \int_{\Omega} \int_{\Omega} |\chi_E(y) - \chi_E(x)| J(x - y) dy dx.$$

Tingueu en compte que, en general, $P_{m^{J,\Omega}}(E) \neq P_{m^J}(E)$ (recordeu la definició de $m^{J,\Omega}$ donada a l'Exemple 0.42).

Endemés,

$$\begin{aligned} P_{m^{J,\Omega}}(E) &= \mathcal{L}^N(E) - \int_E \int_E dm_x^{J,\Omega}(y) dx \\ &= \mathcal{L}^N(E) - \int_E \int_E J(x - y) dy dx - \int_E \left(\int_{\mathbb{R}^N \backslash \Omega} J(x - z) dz \right) dx \end{aligned}$$

i, per tant,

$$P_{m^{J,\Omega}}(E) = P_{m^J}(E) - \int_E \left(\int_{\mathbb{R}^N \backslash \Omega} J(x - z) dz \right) dx, \quad \forall\, E \subset \Omega.$$

EXAMPLE 0.56. Sigui $[V(G), d_G, m^G, \nu_G]$ l'espai mètric de passeig aleatori associat (com a l'Exemple 0.38) a un graf ponderat i finit G. Donats $A, B \subset V(G)$, $\mathrm{Cut}(A, B)$ es defineix com

$$\mathrm{Cut}(A, B) := \sum_{x \in A, y \in B} w_{xy} = L_{m^G}(A, B),$$

i el perímetre d'un conjunt $E \subset V(G)$ ve donat per

$$|\partial E| := \mathrm{Cut}(E, E^c) = \sum_{x \in E, y \in V \backslash E} w_{xy}.$$

En conseqüència, tenim que

$$|\partial E| = P_{m^G}(E) \quad \text{per a tot } E \subset V(G).$$

Ara donem algunes propietats del m-perímetre.

Proposition 0.57. *Sigui* $[X, \mathcal{B}, m, \nu]$ *un espai de passeig aleatori reversible i siguin* A, $B \in \mathcal{B}$ *conjunts amb* m*-perímetre finit tal que* $\nu(A \cap B) = 0$. *Llavors,*

$$P_m(A \cup B) = P_m(A) + P_m(B) - 2L_m(A, B).$$

Proof.

$$\begin{aligned} P_m(A \cup B) &= \int_{A\cup B} \left(\int_{X\setminus(A\cup B)} dm_x(y) \right) d\nu(x) \\ &= \int_A \left(\int_{X\setminus(A\cup B)} dm_x(y) \right) d\nu(x) + \int_B \left(\int_{X\setminus(A\cup B)} dm_x(y) \right) d\nu(x) \\ &= \int_A \left(\int_{X\setminus A} dm_x(y) - \int_B dm_x(y) \right) d\nu(x) \\ &\quad + \int_B \left(\int_{X\setminus B} dm_x(y) - \int_A dm_x(y) \right) d\nu(x), \end{aligned}$$

així doncs, per la reversibilitat de ν respecte a m,

$$P_m(A \cup B) = P_m(A) + P_m(B) - 2\int_A \left(\int_B dm_x(y) \right) d\nu(x).$$ □

Corollary 0.58. *Sigui* $[X, \mathcal{B}, m, \nu]$ *un espai de passeig aleatori reversible i siguin* A, B, $C \in \mathcal{B}$ *conjunts amb interseccions* ν*-nules per parelles. Aleshores,*

$$P_m(A \cup B \cup C) = P_m(A \cup B) + P_m(A \cup C) + P_m(B \cup C) - P_m(A) - P_m(B) - P_m(C).$$

Proposition 0.59 (Submodularity). *Sigui* $[X, \mathcal{B}, m, \nu]$ *un espai de passeig aleatori reversible i siguin* A, $B \in \mathcal{B}$. *Llavors,*

$$P_m(A \cup B) + P_m(A \cap B) = P_m(A) + P_m(B) - 2L_m(A\setminus B, B\setminus A).$$

Per conseqüent,

$$P_m(A \cup B) + P_m(A \cap B) \leqslant P_m(A) + P_m(B).$$

Proof. Per la Proposició 0.58,

$$\begin{aligned} P_m(A \cup B) &= P_m((A\setminus B) \cup (B\setminus A) \cup (A \cap B)) \\ &= P_m((A\setminus B) \cup (B\setminus A)) + P_m(A) + P_m(B) \\ &\quad - P_m(A\setminus B) - P_m(B\setminus A) - P_m(A \cap B). \end{aligned}$$

Aleshores,

$$\begin{aligned} P_m(A \cup B) + P_m(A \cap B) &= P_m(A) + P_m(B) + P_m((A\setminus B) \cup (B\setminus A)) \\ &\quad - P_m(A\setminus B) - P_m(B\setminus A). \end{aligned}$$

Ara bé, per la Proposició 0.57,

$$P_m((A\setminus B) \cup (B\setminus A)) - P_m(A\setminus B) - P_m(B\setminus A) = -2L_m(A\setminus B, B\setminus A).$$ □

Definition 0.60. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $E \in \mathcal{B}$. Per a $x \in X$ definim la *m-curvatura mitjana de ∂E a x* com

$$\mathcal{H}^m_{\partial E}(x) := m_x(X\setminus E) - m_x(E) = 1 - 2m_x(E).$$

Observeu que $\mathcal{H}^m_{\partial E}(x)$ es pot calcular per a tot $x \in X$, no només per als punts de ∂E. A més, si $\nu(E) < \infty$,

$$\int_E \mathcal{H}^m_{\partial E}(x) d\nu(x) = \int_E \left(1 - 2\int_E dm_x(y) \right) d\nu(x) = \nu(E) - 2\int_E \int_E dm_x(y) d\nu(x),$$

per tant, tenint present (0.4), obtenim que

$$\int_E \mathcal{H}^m_{\partial E}(x)d\nu(x) = 2P_m(E) - \nu(E). \tag{0.5}$$

Tingueu en compte també que

$$\mathcal{H}^m_{\partial E}(x) = -\mathcal{H}^m_{\partial(X\setminus E)}(x).$$

REMARK 0.61. Sigui $[\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega]$ l'espai de passeig aleatori donat a l'Exemple 0.42. Aleshores,

$$\mathcal{H}^{m^\Omega}_{\partial E}(x) = m_x(\Omega\setminus E) + m_x(X\setminus\Omega)\delta_x(\Omega\setminus E) - m_x(E) - m_x(X\setminus\Omega)\delta_x(E),$$

i, per tant,

$$\mathcal{H}^{m^\Omega}_{\partial E}(x) = \begin{cases} m_x(\Omega\setminus E) - m_x(E) + m_x(X\setminus\Omega) & \text{si } x \in \Omega\setminus E, \\ m_x(\Omega\setminus E) - m_x(E) - m_x(X\setminus\Omega) & \text{si } x \in E. \end{cases}$$

En particular, per a l'espai de passeig aleatori $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N]$ (vegeu també l'Exemple 0.42), obtenim que

$$\mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) = \begin{cases} \displaystyle\int_{\Omega\setminus E} J(x-y)dy - \int_E J(x-y)dy + \int_{\mathbb{R}^N\setminus\Omega} J(x-y)dy & \text{si } x \in \Omega\setminus E, \\ \displaystyle\int_{\Omega\setminus E} J(x-y)dy - \int_E J(x-y)dy - \int_{\mathbb{R}^N\setminus\Omega} J(x-y)dy & \text{si } x \in E. \end{cases}$$

Finalment, al teorema 0.63 donarem una altra caracterització de l'ergodicitat de Δ_m en termes de propietats geomètriques.

LEMMA 0.62. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i suposem que ν és una mesura de probabilitat. Aleshores, per a $D \in \mathcal{B}$, les següents afirmacions són equivalents:*
(i) D és ν-invariant.
(ii) $\Delta_m \chi_D = 0$ ν-gairebé pertot.
(iii) $P_m(D) = 0$.
(iv) $\displaystyle\int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) = -\nu(D)$.

PROOF. *(i)* ⇔ *(ii)* Se segueix de la definició d'un conjunt ν-invariant i de l'operador Laplace.

(ii) ⇒ *(iii)* Per hipòtesi, $m_x(D) = M_m\chi_D(x) = \chi_D(x)$ per a ν-quasi tot $x \in X$ i, així doncs, en particular, $m_x(X\setminus D) = 0$ per a ν-quasi tot $x \in D$ i, per tant,

$$P_m(D) = L_m(D, X\setminus D) = 0.$$

(iii) ⇒ *(ii)* Suposem que $P_m(D) = 0$. Llavors, per (0.4), tenim que

$$\nu(D) = \int_D \int_D dm_x(y)d\nu(x) = \int_D m_x(D)d\nu(x);$$

doncs,

$$m_x(D) = 1 \quad \text{per a } \nu\text{-quasi tot } x \in D.$$

A més, per la invariància de ν respecte a m, obtenim que

$$\nu(D) = \int_X m_x(D)d\nu(x) = \int_D m_x(D)d\nu(x) + \int_{X\setminus D} m_x(D)d\nu(x) = \nu(D) + \int_{X\setminus D} m_x(D)d\nu(x);$$

per conseqüent,

$$m_x(D) = 0 \quad \text{per a } \nu\text{-quasi tot } x \in X\setminus D.$$

Aleshores,

$$M_m\chi_D(x) = m_x(D) = \chi_D(x) \quad \text{per a } \nu\text{-quasi tot } x \in X.$$

$(iv) \Leftrightarrow (v)$ Per (0.5), tenim que

$$\int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) = 2P_m(D) - \nu(D),$$

doncs, $P_m(D) = 0$ si, i només si, $\displaystyle\int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) = -\nu(D)$. □

THEOREM 0.63. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i suposem que ν és una mesura de probabilitat. Les següents afirmacions són equivalents:*
(i) Δ_m és ergòdic.
(ii) Per a tot $D \in \mathcal{B}$, $\Delta_m \chi_D = 0$ ν-gairebé pertot $\Rightarrow \nu(D) = 0$ o $\nu(D) = 1$.
(iii) Per a tot $D \in \mathcal{B}$, $0 < \nu(D) < 1 \Rightarrow P_m(D) > 0$.
(iv) Per a tot $D \in \mathcal{B}$,

$$0 < \nu(D) < 1 \Rightarrow \frac{1}{\nu(D)} \int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) > -1$$

PROOF. $(i) \Rightarrow (ii)$ Directe.

$(ii) \Rightarrow (iii)$ Si $P_m(D) = 0$ aleshores, pel Lemma 0.62, $\Delta_m \chi_D = 0$ ν-gairebé pertot, doncs (ii) implica que $\nu(D) = 0$ o $\nu(D) = 1$.

$(iii) \Rightarrow (ii)$ Sigui $D \in \mathcal{B}$. Si $\Delta_m \chi_D = 0$ ν-gairebé pertot aleshores, pel Lemma 0.62, tenim que $P_m(D) = 0$ i, per tant, (iii) implica que $\nu(D) = 0$ o $\nu(D) = 1$.

$(ii) \Rightarrow (i)$ Suposem que Δ_m no és ergòdic. Aleshores, segons el Teorema 0.50, $[X, \mathcal{B}, m, \nu]$ no és m-connex, de manera que existeix $D \in \mathcal{B}$ amb $\nu(D) > 0$ tal que $0 < \nu(N^m_D) < 1$ (recordeu el Corol·lari 0.28). Tanmateix, per la Proposició 0.26, $\Delta_m \chi_{N^m_D}(x) = 0$ i, per hipòtesi, això implica que $\nu(N^m_D) = 0$ o $\nu(N^m_D) = 1$, la qual cosa és una contradicció.

$(iii) \Leftrightarrow (iv)$ Aquesta equivalència se segueix per (0.5) i pel Lemma 0.62. □

0.6. Desigualtats de tipus Poincaré

Les desigualtats de tipus Poincaré com les definides a la Definició 0.66 i la Definició 1.81 (vegeu també l'Observació 1.82) tindran un paper molt important en aquesta tesi. Suposant que es compleixi una desigualtat de tipus Poincaré, podrem obtenir resultats sobre les taxes de convergència tant del flux de la calor com del flux per la variació total. A més, també assumirem que es compleix una desigualtat d'aquest tipus per demostrar l'existència de solucions a alguns dels problemes del capítol 5.

Primer introduïm la següent notació.

DEFINITION 0.64. Sigui $(X, \mathcal{B}, \nu)$ un espai de probabilitat. Denotem la mitjana de $f \in L^1(X, \nu)$ (o el valor esperat de f) respecte a ν per

$$\nu(f) := \mathbb{E}_\nu(f) = \int_X f(x)d\nu(x).$$

Altrament, donada $f \in L^2(X, \nu)$, denotem la seva variància respecte a ν per

$$\mathrm{Var}_\nu(f) := \int_X (f(x) - \nu(f))^2 d\nu(x) = \frac{1}{2}\int_{X \times X} (f(x) - f(y))^2 d\nu(y)d\nu(x).$$

En general, si $\nu(X) < \infty$ també denotarem la mitjana d'una funció $f \in L^1(X, \nu)$ per $\nu(f)$, i.e.,

$$\nu(f) := \frac{1}{\nu(X)} \int_X f(x)d\nu(x).$$

Ara presentem l'equivalent no local de l'energia de Dirichlet.

Definition 0.65. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible. Definim el funcional d'energia $\mathcal{H}_m : L^2(X,\nu) \to [0,+\infty]$ per

$$\mathcal{H}_m(f) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{X\times X} (f(x)-f(y))^2 dm_x(y) d\nu(x) & \text{si } f \in L^1(X,\nu)\cap L^2(X,\nu), \\ +\infty & \text{en cas contrari,} \end{cases}$$

i denotem

$$D(\mathcal{H}_m) := L^1(X,\nu)\cap L^2(X,\nu).$$

Tingueu en compte que, per la Proposició 0.47,

$$\mathcal{H}_m(f) = -\int_X f(x)\Delta_m f(x) d\nu(x) \quad \text{per a tot } f \in D(\mathcal{H}_m).$$

Definition 0.66. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i suposem que ν és una mesura de probabilitat. Diem que $[X, \mathcal{B}, m, \nu]$ satisfà una *desigualtat de Poincaré* si existeix $\lambda > 0$ tal que

$$\lambda \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{per a tot } f \in L^2(X,\nu),$$

o, equivalentment,

$$\lambda \|f\|^2_{L^2(X,\nu)} \leqslant \mathcal{H}_m(f) \quad \text{per a tot } f \in L^2(X,\nu) \text{ amb } \nu(f) = 0.$$

Més generalment, diem que $[X, \mathcal{B}, m, \nu]$ satisfà una (p,q)-*desigualtat de Poincaré* ($p, q \in [1,+\infty[$) si existeix una constant $\Lambda > 0$ tal que, per a qualsevol $u \in L^q(X,\nu)$,

$$\|u\|_{L^p(X,\nu)} \leqslant \Lambda \left(\left(\int_X \int_X |u(y)-u(x)|^q dm_x(y) d\nu(x) \right)^{\frac{1}{q}} + \left| \int_X u \, d\nu \right| \right),$$

o, equivalentment, existeix un $\Lambda > 0$ tal que

$$\|u\|_{L^p(X,\nu)} \leqslant \Lambda \|\nabla u\|_{L^q(X\times X, d(\nu\otimes m_x))} \quad \text{per a tot } u \in L^q(X,\nu) \text{ amb } \nu(u) = 0.$$

Quan a $[X, \mathcal{B}, m, \nu]$ es compleixi una $(p,1)$-desigualtat de Poincaré, direm que $[X, \mathcal{B}, m, \nu]$ satisfà una p-desigualtat de Poincaré.

La bretxa espectral de l'operador de Laplace està estretament relacionada amb la desigualtat de Poincaré.

Definition 0.67. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i suposem que ν és una mesura de probabilitat. La *bretxa espectral* de $-\Delta_m$ es defineix com

$$\begin{aligned} \mathrm{gap}(-\Delta_m) &:= \inf \left\{ \frac{\mathcal{H}_m(f)}{\mathrm{Var}_\nu(f)} \ : \ f \in D(\mathcal{H}_m), \ \mathrm{Var}_\nu(f) \neq 0 \right\} \\ &= \inf \left\{ \frac{\mathcal{H}_m(f)}{\|f\|^2_{L^2(X,\nu)}} \ : \ f \in D(\mathcal{H}_m), \ \|f\|_{L^2(X,\nu)} \neq 0, \ \int_X f d\nu = 0 \right\}. \end{aligned}$$

Tingueu en compte que, tal com s'esmenta a Observació 0.46, atès que ν és una mesura de probabilitat, tenim que

$$D(\mathcal{H}_m) = L^2(X,\nu).$$

Remark 0.68. Si $\mathrm{gap}(-\Delta_m) > 0$, aleshores $[X, \mathcal{B}, m, \nu]$ satisfà una desigualtat de Poincaré amb $\lambda = \mathrm{gap}(-\Delta_m)$:

$$\mathrm{gap}(-\Delta_m)\mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{per a tot } f \in L^2(X,\nu),$$

sent la bretxa espectral la millor constant en la desigualtat de Poincaré.

Per tant, ens interessa estudiar quan la bretxa espectral de $-\Delta_m$ és positiva.

Remark 0.69. Suposem que X és un espai mètric polonès i que ν és reversible respecte a m. Es poden trobar condicions suficients per a l'existència d'una desigualtat de Poincaré a, per exemple, [**134**, Corol·lari 31] o [**151**, Teorema 1]. A [**134**] se suposa la positivitat de la curvatura de Ricci gruixuda (vegeu l'Observació 0.95) mentre que a [**151**] la hipòtesi és la següent condició de Foster Lyapunov:

$$M_m V \leqslant (1-\lambda)V + b\chi_K,$$
$$M_m 1_A(x) \geqslant \alpha\mu(A)\chi_K, \ \forall A \in \mathcal{B}$$

per a una funció positiva $V : \mathbb{R}^d \to [1, \ infty)$, uns nombres $b < \infty$, α, $\lambda > 0$, un conjunt $K \subset X$, i una mesura de probabilitat μ. A més, en el teorema 2.19, en relació amb una altra noció de curvatura de Ricci fitada inferiorment, trobarem altres condicions suficients per a l'existència d'una desigualtat de Poincaré.

Si (X, d, μ) és un espai de longitud, μ és "doubling" i $[X, d, m^{\mu,\varepsilon}, \mu]$ (recordeu l'Exemple 0.41) és un espai mètric de passeig aleatori, es poden trobar condicions suficients per a l'existència d'una desigualtat de Poincaré a [**93**, Section 2.3].

Definition 0.70. Sigui $(X, \mathcal{B}, \nu)$ un espai de probabilitat. Denotem per $H(X,\nu)$ el subespai de $L^2(X,\nu)$ format per les funcions ortogonals a les constants, i.e.,

$$H(X,\nu) := \left\{ f \in L^2(X,\nu) \ : \ \nu(f) = 0 \right\}.$$

Remark 0.71. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i suposem que ν és una mesura de probabilitat. Atés que l'operador $-\Delta_m : H(X,\nu) \to H(X,\nu)$ és autoadjunt i no negatiu, i $\|\Delta_m\| \leqslant 2$ (vegeu el Teorema 2.4), per [**44**, Proposition 6.9] tenim l'espectre $\sigma(-\Delta_m)$ de $-\Delta_m$ en $H(X,\nu)$ satisfà

$$\sigma(-\Delta_m) \subset [\alpha, \beta] \subset [0, 2],$$

on

$$\alpha := \inf \left\{ \langle -\Delta_m u, u \rangle \ : \ u \in H(X,\nu), \ \|u\|_{L^2(X,\nu)} = 1 \right\} \in \sigma(-\Delta_m),$$

i

$$\beta := \sup \left\{ \langle -\Delta_m u, u \rangle \ : \ u \in H(X,\nu), \ \|u\|_{L^2(X,\nu)} = 1 \right\} \in \sigma(-\Delta_m).$$

Vegem que $\operatorname{gap}(-\Delta_m) = \alpha$. Per definició, tenim que $\operatorname{gap}(-\Delta_m) \leqslant \alpha$ (recordeu que $\mathcal{H}_m(u) = \langle -\Delta_m u, u \rangle$). Ara, per la desigualtat oposada, preneu $f \in L^2(X,\nu)$ amb $\operatorname{Var}_\nu(f) \neq 0$. Llavors, $u := f - \nu(f) \neq 0$ pertany a $H(X,\nu)$, doncs

$$\alpha \leqslant \mathcal{H}_m\left(\frac{u}{\|u\|_{L^2(X,\nu)}} \right) = \frac{\mathcal{H}_m(u)}{\|u\|^2_{L^2(X,\nu)}} = \frac{\mathcal{H}_m(f)}{\operatorname{Var}_\nu(f)},$$

i, per tant, $\operatorname{gap}(-\Delta_m) \geqslant \alpha$.

Com a conseqüència, obtenim que

$$\operatorname{gap}(-\Delta_m) > 0 \ \Leftrightarrow \ 0 \notin \sigma(-\Delta_m).$$

Amb aquesta observació a l'abast, podem obtenir el següent resultat.

Proposition 0.72. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i m-connex i suposem que ν és una mesura de probabilitat. Si $-\Delta_m$ és la suma d'un operador invertible i un operador compacte en $H(X, \ nu)$, aleshores* $\operatorname{gap}(-\Delta_m) > 0$.

En conseqüència, si l'operador mitjana M_m és compacte en $H(X,\nu)$ llavors $\operatorname{gap}(-\Delta_m) > 0$.

Proof. Si suposem que $-\Delta_m$ és la suma d'un operador invertible i un operador compacte en $H(X,\nu)$, llavors, si $0 \in \sigma(-\Delta_m)$, pel teorema de la alternativa de Fredholm, tenim que existeix $u \in H(X,\nu)$, $u \neq 0$, de manera que $-\Delta_m u = (I - M_m)u = 0$. Llavors, atès que $[X, d, m, \nu]$ és m-connex, pel Teorema 0.50, obtenim que Δ_m és ergòdic, de manera que u és ν-gairebé pertot igual a una constant. Per tant, com que $u \in H(X,\nu)$, hem de tenir $u = 0$ ν-gairebé pertot, cosa que és una contradicció. □

Example 0.73. Si $G = (V(G), E(G))$ és un graf ponderat, finit i connex, llavors, òbviament, M_{m^G} és compacte i, en conseqüència, $\text{gap}(-\Delta_m^G) > 0$. En aquesta situació, se sap que, si $\sharp(V(G)) = N$, l'espectre de $-\Delta_{m^G}$ és $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{N-1}$ i $0 < \lambda_1 = \text{gap}(-\Delta_m^G)$.

De fet, podem demostrar fàcilment que $[V(G), d_G, m^G, \nu_G]$ compleix una (p, q)-desigualtat de Poincaré per a qualssevol p, $q \in [1, \infty[$. De fet, siguin p, $q \in [1, \infty[$ i suposem que no existeix una (p, q)-desigualtat de Poincaré. Aleshores, existeix una successió $(u_n)_{n\in\mathbb{N}} \subset L^p(V(G), \nu_G)$ amb $\|u_n\|_{L^p(V(G),\nu_G)} = 1$ i $\int_{V(G)} u_n(x) d\nu(x) = 0$ $\forall n \in \mathbb{N}$, de manera que

$$\lim_{n\to\infty} \sum_{x\in V(G)} \sum_{y\sim x} w_{xy} |u_n(x) - u_n(y)|^q = 0.$$

Per tant,

$$\lim_{n\to\infty} |u_n(x) - u_n(y)| = 0 \quad \text{per a tot } x, y \in V(G),\ x \sim y. \tag{0.6}$$

A més, com que $\|u_n\|_{L^p(V(G),\nu_G)} = 1$, tenim que, agafant una subsuccessió si es necessari,

$$\lim_{n\to\infty} u_n(x) = u(x) \in \mathbb{R} \quad \text{per a tot } x \in V(G).$$

Tanmateix, com que el graf és connex, obtenim que, per (0.6), $u(x) = u(y)$ per a tot x, $y \in V(G)$, és a dir, existeix $\lambda \in \mathbb{R}$ tal que $u(x) = \lambda$ per a qualsevol $x \in V(G)$; per tant, $u_n \to \lambda$ a $L^p(V(G), \nu_G)$. Així doncs, atès que $\int_{V(G)} u_n(x) d\nu_G(x) = 0$, obtenim que $\lambda = 0$, cosa que està en contradicció amb $\|u_n\|_{L^p(V(G),\nu_G)} = 1$.

Example 0.74. Sigui Ω un domini fitat a $\mathbb{R}^N$ i sigui J un nucli tal que $J \in C(\mathbb{R}^N, \mathbb{R})$ és no negatiu i radialment simètric, amb $J(0) > 0$ i $\int_{\mathbb{R}^N} J(x)dx = 1$. Penseu en l'espai mètric de passeig aleatori reversible $[\Omega, \mathcal{B}_\Omega, m^{J,\Omega}, \mathcal{L}^N]$ tal com es defineix a l'Exemple 0.42 (recordeu també l'Exemple 0.37).

Aleshores, $-\Delta_{m^{J,\Omega}}$ és la suma d'un operador invertible i un operador compacte. En efecte,

$$-\Delta_{m^{J,\Omega}} f(x) = \int_\Omega J(x-y)dy f(x) - \int_\Omega f(y)J(x-y)dy,\ x \in \Omega,$$

on $f \mapsto \displaystyle\int_\Omega J(\cdot - y)dy f(\cdot)$ és un operador invertible a $H(\Omega, \mathcal{L}^N)$ (J és continu, $J(0) > 0$ i Ω és un domini, per tant $\int_\Omega J(x-y)dy > 0$ per a tot $x \in \Omega$) i $f \mapsto \displaystyle\int_\Omega f(y)J(\cdot - y)dy$ és un operador compacte en $H(\Omega, \mathcal{L}^N)$ (això se segueix pel teorema d'Arzelá-–Ascoli). Per tant, en aquest cas, tenim que $\text{gap}(-\Delta_{m^{J,\Omega}})$ és igual a (vegeu també [**18**])

$$\inf\left\{ \frac{\dfrac{1}{2}\displaystyle\int_{\Omega\times\Omega} J(x-y)(u(y)-u(x))^2 dxdy}{\displaystyle\int_\Omega u(x)^2 dx} \ :\ u \in L^2(\Omega),\ \|u\|_{L^2(\Omega)} > 0,\ \int_\Omega u = 0 \right\} > 0.$$

Assenyalem que la condició $J(0) > 0$ és necessària, ja que, en cas contrari, $\int_\Omega J(\cdot - y)dy$ pot ser 0 en un conjunt de mesura positiva (vegeu [**18**, Remark 6.20]).

Un altre resultat en què proporcionem les condicions suficients per a la positivitat de $\text{gap}(-\Delta_m)$ és el següent.

Proposition 0.75. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i suposem que ν és una mesura de probabilitat. Suposem també que Δ_m és ergòdic i que $m_x \ll \nu$ per a tot $x \in X$. Si existeix $p > 2$ i una constant K tal que*

$$\int_X \left\| \frac{dm_x}{d\nu} \right\|_{L^2(X,\nu)}^p d\nu(x) \leqslant K < \infty,$$

aleshores $\text{gap}(-\Delta_m) > 0$.

Proof. Com a conseqüència d'un resultat de Miclo [129], tenim que $\mathrm{gap}(-\Delta_m) > 0$ si Δ_m és ergòdic i M_m és hiperfitat, és a dir, si hi ha $q > 2$ de manera que M_m sigui fitat de $L^2(X,\nu)$ en $L^q(X,\nu)$. Denotem $f_x := \frac{dm_x}{d\nu} \in L^1(X,\nu)$, $x \in X$. Vegem que M_m és hiperfitat. Donat $u \in L^2(X,\nu)$, per la desigualtat de Cauchy-Schwarz, tenim

$$\begin{aligned} \|M_m u\|_p^p &= \int_X |M_m u(x)|^p d\nu(x) = \int_X \left| \int_X u(y) dm_x(y) \right|^p d\nu(x) \\ &= \int_X \left| \int_X u(y) f_x(y) d\nu(y) \right|^p d\nu(x) \\ &\leqslant \|u\|_{L^2(X,\nu)}^p \int_X \|f_x\|_{L^2(X,\nu)}^p d\nu(x), \end{aligned}$$

per tant

$$\|M_m u\|_p \leqslant K^{\frac{1}{p}} \|u\|_{L^2(X,\nu)}.$$

Per tant, M_m és hiperfitat com volíem. □

En els següents exemples donem espais de passeig aleatori per als quals no se satisfà una desigualtat de Poincaré.

Example 0.76. Sigui $[V(G), d_G, m^G, \nu_G]$ l'espai mètric de passeig aleatori associat al graf ponderat i localment finit G amb conjunt de vèrtexs $V(G) := \{x_3, x_4, x_5 \ldots, x_n, \ldots\}$ i pesos:

$$w_{x_{3n},x_{3n+1}} = \frac{1}{n^3},\ w_{x_{3n+1},x_{3n+2}} = \frac{1}{n^2},\ w_{x_{3n+2},x_{3n+3}} = \frac{1}{n^3},$$

per a $n \geqslant 1$, i $w_{x_i,x_j} = 0$ en cas contrari (recordeu l'Exemple 0.38).

(i) Let

$$f_n(x) := \begin{cases} n & \text{si } \ x = x_{3n+1}, x_{3n+2}, \\ 0 & \text{en cas contrari.} \end{cases}$$

Tingueu en compte que $\nu_G(V(G)) < +\infty$ (evitem la seva normalització a una mesura de probabilitat per simplificar). Ara bé,

$$\begin{aligned} 2\mathcal{H}_m(f_n) &= \int_{V(G)} \int_{V(G)} (f_n(x) - f_n(y))^2 dm_x^G(y) d\nu_G(x) \\ &= d_{x_{3n}} \int_{V(G)} (f_n(x_{3n}) - f_n(y))^2 dm_{x_{3n}}^G(y) \\ &\quad + d_{x_{3n+1}} \int_{V(G)} (f_n(x_{3n+1}) - f_n(y))^2 dm_{x_{3n+1}}^G(y) \\ &\quad + d_{x_{3n+2}} \int_{V(G)} (f_n(x_{3n+2}) - f_n(y))^2 dm_{x_{3n+2}}^G(y) \\ &\quad + d_{x_{3n+3}} \int_{V(G)} (f_n(x_{3n+3}) - f_n(y))^2 dm_{x_{3n+3}}^G(y) \\ &= d_{x_{3n}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n}}} + d_{x_{3n+1}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n+1}}} + d_{x_{3n+2}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n+2}}} + d_{x_{3n+3}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n+3}}} \\ &= \frac{4}{n}. \end{aligned}$$

No obstant això, tenim

$$\int_{V(G)} f_n(x) d\nu_G(x) = n(d_{x_{3n+1}} + d_{x_{3n+2}}) = 2n \left(\frac{1}{n^2} + \frac{1}{n^3} \right) = \frac{2}{n} \left(1 + \frac{1}{n} \right),$$

doncs

$$\nu_G(f_n) = \frac{\frac{2}{n}\left(1 + \frac{1}{n}\right)}{\nu_G(V(G))} = \widetilde{O}\left(\frac{1}{n}\right),$$

on fem servir la notació

$$\varphi(n) = \widetilde{O}(\psi(n)) \Leftrightarrow \limsup_{n\to\infty} \left|\frac{\varphi(n)}{\psi(n)}\right| = C \neq 0.$$

Conseqüentment,

$$(f_n(x) - \nu(f_n))^2 = \begin{cases} \widetilde{O}(n^2) & \text{si } \ x = x_{3n+1}, x_{3n+2}, \\ \widetilde{O}\left(\frac{1}{n^2}\right) & \text{en cas contrari.} \end{cases}$$

Finalment,

$$\begin{aligned} \mathrm{Var}_{\nu_G}(f_n) &= \int_{V(G)} (f_n(x) - \nu_G(f_n))^2 d\nu_G(x) \\ &= \widetilde{O}\left(\frac{1}{n^2}\right) \sum_{x \neq x_{3n+1}, x_{3n+2}} d_x + \widetilde{O}(n^2)(d_{x_{3n+1}} + d_{x_{3n+2}}) \\ &= \widetilde{O}\left(\frac{1}{n^2}\right) + 2\widetilde{O}(n^2)\left(\frac{1}{n^2} + \frac{1}{n^3}\right) = \widetilde{O}(1). \end{aligned}$$

Així doncs, $[V(G), d_G, m^G, \nu_G]$ no satisfà una desigualtat de Poincaré.
(ii) Sigui

$$f_n(x) := \begin{cases} n^2 & \text{si } \ x = x_{3n+1}, x_{3n+2}, \\ 0 & \text{en cas contrari.} \end{cases}$$

Amb càlculs similars (vegeu també [**124**, Example 4.7]), aconseguim

$$\int_{V(G)} \int_{V(G)} |f_n(x) - f_n(y)| dm_x^G(y) d\nu_G(x) = \frac{4}{n}$$

i

$$\int_{V(G)} |f_n(x) - \nu_G(f_n)| d\nu_G(x) = \widetilde{O}(1).$$

Per tant, $[V(G), d_G, m^G, \nu_G]$ no satisfà una 1-desigualtat de Poincaré.

Example 0.77. Considerem l'espai mètric de passeig aleatori $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ (recordeu l'Exemple 0.37), on d és la distància euclidiana i $J(x) = \frac{1}{2}\chi_{[-1,1]}$. Definim, per a $n \in \mathbb{N}$,

$$u_n = \frac{1}{2^{n+1}} \chi_{[2^n, 2^{n+1}]} - \frac{1}{2^{n+1}} \chi_{[-2^{n+1}, -2^n]}.$$

Llavors, $\|u_n\|_1 := 1$, $\int_{\mathbb{R}} u_n(x)dx = 0$ i és fàcil veure que, per n prou gran,

$$\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |u_n(y) - u_n(x)| dm_x^J(y) dx = \frac{1}{2^{n+1}}.$$

Aleshores, $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ no satisfà una 1-desigualtat de Poincaré.

Vegem ara que, si $\mathrm{gap}(-\Delta_m) > 0$, Δ_m és ergòdic.

Proposition 0.78. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i suposem que ν és una mesura de probabilitat. Si $[X, \mathcal{B}, m, \nu]$ satisfà una desigualtat de Poincaré, llavors Δ_m és ergòdic (i.e., $[X, \mathcal{B}, m, \nu]$ és m-connex).*

Proof. Sigui $f \in D(\Delta_m)$ tal que $\Delta_m(f) = 0$ ν-gairebé pertot. Així,

$$\mathcal{H}_m(f) = -\int_X f(x) \Delta_m f(x) d\nu(x) = 0$$

i, per tant, si $[X, \mathcal{B}, m, \nu]$ satisfà una desigualtat de Poincaré, tenim que

$$\mathrm{Var}_\nu(f) := \int_X (f(x) - \nu(f))^2 d\nu(x) = 0,$$

doncs f és ν-gairebé pertot igual a una constant:

$$f(x) = \int_X f(x) d\nu(x) \quad \text{per a } \nu\text{-quasi tot } x \in X. \qquad \square$$

L'Exemple 0.76 mostra que la implicació inversa no es compleix en general. Finalment, donem el resultat següent que pot ajudar a trobar fites inferiors per a $\text{gap}(-\Delta_m)$.

Theorem 0.79. *Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible i suposem que ν és una mesura de probabilitat. Suposem també que Δ_m és ergòdic. Aleshores,*

$$\text{gap}(-\Delta_m) = \sup\left\{\lambda \geqslant 0 \,:\, \lambda \mathcal{H}_m(f) \leqslant \int_X (-\Delta_m f)^2 d\nu \quad \forall f \in L^2(X,\nu)\right\}.$$

Proof. Per l'Observació 0.71 sabem que $\text{gap}(-\Delta_m) = \alpha$, on

$$\alpha := \inf\left\{\langle -\Delta_m u, u\rangle \,:\, u \in H(X,\nu),\ \|u\|_{L^2(X,\nu)} = 1\right\} \in \sigma(-\Delta_m).$$

Denotem també, com en eixa observació,

$$\beta := \sup\left\{\langle -\Delta_m u, u\rangle \,:\, u \in H(X,\nu),\ \|u\|_{L^2(X,\nu)} = 1\right\} \in \sigma(-\Delta_m).$$

Definim

$$A := \sup\left\{\lambda \geqslant 0 \,:\, \lambda \mathcal{H}_m(f) \leqslant \int_X (-\Delta_m f)^2 d\nu \quad \forall f \in L^2(X,\nu)\right\}.$$

Vegem que $\alpha \leqslant A$. Sigui $(P_\lambda)_{\lambda \geqslant 0}$ la projecció espectral de l'operador autoadjunt i positiu $-\Delta_m : H(X,\nu) \to H(X,\nu)$. Pel teorema espectral [**140**, Theorem VIII. 6], tenim que, per a qualsevol $f \in H(X,\nu)$,

$$\mathcal{H}_m(f) = \langle -\Delta_m f, f\rangle = \int_\alpha^\beta \lambda d\langle P_\lambda f, f\rangle$$

i

$$\int_X (-\Delta_m f)^2 d\nu = \langle -\Delta_m f, -\Delta_m f\rangle = \int_\alpha^\beta \lambda^2 d\langle P_\lambda f, f\rangle.$$

Per tant, per a tot $f \in H(X,\nu)$,

$$\int_X (-\Delta_m f)^2 d\nu \geqslant \alpha \int_\alpha^\beta \lambda d\langle P_\lambda f, f\rangle = \alpha \mathcal{H}_m(f),$$

i obtenim que $\alpha \leqslant A$ (observeu que, per a qualsevol $f \in L^2(X,\nu)$, podem agafar $g := f - \nu(f) \in H(X,\nu)$ i es compleix que $\Delta_m(g) = \Delta_m(f)$).

Finalment, vegem que $\alpha \geqslant A$. Com que $\alpha \in \sigma(\Delta_m)$, donat $\epsilon > 0$, existeix $0 \neq f \in \text{Rang}(P_{\alpha+\epsilon})$ i, en conseqüència, $P_\lambda f = f$ per a $\lambda \geqslant \alpha + \epsilon$. Així doncs, com que Δ_m és ergòdic, $-\Delta_m(f) \neq 0$ ($0 \neq f \in H(X,\nu)$ no és ν-gairebé pertot igual a una constant), doncs

$$\begin{aligned} 0 < \int_X (-\Delta_m f)^2 d\nu &= \int_\alpha^{\alpha+\epsilon} \lambda^2 d\langle P_\lambda f, f\rangle \leqslant (\alpha+\epsilon) \int_\alpha^{\alpha+\epsilon} \lambda d\langle P_\lambda f, f\rangle = (\alpha+\epsilon)\mathcal{H}_m(f) \\ &< (\alpha + 2\epsilon)\mathcal{H}_m(f). \end{aligned}$$

Això implica que $\alpha + 2\epsilon$ no pertany al conjunt

$$\left\{\lambda \geqslant 0 \,:\, \lambda \mathcal{H}_m(f) \leqslant \int_X (-\Delta_m f)^2 d\nu \quad \forall f \in L^2(X,\nu)\right\},$$

així doncs, $A < \alpha + 2\epsilon$. Per tant, com que $\epsilon > 0$ era arbitrari, tenim que

$$A \leqslant \alpha. \qquad \square$$

0.6.1. Desigualtats de tipus Poincaré en subconjunts. Considerem ara desigualtats de tipus Poincaré en subconjunts.

DEFINITION 0.80. Siguin $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori i $A, B \in \mathcal{B}$ conjunts disjunts tals que $\nu(A) > 0$. Denotem $Q := ((A \cup B) \times (A \cup B)) \backslash (B \times B)$. Diem que $[X, \mathcal{B}, m, \nu]$ satisfà una (p, q)*-desigualtat de tipus Poincaré generalitzada* ($p, q \in [1, +\infty[$) *en* (A, B), si, donat $0 < l \leqslant \nu(A \cup B)$, existeix una constant $\Lambda > 0$ tal que, per a qualsevol $u \in L^q(A \cup B, \nu)$ i qualsevol $Z \in \mathcal{B}_{A \cup B}$ amb $\nu(Z) \geqslant l$,

$$\|u\|_{L^p(A \cup B, \nu)} \leqslant \Lambda \left(\left(\int_Q |u(y) - u(x)|^q dm_x(y) d\nu(x) \right)^{\frac{1}{q}} + \left| \int_Z u \, d\nu \right| \right).$$

REMARK 0.81. Aquestes notacions ens permeten cobrir moltes situacions. Per exemple,
(i) Si $A = X$, $B = \emptyset$ i $[X, \mathcal{B}, m, \nu]$ satisfà una $(2, 2)$-desigualtat de tipus Poincaré generalitzada en $(X, \emptyset)$ llavors $[X, \mathcal{B}, m, \nu]$ satisfà una desigualtat de Poincaré tal com es defineix a la Definició 0.66.
(ii) Sigui $\Omega \in \mathcal{B}$. Si $A := \Omega$, $B := \partial_m \Omega$ i suposem que se satisfà una (p, p)-desigualtat de tipus Poincaré generalitzada en (A, B) llavors la desigualtat adopta la forma següent:

$$\|u\|_{L^p(\Omega_m, \nu)} \leqslant \Lambda \left(\left(\int_{(\Omega_m \times \Omega_m) \backslash (\partial_m \Omega \times \partial_m \Omega)} |u(y) - u(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + \left| \int_Z u \, d\nu \right| \right)$$

que s'utilitzarà extensament al capítol 5. A més, si $A := \Omega_m$ i $B := \emptyset$ obtindrem

$$\|u\|_{L^p(\Omega_m, \nu)} \leqslant \Lambda \left(\left(\int_{(\Omega_m \times \Omega_m)} |u(y) - u(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + \left| \int_Z u \, d\nu \right| \right)$$

que també s'utilitzarà àmpliament al capítol 5.

Al Teorema 0.83 donem condicions suficients perquè un espai de passeig aleatori satisfaci desigualtats d'aquest tipus. Primer demostrem el següent lema.

LEMMA 0.82. *Sigui* $[X, \mathcal{B}, m, \nu]$ *un espai de passeig aleatori reversible. Siguin* $A, B \in \mathcal{B}$ *conjunts disjunts tals que* $B \subset \partial_m A$, $\nu(A) > 0$ *i* A *és* m*-connex (recordeu la Definició 0.32). Suposem que* $\nu(A \cup B) < \infty$ *i*

$$\nu\left(\{x \in A \cup B \, : \, (m_x \llcorner A) \perp (\nu \llcorner A)\}\right) = 0.$$

Donat $q \geqslant 1$, *sigui* $\{u_n\}_n \subset L^q(A \cup B, \nu)$ *una successió fitada en* $L^1(A \cup B, \nu)$ *tal que*

$$\lim_{n \to \infty} \int_Q |u_n(y) - u_n(x)|^q dm_x(y) d\nu(x) = 0 \tag{0.7}$$

on, com a la Definició 0.80, $Q = ((A \cup B) \times (A \cup B)) \backslash (B \times B)$. *Llavors, existeix* $\lambda \in \mathbb{R}$ *tal que*

$$u_n(x) \to \lambda \quad \textit{per a } \nu\textit{-quasi tot } x \in A \cup B,$$
$$\|u_n - \lambda\|_{L^q(A, m_x)} \to 0 \quad \textit{per a } \nu\textit{-quasi tot } x \in A \cup B,$$

i

$$\|u_n - \lambda\|_{L^q(A \cup B, m_x)} \to 0 \quad \textit{per a } \nu\textit{-quasi tot } x \in A.$$

PROOF. Si $B = \emptyset$ (o $\nu(B) = 0$) es poden ometre alguns passos de la prova. Sigui

$$F_n(x, y) = |u_n(y) - u_n(x)|, \quad (x, y) \in Q,$$

$$f_n(x) = \int_A |u_n(y) - u_n(x)|^q \, dm_x(y), \quad x \in A \cup B,$$

i

$$g_n(x) = \int_{A \cup B} |u_n(y) - u_n(x)|^q \, dm_x(y), \quad x \in A.$$

Preneu
$$\mathcal{N}_\perp := \{x \in A \cup B \,:\, (m_x \llcorner A) \perp (\nu \llcorner A)\}\,.$$
De (0.7), se segueix que
$$f_n \to 0 \quad \text{en } L^1(A \cup B, \nu)$$
i
$$g_n \to 0 \quad \text{en } L^1(A, \nu).$$
Passant a una subsuccessió si cal, podem suposar que

$$(0.8) \qquad f_n(x) \to 0 \quad \text{per a tot } x \in (A \cup B)\backslash N_f, \quad \text{on } N_f \subset A \cup B \text{ és } \nu\text{-nul}$$

i

$$(0.9) \qquad g_n(x) \to 0 \quad \text{per a tot } x \in A\backslash N_g, \quad \text{on } N_g \subset A \text{ és } \nu\text{-nul.}$$

D'altra banda, per (0.7), també tenim que
$$F_n \to 0 \quad \text{en } L^q(Q, \nu \otimes m_x).$$
Per tant, podem suposar que, agafant una subsuccessió si cal,

$$(0.10) \qquad F_n(x,y) \to 0 \quad \text{per a tot } (x,y) \in Q\backslash C, \quad \text{on } C \subset Q \text{ és } \nu \otimes m_x\text{-nul.}$$

Sigui $N_1 \subset A$ un conjunt ν-nul que satisfaci
$$\text{per a tot } x \in A\backslash N_1, \text{ la secció } C_x := \{y \in A \cup B : (x,y) \in C\} \text{ de } C \text{ és } m_x\text{-nul·la},$$
i $N_2 \subset A \cup B$ un conjunt ν-nul que satisfaci
$$\text{per a tot } x \in (A \cup B)\backslash N_2, \text{ la secció } C'_x := \{y \in A : (x,y) \in C\} \text{ de } C \text{ és } m_x\text{-nul·la.}$$

Ara bé, com que A és m-connex i $B \subset \partial_m A$, tenim que
$$D := \{x \in A \cup B : m_x(A) = 0\}$$
és ν-nul. De fet, per la definició de D, tenim que $L_m(A \cap D, A) = 0$ per tant, atès que A és m-connex, s'ha de satisfer que $\nu(A \cap D) = 0$. A més, com que $B \subset \partial_m A$, $m_x(A) > 0$ per a tot $x \in B$, per tant $\nu(B \cap D) = 0$.

Denoteu $N := \mathcal{N}_\perp \cup N_f \cup N_g \cup N_1 \cup N_2 \cup D$ (tingueu en compte que $\nu(N) = 0$). Fixeu $x_0 \in A\backslash N$. Agafant una subsuccessió si cal, tenim que $u_n(x_0) \to \lambda$ per algun $\lambda \in [-\infty, +\infty]$; sigui
$$S := \{x \in A \cup B \,:\, u_n(x) \to \lambda\}$$
i vegem que $\nu((A \cup B)\backslash S) = 0$.

Per (0.10), com que $u_n(x_0) \to \lambda$, també tenim que $u_n(y) \to \lambda$ per a tot $y \in (A \cup B)\backslash C_{x_0}$. Tanmateix, com que $x_0 \notin \mathcal{N}_\perp$ i $m_{x_0}(C_{x_0}) = 0$, s'ha de satisfer que $\nu(A\backslash C_{x_0}) > 0$; per tant $\nu(A \cap S) \geqslant \nu(A\backslash C_{x_0}) > 0$. Tingueu en compte que, si $x \in (A \cap S)\backslash N$, per (1.18) de nou, $(A \cup B)\backslash C_x \subset S$, així doncs $m_x((A \cup B)\backslash S) \leqslant m_x(C_x) = 0$; llavors,
$$L_m(A \cap S, (A \cup B)\backslash S) = 0.$$
En particular, $L_m(A \cap S, A\backslash S) = 0$, però, com que A és m-connex i $\nu(A \cap S) > 0$, s'ha de tenir que $\nu(A\backslash S) = 0$, és a dir, $\nu(A) = \nu(A \cap S)$.

Ara, suposem que $\nu(B\backslash S) > 0$. Preneu $x \in B\backslash(S \cup N)$. Per (0.10), tenim que $A\backslash C'_x \subset A\backslash S$, és a dir, $A \cap S \subset C'_x$, per tant $m_x(A \cap S) = 0$. Conseqüentment, com que $x \notin \mathcal{N}_\perp$, hem de tenir $\nu(A\backslash S) > 0$, cosa que contradiu el que ja hem obtingut. En conseqüència, hem obtingut que u_n convergeix ν-gairebé pertot en $A \cup B$ a λ:
$$u_n(x) \to \lambda \quad \text{per a tot } x \in S, \; \nu((A \cup B)\backslash S) = 0.$$

Ja que $\{\|u_n\|_{L^1(A \cup B, \nu)}\}_n$ està fitada, pel Lema de Fatou hem de tenir que $\lambda \in \mathbb{R}$. D'altra banda, per (0.8),
$$F_n(x, \cdot) \to 0 \quad \text{in } L^q(A, m_x)\;,$$
per a tot $x \in \Omega\backslash N_f$. En altres paraules, $\|u_n(\cdot) - u_n(x)\|_{L^q(A, m_x)} \to 0$, doncs
$$\|u_n - \lambda\|_{L^q(A, m_x)} \to 0 \quad \text{per a } \nu\text{-quasi tot } x \in A \cup B.$$

De la mateixa manera, per (0.9),

$$\|u_n - \lambda\|_{L^q(A\cup B, m_x)} \to 0 \quad \text{per a } \nu\text{-quasi tot } x \in A. \qquad \square$$

THEOREM 0.83. *Siguin $p \geqslant 1$ i $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori reversible. Suposem que $A, B \in \mathcal{B}$ són conjunts disjunts tals que $B \subset \partial_m A$, $\nu(A) > 0$ i A és m-connex. Assumiu que $\nu(A \cup B) < \infty$ i*

$$\nu\left(\{x \in A \cup B \,:\, (m_x \llcorner A) \perp (\nu \llcorner A)\}\right) = 0.$$

Suposem també que donat qualsevol conjunt ν-nul $N \subset A$, existeixen $x_1, x_2, \ldots, x_L \in A \backslash N$ i una constant $C > 0$ tal que $\nu \llcorner (A \cup B) \leqslant C(m_{x_1} + \cdots + m_{x_L}) \llcorner (A \cup B)$. Aleshores, $[X, \mathcal{B}, m, \nu]$ satisfà una (p,p)-desigualtat de tipus Poincaré generalitzada en (A, B).

PROOF. Siguin $p \geqslant 1$ i $0 < l \leqslant \nu(A \cup B)$. Volem demostrar que existeix una constant $\Lambda > 0$ tal que

$$\|u\|_{L^p(A\cup B,\nu)} \leqslant \Lambda \left(\left(\int_Q |u(y) - u(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + \left| \int_Z u \, d\nu \right| \right)$$

per a tot $u \in L^p(A \cup B, \nu)$ i tot $Z \in \mathcal{B}_{A\cup B}$ amb $\nu(Z) \geqslant l$. Suposem que aquesta desigualtat no es compleix per a cap Λ. Llavors, existeix una successió $\{u_n\}_{n\in\mathbb{N}} \subset L^p(A \cup B, \nu)$, amb $\|u_n\|_{L^p(A\cup B,\nu)} = 1$, i una successió $Z_n \in \mathcal{B}_{A\cup B}$ amb $\nu(Z_n) \geqslant l$, $n \in \mathbb{N}$, que satisfan

$$\lim_n \int_Q |u_n(y) - u_n(x)|^p dm_x(y) d\nu(x) = 0$$

i

$$\lim_n \int_{Z_n} u_n \, d\nu = 0.$$

Per tant, pel Lemma 0.82, existeixen $\lambda \in \mathbb{R}$ i un conjunt ν-nul $N \subset A$ tal que

$$\|u_n - \lambda\|_{L^p(A\cup B, m_x)} \xrightarrow{n} 0 \quad \text{per a tot } x \in A \backslash N.$$

Ara, per hipòtesi, existeixen $x_1, x_2, \ldots, x_L \in A \backslash N$ i $C > 0$ tal que $\nu \llcorner (A \cup B) \leqslant C(m_{x_1} + \cdots + m_{x_L})$. Així doncs,

$$\|u_n - \lambda\|^p_{L^p(A\cup B,\nu)} \leqslant C \sum_{i=1}^{L} \|u_n - \lambda\|^p_{L^p(A\cup B, m_{x_i})} \xrightarrow{n} 0.$$

A més, atès que $\{\, 1_{Z_n}\}_n$ està fitada en $L^{p'}(A \cup B, \nu)$, existeix $\phi \in L^{p'}(A \cup B, \nu)$ tal que, agafant una subsuccessió si cal, $\chi_{Z_n} \rightharpoonup \phi$ dèbil en $L^{p'}(A \cup B, \nu)$ (dèbil-$*$ en $L^\infty(A \cup B, \nu)$ en el cas $p = 1$)[8]. A més a més, $\phi \geqslant 0$ ν-gairebé pertot en $A \cup B$ i

$$0 < l \leqslant \lim_{n\to+\infty} \nu(Z_n) = \lim_{n\to+\infty} \int_{A\cup B} \chi_{Z_n} d\nu = \int_{A\cup B} \phi d\nu.$$

Llavors, com que $u_n \xrightarrow{n} \lambda$ en $L^p(A \cup B, \nu)$ i $\chi_{Z_n} \overset{n}{\rightharpoonup} \phi$ dèbil en $L^{p'}(A \cup B, \nu)$ (dèbil-$*$ en $L^\infty(A \cup B, \nu)$ en el cas $p = 1$),

$$0 = \lim_{n\to+\infty} \int_{Z_n} u_n = \lim_{n\to+\infty} \int_{A\cup B} \chi_{Z_n} u_n = \lambda \int_{A\cup B} \phi d\nu,$$

doncs $\lambda = 0$. Açò està en contradicció amb $||u_n||_{L^p(A\cup B,\nu)} = 1$ $\forall n \in \mathbb{N}$, atès que $u_n \xrightarrow{n} \lambda$ en $L^p(A \cup B, \nu)$. $\square$

[8]Tingueu en compte que, atès que ν és una mesura σ-finita i $\mathcal{B}$ es genera de manera contable, tenim que $L^1(X, \nu)$ és separable.

Remark 0.84. Observeu que el supòsit

$$\nu\left(\{x \in A \cup B \,:\, (m_x \llcorner A) \perp (\nu \llcorner A)\}\right) = 0$$

significa que ens trobarem en el cas (i) de la proposició 0.22, és a dir, sense tenir en compte un conjunt ν-nul, el passeig aleatori és ν-irreductible.

Remark 0.85.
(i) La suposició que, donat un conjunt ν-nul $N \subset A$, existeixen $x_1, x_2, \ldots, x_L \in A \backslash N$ i $C > 0$ de manera que $\nu \llcorner (A \cup B) \leqslant C(m_{x_1} + \cdots + m_{x_L}) \llcorner (A \cup B)$ no és tan forta com sembla. De fet, això es compleix de manera trivial en grafs ponderats, localment finits i connexos; i també es compleix a $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ (recordeu l'Exemple 0.37) si, per a un domini $A \subset \mathbb{R}^N$, prenem $B \subset \partial_{m^J} A$ tal que $\text{dist}(B, \mathbb{R}^N \backslash A_{m^J}) > 0$. A més, en l'exemple següent veiem que si eliminem aquesta hipòtesi, la tesi no és certa en general.

Considereu l'espai mètric de passeig aleatori $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ on d és la distància euclidiana i $J := \frac{1}{2}\chi_{[-1,1]}$ (recordeu l'Exemple 0.37). Sigui $A := [-1,1]$ i $B := \partial_{m^J} A = [-2,2] \backslash A$. Aleshores, si $N = \{-1, 1\}$, és possible que no trobem punts a $A \backslash N$ que compleixin el supòsit esmentat. De fet, la tesi del teorema no és certa per a cap $p \geqslant 1$ com es pot veure prenent $u_n := \frac{1}{2} n^{\frac{1}{p}} \left(\chi_{[-2,-2+\frac{1}{n}]} - \chi_{[2-\frac{1}{n},2]}\right)$ i $Z := A \cup B$. De fet, noteu primer que $\|u_n\|_{L^p([-2,2],\mathcal{L}^1)} = 1$ i $\int_{[-2,2]} u_n d\mathcal{L}^1 = 0$ per a tot $n \in \mathbb{N}$. Ara, $\text{supp}(m_x^J) = [x-1, x+1]$ per a tot $x \in [-1,1]$ i, per tant, per a qualsevol $x \in [-1,1]$,

$$\begin{aligned}
\int_{[-2,2]} |u_n(y) - u_n(x)|^p dm_x^J(y) &= \int_{[-2,-2+\frac{1}{n}] \cap [x-1,x+1]} n \, dm_x^J(y) \\
&\quad + \int_{[2-\frac{1}{n},2] \cap [x-1,x+1]} n \, dm_x^J(y) \\
&= 2n\chi_{[1-\frac{1}{n},1]}(x) \int_{[2-\frac{1}{n},x+1]} dm_x^J(y) \\
&= 2n\left(x - 1 + \frac{1}{n}\right)\chi_{[1-\frac{1}{n},1]}(x).
\end{aligned}$$

Conseqüentment,

$$\begin{aligned}
\int_{[-1,1]} \int_{[-2,2]} |u_n(y) - u_n(x)|^p dm_x^J(y) d\mathcal{L}^1(x) &= 2n \int_{[1-\frac{1}{n},1]} \left(x - 1 + \frac{1}{n}\right) d\mathcal{L}^1(x) \\
&= 2n\left(\frac{1}{2} - \frac{(1-\frac{1}{n})^2}{2} - \frac{1}{n} + \frac{1}{n^2}\right) = \frac{1}{n}.
\end{aligned}$$

Finalment, per la reversibilitat de $\mathcal{L}^1$ respecte a m^J,

$$\int_{[-2,2]} \int_{[-1,1]} |u_n(y) - u_n(x)|^p dm_x^J(y) d\mathcal{L}^1(x) = \frac{1}{n},$$

doncs

$$\int_Q |u_n(y) - u_n(x)|^p dm_x^J(y) d\mathcal{L}^1(x) \leqslant \frac{2}{n} \xrightarrow{n} 0.$$

(ii) Tanmateix, en aquest exemple, com hem esmentat anteriorment, podem prendre $B \subset \partial_m A$ tal que $\text{dist}(B, \mathbb{R} \backslash [-2,2]) > 0$ per evitar aquest problema i assegurar-se que es compleixen les hipòtesis del teorema; de manera que $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ compleixi una (p,p)-desigualtat de tipus Poincaré generalitzada en (A, B).

En l'exemple següent, l'espai mètric de passeig aleatori $[X, d, m, \nu]$ definit, satisfà que $m_x \perp \nu$ per a tot $x \in X$ (caient així en el cas *(ii)* de la Proposició 0.22) i no compleix una desigualtat de tipus Poincaré.

EXAMPLE 0.86. Sigui $p > 1$. Denoteu la circumferència com $S^1 = \{e^{2\pi i\alpha} : \alpha \in [0, 1)\}$ i prenem $T_\theta : S^1 \longrightarrow S^1$ la funció de rotació irracional $T_\theta(x) = xe^{2\pi i\theta}$ on θ és un nombre irracional. A S^1 considerem la σ-àlgebra Borel $\mathcal{B}$ i la mesura de Hausdorff 1-dimensional $\nu := \mathcal{H}_1 \llcorner S^1$. És ben sabut que T_θ és una transformació unívocament ergòdica sobre $(S^1, \mathcal{B}, \nu)$.

Ara, definim $X := S^1$ i $m_x := \frac{1}{2}\delta_{T_{-\theta}(x)} + \frac{1}{2}\delta_{T_\theta(x)}$, $x \in X$. Llavors, $[X, d, m, \nu]$ és un espai de passeig aleatori reversible, on d és la mètrica donada per la longitud d'arc. De fet, donada una funció mesurable i fitada f a $(X \times X, \mathcal{B} \times \mathcal{B})$, tenim que

$$\begin{aligned}
\int_{S^1}\int_{S^1} f(x,y)dm_x(y)d\nu(x) &= \frac{1}{2}\int_{S^1} f(x, T_{-\theta}(x))d\nu(x) + \frac{1}{2}\int_{S^1} f(x, T_\theta(x))d\nu(x) \\
&= \frac{1}{2}\int_{S^1} f(T_\theta(x), x)d\nu(x) + \frac{1}{2}\int_{S^1} f(T_{-\theta}(x), x)d\nu(x) \\
&= \int_{S^1}\int_{S^1} f(y,x)dm_x(y)d\nu(x).
\end{aligned}$$

Vegem que $[X, d, m, \nu]$ és m-connex. Noteu primer que, per a tot $x \in X$,

$$m_x^{*2} := \frac{1}{2}\delta_x + \frac{1}{4}\delta_{T^2_{-\theta}(x)} + \frac{1}{4}\delta_{T^2_\theta(x)} \geqslant \frac{1}{4}\delta_{T^2_\theta(x)}$$

i, per inducció, és fàcil veure que

$$m_x^{*n} \geqslant \frac{1}{2^n}\delta_{T^n_\theta(x)}\ .$$

Ara, sigui $A \in \mathcal{B}$ tal que $\nu(A) > 0$. Pel teorema ergòdic puntual obtenim que

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_A\left(T^k_\theta(x)\right) = \frac{\nu(A)}{\nu(X)} > 0$$

per a ν-quasi tot $x \in X$. En conseqüència, per a ν-quasi tot $x \in X$, existeix $k \in \mathbb{N}$ de manera que

$$m_x^{*k}(A) \geqslant \frac{1}{2^k}\delta_{T^k_\theta(x)}(A) = \frac{1}{2^k}\chi_A\left(T^k_\theta(x)\right) > 0,$$

doncs $[X, d, m, \nu]$ és m-connex.

Vegem que $[X, d, m, \nu]$ no satisfà una (p, p)-desigualtat de Poincaré. Per a cada $n \in \mathbb{N}$ definim

$$I^n_k := \left\{e^{2\pi i\alpha} : k\theta - \delta(n) < \alpha < k\theta + \delta(n)\right\}, \quad -1 \leqslant k \leqslant 2n,$$

on $\delta(n) > 0$ es tria de manera que

$$I^n_{k_1} \cap I^n_{k_2} = \emptyset \ \text{ per a tot } -1 \leqslant k_1, k_2 \leqslant 2n,\ k_1 \neq k_2$$

(noteu que $e^{2\pi i(k_1\theta-\delta(n))} \neq e^{2\pi i(k_2\theta-\delta(n))}$ per a qualsevol $k_1 \neq k_2$ ja que T_θ és ergòdic). Considereu la següent successió de funcions:

$$u_n := \sum_{k=0}^{n-1} \chi_{I^n_k} - \sum_{k=n}^{2n-1} \chi_{I^n_k},\ n \in \mathbb{N}.$$

Així,

$$\int_X u_n d\nu = 0, \quad \text{per a tot } n \in \mathbb{N},$$

i

$$\int_X |u_n|^p d\nu = 4n\delta(n), \quad \text{per a tot } n \in \mathbb{N}.$$

Fixat $n \in \mathbb{N}$, vegem què passa amb

$$\int_X\int_X |u_n(y) - u_n(x)|^p dm_x(y)d\nu(x).$$

Si $1 \leqslant k \leqslant n-2$ o $n+1 \leqslant k \leqslant 2n-2$ i $x \in I_k^n$ llavors

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = 0$$

atès que $T_{-\theta}(x) \in I_{k-1}^n$ i $T_\theta(x) \in I_{k+1}^n$. Ara, si $x \in I_0^n$, aleshores $T_{-\theta}(x) \in I_{-1}^n$ doncs

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = \frac{1}{2}|-1|^p = \frac{1}{2}$$

i el mateix val si $x \in I_{2n-1}^n$ (llavors $T_\theta(x) \in I_{2n}^n$). Per a qualsevol $x \in I_{n-1}$ tenim que $T_\theta(x) \in I_n^n$ i, per tant,

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = \frac{1}{2}|-2|^p = 2^{p-1}$$

i s'obté el mateix resultat per a $x \in I_{n+1}^n$. De la mateixa manera, si $x \in I_{-1}^n$ o $x \in I_{2n}^n$,

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = \frac{1}{2}.$$

Finalment, si $x \notin \cup_{k=-1}^{2n} I_k^n$ llavors $T_{-\theta}(x), T_\theta(x) \notin \cup_{k=0}^{2n-1} I_k^n$ doncs

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = 0.$$

Conseqüentment,

$$\int_X \int_X |u_n(y) - u_n(x)|^p dm_x(y) d\nu(x) = \frac{1}{2}(4 \cdot 2\delta(n)) + 2^{p-1}(2 \cdot 2\delta(n)) = (4 + 2^{p+1})\delta(n).$$

Per tant, no existeix $\Lambda > 0$ tal que

$$\left\| u_n - \frac{1}{2\pi} \int_X u_n d\nu \right\|_{L^p(X,\nu)}^p \leqslant \Lambda \int_X \int_X |u_n(y) - u_n(x)|^p dm_x(y) d\nu(x), \ \forall n \in \mathbb{N}$$

ja que això implicaria

$$4n\delta(n) \leqslant \Lambda(4 + 2^{p+1})\delta(n) \Longrightarrow n \leqslant \Lambda + 2^{p-1}, \ \forall n \in \mathbb{N}.$$

Finalment, proporcionem un altre resultat en què donem condicions suficients perquè se satisfaci una (p,q)-desigualtat de Poincaré generalitzada.

Theorem 0.87. *Siguin $1 \leqslant p < q$ i considerem un espai de passeig aleatori reversible $[X, \mathcal{B}, m, \nu]$. Siguin $A, B \in \mathcal{B}$ conjunts disjunts tals que $B \subset \partial_m A$, $\nu(A) > 0$ i A és m-connex. Suposem també que $\nu(A \cup B) < \infty$ i $m_x \ll \nu$ per a tot $x \in A \cup B$. Assumim a més que, donat un conjunt ν-nul $N \subset A$, existeixen $x_1, x_2, \ldots, x_L \in A \backslash N$ i $\Omega_1, \Omega_2, \ldots, \Omega_L \in \mathcal{B}_{A \cup B}$, tal que $A \cup B = \bigcup_{i=1}^{L} \Omega_i$ i, si $g_i := \dfrac{dm_{x_i}}{d\nu}$ en Ω_i, llavors $g_i^{-\frac{p}{q-p}} \in L^1(\Omega_i, \nu)$, $i = 1, 2, ..., L$. Aleshores, $[X, \mathcal{B}, m, \nu]$ satisfà una (p,q)-desigualtat de tipus Poincaré generalitzada en (A, B).*

Proof. Sigui $0 < l \leqslant \nu(A \cup B)$. Començant com en la demostració del Teorema 0.83, si suposem que no se satisfà una (p,q)-desigualtat de tipus Poincaré generalitzada en (A, B), llavors existeix una successió $\{u_n\}_{n \in \mathbb{N}} \subset L^q(A \cup B, \nu)$, amb $\|u_n\|_{L^p(A \cup B, \nu)} = 1$, i una successió $Z_n \in \mathcal{B}_{A \cup B}$ amb $\nu(Z_n) \geqslant l$, $n \in \mathbb{N}$, tal que

$$\lim_n \int_Q |u_n(y) - u_n(x)|^q dm_x(y) d\nu(x) = 0$$

i

$$\lim_n \int_{Z_n} u_n \, d\nu = 0.$$

Per tant, pel Lemma 0.82, existeixen $\lambda \in \mathbb{R}$ i un conjunt ν-nul $N \subset A$ tal que

$$\|u_n - \lambda\|_{L^q(A \cup B, m_x)} \xrightarrow{n} 0 \quad \text{per a tot } x \in A \backslash N.$$

Ara, per hipòtesi, existeixen $x_1, x_2, \ldots, x_L \in A\backslash N$ i $\Omega_1, \Omega_2, \ldots, \Omega_L \in \mathcal{B}_{A\cup B}$, de manera que $A \cup B = \bigcup_{i=1}^{L} \Omega_i$ i, si $g_i := \frac{dm_{x_i}}{d\nu}$ on Ω_i, aleshores $g_i^{-\frac{p}{q-p}} \in L^1(\Omega_i, \nu)$, $i = 1, 2, ..., L$. Per tant,

$$
\begin{aligned}
\|u_n - \lambda\|^p_{L^p(A\cup B,\nu)} &\leqslant \sum_{i=1}^{L} \int_{\Omega_i} |u_n(y) - \lambda|^p d\nu(y) \\
&= \sum_{i=1}^{L} \int_{\Omega_i} |u_n(y) - \lambda|^p \frac{g_i(y)^{\frac{p}{q}}}{g_i(y)^{\frac{p}{q}}} d\nu(y) \\
&\leqslant \sum_{i=1}^{L} \left(\int_{\Omega_i} |u_n(y) - \lambda|^q g_i(y) d\nu(y) \right)^{\frac{p}{q}} \left(\int_{\Omega_i} \frac{1}{g_i(y)^{\frac{p}{q-p}}} d\nu(y) \right)^{\frac{q-p}{q}} \\
&= \sum_{i=1}^{L} \left(\int_{\Omega_i} |u_n(y) - \lambda|^q dm_{x_i}(y) \right)^{\frac{p}{q}} \left\| \frac{1}{g_i(y)^{\frac{p}{q-p}}} \right\|^{\frac{q-p}{q}}_{L^1(\Omega_i,\nu)} \\
&= \sum_{i=1}^{L} \|u_n - \lambda\|^p_{L^q(\Omega_i, m_{x_i})} \left\| \frac{1}{g_i^{\frac{p}{q-p}}} \right\|^{\frac{q-p}{q}}_{L^1(\Omega_i,\nu)} \xrightarrow{n} 0.
\end{aligned}
$$

Acabem la prova de la mateixa manera que per al Teorema 0.83. □

0.6.2. Desigualtat isoperimètrica. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori on ν és una mesura de probabilitat. Suposem que $[X, \mathcal{B}, m, \nu]$ compleix una desigualtat de Poincaré, és a dir, existeix $\lambda > 0$ tal que

$$\lambda \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{per a tot } f \in L^2(X, \nu).$$

Donat $D \in \mathcal{B}$, sigui $f := \chi_D$. Llavors, la desigualtat de Poincaré implica que

$$\lambda \mathrm{Var}_\nu(\chi_D) \leqslant \mathcal{H}_m(\chi_D),$$

equació que, recordant

$$P_m(D) = \int_X \int_X |\nabla \chi_D(x,y)| dm_x(y) d\nu(x) = \int_X \int_X \nabla \chi_D(x,y)^2 dm_x(y) d\nu(x),$$

es pot reescriure com

$$\lambda\, \nu(D)(1 - \nu(D)) \leqslant P_m(D) \quad \text{per a tot } D \in \mathcal{B} \tag{0.11}$$

(observeu que, pel teorema 0.63, açò implica, en particular, que Δ_m és ergòdic). Així doncs, com que

$$\min\{x, 1-x\} \leqslant 2x(1-x) \leqslant 2\min\{x, 1-x\} \quad \text{per a qualsevol } 0 \leqslant x \leqslant 1,$$

la desigualtat (0.11) produeix la següent desigualtat isoperimètrica (vegeu [**7**, Theorem 3.46]):

$$\min\{\nu(D), 1 - \nu(D)\} \leqslant \frac{2}{\lambda} P_m(D) \quad \text{per a tot } D \in \mathcal{B}; \tag{0.12}$$

i, al contrari, la desigualtat isoperimètrica (0.12) implica que

$$\frac{\lambda}{2} \nu(D)(1 - \nu(D)) \leqslant P_m(D) \quad \text{per a tot } D \in \mathcal{B}.$$

Definition 0.88. Sigui $[X, \mathcal{B}, m, \nu]$ un espai de passeig aleatori. Si existeix $\lambda > 0$ tal que (0.12) es compleix, direm que $[X, \mathcal{B}, m, \nu]$ compleix una *desigualtat isoperimètrica*.

0.7. La curvatura d'Ollivier-Ricci

Una eina important per a l'estudi de la velocitat de convergència del flux de la calor és la desigualtat de Poincaré (vegeu [**22**]). En el cas de les varietats riemannianes i els semigrups de difusió de Markov, una condició necessària habitual per obtenir aquesta desigualtat funcional és la positivitat de la corresponent curvatura de Ricci de l'espai subjacent (vegeu [**22**] i [**160**]). A [**21**], Bakry i Émery van trobar una manera de definir una noció de curvatura de Ricci fitada inferiorment a través del flux de la calor. A més, Renesse i Sturm [**141**] van demostrar que, en una varietat riemanniana M, la curvatura de Ricci està fitada inferiorment per una constant $K \in \mathbb{R}$ si, i només si, l'entropia de Boltzmann-Shannon és K-convexa al llarg de les geodèsiques de l'espai 2-Wasserstein de mesures de probabilitat en M. Aquesta va ser l'observació clau, utilitzada simultàniament per Lott i Villani [**113**] i Sturm [**147**], per donar una noció de curvatura de Ricci fitada inferiorment en el context general dels espais mètrics mesurables de longitud. En aquests espais, Ambrosio, Gigli i Savaré ([**8**]) van obtenir una relació entre la condició de curvatura-dimensió de Bakry-Émery i la noció de la curvatura de Ricci introduïda per Lott-Villani-Sturm, de fet, van demostrar que aquestes dues nocions de curvatura de Ricci coincideixen sota certs supòsits sobre l'espai mètric mesurable.

Quan l'espai considerat és discret, per exemple, en el cas d'un graf, el concepte anterior de curvatura de Ricci fitada inferiorment no es pot aplicar com en el cas continu. De fet, la definició de Lott-Sturm-Villani no és aplicable si l'espai 2-Wasserstein de l'espai mètric mesurable no conté geodèsiques. Malauradament, aquest és el cas si l'espai subjacent és discret. Per tant, utilitzarem el concepte de curvatura de Ricci fitada inferiorment donat per Y. Ollivier a [**134**] que s'adapta bé al cas discret. Ens referim a [**132**] i les seves referències per obtenir una visió general sobre el camp de recerca de la curvatura discreta.

Per introduir la curvatura de Ricci gruixuda definida per Y. Ollivier a [**134**], primer recordem el problema del transport de masses de Monge-Kantorovich. Sigui (X, d) un espai mètric polonès i μ, $\nu \in \mathcal{P}(X)$[9]. El problema de Monge-Kantorovich és el problema de minimització

$$\min \left\{ \int_{X \times X} d(x,y)\, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

on $\Pi(\mu,\nu) := \{\gamma \in \mathcal{P}(X \times X) \ : \ \pi_0 \# \gamma = \mu,\ \pi_1 \# \gamma = \nu\}$[10] i $\pi_\alpha : X \times X \to X$ està definit per $\pi_\alpha(x,y) := x + \alpha(y-x)$ per a $\alpha \in \{0,1\}$.

Per a $1 \leqslant p < \infty$, la *p-distància de Wasserstein* entre μ i ν es defineix com

$$W_p^d(\mu,\nu) := \left(\min \left\{ \int_{X \times X} d(x,y)^p\, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\} \right)^{\frac{1}{p}}.$$

El problema Monge-Kantorovich té una formulació dual que donem a continuació (vegeu, per exemple, [**159**, Theorem 1.14]).

Teorema de Kantorovich-Rubinstein. *Siguin $\mu, \nu \in \mathcal{P}(X)$. Llavors,*

$$\begin{aligned} W_1^d(\mu,\nu) &= \sup \left\{ \int_X u\, d(\mu - \nu) \ : \ u \in K_d(X) \right\} \\ &= \sup \left\{ \int_X u\, d(\mu - \nu) \ : \ u \in K_d(X) \cap L^\infty(X,\nu) \right\} \end{aligned}$$

on

$$K_d(X) := \{u : X \to \mathbb{R} \ : \ |u(y) - u(x)| \leqslant d(y,x)\}.$$

En geometria riemanniana, la positivitat de la curvatura de Ricci es pot caracteritzar pel fet que "les boles petites estan més a prop, en la 1-distància de Wasserstein, que els seus

[9] $\mathcal{P}(X)$ denota el conjunt de les mesures de probabilitat en X.

[10] $\pi_\alpha \# \gamma$ denota el pushforward de γ per π_α, doncs $\pi_0 \# \gamma$ és el marginal de γ sobre el primer component i $\pi_1 \# \gamma$ és el marginal de γ sobre el segon component.

centres" (vegeu [**141**]). En el marc dels espais mètrics de passeig aleatori, inspirat per això, Y. Ollivier [**134**] va introduir el concepte de *curvatura de Ricci gruixuda, substituint les boles per les mesures* m_x *i utilitzant la* 1*-distància de Wasserstein per mesurar la distància entre elles.*

Definition 0.89. Donat un passeig aleatori m en un espai mètric polonès $[X, d]$ tal que cada mesura m_x té primer moment finit, per a qualsevol parella de punts diferents $x, y \in X$, la *curvatura d'Ollivier-Ricci (o curvatura de Ricci gruixuda) de* $[X, d, m]$ *al llarg de* (x, y) es defineix com

$$\kappa_m(x, y) := 1 - \frac{W_1^d(m_x, m_y)}{d(x, y)}.$$

La *curvatura d'Ollivier-Ricci de* $[X, d, m]$ està definida per

$$\kappa_m := \inf_{\substack{x, y \in X \\ x \neq y}} \kappa_m(x, y).$$

Escriurem $\kappa(x, y)$ en lloc de $\kappa_m(x, y)$, i $\kappa = \kappa_m$, si el context no permet cap confusió.

Tingueu en compte que, en principi, la mètrica d *i el passeig aleatori* m *d'un espai mètric de passeig aleatori* $[X, d, m, \nu]$ *no tenen cap relació a part del fet que* m *està definida a la* σ*-àlgebra de Borel associada a* d *i que cada* m_x, $x \in X$, *té primer moment finit. Per tant, no podem esperar obtenir resultats forts sobre les propietats de* m *imposant condicions només en termes de* d*. Per exemple, com veurem a l'exemple 3.36, les boles en espais mètrics de passeig aleatori no són necessàriament* m*-calibrables (vegeu la Definició 3.33). Tanmateix, imposar condicions sobre* κ*, com* $\kappa > 0$*, crea una relació forta entre el passeig aleatori i la mètrica, que ens permet demostrar resultats com el Teorema 0.93.*

Remark 0.90. Si (X, d, μ) és una varietat riemanniana completa i suau, i $(m_x^{\mu,\epsilon})$ és el passeig aleatori de pas ϵ associat a μ donat a l'Exemple 0.41, llavors està demostrat a [**141**] (vegeu també [**134**]) que, escalant per ϵ^2, $\kappa_{m^{\mu,\epsilon}}(x, y)$ convergeix a la curvatura ordinària de Ricci quan $\epsilon \to 0$.

Example 0.91. Sigui $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ l'espai mètric de passeig aleatori donat a l'exemple 0.37. Vegem que $\kappa(x, y) = 0$. Donats $x, y \in \mathbb{R}^N$, $x \neq y$, pel Teorema de Kantorovich-Rubinstein, tenim que

$$\begin{aligned} W_1^d(m_x^J, m_y^J) &= \sup \left\{ \int_{\mathbb{R}^N} u(z)(J(x - z) - J(y - z))\, dz \ : \ u \in K_d(\mathbb{R}^N) \right\} \\ &= \sup \left\{ \int_{\mathbb{R}^N} (u(x + z) - u(y + z))J(z)\, dz \ : \ u \in K_d(\mathbb{R}^N) \right\}. \end{aligned}$$

Ara, per a tot $u \in K_d(\mathbb{R}^N)$, es compleix que

$$\int_{\mathbb{R}^N} (u(x + z) - u(y + z))J(z)\, dz \leqslant \|x - y\|,$$

doncs $W_1^d(m_x^J, m_y^J) \leqslant \|x - y\|$. D'altra banda, si $u(z) := \frac{\langle z, x-y \rangle}{\|x-y\|}$, aleshores $u \in K_d(\mathbb{R}^N)$ i, per tant,

$$W_1^d(m_x^J, m_y^J) \geqslant \int_{\mathbb{R}^N} (u(x + z) - u(y + z))J(z)\, dz = \|x - y\|.$$

Així doncs,

$$W_1^d(m_x^J, m_y^J) = \|x - y\|,$$

i, en conseqüència, $\kappa(x, y) = 0$.

Example 0.92. Sigui $[V(G), d_G, m^G, \nu_G]$ l'espai mètric de passeig aleatori associat a un graf ponderat i localment finit $G = (V(G), E(G))$ tal com es defineix a l'Exemple 0.38 i recordeu que $N_G(x) := \{z \in V(G) : z \sim x\}$ per a $x \in V(G)$. Llavors, la curvatura d'Ollivier-Ricci al llarg de $(x, y) \in E(G)$ és

$$\kappa(x, y) = 1 - \frac{W_1^{d_G}(m_x, m_y)}{d_G(x, y)},$$

on

$$W_1^{d_G}(m_x, m_y) = \inf_{\mu \in \mathcal{A}} \sum_{z_1 \sim x} \sum_{z_2 \sim y} \mu(z_1, z_2) d_G(z_1, z_2),$$

sent $\mathcal{A}$ el conjunt de totes les matrius amb entrades indexades per $N_G(x) \times N_G(y)$ tal que $\mu(z_1, z_2) \geqslant 0$ i

$$\sum_{z_2 \sim y} \mu(z_1, z_2) = \frac{w_{xz_1}}{d_x}, \quad \sum_{z_1 \sim x} \mu(z_1, z_2) = \frac{w_{yz_2}}{d_y}, \quad \text{per a } (z_1, z_2) \in N_G(x) \times N_G(y).$$

Hi ha una extensa literatura sobre la curvatura d'Ollivier-Ricci en grafs discrets (vegeu, per exemple, [**27**], [**35**], [**62**], [**95**], [**105**], [**112**], [**134**], [**135**], [**136**] *i* [**138**]*).*

En el següent resultat, veiem que els espais mètrics de passeig aleatori amb curvatura d'Ollivier-Ricci positiva són m-connexos.

THEOREM 0.93. *Sigui $[X, d, m, \nu]$ un espai mètric de passeig aleatori tal que ν és una mesura de probabilitat i cada mesura m_x té primer moment finit. Suposem que la curvatura d'Ollivier-Ricci κ satisfà $\kappa > 0$. Aleshores, $[X, d, m, \nu]$ és m-connex.*

PROOF. Sota la hipòtesi $\kappa > -\infty$ (recordeu que $\kappa \leqslant 1$ per definició) Y. Ollivier en [**134**, Proposition 20] demostra la següent propietat de contracció en W_1:

Sigui $[X, d, m, \nu]$ un espai mètric de passeig aleatori. Llavors, per a dues distribucions de probabilitat qualssevol μ i μ',

$$W_1^d(\mu * m^{*n}, \mu' * m^{*n}) \leqslant (1 - \kappa)^n W_1^d(\mu, \mu'). \tag{0.13}$$

Per tant, sota la hipòtesi $\kappa > 0$, en [**134**, Corollary 21] Y. Ollivier demostra que la mesura invariant ν (existeix i) és única llevat de constants multiplicatives i que, si $\nu \in \mathcal{P}(X)$, es compleixen les següents afirmacions:

$$\begin{aligned} &(i) \qquad W_1^d(\mu * m^{*n}, \nu) \leqslant (1 - \kappa)^n W_1^d(\mu, \nu) \quad \forall n \in \mathbb{N},\ \forall \mu \in \mathcal{P}(X), \\ &(ii) \qquad W_1^d(m_x^{*n}, \nu) \leqslant (1 - \kappa)^n \frac{W_1^d(\delta_x, m_x)}{\kappa} \quad \forall n \in \mathbb{N},\ \forall x \in X. \end{aligned} \tag{0.14}$$

Per (0.14) i [**160**, Theorem 6.9][11], tenim que

$$\mu * m^{*n} \rightharpoonup \nu \quad \text{dèbilment com a mesures, } \forall \mu \in \mathcal{P}(X), \tag{0.15}$$

així, prenent $\mu = \delta_x$, obtenim que

$$m_x^{*n} \rightharpoonup \nu \quad \text{dèbilment com a mesures, per a tot } x \in X.$$

Vegem ara que $[X, d, m, \nu]$ és m-connex si $\kappa > 0$. Sigui $D \subset X$ un conjunt de Borel amb $\nu(D) > 0$ i suposem que $\nu(N_D^m) > 0$. Pel Corol·lari 0.28, tenim que $\nu(H_D^m) > 0$. Definim

$$\mu := \frac{1}{\nu(H_D^m)} \nu \llcorner H_D^m \in \mathcal{P}(X),$$

i

$$\mu' := \frac{1}{\nu(N_D^m)} \nu \llcorner N_D^m \in \mathcal{P}(X).$$

Ara, per la Proposició 0.27,

$$\mu * m^{*n} = \mu,$$

[11]**Teorema** Sigui (X, d) un espai polonès; la distància Wassertein W_1 metritza la convergència dèbil en $P_1(X) := \{\mu \in P(X) : \int_X d(x_0, x)\mu(dx) < +\infty\}$, on $x_0 \in X$ és arbitrari.

i
$$\mu' * m^{*n} = \mu',$$
però llavors, per (0.13), obtenim que
$$W_1(\mu,\mu') = W_1(\mu * m^{*n}, \mu' * m^{*n}) \leqslant (1-\kappa)^n W_1(\mu,\mu')$$
cosa que només és possible si $W_1(\mu,\mu') = 0$ ja que $1-\kappa < 1$. Aleshores,
$$\mu = \mu',$$
i això implica que $1 = \mu'(N_D^m) = \mu(N_D^m) = 0$ la qual cosa és una contradicció. Per tant, $\nu(N_D^m) = 0$ com volíem. □

Remark 0.94. Per la Proposició 0.13, la unicitat de la mesura de probabilitat invariant implica la seva ergodicitat. En conseqüència, el Teorema 0.93 se segueix de [**134**, Corollary 21] (vegeu també el Teorema 0.34). Tot i això, hem presentat el resultat per exhaustivitat i per a utilitzar el concepte de m-connexió.

Tingueu en compte que, si D és obert i $\nu(D) > 0$, llavors $N_D^m = \emptyset$, i.e.
$$\sum_{n=1}^{\infty} m_x^{*n}(D) > 0 \quad \text{per a tot } x \in X.$$
De fet, per a tot $x \in N_D^m$, per (0.15), tenim que
$$0 < \nu(D) \leqslant \liminf_n m_x^{*n}(D) = 0.$$

Remark 0.95. Per a un espai mètric de passeig aleatori reversible $[X,d,m,\nu]$, Y. Ollivier en [**134**, Corollary 31] prova que, sota el supòsit
$$\int\int\int d(y,z)^2 dm_x(y) dm_x(z) d\nu(x) < +\infty,$$
si la curvatura d'Ollivier-Ricci κ és positiva i ν és ergòdica[12], aleshores $[X,d,m,\nu]$ satisfà la desigualtat de Poincaré
$$\kappa \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{per a tot } f \in L^2(X,\nu),$$
i, per tant,
$$\kappa \leqslant \mathrm{gap}(-\Delta_m).$$

[12]Pel Teorema 0.93 (vegeu també l'Observació 0.94), aquesta suposició és, de fet, redundant.

CHAPTER 1

Random walks

The main character of the framework spaces on which the thesis is developed is the random walk.

Definition 1.1. Let $(X, \mathcal{B})$ be a measurable space such that the σ-field $\mathcal{B}$ is countably generated. A random walk on $(X, \mathcal{B})$ is a family of probability measures $(m_x)_{x \in X}$ on $\mathcal{B}$ such that $x \mapsto m_x(B)$ is a measurable function on X for each fixed $B \in \mathcal{B}$.

The notation and terminology chosen in this definition comes from [**134**]*, but this notion can be traced back to Neveu's concept of φ-essential irreducibility given in* [**133**] *(see Section 1.1.1). As noted in the paper by Y. Ollivier* [**134**]*, geometers may think of m_x as a replacement for the notion of balls around x, while in probabilistic terms we can rather think of these probability measures as defining a Markov chain whose transition probability from x to y in n steps is*

$$dm_x^{*n}(y) := \int_{z \in X} dm_z(y) dm_x^{*(n-1)}(z), \quad n \geqslant 1$$

and $m_x^{*0} = \delta_x$, the dirac measure at x. In the next section we will deepen on this latter perspective which will be the main one along our work, the former perspective will play an important role in Section 1.7. We therefore take a momentary break from the construction of what will be our framework space for the thesis to recall some results of the classic theory of Markov chains which we believe will aid in providing motivation.

1.1. Markov chains

In this section we immerse ourselves in the probabilistic terminology which houses our work. This will be particularly useful for readers with a probabilistic background as it will serve to clarify where exactly our work falls in this field. Furthermore, we recall well known results to provide further insight into the nature of random walks. To this aim we start by giving the definition of a discrete-time, time-homogeneous Markov Chain and the Markov property which it satisfies. The results in this section can be found in [**76**]*,* [**100**] *or* [**128**]*.*

Definition 1.2. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $(X, \mathcal{B})$ a measurable space such that the σ-field $\mathcal{B}$ is countably generated. A discrete-time, time-homogeneous Markov Chain is a sequence of X-valued random variables $\{X_n : n = 0, 1, 2, \ldots\}$ defined on Ω such that

$$\mathcal{P}(X_{n+1} \in B \mid X_0, X_1, \ldots, X_n) = \mathcal{P}(X_{n+1} \in B \mid X_n) \quad \forall B \in \mathcal{B}, n = 0, 1, \ldots \tag{1.1}$$

The identity (1.1) is called the Markov property.

The Markov property indicates that the future of the process is independent on the past given its present value so that, intuitively, we can say that it is "memoryless". A large variety of examples of Markov chains may be found in [**76**]*,* [**100**] *or* [**128**]*.*

In this thesis we will never work directly with the random variables, instead we will use a different, but equivalent, approach to Markov chains. For each $x \in X$ and $B \in \mathcal{B}$, let

$$P(x, B) := \mathcal{P}(X_{n+1} \in B | X_n = x). \tag{1.2}$$

This defines a stochastic kernel on X, which means that

- *$P(x, \cdot)$ is a probability measure on $\mathcal{B}$ for each fixed $x \in X$, and*
- *$P(\cdot, B)$ is a measurable function on X for each fixed $B \in \mathcal{B}$.*

The stochastic kernel P is also known as a (Markov) transition probability kernel. The "time-homogeneity" of the Markov chain refers to the fact that P as defined in (1.2) is independent of n. This stochastic kernel is what we have previously called a random walk, so that, in our terminology, $m_x(B) = P(x,B)$. Similarly, we denote by m_x^{*n} the n-step transition probability kernel $P^n(x,B) := \mathcal{P}^n(X_{n+1} \in B \mid X_0 = x)$, $x \in X$, $B \in \mathcal{B}$ which, as before, can be recursively defined by

$$P^n(x,B) = \int_X P^{n-1}(y,B)P(x,dy) = \int_X P(y,B)P^{n-1}(x,dy)$$

for $B \in \mathcal{B}$, $x \in X$ and $n = 1,2,\ldots$, with $P^0(x,\cdot) = \delta_x(\cdot)$. Note that, as can be easily proved by induction,

$$dm_x^{*(n+k)}(y) = \int_{z \in X} dm_x^{*k}(z) dm_z^{*n}(y)$$

for every n, $k \in \mathbb{N}$.

*It can then be shown that, in fact, starting with a random walk (or stochastic kernel) on X, we can construct a Markov chain (X_n) such that its transition probability kernels coincide with the given random walk (see [**128**, Theorem 3.4.1]). In particular, it is proven that for a given initial probability distribution μ on $\mathcal{B}$ one can construct the probability measure P_μ on $\mathcal{F}$ so that $P_\mu(X_0 \in B) = \mu(B)$ for $B \in \mathcal{B}$ and, moreover, for every $n = 0,1,\ldots$, $x \in X$ and $B \in \mathcal{B}$,*

$$P_\mu(X_{n+1} \in B \mid X_n = x) = m_x(B).$$

When μ is the Dirac measure at $x \in X$ we denote P_μ by P_x. In the same way, the corresponding expectation operators are denoted by E_μ and E_x, respectively.

We will now define some general concepts of stability for Markov Chains that will be used all along this work. These notions will in some way offer an insight into the long term behaviour of the process as it evolves with time. To this aim let us introduce the following concepts.

Definition 1.3.
(i) Let $A \in \mathcal{B}$, the occupation time η_A is the number of visits of the Markov chain to A after time zero[1]:

$$\eta_A := \sum_{n=1}^{\infty} \chi_{\{X_n \in A\}}.$$

(ii) For any set $A \in \mathcal{B}$, the random variable

$$\tau_A := \min\{n \geqslant 1 \,:\, X_n \in A\},$$

is called the first return time on A. We assume that $\inf \emptyset = \infty$.

The first and least restrictive concept of stability is that of φ-irreducibility, where φ is some measure on $\mathcal{B}$. With this concept we are requiring that, no matter what the starting point, we will be able to reach any "important" set in a finite number of jumps. The "important" sets will be understood as those which have positive measure with respect to the measure φ. We may also understand that we are asking that the Markov chain does not in truth consist of two chains. Let

$$\begin{aligned} U(x,A) &:= \sum_{n=1}^{\infty} m_x^{*n}(A) \\ &= E_x[\eta_A]. \end{aligned}$$

[1]We denote the indicator function (or characteristic function) of a subset A of a set X by χ_A, i.e., $\chi_A : X \to \{0,1\}$ is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

DEFINITION 1.4. Let φ be a measure on $\mathcal{B}$. A random walk m is φ-irreducible if, for every $x \in X$,

$$\varphi(A) > 0 \Rightarrow U(x, A) > 0.$$

Alternatively (see [**128**, Proposition 4.2.1]*), we may also understand this concept by using* τ_A*. Then, m is* φ*-irreducible if, for every* $x \in X$*,*

$$\varphi(A) > 0 \Rightarrow P_x(\tau_A < \infty) > 0,$$

i.e., starting from any point $x \in X$ *we have a positive probability of arriving to any set of positive measure in finite time.*

Instead of taking some possibly arbitrary measure to define the irreducibility of the random walk, we can take the maximal irreducibility measure, which defines the range of the chain more completely. This is done through the following proposition. Let

$$K_{a_{\frac{1}{2}}}(x, A) := \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} m_x^{*n}(x, A), \quad x \in X, A \in \mathcal{B}(X);$$

this kernel defines for each $x \in X$ *a probability measure equivalent*[2] *to* $m_x^{*0}(\cdot) + U(x, \cdot)$ *(which may be infinite for many sets).*

PROPOSITION 1.5. *(*[**128**, Proposition 4.2.2]*) Given a* φ*-irreducible random walk m on* $(X, \mathcal{B})$*, there exists a probability measure* ψ *on* $\mathcal{B}$ *such that*
(i) m is ψ*-irreducible;*
(ii) for any other measure φ'*, the random walk m is* φ'*-irreducible if and only if* $\varphi' \ll \psi$*;*
(iii) if $\psi(A) = 0$*, then* $\psi(\{y : P_y(\tau_A < \infty) > 0\}) = 0$*;*
(iv) the probability measure ψ *is equivalent to*

$$\psi'(A) := \int_X K_{a_{\frac{1}{2}}}(y, A) d\varphi'(y),$$

for any finite irreducibility measure φ'*.*

A measure satisfying the conditions of the previous proposition is called a maximal irreducibility measure. For notation convenience we will say that m is ψ*-irreducible if it is* φ*-irreducible for some measure* φ *and the measure* ψ *is a maximal irreducibility measure.*

A stronger stability notion can be obtained by asking, not only that $U(x, A) > 0$*, but that* $U(x, A) = \infty$ *for every* $x \in X$ *and every set A with* $\varphi(A) > 0$*. Alternatively, we may strengthen the requirement that there is a positive probability of reaching every set of* φ*-positive measure wherever we start from, and instead require that, in fact, this has to eventually happen. These approaches lead to the various concepts of recurrence.*

DEFINITION 1.6.
(i) Let $A \in \mathcal{B}$. The set A is called recurrent if $U(x, A) = E_x[\eta_A] = \infty$ for all $x \in A$.
(ii) The Markov chain is called recurrent if it is ψ-irreducible and $U(x, A) = E_x[\eta_A] = \infty$ for every $x \in X$ and every $A \in \mathcal{B}(X)$ with $\psi(A) > 0$.
(iii) Let $A \in \mathcal{B}$. The set A is called Harris recurrent if

$$P_x[\eta_A = \infty] = 1$$

for all $x \in A$.
(iv) The chain is Harris recurrent if it is ψ-irreducible and every $A \in \mathcal{B}(X)$ with $\psi(A) > 0$ is Harris recurrent.

It follows that any Harris recurrent set is recurrent. Indeed, for recurrence we require that the expected number of visits is infinite, meanwhile Harris recurrence implies that the

[2]Two measures μ and ν are equivalent if $\mu \ll \nu$ and $\nu \ll \mu$, i.e., if they agree on which sets have measure zero.

number of visits is infinite almost surely. Notably, by [**128**, Theorem 9.0.1] *we have that a recurrent chain differs by a ψ-null set from a Harris recurrent chain.*

Moreover, it is proved that there is a dichotomy in the sense that irreducible Markov chains cannot be "partially stable", they either posses these stability properties uniformly in x, or the chain is unstable in a well-defined way (we will however not enter into this, see [**128**] *for details).*

Another stability property that we will use is given by the existence of an invariant measure. This is a measure which provides a distribution such that, if the chain starts distributed in this way, then it remains like this. Furthermore, these measures turn out to be the ones which define the long term behaviour of the chain.

Definition 1.7. A σ-finite measure ν on $\mathcal{B}$ is invariant with respect to the random walk m if

$$\nu(A) := \int_X m_x(A) d\nu(x) \quad \text{for every } A \in \mathcal{B}.$$

Of course, if an invariant measure is finite, then it can and will be normalized to a (stationary) probability measure. Therefore, all along this thesis, whenever we require the finiteness of the invariant measure we will directly consider an invariant probability measure.

Remark 1.8. Note that, if m is a random walk on $(X, \mathcal{B})$, then m^{*n} is also a random walk on $(X, \mathcal{B})$ for every $n \in \mathbb{N}$. Moreover, if ν is invariant with respect to m then ν is invariant with respect to m^{*n} for every $n \in \mathbb{N}$.

Bringing irreducibility and the existence of an invariant measure together we get the following results.

Proposition 1.9. *(*[**128**, Proposition 10.0.1]*) If the random walk m is recurrent then it admits a unique (up to constant multiples) invariant measure.*

Proposition 1.10. *(*[**128**, Proposition 10.1.1]*) If the random walk m is ψ-irreducible and admits an invariant probability measure then it is recurrent; thus, in particular, the invariant probability measure is unique.*

Furthermore, we give the following theorem relating the invariant and maximal irreducibility measures for recurrent chains.

Theorem 1.11. *(*[**128**, Theorem 10.4.9]*) If the random walk m is recurrent (thus, in particular, ψ-irreducible) then the unique (up to constant multiples) invariant measure ν with respect to m is equivalent to ψ (thus ν is a maximal irreducibility measure).*

Another well known concept is that of an ergodic measure.

Definition 1.12. A set $B \in \mathcal{B}$ is said to be invariant (or absorbing or stochastically closed) (with respect to m) if $m_x(B) = 1$ for every $x \in B$.

An invariant probability measure ν is said to be ergodic (with respect to m) if $\nu(B) = 0$ or $\nu(B) = 1$ for every invariant set $B \in \mathcal{B}$.

A profound study of ergodic theory for Markov chains can be found in [**76**, Chapter 5]*. There we can find the construction of a dynamical system associated with a Markov chain such that the previous notion of ergodicity is equivalent to the classical notion of ergodicity for this dynamical system (*[**76**, Theorem 5.2.11]*). The following result ensures that uniqueness of the invariant measure implies its ergodicity.*

Proposition 1.13. *(*[**100**, Proposition 2.4.3]*) Let m be a random walk on X. If m has a unique invariant probability measure ν, then ν is ergodic.*

Which together with Proposition 1.10 and Theorem 1.11 implies the following.

Corollary 1.14. *If m is a ψ-irreducible random walk that admits an invariant probability measure then the invariant probability measure is unique, ergodic and equivalent to ψ.*

Finally, the last property that we introduce is the existence of a measure which is reversible with respect to the Markov chain. This reversibility condition on a measure is stronger than the invariance condition. We first define the tensor product of a σ-finite measure and a stochastic kernel.

Definition 1.15. If ν is a σ-finite measure on $(X, \mathcal{B})$ and N is a stochastic kernel on $(X, \mathcal{B})$, we define the tensor product of ν and N, denoted by $\nu \otimes N$, which is a measure on $(X \times X, \mathcal{B} \times \mathcal{B})$, by

$$\nu \otimes N(A \times B) = \int_A N(x, B) d\nu(x), \ (A, B) \in \mathcal{B} \times \mathcal{B}.$$

Using our notation m for the random walk we denote the tensor product of ν and m by $\nu \otimes m_x$.

A σ-finite measure ν on $\mathcal{B}$ is reversible with respect to the random walk m if the measure $\nu \otimes m_x$ on $\mathcal{B} \times \mathcal{B}$ is symmetric, i.e., for all $(A, B) \in \mathcal{B} \times \mathcal{B}$,

$$\nu \otimes m_x(A \times B) = \nu \otimes m_x(B \times A).$$

Equivalently, ν is reversible with respect to m if, for all bounded measurable functions f defined on $(X \times X, \mathcal{B} \times \mathcal{B})$,

$$\int_X \int_X f(x, y) dm_x(y) d\nu(x) = \int_X \int_X f(y, x) dm_x(y) d\nu(x).$$

Note that, if ν is reversible with respect to m, then it is invariant with respect to m (see [**76**], Proposition 1.5.2]*).*

Associated with a Markov chain we can define the following operator which will play a very important role in many of our developments.

Definition 1.16. If ν is an invariant measure with respect to m, we define the linear operator M_m on $L^1(X, \nu)$ into itself as follows

$$M_m f(x) := \int_X f(y) dm_x(y), \quad f \in L^1(X, \nu).$$

M_m is called the *averaging operator* on $[X, \mathcal{B}, m]$ (see, for example, [**134**]).

Note that, if $f \in L^1(X, \nu)$ then, using the invariance of ν with respect to m,

$$\int_X \int_X |f(y)| dm_x(y) d\nu(x) = \int_X |f(x)| d\nu(x) < \infty,$$

so $f \in L^1(X, m_x)$ for ν-a.e. $x \in X$, thus M_m is well defined from $L^1(X, \nu)$ into itself.

Remark 1.17. Let ν be an invariant measure with respect to m. It follows that

$$\|M_m f\|_{L^1(X,\nu)} \leqslant \|f\|_{L^1(X,\nu)} \quad \forall f \in L^1(X, \nu),$$

so that M_m is a contraction on $L^1(X, \nu)$. In fact, since $M_m f \geqslant 0$ if $f \geqslant 0$, we have that M_m is a positive contraction on $L^1(X, \nu)$.

Moreover, by Jensen's inequality, we have that, for $f \in L^1(X, \nu) \cap L^2(X, \nu)$,

$$\begin{aligned} \|M_m f\|^2_{L^2(X,\nu)} &= \int_X \left(\int_X f(y) dm_x(y) \right)^2 d\nu(x) \\ &\leqslant \int_X \int_X f^2(y) dm_x(y) d\nu(x) \\ &= \int_X f^2(x) d\nu(x) = \|f\|^2_{L^2(X,\nu)}. \end{aligned}$$

Therefore, M_m is a linear operator in $L^2(X, \nu)$ with domain

$$D(M_m) = L^1(X, \nu) \cap L^2(X, \nu).$$

Consequently, if ν is a probability measure, M_m is a bounded linear operator from $L^2(X,\nu)$ into itself satisfying $\|M_m\|_{\mathcal{B}(L^2(X,\nu),L^2(X,\nu))} \leqslant 1$.

Note that, making use of this operator, we have that $B \in \mathcal{B}$ is invariant with respect to m (Definition 1.12) if, and only if, $M_m\chi_B \geqslant \chi_B$. We may slightly weaken this notion as follows.

Definition 1.18. We say that $B \in \mathcal{B}$ is ν-invariant (with respect to m) if $M_m\chi_B = \chi_B$ ν-a.e.

Similarly, we define the notion of a harmonic (or ν-invariant) function.

Definition 1.19. A function $f \in L^1(X,\nu)$ is said to be harmonic (with respect to m) if $M_m f = f$ ν-a.e.

We may therefore recall a classic result which characterises the ergodicity of ν (see, for example, [**100**, Lemma 5.3.2]*).*

Proposition 1.20. *Let ν be an invariant probability measure. Then ν is ergodic if, and only if, every harmonic function is a constant ν-a.e.*

1.1.1. φ-Essential Irreducibility. *A different direction may be taken to define the irreducibility of a random walk (see* [**133**] *or* [**142**, Definition 4.4]*).*

Definition 1.21. Let φ be a measure on $\mathcal{B}$. A random walk m is φ-essentially irreducible if, for φ-a.e. $x \in X$,

$$\varphi(A) > 0 \Rightarrow U(x,A) > 0.$$

Whilst this change may appear small it actually leads to a wider and wilder class of "irreducible" models. However, there is a nice dichotomy result:

Proposition 1.22. *(*[**133**, Proposition 2]*) Let ν be an invariant measure with respect to the random walk m such that m is ν-essentially irreducible, then only one of the following two cases may happen:*
(i) there exists $X_1 \in \mathcal{B}$ such that $\nu(X \setminus X_1) = 0$, X_1 is invariant with respect to m and

$$\nu \ll U(x,.) \quad \text{for every } x \in X_1$$

i.e., the restriction of the Markov chain to X_1 is ν-irreducible;
(ii) there exists $X_2 \in \mathcal{B}$ such that $\nu(X \setminus X_2) = 0$, X_2 is invariant with respect to m and

$$\nu \perp U(x,.) \quad \text{for every } x \in X_2.$$

Since most of the examples of application of our results will actually fall into the first category in this theorem, the previous results in this section will be applicable. However, some extremal examples will actually fall into the second category, a case which classic literature does not usually cover (see [**142**, Chapter 4] *for a discussion of some of the results of using this weakened form). Therefore, we will now proceed to develop some of the results that we have given for φ-irreducible Markov chains but for φ-essentially irreducible Markov chains (assuming, in addition, that φ is an invariant measure). At this point we return to the work of the thesis and recover our terminology in which Markov chains are referred to as random walks and the notion of φ-essential irreducibility will be called m-connectedness.*

1.2. Random walk spaces

We continue developing the spaces in which we will work.

Definition 1.23. Let $(X,\mathcal{B})$ be a measurable space where the σ-field $\mathcal{B}$ is countably generated. Let m be a random walk on $(X,\mathcal{B})$ and ν an invariant measure with respect to m. The measurable space together with m and ν is then called a random walk space and denoted by $[X,\mathcal{B},m,\nu]$.

Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. If (X, d) is a Polish metric space (separable completely metrizable topological space), $\mathcal{B}$ is its Borel σ-algebra and ν is a Radon measure (i.e., ν is inner regular[3] and locally finite[4]) then we say that $[X, \mathcal{B}, m, \nu]$ is a metric random walk space and we denote it by $[X, d, m, \nu]$. Moreover, as is done in [**134**], when necessary, we will also assume that each measure m_x has finite first moment, i.e. for some (hence any) $z \in X$, and for any $x \in X$ one has $\int_X d(z, y) dm_x(y) < +\infty$.

Definition 1.24. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. We say that $[X, \mathcal{B}, m, \nu]$ is m-connected if, for every $D \in \mathcal{B}$ with $\nu(D) > 0$ and ν-a.e. $x \in X$,

$$\sum_{n=1}^{\infty} m_x^{*n}(D) > 0,$$

i.e., m is ν-essentially irreducible.

Note that, in this definition, we are requiring that the random walk is ν-essentially irreducible with the additional requirement that ν is actually an invariant measure (as was done in Proposition 1.22). However, the irreducibility measure and the invariant measure are usually introduced separately as seen in the previous section. Nonetheless, this provides a simpler all-in-one notion whose choice is moreover justified by Theorem 1.11. Note that, as somewhat mentioned in the previous section, the fundamental concept is that all parts of the space can be reached after a certain number of jumps, no matter what the starting point (except for, at most, a ν-null set of points).

We will now recall how this notion was originally introduced in [**123**]. *This will serve to introduce notation which we will use in some results.*

Definition 1.25. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. For a ν-measurable set D, we set

$$N_D^m := \{x \in X \,:\, m_x^{*n}(D) = 0, \ \forall n \in \mathbb{N}\}.$$

For $n \in \mathbb{N}$, we also define

$$H_{D,n}^m := \{x \in X \,:\, m_x^{*n}(D) > 0\},$$

and

$$H_D^m := \bigcup_{n \in \mathbb{N}} H_{D,n}^m = \left\{ x \in X \,:\, \sum_{n=1}^{\infty} m_x^{*n}(D) > 0 \right\}.$$

With this notation we have that $[X, \mathcal{B}, m, \nu]$ is m-connected if, and only if, $\nu(N_D^m) = 0$ for every $D \in \mathcal{B}$ such that $\nu(D) > 0$. Note that N_D^m and H_D^m are disjoint and

$$X = N_D^m \cup H_D^m.$$

Observe also that N_D^m, $H_{D,n}^m$ and H_D^m belong to $\mathcal{B}$. In the next result we see that N_D^m is invariant and H_D^m is ν-invariant (recall Definitions 1.12 and 1.18).

Proposition 1.26. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let $D \in \mathcal{B}$. If $N_D^m \neq \emptyset$ then:*

(i)

$$m_x^{*n}(H_D^m) = 0 \ \ (\text{thus } m_x^{*n}(N_D^m) = 1) \quad \text{for every } x \in N_D^m \text{ and } n \in \mathbb{N},$$

i.e., N_D^m is invariant with respect to m.

(ii)

$$m_x^{*n}(H_D^m) = 1 \ \ (\text{thus } m_x^{*n}(N_D^m) = 0) \quad \text{for } \nu\text{-almost every } x \in H_D^m, \text{ and for all } n \in \mathbb{N}.$$

i.e., H_D^m is ν-invariant with respect to m.

---

[3] ν is inner regular if, for any open set U, $\nu(U)$ is the supremum of $\nu(K)$ over all compact subsets K of U.

[4] ν is locally finite if every point of X has a neighborhood U for which $\nu(U)$ is finite.

Consequently, for every $x \in N_D^m$ and ν-a.e. $y \in H_D^m$ we have $m_x \perp m_y$, i.e. m_x and m_y are mutually singular[5].

PROOF. *(i)*: Suppose that $m_x^{*k}(H_D^m) > 0$ for some $x \in N_D^m$ and $k \in \mathbb{N}$, then, since $H_D^m = \cup_n H_{D,n}^m$ there exists $n \in \mathbb{N}$ such that $m_x^{*k}(H_{D,n}^m) > 0$ but in that case we have

$$m_x^{*(n+k)}(D) = \int_{z\in X} m_z^{*n}(D) dm_x^{*k}(z) \geqslant \int_{z\in H_{D,n}^m} m_z^{*n}(D) dm_x^{*k}(z) > 0$$

since $m_z^{*n}(D) > 0$ for every $z \in H_{D,n}^m$, and this contradicts that $x \in N_D^m$.

(ii): Fix $n \in \mathbb{N}$. Using the invariance of ν with respect to m^{*n} and statement *(i)* we have that

$$\begin{aligned} \nu(N_D^m) &= \int_X m_x^{*n}(N_D^m) d\nu(x) = \int_{H_D^m} m_x^{*n}(N_D^m) d\nu(x) + \int_{N_D^m} d\nu(x) \\ &= \int_{H_D^m} m_x^{*n}(N_D^m) d\nu(x) + \nu(N_D^m). \end{aligned}$$

Consequently, $m_x^{*n}(N_D^m) = 0$ for ν-a.e. $x \in H_D^m$.

□

This result exemplifies how a random walk m which is not m-connected is in reality composed of two (or more) separate random walks, one whose jumps occur in H_D^m and the other in N_D^m. Moreover, we may restrict the invariant measure to any of these subsets in order to obtain invariant measures for the restricted random walks as seen in the following result.

PROPOSITION 1.27. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let $D \in \mathcal{B}$. For every $n \in \mathbb{N}$ and $A \in \mathcal{B}$,*

$$\nu(A \cap H_D^m) = \int_{H_D^m} m_x^{*n}(A) d\nu(x),$$

and

$$\nu(A \cap N_D^m) = \int_{N_D^m} m_x^{*n}(A) d\nu(x).$$

PROOF. By the invariance of ν with respect to m^{*n} and Proposition 1.26 we have that, for any $A \in \mathcal{B}$,

$$\nu(A \cap H_D^m) = \int_X m_x^{*n}(A \cap H_D^m) d\nu(x) = \int_{H_D^m} m_x^{*n}(A \cap H_D^m) d\nu(x) = \int_{H_D^m} m_x^{*n}(A) d\nu(x).$$

Similarly, one proves the other statement. □

In the following result we see that, given a random walk space $[X, \mathcal{B}, m, \nu]$, if we start at ν-almost any point x in a set $D \in \mathcal{B}$ of ν-positive measure, then there is a positive probability that we will eventually return to D. In the terms of the previous section we have that $P_x(\tau_D < \infty) > 0$ for ν-a.e. $x \in D$.

COROLLARY 1.28. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. For any $D \in \mathcal{B}$, we have that*

$$\nu(D \cap N_D^m) = 0.$$

Consequently, if $\nu(D) > 0$, then $D \subset H_D^m$ up to a ν-null set; therefore, for ν-a.e. $x \in D$ there exists $n \in \mathbb{N}$ such that $m_x^{*n}(D) > 0$.

PROOF. By Proposition 1.27,

$$\nu(D \cap N_D^m) = \int_{N_D^m} m_x^{*n}(D) d\nu(x) = 0.$$ □

[5]Two positive measures μ and ν are mutually singular if there exist two disjoint sets A and B in $\mathcal{B}$ whose union is X such that μ is zero on all measurable subsets of B while ν is zero on all measurable subsets of A.

Finally, we will now give another approach to define an m-connected random walk space. This approach requires the notion of m-interaction between sets and is very good at providing intuition not only for the concept of m-connectedness but also for the reversibility condition of a measure.

Definition 1.29. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let A, $B \in \mathcal{B}$. We define the *m-interaction* between A and B as

$$L_m(A,B) := \int_A \int_B dm_x(y) d\nu(x) = \int_A m_x(B) d\nu(x).$$

Note that, whenever $L_m(A,B) < +\infty$, if ν is reversible with respect to m, we have that

$$L_m(A,B) = L_m(B,A).$$

A possible interpretation of this notion is the following: for a population which is originally distributed according to ν and which moves according to the law provided by the random walk m, $L_m(A,B)$ measures how many individuals are moving from A to B in one jump. Then, if ν is reversible with respect to m, this is equal to the amount of individuals moving from B to A in one jump.

In order to facilitate notation we make the following definition.

Definition 1.30. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. We say that $[X, \mathcal{B}, m, \nu]$ is a reversible random walk space if ν is reversible with respect to m. Moreover, if $[X, d, m, \nu]$ is a metric random walk space and ν is reversible with respect to m then we will say that $[X, d, m, \nu]$ is a reversible metric random walk space.

The following result gives a characterization of m-connectedness in terms of the m-interaction between sets.

Proposition 1.31. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. The following statements are equivalent:*

(i) $[X, \mathcal{B}, m, \nu]$ is m-connected.

(ii) If $A, B \in \mathcal{B}$ satisfy $A \cup B = X$ and $L_m(A,B) = 0$, then either $\nu(A) = 0$ or $\nu(B) = 0$.

(iii) If $A \in \mathcal{B}$ is a ν-invariant set then either $\nu(A) = 0$ or $\nu(X \setminus A) = 0$.

Proof. *(i) $\Rightarrow$ (ii)*: Assume that $[X, \mathcal{B}, m, \nu]$ is m-connected and let A, B be as in statement *(ii)*. If

$$0 = L_m(A,B) = \int_A \int_B dm_x(y) d\nu(x),$$

then there exists $N_1 \in \mathcal{B}$, $\nu(N_1) = 0$, such that

$$m_x(B) = 0 \quad \text{for every } x \in A \setminus N_1.$$

Now, since ν is invariant with respect to m,

$$0 = \nu(N_1) = \int_X m_x(N_1) d\nu(x),$$

and, consequently, there exists $N_2 \in \mathcal{B}$, $\nu(N_2) = 0$, such that

$$m_x(N_1) = 0 \quad \forall x \in X \setminus N_2.$$

Hence, for $x \in A\backslash(N_1 \cup N_2)$,

$$\begin{aligned} m_x^{*2}(B) &= \int_X \chi_B(y)dm_x^{*2}(y) = \int_X \left(\int_X \chi_B(y)dm_z(y)\right) dm_x(z) \\ &= \int_X m_z(B)dm_x(z) = \int_A m_z(B)dm_x(z) + \underbrace{\int_B m_z(B)dm_x(z)}_{=0,\text{ since } x \in A\backslash N_1.} \\ &= \underbrace{\int_{A\backslash N_1} m_z(B)dm_x(z)}_{=0,\text{ since } z \in A\backslash N_1.} + \underbrace{\int_{N_1} m_z(B)dm_x(z)}_{=0,\text{ since } x \in A\backslash N_2.} = 0 \end{aligned}$$

Working as above, we find $N_3 \in \mathcal{B}$, $\nu(N_3) = 0$, such that

$$m_x(N_1 \cup N_2) = 0 \quad \forall x \in X\backslash N_3.$$

Hence, for $x \in A\backslash(N_1 \cup N_2 \cup N_3)$, we have that

$$\begin{aligned} m_x^{*3}(B) &= \int_X \chi_B(y)dm_x^{*3}(y) = \int_X \left(\int_X \chi_B(y)dm_z^{*2}(y)\right) dm_x(z) \\ &= \int_X m_z^{*2}(B)dm_x(z) \leqslant \int_A m_z^{*2}(B)dm_x(z) + \underbrace{\int_B m_z^{*2}(B)dm_x(z)}_{=0,\text{ since } x \in A\backslash(N_1 \cup N_2).} \\ &\leqslant \underbrace{\int_{A\backslash(N_1\cup N_2)} m_z^{*2}(B)dm_x(z)}_{=0,\text{ since } z \in A\backslash(N_1 \cup N_2).} + \underbrace{\int_{N_1\cup N_2} m_z^{*2}(B)dm_x(z)}_{=0,\text{ since } x \in A\backslash N_3.} = 0. \end{aligned}$$

Inductively, we obtain that

$$m_x^{*n}(B) = 0 \quad \text{for } \nu\text{-a.e } x \in A \text{ and every } n \in \mathbb{N}.$$

Consequently,

$$A \subset N_B^m$$

up to a ν-null set thus $\nu(B) > 0$ implies that $\nu(A) = 0$.

(ii) $\Rightarrow$ *(iii)*: Note that, if A is ν-invariant, then $L_m(A, X\backslash A) = 0$.

(iii) $\Rightarrow$ *(i)*: Let $D \in \mathcal{B}$ with $\nu(D) > 0$. Then, by Proposition 1.26, we have that H_D^m is ν-invariant but, by Corollary 1.28, $\nu(H_D^m) \geqslant \nu(D) > 0$ thus $\nu(N_D^m) = 0$. □

Note that this result also justifies the choice of the terminology used since the characterisation of m-connectedness given is in some way reminiscent of the definition of a connected topological space.

Let us also use this moment to introduce the notion of m-connectedness for a subset of a reversible random walk space.

Definition 1.32. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space, and let $\Omega \in \mathcal{B}$ with $\nu(\Omega) > 0$. Let $\mathcal{B}_\Omega$ be the following σ-algebra

$$\mathcal{B}_\Omega := \{B \in \mathcal{B} \,:\, B \subset \Omega\}.$$

We say that Ω is *m-connected (with respect to ν)* if $L_m(A, B) > 0$ for every pair of non-ν-null sets A, $B \in \mathcal{B}_\Omega$ such that $A \cup B = \Omega$.

If a random walk space $[X, \mathcal{B}, m, \nu]$ is not m-connected, then we may obtain non-trivial decompositions of X as the following.

Definition 1.33. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Suppose that $\Omega_1, \Omega_2 \in \mathcal{B}$ satisfy: $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, $\nu(\Omega_1) > 0$, $\nu(\Omega_2) > 0$ and $L_m(\Omega_1, \Omega_2) = 0$. Then, we will write $\Omega = \Omega_1 \sqcup_m \Omega_2$.

We are now able to characterise the m-connectedness of a random walk space in terms of the ergodicity of the invariant measure (recall Corollary 1.14).

THEOREM 1.34. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and suppose that ν is a probability measure. Then*

$$[X, \mathcal{B}, m, \nu] \text{ is } m\text{-connected} \Leftrightarrow \nu \text{ is ergodic with respect to } m.$$

PROOF. ($\Rightarrow$). Let $B \in \mathcal{B}$ be invariant. Then B is ν-invariant thus, by Proposition 1.31, we have that $\nu(B) = 0$ or $\nu(B) = 1$.
($\Leftarrow$). Let $D \in \mathcal{B}$ with $\nu(D) > 0$. By Proposition 1.26, we have that N_D^m is invariant with respect to m. Then, since ν is ergodic, we have that $\nu(N_D^m) = 0$ or $\nu(N_D^m) = 1$. Now, since $\nu(D) > 0$, by Corollary 1.28, we have that $\nu(N_D^m) = 0$ and, consequently, $[X, \mathcal{B}, m, \nu]$ is m-connected. □

Finally, let us give a sufficient condition for the φ-irreducibility of a random walk. This involves the following definition (see, for example, [**100**, Section 7.2]*).*

DEFINITION 1.35. Let $[X, d, m, \nu]$ be a metric random walk space. We say that $[X, d, m, \nu]$ has the *strong-Feller property at $x_0 \in X$* if

$$m_{x_0}(A) = \lim_{n \to +\infty} m_{x_n}(A) \quad \text{for every Borel set} \quad A \subset X$$

whenever $x_n \to x_0$ in (X, d) as $n \to +\infty$.

We say that $[X, d, m, \nu]$ has the *strong-Feller property* if it has the strong-Feller property at every point in X.

PROPOSITION 1.36. *Let $[X, d, m, \nu]$ be a metric random walk space such that $\operatorname{supp} \nu = X$. Suppose further that $[X, d, m, \nu]$ has the strong-Feller property and that (X, d) is connected. Then, m is ν-irreducible (thus m-connected).*

PROOF. Recall that setwise convergence of a sequence of probability measures is equivalent to the convergence of the integrals against bounded measurable functions (in fact, by [**82**, Theorem 2.3], convergence on open or closed sets is enough). Therefore, since $[X, d, m, \nu]$ has the strong-Feller property and

$$m_x^{*k}(A) = \int_X m_y^{*(k-1)}(A) dm_x(y), \quad x \in X, A \in \mathcal{B},$$

$[X, d, m^{*k}, \nu]$ also has the strong-Feller property for any $k \in \mathbb{N}$.

Let $D \in \mathcal{B}$ with $\nu(D) > 0$. Let us see first that H_D^m is open or, equivalently, that N_D^m is closed. If $(x_n)_{n \geqslant 1} \subset N_D^m$ is a sequence such that $\lim_{n \to \infty} x_n = x \in X$, then

$$m_x^{*k}(D) = \lim_{n \to \infty} m_{x_n}^{*k}(D) = 0$$

for any $k \in \mathbb{N}$, thus $x \in N_D^m$ as desired.

However, H_D^m is also closed. Indeed, if $m_x(H_D^m) < 1$ for some $x \in H_D^m$, since $[X, d, m, \nu]$ has the strong-Feller property, there exists $r > 0$ such that $m_y(H_D^m) < 1$ for every $y \in B_r(x) \subset H_D^m$. Therefore, by Proposition 1.26, $\nu(B_r(x)) = 0$, which is in contradiction with $\operatorname{supp} \nu = X$. Hence,

$$m_x(H_D^m) = 1 \Leftrightarrow x \in H_D^m.$$

Then, given $(x_n)_{n \geqslant 1} \subset H_D^m$ such that $\lim_{n \to \infty} x_n = x \in X$, we have

$$m_x(H_D^m) = \lim_{n \to \infty} m_{x_n}(H_D^m) = 1,$$

so $x \in H_D^m$. Consequently, H_D^m is closed and, therefore, since X is connected, we have that $X = H_D^m$ which implies that $N_D^m = \emptyset$. □

Note that this result gives a relation between the topological connectedness and the m-connectedness of a metric random walk space.

1.3. Examples

Example 1.37. Consider the metric measure space $(\mathbb{R}^N, d, \mathcal{L}^N)$, where d is the Euclidean distance and $\mathcal{L}^N$ the Lebesgue measure on $\mathbb{R}^N$. For simplicity, we will write dx instead of $d\mathcal{L}^N(x)$. Let $J : \mathbb{R}^N \to [0, +\infty[$ be a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} J(x)dx = 1$. Let m^J be the following random walk on $(\mathbb{R}^N, d)$:

$$m_x^J(A) := \int_A J(x-y)dy \quad \text{for every } x \in \mathbb{R}^N \text{ and every Borel set } A \subset \mathbb{R}^N.$$

Then, applying Fubini's Theorem it is easy to see that the Lebesgue measure $\mathcal{L}^N$ is reversible with respect to m^J. Therefore, $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ is a reversible metric random walk space.

An interpretation, similar to the one given in Section 1.2 for the m-interaction between sets, can be given for m^J. In this case, if in $\mathbb{R}^N$ we consider a population such that each individual starting at location x jumps to location y according to the probability distribution $J(x-y)$, then, for a Borel set A in $\mathbb{R}^N$, $m_x^J(A)$ is measuring the proportion of individuals who started at x and are arriving at A after one jump.

Example 1.38. Consider a locally finite[6] weighted discrete graph $G = (V(G), E(G))$, where $V(G)$ is the vertex set, $E(G)$ is the edge set and each edge $(x, y) \in E(G)$ (we will write $x \sim y$ if $(x, y) \in E(G)$) has a positive weight $w_{xy} = w_{yx}$ assigned. Suppose further that $w_{xy} = 0$ if $(x, y) \notin E(G)$.

A finite sequence $\{x_k\}_{k=0}^n$ of vertices of the graph is called a *path* if $x_k \sim x_{k+1}$ for all $k = 0, 1, ..., n-1$. The *length* of a path $\{x_k\}_{k=0}^n$ is defined as the number n of edges in the path. With this terminology, $G = (V(G), E(G))$ is said to be *connected* if, for any two vertices $x, y \in V$, there is a path connecting x and y, that is, a path $\{x_k\}_{k=0}^n$ such that $x_0 = x$ and $x_n = y$. Finally, if $G = (V(G), E(G))$ is connected, the *graph distance* $d_G(x, y)$ between any two distinct vertices x, y is defined as the minimum of the lengths of the paths connecting x and y. Note that this metric is independent of the weights.

For $x \in V(G)$ we define the weight at x as

$$d_x := \sum_{y \sim x} w_{xy} = \sum_{y \in V(G)} w_{xy},$$

and the neighbourhood of x as $N_G(x) := \{y \in V(G) : x \sim y\}$. Note that, by definition of locally finite graph, the sets $N_G(x)$ are finite. When all the weights are 1, d_x coincides with the degree of the vertex x in a graph, that is, the number of edges containing x.

For each $x \in V(G)$ we define the following probability measure

$$m_x^G := \frac{1}{d_x} \sum_{y \sim x} w_{xy}\, \delta_y.$$

It is not difficult to see that the measure ν_G defined as

$$\nu_G(A) := \sum_{x \in A} d_x, \quad A \subset V(G),$$

is a reversible measure with respect to this random walk. Therefore, $[V(G), \mathcal{B}, m^G, \nu_G]$ is a reversible random walk space ($\mathcal{B}$ is the σ-algebra of all subsets of $V(G)$) and $[V(G), d_G, m^G, \nu_G]$ is a reversible metric random walk space.

Proposition 1.39. *Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with a connected locally finite weighted discrete graph $G = (V(G), E(G))$. Then m^G is ν_G-irreducible.*

[6] A graph is locally finite if every vertex is only contained in a finite number of edges.

Proof. Take $D \subset V(G)$ with $\nu_G(D) > 0$, and let us see that $N_D^{m^G} = \emptyset$. Suppose that there exists $y \in N_D^{m^G}$, this implies that

$$(m^G)_y^{*n}(D) = 0 \quad \forall\, n \in \mathbb{N}. \tag{1.3}$$

Now, given $x \in D$, there exists a path $\{x, z_1, z_2, \ldots, z_{k-1}, y\}$ ($x \sim z_1 \sim z_2 \sim \cdots \sim z_{k-1} \sim y$) of length k connecting x and y, and, therefore,

$$(m^G)_y^{*k}(\{x\}) \geqslant \frac{w_{yz_{k-1}} w_{z_{k-1}z_{k-2}} \cdots w_{z_2 z_1} w_{z_1 x}}{d_y d_{z_{k-1}} d_{z_{k-2}} \cdots d_{z_2} d_{z_1}} > 0,$$

which is in contradiction with (1.3). □

In Machine Learning Theory ([87], [88]), an example of a weighted discrete graph is a point cloud in $\mathbb{R}^N$, $V = \{x_1, \ldots, x_n\}$, with edge weights w_{x_i,x_j} given by

$$w_{x_i,x_j} := \eta(|x_i - x_j|), \quad 1 \leqslant i, j \leqslant n,$$

where the kernel $\eta : [0,\infty) \to [0,\infty)$ is a radial profile satisfying

(i) $\eta(0) > 0$, and η is continuous at 0,
(ii) η is non-decreasing,
(iii) and the integral $\int_0^\infty \eta(r) r^N dr$ is finite.

Example 1.40. Let $K : X \times X \to \mathbb{R}$ be a Markov kernel on a countable space X, i.e.,

$$K(x,y) \geqslant 0 \quad \forall x, y \in X, \qquad \sum_{y \in X} K(x,y) = 1 \quad \forall x \in X.$$

Then, if

$$m_x^K(A) := \sum_{y \in A} K(x,y), \quad x \in X,\ A \subset X$$

and $\mathcal{B}$ is the σ-algebra of all subsets of X, m^K is a random walk on $(X, \mathcal{B})$.

Recall that, in discrete Markov chain theory terminology, a measure π on X satisfying

$$\sum_{x \in X} \pi(x) = 1 \quad \text{and} \quad \pi(y) = \sum_{x \in X} \pi(x) K(x,y) \quad \forall y \in X,$$

is called a stationary probability measure (or steady state) on X. Of course, π is a stationary probability measure if, and only if, π is and invariant probability measure with respect to m^K. Consequently, if π is a stationary probability measure on X, then $[X, \mathcal{B}, m^K, \pi]$ is a random walk space.

Furthermore, a stationary probability measure π is said to be reversible for K if the following detailed balance equation holds:

$$K(x,y)\pi(x) = K(y,x)\pi(y) \ \text{ for } x, y \in X.$$

This balance condition is equivalent to

$$dm_x^K(y) d\pi(x) = dm_y^K(x) d\pi(y) \ \text{ for } x, y \in X.$$

Note that, given a locally finite weighted discrete graph $G = (V(G), E(G))$ as in Example 1.38, there is a natural definition of a Markov chain on the vertices. Indeed, define the Markov kernel $K_G : V(G) \times V(G) \to \mathbb{R}$ as

$$K_G(x,y) := \frac{1}{d_x} w_{xy}.$$

Then, m^G and m^{K_G} define the same random walk. If $\nu_G(V(G))$ is finite, the unique reversible probability measure with respect to m^G is given by

$$\pi_G(x) := \frac{1}{\nu_G(V(G))} \sum_{z \in V(G)} w_{xz}.$$

Example 1.41. From a metric measure space (X, d, μ) we can obtain a random walk, the so called *ϵ-step random walk associated with* μ, as follows. Assume that balls in X have finite measure and that $\text{Supp}(\mu) = X$. Given $\epsilon > 0$, the ϵ-step random walk on X, starting at $x \in X$, consists in randomly jumping in the ball of radius ϵ around x, with probability proportional to μ; namely

$$m_x^{\mu,\epsilon} := \frac{\mu \llcorner B(x,\epsilon)}{\mu(B(x,\epsilon))}$$

where $\mu \llcorner B(x,\epsilon)$ denotes the restriction of μ to $B(x,\varepsilon)$ (or, more precisely, to $\mathcal{B}_{B(x,\varepsilon)}$, where $\mathcal{B}$ is the Borel σ-algebra associated with (X, d).).

If $\mu(B(x,\varepsilon)) = \mu(B(y,\varepsilon))$ for every x, $y \in X$, then μ is reversible with respect to $m^{\mu,\epsilon}$, thus $[X, d, m^{\mu,\epsilon}, \mu]$ is a reversible metric random walk space.

Example 1.42. Given a random walk space $[X, \mathcal{B}, m, \nu]$ and $\Omega \in \mathcal{B}$ with $\nu(\Omega) > 0$, let

$$m_x^{\Omega}(A) := \int_A dm_x(y) + \left(\int_{X \setminus \Omega} dm_x(y) \right) \delta_x(A) \quad \text{ for every } A \in \mathcal{B}_\Omega \text{ and } x \in \Omega.$$

Then, m^Ω is a random walk on $(\Omega, \mathcal{B}_\Omega)$ and it easy to see that $\nu \llcorner \Omega$ is invariant with respect to m^Ω. Therefore, $[\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega]$ is a random walk space. Moreover, if ν is reversible with respect to m then $\nu \llcorner \Omega$ is reversible with respect to m^Ω. Of course, if ν is a probability measure we may normalize $\nu \llcorner \Omega$ to obtain the random walk space

$$\left[\Omega, \mathcal{B}_\Omega, m^\Omega, \frac{1}{\nu(\Omega)} \nu \llcorner \Omega \right].$$

Note that, if $[X, d, m, \nu]$ is a metric random walk space and Ω is closed, then $[\Omega, d, m^\Omega, \nu \llcorner \Omega]$ is also a metric random walk space, where we abuse notation and denote by d the restriction of d to Ω.

In particular, in the context of Example 1.37, if Ω is a closed and bounded subset of $\mathbb{R}^N$, we obtain the metric random walk space $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N \llcorner \Omega]$ where $m^{J,\Omega} := (m^J)^\Omega$; that is,

$$m_x^{J,\Omega}(A) := \int_A J(x-y)dy + \left(\int_{\mathbb{R}^n \setminus \Omega} J(x-z)dz \right) d\delta_x \quad \text{ for every Borel set } A \subset \Omega \text{ and } x \in \Omega.$$

Using this last example we can characterise m-connected sets as follows (recall Definition 1.32).

Proposition 1.43. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space, and let $\Omega \in \mathcal{B}$ with $\nu(\Omega) > 0$. Then,*

$$\Omega \text{ is } m\text{-connected} \Leftrightarrow [\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega] \text{ is } m^\Omega\text{-connected.}$$

Proof. Let A, $B \in \mathcal{B}_\Omega$ be disjoint sets. Then, for $x \in A$,

$$m_x^\Omega(B) = m_x(B) + m_x(X \setminus \Omega)\delta_x(B) = m_x(B)$$

thus $L_{m^\Omega}(A, B) = L_m(A, B)$. Consequently, the result follows by Proposition 1.31. □

1.4. The nonlocal gradient, divergence and Laplace operator

Let us introduce the nonlocal counterparts of some classical concepts.

Definition 1.44. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. Given a function $u : X \to \mathbb{R}$ we define its nonlocal gradient $\nabla u : X \times X \to \mathbb{R}$ as

$$\nabla u(x, y) := u(y) - u(x) \quad \forall\, x, y \in X.$$

Moreover, given $\mathbf{z} : X \times X \to \mathbb{R}$, its *$m$-divergence* $\text{div}_m \mathbf{z} : X \to \mathbb{R}$ is defined as

$$(\text{div}_m \mathbf{z})(x) := \frac{1}{2} \int_X (\mathbf{z}(x,y) - \mathbf{z}(y,x)) dm_x(y).$$

We define the (nonlocal) Laplace operator as follows (recall the definition of the averaging operator given in Definition 1.16).

Definition 1.45. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space, we define the *Laplace operator* (or *Laplacian*) from $L^1(X, \nu)$ into itself as $\Delta_m := M_m - I$, i.e.,

$$\Delta_m u(x) = \int_X u(y) dm_x(y) - u(x) = \int_X (u(y) - u(x)) dm_x(y), \quad u \in L^1(X, \nu).$$

The Laplace operator is also called the drift operator (see [**128**, Chapter 8]*). Note that*

$$\Delta_m f(x) = \mathrm{div}_m(\nabla f)(x).$$

Remark 1.46. We have that $\|\Delta_m f\|_{L^1(X,\nu)} \leqslant 2\|f\|_{L^1(X,\nu)}$ and

$$\int_X \Delta_m f(x) d\nu(x) = 0 \quad \forall\, f \in L^1(X, \nu). \tag{1.4}$$

As in Remark 1.17, we obtain that Δ_m is a linear operator in $L^2(X, \nu)$ with domain

$$D(\Delta_m) = L^1(X, \nu) \cap L^2(X, \nu).$$

Moreover, if ν is a probability measure, Δ_m is a bounded linear operator in $L^2(X, \nu)$ satisfying $\|\Delta_m f\|_{L^2(X,\nu)} \leqslant 2\|f\|_{L^2(X,\nu)}$ for every $f \in L^2(X, \nu)$.

In the case of the random walk space associated with a locally finite weighted discrete graph $G = (V, E)$ (as defined in Example 1.38), the Laplace operator coincides with the graph Laplacian (usually called the normalized graph Laplacian) studied by many authors (see, for example, [**26**]*,* [**27**]*,* [**75**] *or* [**105**]*):*

$$\Delta u(x) := \frac{1}{d_x} \sum_{y \sim x} w_{xy}(u(y) - u(x)), \quad u \in L^2(V, \nu_G), \ x \in V.$$

Proposition 1.47. *(Integration by parts formula) Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Then,*

$$\int_X f(x) \Delta_m g(x) d\nu(x) = -\frac{1}{2} \int_{X \times X} \nabla f(x, y) \nabla g(x, y) d(\nu \otimes m_x)(x, y) \tag{1.5}$$

for $f, g \in L^1(X, \nu) \cap L^2(X, \nu)$.

Proof. Since, by the reversibility of ν with respect to m,

$$\int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) = \int_X \int_X f(y)(g(x) - g(y)) dm_x(y) d\nu(x)$$

we get that

$$\begin{aligned} &\int_X f(x) \Delta_m g(x) d\nu(x) = \int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) \\ &= \frac{1}{2} \int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) + \frac{1}{2} \int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) \\ &= \frac{1}{2} \int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) + \frac{1}{2} \int_X \int_X f(y)(g(x) - g(y)) dm_x(y) d\nu(x) \\ &= -\frac{1}{2} \int_{X \times X} \nabla f(x, y) \nabla g(x, y) d(\nu \otimes m_x)(x, y). \end{aligned}$$

□

In fact, we may prove, in the same way, the following more general result which will be useful in Chapter 5.

Lemma 1.48. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. Let $q \geqslant 1$. If $Q \subset X \times X$ is a symmetric set (i.e., $(x,y) \in Q \iff (y,x) \in Q$) and $\Psi : Q \to \mathbb{R}$ is a $\nu \otimes m_x$-a.e. antisymmetric function (i.e., $\Psi(x,y) = -\Psi(y,x)$ for $\nu \otimes m_x$-a.e. $(x,y) \in Q$) with $\Psi \in L^q(Q, \nu \otimes m_x)$ and $u \in L^{q'}(X,\nu)$ then*

$$\int_Q \Psi(x,y)u(x)d(\nu \otimes m_x)(x,y) = -\frac{1}{2}\int_Q \Psi(x,y)(u(y)-u(x))d(\nu \otimes m_x)(x,y).$$

In particular, if $\Psi \in L^1(Q,\nu \otimes m_x)$,

$$\int_Q \Psi(x,y)d(\nu \otimes m_x)(x,y) = 0.$$

We are now able to characterise the m-connectedness of a random walk space in terms of the ergodicity of the Laplace operator. Following Bakry, Gentil and Ledoux [**22**]*, we give the following definition.*

Definition 1.49. Let $[X,\mathcal{B},m,\nu]$ be a random walk space. We say that Δ_m is *ergodic* if, for $u \in D(\Delta_m)$,

$$\Delta_m u = 0 \ \nu\text{-a.e.} \Rightarrow \ u \text{ is a constant } \nu\text{-a.e.}$$

(being this constant 0 if ν is not finite), i.e., every harmonic function in $D(\Delta_m)$ (recall Definition 1.19) is a constant ν-a.e.

Theorem 1.50. *Let $[X,\mathcal{B},m,\nu]$ be a random walk space and suppose that ν is a probability measure. Then,*

$$[X,\mathcal{B},m,\nu] \text{ is } m\text{-connected} \Leftrightarrow \Delta_m \text{ is ergodic}$$

Proof. ($\Leftarrow$): Let $D \in \mathcal{B}$ with $\nu(D) > 0$ and recall that, by Corollary 1.28, $\nu(H_D^m) \geqslant \nu(D) > 0$. Consider the function

$$u(x) := \chi_{H_D^m}(x), \quad x \in X,$$

and note that, since ν is finite, $u \in L^2(X,\nu) = D(\Delta_m)$. Now, since, by Proposition 1.26, H_D^m is ν-invariant we have that $\Delta_m u = 0$ ν-a.e. Thus, by the ergodicity of Δ_m and recalling that $\nu(H_D^m) > 0$ we must have that $u = \chi_{H_D^m} = 1$ ν-a.e. and, therefore, $\nu(N_D^m) = 0$.

($\Rightarrow$): Suppose now that $[X,\mathcal{B},m,\nu]$ is m-connected and let $u \in L^2(X,\nu) = D(\Delta_m)$ such that u is not ν-a.e. a constant, let's see that $\Delta_m u$ is not ν-a.e. 0.

We may find $U \in \mathcal{B}$ with $0 < \nu(U) < 1$ such that $u(x) < u(y)$ for every $x \in U$ and $y \in X \backslash U$. Then, since $L_m(U, X\backslash U) > 0$,

$$\frac{1}{2}\int_X\int_X \nabla u(x,y)^2 dm_x(y)d\nu(x) \geqslant \frac{1}{2}\int_U\int_{X\backslash U} \nabla u(x,y)^2 dm_x(y)d\nu(x) > 0$$

but $\frac{1}{2}\int_X\int_X \nabla u(x,y)^2 dm_x(y)d\nu(x) = -\int_X u(x)\Delta_m u(x)d\nu(x)$ so $\Delta_m u$ is not ν-a.e. 0. □

This result together with Theorem 1.34 shows that both concepts of ergodicity, the one for the invariant measure and the one for the Laplace operator, are equivalent. Such a relation recovers Proposition 1.20.

1.5. The nonlocal boundary, perimeter and mean curvature

In this section we introduce the nonlocal counterparts of the notions of boundary, perimeter and mean curvature.

The following notion of nonlocal boundary will play the role of the classical boundary when we consider the nonlocal counterparts of classical equations in Chapter 5, that is, boundary conditions will be imposed on this set.

Definition 1.51. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and $\Omega \in \mathcal{B}$. We define the *m-boundary of* Ω by

$$\partial_m \Omega := \{x \in X \backslash \Omega \,:\, m_x(\Omega) > 0\}$$

and its *m-closure* as

$$\Omega_m := \Omega \cup \partial_m \Omega.$$

In Chapter 3 the following notion of nonlocal perimeter will be widely used.

Definition 1.52. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and $E \in \mathcal{B}$. The m-perimeter of E is defined by

$$P_m(E) := L_m(E, X \backslash E) = \int_E \int_{X \backslash E} dm_x(y) d\nu(x).$$

In regards to the interpretation given for the m-interaction between sets (below Definition 1.29) this notion of perimeter can be interpreted as measuring the total flux of individuals that cross the "boundary" (in a very weak sense) of a set in one jump.

Lemma 1.53. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and $E \in \mathcal{B}$ with $\nu(E) < \infty$. Then,*

$$P_m(E) = \nu(E) - \int_E \int_E dm_x(y) d\nu(x). \tag{1.6}$$

Furthermore, $P_m(E) = P_m(X \backslash E)$ and

$$P_m(E) = \frac{1}{2} \int_X \int_X |\chi_E(y) - \chi_E(x)| dm_x(y) d\nu(x) = \frac{1}{2} \int_X \int_X |\nabla \chi_E(x, y)| dm_x(y) d\nu(x).$$

Proof. Equation (1.6) is straightforward. Now,

$$\begin{aligned}
P_m(E) - P_m(X \backslash E) &= \int_E \int_{X \backslash E} dm_x(y) d\nu(x) - \int_{X \backslash E} \int_E dm_x(y) d\nu(x) \\
&= \int_E \int_X dm_x(y) d\nu(x) - \int_E \int_E dm_x(y) d\nu(x) - \left(\int_X \int_E dm_x(y) d\nu(x) - \int_E \int_E dm_x(y) d\nu(x) \right) \\
&= \int_E d\nu(x) - \int_X \int_X \chi_E(y) dm_x(y) d\nu(x) \\
&= \nu(E) - \int_X \chi_E(x) d\nu(x) \\
&= \nu(E) - \nu(E) = 0.
\end{aligned}$$

For the last statement note that

$$\int_X \int_X |\chi_E(y) - \chi_E(x)| dm_x(y) d\nu(x) = P_m(E) + P_m(X \backslash E) = 2 P_m(E). \qquad \square$$

The notion of m-perimeter can be localized to a subset as follows.

Definition 1.54. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and $\Omega \in \mathcal{B}$ with $\nu(\Omega) < \infty$. Then, for $E \in \mathcal{B}$, we define

$$P_m(E, \Omega) := L_m(E \cap \Omega, X \backslash E) + L_m(E \backslash \Omega, \Omega \backslash E).$$

Observe that

$$L_m(E, X \backslash E) = L_m(E \cap \Omega, X \backslash E) + L_m(E \backslash \Omega, \Omega \backslash E) + L_m(E \backslash \Omega, X \backslash (E \cup \Omega))$$

and, consequently, we have

$$P_m(E, \Omega) = \int_E \int_{X \backslash E} dm_x(y) d\nu(x) - \int_{E \backslash \Omega} \int_{X \backslash (E \cup \Omega)} dm_x(y) d\nu(x),$$

when both integrals are finite.

Example 1.55. Let $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ be the metric random walk space given in Example 1.37. Then,

$$P_{m^J}(E) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\chi_E(y) - \chi_E(x)| J(x-y)dydx,$$

which coincides with the concept of J-perimeter introduced in [120]. Furthermore,

$$P_{m^{J,\Omega}}(E) = \frac{1}{2}\int_{\Omega}\int_{\Omega} |\chi_E(y) - \chi_E(x)| J(x-y)dydx.$$

Note that, in general, $P_{m^{J,\Omega}}(E) \neq P_{m^J}(E)$ (recall the definition of $m^{J,\Omega}$ given in Example 1.42).

Moreover,

$$\begin{aligned} P_{m^{J,\Omega}}(E) &= \mathcal{L}^N(E) - \int_E\int_E dm_x^{J,\Omega}(y)dx \\ &= \mathcal{L}^N(E) - \int_E\int_E J(x-y)dydx - \int_E\left(\int_{\mathbb{R}^N\setminus\Omega} J(x-z)dz\right)dx \end{aligned}$$

and, therefore,

$$P_{m^{J,\Omega}}(E) = P_{m^J}(E) - \int_E\left(\int_{\mathbb{R}^N\setminus\Omega} J(x-z)dz\right)dx, \quad \forall\, E \subset \Omega. \tag{1.7}$$

Example 1.56. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated (as in Example 1.38) to a finite weighted discrete graph G. Given $A, B \subset V(G)$, $\mathrm{Cut}(A,B)$ is defined as

$$\mathrm{Cut}(A,B) := \sum_{x\in A,\, y\in B} w_{xy} = L_{m^G}(A,B),$$

and the perimeter of a set $E \subset V(G)$ is given by

$$|\partial E| := \mathrm{Cut}(E, E^c) = \sum_{x\in E,\, y\in V\setminus E} w_{xy}.$$

Consequently, we have that

$$|\partial E| = P_{m^G}(E) \quad \text{for all } E \subset V(G). \tag{1.8}$$

We now give some properties of the m-perimeter.

Proposition 1.57. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and let $A,\ B \in \mathcal{B}$ be sets with finite m-perimeter such that $\nu(A\cap B) = 0$. Then,*

$$P_m(A\cup B) = P_m(A) + P_m(B) - 2L_m(A,B).$$

Proof.

$$\begin{aligned} P_m(A\cup B) &= \int_{A\cup B}\left(\int_{X\setminus(A\cup B)} dm_x(y)\right)d\nu(x) \\ &= \int_A\left(\int_{X\setminus(A\cup B)} dm_x(y)\right)d\nu(x) + \int_B\left(\int_{X\setminus(A\cup B)} dm_x(y)\right)d\nu(x) \\ &= \int_A\left(\int_{X\setminus A} dm_x(y) - \int_B dm_x(y)\right)d\nu(x) \\ &\quad + \int_B\left(\int_{X\setminus B} dm_x(y) - \int_A dm_x(y)\right)d\nu(x), \end{aligned}$$

thus, by the reversibility of ν with respect to m,

$$P_m(A\cup B) = P_m(A) + P_m(B) - 2\int_A\left(\int_B dm_x(y)\right)d\nu(x).$$

□

Corollary 1.58. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and let A, B, $C \in \mathcal{B}$ be sets with pairwise ν-null intersections. Then,*

$$P_m(A \cup B \cup C) = P_m(A \cup B) + P_m(A \cup C) + P_m(B \cup C) - P_m(A) - P_m(B) - P_m(C).$$

Proposition 1.59 (Submodularity). *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and let A, $B \in \mathcal{B}$. Then*

$$P_m(A \cup B) + P_m(A \cap B) = P_m(A) + P_m(B) - 2L_m(A \backslash B, B \backslash A).$$

Consequently,

$$P_m(A \cup B) + P_m(A \cap B) \leqslant P_m(A) + P_m(B).$$

Proof. By Corollary 1.58,

$$\begin{aligned} P_m(A \cup B) &= P_m((A \backslash B) \cup (B \backslash A) \cup (A \cap B)) \\ &= P_m((A \backslash B) \cup (B \backslash A)) + P_m(A) + P_m(B) \\ &\quad - P_m(A \backslash B) - P_m(B \backslash A) - P_m(A \cap B). \end{aligned}$$

Hence,

$$\begin{aligned} P_m(A \cup B) + P_m(A \cap B) &= P_m(A) + P_m(B) + P_m((A \backslash B) \cup (B \backslash A)) \\ &\quad - P_m(A \backslash B) - P_m(B \backslash A). \end{aligned}$$

Now, by Proposition 1.57,

$$P_m((A \backslash B) \cup (B \backslash A)) - P_m(A \backslash B) - P_m(B \backslash A) = -2L_m(A \backslash B, B \backslash A).$$ □

The concept of mean curvature can be defined in our nonlocal context as follows.

Definition 1.60. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let $E \in \mathcal{B}$. For a point $x \in X$ we define the *m-mean curvature of ∂E at x* as

$$\mathcal{H}^m_{\partial E}(x) := m_x(X \backslash E) - m_x(E) = 1 - 2m_x(E). \tag{1.9}$$

Intuitively, at points from where it is "easier to jump" to $X \backslash E$ than to E, the m-mean curvature of ∂E is positive. Some examples can be found in [**121**, Example 3.6].

Note that $\mathcal{H}^m_{\partial E}(x)$ can be computed for every $x \in X$, not only for points in ∂E. Furthermore, if $\nu(E) < \infty$,

$$\int_E \mathcal{H}^m_{\partial E}(x) d\nu(x) = \int_E \left(1 - 2\int_E dm_x(y)\right) d\nu(x) = \nu(E) - 2\int_E \int_E dm_x(y) d\nu(x), \tag{1.10}$$

hence, having in mind (1.6), we obtain that

$$\int_E \mathcal{H}^m_{\partial E}(x) d\nu(x) = 2P_m(E) - \nu(E). \tag{1.11}$$

Observe also that

$$\mathcal{H}^m_{\partial E}(x) = -\mathcal{H}^m_{\partial (X \backslash E)}(x). \tag{1.12}$$

Remark 1.61. Let $[\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega]$ be the random walk space given in Example 1.42. Then,

$$\mathcal{H}^{m^\Omega}_{\partial E}(x) = m_x(\Omega \backslash E) + m_x(X \backslash \Omega)\delta_x(\Omega \backslash E) - m_x(E) - m_x(X \backslash \Omega)\delta_x(E),$$

thus,

$$\mathcal{H}^{m^\Omega}_{\partial E}(x) = \begin{cases} m_x(\Omega \backslash E) - m_x(E) + m_x(X \backslash \Omega) & \text{if } x \in \Omega \backslash E, \\ m_x(\Omega \backslash E) - m_x(E) - m_x(X \backslash \Omega) & \text{if } x \in E. \end{cases}$$

In particular, for the random walk space $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N]$ (see also Example 1.42), we get

$$\mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) = \begin{cases} \displaystyle\int_{\Omega\backslash E} J(x-y)dy - \int_E J(x-y)dy + \int_{\mathbb{R}^N\backslash\Omega} J(x-y)dy & \text{if } x \in \Omega\backslash E, \\ \displaystyle\int_{\Omega\backslash E} J(x-y)dy - \int_E J(x-y)dy - \int_{\mathbb{R}^N\backslash\Omega} J(x-y)dy & \text{if } x \in E. \end{cases}$$

In Theorem 1.63 we will give another characterization of the ergodicity of Δ_m in terms of geometric properties.

Lemma 1.62. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and suppose that ν is a probability measure. Then, for $D \in \mathcal{B}$, the following statements are equivalent*
(i) D is ν-invariant.
(ii) $\Delta_m \chi_D = 0$ ν-a.e.
(iii) $P_m(D) = 0$.
(iv) $\displaystyle\int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) = -\nu(D)$.

Proof. $(i) \Leftrightarrow (ii)$ By definition of a ν-invariant set and of the Laplace operator.

$(ii) \Rightarrow (iii)$ By hypothesis, $m_x(D) = M_m\chi_D(x) = \chi_D(x)$ for ν-a.e. $x \in X$, thus, in particular, $m_x(X\backslash D) = 0$ for ν-a.e. $x \in D$ and, therefore,

$$P_m(D) = L_m(D, X\backslash D) = 0.$$

$(iii) \Rightarrow (ii)$ Suppose that $P_m(D) = 0$. Then, by (1.6), we have that

$$\nu(D) = \int_D \int_D dm_x(y)d\nu(x) = \int_D m_x(D)d\nu(x)$$

thus

$$m_x(D) = 1 \quad \text{for } \nu\text{-a.e. } x \in D.$$

Moreover, by the invariance of ν with respect to m, we have that

$$\nu(D) = \int_X m_x(D)d\nu(x) = \int_D m_x(D)d\nu(x) + \int_{X\backslash D} m_x(D)d\nu(x) = \nu(D) + \int_{X\backslash D} m_x(D)d\nu(x),$$

thus

$$m_x(D) = 0 \quad \text{for } \nu\text{-a.e. } x \in X\backslash D.$$

Consequently,

$$M_m\chi_D(x) = m_x(D) = \chi_D(x) \quad \text{for } \nu\text{-a.e. } x \in X$$

as desired.

$(iv) \Leftrightarrow (iii)$ By (1.11), we have that

$$\int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) = 2P_m(D) - \nu(D),$$

thus $P_m(D) = 0$ if, and only if, $\displaystyle\int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) = -\nu(D)$. □

Theorem 1.63. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and suppose that ν is a probability measure. The following statements are equivalent:*
(i) Δ_m is ergodic.
(ii) For every $D \in \mathcal{B}$, $\Delta_m\chi_D = 0$ ν-a.e. $\Rightarrow \nu(D) = 0$ or $\nu(D) = 1$.
(iii) For every $D \in \mathcal{B}$, $0 < \nu(D) < 1 \Rightarrow P_m(D) > 0$.
(iv) For every $D \in \mathcal{B}$,

$$0 < \nu(D) < 1 \Rightarrow \frac{1}{\nu(D)} \int_D \mathcal{H}^m_{\partial D}(x)d\nu(x) > -1$$

Proof. $(i) \Rightarrow (ii)$ Straightforward.

$(ii) \Rightarrow (iii)$ If $P_m(D) = 0$ then, by Lemma 1.62, $\Delta_m \chi_D = 0$ ν-a.e. thus (ii) implies that $\nu(D) = 0$ or $\nu(D) = 1$.

$(iii) \Rightarrow (ii)$ Let $D \in \mathcal{B}$. If $\Delta_m \chi_D = 0$ ν-a.e. then, by Lemma 1.62, we have that $P_m(D) = 0$ thus (iii) implies that $\nu(D) = 0$ or $\nu(D) = 1$.

$(ii) \Rightarrow (i)$ Suppose that Δ_m is not ergodic. Then, by Theorem 1.50, $[X, \mathcal{B}, m, \nu]$ is not m-connected so there exists $D \in \mathcal{B}$ with $\nu(D) > 0$ such that $0 < \nu(N_D^m) < 1$ (recall Corollary 1.28). However, by Proposition 1.26, $\Delta_m \chi_{N_D^m}(x) = 0$ and, by hypothesis, this implies that $\nu(N_D^m) = 0$ or $\nu(N_D^m) = 1$, which is a contradiction.

$(iii) \Leftrightarrow (iv)$ This equivalence follows by (1.11) and Lemma 1.62. $\square$

We conclude this section with the following result, which will be used in Chapter 4.

Lemma 1.64. *Let $[X, d, m, \nu]$ be a metric random walk space, and let $x \in X$. Suppose that $[X, d, m, \nu]$ has the strong-Feller property at x. Then, for a sequence of measurable sets $A_n \subset B(x, \frac{1}{n})$ with $\nu(A_n) > 0$, $n \in \mathbb{N}$, we have:*

$$\lim_{n\to\infty} \frac{1}{\nu(A_n)} \int_{A_n} m_y(E) d\nu(y) = m_x(E) \text{ for every measurable set } E \subset X.$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\nu(A_n)} \int_{A_n} \mathcal{H}^m_{\partial E}(y) d\nu(y) = \mathcal{H}^m_{\partial E}(x) \text{ for every measurable set } E \subset X.$$

Proof. Let $E \subset X$ be a measurable set. Since the random walk has the strong-Feller property at x, there exists $n_0 \in \mathbb{N}$ such that $|m_y(E) - m_x(E)| < \varepsilon$ for every $y \in B(x, \frac{1}{n_0})$. Then, for $n \geqslant n_0$, since $A_n \subset B(x, \frac{1}{n})$, we have

$$\left| \frac{1}{\nu(A_n)} \int_{A_n} m_y(E) d\nu(y) - m_x(E) \right| \leqslant \frac{1}{\nu(A_n)} \int_{A_n} |m_y(E) - m_x(E)| d\nu(y) < \varepsilon. \qquad \square$$

1.6. Poincaré type inequalities

Poincaré type inequalities like those defined in Definition 1.67 and Definition 1.81 (see also Remark 1.82) will play a very important role in this thesis. Assuming that a Poincaré type inequality is satisfied we will be able to obtain results on the rates of convergence of both the heat flow and the total variation flow. Moreover, we will also assume that an inequality of this type holds in order to prove existence of solutions to some of the problems in Chapter 5.

We first introduce some notation.

Definition 1.65. Let $(X, \mathcal{B}, \nu)$ be a probability space. We denote the mean value of $f \in L^1(X, \nu)$ (or the expected value of f) with respect to ν by

$$\nu(f) := \mathbb{E}_\nu(f) = \int_X f(x) d\nu(x).$$

Moreover, given $f \in L^2(X, \nu)$, we denote its variance with respect to ν by

$$\mathrm{Var}_\nu(f) := \int_X (f(x) - \nu(f))^2 d\nu(x) = \frac{1}{2} \int_{X\times X} (f(x) - f(y))^2 d\nu(y) d\nu(x).$$

In general, if $\nu(X) < \infty$ we will also denote the mean of a function $f \in L^1(X, \nu)$ by $\nu(f)$, i.e.,

$$\nu(f) := \frac{1}{\nu(X)} \int_X f(x) d\nu(x).$$

We now introduce the nonlocal counterpart of the Dirichlet energy.

Definition 1.66. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. We define the energy functional $\mathcal{H}_m : L^2(X,\nu) \to [0,+\infty]$ by

$$\mathcal{H}_m(f) := \begin{cases} \displaystyle\frac{1}{2}\int_{X\times X} (f(x)-f(y))^2 dm_x(y) d\nu(x) & \text{if } f \in L^1(X,\nu)\cap L^2(X,\nu), \\ +\infty & \text{else,} \end{cases}$$

and denote

$$D(\mathcal{H}_m) := L^1(X,\nu)\cap L^2(X,\nu).$$

Note that, by Proposition 1.47,

$$\mathcal{H}_m(f) = -\int_X f(x)\Delta_m f(x) d\nu(x) \quad \textit{for every } f \in D(\mathcal{H}_m).$$

Definition 1.67. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and suppose that ν is a probability measure. We say that $[X, \mathcal{B}, m, \nu]$ satisfies a *Poincaré inequality* if there exists $\lambda > 0$ such that

$$\lambda \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu),$$

or, equivalently,

$$\lambda \|f\|^2_{L^2(X,\nu)} \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu) \text{ with } \nu(f) = 0.$$

More generally, we say that $[X, \mathcal{B}, m, \nu]$ satisfies a (p,q)-*Poincaré inequality* ($p, q \in [1,+\infty[$) if there exists a constant $\Lambda > 0$ such that, for any $u \in L^q(X,\nu)$,

$$\|u\|_{L^p(X,\nu)} \leqslant \Lambda \left(\left(\int_X \int_X |u(y)-u(x)|^q dm_x(y) d\nu(x) \right)^{\frac{1}{q}} + \left| \int_X u \, d\nu \right| \right),$$

or, equivalently, there exists a $\Lambda > 0$ such that

$$\|u\|_{L^p(X,\nu)} \leqslant \Lambda \|\nabla u\|_{L^q(X\times X, d(\nu\otimes m_x))} \quad \text{for all } u \in L^q(X,\nu) \text{ with } \nu(u) = 0.$$

When $[X, \mathcal{B}, m, \nu]$ satisfies a $(p,1)$-Poincaré inequality, we will say that $[X, \mathcal{B}, m, \nu]$ satisfies a p-Poincaré inequality.

The spectral gap of the Laplace operator is closely related to the Poincaré inequality.

Definition 1.68. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and suppose that ν is a probability measure. The *spectral gap* of $-\Delta_m$ is defined as

$$\begin{aligned} \mathrm{gap}(-\Delta_m) &:= \inf \left\{ \frac{\mathcal{H}_m(f)}{\mathrm{Var}_\nu(f)} : f \in D(\mathcal{H}_m), \ \mathrm{Var}_\nu(f) \neq 0 \right\} \\ &= \inf \left\{ \frac{\mathcal{H}_m(f)}{\|f\|^2_{L^2(X,\nu)}} : f \in D(\mathcal{H}_m), \ \|f\|_{L^2(X,\nu)} \neq 0, \ \int_X f d\nu = 0 \right\}. \end{aligned} \tag{1.13}$$

Observe that, as mentioned in Remark 1.46, since ν is a probability measure, we have that

$$D(\mathcal{H}_m) = L^2(X,\nu).$$

Remark 1.69. If $\mathrm{gap}(-\Delta_m) > 0$, then $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality with $\lambda = \mathrm{gap}(-\Delta_m)$:

$$\mathrm{gap}(-\Delta_m)\mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu),$$

being the spectral gap the best constant in the Poincaré inequality.

Therefore, we are interested in studying when the spectral gap of $-\Delta_m$ is positive.

Remark 1.70. Suppose that X is a Polish metric space and that ν is reversible with respect to m. Sufficient conditions for the existence of a Poincaré inequality can be found in, for example, [**134**, Corollary 31] or [**151**, Theorem 1]. In [**134**] the positivity of the coarse Ricci curvature (see Remark 1.96) is assumed while in [**151**] the hypothesis is the following Foster Lyapunov condition:

$$M_m V \leqslant (1-\lambda)V + b\chi_K,$$
$$M_m 1_A(x) \geqslant \alpha\mu(A)\chi_K \quad \forall A \in \mathcal{B},$$

for a positive function $V : \mathbb{R}^d \to [1,\infty)$, numbers $b < \infty$, α, $\lambda > 0$, a set $K \subset X$, and a probability measure μ. Moreover, in Theorem 2.19, in relation to another notion of Ricci curvature bounded from below, we will find further sufficient conditions for the existence of a Poincaré inequality.

If (X, d, μ) is a length space, μ is doubling[7] and $[X, d, m^{\mu,\varepsilon}, \mu]$ (recall Example 1.41) is a metric random walk space, sufficient conditions for the existence of a Poincaré inequality can be found in [**93**, Section 2.3].

Definition 1.71. Let $(X, \mathcal{B}, \nu)$ be a probability space. We denote by $H(X,\nu)$ the subspace of $L^2(X,\nu)$ consisting of the functions which are orthogonal to the constants, i.e.,

$$H(X,\nu) := \left\{ f \in L^2(X,\nu) \,:\, \nu(f) = 0 \right\}.$$

Remark 1.72. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and suppose that ν is a probability measure. Since the operator $-\Delta_m : H(X,\nu) \to H(X,\nu)$ is self-adjoint and non-negative, and $\|\Delta_m\| \leqslant 2$ (see Theorem 2.4), by [**44**, Proposition 6.9] we have that the spectrum $\sigma(-\Delta_m)$ of $-\Delta_m$ in $H(X,\nu)$ satisfies

$$\sigma(-\Delta_m) \subset [\alpha, \beta] \subset [0,2],$$

where

$$\alpha := \inf \left\{ \langle -\Delta_m u, u \rangle \,:\, u \in H(X,\nu),\ \|u\|_{L^2(X,\nu)} = 1 \right\} \in \sigma(-\Delta_m),$$

and

$$\beta := \sup \left\{ \langle -\Delta_m u, u \rangle \,:\, u \in H(X,\nu),\ \|u\|_{L^2(X,\nu)} = 1 \right\} \in \sigma(-\Delta_m).$$

Let's see that $\operatorname{gap}(-\Delta_m) = \alpha$. By definition we have that $\operatorname{gap}(-\Delta_m) \leqslant \alpha$ (recall that $\mathcal{H}_m(u) = \langle -\Delta_m u, u \rangle$). Now, for the opposite inequality, let $f \in L^2(X,\nu)$ with $\operatorname{Var}_\nu(f) \neq 0$. Then, $u := f - \nu(f) \neq 0$ belongs to $H(X,\nu)$, so

$$\alpha \leqslant \mathcal{H}_m\left(\frac{u}{\|u\|_{L^2(X,\nu)}} \right) = \frac{\mathcal{H}_m(u)}{\|u\|^2_{L^2(X,\nu)}} = \frac{\mathcal{H}_m(f)}{\operatorname{Var}_\nu(f)},$$

and, therefore, $\operatorname{gap}(-\Delta_m) \geqslant \alpha$.

As a consequence, we obtain that

$$\operatorname{gap}(-\Delta_m) > 0 \Leftrightarrow 0 \notin \sigma(-\Delta_m).$$

With this remark at hand we are able to obtain the following result.

Proposition 1.73. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and suppose that ν is a probability measure. If $-\Delta_m$ is the sum of an invertible and a compact operator in $H(X,\nu)$, then* $\operatorname{gap}(-\Delta_m) > 0$.

Consequently, if the averaging operator M_m is compact in $H(X,\nu)$ then $\operatorname{gap}(-\Delta_m) > 0$.

Proof. If we assume that $-\Delta_m$ is the sum of an invertible and a compact operator in $H(X,\nu)$, then, if $0 \in \sigma(-\Delta_m)$, by Fredholm's alternative Theorem, we have that there exists $u \in H(X,\nu)$, $u \neq 0$, such that $-\Delta_m u = (I - M_m)u = 0$. Then, since $[X, d, m, \nu]$ is m-connected, by Theorem 1.50, we obtain that Δ_m is ergodic so u is ν-a.e. a constant. Therefore, since $u \in H(X,\nu)$, we must have $u = 0$ ν-a.e., which is a contradiction. □

[7]A nontrivial measure μ on a metric space X is said to be doubling if there exists a constant $C > 0$ such that $0 < \mu(B(x,2r)) \leqslant C\mu(B(x,r)) < \infty$ for all $x \in X$ and $r > 0$.

Example 1.74. If $G = (V(G), E(G))$ is a finite connected weighted discrete graph, then, obviously, M_{m^G} is compact and, consequently, $\mathrm{gap}(-\Delta_m^G) > 0$. In this situation, it is well known that, if $\sharp(V(G)) = N$, the spectrum of $-\Delta_{m^G}$ is $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{N-1}$ and $0 < \lambda_1 = \mathrm{gap}(-\Delta_m^G)$ (see [**63**]).

In fact, we can easily prove that $[V(G), d_G, m^G, \nu_G]$ satisfies a (p,q)-Poincaré inequality for any p, $q \in [1, \infty[$. Indeed, let p, $q \in [1, \infty[$ and suppose that a (p,q)-Poincaré inequality does not hold. Then, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset L^p(V(G), \nu_G)$ with $\|u_n\|_{L^p(V(G),\nu_G)} = 1$ and $\int_{V(G)} u_n(x) d\nu(x) = 0$ $\forall n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sum_{x\in V(G)} \sum_{y\sim x} w_{xy} |u_n(x) - u_n(y)|^q = 0.$$

Hence,

$$\lim_{n\to\infty} |u_n(x) - u_n(y)| = 0 \quad \text{for every } x, y \in V(G),\ x \sim y. \tag{1.14}$$

Moreover, since $\|u_n\|_{L^p(V(G),\nu_G)} = 1$, we have that, up to a subsequence,

$$\lim_{n\to\infty} u_n(x) = u(x) \in \mathbb{R} \quad \text{for every } x \in V(G).$$

However, since the graph is connected, we get, by (1.14), $u(x) = u(y)$ for every x, $y \in V(G)$, i.e., there exists $\lambda \in \mathbb{R}$ such that $u(x) = \lambda$ for every $x \in V(G)$; thus $u_n \to \lambda$ in $L^p(V(G), \nu_G)$. Therefore, since $\int_{V(G)} u_n(x) d\nu_G(x) = 0$, we get that $\lambda = 0$, which is in contradiction with $\|u_n\|_{L^p(V(G),\nu_G)} = 1$.

Example 1.75. Let Ω be a bounded domain in $\mathbb{R}^N$ and let J be a kernel such that $J \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative and radially symmetric, with $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$. Consider the reversible metric random walk space $[\Omega, \mathcal{B}_\Omega, m^{J,\Omega}, \mathcal{L}^N]$ as defined in Example 1.42 (recall also Example 1.37).

Then, $-\Delta_{m^{J,\Omega}}$ is the sum of an invertible and a compact operator. Indeed,

$$-\Delta_{m^{J,\Omega}} f(x) = \int_\Omega J(x-y)dy f(x) - \int_\Omega f(y)J(x-y)dy,\ x \in \Omega,$$

where $f \mapsto \displaystyle\int_\Omega J(\cdot - y)dy f(\cdot)$ is an invertible operator in $H(\Omega, \mathcal{L}^N)$ (J is continuous, $J(0) > 0$ and Ω is a domain thus $\int_\Omega J(x-y)dy > 0$ for every $x \in \Omega$) and $f \mapsto \displaystyle\int_\Omega f(y)J(\cdot - y)dy$ is a compact operator in $H(\Omega, \mathcal{L}^N)$ (this follows by the Arzelà-–Ascoli theorem). Hence, in this case, we have that $\mathrm{gap}(-\Delta_{m^{J,\Omega}})$ is equal to (see also [**18**])

$$\inf \left\{ \frac{\frac{1}{2}\int_{\Omega\times\Omega} J(x-y)(u(y)-u(x))^2 dxdy}{\int_\Omega u(x)^2 dx} \ :\ u \in L^2(\Omega),\ \|u\|_{L^2(\Omega)} > 0,\ \int_\Omega u = 0 \right\} > 0.$$

Let us point out that the condition $J(0) > 0$ is necessary since, otherwise, $\int_\Omega J(\cdot - y)dy$ may be 0 on a set of positive measure (see [**18**, Remark 6.20]).

Another result in which we provide sufficient conditions for the positivity of $\mathrm{gap}(-\Delta_m)$ *is the following. In the proof we will use that, as a consequence of a result by Miclo* [**129**], $\mathrm{gap}(-\Delta_m) > 0$ *if* Δ_m *is ergodic and* M_m *is hyperbounded, that is, if there exists* $q > 2$ *such that* M_m *is bounded from* $L^2(X, \nu)$ *to* $L^q(X, \nu)$.

Proposition 1.76. *Let* $[X, \mathcal{B}, m, \nu]$ *be a reversible random walk space and suppose that* ν *is a probability measure. Assume that* Δ_m *is ergodic and that* $m_x \ll \nu$ *for every* $x \in X$*. If there exists* $p > 2$ *and a constant* K *such that*

$$\int_X \left\| \frac{dm_x}{d\nu} \right\|_{L^2(X,\nu)}^p d\nu(x) \leqslant K < \infty,$$

then $\text{gap}(-\Delta_m) > 0$.

Proof. Let $f_x := \frac{dm_x}{d\nu} \in L^1(X,\nu)$, $x \in X$. Let's see that M_m is hyperbounded. Given $u \in L^2(X,\nu)$, by the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \|M_m u\|_p^p &= \int_X |M_m u(x)|^p d\nu(x) = \int_X \left| \int_X u(y) dm_x(y) \right|^p d\nu(x) \\ &= \int_X \left| \int_X u(y) f_x(y) d\nu(y) \right|^p d\nu(x) \\ &\leqslant \|u\|_{L^2(X,\nu)}^p \int_X \|f_x\|_{L^2(X,\nu)}^p d\nu(x), \end{aligned}$$

hence

$$\|M_m u\|_p \leqslant K^{\frac{1}{p}} \|u\|_{L^2(X,\nu)}.$$

Therefore, M_m is hyperbounded as desired. □

In the next examples we give random walk spaces for which a Poincaré inequality does not hold.

Example 1.77. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with the locally finite weighted discrete graph G with vertex set $V(G) := \{x_3, x_4, x_5 \ldots, x_n, \ldots\}$ and weights:

$$w_{x_{3n},x_{3n+1}} = \frac{1}{n^3},\ w_{x_{3n+1},x_{3n+2}} = \frac{1}{n^2},\ w_{x_{3n+2},x_{3n+3}} = \frac{1}{n^3},$$

for $n \geqslant 1$, and $w_{x_i,x_j} = 0$ otherwise (recall Example 1.38).

(i) Let

$$f_n(x) := \begin{cases} n & \text{if } \ x = x_{3n+1}, x_{3n+2}, \\ 0 & \text{else.} \end{cases}$$

Note that $\nu_G(V(G)) < +\infty$ (we avoid its normalization to a probability measure for simplicity). Now,

$$\begin{aligned} 2\mathcal{H}_m(f_n) &= \int_{V(G)} \int_{V(G)} (f_n(x) - f_n(y))^2 dm_x^G(y) d\nu_G(x) \\ &= d_{x_{3n}} \int_{V(G)} (f_n(x_{3n}) - f_n(y))^2 dm_{x_{3n}}^G(y) \\ &\quad + d_{x_{3n+1}} \int_{V(G)} (f_n(x_{3n+1}) - f_n(y))^2 dm_{x_{3n+1}}^G(y) \\ &\quad + d_{x_{3n+2}} \int_{V(G)} (f_n(x_{3n+2}) - f_n(y))^2 dm_{x_{3n+2}}^G(y) \\ &\quad + d_{x_{3n+3}} \int_{V(G)} (f_n(x_{3n+3}) - f_n(y))^2 dm_{x_{3n+3}}^G(y) \\ &= d_{x_{3n}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n}}} + d_{x_{3n+1}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n+1}}} + d_{x_{3n+2}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n+2}}} + d_{x_{3n+3}} n^2 \frac{\frac{1}{n^3}}{d_{x_{3n+3}}} \\ &= \frac{4}{n}. \end{aligned}$$

However, we have

$$\int_{V(G)} f_n(x) d\nu_G(x) = n(d_{x_{3n+1}} + d_{x_{3n+2}}) = 2n\left(\frac{1}{n^2} + \frac{1}{n^3}\right) = \frac{2}{n}\left(1 + \frac{1}{n}\right),$$

thus

$$\nu_G(f_n) = \frac{\frac{2}{n}\left(1+\frac{1}{n}\right)}{\nu_G(V(G))} = \widetilde{O}\left(\frac{1}{n}\right),$$

where we use the notation

$$\varphi(n) = \widetilde{O}(\psi(n)) \Leftrightarrow \limsup_{n\to\infty}\left|\frac{\varphi(n)}{\psi(n)}\right| = C \neq 0.$$

Therefore,

$$(f_n(x) - \nu(f_n))^2 = \begin{cases} \widetilde{O}(n^2) & \text{if } x = x_{3n+1}, x_{3n+2}, \\ \widetilde{O}\left(\frac{1}{n^2}\right) & \text{otherwise.} \end{cases}$$

Finally,

$$\begin{aligned} \operatorname{Var}_{\nu_G}(f_n) &= \int_{V(G)} (f_n(x) - \nu_G(f_n))^2 d\nu_G(x) \\ &= \widetilde{O}\left(\frac{1}{n^2}\right) \sum_{x\neq x_{3n+1}, x_{3n+2}} d_x + \widetilde{O}(n^2)(d_{x_{3n+1}} + d_{x_{3n+2}}) \\ &= \widetilde{O}\left(\frac{1}{n^2}\right) + 2\widetilde{O}(n^2)\left(\frac{1}{n^2} + \frac{1}{n^3}\right) = \widetilde{O}(1). \end{aligned}$$

Consequently, $[V(G), d_G, m^G, \nu_G]$ does not satisfy a Poincaré inequality.
(ii) Let

$$f_n(x) := \begin{cases} n^2 & \text{if } x = x_{3n+1}, x_{3n+2}, \\ 0 & \text{else.} \end{cases}$$

With similar computations (see also [**124**, Example 4.7]), we get that

$$\int_{V(G)}\int_{V(G)} |f_n(x) - f_n(y)| dm_x^G(y) d\nu_G(x) = \frac{4}{n}$$

and

$$\int_{V(G)} |f_n(x) - \nu_G(f_n)| d\nu_G(x) = \widetilde{O}(1).$$

Therefore, $[V(G), d_G, m^G, \nu_G]$ does not satisfy a 1-Poincaré inequality.

Example 1.78. Consider the metric random walk space $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ (recall Example 1.37), where d is the Euclidean distance and $J(x) = \frac{1}{2}\chi_{[-1,1]}$. Define, for $n \in \mathbb{N}$,

$$u_n = \frac{1}{2^{n+1}}\chi_{[2^n, 2^{n+1}]} - \frac{1}{2^{n+1}}\chi_{[-2^{n+1}, -2^n]}.$$

Then, $\|u_n\|_1 := 1$, $\displaystyle\int_{\mathbb{R}} u_n(x)dx = 0$ and it is easy to see that, for n large enough,

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |u_n(y) - u_n(x)| dm_x^J(y) dx = \frac{1}{2^{n+1}}.$$

Therefore, $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ does not satisfy a 1-Poincaré inequality.

Let's now see that, if $gap(-\Delta_m) > 0$, then Δ_m is ergodic.

Proposition 1.79. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and assume that ν is a probability measure. If $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality, then Δ_m is ergodic (i.e., $[X, \mathcal{B}, m, \nu]$ is m-connected).*

Proof. Let $f \in D(\Delta_m)$ such that $\Delta_m(f) = 0$ ν-a.e. Then,

$$\mathcal{H}_m(f) = -\int_X f(x)\Delta_m f(x) d\nu(x) = 0$$

and, therefore, if $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality, we have that

$$\mathrm{Var}_\nu(f) := \int_X (f(x) - \nu(f))^2 d\nu(x) = 0$$

thus f is ν-a.e. a constant:

$$f(x) = \int_X f(x) d\nu(x) \quad \text{for } \nu\text{-a.e. } x \in X. \qquad \square$$

Example 1.77 shows that the reverse implication does not hold in general. Finally, we give the following result which may aid in finding lower bounds for gap$(-\Delta_m)$.

Theorem 1.80. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space such that ν is a probability measure. Assume that Δ_m is ergodic. Then,*

$$\mathrm{gap}(-\Delta_m) = \sup\left\{\lambda \geqslant 0 \,:\, \lambda\mathcal{H}_m(f) \leqslant \int_X (-\Delta_m f)^2 d\nu \ \ \forall f \in L^2(X, \nu)\right\}.$$

Proof. By Remark 1.72 we know that $\mathrm{gap}(-\Delta_m) = \alpha$, where

$$\alpha := \inf\left\{\langle -\Delta_m u, u\rangle \,:\, u \in H(X, \nu),\ \|u\|_{L^2(X,\nu)} = 1\right\} \in \sigma(-\Delta_m).$$

Let also, as in that remark,

$$\beta := \sup\left\{\langle -\Delta_m u, u\rangle \,:\, u \in H(X, \nu),\ \|u\|_{L^2(X,\nu)} = 1\right\} \in \sigma(-\Delta_m).$$

Set

$$A := \sup\left\{\lambda \geqslant 0 \,:\, \lambda\mathcal{H}_m(f) \leqslant \int_X (-\Delta_m f)^2 d\nu \ \ \forall f \in L^2(X, \nu)\right\}.$$

Let's see that $\alpha \leqslant A$. Let $(P_\lambda)_{\lambda \geqslant 0}$ be the spectral projection of the self-adjoint and positive operator $-\Delta_m : H(X, \nu) \to H(X, \nu)$. By the spectral theorem [**140**, Theorem VIII. 6], we have, for any $f \in H(X, \nu)$,

$$\mathcal{H}_m(f) = \langle -\Delta_m f, f\rangle = \int_\alpha^\beta \lambda d\langle P_\lambda f, f\rangle$$

and

$$\int_X (-\Delta_m f)^2 d\nu = \langle -\Delta_m f, -\Delta_m f\rangle = \int_\alpha^\beta \lambda^2 d\langle P_\lambda f, f\rangle.$$

Hence, for any $f \in H(X, \nu)$,

$$\int_X (-\Delta_m f)^2 d\nu \geqslant \alpha \int_\alpha^\beta \lambda d\langle P_\lambda f, f\rangle = \alpha\mathcal{H}_m(f),$$

and we get $\alpha \leqslant A$ (note that, for any $f \in L^2(X, \nu)$, we may take $g := f - \nu(f) \in H(X, \nu)$ and we have that $\Delta_m(g) = \Delta_m(f)$).

Finally, let us see that $\alpha \geqslant A$. Since $\alpha \in \sigma(\Delta_m)$, given $\epsilon > 0$, there exists $0 \neq f \in \mathrm{Range}(P_{\alpha+\epsilon})$ and, consequently, $P_\lambda f = f$ for $\lambda \geqslant \alpha+\epsilon$. Then, since Δ_m is ergodic, $-\Delta_m(f) \neq 0$ ($0 \neq f \in H(X, \nu)$ is not ν-a.e. a constant), thus

$$\begin{aligned} 0 < \int_X (-\Delta_m f)^2 d\nu &= \int_\alpha^{\alpha+\epsilon} \lambda^2 d\langle P_\lambda f, f\rangle \leqslant (\alpha + \epsilon) \int_\alpha^{\alpha+\epsilon} \lambda d\langle P_\lambda f, f\rangle = (\alpha + \epsilon)\mathcal{H}_m(f) \\ &< (\alpha + 2\epsilon)\mathcal{H}_m(f). \end{aligned}$$

This implies that $\alpha + 2\epsilon$ does not belong to the set

$$\left\{\lambda \geqslant 0 \,:\, \lambda\mathcal{H}_m(f) \leqslant \int_X (-\Delta_m f)^2 d\nu \ \ \forall f \in L^2(X, \nu)\right\},$$

thus $A < \alpha + 2\epsilon$. Therefore, since $\epsilon > 0$ was arbitrary, we have

$$A \leqslant \alpha. \qquad \square$$

1.6.1. Poincaré type inequalities on subsets. *Let us now consider Poincaré type inequalities on subsets.*

DEFINITION 1.81. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let $A, B \in \mathcal{B}$ be disjoint sets such that $\nu(A) > 0$. Let $Q := ((A \cup B) \times (A \cup B)) \setminus (B \times B)$. We say that $[X, \mathcal{B}, m, \nu]$ satisfies a *generalised (p, q)-Poincaré type inequality ($p, q \in [1, +\infty[$) on* (A, B), if, given $0 < l \leqslant \nu(A \cup B)$, there exists a constant $\Lambda > 0$ such that, for any $u \in L^q(A \cup B, \nu)$ and any $Z \in \mathcal{B}_{A\cup B}$ with $\nu(Z) \geqslant l$,

$$\|u\|_{L^p(A\cup B,\nu)} \leqslant \Lambda \left(\left(\int_Q |u(y) - u(x)|^q dm_x(y) d\nu(x) \right)^{\frac{1}{q}} + \left| \int_Z u \, d\nu \right| \right).$$

REMARK 1.82. These notations allows us to cover many situations. For example,
(i) If $A = X$, $B = \emptyset$ and $[X, \mathcal{B}, m, \nu]$ satisfies a generalised $(2, 2)$-Poincaré type inequality on $(X, \emptyset)$ then $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality as defined in Definition 1.67.
(ii) Let $\Omega \in \mathcal{B}$. If $A := \Omega$, $B := \partial_m \Omega$ and we assume that a (p, p)-Poincaré type inequality on (A, B) holds then the inequality takes the following form:

$$\|u\|_{L^p(\Omega_m,\nu)} \leqslant \Lambda \left(\left(\int_{(\Omega_m \times \Omega_m) \setminus (\partial_m \Omega \times \partial_m \Omega)} |u(y) - u(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + \left| \int_Z u \, d\nu \right| \right)$$

which will be extensively used in Chapter 5. Moreover, if $A := \Omega_m$ and $B := \emptyset$ we obtain

$$\|u\|_{L^p(\Omega_m,\nu)} \leqslant \Lambda \left(\left(\int_{(\Omega_m \times \Omega_m)} |u(y) - u(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + \left| \int_Z u \, d\nu \right| \right)$$

which will also be widely used in Chapter 5.

In Theorem 1.84 we give sufficient conditions for a random walk space to satisfy inequalities of this kind. Let's first prove the following lemma.

LEMMA 1.83. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $A, B \in \mathcal{B}$ be disjoint sets such that $B \subset \partial_m A$, $\nu(A) > 0$ and A is m-connected (recall Definition 1.32). Suppose that $\nu(A \cup B) < \infty$ and that*

$$\nu\left(\{x \in A \cup B \, : \, (m_x \llcorner A) \perp (\nu \llcorner A)\}\right) = 0.$$

Let $q \geqslant 1$. Let $\{u_n\}_n \subset L^q(A \cup B, \nu)$ be a bounded sequence in $L^1(A \cup B, \nu)$ satisfying

$$\lim_{n\to\infty} \int_Q |u_n(y) - u_n(x)|^q dm_x(y) d\nu(x) = 0 \tag{1.15}$$

where, as in Definition 1.81, $Q = ((A \cup B) \times (A \cup B)) \setminus (B \times B)$. Then, there exists $\lambda \in \mathbb{R}$ such that

$$u_n(x) \to \lambda \quad \textit{for } \nu\textit{-a.e. } x \in A \cup B,$$

$$\|u_n - \lambda\|_{L^q(A,m_x)} \to 0 \quad \textit{for } \nu\textit{-a.e. } x \in A \cup B,$$

and

$$\|u_n - \lambda\|_{L^q(A\cup B,m_x)} \to 0 \quad \textit{for } \nu\textit{-a.e. } x \in A.$$

PROOF. If $B = \emptyset$ (or $\nu(B) = 0$) one can skip some steps of the proof. Let

$$F_n(x, y) = |u_n(y) - u_n(x)|, \quad (x, y) \in Q,$$

$$f_n(x) = \int_A |u_n(y) - u_n(x)|^q \, dm_x(y), \quad x \in A \cup B,$$

and
$$g_n(x) = \int_{A\cup B} |u_n(y) - u_n(x)|^q \, dm_x(y), \quad x \in A.$$
Let
$$\mathcal{N}_\perp := \{x \in A \cup B \,:\, (m_x \llcorner A) \perp (\nu \llcorner A)\}.$$
From (1.15), it follows that
$$f_n \to 0 \quad \text{in } L^1(A \cup B, \nu)$$
and
$$g_n \to 0 \quad \text{in } L^1(A, \nu).$$
Passing to a subsequence if necessary, we can assume that

(1.16) $$f_n(x) \to 0 \quad \text{for every } x \in (A \cup B)\backslash N_f, \quad \text{where } N_f \subset A \cup B \text{ is } \nu\text{-null}$$

and

(1.17) $$g_n(x) \to 0 \quad \text{for every } x \in A\backslash N_g, \quad \text{where } N_g \subset A \text{ is } \nu\text{-null}.$$

On the other hand, by (1.15), we also have that
$$F_n \to 0 \quad \text{in } L^q(Q, \nu \otimes m_x).$$
Therefore, we can suppose that, up to a subsequence,

(1.18) $$F_n(x,y) \to 0 \quad \text{for every } (x,y) \in Q\backslash C, \quad \text{where } C \subset Q \text{ is } \nu \otimes m_x\text{-null}.$$

Let $N_1 \subset A$ be a ν-null set satisfying that,
$$\text{for all } x \in A\backslash N_1, \text{ the section } C_x := \{y \in A \cup B : (x,y) \in C\} \text{ of } C \text{ is } m_x\text{-null},$$
and $N_2 \subset A \cup B$ be a ν-null set satisfying that,
$$\text{for all } x \in (A \cup B)\backslash N_2, \text{ the section } C'_x := \{y \in A : (x,y) \in C\} \text{ of } C \text{ is } m_x\text{-null}.$$

Now, since A is m-connected and $B \subset \partial_m A$, we have that
$$D := \{x \in A \cup B : m_x(A) = 0\}$$
is ν-null. Indeed, by the definition of D, we have that $L_m(A \cap D, A) = 0$ thus, since A is m-connected, we must have $\nu(A \cap D) = 0$. Now, since $B \subset \partial_m A$, $m_x(A) > 0$ for every $x \in B$, thus $\nu(B \cap D) = 0$.

Set $N := \mathcal{N}_\perp \cup N_f \cup N_g \cup N_1 \cup N_2 \cup D$ (note that $\nu(N) = 0$). Fix $x_0 \in A\backslash N$. Up to a subsequence we have that $u_n(x_0) \to \lambda$ for some $\lambda \in [-\infty, +\infty]$; let
$$S := \{x \in A \cup B \,:\, u_n(x) \to \lambda\}$$
and let's see that $\nu((A \cup B)\backslash S) = 0$.

By (1.18), since $u_n(x_0) \to \lambda$, we also have that $u_n(y) \to \lambda$ for every $y \in (A \cup B)\backslash C_{x_0}$. However, since $x_0 \notin \mathcal{N}_\perp$ and $m_{x_0}(C_{x_0}) = 0$, we must have that $\nu(A\backslash C_{x_0}) > 0$; thus $\nu(A \cap S) \geqslant \nu(A\backslash C_{x_0}) > 0$. Note that, if $x \in (A \cap S)\backslash N$ then, by (1.18) again, $(A \cup B)\backslash C_x \subset S$ thus $m_x((A \cup B)\backslash S) \leqslant m_x(C_x) = 0$; therefore,
$$L_m(A \cap S, (A \cup B)\backslash S) = 0.$$
In particular, $L_m(A \cap S, A\backslash S) = 0$, but, since A is m-connected and $\nu(A \cap S) > 0$, we must have $\nu(A\backslash S) = 0$, i.e. $\nu(A) = \nu(A \cap S)$.

Now, suppose that $\nu(B\backslash S) > 0$. Let $x \in B\backslash(S \cup N)$. By (1.18), we have that $A\backslash C'_x \subset A\backslash S$, i.e., $A \cap S \subset C'_x$, thus $m_x(A \cap S) = 0$. Therefore, since $x \notin \mathcal{N}_\perp$, we must have $\nu(A\backslash S) > 0$ which is in contradiction with what we have already obtained. Consequently, we have obtained that u_n converges ν-a.e. in $A \cup B$ to λ:
$$u_n(x) \to \lambda \quad \text{for every } x \in S, \; \nu((A \cup B)\backslash S) = 0.$$

Since $\{\|u_n\|_{L^1(A\cup B,\nu)}\}_n$ is bounded, by Fatou's Lemma we must have that $\lambda \in \mathbb{R}$. On the other hand, by (1.16),
$$F_n(x,\cdot) \to 0 \quad \text{in } L^q(A, m_x),$$

for every $x \in \Omega \backslash N_f$. In other words, $\|u_n(\cdot) - u_n(x)\|_{L^q(A,m_x)} \to 0$, thus

$$\|u_n - \lambda\|_{L^q(A,m_x)} \to 0 \quad \text{for } \nu\text{-a.e. } x \in A \cup B.$$

Similarly, by (1.17),

$$\|u_n - \lambda\|_{L^q(A\cup B,m_x)} \to 0 \quad \text{for } \nu\text{-a.e. } x \in A. \qquad \square$$

THEOREM 1.84. *Let $p \geqslant 1$. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $A, B \in \mathcal{B}$ be disjoint sets such that $B \subset \partial_m A$, $\nu(A) > 0$ and A is m-connected. Suppose that $\nu(A \cup B) < \infty$ and that*

$$\nu\left(\{x \in A \cup B : (m_x \llcorner A) \perp (\nu \llcorner A)\}\right) = 0.$$

Assume further that, given a ν-null set $N \subset A$, there exist $x_1, x_2, \ldots, x_L \in A \backslash N$ and a constant $C > 0$ such that $\nu \llcorner (A \cup B) \leqslant C(m_{x_1} + \cdots + m_{x_L}) \llcorner (A \cup B)$. Then, $[X, \mathcal{B}, m, \nu]$ satisfies a generalised (p,p)-Poincaré type inequality on (A, B).

PROOF. Let $p \geqslant 1$ and $0 < l \leqslant \nu(A \cup B)$. We want to prove that there exists a constant $\Lambda > 0$ such that

$$\|u\|_{L^p(A\cup B,\nu)} \leqslant \Lambda \left(\left(\int_Q |u(y) - u(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + \left| \int_Z u \, d\nu \right| \right)$$

for every $u \in L^p(A \cup B, \nu)$ and every $Z \in \mathcal{B}_{A\cup B}$ with $\nu(Z) \geqslant l$. Suppose that this inequality is not satisfied for any Λ. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^p(A \cup B, \nu)$, with $\|u_n\|_{L^p(A\cup B,\nu)} = 1$, and a sequence $Z_n \in \mathcal{B}_{A\cup B}$ with $\nu(Z_n) \geqslant l$, $n \in \mathbb{N}$, satisfying

$$\lim_n \int_Q |u_n(y) - u_n(x)|^p dm_x(y) d\nu(x) = 0$$

and

$$\lim_n \int_{Z_n} u_n \, d\nu = 0.$$

Therefore, by Lemma 1.83, there exist $\lambda \in \mathbb{R}$ and a ν-null set $N \subset A$ such that

$$\|u_n - \lambda\|_{L^p(A\cup B,m_x)} \xrightarrow{n} 0 \quad \text{for every } x \in A \backslash N.$$

Now, by hypothesis, there exist $x_1, x_2, \ldots, x_L \in A \backslash N$ and $C > 0$ such that $\nu \llcorner (A \cup B) \leqslant C(m_{x_1} + \cdots + m_{x_L})$. Therefore,

$$\|u_n - \lambda\|^p_{L^p(A\cup B,\nu)} \leqslant C \sum_{i=1}^{L} \|u_n - \lambda\|^p_{L^p(A\cup B,m_{x_i})} \xrightarrow{n} 0.$$

Moreover, since $\{\chi_{Z_n}\}_n$ is bounded in $L^{p'}(A \cup B, \nu)$, there exists $\phi \in L^{p'}(A \cup B, \nu)$ such that, up to a subsequence, $\chi_{Z_n} \rightharpoonup \phi$ weakly in $L^{p'}(A \cup B, \nu)$ (weakly-$*$ in $L^\infty(A \cup B, \nu)$ in the case $p = 1$)[8]. In addition, $\phi \geqslant 0$ ν-a.e. in $A \cup B$ and

$$0 < l \leqslant \lim_{n\to+\infty} \nu(Z_n) = \lim_{n\to+\infty} \int_{A\cup B} \chi_{Z_n} d\nu = \int_{A\cup B} \phi d\nu.$$

Then, since $u_n \xrightarrow{n} \lambda$ in $L^p(A \cup B, \nu)$ and $\chi_{Z_n} \overset{n}{\rightharpoonup} \phi$ weakly in $L^{p'}(A \cup B, \nu)$ (weakly-$*$ in $L^\infty(A \cup B, \nu)$ in the case $p = 1$),

$$0 = \lim_{n\to+\infty} \int_{Z_n} u_n = \lim_{n\to+\infty} \int_{A\cup B} \chi_{Z_n} u_n = \lambda \int_{A\cup B} \phi d\nu,$$

thus $\lambda = 0$. This is in contradiction with $||u_n||_{L^p(A\cup B,\nu)} = 1$ $\forall n \in \mathbb{N}$, since $u_n \xrightarrow{n} \lambda$ in $L^p(A \cup B, \nu)$. $\square$

[8]Note that, since ν is a σ-finite measure and $\mathcal{B}$ is countably generated, we have that $L^1(X, \nu)$ is separable.

Remark 1.85. Note that the assumption

$$\nu\left(\{x \in A \cup B \,:\, (m_x \llcorner A) \perp (\nu \llcorner A)\}\right) = 0$$

means that we will find ourselves in case (i) of Proposition 1.22, i.e., disregarding a ν-null set the random walk is ν-irreducible.

Remark 1.86.
(i) The assumption that, given a ν-null set $N \subset A$, there exist $x_1, x_2, \ldots, x_L \in A\backslash N$ and $C > 0$ such that $\nu \llcorner (A \cup B) \leqslant C(m_{x_1} + \cdots + m_{x_L}) \llcorner (A \cup B)$ is not as strong as it seems. Indeed, this is trivially satisfied by connected locally finite weighted discrete graphs and is also satisfied by $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ (recall Example 1.37) if, for a domain $A \subset \mathbb{R}^N$, we take $B \subset \partial_{m^J} A$ such that $\mathrm{dist}(B, \mathbb{R}^N \backslash A_{m^J}) > 0$. Moreover, in the following example we see that if we remove this hypothesis then the thesis is not true in general.

Consider the metric random walk space $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ where d is the Euclidean distance and $J := \frac{1}{2}\chi_{[-1,1]}$ (recall Example 1.37). Let $A := [-1,1]$ and $B := \partial_{m^J} A = [-2,2]\backslash A$. Then, if $N = \{-1,1\}$, we may not find points in $A\backslash N$ satisfying the aforementioned assumption. In fact, the thesis of the theorem does not hold for any $p \geqslant 1$ as can be seen by taking $u_n := \frac{1}{2} n^{\frac{1}{p}} \left(\chi_{[-2,-2+\frac{1}{n}]} - \chi_{[2-\frac{1}{n},2]}\right)$ and $Z := A \cup B$. Indeed, first note that $\|u_n\|_{L^p([-2,2],\mathcal{L}^1)} = 1$ and $\int_{[-2,2]} u_n d\mathcal{L}^1 = 0$ for every $n \in \mathbb{N}$. Now, $\mathrm{supp}(m^J_x) = [x-1, x+1]$ for $x \in [-1,1]$ and, therefore, for $x \in [-1,1]$,

$$\begin{aligned}
\int_{[-2,2]} |u_n(y) - u_n(x)|^p dm^J_x(y) &= \int_{[-2,-2+\frac{1}{n}] \cap [x-1,x+1]} n\, dm^J_x(y) \\
&\quad + \int_{[2-\frac{1}{n},2] \cap [x-1,x+1]} n\, dm^J_x(y) \\
&= 2n\chi_{[1-\frac{1}{n},1]}(x) \int_{[2-\frac{1}{n},x+1]} dm^J_x(y) \\
&= 2n\left(x - 1 + \frac{1}{n}\right)\chi_{[1-\frac{1}{n},1]}(x).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{[-1,1]} \int_{[-2,2]} |u_n(y) - u_n(x)|^p dm^J_x(y) d\mathcal{L}^1(x) &= 2n \int_{[1-\frac{1}{n},1]} \left(x - 1 + \frac{1}{n}\right) d\mathcal{L}^1(x) \\
&= 2n\left(\frac{1}{2} - \frac{(1-\frac{1}{n})^2}{2} - \frac{1}{n} + \frac{1}{n^2}\right) = \frac{1}{n}.
\end{aligned}$$

Finally, by the reversibility of $\mathcal{L}^1$ with respect to m^J,

$$\int_{[-2,2]} \int_{[-1,1]} |u_n(y) - u_n(x)|^p dm^J_x(y) d\mathcal{L}^1(x) = \frac{1}{n},$$

thus

$$\int_Q |u_n(y) - u_n(x)|^p dm^J_x(y) d\mathcal{L}^1(x) \leqslant \frac{2}{n} \xrightarrow{n} 0.$$

(ii) However, in this example, as we mentioned before, we can take $B \subset \partial_m A$ such that $\mathrm{dist}(B, \mathbb{R}\backslash[-2,2]) > 0$ to avoid this problem and to ensure that the hypotheses of the theorem are satisfied so that $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ satisfies a generalised (p,p)-Poincaré type inequality on (A,B).

In the following example, the metric random walk space $[X, d, m, \nu]$ defined satisfies that $m_x \perp \nu$ for every $x \in X$ (thus falling into the case (ii) of Proposition 1.22), and a Poincaré type inequality does not hold.

Example 1.87. Let $p > 1$. Let $S^1 = \{e^{2\pi i\alpha} : \alpha \in [0,1)\}$ and let $T_\theta : S^1 \longrightarrow S^1$ denote the irrational rotation map $T_\theta(x) = xe^{2\pi i\theta}$ where θ is an irrational number. On S^1 consider the Borel σ-algebra $\mathcal{B}$ and the 1-dimensional Hausdorff measure $\nu := \mathcal{H}_1 \llcorner S^1$. It is well known that T_θ is a uniquely ergodic measure-preserving transformation on $(S^1, \mathcal{B}, \nu)$.

Now, denote $X := S^1$ and let $m_x := \frac{1}{2}\delta_{T_{-\theta}(x)} + \frac{1}{2}\delta_{T_\theta(x)}$, $x \in X$. Then, $[X, d, m, \nu]$ is a reversible metric random walk space, where d is the metric given by the arclength. Indeed, let f be a bounded measurable function on $(X \times X, \mathcal{B} \times \mathcal{B})$, then

$$\begin{aligned}\int_{S^1}\int_{S^1} f(x,y)dm_x(y)d\nu(x) &= \frac{1}{2}\int_{S^1} f(x, T_{-\theta}(x))d\nu(x) + \frac{1}{2}\int_{S^1} f(x, T_\theta(x))d\nu(x) \\ &= \frac{1}{2}\int_{S^1} f(T_\theta(x), x)d\nu(x) + \frac{1}{2}\int_{S^1} f(T_{-\theta}(x), x)d\nu(x) \\ &= \int_{S^1}\int_{S^1} f(y,x)dm_x(y)d\nu(x).\end{aligned}$$

Let's see that $[X, d, m, \nu]$ is m-connected. First note that, for $x \in X$,

$$m_x^{*2} := \frac{1}{2}\delta_x + \frac{1}{4}\delta_{T^2_{-\theta}(x)} + \frac{1}{4}\delta_{T^2_\theta(x)} \geqslant \frac{1}{4}\delta_{T^2_\theta(x)}$$

and, by induction, it is easy to see that

$$m_x^{*n} \geqslant \frac{1}{2^n}\delta_{T^n_\theta(x)} .$$

Now, let $A \in \mathcal{B}$ such that $\nu(A) > 0$. By the pointwise ergodic theorem we have that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_A\left(T^k_\theta(x)\right) = \frac{\nu(A)}{\nu(X)} > 0$$

for ν-a.e. $x \in X$. Consequently, for ν-a.e. $x \in X$, there exists $k \in \mathbb{N}$ such that

$$m_x^{*k}(A) \geqslant \frac{1}{2^k}\delta_{T^k_\theta(x)}(A) = \frac{1}{2^k}\chi_A\left(T^k_\theta(x)\right) > 0,$$

thus $[X, d, m, \nu]$ is m-connected.

Let's see that $[X, d, m, \nu]$ does not satisfy a (p,p)-Poincaré inequality. For $n \in \mathbb{N}$ let

$$I^n_k := \left\{e^{2\pi i\alpha} : k\theta - \delta(n) < \alpha < k\theta + \delta(n)\right\}, \quad -1 \leqslant k \leqslant 2n,$$

where $\delta(n) > 0$ is chosen so that

$$I^n_{k_1} \cap I^n_{k_2} = \varnothing \quad \text{for every } -1 \leqslant k_1, k_2 \leqslant 2n,\ k_1 \neq k_2$$

(note that $e^{2\pi i(k_1\theta - \delta(n))} \neq e^{2\pi i(k_2\theta - \delta(n))}$ for every $k_1 \neq k_2$ since T_θ is ergodic). Consider the following sequence of functions:

$$u_n := \sum_{k=0}^{n-1} \chi_{I^n_k} - \sum_{k=n}^{2n-1} \chi_{I^n_k},\ n \in \mathbb{N}.$$

Then

$$\int_X u_n d\nu = 0, \quad \text{for every } n \in \mathbb{N},$$

and

$$\int_X |u_n|^p d\nu = 4n\delta(n), \quad \text{for every } n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$, let's see what happens with

$$\int_X\int_X |u_n(y) - u_n(x)|^p dm_x(y)d\nu(x).$$

If $1 \leqslant k \leqslant n-2$ or $n+1 \leqslant k \leqslant 2n-2$ and $x \in I_k^n$ then

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = 0$$

since $T_{-\theta}(x) \in I_{k-1}^n$ and $T_\theta(x) \in I_{k+1}^n$. Now, if $x \in I_0^n$ then $T_{-\theta}(x) \in I_{-1}^n$ thus

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = \frac{1}{2}|-1|^p = \frac{1}{2}$$

and the same holds if $x \in I_{2n-1}^n$ (then $T_\theta(x) \in I_{2n}^n$). For $x \in I_{n-1}$ we have $T_\theta(x) \in I_n^n$ thus

$$\int_X |u_n(y)-u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x))-u_n(x)|^p+\frac{1}{2}|u_n(T_\theta(x))-u_n(x)|^p = \frac{1}{2}|-2|^p = 2^{p-1}$$

and the same result is obtained for $x \in I_{n+1}^n$. Similarly, if $x \in I_{-1}^n$ or $x \in I_{2n}^n$,

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = \frac{1}{2}.$$

Finally, if $x \notin \cup_{k=-1}^{2n} I_k^n$ then $T_{-\theta}(x), T_\theta(x) \notin \cup_{k=0}^{2n-1} I_k^n$ thus

$$\int_X |u_n(y) - u_n(x)|^p dm_x(y) = \frac{1}{2}|u_n(T_{-\theta}(x)) - u_n(x)|^p + \frac{1}{2}|u_n(T_\theta(x)) - u_n(x)|^p = 0.$$

Consequently,

$$\int_X \int_X |u_n(y) - u_n(x)|^p dm_x(y) d\nu(x) = \frac{1}{2}(4 \cdot 2\delta(n)) + 2^{p-1}(2 \cdot 2\delta(n)) = (4 + 2^{p+1})\delta(n).$$

Therefore, there is no $\Lambda > 0$ such that

$$\left\| u_n - \frac{1}{2\pi}\int_X u_n d\nu \right\|_{L^p(X,\nu)}^p \leqslant \Lambda \int_X \int_X |u_n(y) - u_n(x)|^p dm_x(y) d\nu(x), \ \forall n \in \mathbb{N}$$

since this would imply

$$4n\delta(n) \leqslant \Lambda(4 + 2^{p+1})\delta(n) \Longrightarrow n \leqslant \Lambda + 2^{p-1}, \ \forall n \in \mathbb{N}.$$

Finally, we provide another result in which we give sufficient conditions for a generalised (p,q)-Poincaré inequality to hold.

Theorem 1.88. *Let $1 \leqslant p < q$. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $A, B \in \mathcal{B}$ be disjoint sets such that $B \subset \partial_m A$, $\nu(A) > 0$ and A is m-connected. Suppose that $\nu(A \cup B) < \infty$ and $m_x \ll \nu$ for every $x \in A \cup B$. Assume further that, given a ν-null set $N \subset A$, there exist $x_1, x_2, \ldots, x_L \in A\backslash N$ and $\Omega_1, \Omega_2, \ldots, \Omega_L \in \mathcal{B}_{A\cup B}$, such that $A \cup B = \bigcup_{i=1}^L \Omega_i$ and, if $g_i := \dfrac{dm_{x_i}}{d\nu}$ on Ω_i, then $g_i^{-\frac{p}{q-p}} \in L^1(\Omega_i, \nu)$, $i = 1, 2, ..., L$. Then, $[X, \mathcal{B}, m, \nu]$ satisfies a generalised (p,q)-Poincaré type inequality on (A, B).*

Proof. Let $0 < l \leqslant \nu(A \cup B)$. Starting as in the proof of Theorem 1.84, if we suppose that a generalised (p,q)-Poincaré type inequality on (A, B) does not hold, then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^q(A \cup B, \nu)$, with $\|u_n\|_{L^p(A\cup B,\nu)} = 1$, and a sequence $Z_n \in \mathcal{B}_{A\cup B}$ with $\nu(Z_n) \geqslant l$, $n \in \mathbb{N}$, satisfying

$$\lim_n \int_Q |u_n(y) - u_n(x)|^q dm_x(y) d\nu(x) = 0$$

and

$$\lim_n \int_{Z_n} u_n \, d\nu = 0.$$

Therefore, by Lemma 1.83, there exist $\lambda \in \mathbb{R}$ and a ν-null set $N \subset A$ such that

$$\|u_n - \lambda\|_{L^q(A\cup B, m_x)} \xrightarrow{n} 0 \quad \text{for every } x \in A\backslash N.$$

Now, by hypothesis, there exist $x_1, x_2, \ldots, x_L \in A \backslash N$ and $\Omega_1, \Omega_2, \ldots, \Omega_L \in \mathcal{B}_{A\cup B}$, such that $A \cup B = \bigcup_{i=1}^{L} \Omega_i$ and, if $g_i := \frac{dm_{x_i}}{d\nu}$ on Ω_i, then $g_i^{-\frac{p}{q-p}} \in L^1(\Omega_i, \nu)$, $i = 1, 2, ..., L$. Therefore,

$$\begin{aligned}
\|u_n - \lambda\|^p_{L^p(A\cup B,\nu)} &\leqslant \sum_{i=1}^{L} \int_{\Omega_i} |u_n(y) - \lambda|^p d\nu(y) \\
&= \sum_{i=1}^{L} \int_{\Omega_i} |u_n(y) - \lambda|^p \frac{g_i(y)^{\frac{p}{q}}}{g_i(y)^{\frac{p}{q}}} d\nu(y) \\
&\leqslant \sum_{i=1}^{L} \left(\int_{\Omega_i} |u_n(y) - \lambda|^q g_i(y) d\nu(y) \right)^{\frac{p}{q}} \left(\int_{\Omega_i} \frac{1}{g_i(y)^{\frac{p}{q-p}}} d\nu(y) \right)^{\frac{q-p}{q}} \\
&= \sum_{i=1}^{L} \left(\int_{\Omega_i} |u_n(y) - \lambda|^q dm_{x_i}(y) \right)^{\frac{p}{q}} \left\| \frac{1}{g_i(y)^{\frac{p}{q-p}}} \right\|^{\frac{q-p}{q}}_{L^1(\Omega_i,\nu)} \\
&= \sum_{i=1}^{L} \|u_n - \lambda\|^p_{L^q(\Omega_i, m_{x_i})} \left\| \frac{1}{g_i^{\frac{p}{q-p}}} \right\|^{\frac{q-p}{q}}_{L^1(\Omega_i,\nu)} \xrightarrow{n} 0.
\end{aligned}$$

We now finish the proof in the same way as for Theorem 1.84. □

1.6.2. Isoperimetric Inequality. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space such that ν is a probability measure. Suppose that $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality, i.e., there exists $\lambda > 0$ such that*

$$\lambda \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu).$$

Let $D \in \mathcal{B}$ and take $f = \chi_D$. Then, the Poincaré inequality implies that

$$\lambda \mathrm{Var}_\nu(\chi_D) \leqslant \mathcal{H}_m(\chi_D)$$

which, recalling that

$$P_m(D) = \frac{1}{2} \int_X \int_X |\nabla \chi_D(x,y)| dm_x(y) d\nu(x) = \frac{1}{2} \int_X \int_X \nabla \chi_D(x,y)^2 dm_x(y) d\nu(x),$$

can be rewritten as

$$\lambda\, \nu(D)(1 - \nu(D)) \leqslant P_m(D) \quad \text{for every } D \in \mathcal{B} \tag{1.19}$$

(observe that, by Theorem 1.63, this implies, in particular, that Δ_m is ergodic). Hence, since

$$min\{x, 1-x\} \leqslant 2x(1-x) \leqslant 2min\{x, 1-x\} \quad \text{for } 0 \leqslant x \leqslant 1,$$

inequality (1.19) yields the following isoperimetric inequality (see [7, Theorem 3.46] for the local case):

$$min\{\nu(D), 1 - \nu(D)\} \leqslant \frac{2}{\lambda} P_m(D) \quad \text{for every } D \in \mathcal{B}; \tag{1.20}$$

and, conversely, the isoperimetric inequality (1.20) implies that

$$\frac{\lambda}{2} \nu(D)(1 - \nu(D)) \leqslant P_m(D) \quad \text{for every } D \in \mathcal{B}.$$

Definition 1.89. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. If there exists $\lambda > 0$ such that (1.20) is satisfied we will say that $[X, \mathcal{B}, m, \nu]$ satisfies an *isoperimetric inequality*.

1.7. Ollivier-Ricci curvature

An important tool in the study of the speed of convergence of the heat flow to the equilibrium is the Poincaré inequality (see [**22**]*). In the case of Riemannian manifolds and Markov diffusion semigroups, a usual condition required to obtain this functional inequality is the positivity of the corresponding Ricci curvature of the underlying space (see* [**22**] *and* [**160**]*). In* [**21**]*, Bakry and Émery found a way to define the lower Ricci curvature bound through the heat flow. Moreover, Renesse and Sturm* [**141**] *proved that, on a Riemannian manifold M, the Ricci curvature is bounded from below by some constant $K \in \mathbb{R}$ if, and only if, the Boltzmann-Shannon entropy is K-convex along geodesics in the 2-Wasserstein space of probability measures on M. This was the key observation, used simultaneously by Lott and Villani* [**113**] *and Sturm* [**147**]*, to give a notion of a lower Ricci curvature bound in the general context of length metric measure spaces. In these spaces, a relation between the Bakry-Émery curvature-dimension condition and the notion of the Ricci curvature bound introduced by Lott-Villani-Sturm was obtained by Ambrosio, Gigli and Savaré in* [**8**]*, where they proved that these two notions of Ricci curvature coincide under certain assumptions on the metric measure space.*

When the space under consideration is discrete, for instance, in the case of a graph, the previous concept of a Ricci curvature bound is not as clearly applicable as in the continuous setting. Indeed, the definition by Lott-Sturm-Villani does not apply if the 2-Wasserstein space over the metric measure space does not contain geodesics. Unfortunately, this is the case if the underlying space is discrete. Therefore, we will use the concept of a Ricci curvature bound introduced by Y. Ollivier in [**134**] *which is well suited for the discrete case. We refer to* [**132**] *and the references therein for the vibrant research field of discrete curvature.*

In order to introduce the coarse Ricci curvature defined by Y. Ollivier in [**134**] *we first recall the Monge-Kantorovich transportation problem. Let (X, d) be a Polish metric space and μ, $\nu \in \mathcal{P}(X)$*[9]*. The Monge-Kantorovich problem is the minimization problem*

$$\min\left\{\int_{X\times X} d(x,y)\,d\gamma(x,y)\ :\ \gamma\in\Pi(\mu,\nu)\right\},$$

where $\Pi(\mu,\nu) := \{\gamma\in\mathcal{P}(X\times X)\ :\ \pi_0\#\gamma=\mu,\ \pi_1\#\gamma=\nu\}$[10] *and $\pi_\alpha : X\times X\to X$ is defined by $\pi_\alpha(x,y) := x+\alpha(y-x)$ for $\alpha\in\{0,1\}$.*

For $1\leqslant p<\infty$, the p-Wasserstein distance between μ and ν is defined as

$$W_p^d(\mu,\nu) := \left(\min\left\{\int_{X\times X} d(x,y)^p\,d\gamma(x,y)\ :\ \gamma\in\Pi(\mu,\nu)\right\}\right)^{\frac{1}{p}}. \tag{1.21}$$

The Monge-Kantorovich problem has a dual formulation that can be stated as follows (see for instance [**159**, Theorem 1.14]*).*

Kantorovich-Rubinstein's Theorem. *Let $\mu,\nu\in\mathcal{P}(X)$. Then,*

$$\begin{aligned} W_1^d(\mu,\nu) &= \sup\left\{\int_X u\,d(\mu-\nu)\ :\ u\in K_d(X)\right\}\\ &= \sup\left\{\int_X u\,d(\mu-\nu)\ :\ u\in K_d(X)\cap L^\infty(X,\nu)\right\}\end{aligned}$$

where

$$K_d(X) := \{u : X\to\mathbb{R}\ :\ |u(y)-u(x)|\leqslant d(y,x)\}.$$

In Riemannian geometry, positive Ricci curvature is characterized by the fact that "small balls are closer, in the 1-Wasserstein distance, than their centers are" (see [**141**]*). In the framework of metric random walk spaces, inspired by this, Y. Ollivier* [**134**] *introduced the*

[9] $\mathcal{P}(X)$ denotes the set of probability measures on X.

[10] $\pi_\alpha\#\gamma$ denotes the pushforward of γ by π_α, thus $\pi_0\#\gamma$ is the marginal of γ on the first component and $\pi_1\#\gamma$ is the marginal of γ on the second component.

concept of coarse Ricci curvature, substituting the balls by the measures m_x and using the 1-Wasserstein distance to measure the distance between them.

DEFINITION 1.90. Given random walk m on a Polish metric space $[X,d]$ such that each measure m_x has finite first moment, for any two distinct points $x, y \in X$, the *Ollivier-Ricci curvature (or coarse Ricci curvature) of* $[X,d,m]$ *along* (x,y) is defined as

$$\kappa_m(x,y) := 1 - \frac{W_1^d(m_x, m_y)}{d(x,y)}.$$

The *Ollivier-Ricci curvature of* $[X,d,m]$ is defined by

$$\kappa_m := \inf_{\substack{x,y \in X \\ x \neq y}} \kappa_m(x,y).$$

We will write $\kappa(x,y)$ instead of $\kappa_m(x,y)$, and $\kappa = \kappa_m$, if the context allows no confusion.

Observe that, in principle, the metric d and the random walk m of a metric random walk space $[X,d,m,\nu]$ have no relation between them aside from the fact that m is defined on the Borel σ-algebra associated with d and that each m_x, $x \in X$, has finite first moment. Therefore, we can not expect to obtain strong results on the properties of m by imposing conditions only in terms of d. For example, as we will see in Example 3.36 balls in metric random walk spaces are not necessarily m-calibrable (see Definition 3.33). However, imposing conditions on κ, like $\kappa > 0$, effectively creates a strong relation between the random walk and the metric which allows us to prove results like Theorem 1.94.

REMARK 1.91. If (X,d,μ) is a smooth complete Riemannian manifold and $(m_x^{\mu,\epsilon})$ is the ϵ-step random walk associated with μ given in Example 1.41, then it is proved in [**141**] (see also [**134**]) that, up to scaling by ϵ^2, $\kappa_{m^{\mu,\epsilon}}(x,y)$ gives back the ordinary Ricci curvature when $\epsilon \to 0$.

EXAMPLE 1.92. Let $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ be the metric random walk space given in Example 1.37. Let us see that $\kappa(x,y) = 0$. Given $x, y \in \mathbb{R}^N$, $x \neq y$, by Kantorovich-Rubinstein's Theorem, we have

$$\begin{aligned} W_1^d(m_x^J, m_y^J) &= \sup\left\{ \int_{\mathbb{R}^N} u(z)(J(x-z) - J(y-z))\, dz \;:\; u \in K_d(\mathbb{R}^N) \right\} \\ &= \sup\left\{ \int_{\mathbb{R}^N} (u(x+z) - u(y+z))J(z)\, dz \;:\; u \in K_d(\mathbb{R}^N) \right\}. \end{aligned}$$

Now, for $u \in K_d(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} (u(x+z) - u(y+z))J(z)\, dz \leqslant \|x-y\|,$$

thus $W_1^d(m_x^J, m_y^J) \leqslant \|x-y\|$. On the other hand, if $u(z) := \frac{\langle z, x-y \rangle}{\|x-y\|}$, then $u \in K_d(\mathbb{R}^N)$, hence

$$W_1^d(m_x^J, m_y^J) \geqslant \int_{\mathbb{R}^N} (u(x+z) - u(y+z))J(z)\, dz = \|x-y\|.$$

Therefore,

$$W_1^d(m_x^J, m_y^J) = \|x-y\|,$$

and, consequently, $\kappa(x,y) = 0$.

EXAMPLE 1.93. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with a locally finite weighted discrete graph $G = (V(G), E(G))$ as defined in Example 1.38 and recall that $N_G(x) := \{z \in V(G) \;:\; z \sim x\}$ for $x \in V(G)$. Then, the Ollivier-Ricci curvature along $(x,y) \in E(G)$ is

$$\kappa(x,y) = 1 - \frac{W_1^{d_G}(m_x, m_y)}{d_G(x,y)},$$

where

$$W_1^{d_G}(m_x, m_y) = \inf_{\mu \in \mathcal{A}} \sum_{z_1 \sim x} \sum_{z_2 \sim y} \mu(z_1, z_2) d_G(z_1, z_2),$$

being $\mathcal{A}$ the set of all matrices with entries indexed by $N_G(x) \times N_G(y)$ such that $\mu(z_1, z_2) \geqslant 0$ and

$$\sum_{z_2 \sim y} \mu(z_1, z_2) = \frac{w_{xz_1}}{d_x}, \quad \sum_{z_1 \sim x} \mu(z_1, z_2) = \frac{w_{yz_2}}{d_y}, \quad \text{for } (z_1, z_2) \in N_G(x) \times N_G(y).$$

There is an extensive literature about Ollivier-Ricci curvature on discrete graphs (see for instance, [**27**], [**35**], [**62**], [**95**], [**105**], [**112**], [**134**], [**135**], [**136**] *and* [**138**]*).*

In the next result we see that metric random walk spaces with positive Ollivier-Ricci curvature are m-connected.

THEOREM 1.94. *Let $[X, d, m, \nu]$ be a metric random walk space such that ν is a probability measure and each measure m_x has finite first moment. Assume that the Ollivier-Ricci curvature κ satisfies $\kappa > 0$. Then, $[X, d, m, \nu]$ is m-connected.*

PROOF. Under the hypothesis $\kappa > -\infty$ (recall that $\kappa \leqslant 1$ by definition) Y. Ollivier in [**134**, Proposition 20] proves the following W_1 contraction property:

Let $[X, d, m, \nu]$ be a metric random walk space. Then, for any two probability distributions, μ and μ',

$$W_1^d(\mu * m^{*n}, \mu' * m^{*n}) \leqslant (1-\kappa)^n W_1^d(\mu, \mu'). \tag{1.22}$$

Hence, under the hypothesis $\kappa > 0$, Y. Ollivier in [**134**, Corollary 21] proves that the invariant measure ν (exists and) is unique up to a multiplicative constant, and that, if $\nu \in \mathcal{P}(X)$, the following hold:

$$\begin{aligned} &(i) \qquad W_1^d(\mu * m^{*n}, \nu) \leqslant (1-\kappa)^n W_1^d(\mu, \nu) \quad \forall n \in \mathbb{N},\ \forall \mu \in \mathcal{P}(X), \\ &(ii) \qquad W_1^d(m_x^{*n}, \nu) \leqslant (1-\kappa)^n \frac{W_1^d(\delta_x, m_x)}{\kappa} \quad \forall n \in \mathbb{N},\ \forall x \in X. \end{aligned} \tag{1.23}$$

By (1.23) and [**160**, Theorem 6.9][11], we have that

$$\mu * m^{*n} \rightharpoonup \nu \quad \text{weakly as measures, } \forall \mu \in \mathcal{P}(X), \tag{1.24}$$

thus, taking $\mu = \delta_x$, we obtain that

$$m_x^{*n} \rightharpoonup \nu \quad \text{weakly as measures, for every } x \in X.$$

Let us now see that $[X, d, m, \nu]$ is m-connected if $\kappa > 0$. Take $D \subset X$ a Borel set with $\nu(D) > 0$ and suppose that $\nu(N_D^m) > 0$. By Proposition 1.28, we have $\nu(H_D^m) > 0$. Let

$$\mu := \frac{1}{\nu(H_D^m)} \nu \llcorner H_D^m \in \mathcal{P}(X),$$

and

$$\mu' := \frac{1}{\nu(N_D^m)} \nu \llcorner N_D^m \in \mathcal{P}(X).$$

Now, by Proposition 1.27,

$$\mu * m^{*n} = \mu,$$

and

$$\mu' * m^{*n} = \mu',$$

but then, by (1.22), we get

$$W_1(\mu, \mu') = W_1(\mu * m^{*n}, \mu' * m^{*n}) \leqslant (1-\kappa)^n W_1(\mu, \mu')$$

[11]**Theorem** Let (X, d) be a Polish space; then the Wassertein distance W_1 metrizes the weak convergence in $P_1(X) := \{\mu \in P(X) : \int_X d(x_0, x)\mu(dx) < +\infty\}$, where $x_0 \in X$ is arbitrary.

which is only possible if $W_1(\mu, \mu') = 0$ since $1 - \kappa < 1$. Hence,

$$\mu = \mu',$$

and this implies $1 = \mu'(N_D^m) = \mu(N_D^m) = 0$ which is a contradiction. Therefore, $\nu(N_D^m) = 0$ as desired. □

Remark 1.95. By Proposition 1.13 uniqueness of the invariant probability measure implies its ergodicity. Consequently, Theorem 1.94 follows from [**134**, Corollary 21] (recall also Theorem 1.34). We have presented the result for the sake of completeness and using the framework of m-connectedness.

Observe that, if D is open and $\nu(D) > 0$ then $N_D^m = \emptyset$, i.e.

$$\sum_{n=1}^{\infty} m_x^{*n}(D) > 0 \quad \text{for every } x \in X.$$

Indeed, for $x \in N_D^m$, by (1.24), we have

$$0 < \nu(D) \leqslant \liminf_n m_x^{*n}(D) = 0.$$

Remark 1.96. For a reversible metric random walk space $[X, d, m, \nu]$, Y. Ollivier in [**134**, Corollary 31] proves, under the assumption

$$\int\int\int d(y,z)^2 dm_x(y) dm_x(z) d\nu(x) < +\infty,$$

that, if the Ollivier-Ricci curvature κ is positive and ν is ergodic[12], then $[X, d, m, \nu]$ satisfies the Poincaré inequality

$$\kappa \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),$$

and, consequently,

$$\kappa \leqslant \mathrm{gap}(-\Delta_m).$$

[12]By Theorem 1.94 (see also Remark 1.95), this assumption is actually redundant.

CHAPTER 2

The heat flow

Our objective in this chapter is to define and study the heat flow in random walk spaces and, in doing so, to unify into a broad framework the study of the heat flow in a variety of models. For example, this study will cover the heat flow in graphs or in nonlocal models in $\mathbb{R}^N$ *associated with a nonsingular kernel (see* [**18**]*).*

Let us make a short summary of the results that we will obtain in this chapter. In Theorem 2.4 we prove that, if $[X, \mathcal{B}, m, \nu]$ *is a reversible random walk space, the operator* $-\Delta_m$ *generates a Markovian semigroup* $(e^{t\Delta_m})_{t\geqslant 0}$ *in* $L^2(X,\nu)$ *called the heat flow in* $[X, \mathcal{B}, m, \nu]$*. Then, in consideration of the great importance that understanding the behaviour of the semigroup* $(e^{t\Delta_m})_{t\geqslant 0}$ *as* $t \to \infty$ *has in many applications, we study the asymptotic behaviour of the heat flow. In this regard, we prove that, if* ν *is a probability measure and* $[X, \mathcal{B}, m, \nu]$ *satisfies a Poincaré inequality, then the heat flow converges to the mean of the initial datum with exponential rate. Moreover, we prove that the infinite speed of propagation of the heat flow is equivalent to the m-connectedness of* $[X, \mathcal{B}, m, \nu]$ *(see Theorem 2.9).*

2.1. The heat flow

Let's start by defining the following symmetric form on $L^2(X,\nu)$*.*

Definition 2.1. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. In $L^2(X,\nu)$ we consider the symmetric form given by (recall Proposition 1.47)

$$\mathcal{E}_m(f,g) = -\int_X f(x)\Delta_m g(x) d\nu(x) = \frac{1}{2}\int_{X\times X} \nabla f(x,y)\nabla g(x,y) dm_x(y) d\nu(x),$$

with domain $D(\mathcal{E}_m) = D(\mathcal{H}_m) = L^1(X,\nu) \cap L^2(X,\nu)$ for both variables (which is a dense linear subspace of $L^2(X,\nu)$).

Note that $\mathcal{H}_m(f) = \mathcal{E}_m(f,f)$*.*

Following the notation given in [**86**, Chapter 1]*, we make the following definitions.*

Definition 2.2. Let H be a real Hilbert space with inner product $(\cdot,\cdot)$. A non-negative symmetric bilinear form $\mathcal{E}$ which is densely defined on H is called a *symmetric form on* H. Moreover, $\mathcal{E}$ is said to be *closed* if, for every sequence $(f_n)_n \subset D(\mathcal{E})$ such that

$$\mathcal{E}(f_n - f_k, f_n - f_k) + (f_n - f_k, f_n - f_k) \to 0 \quad \text{as } n,\, m \to \infty,$$

there exists $f \in D(\mathcal{E})$ such that

$$\mathcal{E}(f_n - f, f_n - f) + (f_n - f, f_n - f) \to 0 \quad \text{as } n \to \infty,$$

Let $(X, \mathcal{B}, \nu)$ be a σ-finite measure space. A symmetric form $\mathcal{E}$ on $L^2(X,\nu)$ is said to be a Markovian symmetric form if, for each $\varepsilon > 0$, there exists a real function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

$$\phi_\varepsilon(t) = t,\ \forall t \in [0,1],\ -\varepsilon \leqslant \phi_\varepsilon(t) \leqslant 1+\varepsilon,\ \forall t \in \mathbb{R},\ 0 \leqslant \phi_\varepsilon(t') - \phi_\varepsilon(t) \leqslant t' - t \ \text{ for every } t < t'$$

and, for every $f \in D(\mathcal{E})$, we have that

$$\phi_\varepsilon(f) \in D(\mathcal{E}) \ \text{ and } \ \mathcal{E}(\phi_\varepsilon(f), \phi_\varepsilon(f)) \leqslant \mathcal{E}(f,f).$$

Furthermore, a closed Markovian symmetric form on $L^2(X,\nu)$ is called a *Dirichlet form*.

A function $f \in L^2(X,\nu)$ is called a *normal contraction* of a function $g \in L^2(X,\nu)$ if

$$|g(x) - g(y)| \leqslant |f(x) - f(y)|, \text{ for } \nu\text{-a.e. } x, y \in X, \ \text{ and } \ |g(x)| \leqslant |f(x)|, \text{ for } \nu\text{-a.e. } x \in X.$$

REMARK 2.3. As proved in [**86**, §1.4], a closed symmetric form on $L^2(X,\nu)$ is Markovian if, and only if, for every $f \in D(\mathcal{E})$ and every normal contraction g of f we have that $g \in D(\mathcal{E})$ and $\mathcal{E}(g,g) \leqslant \mathcal{E}(f,f)$.

THEOREM 2.4. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. Then, $-\Delta_m$ is a non-negative self-adjoint operator in $L^2(X,\nu)$ with associated closed symmetric form $\mathcal{E}_m$. In fact, $\mathcal{E}_m$ is Markovian and, therefore, a Dirichlet form.*

PROOF. By Proposition 1.47 we have that $-\Delta_m$ is a self-adjoint operator in $L^2(X,\nu)$ and

$$\int_X f(x)(-\Delta_m f)(x)d\nu(x) = \mathcal{H}_m(f) \geqslant 0 \quad \text{for every } f \in D(\Delta_m).$$

Let's prove that $\mathcal{E}_m$ is closed. Consider $f_n \in D(\mathcal{E}_m)$ such that

$$\mathcal{E}_m(f_n - f_k, f_n - f_k) + \|f_n - f_k\|_{L^2(X,\nu)} \to 0, \quad \text{when } n,k \to +\infty.$$

Since $f_n \xrightarrow{n} f \in L^2(X,\nu)$, we can assume that there exists a ν-null set N such that $f_n(x) \to f(x)$ for all $x \in X\backslash N$. Then, $(f_n(x) - f_n(y))^2 \to (f(x) - f(y))^2$ for all $(x,y) \in (X\backslash N) \times (X\backslash N) = (X \times X)\backslash[(N \times X) \cup (X \times N)]$. Now, since ν is invariant with respect to m, we have that

$$\begin{aligned} \nu \otimes m_x([(N \times X) \cup (X \times N)]) &= \int_N \left(\int_X dm_x(y) \right) d\nu(x) + \int_X \left(\int_X \chi_N(y) dm_x(y) \right) d\nu(x) \\ &= \nu(N) + \int_X \chi_N(y) d\nu(y) = 2\nu(N) = 0. \end{aligned}$$

Then, Fatou's Lemma yields that

$$\begin{aligned} 0 \leqslant \lim_{n\to\infty} \mathcal{E}_m(f_n - f, f_n - f) &= \lim_{n\to+\infty} \frac{1}{2} \int_{X\times X} (\nabla(f_n - f)(x,y))^2 d(\nu \otimes m_x)(x,y) \\ &= \lim_{n\to+\infty} \frac{1}{2} \int_{X\times X} \liminf_{k\to+\infty} (\nabla(f_n - f_k)(x,y))^2 d(\nu \otimes m_x)(x,y) \\ &\leqslant \lim_{n\to+\infty} \liminf_{k\to+\infty} \mathcal{E}_m(f_n - f_k, f_n - f_k) = 0. \end{aligned}$$

Therefore, $\mathcal{E}_m$ is closed. Moreover, given $f \in D(\mathcal{E}_m)$, if g is a normal contraction of f, then

$$g \in D(\mathcal{E}_m) \quad \text{and} \quad \mathcal{E}_m(g,g) \leqslant \mathcal{E}_m(f,f),$$

thus $\mathcal{E}_m$ is a Markovian symmetric form. □

By Theorem 2.4 and as a consequence of the results obtained in [**86**, Chapter 1]*, we have that if $(T_t^m)_{t\geqslant 0}$ is the strongly continuous semigroup (see Definition A.8) associated with $\mathcal{E}_m$, then $(T_t^m)_{t\geqslant 0}$ is a positivity preserving (i.e., $T_t^m f \geqslant 0$ if $f \geqslant 0$) Markovian semigroup (i.e., $0 \leqslant T_t^m f \leqslant 1$ ν-a.e. whenever $f \in L^2(X,\nu)$ and $0 \leqslant f \leqslant 1$ ν-a.e.). Moreover, Δ_m is the infinitesimal generator of $(T_t^m)_{t\geqslant 0}$, that is,*

$$\Delta_m f = \lim_{t\downarrow 0} \frac{T_t^m f - f}{t} \qquad \forall f \in D(\Delta_m).$$

DEFINITION 2.5. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. We denote $e^{t\Delta_m} := T_t^m$ and say that $\{e^{t\Delta_m} : t \geqslant 0\}$ is the *heat flow in the random walk space* $[X,\mathcal{B},m,\nu]$.

Therefore, we have that, for every $u_0 \in L^2(X,\nu)$, $u(t) := e^{t\Delta_m}u_0$ is the unique solution of the heat equation

$$\begin{cases} u'(t) = \Delta_m u(t) & \textit{for every } t \in (0,+\infty), \\ u(0) = u_0, \end{cases}$$

in the sense that $u \in C([0,+\infty) : L^2(X,\nu)) \cap C^1((0,+\infty) : L^2(X,\nu))$ *and satisfies*

$$\begin{cases} \dfrac{du}{dt}(t)(x) = \displaystyle\int_X (u(t)(y) - u(t)(x))dm_x(y) & \text{for every } t > 0 \text{ and } \nu\text{-a.e. } x \in X, \\ u(0) = u_0. \end{cases}$$

By the Hille-Yosida exponential formula (see Theorem A.20) we have that

$$e^{t\Delta_m}u_0 = \lim_{n\to+\infty} \left[\left(I - \frac{t}{n}\Delta_m \right)^{-1} \right]^n u_0.$$

Moreover, as a consequence of (1.4), if ν *is a probability measure, we have that the semigroup* $(e^{t\Delta_m})_{t\geqslant 0}$ *conserves the mass. In fact,*

$$\frac{d}{dt}\int_X e^{t\Delta_m}u_0(x)d\nu(x) = \int_X \Delta_m u_0(x)d\nu(x) = 0,$$

and, therefore,

$$\int_X e^{t\Delta_m}u_0(x)d\nu(x) = \int_X u_0(x)d\nu(x), \ \forall t > 0. \tag{2.1}$$

Remark 2.6. It is easy to see that the functional $\mathcal{H}_m$ is convex and, moreover, with a proof similar to the proof of closedness in Theorem 2.4, we get that the functional $\mathcal{H}_m$ is closed and lower semi-continuous in $L^2(X,\nu)$. Moreover, it is not difficult to see that $\partial\mathcal{H}_m = -\Delta_m$. Consequently, $-\Delta_m$ is a maximal monotone operator in $L^2(X,\nu)$ (see [**43**]). In particular,

$$R(I + \lambda\partial\mathcal{H}_m) = L^2(X,\nu) \quad \forall\lambda > 0. \tag{2.2}$$

Furthermore, we can also consider the heat flow in $L^1(X,\nu)$. Indeed, if we define in $L^1(X,\nu)$ the operator A as $Au = v \Leftrightarrow v(x) = -\Delta_m u(x)$ for all $x \in X$, then A is completely accretive (see section A.7 of Appendix A). Indeed, let

$$P_0 := \{q \in C^\infty(\mathbb{R}) \,:\, 0 \leqslant q' \leqslant 1, \ \text{supp}(q') \text{ is compact and } 0 \notin \text{supp}(q)\}.$$

Given $f \in L^1(X,\nu)$, and $q \in P_0$, applying (1.5), we have

$$\int_X q(f(x))Af(x)d\nu(x) = \frac{1}{2}\int_{X\times X} (q(f(y)) - q(f(x)))(f(y) - f(x))dm_x(y)d\nu(x) \geqslant 0.$$

Then, by Corollary A.36 ([**31**, Proposition 2.2]), we have that A is a completely accretive operator.

Moreover, A is m-completely accretive in $L^1(X,\nu)$. Indeed, by (A.8), (2.2) and having in mind that A is closed, we have that

$$L^1(X,\nu) = \overline{L^1(X,\nu) \cap L^2(X,\nu)}^{L^1(X,\nu)} \subset \overline{R(I + A)}^{L^1(X,\nu)} \subset R\left(I + \overline{A}^{L^1(X,\nu)}\right) = R(I + A).$$

Consequently, by Proposition A.43, A generates a C_0-semigroup $(S(t))_{t\geqslant 0}$ in $L^1(X,\nu)$ satisfying

$$\|S(t)u_0\|_{L^p(X,\nu)} \leqslant \|u_0\|_{L^p(X,\nu)} \quad \forall u_0 \in L^1(X,\nu) \cap L^p(X,\nu), \quad 1 \leqslant p \leqslant +\infty. \tag{2.3}$$

Moreover, if $\nu(X) < \infty$, by Proposition A.41, we have that $S(t)$ is an extension to $L^1(X,\nu)$ of the heat flow $e^{t\Delta_m}$ in $L^2(X,\nu)$ (that we will denote equally) and, by Corollary A.45, for every $u_0 \in D(A) = L^1(X,\nu)$, the mild solution $u(t) = e^{-tA}u_0$ of the problem

$$\begin{cases} \dfrac{du}{dt} + Au \ni 0, \\ u(0) = u_0, \end{cases}$$

is a strong solution.

Example 2.7. (1) Consider the random walk space $[X, \mathcal{B}, m^K, \pi]$ associated with a Markov kernel K (as in Example 1.40) and assume that the stationary probability measure π is reversible. Then, the Laplacian Δ_{m^K} is given by

$$\Delta_{m^K} f(x) := \int_X f(y) dm_x^K(y) - f(x) = \sum_{y \in X} K(x,y) f(y) - f(x), \quad x \in X, \ f \in L^2(X,\pi).$$

Consequently, given $u_0 \in L^2(X,\pi)$, $u(t) := e^{t\Delta_{m^K}} u_0$ is the solution of the equation

$$\begin{cases} \dfrac{du}{dt}(t,x) = \displaystyle\sum_{y \in X} K(x,y) u(t)(y) - u(t)(x) \quad \text{on} \quad (0,+\infty) \times X, \\ u(0) = u_0. \end{cases}$$

Therefore, $e^{t\Delta_{m^K}} = e^{t(K-I)}$ is the *heat semigroup* on X with respect to the geometry determined by the Markov kernel K. In the case that X is a finite set, we have

$$e^{t\Delta_{m^K}} = e^{t(K-I)} = e^{-t} \sum_{n=0}^{+\infty} \frac{t^n K^n}{n!}.$$

(2) If we consider the metric random walk space $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ as defined in Example 1.37, the Laplacian is given by

$$\Delta_{m^J} f(x) := \int_{\mathbb{R}^N} (f(y) - f(x)) J(x-y) dy.$$

Then, given $u_0 \in L^2(\mathbb{R}^N, \mathcal{L}^N)$ we have that $u(t) := e^{t\Delta_{m^J}} u_0$ is the solution of the J-nonlocal heat equation

$$\begin{cases} \dfrac{du}{dt}(t,x) = \displaystyle\int_{\mathbb{R}^N} (u(t)(y) - u(t)(x)) J(x-y) dy \quad \text{in} \quad \mathbb{R}^N \times (0,+\infty), \\ u(0) = u_0. \end{cases} \tag{2.4}$$

If Ω is a closed bounded subset of $\mathbb{R}^N$ and we consider the metric random walk space $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N \llcorner \Omega]$ (see Example 1.42), we have that

$$\Delta_{m^{J,\Omega}} f(x) = \int_\Omega (f(y) - f(x)) dm_x^{J,\Omega}(y) = \int_\Omega J(x-y)(f(y) - f(x)) dy.$$

Then, we have that $u(t) := e^{t\Delta_{m^{J,\Omega}}} u_0$ is the solution of the homogeneous Neumann problem for the J-nonlocal heat equation:

$$\begin{cases} \dfrac{du}{dt}(t,x) = \displaystyle\int_\Omega (u(t)(y) - u(t)(x)) J(x-y) dx \quad \text{in} \quad (0,+\infty) \times \Omega, \\ u(0) = u_0. \end{cases} \tag{2.5}$$

See [**18**] for a comprehensive study of problems (2.4) and (2.5).

Observe that, in general, given a reversible random walk space $[X, \mathcal{B}, m, \nu]$ and $\Omega \in \mathcal{B}$, we have that $u(t) := e^{t\Delta_{m^\Omega}} u_0$ is the solution of

$$\begin{cases} \dfrac{du}{dt}(t)(x) = \displaystyle\int_\Omega (u(t)(y) - u(t)(x)) dm_x(y) \quad \text{in} \quad (0,+\infty) \times \Omega, \\ u(0) = u_0, \end{cases}$$

which, like (2.5), is an homogeneous Neumann problem for the m-heat equation.

In [**122**], *it is shown, by means of the Fourier transform, that if $D \subset \mathbb{R}^N$ has $\mathcal{L}^N$-finite measure, then*

$$e^{t\Delta_{m^J}} \chi_D(x) = e^{-t} \sum_{n=0}^{\infty} \int_D (J*)^n (x-y) dy \frac{t^n}{n!} \quad \textit{for every } x \in \mathbb{R}^N \textit{ and } t > 0, \tag{2.6}$$

where $(J*)^1 := J$, $(J*)^2 := J * J$ (the convolution of J and J) and $(J*)^{n+1}$ is defined inductively by $(J*)^{n+1} := (J*)^n * J$ for $n \geqslant 2$. In the next result we generalize (2.6) for general random walk spaces.

THEOREM 2.8. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $u_0 \in L^1(X,\nu) \cap L^2(X,\nu)$. Then,*

$$e^{t\Delta_m} u_0(x) = e^{-t} \sum_{n=0}^{+\infty} \int_X u_0(y) dm_x^{*n}(y) \frac{t^n}{n!} \quad \text{for every } x \in X \text{ and } t > 0. \tag{2.7}$$

In particular, for $D \in \mathcal{B}$ with $\nu(D) < +\infty$, we have

$$e^{t\Delta_m} \chi_D(x) = e^{-t} \sum_{n=0}^{+\infty} m_x^{*n}(D) \frac{t^n}{n!} \quad \text{for every } x \in X \text{ and } t > 0.$$

PROOF. Recall that $m_x^{*0} = \delta_x$, $x \in X$. Let $u_0 \in L^1(X,\nu) \cap L^2(X,\nu)$ and

$$u(t)(x) := e^{-t} \sum_{n=0}^{+\infty} \int_X u_0(y) dm_x^{*n}(y) \frac{t^n}{n!} \quad \text{for } x \in X \text{ and } t > 0.$$

Let's see that u is well defined. Recall that, by the invariance of ν with respect to m_x^{*n}, since $u_0 \in L^1(X,\nu)$, we have that $u_0 \in L^1(X, m_x^{*n})$ for ν-a.e. $x \in X$ and every $n \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned} &\int_X \sum_{n=0}^{k} \left| \int_X u_0(y) dm_x^{*n}(y) \right| \frac{t^n}{n!} d\nu(x) = \sum_{n=0}^{k} \int_X \left| \int_X u_0(y) dm_x^{*n}(y) \right| \frac{t^n}{n!} d\nu(x) \\ &\leqslant \sum_{n=0}^{k} \int_X \int_X |u_0(y)|\, dm_x^{*n}(y) d\nu(x) \frac{t^n}{n!} = \sum_{n=0}^{k} \int_X |u_0(x)|\, d\nu(x) \frac{t^n}{n!} \leqslant e^t \|u_0\|_{L^1(X,\nu)}. \end{aligned}$$

Then, if

$$f_k(x,t) := \sum_{n=0}^{k} \left| \int_X u_0(y) dm_x^{*n}(y) \right| \frac{t^n}{n!}, \quad (x,t) \in X \times (0, +\infty),$$

we have that $0 \leqslant f_k(x,t) \leqslant f_{k+1}(x,t)$ and $\displaystyle\int_X f_k(x,t) d\nu(x) \leqslant e^t \|u_0\|_{L^1(X,\nu)}$ for every $k \in \mathbb{N}$, $x \in X$ and $t > 0$. Therefore, we may apply the monotone convergence theorem to get that the function

$$x \mapsto \sum_{n=0}^{+\infty} \left| \int_X u_0(y) dm_x^{*n}(y) \right| \frac{t^n}{n!}, \quad x \in X,$$

belongs to $L^1(X,\nu)$ for each $t > 0$ (hence, in particular, it is ν-a.e finite) with

$$\int_X \sum_{n=0}^{+\infty} \left| \int_X u_0(y) dm_x^{*n}(y) \right| \frac{t^n}{n!} d\nu(x) \leqslant e^t \|u_0\|_{L^1(X,\nu)}, \quad t > 0.$$

Note that the same applies to the function

$$x \mapsto \sum_{n=0}^{+\infty} \int_X |u_0(y)|\, dm_x^{*n}(y) \frac{t^n}{n!}, \quad x \in X.$$

From this we get that $u(t)(x)$ is well defined and also the uniform convergence of the series for t in compact subsets of $[0, +\infty)$. Hence,

$$\frac{du}{dt}(t)(x) = -u(t)(x) + e^{-t} \sum_{n=1}^{+\infty} \int_X u_0(y) dm_x^{*n}(y) \frac{t^{n-1}}{(n-1)!}.$$

Therefore, to prove (2.7), we only need to show that

$$e^{-t}\sum_{n=1}^{+\infty}\int_X u_0(y)dm_x^{*n}(y)\frac{t^{n-1}}{(n-1)!}=\int_X u(z,t)dm_x(z).$$

Recall that, for every $x\in X$ and $n\in\mathbb{N}$,

$$\int_X u_0(y)dm_x^{*n}(y)=\int_X\left(\int_X u_0(y)dm_z^{*(n-1)}(y)\right)dm_x(z).$$

Consequently,

$$e^{-t}\sum_{n=1}^{+\infty}\int_X u_0(y)dm_x^{*n}(y)\frac{t^{n-1}}{(n-1)!}=e^{-t}\sum_{n=1}^{+\infty}\int_X\left(\int_X u_0(y)dm_z^{*(n-1)}(y)\right)dm_x(z)\frac{t^{n-1}}{(n-1)!}$$
$$=\int_{z\in X}\left(e^{-t}\sum_{n=1}^{+\infty}\int_X u_0(y)dm_z^{*(n-1)}(y)\frac{t^{n-1}}{(n-1)!}\right)dm_x(z)=\int_X u(t)(z)dm_x(z),$$

where we have interchanged the series and integral thanks to the dominated convergence theorem since

$$\left|e^{-t}\sum_{n=1}^{k}\int_X u_0(y)dm_z^{*(n-1)}(y)\frac{t^{n-1}}{(n-1)!}\right|\leqslant e^{-t}\sum_{n=1}^{+\infty}\int_X |u_0(y)|dm_z^{*(n-1)}(y)\frac{t^{n-1}}{(n-1)!}=:F(z,t)$$

and $F(\cdot,t)$ belongs to $L^1(X,\nu)$, thus to $L^1(X,m_x)$ for ν-a.e. $x\in X$ and every $t>0$. □

2.1.1. Infinite speed of propagation. *Let us see that the infinite speed of propagation of the heat flow is equivalent to the m-connectedness of the random walk space.*

THEOREM 2.9. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. $[X,\mathcal{B},m,\nu]$ is m-connected if, and only if, for any non-ν-null $0\leqslant u_0\in L^1(X,\nu)\cap L^2(X,\nu)$, we have that $e^{t\Delta_m}u_0>0$ for ν-a.e. $x\in X$ and all $t>0$.*

PROOF. ($\Rightarrow$): Given a non-ν-null $0\leqslant u_0\in L^1(X,\nu)\cap L^2(X,\nu)$, there exist $D\in\mathcal{B}$ with $\nu(D)>0$ and $\alpha>0$, such that $u_0\geqslant\alpha\chi_D$. Therefore, by Theorem 2.8 and the m-connectedness of $[X,\mathcal{B},m,\nu]$,

$$e^{t\Delta_m}u_0(x)\geqslant\alpha e^{t\Delta_m}\chi_D(x)=\alpha e^{-t}\sum_{n=0}^{\infty}m_x^{*n}(D)\frac{t^n}{n!}>0\quad\text{for }\nu\text{-a.e. }x\in X\text{ and every }t>0.$$

Indeed, since $[X,\mathcal{B},m,\nu]$ is m-connected we have that $\sum_{n=1}^{\infty}m_x^{*n}(D)>0$ ν-a.e.

($\Leftarrow$): Take $D\in\mathcal{B}$ with $\nu(D)>0$. By Theorem 2.8, we have that

$$e^{t\Delta_m}\chi_D(x)=e^{-t}\sum_{n=0}^{\infty}m_x^{*n}(D)\frac{t^n}{n!}>0\quad\text{for }\nu\text{-a.e. }x\in X\text{ and every }t>0.$$

Therefore, since $m_x^{*0}=\delta_x$ we have that $\sum_{n=1}^{\infty}m_x^{*n}(D)>0$ for ν-a.e. $x\in X\backslash D$. However, by Corollary 1.28, we already have that $\sum_{n=1}^{\infty}m_x^{*n}(D)>0$ for ν-a.e. $x\in D$. □

REMARK 2.10. In the preceding proof, if m is ν-irreducible (Definition 1.4) we obtain that, in fact,

$$e^{t\Delta_m}u_0(x)>0\quad\text{for all }x\in X\text{ and for all }t>0\,.$$

Moreover, when X is a topological space, we introduce a weaker notion of m-connectedness that will serve to characterise the infinite speed of propagation of the heat flow for continuous initial data.

DEFINITION 2.11. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. Assume that X is equipped with a topology and $\mathcal{B}$ is the associated Borel σ-algebra. We say that $[X, \mathcal{B}, m, \nu]$ is weakly m-connected if, for every open set $D \in \mathcal{B}$ with $\nu(D) > 0$ and ν-a.e. $x \in X$,

$$\sum_{n=1}^{\infty} m_x^{*n}(D) > 0,$$

i.e., if for every open set $D \in \mathcal{B}$ with $\nu(D) > 0$, we have that $\nu(N_D^m) = 0$.

In the same way as m-connectedness is reminiscent of φ-irreducibility (in fact, φ-essential irreducibility), this weaker notion of m-connectedness for topological spaces is evocative of classical notions like that of open set irreducibility (see [**128**, Chapter 6.1.2]*).*

THEOREM 2.12. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Assume that X is a normal space*[1]*, $\mathcal{B}$ is the associated Borel σ-algebra and ν is inner regular. Then, $[X, \mathcal{B}, m, \nu]$ is weakly-m-connected if, and only if, for every non-ν-null $0 \leqslant u_0 \in L^1(X,\nu) \cap L^2(X,\nu) \cap C(X)$, we have that $e^{t\Delta_m} u_0 > 0$ ν-a.e. for all $t > 0$.*

PROOF. ($\Rightarrow$): Similar to the proof of the left to right implication in Theorem 2.9.
($\Leftarrow$): Take $D \in \mathcal{B}$ open with $\nu(D) > 0$, since ν is inner regular there exists a compact set $K \subset D$ with $\nu(K) > 0$. By Urysohn's lemma we may find a continuous function $0 \leqslant u_0 \leqslant 1$ such that $u_0 = 0$ on $X \setminus D$ and $u_0 = 1$ on K, thus $u_0 \leqslant \chi_D$. Hence

$$e^{-t} \sum_{n=0}^{\infty} m_x^{*n}(D) \frac{t^n}{n!} \geqslant e^{t\Delta_m} u_0(x) > 0 \quad \text{for } \nu\text{-a.e. } x \in X \text{ and every } t > 0.$$

So we conclude as in Theorem 2.9. □

2.2. Asymptotic behaviour

Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. It is easy to see that Δ_m is ergodic if, and only if,

$$f \in L^1(X,\nu) \cap L^2(X,\nu), \quad e^{t\Delta_m} f = f \quad \forall t \geqslant 0 \;\Rightarrow\; f \text{ is } \nu\text{-a.e. a constant.}$$

Moreover, we have the following result.

PROPOSITION 2.13. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. For every $f \in L^2(X,\nu)$,*

$$\lim_{t \to \infty} e^{t\Delta_m} f = f_\infty \in \{u \in L^2(X,\nu) \,:\, \Delta_m u = 0\}.$$

Suppose that $[X, \mathcal{B}, m, \nu]$ is m-connected, then,

(i) if $\nu(X) = +\infty$, $f_\infty = 0$ ν-a.e.

(ii) if ν is a probability measure, $f_\infty = \int_X f(x) d\nu(x)$ ν-a.e.

PROOF. Since $\mathcal{H}_m$ is a proper and lower semicontinuous function in X attaining the minimum at the zero function and, moreover, $\mathcal{H}_m$ is even, by [**46**, Theorem 5], we have that the strong limit in $L^2(X,\nu)$ of $e^{t\Delta_m} f$ exists and is a minimum point of $\mathcal{H}_m$, i.e.,

$$u_\infty \in \{u \in L^1(X,\nu) \cap L^2(X,\nu) \,:\, 0 \in \Delta_m(u)\}.$$

The second part is a consequence of the ergodicity of Δ_m (recall Theorem 1.50) and the conservation of mass (2.1). See also [**22**, Proposition 3.1.13]. □

If a Poincaré inequality holds it follows, with a similar proof to the one done in the continuous setting (see, for instance, [**22**]*), that, if* $\text{gap}(-\Delta_m) > 0$*, then $e^{t\Delta_m} u_0$ converges to $\nu(u_0)$ with exponential rate* $\text{gap}(-\Delta_m)$*.*

[1] A topological space X is a normal space if, given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F that are also disjoint.

THEOREM 2.14. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space such that ν is a probability measure. The following statements are equivalent:*

(i) There exists $\lambda > 0$ such that

$$\lambda \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu).$$

(ii) For every $f \in L^2(X,\nu)$

$$\|e^{t\Delta_m} f - \nu(f)\|_{L^2(X,\nu)} \leqslant e^{-\lambda t}\|f - \nu(f)\|_{L^2(X,\nu)} \quad \text{for all } t \geqslant 0;$$

or, equivalently, for every $f \in L^2(X,\nu)$ with $\nu(f) = 0$,

$$\|e^{t\Delta_m} f\|_{L^2(X,\nu)} \leqslant e^{-\lambda t}\|f\|_{L^2(X,\nu)} \quad \text{for all } t \geqslant 0.$$

REMARK 2.15. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and assume that ν is a probability measure. Let μ_1, $\mu_2 \in \mathcal{P}(X)$. We denote by $\|\mu_1 - \mu_2\|_{TV}$ the total variation distance between μ_1 and μ_2, i.e.,

$$\|\mu_1 - \mu_2\|_{TV} := \sup\{|\mu_1(A) - \mu_2(A)| \,:\, A \in \mathcal{B}\}.$$

Then, for $f \in L^2(X,\nu)$ and $\mu_t = \left(e^{t\Delta_m} f\right)\nu$, we have

$$\|\mu_t - \nu\|_{TV} \leqslant \|f - 1\|_{L^2(X,\nu)}\, e^{-\mathrm{gap}(-\Delta_m)t}.$$

Indeed, by Theorem 2.14, for any $A \in \mathcal{B}$,

$$\begin{aligned}\left|\int_A e^{t\Delta_m} f d\nu - \nu(A)\right| &\leqslant \int_A \left|e^{t\Delta_m} f - 1\right| d\nu \leqslant \left(\int_X \left|e^{t\Delta_m} f - 1\right|^2 d\nu\right)^{\frac{1}{2}} \\ &\leqslant e^{-\mathrm{gap}(-\Delta_m)t}\|f - 1\|_{L^2(X,\nu)}.\end{aligned}$$

2.3. The Bakry-Émery curvature-dimension condition

The use of the Bakry-Émery curvature-dimension condition to obtain a valid definition of a Ricci curvature bound in Markov chains was first considered in 1998 by Schmuckenschlager [**144**]. *Moreover, in 2010, Lin and Yau* [**112**] *applied this idea to graphs. Subsequently, this concept of curvature in the discrete setting has been frequently used (see* [**107**] *and the references therein). Note that, to deal with the Bakry-Émery curvature-dimension condition, one needs to make use of a carré du champ operator Γ (see* [**22**, Section 1.4.2]*) . In the framework of Markov diffusion semigroups, in order to get good inequalities from this curvature-dimension condition, it is essential that the generator A of the semigroup satisfies the chain rule formula:*

$$A(\Phi(f)) = \Phi'(f)A(f) + \Phi''(f)\Gamma(f) \quad \text{for } f \in D(A) \text{ and smooth } \Phi : \mathbb{R} \to \mathbb{R},$$

which characterizes diffusion operators in the continuous setting (see [**22**]*). Unfortunately, this chain rule does not hold in the discrete setting, and this is one of the main difficulties that arises when working with this curvature-dimension condition in metric random walk spaces.*

Following [**22**, Definition 1.4.2], *we make the following definition.*

DEFINITION 2.16. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. The bilinear map

$$\Gamma(f,g)(x) := \frac{1}{2}\Big(\Delta_m(fg)(x) - f(x)\Delta_m g(x) - g(x)\Delta_m f(x)\Big), \quad x \in X,\ f,g \in L^2(X,\nu),$$

is called the *carré du champ operator* of Δ_m.

With this notion, and following the theory developed in [**22**], *we can study the Bakry-Émery curvature-dimension condition in reversible random walk spaces. In particular, we will study its relation with the spectral gap.*

*According to Bakry and Émery [**21**], we define the Ricci curvature operator Γ_2 by iterating Γ as follows:*

$$\Gamma_2(f,g) := \frac{1}{2}\Big(\Delta_m\Gamma(f,g) - \Gamma(f,\Delta_m g) - \Gamma(\Delta_m f, g)\Big),$$

which is well defined for $f, g \in L^2(X,\nu)$. We will write, for $f \in L^2(X,\nu)$,

$$\Gamma(f) := \Gamma(f,f) = \frac{1}{2}\Delta_m(f^2) - f\Delta_m f$$

and

$$\Gamma_2(f) := \Gamma_2(f,f) = \frac{1}{2}\Delta_m\Gamma(f) - \Gamma(f,\Delta_m f).$$

It is easy to see that, for $x \in X$,

$$\Gamma(f,g)(x) = \frac{1}{2}\int_X \nabla f(x,y)\nabla g(x,y)dm_x(y) \quad \text{and} \quad \Gamma(f)(x) = \frac{1}{2}\int_X |\nabla f(x,y)|^2 dm_x(y).$$

Consequently (recall Definitions 1.66 and 2.1),

$$\int_X \Gamma(f,g)(x)d\nu(x) = \mathcal{E}_m(f,g) \quad \text{and} \quad \int_X \Gamma(f)(x)d\nu(x) = \mathcal{H}_m(f). \tag{2.8}$$

Furthermore, by (1.4) and (2.8), we get

$$\int_X \Gamma_2(f)\,d\nu = \frac{1}{2}\int_X (\Delta_m\Gamma(f) - 2\Gamma(f,\Delta_m f))\,d\nu = -\int_X \Gamma(f,\Delta_m f)\,d\nu = -\mathcal{E}_m(f,\Delta_m f),$$

thus

$$\int_X \Gamma_2(f)\,d\nu = \int_X (\Delta_m f)^2\,d\nu. \tag{2.9}$$

Definition 2.17. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. The operator Δ_m satisfies the *Bakry-Émery curvature-dimension condition $BE(K,n)$* for $n \in (1,+\infty)$ and $K \in \mathbb{R}$ if

$$\Gamma_2(f) \geqslant \frac{1}{n}(\Delta_m f)^2 + K\Gamma(f) \quad \forall f \in L^2(X,\nu). \tag{2.10}$$

The constant n is the *dimension of the operator* Δ_m, and K is the *lower bound of the Ricci curvature of the operator* Δ_m. If there exists $K \in \mathbb{R}$ such that

$$\Gamma_2(f) \geqslant K\Gamma(f) \quad \forall f \in L^2(X,\nu), \tag{2.11}$$

then it is said that the operator Δ_m satisfies the *Bakry-Émery curvature-dimension condition $BE(K,\infty)$*.

Observe that, if Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K,n)$, then it also satisfies the Bakry-Émery curvature-dimension condition $BE(K,m)$ for $m > n$.

Definition 2.17 is motivated by the well known fact that on a complete n-dimensional Riemannian manifold (M,g), the Laplace-Beltrami operator Δ_g satisfies $BE(K,n)$ if, and only if, the Ricci curvature of the Riemannian manifold is bounded from below by K (see, for example, [**22**, Appendix C.6]*).*

As mentioned at the beginning of this section, the use of the Bakry-Émery curvature-dimension condition as a possible definition of a Ricci curvature bound in Markov chains was first considered in 1998 [**144**]. *More recently, following the work by Lin and Yau* [**112**]*, this concept of Ricci curvature has been commonly used in the discrete setting (see* [**107**] *and the references therein).*

Integrating (2.10) over X with respect to ν yields

$$\int_X \Gamma_2(f)\,d\nu \geqslant \frac{1}{n}\int_X (\Delta_m f)^2\,d\nu + K\int_X \Gamma(f)\,d\nu.$$

Now, by (2.8) and (2.9), this inequality can be rewritten as

$$\int_X (\Delta_m f)^2 \, d\nu \geqslant \frac{1}{n} \int_X (\Delta_m f)^2 \, d\nu + K\mathcal{H}_m(f),$$

or, equivalently, as

$$K\frac{n}{n-1}\mathcal{H}_m(f) \leqslant \int_X (\Delta_m f)^2 \, d\nu. \tag{2.12}$$

Similarly, integrating (2.11) over X with respect to ν we get

$$K\mathcal{H}_m(f) \leqslant \int_X (\Delta_m f)^2 \, d\nu. \tag{2.13}$$

Definition 2.18. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $n \in (1, +\infty)$ and $K \in \mathbb{R}$. The operator Δ_m satisfies the *integrated Bakry-Émery curvature-dimension condition $IBE(K,n)$* if the inequality (2.12) holds for every $f \in L^2(X,\nu)$. Moreover, if (2.13) holds for every $f \in L^2(X,\nu)$ we will say that Δ_m satisfies the *integrated Bakry-Émery curvature-dimension condition $IBE(K,\infty)$*.

On account of Theorem 1.80, we can rewrite the Poincaré inequality via the integrated Bakry-Émery curvature-dimension conditions as follows (see [**22**, Theorem 4.8.4]*; see also* [**25**, Theorem 2.1]*).*

Theorem 2.19. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that Δ_m is ergodic (or, equivalently, that $[X, d, m, \nu]$ is m-connected). Let $n \in (1, +\infty)$ and $K > 0$. Then,*

(1) Δ_m satisfies the integrated Bakry-Émery curvature-dimension condition $IBE(K,n)$ if, and only if, $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality with constant $K\frac{n}{n-1}$.

(2) Δ_m satisfies the integrated Bakry-Émery curvature-dimension condition $IBE(K,\infty)$ if, and only if, $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality with constant K.

Therefore,

(1) if Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K,n)$, then

$$\operatorname{gap}(-\Delta_m) \geqslant K\frac{n}{n-1}. \tag{2.14}$$

(2) if Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K,\infty)$, then

$$\operatorname{gap}(-\Delta_m) \geqslant K. \tag{2.15}$$

In the next example we will see that, in general, an integrated Bakry-Émery curvature-dimension condition $IBE(K,n)$ with $K > 0$ does not imply a Bakry-Émery curvature-dimension condition $BE(K,n)$ with $K > 0$.

Example 2.20. Consider the weighted discrete graph $G = (V(G), E(G))$ with vertex set $V(G) = \{a, b, c\}$ and weights: $w_{a,b} = w_{b,c} = 1$ and $w_{i,j} = 0$ otherwise. Let $[V(G), d_G, m^G, \nu_G]$ be the associated metric random walk space and let $\Delta := \Delta_{m^G}$. A simple calculation gives

$$\Gamma(f)(a) = \frac{1}{2}(f(b) - f(a))^2 = \frac{1}{2}(\Delta f(a))^2,$$

$$\Gamma(f)(c) = \frac{1}{2}(f(b) - f(c))^2 = \frac{1}{2}(\Delta f(c))^2,$$

and

$$\begin{aligned} \Gamma(f)(b) &= \frac{1}{4}(f(b) - f(a))^2 + \frac{1}{4}(f(b) - f(c))^2 = \frac{1}{4}\left((\Delta f(a))^2 + (\Delta f(c))^2\right) \\ &= \frac{1}{2}\left(\Gamma(f)(a) + \Gamma(f)(c)\right). \end{aligned}$$

Moreover,

$$\Gamma_2(f)(a) = \frac{1}{8}(\Delta f(c))^2 + \frac{5}{8}(\Delta f(a))^2 + \frac{1}{4}\Delta f(a)\Delta f(c) \tag{2.16}$$

and

$$\Gamma_2(f)(c) = \frac{1}{8}(\Delta f(a))^2 + \frac{5}{8}(\Delta f(c))^2 + \frac{1}{4}\Delta f(a)\Delta f(c). \tag{2.17}$$

Having in mind (2.16) and (2.17), the inequality

$$\Gamma_2(f)(v) \geqslant \frac{1}{n}(\Delta f)^2(v) + K\Gamma(f)(v)$$

holds true for $v \in \{a, c\}$ and every $f \in L^2(V(G), \nu_G)$ if, and only if,

$$\frac{1}{4}y^2 + \frac{5}{4}x^2 + \frac{1}{2}xy \geqslant \frac{2}{n}x^2 + Kx^2 \qquad \forall x, y \in \mathbb{R}. \tag{2.18}$$

Now, since (2.18) is true for $x = 0$, (2.18) holds if, and only if,

$$K \leqslant \inf_{x \neq 0,\, y} \frac{\frac{1}{4}y^2 + \frac{5}{4}x^2 + \frac{1}{2}xy - \frac{2}{n}x^2}{x^2}.$$

Moreover, taking $y = \lambda x$, we obtain that the following inequality must be satisfied

$$K \leqslant \inf_{\lambda} \left(\frac{1}{4}\lambda^2 + \frac{5}{4} + \frac{1}{2}\lambda - \frac{2}{n} \right) = 1 - \frac{2}{n}.$$

In fact, it is easy to see that (2.18) is true for any $K \leqslant 1 - \frac{2}{n}$.

On the other hand, for $f \in L^2(V(G), \nu_G)$,

$$\Gamma_2(f)(b) = \frac{1}{2}(\Delta f(b))^2 + \Gamma(f)(b),$$

and it is easy to see that

$$\Gamma_2(f)(b) \geqslant \frac{1}{n}(\Delta f(b))^2 + K\Gamma(f)(b) \quad \text{for every } n > 1 \text{ and } \; K \leqslant 1 - \frac{2}{n}.$$

Therefore, we have that this graph Laplacian satisfies the Bakry-Émery curvature-dimension condition

$$BE\left(1 - \frac{2}{n}, n\right) \quad \text{for every } n > 1,$$

being $K = 1 - \frac{2}{n}$ the best constant for a fixed $n > 1$.

Now, it is easy to see that $\mathrm{gap}(-\Delta) = 1$ thus, by Theorem 2.19, we have that Δ satisfies the *integrated Bakry-Émery curvature-dimension condition* $IBE(K, n)$ with $K = 1 - \frac{1}{n} > 1 - \frac{2}{n}$.

Note that Δ satisfies the Bakry-Émery curvature-dimension condition $BE(1, \infty)$ and hence, in this example, the bound in (2.15) is sharp but there is a gap in the bound (2.14).

It is well known that in the case of diffusion semigroups, the Bakry-Émery curvature-dimension condition $BE(K, \infty)$ of its generator is characterized by gradient estimates on the semigroup (see, for instance, [**20**] *or* [**22**]*). The same characterization is also true for locally finite weighted discrete graphs (see, for example,* [**54**] *and* [**107**]*). With a similar proof we have that in the general context of metric random walk spaces this characterization is also true.*

THEOREM 2.21. *Let $[X, d, m, \nu]$ be a reversible metric random walk space and let $(T_t)_{t>0} = (e^{t\Delta_m})_{t>0}$ be the heat semigroup. Then, Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K, \infty)$ with $K > 0$ if, and only if,*

$$\Gamma(T_t f) \leqslant e^{-2Kt} T_t(\Gamma(f)) \quad \forall\, t \geqslant 0 \text{ and } f \in L^2(X, \nu). \tag{2.19}$$

Proof. Fix $t > 0$. For $s \in [0, t)$, we define the function

$$g(s, x) := e^{-2Ks} T_s(\Gamma(T_{t-s}f))(x), \quad x \in X.$$

The same computations as in [**107**] show that

$$\frac{\partial g}{\partial s}(s, x) = 2e^{-2Ks} T_s \left(\Gamma_2(T_{t-s}f) - K\Gamma(T_{t-s}f)\right)(x).$$

Then, if Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K, \infty)$ with $K > 0$, we have that $\frac{\partial g}{\partial s}(s, x) \geqslant 0$ which is equivalent to (2.19).

On the other hand, if (2.19) holds, then $\frac{\partial g}{\partial s}(0, x) \geqslant 0$, which is equivalent to

$$\Gamma_2(T_t f) - K\Gamma(T_t f) \geqslant 0.$$

Then, letting $t \to 0$, we get $\Gamma_2(f) - K\Gamma(f) \geqslant 0$. □

2.4. Transport inequalities

*Following the papers by Marton and Talagrand ([**118**], [**152**]) about transport inequalities, which relate the Wasserstein distances to entropy and information, this research topic has had a great development (see the survey [**94**]). One of the keystones of this theory was the discovery in 1986 by Marton [**117**] of the link between transport inequalities and the concentration of measure. Note that concentration of measure inequalities can be obtained by means of other functional inequalities, such as isoperimetric and logarithmic Sobolev inequalities (see Ledoux's textbook [**110**]). In this section we show that, under the positivity of the Bakry-Émery curvature-dimension condition or the Ollivier-Ricci curvature, a transport-information inequality holds (Theorems 2.27 and 2.34). Moreover, we prove that if a transport-information inequality holds then a transport-entropy inequality is also satisfied (Theorem 2.31) and that, in general, the converse implication does not hold.*

Definition 2.22. Let $[X, d, m, \nu]$ be a metric random walk space. We define (recall (1.21))

$$\Theta(x) := \frac{1}{2}\left(W_2^d(\delta_x, m_x)\right)^2 = \frac{1}{2}\int_X d(x, y)^2 dm_x(y), \quad x \in X,$$

and

$$\Theta_m := \operatorname*{ess\,sup}_{x \in X} \Theta(x).$$

Note that, if diam(X) is finite then, since $\Theta(x) \leqslant \frac{1}{2}\left(diam(\mathrm{supp}(m_x)\right)^2$, we have $\Theta_m \leqslant \frac{1}{2}(diam(X))^2$. Observe also that

$$\|\Gamma(f)\|_\infty = \sup_{x \in X} \frac{1}{2}\int_X (f(x) - f(y))^2 dm_x(y) \leqslant \Theta_m \|f\|_{Lip}^2. \tag{2.20}$$

Example 2.23. Given a metric measure space (X, d, μ) as in Example 1.41, if $m^{\mu,\epsilon}$ is the ϵ-step random walk associated with μ, that is

$$m_x^{\mu,\epsilon} := \frac{\mu \llcorner B(x, \epsilon)}{\mu(B(x, \epsilon))} \quad \text{for } x \in X,$$

then

$$\Theta_{m^{\mu,\epsilon}} \leqslant \frac{1}{2}\epsilon^2.$$

*Following [**134**] we define the jump of a random walk as follows.*

Definition 2.24. Let $[X, d, m, \nu]$ be a metric random walk space. The *jump* of the random walk at x is defined by

$$J_m(x) := W_1^d(\delta_x, m_x) = \int_X d(x, y) dm_x(y).$$

EXAMPLE 2.25. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with a locally finite discrete graph as in Example 1.38. Then, for $x \in V(G)$,

$$J_{m^G}(x) = \frac{1}{d_x} \sum_{y \sim x,\, y \neq x} w_{xy} \leqslant 1$$

thus

$$\Theta(x) = \frac{1}{2} J_{m^G}(x) = \frac{1}{2d_x} \sum_{x \sim y,\ x \neq y} w_{xy} \leqslant \frac{1}{2}.$$

If $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N \llcorner \Omega]$ is the metric random walk space given in Example 1.42 with $J(x) = \frac{1}{|B_r(0)|} \chi_{B_r(0)}(x)$ then, for $x \in \Omega$,

$$\Theta(x) \leqslant \frac{N}{2(N+2)} r^2.$$

DEFINITION 2.26. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and let μ be a probability measure on X. The *Fisher-Donsker-Varadhan information* of μ with respect to ν is defined by

$$I_\nu(\mu) := \begin{cases} 2\mathcal{H}_m(\sqrt{f}) & \text{if } \mu = f\nu,\ f \geqslant 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Observe that

$$D(I_\nu) = \{\mu \in \mathcal{P}(X) : \mu = f\nu,\ f \in L^1(X,\nu)^+\}$$

since $\sqrt{f} \in L^2(X,\nu) = D(\mathcal{H}_m)$ *whenever* $f \in L^1(X,\nu)^+$ *(we use the notation* $L^p(X,\nu)^+ := \{f \in L^p(X,\nu) : f \geqslant 0\ \nu\text{-a.e.}\}$*).*

In the next result we show that the Bakry-Émery curvature-dimension condition $BE(K,\infty)$ *with* $K > 0$ *implies a transport-information inequality, result that was obtained for the particular case of Markov chains in discrete spaces in* [**81**].

THEOREM 2.27. *Let* $[X, d, m, \nu]$ *be a reversible metric random walk space such that* ν *is a probability measure and assume that* Θ_m *is finite. If* Δ_m *satisfies the Bakry-Émery curvature-dimension condition* $BE(K,\infty)$ *with* $K > 0$*, then* ν *satisfies the transport-information inequality*

$$W_1^d(\mu,\nu) \leqslant \frac{\sqrt{\Theta_m}}{K} \sqrt{I_\nu(\mu)}, \quad \text{for all probability measures } \mu \ll \nu. \tag{2.21}$$

PROOF. Let $\mu \in \mathcal{P}(X)$ such that $\mu \ll \nu$ and set $\mu = f\nu$. By the Kantorovich-Rubinstein Theorem we have that

$$W_1^d(\mu,\nu) = \sup\left\{ \int_X g(x)(f(x)-1)d\nu(x) \ :\ \|g\|_{Lip} \leqslant 1 \text{ and } g \in L^\infty(X,\nu) \right\}.$$

Let $T_t = e^{t\Delta_m}$ be the heat semigroup. Given $g \in L^\infty(X,\nu)$ with $\|g\|_{Lip} \leqslant 1$ and having in mind Proposition 2.13, we get

$$\begin{aligned} \int_X g(x)(f(x)-1)d\nu(x) &= -\int_0^{+\infty} \frac{d}{dt} \int_X (T_t g)(x) f(x) d\nu(x) dt \\ &= -\int_0^{+\infty} \int_X \Delta_m (T_t g)(x) f(x) d\nu(x) dt \\ &= \int_0^{+\infty} \mathcal{E}_m(T_t g, f) dt = \int_0^{+\infty} \int_X \Gamma(T_t g, f)(x) d\nu(x) dt. \end{aligned}$$

Now, using the Cauchy-Schwartz inequality, the reversibility of ν with respect to m and that

$$(\sqrt{f}(y) + \sqrt{f}(x))^2 \leqslant 2((f(x) + f(y)),$$

we obtain the following:

$$\int_X \Gamma(T_t g, f)(x)d\nu(x) = \frac{1}{2}\int_{X\times X}((T_t g)(y) - (T_t g)(x))(f(y) - f(x))dm_x(y)d\nu(x)$$

$$= \frac{1}{2}\int_{X\times X}((T_t g)(y) - (T_t g)(x))(\sqrt{f}(y) - \sqrt{f}(x))(\sqrt{f}(y) + \sqrt{f}(x))dm_x(y)d\nu(x)$$

$$\leqslant \left(\int_{X\times X}\frac{1}{4}(\sqrt{f}(y) - \sqrt{f}(x))^2 dm_x(y)d\nu(x)\right)^{\frac{1}{2}}$$

$$\times\left(\int_{X\times X}((T_t g)(y) - (T_t g)(x))^2(\sqrt{f}(y) + \sqrt{f}(x))^2 dm_x(y)d\nu(x)\right)^{\frac{1}{2}}$$

$$\leqslant \left(\frac{1}{2}\int_X \Gamma(\sqrt{f})(x)d\nu(x)\right)^{\frac{1}{2}}\left(4\int_X\left(\int_X((T_t g)(y) - (T_t g)(x))^2 dm_x(y)\right)f(x)d\nu(x)\right)^{\frac{1}{2}}.$$

Then, applying Theorem 2.21, we get

$$\int_X g(x)(f(x)-1)d\nu(x) \leqslant \left(\frac{1}{2}\mathcal{H}_m(\sqrt{f})\right)^{\frac{1}{2}}\int_0^{+\infty}\left(4\int_X\Gamma((T_t g)(x))f(x)d\nu(x)\right)^{\frac{1}{2}}dt$$

$$\leqslant \left(2\mathcal{H}_m(\sqrt{f})\right)^{\frac{1}{2}}\int_0^{+\infty}\left(e^{-2Kt}\int_X T_t(\Gamma(g))(x)f(x)d\nu(x)\right)^{\frac{1}{2}}dt.$$

Now, by (2.3) and (2.20), we have

$$|T_t(\Gamma(g))(x)| \leqslant \|T_t(\Gamma(g))\|_\infty \leqslant \|\Gamma(g)\|_\infty \leqslant \Theta_m.$$

Hence,

$$\int_X g(x)(f(x)-1)d\nu(x) \leqslant \left(2\mathcal{H}_m(\sqrt{f})\right)^{\frac{1}{2}}\int_0^{+\infty}\left(e^{-2Kt}\Theta_m\int_X f(x)d\nu(x)\right)^{\frac{1}{2}}dt$$

$$\leqslant \frac{\sqrt{\Theta_m}}{K}\left(2\mathcal{H}_m(\sqrt{f})\right)^{\frac{1}{2}}.$$

Finally, taking the supremum over all functions $g \in L^\infty(X,\nu)$ with $\|g\|_{Lip} \leqslant 1$ we get (2.21). □

REMARK 2.28. If ν satisfies the transport-information inequality

$$W_1^d(\mu,\nu) \leqslant \lambda\sqrt{2\mathcal{H}_m(\sqrt{f})} \qquad \text{for all } \mu = f\nu \text{ with } f \in L^1(X,\nu)^+, \tag{2.22}$$

then ν is ergodic. Indeed, if ν is not ergodic then, by Theorem 1.63, there exists $D \in \mathcal{B}$ with $0 < \nu(D) < 1$ such that $\Delta_m \chi_D = 0$ ν-a.e. Now, if $\mu := \frac{1}{\nu(D)}\chi_D\nu$ then $\mu \neq \nu$ and, therefore, by (2.22), $\mathcal{H}_m(\chi_D) > 0$, which is in contradiction with $\Delta_m\chi_D = 0$.

As a consequence of the previous Remark and Theorem 2.27, we have that the positivity of the Bakry-Émery curvature-dimension condition implies ergodicity of Δ_m. Therefore, by Theorem 2.19, we have the following result.

THEOREM 2.29. *Let $[X, d, m, \nu]$ be a reversible metric random walk space such that ν is a probability measure and assume that Θ_m is finite. Then,*
(1) if Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K, n)$,

$$\text{gap}(-\Delta_m) \geqslant K\frac{n}{n-1}.$$

(2) if Δ_m satisfies the Bakry-Émery curvature-dimension condition $BE(K,\infty)$,

$$\text{gap}(-\Delta_m) \geqslant K.$$

DEFINITION 2.30. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. The *relative entropy* of $0 \leqslant \mu \in \mathcal{M}(X)$[2] with respect to ν is defined by

$$\mathrm{Ent}_\nu(\mu) := \begin{cases} \displaystyle\int_X f \log f d\nu - \nu(f) \log(\nu(f)) & \text{if } \mu = f\nu, \ f \geqslant 0, \ f \log f \in L^1(X, \nu), \\ +\infty, & \text{otherwise,} \end{cases}$$

with the usual convention that $f(x) \log f(x) = 0$ if $f(x) = 0$.

*The next result shows that a transport-information inequality implies a transport-entropy inequality and, therefore, normal concentration (see, for example, [**37, 110**]).*

THEOREM 2.31. *Let $[X, d, m, \nu]$ be a reversible metric random walk space such that ν is a probability measure. Suppose that Θ_m is finite and that there exists some $x_0 \in X$ such that $\int d(x, x_0) d\nu(x) < \infty$. Then, the transport-information inequality*

$$W_1^d(\mu, \nu) \leqslant \frac{1}{K}\sqrt{I_\nu(\mu)} \quad \textit{for all } \mu \in \mathcal{P}(X) \textit{ such that } \mu \ll \nu, \tag{2.23}$$

implies the transport-entropy inequality

$$W_1^d(\mu, \nu) \leqslant \sqrt{\frac{\sqrt{2\Theta_m}}{K} \mathrm{Ent}_\nu(\mu)} \quad \textit{for all } \mu \in \mathcal{P}(X) \textit{ such that } \mu \ll \nu. \tag{2.24}$$

PROOF. By [**37**, Theorem 1.3], (2.24) holds if, and only if,

$$\int_X e^{\lambda f(x)} d\nu(x) \leqslant e^{\lambda^2 \frac{\sqrt{\Theta_m}}{2\sqrt{2}K}} \tag{2.25}$$

for every function f with $\|f\|_{Lip} \leqslant 1$ and $\nu(f) = 0$, and every $\lambda \in \mathbb{R}$.

Let $f \in L^\infty(X, \nu)$ with $\|f\|_{Lip} \leqslant 1$ and $\nu(f) = 0$, we define $\Lambda : \mathbb{R} \to \mathbb{R}$ by

$$\Lambda(\lambda) := \int_X e^{\lambda f(x)} d\nu(x),$$

and the probability measures μ_λ, $\lambda \in \mathbb{R}$, as follows:

$$\mu_\lambda := \frac{1}{\Lambda(\lambda)} e^{\lambda f} d\nu.$$

By the Kantorovich-Rubinstein Theorem and the assumption (2.23), we have

$$\begin{aligned} \frac{d}{d\lambda} \log(\Lambda(\lambda)) &= \frac{1}{\Lambda(\lambda)} \int_X f(x) e^{\lambda f(x)} d\nu(x) = \int_X f(x)(d\mu_\lambda(x) - d\nu(x)) \leqslant W_1^d(\mu_\lambda, \nu) \\ &\leqslant \frac{1}{K}\sqrt{2\mathcal{H}_m\left(\sqrt{\frac{1}{\Lambda(\lambda)} e^{\lambda f}}\right)} = \frac{\sqrt{2}}{K}\sqrt{\int_X \Gamma\left(\sqrt{\frac{1}{\Lambda(\lambda)} e^{\lambda f}}\right)(x) d\nu(x)} \\ &= \frac{\sqrt{2}}{K}\sqrt{\int_X \frac{1}{\Lambda(\lambda)} \Gamma\left(e^{\frac{\lambda f}{2}}\right)(x) d\nu(x)}. \end{aligned}$$

Now, since $1 - \frac{1}{a} \leqslant \log a$ for every $a \geqslant 1$ and having in mind the reversibility of ν, we get that, for any $g \in L^2(X, \nu)$,

$$\int_X \Gamma(g)(x) d\nu(x) \leqslant \int_X g^2(x) \Gamma(\log g)(x) d\nu(x),$$

[2] $\mathcal{M}(X)$ is the set of Radon measures on X.

and, consequently, (2.20) yields

$$\begin{aligned}\frac{d}{d\lambda}\log(\Lambda(\lambda)) &\leqslant \frac{\sqrt{2}}{K}\sqrt{\int_X \frac{1}{\Lambda(\lambda)}e^{\lambda f(x)}\Gamma\left(\frac{\lambda f}{2}\right)(x)d\nu(x)} \\ &= \frac{\lambda}{\sqrt{2}K}\sqrt{\int_X \frac{1}{\Lambda(\lambda)}e^{\lambda f(x)}\Gamma(f)(x)d\nu(x)} \\ &= \frac{\lambda}{\sqrt{2}K}\sqrt{\int_X \Gamma(f)(x)d\mu_\lambda(x)} \leqslant \frac{\sqrt{\Theta_m}}{\sqrt{2}K}\lambda.\end{aligned}$$

Then, integrating this inequality we get (2.25).

Now, if $f \notin L^\infty(X,\nu)$, let $f_n := f \wedge n - \nu(f \wedge n) \in L^\infty(X,\nu)$ for $n \in \mathbb{N}$, which satisfy $\|f_n\|_{Lip} \leqslant 1$ and $\nu(f_n) = 0$ for every $n \in \mathbb{N}$. Then, Fatou's Lemma yields:

$$\int_X e^{\lambda f(x)}d\nu(x) \leqslant \liminf_n \int_X e^{\lambda f_n(x)}d\nu(x) \leqslant e^{\lambda^2\frac{\sqrt{\Theta_m}}{2\sqrt{2}K}},$$

as desired. □

In the next example we see that, in general, a transport-entropy inequality does not imply a transport-information inequality.

Example 2.32. Let $\Omega := [-1,0] \cup [2,3]$ and consider the metric random walk space $[\Omega, d, m^{J,\Omega}, \frac{1}{2}\mathcal{L}^1 \llcorner \Omega]$, with d the Euclidean distance and $J = \frac{1}{2}\chi_{[-1,1]}$ (see Example 1.42). By the Gaussian integrability criterion [**73**, Theorem 2.3], ν satisfies a transport-entropy inequality. However, ν does not satisfy a transport-information inequality since this would imply that ν is ergodic (see Remark 2.28) and it is easy to see that $[\Omega, d, m^{J,\Omega}, \frac{1}{2}\mathcal{L}^1 \llcorner \Omega]$ is not m-connected (thus, by Theorem 1.34, ν is not ergodic).

By Theorems 1.34 and 1.94, we have that the metric random walk space of Example 2.32 has non-positive Ollivier-Ricci curvature. In the next theorem we will see that, under positive Ollivier-Ricci curvature, a transport-information inequality holds. First, we need the following result.

Lemma 2.33. *Let $[X, d, m, \nu]$ be a reversible metric random walk space such that ν is a probability measure. If $f \in L^2(X,\nu)$ with $\|f\|_{Lip} \leqslant 1$, then $\|e^{t\Delta_m}f\|_{Lip} \leqslant e^{-t\kappa_m}$ (recall Definition 1.90 for the definition of κ_m.).*

Proof. By [**134**, Proposition 25], we have that

$$\kappa_{m^{*(n+l)}} \geqslant \kappa_{m^{*n}} + \kappa_{m^{*l}} - \kappa_{m^{*n}}\kappa_{m^{*l}} \quad \forall\, n,l \in \mathbb{N},$$

where $\kappa_{m^{*1}} = \kappa_m$. Hence,

$$1 - \kappa_{m^{*n}} \leqslant (1-\kappa_m)^n \quad \forall\, n \in \mathbb{N}. \tag{2.26}$$

By Theorem 2.8 and equation (2.26), we have

$$\begin{aligned}|e^{t\Delta_m}f(x) - e^{t\Delta_m}f(y)| &= \left| e^{-t}\sum_{n=0}^{+\infty}\int_X f(z)(dm_x^{*n}(z) - dm_y^{*n}(z))\frac{t^n}{n!}\right| \\ &\leqslant e^{-t}\sum_{n=0}^{+\infty} W_1^d(m_x^{*n}, m_y^{*n})\frac{t^n}{n!} \\ &\leqslant e^{-t}\sum_{n=0}^{+\infty}(1-\kappa_{m^{*n}})d(x,y)\frac{t^n}{n!} \leqslant e^{-t}\sum_{n=0}^{+\infty}(1-\kappa_m)^n\frac{t^n}{n!}d(x,y) \\ &= e^{-t}e^{t(1-\kappa_m)}d(x,y) = e^{-t\kappa_m}d(x,y),\end{aligned}$$

it follows from this that $\|e^{t\Delta_m}f\|_{Lip} \leqslant e^{-t\kappa_m}$. □

THEOREM 2.34. *Let $[X, d, m, \nu]$ be a reversible metric random walk space such that ν is a probability measure and each measure m_x has finite first moment. Assume further that Θ_m is finite. If $\kappa_m > 0$ then the following transport-information inequality holds:*

$$W_1^d(\mu,\nu) \leqslant \frac{\sqrt{2\Theta_m}}{\kappa_m}\sqrt{I_\nu(\mu)}, \quad \text{for all } \mu \in \mathcal{P}(X) \text{ such that } \mu \ll \nu.$$

PROOF. Let $T_t = e^{t\Delta_m}$ be the heat semigroup and $\mu = f\nu$ be a probability measure in X. We use, as in the proof of Theorem 2.27, the Kantorovich-Rubinstein Theorem. Let $g \in L^\infty(X,\nu)$ with $\|g\|_{Lip} \leqslant 1$. Having in mind Lemma 2.33, we have

$$\begin{aligned}
&\int_X g(x)(f(x)-1)d\nu(x) = -\int_0^{+\infty}\frac{d}{dt}\int_X (T_tg)(x)f(x)d\nu(x)dt \\
&= -\int_0^{+\infty}\int_X \Delta_m(T_tg)(x)f(x)d\nu(x)dt \\
&= \int_0^{+\infty}\frac{1}{2}\int_{X\times X}((T_tg)(y)-(T_tg)(x))(f(y)-f(x))dm_x(y)d\nu(x)dt \\
&\leqslant \int_0^{+\infty}\|T_tg\|_{Lip}\frac{1}{2}\int_{X\times X} d(x,y)|f(y)-f(x)|dm_x(y)d\nu(x)dt \\
&\leqslant \int_0^{+\infty} e^{-t\kappa_m}\frac{1}{2}\int_{X\times X} d(x,y)|f(y)-f(x)|dm_x(y)d\nu(x)dt \\
&= \frac{1}{2\kappa_m}\int_{X\times X} d(x,y)|f(y)-f(x)|dm_x(y)d\nu(x) \\
&= \frac{1}{2\kappa_m}\int_{X\times X} d(x,y)|\sqrt{f}(y)-\sqrt{f}(x)|\left(\sqrt{f}(y)+\sqrt{f}(x)\right)dm_x(y)d\nu(x) \\
&\leqslant \frac{\sqrt{2}}{2\kappa_m}\sqrt{\mathcal{H}_m(\sqrt{f})}\sqrt{\int_{X\times X} d^2(x,y)\left(\sqrt{f}(y)+\sqrt{f}(x)\right)^2 dm_x(y)d\nu(x)}.
\end{aligned}$$

Now, using reversibility of ν with respect to m,

$$\begin{aligned}
&\int_{X\times X} d^2(x,y)\left(\sqrt{f}(y)+\sqrt{f}(x)\right)^2 dm_x(y)d\nu(x) \\
&= \int_{X\times X} d^2(x,y)\left(2f(x)+2f(y)-\left(\sqrt{f}(y)-\sqrt{f}(x)\right)^2\right)dm_x(y)d\nu(x) \\
&\leqslant 2\int_{X\times X} d^2(x,y)\left(f(x)+f(y)\right)dm_x(y)d\nu(x) \leqslant 8\Theta_m.
\end{aligned}$$

Therefore, we get

$$\int_X g(x)(f(x)-1)d\nu(x) \leqslant \frac{2\sqrt{\Theta_m}}{\kappa_m}\sqrt{\mathcal{H}_m(\sqrt{f})},$$

thus taking the supremum over g yields

$$W_1^d(\mu,\nu) \leqslant \frac{\sqrt{2\Theta_m}}{\kappa_m}\sqrt{2\mathcal{H}_m(\sqrt{f})} = \frac{\sqrt{2\Theta_m}}{\kappa_m}\sqrt{I_\nu(\mu)}.$$

□

CHAPTER 3

The total variation flow

*Since its introduction as a means of solving the denoising problem in the seminal work by Rudin, Osher and Fatemi ([**143**]), the total variation flow has remained one of the most popular tools in Image Processing*[1]*. Furthermore, the use of neighbourhood filters by Buades, Coll and Morel in [**47**], that was originally proposed by P. Yaroslavsky ([**161**]), has led to an extensive literature in nonlocal models in image processing (see for instance [**48**], [**92**], [**109**], [**114**] and the references therein). Consequently, there is great interest in studying the total variation flow in the nonlocal context. Moreover, a different line of research considers an image as a weighted discrete graph, where the pixels are taken as the vertices and the "similarity" between pixels as the weights*[2]*. Therefore, the study of the 1-Laplacian operator and the total variation flow in random walk spaces has a potentially broad scope of application.*

*Further motivation for the study of the 1-Laplacian operator comes from spectral clustering. Partitioning data into sensible groups is a fundamental problem in machine learning, computer science, statistics and science in general. In these fields, it is usual to face large amounts of empirical data, and getting a first impression of these data by identifying groups with similar properties has proved to be very useful. One of the most popular approaches to this problem is to find the best balanced cut of a graph representing the data, such as the Cheeger ratio cut ([**61**]) which we will now introduce. Consider a finite weighted connected graph* $G = (V, E)$*, where* $V = \{x_1, \ldots, x_n\}$ *is the set of vertices (or nodes) and* E *the set of edges, which are weighted by a function* $w_{ji} = w_{ij} \geqslant 0$*,* $(i, j) \in E$*. The degree of the vertex* x_i *is denoted by* $d_i := \sum_{j=1}^{n} w_{ij}$*,* $i = 1, \ldots, n$*. In this context, the Cheeger cut value of a partition* $\{S, S^c\}$ *(*$S^c := V \setminus S$*) of* V *is defined as*

$$\mathcal{C}(S) := \frac{\mathrm{Cut}(S, S^c)}{\min\{\mathrm{vol}(S), \mathrm{vol}(S^c)\}},$$

where

$$\mathrm{Cut}(A, B) = \sum_{i \in A, j \in B} w_{ij},$$

and $\mathrm{vol}(S)$ *is the volume of* S*, defined as* $\mathrm{vol}(S) := \sum_{i \in S} d_i$*. Then,*

$$h(G) := \min_{S \subset V} \mathcal{C}(S)$$

is called the Cheeger constant, and a partition $\{S, S^c\}$ *of* V *is called a Cheeger cut of* G *if* $h(G) = \mathcal{C}(S)$*. Unfortunately, the Cheeger minimization problem of computing* $h(G)$ *is NP-hard ([**99**], [**148**]). However, it turns out that* $h(G)$ *can be approximated by the second eigenvalue* λ_2 *of the graph Laplacian thanks to the following Cheeger inequality ([**63**]):*

$$\frac{\lambda_2}{2} \leqslant h(G) \leqslant \sqrt{2\lambda_2}. \tag{3.1}$$

*This motivates the spectral clustering method ([**115**]), which, in its simplest form, thresholds the second eigenvalue of the graph Laplacian to get an approximation to the Cheeger constant and, moreover, to a Cheeger cut. In order to achieve a better approximation than the one provided by the classical spectral clustering method, a spectral clustering based on the graph*

[1]From the mathematical point of view, the study of the total variation flow in $\mathbb{R}^N$ was established in [**12**].

[2]The way in which these weights are defined depends on the problem at hand, see, for instance, [**79**] and [**114**].

p-Laplacian was developed in [**51**]*, where it is showed that the second eigenvalue of the graph p-Laplacian tends to the Cheeger constant $h(G)$ as $p \to 1^+$. In* [**148**] *the idea was further developed by directly considering the variational characterization of the Cheeger constant $h(G)$*

$$h(G) = \min_{u \in L^1} \frac{|u|_{TV}}{\|u - \text{median}(u))\|_1}, \tag{3.2}$$

where

$$|u|_{TV} := \frac{1}{2} \sum_{i,j=1}^{n} w_{ij} |u(x_i) - u(x_j)|.$$

The subdifferential of the energy functional $|\cdot|_{TV}$ is the 1-Laplacian in graphs Δ_1. Using the nonlinear eigenvalue problem $\lambda\,\text{sign}(u) \in -\Delta_1 u$, the theory of 1-Spectral Clustering is developed in [**57**], [**58**], [**59**] *and* [**99**].

Accordingly, the aim of this chapter is to study the total variation flow in reversible random walk spaces, obtaining general results that can be applied, as aforementioned, to the different points of view in image processing. In this regard, we introduce the 1-Laplacian operator associated with a random walk space (see Definition 3.16) and obtain various characterizations (see Theorem 3.13). In doing so, we generalize results obtained in [**120**] *and* [**121**] *for the particular case of $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$, and, moreover, generalize results in graph theory. We then proceed to prove existence and uniqueness of solutions of the total variation flow in random walk spaces and to study its asymptotic behaviour with the help of Poincaré type inequalities. Furthermore, we introduce the concepts of Cheeger and calibrable sets in random walk spaces and characterize calibrability by using the 1-Laplacian operator. Moreover, in Section 3.5, in connection with the 1-Spectral Clustering, we study the eigenvalue problem of the 1-Laplacian Δ_1^m and then relate it to the optimal Cheeger cut problem. Then again, these results apply, in particular, to locally finite weighted connected graphs, complementing the results obtained in the previously mentioned papers* [**57**], [**58**], [**59**] *and* [**99**]. *Lastly, in Section 3.6, we obtain a generalization of the Cheeger inequality* (3.1) *and of the variational characterization of the Cheeger constant* (3.2).

The Cheeger problem in the fractional case is studied in [**41**].

3.1. The nonlocal total variation

DEFINITION 3.1. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. We define the space of functions $BV_m(X, \nu)$ as follows

$$BV_m(X,\nu) := \left\{ u : X \to \mathbb{R} \ \text{ measurable } : \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) < \infty \right\}.$$

Moreover, the *m-total variation* of a function $u \in BV_m(X, \nu)$ is defined by

$$TV_m(u) := \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) = \frac{1}{2} \int_{X \times X} |u(y) - u(x)| d(\nu \otimes m_x)(x, y).$$

Note that $L^1(X, \nu) \subset BV_m(X, \nu)$ (in fact, by the invariance of ν with respect to m, $TV_m(u) \leqslant \|u\|_{L^1(X,\nu)}$ for every $u \in L^1(X, \nu)$). Observe also that, by Lemma 1.53,

$$P_m(E) = TV_m(\chi_E). \tag{3.3}$$

The space $BV_m(X, \nu)$ is the nonlocal counterpart of classical local bounded variation spaces. Note further that, in the local context, given a Lebesgue measurable set $E \subset \mathbb{R}^n$, its perimeter is equal to the total variation of its characteristic function (see [**7**]*) and the above equation* (3.3) *provides the nonlocal analogue.*

However, although they represent analogous concepts in different settings, the local classical BV-spaces and the nonlocal BV-spaces are of a different nature. For example, in our nonlocal framework $L^1(X, \nu) \subset BV_m(X, \nu)$ in contrast with classical local bounded variation spaces that are, by definition, contained in L^1.

Example 3.2. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space given in Example 1.38. Then,

$$\begin{aligned} TV_{m^G}(u) &= \frac{1}{2}\int_{V(G)}\int_{V(G)} |u(y)-u(x)|dm^G_x(y)d\nu_G(x) \\ &= \frac{1}{2}\int_{V(G)} \frac{1}{d_x}\left(\sum_{y\in V(G)} |u(y)-u(x)|w_{xy}\right)d\nu_G(x) \\ &= \frac{1}{2}\sum_{x\in V(G)} d_x\left(\frac{1}{d_x}\sum_{y\in V(G)}|u(y)-u(x)|w_{xy}\right) = \frac{1}{2}\sum_{x\in V(G)}\sum_{y\in V(G)}|u(y)-u(x)|w_{xy}, \end{aligned}$$

which coincides with the anisotropic total variation defined in [89].

In the following results we give some properties of the m-total variation.

Proposition 3.3. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and $\phi : \mathbb{R} \to \mathbb{R}$ a Lipschitz continuous function. If $u \in BV_m(X,\nu)$, then $\phi(u) \in BV_m(X,\nu)$ and*

$$TV_m(\phi(u)) \leqslant \|\phi\|_{Lip} TV_m(u).$$

Proof.

$$\begin{aligned} TV_m(\phi(u)) &= \frac{1}{2}\int_X\int_X |\phi(u)(y)-\phi(u)(x)|dm_x(y)d\nu(x) \\ &\leqslant \|\phi\|_{Lip}\frac{1}{2}\int_X\int_X |u(y)-u(x)|dm_x(y)d\nu(x) = \|\phi\|_{Lip}TV_m(u). \end{aligned}$$

□

Proposition 3.4. *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. Then, TV_m is convex and 1-Lipschitz continuous in $L^1(X,\nu)$.*

Proof. The convexity of TV_m follows easily. Let us see that it is 1-Lipschitz continuous. Let u, $v \in L^1(X,\nu)$. Since ν is invariant with respect to m, we have that

$$\begin{aligned} |TV_m(v) - TV_m(u)| &= \frac{1}{2}\left|\int_X\int_X (|v(y)-v(x)| - |u(y)-u(x)|)\, dm_x(y)d\nu(x)\right| \\ &\leqslant \frac{1}{2}\left(\int_X\int_X |v(y)-u(y)|dm_x(y)d\nu(x) + \int_X |v(x)-u(x)|d\nu(x)\right) \\ &= \frac{1}{2}\left(\int_X |v(y)-u(y)|d\nu(y) + \int_X |v(x)-u(x)|d\nu(x)\right) \\ &= \|v-u\|_{L^1(X,\nu)}. \end{aligned}$$

□

As in the local case, we have a coarea formula relating the m-total variation of a function with the m-perimeter of its superlevel sets.

Theorem 3.5 (**Coarea formula**). *Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. For any $u \in L^1(X,\nu)$, let $E_t(u) := \{x \in X \ : \ u(x) > t\}$. Then,*

$$TV_m(u) = \int_{-\infty}^{+\infty} P_m(E_t(u))\, dt.$$

Proof. Let $u \in L^1(X,\nu)$. Since

$$u(x) = \int_0^{+\infty} \chi_{E_t(u)}(x)\, dt - \int_{-\infty}^0 (1-\chi_{E_t(u)}(x))\, dt \quad \forall x \in X,$$

we have

$$u(y) - u(x) = \int_{-\infty}^{+\infty} \chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)\, dt \quad \forall x, y \in X.$$

Moreover, since $u(y) \geqslant u(x)$ implies $\chi_{E_t(u)}(y) \geqslant \chi_{E_t(u)}(x)$, we obtain that

$$|u(y) - u(x)| = \int_{-\infty}^{+\infty} |\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)|\, dt.$$

Therefore, we get

$$\begin{aligned} TV_m(u) &= \frac{1}{2}\int_X \int_X |u(y)-u(x)| dm_x(y) d\nu(x) \\ &= \frac{1}{2}\int_X \int_X \left(\int_{-\infty}^{+\infty} |\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)| dt \right) dm_x(y) d\nu(x) \\ &= \int_{-\infty}^{+\infty} \left(\frac{1}{2}\int_X \int_X |\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)| dm_x(y) d\nu(x) \right) dt = \int_{-\infty}^{+\infty} P_m(E_t(u)) dt, \end{aligned}$$

where Tonelli-Hobson's Theorem is used in the third equality. □

LEMMA 3.6. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected random walk space. Then,*

$$TV_m(u) = 0 \Leftrightarrow u \text{ is a constant } \nu\text{-a.e.}$$

PROOF. ($\Leftarrow$) Suppose that u is ν-a.e. equal to a constant $k \in \mathbb{R}$, then, since ν is invariant with respect to m, we have

$$\begin{aligned} TV_m(u) &= \frac{1}{2}\int_X \int_X |u(y)-u(x)| dm_x(y) d\nu(x) \\ &= \int_X \int_X |u(y) - k| dm_x(y) d\nu(x) \\ &= \int_X |u(x) - k| d\nu(x) = 0. \end{aligned}$$

($\Rightarrow$) Suppose that

$$0 = TV_m(u) = \frac{1}{2}\int_X \int_X |u(y)-u(x)| dm_x(y) d\nu(x).$$

Then, $\displaystyle\int_X |u(y)-u(x)| dm_x(y) = 0$ for ν-a.e. $x \in X$, thus

$$|\Delta_m u(x)| = \left| \int_X (u(y)-u(x)) dm_x(y) \right| \leqslant \int_X |u(y)-u(x)| dm_x(y) = 0 \quad \text{for } \nu\text{-a.e. } x \in X,$$

so we conclude by Theorem 1.50. □

3.2. The 1-Laplacian and the total variation flow

DEFINITION 3.7. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. For $p \geqslant 1$, we denote

$$X_m^p(X, \nu) := \left\{ \mathbf{z} \in L^\infty(X \times X, \nu \otimes m_x) \ : \ \mathrm{div}_m \mathbf{z} \in L^p(X, \nu) \right\}.$$

The following proposition follows similarly to Proposition 1.47.

PROPOSITION 3.8 (Green's Formula). *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $1 \leqslant p \leqslant \infty$, $u \in BV_m(X, \nu) \cap L^{p'}(X, \nu)$ and $\mathbf{z} \in X_m^p(X, \nu)$, then*

$$\int_X u(x) (\mathrm{div}_m \mathbf{z})(x) d\nu(x) = -\frac{1}{2} \int_{X \times X} \nabla u(x,y) \mathbf{z}(x,y) d(\nu \otimes m_x)(x,y). \tag{3.4}$$

In the next result we characterize the m-total variation and the m-perimeter using the m-divergence operator (see, for example, [7, Proposition 3.6] *for the analogous result in the* *local case). Let us denote by* $sign_0(r)$ *the usual sign function and by* $sign(r)$ *the multivalued sign function:*

$$(3.5)\qquad \operatorname{sign}_0(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0; \end{cases} \qquad \operatorname{sign}(r) := \begin{cases} 1 & \text{if } r > 0, \\ [-1,1] & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

PROPOSITION 3.9. *Let* $[X, \mathcal{B}, m, \nu]$ *be a reversible random walk space and* $1 \leqslant p \leqslant \infty$. *For* $u \in BV_m(X,\nu) \cap L^{p'}(X,\nu)$, *we have*

$$(3.6)\qquad TV_m(u) = \sup\left\{ \int_X u(x)(\operatorname{div}_m \mathbf{z})(x) d\nu(x) \ : \ \mathbf{z} \in X_m^p(X,\nu), \ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1 \right\}.$$

In particular, for any $E \in \mathcal{B}$ *with* $\nu(E) < \infty$, *we have*

$$P_m(E) = \sup\left\{ \int_E (\operatorname{div}_m \mathbf{z})(x) d\nu(x) \ : \ \mathbf{z} \in X_m^1(X,\nu), \ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1 \right\}.$$

PROOF. Let $u \in BV_m(X,\nu) \cap L^{p'}(X,\nu)$. Given $\mathbf{z} \in X_m^p(X,\nu)$ with $\|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$, Green's formula (3.4) yields

$$\begin{aligned} \int_X u(x)(\operatorname{div}_m \mathbf{z})(x) d\nu(x) &= -\frac{1}{2}\int_{X\times X} \nabla u(x,y)\mathbf{z}(x,y) d(\nu\otimes m_x)(x,y) \\ &\leqslant \frac{1}{2}\int_{X\times X} |u(y)-u(x)| dm_x(y) d\nu(x) = TV_m(u). \end{aligned}$$

Therefore,

$$\sup\left\{ \int_X u(x)(\operatorname{div}_m \mathbf{z})(x) dx \ : \ \mathbf{z} \in X_m^p(X,\nu), \ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1 \right\} \leqslant TV_m(u).$$

On the other hand, since ν is σ-finite, there exists a sequence of ν-measurable sets $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$ of ν-finite measure, such that $X = \cup_{n=1}^\infty K_n$. Then, if we define $\mathbf{z}_n(x,y) := \operatorname{sign}_0(u(y)-u(x))\chi_{K_n\times K_n}(x,y)$, we have that $\mathbf{z}_n \in X_m^p(X,\nu)$ with $\|\mathbf{z}_n\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$ and

$$\begin{aligned} TV_m(u) &= \frac{1}{2}\int_{X\times X} |u(y)-u(x)| d(\nu\otimes m_x)(x,y) \\ &= \lim_{n\to\infty} \frac{1}{2}\int_{K_n\times K_n} |u(y)-u(x)| d(\nu\otimes m_x)(x,y) \\ &= \lim_{n\to\infty} \frac{1}{2}\int_{X\times X} \nabla u(x,y)\mathbf{z}_n(x,y) d(\nu\otimes m_x)(x,y) \\ &= \lim_{n\to\infty} \int_X u(x)(\operatorname{div}_m(-\mathbf{z}_n))(x) d\nu(x)4 \\ &\leqslant \sup\left\{ \int_X u(x)(\operatorname{div}_m(\mathbf{z}))(x) d\nu(x) \ : \ \mathbf{z} \in X_m^p(X,\nu), \ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1 \right\}. \qquad \square \end{aligned}$$

COROLLARY 3.10. *Let* $[X, \mathcal{B}, m, \nu]$ *be a reversible random walk space. Then,* TV_m *is lower semi-continuous with respect to the weak convergence in* $L^2(X,\nu)$.

PROOF. If $u_n \rightharpoonup u$ weakly in $L^2(X,\nu)$ then, given $\mathbf{z} \in X_m^2(X,\nu)$ with $\|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$, we have that

$$\int_X u(x)(\operatorname{div}_m \mathbf{z})(x) d\nu(x) = \lim_{n\to\infty} \int_X u_n(x)(\operatorname{div}_m \mathbf{z})(x) d\nu(x) \leqslant \liminf_{n\to\infty} TV_m(u_n)$$

by Proposition 3.9. Now, taking the supremum over $\mathbf{z}$ in this inequality (and by Proposition 3.9 again), we get

$$TV_m(u) \leqslant \liminf_{n\to\infty} TV_m(u_n). \qquad \square$$

We will now introduce the 1-Laplacian operator in random walk spaces. To this aim we will first prove Theorem 3.13 which requires the following definitions.

Definition 3.11. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. We define $\mathcal{F}_m : L^2(X, \nu) \to]-\infty, +\infty]$ by

$$\mathcal{F}_m(u) := \begin{cases} TV_m(u) & \text{if } u \in L^2(X,\nu) \cap BV_m(X,\nu), \\ +\infty & \text{if } u \in L^2(X,\nu) \backslash BV_m(X,\nu). \end{cases}$$

Consider the formal nonlocal evolution equation

$$u_t(x,t) = \int_X \frac{u(y,t) - u(x,t)}{|u(y,t) - u(x,t)|} dm_x(y), \quad x \in X, t \geqslant 0. \tag{3.7}$$

*In order to study the Cauchy problem associated with this equation, we will see in Theorem 3.19 that we can rewrite it as the gradient flow in $L^2(X,\nu)$ of the functional $\mathcal{F}_m$ which is convex and lower semi-continuous. Following the method used in [**12**] we will characterize the subdifferential of the functional $\mathcal{F}_m$.*

Definition 3.12. Let (X, ν) be a measure space. Given a functional $\Phi : L^2(X,\nu) \to [0, \infty]$, we define $\widetilde{\Phi} : L^2(X,\nu) \to [0,\infty]$ as

$$\widetilde{\Phi}(v) := \sup \left\{ \frac{\displaystyle\int_X v(x) w(x) d\nu(x)}{\Phi(w)} \; : \; w \in L^2(X,\nu) \right\}$$

with the convention that $\frac{0}{0} = \frac{0}{\infty} = 0$. Obviously, if $\Phi_1 \leqslant \Phi_2$, then $\widetilde{\Phi}_2 \leqslant \widetilde{\Phi}_1$.

Theorem 3.13. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Let $u \in L^2(X,\nu)$ and $v \in L^2(X,\nu)$. The following assertions are equivalent:*
(i) $v \in \partial \mathcal{F}_m(u)$;
(ii) there exists $\mathbf{z} \in X_m^2(X,\nu)$ with $\|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$ such that

$$v = -\text{div}_m \mathbf{z} \tag{3.8}$$

and

$$\int_X u(x) v(x) d\nu(x) = \mathcal{F}_m(u);$$

(iii) there exists $\mathbf{z} \in X_m^2(X,\nu)$ with $\|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$ such that (3.8) holds and

$$\frac{1}{2} \int_{X\times X} \nabla u(x,y) \mathbf{z}(x,y) d(\nu \otimes m_x)(x,y) = \mathcal{F}_m(u);$$

(iv) there exists $\mathbf{g} \in L^\infty(X\times X, \nu\otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$ such that

$$v(x) = -\int_X \mathbf{g}(x,y) dm_x(y) \quad \textit{for } \nu\textit{-a.e } x \in X, \tag{3.9}$$

and

$$-\int_X \int_X \mathbf{g}(x,y) dm_x(y)\, u(x) d\nu(x) = \mathcal{F}_m(u). \tag{3.10}$$

(v) there exists $\mathbf{g} \in L^\infty(X\times X, \nu\otimes m_x)$ antisymmetric, satisfying (3.9) and

$$\mathbf{g}(x,y) \in \text{sign}(u(y) - u(x)) \quad \textit{for } (\nu\otimes m_x)\textit{-a.e. } (x,y) \in X \times X. \tag{3.11}$$

PROOF. Since $\mathcal{F}_m$ is convex, lower semi-continuous and positive homogeneous of degree 1, by [**12**, Theorem 1.8], we have

$$\partial\mathcal{F}_m(u) = \left\{ v \in L^2(X,\nu) \ : \ \widetilde{\mathcal{F}_m}(v) \leqslant 1, \ \int_X u(x)v(x)d\nu(x) = \mathcal{F}_m(u) \right\}. \tag{3.12}$$

We define, for $v \in L^2(X,\nu)$,

$$\Psi(v) := \inf\left\{ \|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \ : \ \mathbf{z} \in X_m^2(X,\nu), \ v = -\mathrm{div}_m \mathbf{z} \right\}. \tag{3.13}$$

Observe that Ψ is convex, lower semi-continuous and positive homogeneous of degree 1. Moreover, it is easy to see that, if $\Psi(v) < \infty$, the infimum in (3.13) is attained i.e., there exists some $\mathbf{z} \in X_m^2(X,\nu)$ such that $v = -\mathrm{div}_m\mathbf{z}$ and $\Psi(v) = \|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)}$.

Let us see that

$$\Psi = \widetilde{\mathcal{F}_m}.$$

We begin by proving that $\widetilde{\mathcal{F}_m}(v) \leqslant \Psi(v)$. If $\Psi(v) = +\infty$ then this assertion is trivial. Therefore, suppose that $\Psi(v) < +\infty$. Let $\mathbf{z} \in L^\infty(X\times X,\nu\otimes m_x)$ such that $v = -\mathrm{div}_m\mathbf{z}$. Then, for $w \in L^2(X,\nu)$, we have

$$\int_X w(x)v(x)d\nu(x) = \frac{1}{2}\int_{X\times X} \nabla w(x,y)\mathbf{z}(x,y)d(\nu\otimes m_x)(x,y) \leqslant \|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)}\mathcal{F}_m(w).$$

Taking the supremum over w we obtain that $\widetilde{\mathcal{F}_m}(v) \leqslant \|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)}$. Now, taking the infimum over $\mathbf{z}$, we get $\widetilde{\mathcal{F}_m}(v) \leqslant \Psi(v)$.

To prove the opposite inequality let us denote

$$D := \{\mathrm{div}_m\mathbf{z} \ : \ \mathbf{z} \in X_m^2(X,\nu)\}.$$

Then, by (3.6), we have that, for $v \in L^2(X,\nu)$,

$$\begin{aligned} \widetilde{\Psi}(v) &= \sup_{w\in L^2(X,\nu)} \frac{\displaystyle\int_X w(x)v(x)d\nu(x)}{\Psi(w)} \geqslant \sup_{w\in D} \frac{\displaystyle\int_X w(x)v(x)d\nu(x)}{\Psi(w)} \\ &= \sup_{\mathbf{z}\in X_m^2(X,\nu)} \frac{\displaystyle\int_X \mathrm{div}_m\mathbf{z}(x)v(x)d\nu(x)}{\|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)}} = \mathcal{F}_m(v). \end{aligned}$$

Thus, $\mathcal{F}_m \leqslant \widetilde{\Psi}$, which implies, by [**12**, Proposition 1.6], that $\Psi = \widetilde{\widetilde{\Psi}} \leqslant \widetilde{\mathcal{F}_m}$. Therefore, $\Psi = \widetilde{\mathcal{F}_m}$, and, consequently, from (3.12), we get

$$\begin{aligned} \partial\mathcal{F}_m(u) &= \left\{ v \in L^2(X,\nu) \ : \ \Psi(v) \leqslant 1, \ \int_X u(x)v(x)d\nu(x) = \mathcal{F}_m(u) \right\} \\ &= \left\{ v \in L^2(X,\nu) \ : \ \exists \mathbf{z} \in X_m^2(X,\nu), \ v = -\mathrm{div}_m\mathbf{z}, \ \|\mathbf{z}\|_\infty \leqslant 1, \ \int_X u(x)v(x)d\nu(x) = \mathcal{F}_m(u) \right\}. \end{aligned}$$

Hence, the equivalence between (i) and (ii) follows.

To get the equivalence between (ii) and (iii) we only need to apply Proposition 3.8.

On the other hand, to see that (iii) implies (iv), it is enough to take $\mathbf{g}(x,y) = \frac{1}{2}(\mathbf{z}(x,y) - \mathbf{z}(y,x))$. To see that (iv) implies (ii), it is enough to take $\mathbf{z}(x,y) = \mathbf{g}(x,y)$ (observe that, from (3.9), $-\mathrm{div}_m(\mathbf{g}) = v$, so $\mathbf{g} \in X_m^2(X,\nu)$). Finally, to see that (iv) and (v) are equivalent, we need to show that (3.10) and (3.11) are equivalent. Now, since $\mathbf{g}$ is antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$ and ν is reversible with respect to m, we have

$$-2\int_X\int_X \mathbf{g}(x,y)dm_x(y)\,u(x)d\nu(x) = \int_{X\times X} \mathbf{g}(x,y)(u(y)-u(x))d(\nu\otimes m_x)(x,y),$$

thus the equivalence between (3.10) and (3.11) follows. □

REMARK 3.14. The next space, in its local version, was introduced in [**127**]. Set

$$G_m(X,\nu) := \{f \in L^2(X,\nu) \,:\, \exists \mathbf{z} \in X_m^2(X,\nu) \text{ such that } f = \operatorname{div}_m(\mathbf{z})\}$$

and consider in $G_m(X,\nu)$ the norm

$$\|f\|_{m,*} := \inf\{\|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \,:\, f = \operatorname{div}_m(\mathbf{z})\}.$$

Following the proof of Theorem 3.13 we obtain that

$$\|f\|_{m,*} = \sup\left\{\left|\int_X f(x)u(x)d\nu(x)\right| \,:\, u \in L^2(X,\nu),\ TV_m(u) \leqslant 1\right\}, \tag{3.14}$$

and

$$\partial\mathcal{F}_m(u) = \left\{v \in G_m(X,\nu) \,:\, \|v\|_{m,*} \leqslant 1,\ \ \int_X u(x)v(x)d\nu(x) = \mathcal{F}_m(u)\right\}. \tag{3.15}$$

In particular,

$$\partial\mathcal{F}_m(0) = \{v \in G_m(X,\nu) \,:\, \|v\|_{m,*} \leqslant 1\}.$$

By Theorem 3.13 and Corollary A.36 we obtain the following result.

PROPOSITION 3.15. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. Then, $\partial\mathcal{F}_m$ is an* m-*completely accretive operator in $L^2(X,\nu)$ (see Appendix A).*

PROOF. Suppose that $v_i \in \partial\mathcal{F}_m(u_i)$, $i = 1,2$. Then, by Theorem 3.13, for $i \in \{1,2\}$, there exists $\mathbf{g}_i \in L^\infty(X\times X,\nu\otimes m_x)$ antisymmetric, satisfying

$$v_i(x) = -\int_X \mathbf{g}_i(x,y)dm_x(y) \quad \text{for } \nu\text{-a.e } x \in X,$$

and

$$\mathbf{g}_i(x,y) \in \operatorname{sign}(u_i(y) - u_i(x)) \quad \text{for } (\nu\otimes m_x)\text{-a.e. } (x,y) \in X\times X.$$

Now, recall that

$$P_0 = \{q \in C^\infty(\mathbb{R}) \,:\, 0 \leqslant q' \leqslant 1,\ \operatorname{supp}(q') \text{ is compact and } 0 \notin \operatorname{supp}(q)\}$$

and let $q \in P_0$. Then, using that $\mathbf{g}$ is antisymmetric and the reversibility of ν with respect to m in the second equality, we get

$$\begin{aligned}
&\int_X (v_1(x) - v_2(x))q(u_1(x) - u_2(x))d\nu(x)\\
&= \int_X\int_X (\mathbf{g}_2(x,y) - \mathbf{g}_1(x,y))q(u_1(x) - u_2(x))dm_x(y)d\nu(x)\\
&= \frac{1}{2}\int_X\int_X (\mathbf{g}_2(x,y) - \mathbf{g}_1(x,y))(q(u_1(x) - u_2(x)) - q(u_1(y) - u_2(y)))dm_x(y)d\nu(x)\\
&= \frac{1}{2}\int_{\{(x,y)\,:\,u_1(x)\neq u_1(y),u_2(x)=u_2(y)\}} (\mathbf{g}_2(x,y) - \mathbf{g}_1(x,y))\\
&\qquad\qquad \times(q(u_1(x) - u_2(x)) - q(u_1(y) - u_2(y)))dm_x(y)d\nu(x)\\
&+\frac{1}{2}\int_{\{(x,y)\,:\,u_1(x)=u_1(y),u_2(x)\neq u_2(y)\}} (\mathbf{g}_2(x,y) - \mathbf{g}_1(x,y))\\
&\qquad\qquad \times(q(u_1(x) - u_2(x)) - q(u_1(y) - u_2(y)))dm_x(y)d\nu(x)\\
&+\frac{1}{2}\int_{\{(x,y)\,:\,u_1(x)\neq u_1(y),u_2(x)\neq u_2(y)\}} (\mathbf{g}_2(x,y) - \mathbf{g}_1(x,y))\\
&\qquad\qquad \times(q(u_1(x) - u_2(x)) - q(u_1(y) - u_2(y)))dm_x(y)d\nu(x).
\end{aligned}$$

Consequently, since the last three integrals are nonnegative, we get that

$$\int_X (v_1(x) - v_2(x))q(u_1(x) - u_2(x))d\nu(x) \geqslant 0$$

and it follows from Corollary A.36 that $\partial\mathcal{F}_m$ is a completely accretive operator in $L^2(X,\nu)$. Moreover, since $\mathcal{F}_m$ is convex and lower semi-continuous in the Hilbert space $L^2(X,\nu)$, we have that $\partial\mathcal{F}_m$ is maximal monotone and, therefore, it is actually m-completely accretive in $L^2(X,\nu)$. □

Definition 3.16. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. We define in $L^2(X,\nu)$ the multivalued operator Δ_1^m by

$$(u,v) \in \Delta_1^m \ \text{ if, and only if, } \ -v \in \partial\mathcal{F}_m(u).$$

As usual, we will write $v \in \Delta_1^m u$ for $(u,v) \in \Delta_1^m$.

Chang in [**57**] *and Hein and Bühler in* [**99**] *define a similar operator in the particular case of finite graphs:*

Example 3.17. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk given in Example 1.38. By Theorem 3.13, we have

$$(u,v) \in \Delta_1^{m^G} \Leftrightarrow \begin{cases} \exists \mathbf{g} \in L^\infty(V(G)\times V(G), \nu_G \otimes m_x^G) \ \text{ antisymmetric such that} \\ \|\mathbf{g}\|_{L^\infty(V(G)\times V(G),\nu_G\otimes m_x^G)} \leqslant 1, \\ \displaystyle\frac{1}{d_x}\sum_{y\in V(G)} \mathbf{g}(x,y)w_{xy} = v(x) \quad \forall\, x \in V(G), \text{ and} \\ \mathbf{g}(x,y) \in \operatorname{sign}(u(y)-u(x)) \quad \text{for } (\nu_G\otimes m_x^G)\text{-a.e. } (x,y) \in V(G)\times V(G). \end{cases}$$

The next example shows that the operator $\Delta_1^{m^G}$ is indeed multivalued. Let $V(G) = \{a,b\}$, $0 < p < 1$, $w_{aa} = w_{bb} = p$ and $w_{ab} = w_{ba} = 1-p$. Then,

$$(u,v) \in \Delta_1^{m^G} \Leftrightarrow \begin{cases} \text{there exists } \mathbf{g} \in L^\infty(\{a,b\}\times\{a,b\}, \nu_G\otimes m_x^G) \ \text{ antisymmetric such that} \\ \|\mathbf{g}\|_{L^\infty(\{a,b\}\times\{a,b\},\nu_G\otimes m_x^G)} \leqslant 1, \\ \mathbf{g}(a,a)p + \mathbf{g}(a,b)(1-p) = v(a), \quad \mathbf{g}(b,b)p + \mathbf{g}(b,a)(1-p) = v(b), \text{ and} \\ \mathbf{g}(a,b) \in \operatorname{sign}(u(b)-u(a)). \end{cases}$$

Now, since $\mathbf{g}$ is antisymmetric, we get

$$v(a) = \mathbf{g}(a,b)(1-p), \quad v(b) = -\mathbf{g}(a,b)(1-p) \quad \text{and} \quad \mathbf{g}(a,b) \in \operatorname{sign}(u(b)-u(a)).$$

Proposition 3.18 (Integration by parts). *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. For any $(u,v) \in \Delta_1^m$ it holds that*

$$-\int_X vw\,d\nu \leqslant TV_m(w) \qquad \text{for all } \ w \in BV_m(X,\nu)\cap L^2(X,\nu), \tag{3.16}$$

and

$$-\int_X vu\,d\nu = TV_m(u). \tag{3.17}$$

Proof. Since $-v \in \partial\mathcal{F}_m(u)$, given $w \in BV_m(X,\nu)$, we have that

$$-\int_X vw\,d\nu \leqslant \mathcal{F}_m(u+w) - \mathcal{F}_m(u) \leqslant \mathcal{F}_m(w),$$

so we get (3.16). On the other hand, (3.17) is given in Theorem 3.13. □

*As a consequence of Theorem 3.13 and Proposition 3.15, on account of Theorem A.31 ([**43**, Theoreme 3.6]) and by the complete accretivity of the operator (see the results in Appendix A.7), we can give the following existence and uniqueness result for the Cauchy problem*

$$\begin{cases} u_t - \Delta_1^m u \ni 0 & \text{in } (0,T) \times X \\ u(0,x) = u_0(x) & x \in X, \end{cases} \tag{3.18}$$

which is a rewrite of the formal expression (3.7)*.*

Theorem 3.19. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. For every $u_0 \in L^2(X,\nu)$ and any $T > 0$, there exists a unique solution of the Cauchy problem* (3.18) *in $(0,T)$ in the following sense: $u \in C([0,T]; L^2(X,\nu)) \cap W^{1,1}(0,T; L^2(X,\nu))$ (see section A.2 of Appendix A), $u(0,\cdot) = u_0$ in $L^2(X,\nu)$, and, for almost all $t \in (0,T)$,*

$$u_t(t,\cdot) - \Delta_1^m u(t) \ni 0.$$

Moreover, we have the following contraction and maximum principle in any $L^q(X,\nu)$–space, $1 \leqslant q \leqslant \infty$:

$$\|(u(t) - v(t))^+\|_{L^q(X,\nu)} \leqslant \|(u_0 - v_0)^+\|_{L^q(X,\nu)} \quad \forall\, 0 < t < T,$$

for any pair of solutions u and v of problem (3.18) *with initial datum u_0 and v_0, respectively.*

Definition 3.20. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Given $u_0 \in L^2(X,\nu)$, we denote by $e^{t\Delta_1^m} u_0$ the unique solution of problem (3.18). We call the semigroup $\{e^{t\Delta_1^m}\}_{t \geqslant 0}$ in $L^2(X,\nu)$ the *total variation flow* in $[X, \mathcal{B}, m, \nu]$.

In the next result we give an important property of the total variation flow in random walk spaces.

Proposition 3.21. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. The total variation flow satisfies the mass conservation property: for $u_0 \in L^1(X,\nu) \cap L^2(X,\nu)$,*

$$\int_X e^{t\Delta_1^m} u_0 d\nu = \int_X u_0 d\nu \quad \text{for every } t \geqslant 0.$$

Proof. By Proposition 3.18, we have

$$-\frac{d}{dt} \int_X e^{t\Delta_1^m} u_0 d\nu \leqslant TV_m(1) = 0,$$

and

$$\frac{d}{dt} \int_X e^{t\Delta_1^m} u_0 d\nu \leqslant TV_m(-1) = 0.$$

Hence,

$$\frac{d}{dt} \int_X e^{t\Delta_1^m} u_0 d\nu = 0,$$

and, consequently,

$$\int_X e^{t\Delta_1^m} u_0 d\nu = \int_X u_0 d\nu \quad \text{for any } t \geqslant 0.$$

□

3.3. Asymptotic behaviour

Proposition 3.22. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. For every $u_0 \in L^1(X,\nu) \cap L^2(X,\nu)$, there exists*

$$u_\infty \in \{u \in L^1(X,\nu) \cap L^2(X,\nu) \,:\, 0 \in \Delta_1^m(u)\}$$

such that

$$\lim_{t \to \infty} e^{t\Delta_1^m} u_0 = u_\infty \quad \text{in } L^2(X,\nu).$$

Suppose further that $[X, \mathcal{B}, m, \nu]$ is m-connected, then:

(i) if $\nu(X) = \infty$, $u_\infty = 0$ ν-a.e.

(ii) if ν is a probability measure,

$$u_\infty = \int_X u_0(x) d\nu(x) \quad \nu\text{-a.e.}$$

Proof. Since $\mathcal{F}_m$ is a proper and lower semicontinuous function in X attaining the minimum at the zero function and, moreover, $\mathcal{F}_m$ is even, by [**46**, Theorem 5], we have that the strong limit in $L^2(X,\nu)$ of $e^{t\Delta_1^m}u_0$ exists and is a minimum point of $\mathcal{F}_m$, i.e.,

$$u_\infty \in \{u \in L^1(X,\nu) \cap L^2(X,\nu) \, : \, 0 \in \Delta_1^m(u)\}.$$

Suppose now that $[X,\mathcal{B},m,\nu]$ is m-connected. Then, since $0 \in \Delta_1^m(u_\infty)$, we have that $TV_m(u_\infty) = 0$ thus, by Lemma 3.6, we get that u_∞ is a constant ν-a.e.. Therefore, if $\nu(X) = \infty$, we must have $u_\infty = 0$ ν-a.e. and, if ν is a probability measure, Proposition 3.21 yields

$$u_\infty = \int_X u_0(x) d\nu(x). \qquad \square$$

Let us see that we can specify a rate of convergence of the total variation flow $(e^{t\Delta_1^m})_{t\geqslant 0}$ when a Poincaré type inequality holds.

Definition 3.23. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and $1 \leqslant p, q < +\infty$. Suppose that $[X,\mathcal{B},m,\nu]$ satisfies a (p,q)-Poincaré inequality (recall Definition 1.67). We will denote

$$\lambda_{[X,\mathcal{B},m,\nu]}^{(p,q)} := \inf\left\{ \frac{\|\nabla u\|_{L^q(X\times X,\nu\otimes m_x)}}{\|u\|_{L^p(X,\nu)}} \, : \, \|u\|_{L^p(X,\nu)} \neq 0, \ \int_X u(x)d\nu(x) = 0 \right\}.$$

Moreover, when $[X,\mathcal{B},m,\nu]$ satisfies a p-Poincaré inequality we will write

$$\lambda_{[X,\mathcal{B},m,\nu]}^{p} := \lambda_{[X,\mathcal{B},m,\nu]}^{(p,1)} = \inf\left\{ \frac{TV_m(u)}{\|u\|_{L^p(X,\nu)}} \, : \, \|u\|_{L^p(X,\nu)} \neq 0, \ \int_X u(x)d\nu(x) = 0 \right\}.$$

The following result was proved in [**18**, Theorem 7.11] *for the particular case of the metric random walk space $[\Omega, d, m^{J,\Omega}, \mathcal{L}^N \llcorner \Omega]$ given in Example 1.37, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. We generalise the result to any reversible random walk space. Let us first recall the definition of a Liapunov functional for a continuous semigroup.*

Definition 3.24. Let X be a metric space and S a continuous semigroup on X. A Liapunov functional for T (see [**70**]) is a map $V : X \to \mathbb{R}$ such that $V(S(t)u) \leqslant V(u)$ for any $u \in X$, $t \geqslant 0$.

Theorem 3.25. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and assume that ν is a probability measure. If $[X,\mathcal{B},m,\nu]$ satisfies a 1-Poincaré inequality, then, for any $u_0 \in L^2(X,\nu)$,*

$$\left\| e^{t\Delta_1^m}u_0 - \nu(u_0) \right\|_{L^1(X,\nu)} \leqslant \frac{1}{2\lambda_{[X,\mathcal{B},m,\nu]}^{1}} \frac{\|u_0\|_{L^2(X,\nu)}^2}{t} \quad \textit{for all } t > 0.$$

Proof. Let $u_0 \in L^2(X,\nu)$. The complete accretivity of the operator $-\Delta_1^m$ (recall Definition A.34) implies that

$$\mathcal{L}(u) := \|u - \nu(u_0)\|_{L^1(X,\nu)}, \quad u \in L^2(X,\nu),$$

is a Liapunov functional for the semigroup $\{e^{t\Delta_1^m} : t \geqslant 0\}$. Indeed, by Proposition A.43, $e^{t\Delta_1^m}$ is a complete contraction (see Definition A.32) for $t \geqslant 0$ thus, taking the normal functional $N : L^1(X,\nu) \to (-\infty, +\infty]$ defined by $N(u) := \|u - \nu(u_0)\|_{L^1(X,\nu)}$, since $e^{t\Delta_1^m}0 = 0$, $t \geqslant 0$, we get that

$$\left\| e^{t\Delta_1^m}u - \nu(u_0) \right\|_{L^1(X,\nu)} \leqslant \|u - \nu(u_0)\|_{L^1(X,\nu)}, \quad u \in L^2(X,\nu), \ t \geqslant 0.$$

In particular, if $v(t) := e^{t\Delta_1^m}u_0 - \nu(u_0)$,

$$\|v(t)\|_{L^1(X,\nu)} \leqslant \|v(s)\|_{L^1(X,\nu)} \quad \text{for } t \geqslant s. \tag{3.19}$$

Now, by Proposition 3.21, we have that $\nu(u(t)) = \nu(u_0)$ for all $t \geqslant 0$, so the 1-Poincaré inequality yields[3]

$$\lambda^1_{[X,\mathcal{B},m,\nu]}\|v(s)\|_{L^1(X,\nu)} \leqslant TV_m(v(s)), \quad s \geqslant 0. \tag{3.20}$$

Therefore, by (3.19) and (3.20),

$$t\|v(t)\|_{L^1(X,\nu)} \leqslant \int_0^t \|v(s)\|_{L^1(X,\nu)}ds \leqslant \frac{1}{\lambda^1_{[X,\mathcal{B},m,\nu]}} \int_0^t TV_m(v(s))ds, \quad t \geqslant 0. \tag{3.21}$$

On the other hand, by Proposition 3.18,

$$-\frac{1}{2}\frac{d}{dt}\|e^{t\Delta_1^m}u_0\|^2_{L^2(X,\nu)} = -\int_X e^{t\Delta_1^m}u_0 \frac{d}{dt}e^{t\Delta_1^m}u_0 d\nu = TV_m(e^{t\Delta_1^m}u_0),$$

and then,

$$\frac{1}{2}\|e^{t\Delta_1^m}u_0\|^2_{L^2(X,\nu)} - \frac{1}{2}\|u_0\|^2_{L^2(X,\nu)} = -\int_0^t TV_m(e^{s\Delta_1^m}u_0)ds = -\int_0^t TV_m(v(s))ds,$$

which implies

$$\int_0^t TV_m(v(s))ds \leqslant \frac{1}{2}\|u_0\|^2_{L^2(X,\nu)}, \quad t \geqslant 0.$$

Hence, by (3.21),

$$\|v(t)\|_{L^1(X,\nu)} \leqslant \frac{1}{2\lambda^1_{[X,\mathcal{B},m,\nu]}} \frac{\|u_0\|^2_{L^2(X,\nu)}}{t}, \quad t > 0,$$

which concludes the proof. □

REMARK 3.26. If $\nu(X) = \infty$ and $u_0 \in L^1(X,\nu) \cap L^2(X,\nu)$, then, if $\int_X u_0 d\nu = 0$, we may proceed similarly (substituting $\nu(u_0)$ by 0) to obtain that

$$\left\|e^{t\Delta_1^m}u_0\right\|_{L^1(X,\nu)} \leqslant \frac{1}{2\lambda^1_{[X,\mathcal{B},m,\nu]}} \frac{\|u_0\|^2_{L^2(X,\nu)}}{t} \quad \text{for all } t > 0.$$

On account of Theorem 3.25, we obtain the following result on the asymptotic behaviour of the total variation flow.

COROLLARY 3.27. *Under the hypothesis of Theorem 1.84 with $p = 1$, $A = X$, $B = \emptyset$ and assuming that ν is a probability measure; if $u_0 \in L^2(X,\nu)$, then*

$$\left\|e^{t\Delta_1^m}u_0 - \nu(u_0)\right\|_{L^1(X,\nu)} \leqslant \frac{1}{2\lambda^1_{[X,\mathcal{B},m,\nu]}} \frac{\|u_0\|^2_{L^2(X,\nu)}}{t} \quad \textit{for all } t > 0.$$

Let us see that, when $[X,\mathcal{B},m,\nu]$ satisfies a 2-Poincaré inequality, the solution of the total variation flow reaches the steady state in finite time.

THEOREM 3.28. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and assume that ν is a probability measure. Suppose that $[X,d,m,\nu]$ satisfies a 2-Poincaré inequality. Then, for any $u_0 \in L^2(X,\nu)$,*

$$\|e^{t\Delta_1^m}u_0 - \nu(u_0)\|_{L^2(X,\nu)} \leqslant \left(\|u_0 - \nu(u_0)\|_{L^2(X,\nu)} - \lambda^2_{[X,\mathcal{B},m,\nu]}t\right)^+ \quad \textit{for all } t \geqslant 0.$$

Consequently,

$$e^{t\Delta_1^m}u_0 = \nu(u_0) \qquad \forall\, t \geqslant \hat{t} := \frac{\|u_0-\nu(u_0)\|_{L^2(X,\nu)}}{\lambda^2_{[X,\mathcal{B},m,\nu]}}.$$

[3] This inequality is, of course, true if $\|v(s)\|_{L^1(X,\nu)} = 0$.

Proof. Let $u_0 \in L^2(X,\nu)$, $u(t) := e^{t\Delta_1^m}u_0$ and $v(t) := u(t) - \nu(u_0)$. Since $\Delta_1^m u(t) = \Delta_1^m v(t)$, we have that

$$\frac{d}{dt}v(t) \in \Delta_1^m v(t), \quad t > 0.$$

Note that $v(t) \in BV_m(X,\nu)$ for every $t > 0$. Indeed, since $-\Delta_m^1 = \partial\mathcal{F}_m$ is a maximal monotone operator in $L^2(X,\nu)$, then, by [**43**, Theorem 3.7] with $H = L^2(X,\nu)$, we have that $v(t) \in D(\Delta_m^1) \subset BV_m(X,\nu)$ for every $t > 0$.

Consequently, by Theorem 3.13, there exists $\mathbf{g}_t \in L^\infty(X\times X, \nu\otimes m_x)$ antisymmetric with $\|\mathbf{g}_t\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$ such that

$$\int_X \mathbf{g}_t(x,y)\,dm_x(y) = \frac{d}{dt}v(t)(x) \quad \text{for } \nu\text{-a.e } x\in X \text{ and every } t>0, \tag{3.22}$$

and

$$-\int_X\int_X \mathbf{g}_t(x,y)dm_x(y)\,v(t)(x)d\nu(x) = \mathcal{F}_m(v(t)) = TV_m(u(t)) \quad \text{for every } t>0. \tag{3.23}$$

Then, multiplying (3.22) by $v(t)$, integrating over X with respect to ν and having in mind (3.23), we get

$$\frac{1}{2}\frac{d}{dt}\int_X v(t)^2 d\nu + TV_m(v(t)) = 0.$$

Now, by Proposition 3.21, we have that $\nu(u(t)) = \nu(u_0)$ for all $t \geqslant 0$, and, since $[X,\mathcal{B},m,\nu]$ satisfies a 2-Poincaré inequality, we have

$$\lambda^2_{[X,\mathcal{B},m,\nu]}\|v(t)\|_{L^2(X,\nu)} \leqslant TV_m(v(t)) \quad \text{for all } t\geqslant 0.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(X,\nu)} + \lambda^2_{[X,\mathcal{B},m,\nu]}\|v(t)\|_{L^2(X,\nu)} \leqslant 0 \quad \text{for all } t\geqslant 0.$$

Now, integrating this ordinary differential inequation we get

$$\|v(t)\|_{L^2(X,\nu)} \leqslant \left(\|v(0)\|_{L^2(X,\nu)} - \lambda^2_{[X,\mathcal{B},m,\nu]}t\right)^+ \quad \text{for all } t\geqslant 0,$$

that is,

$$\|u(t)-\nu(u_0)\|_{L^2(X,\nu)} \leqslant \left(\|u_0-\nu(u_0)\|_{L^2(X,\nu)} - \lambda^2_{[X,\mathcal{B},m,\nu]}t\right)^+ \quad \text{for all } t\geqslant 0. \qquad \square$$

Remark 3.29. As before, if $\nu(X) = \infty$ and $u_0 \in L^1(X,\nu)\cap L^2(X,\nu)$ with $\int_X u_0 d\nu = 0$, we obtain that

$$\|e^{t\Delta_1^m}u_0\|_{L^2(X,\nu)} \leqslant \left(\|u_0\|_{L^2(X,\nu)} - \lambda^2_{[X,\mathcal{B},m,\nu]}t\right)^+ \quad \text{for all } t\geqslant 0.$$

Definition 3.30. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and suppose that ν is a probability measure. We define the *extinction time* as

$$T^*(u_0) := \inf\{t>0 \ : \ e^{t\Delta_1^m}u_0 = \nu(u_0)\}, \ u_0\in L^2(X,\nu).$$

Under the conditions of Theorem 3.28, we have that

$$T^*(u_0) \leqslant \frac{\|u_0-\nu(u_0)\|_{L^2(X,\nu)}}{\lambda^2_{[X,\mathcal{B},m,\nu]}}.$$

To obtain a lower bound on the extinction time we will use the norm $\|\cdot\|_{m,*}$ introduced in Remark 3.14.

Theorem 3.31. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and suppose that ν is a probability measure. Then,*

$$T^*(u_0) \geqslant \|u_0-\nu(u_0)\|_{m,*} \quad \forall u_0\in L^2(X,\nu).$$

Proof. We may assume that $T^*(u_0) < \infty$. Let $u_0 \in L^2(X,\nu)$. If $u(t) := e^{t\Delta_1^m} u_0$, we have

$$u_0 - \nu(u_0) = -\int_0^{T^*(u_0)} u'(t)dt.$$

Then, by integration by parts (Proposition 3.18), we get

$$\begin{aligned} \|u_0 - \nu(u_0)\|_{m,*} &= \sup\left\{ \int_X w(u_0 - \nu(u_0))d\nu \ : \ TV_m(w) \leqslant 1 \right\} \\ &= \sup\left\{ \int_X w\left(\int_0^{T^*(u_0)} -u'(t)dt \right) d\nu \ : \ TV_m(w) \leqslant 1 \right\} \\ &= \sup\left\{ \int_0^{T^*(u_0)} \int_X -wu'(t)d\nu dt \ : \ TV_m(w) \leqslant 1 \right\} \\ &\leqslant \sup\left\{ \int_0^{T^*(u_0)} TV_m(w)dt \ : \ TV_m(w) \leqslant 1 \right\} = T^*(u_0). \end{aligned}$$ □

As a consequence of Example 1.74 and Theorem 3.28, we get the following result.

Theorem 3.32. *Let $G = (V(G), E(G))$ be a finite weighted connected discrete graph and let $[V(G), d_G, m^G, \nu_G]$ be the associated metric random walk space (recall Example 1.38). Then,*

$$\|e^{t\Delta_1^{m^G}} u_0 - \nu(u_0)\|_{L^2(V(G),\nu_G)} \leqslant \lambda^2_{[V(G),d_G,m^G,\nu_G]} \left(\hat{t} - t\right)^+,$$

where $\hat{t} := \frac{\|u_0-\nu(u_0)\|_{L^2(V(G),\nu_G)}}{\lambda^2_{[V(G),d_G,m^G,\nu_G]}}$. *Consequently,*

$$e^{t\Delta_1^{m^G}} u_0 = \nu(u_0) \quad \textit{for all} \ \ t \geqslant \hat{t}.$$

3.4. m-Cheeger and m-Calibrable sets

Definition 3.33. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. Given $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$, we define $h_1^m(\Omega)$[4] (the *m-Cheeger constant* of Ω) as

$$h_1^m(\Omega) := \inf\left\{ \frac{P_m(E)}{\nu(E)} \ : \ E \in \mathcal{B}_\Omega, \ \nu(E) > 0 \right\}. \tag{3.24}$$

If a set $E \in \mathcal{B}_\Omega$ minimizes (3.24), then E is said to be an *m-Cheeger set* of Ω. Furthermore, we say that Ω is *m-calibrable* if it is an m-Cheeger set of itself, that is, if

$$h_1^m(\Omega) = \frac{P_m(\Omega)}{\nu(\Omega)}.$$

Note that, by (1.6)*, we have that* $h_1^m(\Omega) \leqslant 1$.

Definition 3.34. Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. For ease of notation, we will denote

$$\lambda_\Omega^m := \frac{P_m(\Omega)}{\nu(\Omega)}.$$

Remark 3.35. (1) Let $[\mathbb{R}^N, \mathcal{B}, m^J, \mathcal{L}^N]$ be the random walk space given in Example 1.37. Then, the concepts of m-Cheeger set and m-calibrable set coincide with the concepts of J-Cheeger set and J-calibrable set introduced in [**120**] (see also [**121**]).

[4] The notation $h_1^m(\Omega)$ is chosen together with the one that we will use for the classical Cheeger constant $h_1(\Omega)$ (see (3.25)). In both of these, the subscript 1 is there to further distinguish them from the upcoming notation $h_m(X)$ for the m-Cheeger constant of X (see (3.45)).

(2) Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with a locally finite weighted discrete graph $G = (V(G), E(G))$ having more than two vertices and no loops (i.e., $w_{xx} = 0$ for all $x \in V$). Then, any subset consisting of two vertices is m^G-calibrable. Indeed, let $\Omega = \{x, y\}$, then, by (1.6), we have

$$\frac{P_{m^G}(\{x\})}{\nu_G(\{x\})} = 1 - \int_{\{x\}} \int_{\{x\}} dm_x^G(z) d\nu_G(z) = 1 \geqslant \frac{P_{m^G}(\Omega)}{\nu_G(\Omega)},$$

and, similarly,

$$\frac{P_{m^G}(\{y\})}{\nu_G(\{y\})} = 1 \geqslant \frac{P_{m^G}(\Omega)}{\nu_G(\Omega)}.$$

Therefore, Ω is m^G-calibrable.

*In [**120**] it is proved that, for the metric random walk space $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ given in Example 1.37, each ball is a J-calibrable set. In the next example we will see that this result is not true in general.*

Example 3.36. Let $G = (V(G), E(G))$ be the finite weighted discrete graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_7\}$ and the following weights: $w_{x_1,x_2} = 2$, $w_{x_2,x_3} = 1$, $w_{x_3,x_4} = 2$, $w_{x_4,x_5} = 2$, $w_{x_5,x_6} = 1$, $w_{x_6,x_7} = 2$ and $w_{x_i,x_j} = 0$ otherwise. Let $[V(G), d_G, m^G, \nu_G]$ be the associated metric random walk space. Then, if $E_1 = B(x_4, \frac{5}{2}) = \{x_2, x_3, \ldots, x_6\}$,

$$\frac{P_{m^G}(E_1)}{\nu_G(E_1)} = \frac{w_{x_1x_2} + w_{x_6x_7}}{d_{x_2} + d_{x_3} + d_{x_4} + d_{x_5} + d_{x_6}} = \frac{1}{4}.$$

However, taking $E_2 = B(x_4, \frac{3}{2}) = \{x_3, x_4, x_5\} \subset E_1$, we have

$$\frac{P_{m^G}(E_2)}{\nu_G(E_2)} = \frac{w_{x_2x_3} + w_{x_5x_6}}{d_{x_3} + d_{x_4} + d_{x_5}} = \frac{1}{5}.$$

Consequently, the ball $B(x_4, \frac{5}{2})$ is not m^G-calibrable.

In the next Example we will see that there exist random walk spaces with sets that do not contain m-Cheeger sets.

Example 3.37. Let $G = (V(G), E(G))$ be the finite weighted discrete graph defined in Example 3.80 (1), i.e., $V(G) = \{x_0, x_1, \ldots, x_n \ldots\}$ and weights:

$$w_{x_{2n}x_{2n+1}} = \frac{1}{2^n}, \quad w_{x_{2n+1}x_{2n+2}} = \frac{1}{3^n} \quad \text{for } n = 0, 1, 2, \ldots,$$

and $w_{x_i,x_j} = 0$ otherwise. If $\Omega := \{x_1, x_2, x_3 \ldots\}$, then $\frac{P_{m^G}(D)}{\nu_G(D)} > 0$ for every $D \subset \Omega$ with $\nu_G(D) > 0$ but, working as in Example 3.80, we get $h_1^m(\Omega) = 0$. Therefore, Ω has no m-Cheeger set.

*It is well known (see [**85**]) that, for a bounded smooth domain $\Omega \subset \mathbb{R}^N$, the classical Cheeger constant*

$$h_1(\Omega) := \inf\left\{ \frac{Per(E)}{|E|} \ : \ E \subset \Omega, \ |E| > 0 \right\}, \tag{3.25}$$

is an optimal Poincaré constant, namely, it coincides with the first eigenvalue of the 1-Laplacian:

$$h_1(\Omega) = \Lambda_1(\Omega) := \inf\left\{ \frac{\displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}}{\|u\|_{L^1(\Omega)}} \ : \ u \in BV(\Omega), \ \|u\|_{L^\infty(\Omega)} = 1 \right\}.$$

In order to get a nonlocal version of this result, we introduce the following constant.

Definition 3.38. Let $[X,\mathcal{B},m,\nu]$ be a random walk space. For $\Omega \in \mathcal{B}$ with $0<\nu(\Omega)<\nu(X)$, we define

$$\Lambda_1^m(\Omega) := \inf\left\{TV_m(u) \ : \ u \in L^1(X,\nu), \ u=0 \text{ in } X\setminus\Omega, \ u \geqslant 0, \ \int_X u(x)d\nu(x)=1\right\}$$

$$= \inf\left\{\frac{TV_m(u)}{\displaystyle\int_X u(x)d\nu(x)} \ : \ u \in L^1(X,\nu)\setminus\{0\}, \ u=0 \text{ in } X\setminus\Omega, \ u\geqslant 0\right\}.$$

Theorem 3.39. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space. Let $\Omega \in \mathcal{B}$ with $0<\nu(\Omega)<\nu(X)$. Then,*

$$h_1^m(\Omega) = \Lambda_1^m(\Omega).$$

Proof. Given $E \in \mathcal{B}$ with $\nu(E)>0$, we have

$$\frac{TV_m(\chi_E)}{\|\chi_E\|_{L^1(X,\nu)}} = \frac{P_m(E)}{\nu(E)}.$$

Therefore, $\Lambda_1^m(\Omega) \leqslant h_1^m(\Omega)$. For the opposite inequality we will follow an idea used in [85]. Given $u \in L^1(X,\nu)\setminus\{0\}$, with $u=0$ in $X\setminus\Omega$ and $u \geqslant 0$, we have

$$\begin{aligned} TV_m(u) &= \int_0^{+\infty} P_m(E_t(u))\,dt = \int_0^{\|u\|_{L^\infty(X,\nu)}} \frac{P_m(E_t(u))}{\nu(E_t(u))}\nu(E_t(u))\,dt \\ &\geqslant h_1^m(\Omega)\int_0^{+\infty}\nu(E_t(u))\,dt = h_1^m(\Omega)\int_X u(x)d\nu(x) \end{aligned}$$

where the first equality follows by the coarea formula (Theorem 3.5) and the last one by Cavalieri's Principle. Taking the infimum over u in the above expression we get $\Lambda_1^m(\Omega) \geqslant h_1^m(\Omega)$. □

Let us recall that, in the local case, a set $\Omega \subset \mathbb{R}^N$ is called calibrable if

$$\frac{Per(\Omega)}{|\Omega|} = \inf\left\{\frac{Per(E)}{|E|} \ : \ E \subset \Omega, \ E \ \text{ with finite perimeter}, \ |E|>0\right\}.$$

The following characterization of convex calibrable sets is proved in [6].

Theorem 3.40. *([6]) Given a bounded convex set $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$, the following assertions are equivalent:*

(a) Ω is calibrable.

(b) χ_Ω satisfies $-\Delta_1\chi_\Omega = \frac{Per(\Omega)}{|\Omega|}\chi_\Omega$, where $\Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right)$.

(c) $(N-1)\operatorname{ess\,sup}_{x\in\partial\Omega}\mathcal{H}_{\partial\Omega}(x) \leqslant \dfrac{Per(\Omega)}{|\Omega|}$.

In the following results, we will see that the nonlocal counterparts of some of the implications in this theorem also hold true in our setting, while others do not.

Remark 3.41. Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space.

(1) Let $\Omega \in \mathcal{B}$ with $0<\nu(\Omega)<\nu(X)$ and assume that there exists a constant $\lambda>0$ and a measurable function $\tau : X \to \mathbb{R}$ such that $\tau(x)=1$ for $x \in \Omega$ and

$$-\lambda\tau \in \Delta_1^m\chi_\Omega \quad \text{in } X.$$

Let us see that $\lambda = \lambda_\Omega^m$. By Theorem 3.13, there exists $\mathbf{g} \in L^\infty(X\times X,\nu\otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$ satisfying

$$-\int_X \mathbf{g}(x,y)\,dm_x(y) = \lambda\tau(x) \quad \text{for } \nu\text{-a.e } x\in X$$

and

$$-\int_X \int_X \mathbf{g}(x,y) dm_x(y)\, \chi_\Omega(x) d\nu(x) = \mathcal{F}_m(\chi_\Omega) = P_m(\Omega).$$

Then,

$$\begin{aligned}\lambda\nu(\Omega) &= \int_X \lambda\tau(x)\chi_\Omega(x)d\nu(x) \\ &= -\int_X \left(\int_X \mathbf{g}(x,y)\, dm_x(y)\right) \chi_\Omega(x) d\nu(x) \\ &= P_m(\Omega)\end{aligned}$$

and, consequently,

$$\lambda = \frac{P_m(\Omega)}{\nu(\Omega)} = \lambda_\Omega^m.$$

(2) Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$, and $\tau : X \to \mathbb{R}$ a measurable function with $\tau(x) = 1$ for $x \in \Omega$. Then,

$$-\lambda_\Omega^m \tau \in \Delta_1^m \chi_\Omega \quad \text{in } X \;\Leftrightarrow\; -\lambda_\Omega^m \tau \in \Delta_1^m 0 \quad \text{in } X. \tag{3.26}$$

Indeed, the left to right implication follows from the fact that, for $u \in L^2(X,\nu)$,

$$\partial\mathcal{F}_m(u) \subset \partial\mathcal{F}_m(0),$$

and, for the converse implication, we have that there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$ and satisfying

$$-\lambda_\Omega^m \tau(x) = \int_X \mathbf{g}(x,y)\, dm_x(y) \quad \text{for } \nu\text{-a.e. } x \in X.$$

Now, multiplying this equation by χ_Ω and integrating over X with respect to ν, since ν is reversible with respect to m and $\mathbf{g}$ is antisymmetric, we get

$$\begin{aligned}P_m(\Omega) = \lambda_\Omega^m \nu(\Omega) &= \lambda_\Omega^m \int_X \tau(x)\chi_\Omega(x)d\nu(x) \\ &= -\int_X \int_X \mathbf{g}(x,y)\chi_\Omega(x)dm_x(y)d\nu(x) \\ &= \frac{1}{2}\int_X \int_X \mathbf{g}(x,y)(\chi_\Omega(y) - \chi_\Omega(x))dm_x(y)d\nu(x) \\ &\leqslant \frac{1}{2}\int_X \int_X |\chi_\Omega(y) - \chi_\Omega(x)|\, dm_x(y)d\nu(x) = P_m(\Omega).\end{aligned}$$

Therefore, the previous inequality is, in fact, an equality, thus

$$\mathbf{g}(x,y) \in \operatorname{sign}(\chi_\Omega(y) - \chi_\Omega(x)) \quad \text{for } (\nu\otimes m_x)\text{-a.e. } (x,y) \in X \times X,$$

and, consequently,

$$-\lambda_\Omega^m \tau \in \Delta_1^m \chi_\Omega \quad \text{in } X.$$

The next result is the nonlocal version of the fact that (a) is equivalent to (b) in Theorem 3.40.

THEOREM 3.42. *Let $[X,\mathcal{B},m,\nu]$ be an m-connected reversible random walk space. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Then, the following assertions are equivalent:*
(i) Ω is m-calibrable,
(ii) there exists $\lambda > 0$ and a measurable function $\tau : X \to \mathbb{R}$ equal to 1 in Ω such that

$$-\lambda\tau \in \Delta_1^m \chi_\Omega \quad \textit{in } X, \tag{3.27}$$

(iii)

$$-\lambda_\Omega^m \tau^* \in \Delta_1^m \chi_\Omega \quad \textit{in } X,$$

for

$$\tau^*(x) = \begin{cases} 1 & \textit{if } x \in \Omega, \\ -\dfrac{1}{\lambda_\Omega^m} m_x(\Omega) & \textit{if } x \in X \setminus \Omega. \end{cases}$$

Proof. Observe that, since $[X, \mathcal{B}, m, \nu]$ is m-connected, by Proposition 1.63, we have $P_m(\Omega) > 0$ and, therefore, $\lambda_\Omega^m > 0$.

$(iii) \Rightarrow (ii)$: Trivial.

$(ii) \Rightarrow (i)$: Suppose that there exists a measurable function $\tau : X \to \mathbb{R}$ equal to 1 in Ω satisfying (3.27). Then, by Remark 3.41(1), $\lambda = \lambda_\Omega^m$. Hence, there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$ satisfying

$$-\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda_\Omega^m \tau(x) \quad \text{for } \nu\text{-a.e. } x \in X$$

and

$$-\int_X \int_X \mathbf{g}(x,y) dm_x(y)\, \chi_\Omega(x) d\nu(x) = P_m(\Omega).$$

Then, if $F \in \mathcal{B}_\Omega$ with $\nu(F) > 0$, since $\mathbf{g}$ is antisymmetric, by using the reversibility of ν with respect to m, we get

$$\begin{aligned} \lambda_\Omega^m \nu(F) &= \lambda_\Omega^m \int_X \tau(x) \chi_F(x) d\nu(x) = -\int_X \int_X \mathbf{g}(x,y) \chi_F(x)\, dm_x(y) d\nu(x) \\ &= \frac{1}{2} \int_X \int_X \mathbf{g}(x,y) (\chi_F(y) - \chi_F(x))\, dm_x(y) d\nu(x) \leqslant P_m(F). \end{aligned}$$

Therefore, $h_1^m(\Omega) = \lambda_\Omega^m$ and, consequently, Ω is m-calibrable.

$(i) \Rightarrow (iii)$ Suppose that Ω is m-calibrable. Let

$$\tau^*(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ -\dfrac{1}{\lambda_\Omega^m} m_x(\Omega) & \text{if } x \in X \setminus \Omega. \end{cases}$$

We claim that $-\lambda_\Omega^m \tau^* \in \Delta_1^m 0$, that is,

$$\lambda_\Omega^m \tau^* \in \partial \mathcal{F}_m(0). \tag{3.28}$$

Indeed, take $w \in L^2(X, \nu)$ with $\mathcal{F}_m(w) < +\infty$. Since

$$w(x) = \int_0^{+\infty} \chi_{E_t(w)}(x) dt - \int_{-\infty}^0 (1 - \chi_{E_t(w)}(x)) dt,$$

and

$$\int_X \tau^*(x) d\nu(x) = \int_\Omega 1 d\nu(x) - \frac{1}{\lambda_\Omega^m} \int_{X\setminus\Omega} m_x(\Omega) d\nu(x) = \nu(\Omega) - \frac{1}{\lambda_\Omega^m} P_m(\Omega) = 0,$$

we have

$$\int_X \lambda_\Omega^m \tau^*(x) w(x) d\nu(x) = \lambda_\Omega^m \int_{-\infty}^{+\infty} \int_X \tau^*(x) \chi_{E_t(w)}(x) d\nu(x) dt. \tag{3.29}$$

Now, using that $\tau^* = 1$ in Ω and that Ω is m-calibrable we have that

$$(3.30)\qquad \begin{aligned} &\lambda_\Omega^m \int_{-\infty}^{+\infty} \int_X \tau^*(x)\chi_{E_t(w)}(x)d\nu(x)dt \\ &= \lambda_\Omega^m \int_{-\infty}^{+\infty} \nu(E_t(w)\cap\Omega)dt + \lambda_\Omega^m \int_{-\infty}^{+\infty}\int_{E_t(w)\setminus\Omega} \tau^*(x)d\nu(x)dt \\ &\leqslant \int_{-\infty}^{+\infty} P_m(E_t(w)\cap\Omega)dt + \lambda_\Omega^m \int_{-\infty}^{+\infty}\int_{E_t(w)\setminus\Omega} \tau^*(x)d\nu(x)dt. \end{aligned}$$

By Proposition 1.57 and the coarea formula (Theorem 3.5) we get

$$\begin{aligned} &\int_{-\infty}^{+\infty} P_m(E_t(w)\cap\Omega)dt \\ &= \int_{-\infty}^{+\infty} P_m(E_t(w)\cap\Omega)dt + \int_{-\infty}^{+\infty} P_m(E_t(w)\setminus\Omega)dt - \int_{-\infty}^{+\infty} 2L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt \\ &\quad - \int_{-\infty}^{+\infty} P_m(E_t(w)\setminus\Omega)dt + \int_{-\infty}^{+\infty} 2L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt \\ &= \int_{-\infty}^{+\infty} P_m(E_t(w))dt - \int_{-\infty}^{+\infty} P_m(E_t(w)\setminus\Omega)dt + \int_{-\infty}^{+\infty} 2L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt \\ &= \mathcal{F}_m(w) - \int_{-\infty}^{+\infty} P_m(E_t(w)\setminus\Omega)dt + \int_{-\infty}^{+\infty} 2L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt. \end{aligned}$$

Hence, if we prove that

$$\begin{aligned} I \ &:= -\int_{-\infty}^{+\infty} P_m(E_t(w)\setminus\Omega)dt + \int_{-\infty}^{+\infty} 2L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt \\ &\quad + \lambda_\Omega^m \int_{-\infty}^{+\infty}\int_{E_t(w)\setminus\Omega} \tau^*(x)d\nu(x)dt \end{aligned}$$

is non-positive then, by (3.29) and (3.30), we get

$$\int_X \lambda_\Omega^m \tau^*(x)w(x)d\nu(x) \leqslant \mathcal{F}_m(w),$$

which proves (3.28). Now, since

$$\begin{aligned} P_m(E_t(w)\setminus\Omega) \ &= L_m(E_t(w)\setminus\Omega, X\setminus(E_t(w)\setminus\Omega)) \\ &= L_m(E_t(w)\setminus\Omega, (E_t(w)\cap\Omega)\ \dot\cup\ (X\setminus E_t(w))) \\ &= L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega) + L_m(E_t(w)\setminus\Omega, X\setminus E_t(w)), \end{aligned}$$

and $\tau^*(x) = -\frac{1}{\lambda_\Omega^m} m_x(\Omega)$ for $x \in X\setminus\Omega$, we have

$$\begin{aligned} I \ &= -\int_{-\infty}^{+\infty} L_m(E_t(w)\setminus\Omega, X\setminus E_t(w))dt + \int_{-\infty}^{+\infty} L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt \\ &\quad - \int_{-\infty}^{+\infty}\int_{E_t(w)\setminus\Omega}\int_\Omega dm_x(y)d\nu(x)dt \\ &\leqslant \int_{-\infty}^{+\infty} L_m(E_t(w)\setminus\Omega, E_t(w)\cap\Omega)dt - \int_{-\infty}^{+\infty} L_m(E_t(w)\setminus\Omega, \Omega)dt \leqslant 0 \end{aligned}$$

as desired

Finally, by (3.26), (3.28) yields

$$-\lambda_\Omega^m \tau^* \in \Delta_1^m \chi_\Omega \quad \text{in } X,$$

and this concludes the proof. □

Even though, in principle, the m-calibrability of a set is a nonlocal concept which may, therefore, depend on the whole of X, in the next result we will see that the m-calibrability of a set depends only on the set itself.

THEOREM 3.43. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Then, Ω is m-calibrable if, and only if, there exists an antisymmetric function $\mathbf{g}$ in $\Omega \times \Omega$ such that*

$$-1 \leqslant \mathbf{g}(x,y) \leqslant 1 \qquad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in \Omega \times \Omega, \tag{3.31}$$

and

$$\lambda_\Omega^m = -\int_\Omega \mathbf{g}(x,y)\, dm_x(y) + 1 - m_x(\Omega), \quad \text{for } \nu\text{-a.e. } x \in \Omega. \tag{3.32}$$

Observe that, on account of (1.6), (3.32) is equivalent to

$$m_x(\Omega) = \frac{1}{\nu(\Omega)} \int_\Omega m_z(\Omega) d\nu(z) - \int_\Omega \mathbf{g}(x,y)\, dm_x(y) \qquad \text{for } \nu\text{-a.e. } x \in \Omega.$$

PROOF. By Theorem 3.42, we have that Ω is m-calibrable if, and only if, there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $g(x,y) \in \operatorname{sign}(\chi_\Omega(y) - \chi_\Omega(x))$ for $\nu \otimes m_x$-a.e. $(x,y) \in \Omega \times \Omega$, satisfying

$$-\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda_\Omega^m \quad \text{for } \nu\text{-a.e. } x \in \Omega$$

and

$$m_x(\Omega) = \int_X \mathbf{g}(x,y)\, dm_x(y) \quad \text{for } \nu\text{-a.e. } x \in X \backslash \Omega.$$

Now, having in mind that $g(x,y) = -1$ if $x \in \Omega$ and $y \in X \backslash \Omega$, we have that, for ν-a.e. $x \in \Omega$,

$$\begin{aligned} \lambda_\Omega^m &= -\int_X \mathbf{g}(x,y)\, dm_x(y) = -\int_\Omega \mathbf{g}(x,y)\, dm_x(y) - \int_{X\backslash\Omega} \mathbf{g}(x,y)\, dm_x(y) \\ &= -\int_\Omega \mathbf{g}(x,y)\, dm_x(y) + m_x(X\backslash\Omega) = -\int_\Omega \mathbf{g}(x,y)\, dm_x(y) + 1 - m_x(\Omega). \end{aligned}$$

Therefore, we have obtained (3.31) and (3.32).

Let us now suppose that we have an antisymmetric function $\mathbf{g}$ in $\Omega \times \Omega$ satisfying (3.31) and (3.32). To check that Ω is m-calibrable we need to find $\tilde{\mathbf{g}}(x,y) \in \operatorname{sign}(\chi_\Omega(y) - \chi_\Omega(x))$ antisymmetric such that

$$\begin{cases} -\lambda_\Omega^m = \displaystyle\int_X \tilde{\mathbf{g}}(x,y) dm_x(y), & x \in \Omega, \\ m_x(\Omega) = \displaystyle\int_X \tilde{\mathbf{g}}(x,y) dm_x(y), & x \in X\backslash\Omega, \end{cases}$$

which is equivalent to

$$\begin{cases} -\lambda_\Omega^m = \displaystyle\int_\Omega \tilde{\mathbf{g}}(x,y) dm_x(y) - m_x(X\backslash\Omega), & x \in \Omega, \\ m_x(\Omega) = \displaystyle\int_{X\backslash\Omega} \tilde{\mathbf{g}}(x,y) dm_x(y) + m_x(\Omega), & x \in X\backslash\Omega, \end{cases}$$

since, necessarily, $\tilde{\mathbf{g}}(x,y) = -1$ for $x \in \Omega$ and $y \in X\backslash\Omega$, and $\tilde{\mathbf{g}}(x,y) = 1$ for $x \in X\backslash\Omega$ and $y \in \Omega$. Now, the second equality in this system is satisfied if we take $\tilde{\mathbf{g}}(x,y) = 0$ for $x, y \in X\backslash\Omega$, and the first one is just a rewrite of (3.32) if we take $\tilde{\mathbf{g}}(x,y) = \mathbf{g}(x,y)$ for $x, y \in \Omega$. □

Corollary 3.44. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space. A set $\Omega \in \mathcal{B}$ is m-calibrable if, and only if, it is m^{Ω_m}-calibrable as a subset of $[\Omega_m, \mathcal{B}_{\Omega_m}, m^{\Omega_m}, \nu \llcorner \Omega_m]$ (recall Example 1.42).*

Remark 3.45. Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space.
(1) Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Observe that, as we have proved,

$$\Omega \text{ is } m\text{-calibrable} \Leftrightarrow -\lambda_\Omega^m \chi_\Omega + m_{(.)}(\Omega)\, \chi_{X\backslash\Omega} \in \Delta_1^m \chi_\Omega\,. \tag{3.33}$$

(2) Suppose that ν is a probability measure and let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < 1$. Let us see that the equation

$$-\lambda_\Omega^m \chi_\Omega \in \Delta_1^m \chi_\Omega \quad \text{in } X$$

does not hold true. Suppose that

$$-\lambda_\Omega^m \chi_\Omega + h\, \chi_{X\backslash\Omega} \in \Delta_1^m \chi_\Omega \quad \text{in } X$$

for some measurable function $h : X \to \mathbb{R}$. Then, there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with

$$\mathbf{g}(x,y) \in \operatorname{sign}(\chi_\Omega(y) - \chi_\Omega(x)) \quad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in X \times X$$

such that

$$-\lambda_\Omega^m \chi_\Omega(x) + h(x)\, \chi_{X\backslash\Omega}(x) = \int_X \mathbf{g}(x,y)\, dm_x(y) \quad \text{for } \nu\text{-a.e } x \in X. \tag{3.34}$$

Now, integrating (3.34) over X with respect to ν yields

$$-\underbrace{\lambda_\Omega^m \nu(\Omega)}_{=P_m(\Omega)} + \int_{X\backslash\Omega} h(x) d\nu(x) = \int_X \int_X \mathbf{g}(x,y)\, dm_x(y) d\nu(x).$$

However, since ν is a probability measure, we have that $\mathbf{g}$ is $\nu \otimes m_x$-integrable over $X \times X$, thus, since $\mathbf{g}$ is antisymmetric, the integral on the right hand side of the above equation is zero. Consequently, we get that

$$\int_{X\backslash\Omega} h(x) d\nu(x) = P_m(\Omega). \tag{3.35}$$

Now, since $[X, \mathcal{B}, m, \nu]$ is m-connected, Theorem 1.63 yields that $P_m(\Omega) > 0$, thus, by (3.35), h is non-ν-null.

We have proven that, if ν is a probability measure (so that, in particular, $\mathbf{g}$ is $\nu \otimes m_x$-integrable), the equation

$$-\lambda_\Omega^m \chi_\Omega \in \Delta_1^m \chi_\Omega \quad \text{in } X \tag{3.36}$$

does not hold true for any set $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < 1$. However, if $\nu(X) = +\infty$, then (3.36) may be satisfied, as shown by the next example.

Example 3.46. Consider the metric random walk space $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ with $J = \frac{1}{2}\chi_{[-1,1]}$ (as defined in Example 1.37). Let us see that

$$-\lambda_{[-1,1]}^{m^J} \chi_{[-1,1]} \in \Delta_1^{m^J} \chi_{[-1,1]},$$

where $\lambda_{[-1,1]}^{m^J} = \frac{1}{4}$. Take $\mathbf{g}(x,y)$ to be antisymmetric and defined as follows for $y < x$:

$$\mathbf{g}(x,y) = -\chi_{\{y<x<y+1<0\}}(x,y) - \frac{1}{2}\chi_{\{-1<y<x<0\}}(x,y) + \frac{1}{2}\chi_{\{0<y<x<1\}}(x,y) + \chi_{\{0<x-1<y<x\}}(x,y).$$

Then, $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m^J_x)$,

$$\mathbf{g}(x,y) \in \operatorname{sign}(\chi_{[-1,1]}(y) - \chi_{[-1,1]}(x)) \quad \text{for } (\nu \otimes m^J_x)\text{-a.e. } (x,y) \in \mathbb{R} \times \mathbb{R},$$

and

$$-\frac{1}{4}\chi_{[-1,1]}(x) = \int_{\mathbb{R}} \mathbf{g}(x,y)\, dm^J_x(y) \quad \text{for } \nu\text{-a.e } x \in \mathbb{R}.$$

Note that $\mathbf{g}$ is not $\nu \otimes m^J_x$ integrable.

Remark 3.47. As a consequence of Theorem 3.40, it holds that (see [**6**, Introduction] or [**12**, Section 4.4]) a bounded convex set $\Omega \subset \mathbb{R}^N$ is calibrable if, and only if, $u(t,x) = \left(1 - \frac{\operatorname{Per}(\Omega)}{|\Omega|} t\right)^+ \chi_\Omega(x)$ is a solution of the Cauchy problem

$$\begin{cases} u_t - \Delta_1 u \ni 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0) = \chi_\Omega. \end{cases}$$

That is, a calibrable set Ω is that for which the gradient descent flow associated with the total variation tends to decrease linearly the height of χ_Ω without distortion of its boundary.

Now, as a consequence of (3.33), we can obtain a similar result in our context if we introduce an absorption term in the corresponding Cauchy problem. The appearance of this term is due to the nonlocality of the diffusion considered. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$, then Ω is m-calibrable if, and only if, $u(t)(x) = (1 - \lambda^m_\Omega t)^+ \chi_\Omega(x)$ is a solution of

$$\begin{cases} u_t(t)(x) - \Delta^m_1 u(t)(x) \ni -m_x(\Omega)\, \chi_{X\setminus\Omega}(x)\chi_{[0,1/\lambda^m_\Omega)}(t) & (t,x) \in (0,\infty) \times X, \\ u(0)(x) = \chi_\Omega(x), & x \in X. \end{cases}$$

Indeed, for any $T > 0$, by Theorem A.31 ([**43**, Theoreme 3.6]) with $f \in L^2(0,T; L^2(X,\nu))$ defined by

$$f(t)(x) := -m_x(\Omega)\chi_{X\setminus\Omega}(x)\chi_{[0,1/\lambda^m_\Omega)}(t),$$

we have that this problem has a unique solution $u \in W^{1,1}(0,T; L^2(X,\nu))$. Then, if $u(t)(x) = (1 - \lambda^m_\Omega t)^+ \chi_\Omega(x)$ we get that, for $t < 1/\lambda^m_\Omega$,

$$\underbrace{-\lambda^m_\Omega \chi_\Omega}_{u_t} - \underbrace{\Delta^m_1 \chi_\Omega}_{\Delta^m_1 u} \ni -\lambda^m_\Omega \chi_\Omega - (-\lambda^m_\Omega \chi_\Omega + m_.(\Omega)\chi_{X\setminus\Omega})$$

which is equivalent to Ω being m-calibrable by (3.33). The only if direction follows by the uniqueness of the solution.

The following result relates the m-calibrability of a set with its m-mean curvature. This is the nonlocal version of one of the implications in the equivalence between (a) and (c) in Theorem 3.40.

Proposition 3.48. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Then,*

$$\Omega \ \ m\text{-calibrable} \ \Rightarrow \ \frac{1}{\nu(\Omega)} \int_\Omega m_x(\Omega) d\nu(x) \leqslant 2\, \nu\text{-}\operatorname*{ess\,inf}_{x\in\Omega} m_x(\Omega). \tag{3.37}$$

Equivalently,

$$\Omega \ \ m\text{-calibrable} \ \Rightarrow \ \nu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{H}^m_{\partial\Omega}(x) \leqslant \lambda^m_\Omega. \tag{3.38}$$

Proof. By Theorem 3.43, there exists an antisymmetric function $\mathbf{g}$ in $\Omega \times \Omega$ such that

$$-1 \leqslant \mathbf{g}(x,y) \leqslant 1 \qquad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in \Omega \times \Omega,$$

and

$$\frac{1}{\nu(\Omega)}\int_\Omega m_z(\Omega)d\nu(z) = m_x(\Omega) + \int_\Omega \mathbf{g}(x,y)\,dm_x(y) \qquad \text{for } \nu\text{-a.e. } x \in \Omega.$$

Hence,

$$\frac{1}{\nu(\Omega)}\int_\Omega m_z(\Omega)d\nu(z) \leqslant 2m_x(\Omega) \quad \text{for } \nu\text{-a.e. } x \in \Omega,$$

thus (3.37) follows.

The equivalent thesis (3.38) follows from (3.37) and the fact that

$$\nu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{H}^m_{\partial\Omega}(x) \leqslant \lambda^m_\Omega \iff \frac{1}{\nu(\Omega)}\int_\Omega m_x(\Omega)d\nu(x) \leqslant 2\,\nu\text{-}\operatorname*{ess\,inf}_{x\in\Omega} m_x(\Omega).$$

For this last equivalence recall from (1.10) that

$$\mathcal{H}^m_{\partial\Omega}(x) = 1 - 2m_x(\Omega)$$

and that (see (1.6))

$$\lambda^m_\Omega = \frac{P_m(\Omega)}{\nu(\Omega)} = 1 - \frac{1}{\nu(\Omega)}\int_\Omega m_x(\Omega)d\nu(x).$$ □

The converse of Proposition 3.48 is not true in general, an example is given in [**120**] *(see also* [**121**]*) for* $[\mathbb{R}^3, d, m^J, \mathcal{L}^3]$, *with* d *the Euclidean distance and* $J = \frac{1}{|B_1(0)|}\chi_{B_1(0)}$. *Let us see an example, in the case of graphs, where the converse of Proposition 3.48 is not true*

Example 3.49. Let $G = (V(G), E(G))$ be a finite weighted discrete graph with $V(G) = \{x_1, x_2, \ldots, x_8\}$ and the following weights: $w_{x_1,x_2} = w_{x_2,x_3} = w_{x_6,x_7} = w_{x_7,x_8} = 2$, $w_{x_3,x_4} = w_{x_4,x_5} = 1$, $w_{x_4,x_5} = 10$ and $w_{x_i,x_j} = 0$ otherwise. Let $[V(G), d_G, m^G, \nu_G]$ be the associated metric random walk space. If $\Omega := \{x_2, x_3, x_4, x_5, x_6, x_7\}$, then

$$\lambda^{m^G}_\Omega = \frac{1}{9} \quad \text{and} \quad \mathcal{H}^{m^G}_{\partial\Omega}(x) \leqslant 0 \qquad \forall\, x \in \Omega.$$

Therefore, (3.38) holds. However, Ω is not m^G-calibrable since, if $A := \{x_4, x_5\}$, we have

$$\frac{P_{m^G}(A)}{\nu_G(A)} = \frac{1}{11}.$$

Proposition 3.50. *Let* $[X, \mathcal{B}, m, \nu]$ *be a reversible random walk space. Let* $\Omega \in \mathcal{B}$ *with* $0 < \nu(\Omega) < \nu(X)$. *Let* Ω_1, $\Omega_2 \in \mathcal{B}$.
(1) If $\Omega = \Omega_1 \sqcup_m \Omega_2$ *(recall Definition 1.33), then*

$$\min\{\lambda^m_{\Omega_1}, \lambda^m_{\Omega_2}\} \leqslant \lambda^m_\Omega.$$

(2) If $\Omega = \Omega_1 \sqcup_m \Omega_2$ *is* m*-calibrable, then each* Ω_i *is* m*-calibrable and*

$$\lambda^m_\Omega = \lambda^m_{\Omega_1} = \lambda^m_{\Omega_2}.$$

Proof. (1) is a direct consequence of Proposition 1.57 and the fact that, for a, b, c, d positive real numbers, $\min\left\{\frac{a}{b}, \frac{c}{d}\right\} \leqslant \frac{a+c}{b+d}$. (2) is a direct consequence of (1) together with the definition of m-calibrability. □

3.5. The eigenvalue problem for the 1-Laplacian

In this section we introduce the eigenvalue problem associated with the 1*-Laplacian* Δ^m_1 *and its relation with the Cheeger minimization problem. For the particular case of finite weighted discrete graphs where the weights are either* 0 *or* 1*, this problem was first studied by Hein and Bühler (*[**99**]*) and a more complete study was subsequently performed by Chang in* [**57**].

Definition 3.51. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. A pair $(\lambda, u) \in \mathbb{R} \times L^2(X, \nu)$ is called an *m-eigenpair* of the 1-Laplacian Δ_1^m on X if $\|u\|_{L^1(X,\nu)} = 1$ and there exists $\xi \in \operatorname{sign}(u)$ (i.e., $\xi(x) \in \operatorname{sign}(u(x))$ for every $x \in X$) such that

$$\lambda \xi \in \partial \mathcal{F}_m(u) = -\Delta_1^m u.$$

The function u is called an *m-eigenfunction of Δ_1^m* and λ an *m-eigenvalue of Δ_1^m* associated with u.

Observe that, if (λ, u) is an m-eigenpair of Δ_1^m, then $(\lambda, -u)$ is also an m-eigenpair of Δ_1^m.

Remark 3.52. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space. By Theorem 3.13, the following statements are equivalent:
(1) (λ, u) is an m-eigenpair of Δ_1^m .
(2) There exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$, such that

$$\begin{cases} -\displaystyle\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda \xi(x) \quad \text{for } \nu\text{-a.e. } x \in X, \\ -\displaystyle\int_X \int_X \mathbf{g}(x,y) dm_x(y)\, u(x) d\nu(x) = TV_m(u). \end{cases} \tag{3.39}$$

(3) There exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$, such that

$$\begin{cases} -\displaystyle\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda \xi(x) \quad \text{for } \nu\text{-a.e. } x \in X, \\ \mathbf{g}(x,y)(u(y) - u(x)) = |u(y) - u(x)| \quad \text{for } \nu \otimes m_x\text{-a.e. } (x,y) \in X \times X; \end{cases} \tag{3.40}$$

(4) There exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1$, such that

$$\begin{cases} -\displaystyle\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda \xi(x) \quad \text{for } \nu\text{-a.e. } x \in X, \\ \lambda = TV_m(u); \end{cases}$$

Remark 3.53. Note that, since $TV_m(u) = \lambda$ for any m-eigenpair (λ, u) of Δ_1^m, then

$$\begin{aligned} \lambda &= TV_m(u) = \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) \\ &\leqslant \frac{1}{2} \int_X \int_X (|u(y)| + |u(x)|) dm_x(y) d\nu(x) = \|u\|_1 = 1, \end{aligned}$$

thus

$$0 \leqslant \lambda \leqslant 1.$$

Example 3.54. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space given in Example 1.38. Then, a pair $(\lambda, u) \in \mathbb{R} \times L^2(V(G), \nu_G)$ is an m^G-eigenpair of $\Delta_1^{m^G}$ if $\|u\|_{L^1(V(G),\nu_G)} = 1$ and there exists $\xi \in \operatorname{sign}(u)$ and $\mathbf{g} \in L^\infty(V(G) \times V(G), \nu_G \otimes m_x^G)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(V(G)\times V(G), \nu_G \otimes m_x^G)} \leqslant 1$ such that

$$\begin{cases} -\displaystyle\sum_{y \in V(G)} \mathbf{g}(x,y) \frac{w_{xy}}{d_x} = \lambda \xi(x) \quad \text{for } \nu_G\text{-a.e. } x \in V(G), \\ \mathbf{g}(x,y) \in \operatorname{sign}(u(y) - u(x)) \quad \text{for } \nu_G \otimes m_x^G\text{-a.e. } (x,y) \in V(G) \times V(G). \end{cases}$$

In [**57**], Chang gives the 1-Laplacian spectrum for some particular graphs like the Petersen graph, the complete graph K_n, the circle graphs with n vertices C_n, etc. We will now provide an example in which the vertices have loops.

EXAMPLE 3.55. Let $G = (V(G), E(G))$ be the following finite connected weighted discrete graph. Take $V(G) = \{a, b\}$, $0 < p < 1$ and the following weights: $w_{aa} = w_{bb} = p$, $w_{ab} = w_{ba} = 1 - p$. Then, $(\lambda, u) \in \mathbb{R} \times L^2(V(G), \nu_G)$ is an m^G-eigenpair of $\Delta_1^{m^G}$ if $|u(a)| + |u(b)| = 1$ and there exists $\xi \in \operatorname{sign}(u)$ and $\mathbf{g} \in L^\infty(V(G) \times V(G), \nu_G \otimes m_x^G)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(V(G)\times V(G), \nu_G \otimes m_x^G)} \leqslant 1$ such that

$$\begin{cases} \mathbf{g}(a,a) = \mathbf{g}(b,b) = 0, \ \mathbf{g}(a,b) = -\mathbf{g}(b,a), \\ -\mathbf{g}(a,b)(1-p) = \lambda\xi(a), \\ \mathbf{g}(a,b)(1-p) = \lambda\xi(b), \\ \mathbf{g}(a,b)(u(b) - u(a)) = |u(b) - u(a)|. \end{cases} \tag{3.41}$$

Now, using a case-by-case argument, it follows easily from the system (3.41) that the m-eigenvalues of $\Delta_1^{m^G}$ are

$$\lambda = 0 \quad \text{and} \quad \lambda = 1 - p,$$

and the following pairs are m-eigenpairs of $\Delta_1^{m^G}$ (observe that the measure ν_G is not normalized to a probability measure):

$$\lambda = 0 \quad \text{and} \quad (u(a), u(b)) = (1/2, 1/2),$$

$$\lambda = 1 - p \quad \text{and} \quad (u(a), u(b)) = (0, -1) + \mu(1, 1) \quad \forall 0 \leqslant \mu \leqslant 1.$$

For example, suppose that (λ, u) is an m-eigenpair with $u(a) = u(b)$. Then, $u(a) = u(b) = \frac{1}{2}$ ($u(a) = u(b) = -\frac{1}{2}$ yields the same eigenvalue) and, therefore, $\xi = 1$ thus $\lambda = 0$. Alternatively, we could have $u(a) > u(b)$ thus $g(a, b) = -1$ and we continue by using (3.41).

Observe that, if a locally finite weighted discrete graph contains a vertex x with no loop, i.e. $w_{x,x} = 0$, then $\left(1, \frac{1}{d_x}\delta_x\right)$ is an m-eigenpair of the 1-Laplacian. Conversely, if 1 is an m-eigenvalue of $\Delta_1^{m^G}$, then there exists at least one vertex in the graph with no loop (this follows easily from Proposition 3.72).

We have the following relation between m-calibrable sets and m-eigenpairs of Δ_1^m.

THEOREM 3.56. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. We have:*

(i) If $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ is an m-eigenpair of Δ_1^m, then Ω is m-calibrable.

(ii) If Ω is m-calibrable and

$$m_x(\Omega) \leqslant \lambda_\Omega^m \quad \text{for } \nu\text{-a.e. } x \in X \setminus \Omega, \tag{3.42}$$

then $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ is an m-eigenpair of Δ_1^m.

PROOF. (i): Since $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ is an m-eigenpair of Δ_1^m, there exists $\xi \in \operatorname{sign}(\chi_\Omega)$ such that $-\lambda_\Omega^m \xi \in \Delta_1^m(\chi_\Omega)$. Then, by Theorem 3.42, we have that Ω is m-calibrable.
(ii): If Ω is m-calibrable, by Theorem 3.42, we have

$$-\lambda_\Omega^m \tau^* \in \Delta_1^m \chi_\Omega \quad \text{in } X$$

for

$$\tau^*(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ -\dfrac{1}{\lambda_\Omega^m} m_x(\Omega) & \text{if } x \in X \setminus \Omega. \end{cases}$$

Now, by (3.42), we have that $\tau^* \in \operatorname{sign}(\chi_\Omega)$ and, consequently, $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of Δ_1^m. $\square$

In the next example we see that, in Theorem 3.56, the reverse implications in (i) and (ii) are false in general.

Example 3.57. (1) Let $G = (V(G), E(G))$ be the weighted discrete graph with vertex set $V(G) = \{a, b, c\}$ and weights $w_{ab} = w_{ac} = w_{bc} = \frac{1}{2}$ and $w_{aa} = w_{bb} = w_{cc} = 0$. Consider the associated metric random walk space $[V(G), d_G, m^G, \nu_G]$. Then, $m_a = \frac{1}{2}\delta_b + \frac{1}{2}\delta_c$, $m_b = \frac{1}{2}\delta_a + \frac{1}{2}\delta_c$, $m_c = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$ and $\nu_G = \delta_a + \delta_b + \delta_c$. By Remark 3.35(2), we have that $\Omega := \{a, b\}$ is m^G-calibrable. However, $\lambda_\Omega^{m^G} = \frac{1}{2}$ and $(\frac{1}{2}, \chi_\Omega)$ is not an m-eigenpair of Δ_1^m since $0 \notin \text{med}_\nu(\chi_\Omega)$ (see Corollary 3.71). Therefore, (3.42) does not hold (it follows by a simple calculation that $m_c^G(\Omega) = 1 > \frac{1}{2} = \lambda_\Omega^{m^G}$).
(2) Consider the metric random walk space $[\mathbb{Z}^2, d_{\mathbb{Z}^2}, m^{\mathbb{Z}^2}, \nu_{\mathbb{Z}^2}]$, where $d_{\mathbb{Z}^2} : \mathbb{Z}^2 \to \mathbb{N} \cup \{0\}$ is defined as

$$d_{\mathbb{Z}^2}((x_1, x_2), (y_1, y_2)) := |x_1 - y_1| + |x_2 - y_2|, \quad (x_1, x_2), (y_1, y_2) \in \mathbb{Z}^2,$$

and the weights are defined as: $w_{xy} = 1$ if $d_{\mathbb{Z}^2}(x, y) = 1$ and $w_{xy} = 0$ otherwise. For ease of notation we denote $m^{\mathbb{Z}^2}$ by m. Let

$$\Omega_k := \{(x_1, x_2) \in \mathbb{Z}^2 : 0 \leqslant x_1, x_2 \leqslant k - 1\} \quad \text{for } k \geqslant 1.$$

It is easy to see that

$$\lambda_{\Omega_k}^m = \frac{1}{k}.$$

Moreover, for $1 \leqslant k \leqslant 4$ these sets are m-calibrable and satisfy (3.42). Therefore, for $1 \leqslant k \leqslant 4$, $\left(\frac{1}{k}, \frac{1}{\nu(\Omega_k)}\chi_{\Omega_k}\right)$ is an m-eigenpair of Δ_1^m and, with the same reasoning, it is also an m_k-eigenpair of $\Delta_1^{m_k}$ in

$$[(\Omega_k)_m, d_{\mathbb{Z}^2}, m_k, \nu_{\mathbb{Z}^2} \llcorner (\Omega_k)_m]$$

(recall Definition 1.51 and Example 1.42; for ease of notation we have denoted $m_k := m^{(\Omega_k)_m}$). In fact, we have that $(m_k)_x(\{y\}) = m_x(\{y\})$ for every $x,\ y \in (\Omega_k)_m$, i.e., the probabilities associated with the jumps between different vertices in $(\Omega_k)_m$ do not vary. The difference is that, when considering m_k, a loop is "appearing" at each vertex of $\partial_m \Omega_k$, i.e., $(m_k)_x(\{x\}) > 0$ for every $x \in \partial_m \Omega_k$ (note that $\partial_m \Omega_k$ is the set of vertices which are at a distance of 1 from Ω_k).

Let us see what happens for

$$\Omega_5 := \{(x_1, x_2) \in \mathbb{Z}^2 : 0 \leqslant x_1, x_2 \leqslant 4\}.$$

In this case,

$$\lambda_{\Omega_5}^m = \frac{1}{5},$$

and an algebraic calculation gives that $\left(\frac{1}{5}, \frac{1}{\nu(\Omega_5)}\chi_{\Omega_5}\right)$ is an m-eigenpair of Δ_1^m (see Figure 1). Moreover, $\left(\frac{1}{5}, \frac{1}{\nu(\Omega_5)}\chi_{\Omega_5}\right)$ is also an m^A-eigenpair of $\Delta_1^{m^A}$ in

$$\left[A, d_{\mathbb{Z}^2}, m^A, \nu_{\mathbb{Z}^2} \llcorner A\right]$$

where $A := \{(x_1, x_2) \in \mathbb{Z}^2 : -2 \leqslant x_1, x_2 \leqslant 6\}$; and the same is true if we take A to be the smaller set shown in Figure 1. However,

$$(m_5)_x(\Omega_5) = \frac{1}{4} \quad \forall x \in \partial_m \Omega_5$$

so (3.42) is not satisfied. Furthermore, $\left(\frac{1}{5}, \frac{1}{\nu(\Omega_5)}\chi_{\Omega_5}\right)$ fails to be an m_5-eigenpair of $\Delta_1^{m_5}$ in

$$[(\Omega_5)_m, d_{\mathbb{Z}^2}, m_5, \nu_{\mathbb{Z}^2} \llcorner (\Omega_5)_m]$$

since the condition on the median given in Corollary 3.71 is not satisfied; nevertheless, Ω_5 is still m_5-calibrable in this setting.

Remark 3.58. Let us give some characterizations of (3.42).

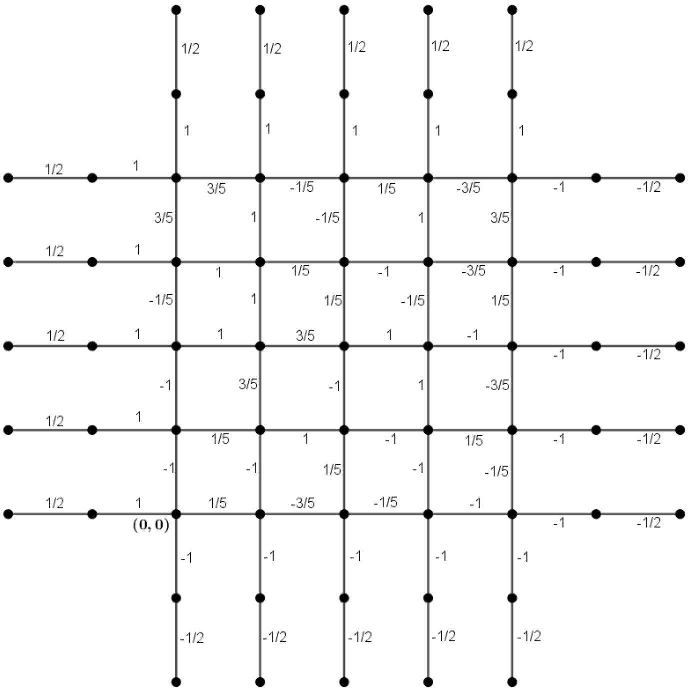


FIGURE 1. The numbers in the graph are the values of a function $g(x,y)$ satisfying (3.52), where x is the vertex to the left of the number and y is the one to the right, or, alternatively, x is the one above and y the one below. Elsewhere, $g(x,y)$ is taken as 0. The vertex $(0,0)$ is labelled in the graph. As an example, $g((0,0),(1,0)) = 1/5$ and $g((0,1),(0,0)) = -1$.

(1) In terms of the m-mean curvature we have that,

$$(3.42) \Leftrightarrow \nu\text{-ess}\sup_{x\in\Omega^c} \mathcal{H}^m_{\partial\Omega^c}(x) \leqslant \frac{1}{\nu(\Omega)}\int_\Omega \mathcal{H}^m_{\partial\Omega}(x)d\nu(x),$$

where $\Omega^c = X\backslash\Omega$. Indeed, (3.42) is equivalent to

$$1 - 2m_x(\Omega) \geqslant 1 - 2\frac{P_m(\Omega)}{\nu(\Omega)} = \frac{\nu(\Omega) - 2P_m(\Omega)}{\nu(\Omega)} \quad \text{for } \nu\text{-a.e. } x \in \Omega^c,$$

and this inequality can be rewritten as

$$-\mathcal{H}^m_{\partial\Omega}(x) \leqslant \frac{1}{\nu(\Omega)}\int_\Omega \mathcal{H}^m_{\partial\Omega}(y)d\nu(y) \quad \text{for } \nu\text{-a.e. } x \in \Omega^c$$

thanks to (1.9) and (1.11). Hence, since $\mathcal{H}^m_{\partial\Omega}(x) = -\mathcal{H}^m_{\partial\Omega^c}(x)$, we are done.
(2) Furthermore, we have that

$$(3.42) \Leftrightarrow \frac{1}{\nu(\Omega)}\int_\Omega m_x(\Omega)d\nu(x) \leqslant \nu\text{-ess}\inf_{x\in\Omega^c} m_x(\Omega^c).$$

Indeed, in this case, on account of (1.6), we rewrite (3.42) as

$$1 - m_x(\Omega^c) \leqslant 1 - \frac{1}{\nu(\Omega)}\int_\Omega m_y(\Omega)d\nu(y) \quad \text{for } \nu\text{-a.e. } x \in \Omega^c,$$

or, equivalently,

$$\frac{1}{\nu(\Omega)}\int_\Omega m_y(\Omega)d\nu(y) \leqslant m_x(\Omega^c) \quad \text{for } \nu\text{-a.e. } x \in \Omega^c,$$

which gives us the characterization.

In the next example we give m-eigenpairs of the 1*-Laplacian for the metric random walk space given in Example 1.37.*

EXAMPLE 3.59. Let $\Omega \subset \mathbb{R}^N$ with $\mathcal{L}^N(\Omega) < \infty$ and consider the metric random walk space $[\Omega, d, m^{J,\Omega}, \nu \llcorner \Omega]$ with $J := \frac{1}{\mathcal{L}^N(B_r(0))}\chi_{B_r(0)}$ (recall Examples 1.37 and 1.42). Moreover, assume that there exists $B_\rho(x_0) \subset \Omega$ such that $\operatorname{dist}(B_\rho(x_0), \mathbb{R}^N \setminus \Omega) > r$. Then, by (1.7), we have

$$P_{m^{J,\Omega}}(B_\rho(x_0)) = P_{m^J}(B_\rho(x_0)),$$

and, since $B_\rho(x_0)$ is m^J-calibrable, we have that $B_\rho(x_0)$ is $m^{J,\Omega}$-calibrable (Corollary 3.44). Assume also that $\mathcal{L}^N(B_\rho(x_0)) < \frac{1}{2}\mathcal{L}^N(B_r(0))$. Let us see that

$$m_x^{J,\Omega}(B_\rho(x_0)) \leqslant \lambda_{B_\rho(x_0)}^{m^{J,\Omega}} \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega \setminus B_\rho(x_0). \tag{3.43}$$

By Remark 3.58, (3.43) is equivalent to

$$\frac{1}{\mathcal{L}^N(B_\rho(x_0))}\int_{B_\rho(x_0)} m_x^{J,\Omega}(B_\rho(x_0))dx \leqslant \mathcal{L}^N\text{-}\operatorname*{ess\,inf}_{x \in \Omega \setminus B_\rho(x_0)} m_x^{J,\Omega}(\Omega \setminus B_\rho(x_0))).$$

Now, for $x \in \Omega$, we have

$$m_x^{J,\Omega}(B_\rho(x_0)) = m_x^J(B_\rho(x_0)) = \frac{1}{\mathcal{L}^N(B_r(0))}\int_{B_\rho(x_0)} \chi_{B_r(0)}(x-y)dy \leqslant \frac{1}{2}.$$

Then, for $x \in \Omega \setminus B_\rho(x_0)$, we have

$$m_x^{J,\Omega}(\Omega \setminus B_\rho(x_0))) = 1 - m_x^{J,\Omega}(B_\rho(x_0)) \geqslant \frac{1}{2} \geqslant \frac{1}{\mathcal{L}^N(B_\rho(x_0))}\int_{B_\rho(x_0)} m_x^{J,\Omega}(B_\rho(x_0))dx.$$

Hence, (3.43) holds. Therefore, by Theorem 3.56, we have that

$$\left(\lambda_{B_\rho(x_0)}^{m^{J,\Omega}}, \frac{1}{\mathcal{L}^N(B_\rho(x_0))}\chi_{B_\rho(x_0)}\right)$$

is an $m^{J,\Omega}$-eigenpair of $\Delta_1^{m^{J,\Omega}}$.

Similarly, for the metric random walk space $[\mathbb{R}^n, d, m^J, \mathcal{L}^N]$ with $J = \frac{1}{\mathcal{L}^N(B_r(0))}\chi_{B_r(0)}$, and for $\mathcal{L}^N(B_\rho(x_0)) < \frac{1}{2}\mathcal{L}^N(B_r(0))$, we have that

$$\left(\lambda_{B_\rho(x_0)}^{m^J}, \frac{1}{\mathcal{L}^N(B_\rho(x_0))}\chi_{B_\rho(x_0)}\right)$$

is an m^J-eigenpair of $\Delta_1^{m^J}$.

3.6. The m-Cheeger constant

In 1969, Jeff Cheeger [**61**] *proved his famous inequality*

$$\frac{h_M^2}{2} \leqslant \lambda_1(\Delta_M),$$

where M is a compact manifold, $\lambda_1(\Delta_M)$ is the first non-trivial eigenvalue of the Laplace Beltrami operator Δ_M on $L^2(M, \mathrm{vol})$ and the Cheeger constant h_M is defined as follows:

$$h_M := \inf \frac{Area(\partial S)}{\min(vol(S), vol(M \setminus S))},$$

where the infimum runs over all $S \subset M$ with sufficiently smooth boundary. This inequality can be traced back to Polya and Szego's paper [**139**].

On graphs, the first results regarding Cheeger's bound for the lowest eigenvalue of the graph Laplacian are due to Dodziuk [**74**] *and Alon and Milmann* [**5**]. *These estimates have been subsequently improved and several variants have been obtained. In a locally finite weighted discrete graph $G = (V(G), E(G))$ the Cheeger constant is defined as*

$$h_G := \inf_{D \subset V(G)} \frac{|\partial D|}{\min\{\nu_G(D), \nu_G(V(G) \setminus D)\}},$$

where (recall Example 1.56)

$$|\partial D| = \sum_{x \in D,\, y \in V \setminus D} w_{xy}.$$

For locally finite connected weighted discrete graphs, the following relation between the Cheeger constant and the first positive eigenvalue $\lambda_1(G)$ of the graph Laplacian has been proved in [**63**] *(see also* [**26**]*):*

$$\frac{h_G^2}{2} \leqslant \lambda_1(G) \leqslant 2h_G. \tag{3.44}$$

In this section, among other results, we will generalise this inequality to reversible random walk spaces.

Definition 3.60. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and suppose that ν is a probability measure. We define the *Cheeger constant* of $[X, \mathcal{B}, m, \nu]$ as

$$h_m(X) := \inf \left\{ \frac{P_m(D)}{\min\{\nu(D), \nu(X \setminus D)\}} \ : \ D \in \mathcal{B},\ 0 < \nu(D) < 1 \right\},$$

or, equivalently,

$$h_m(X) = \inf \left\{ \frac{P_m(D)}{\nu(D)} \ : \ D \in \mathcal{B},\ 0 < \nu(D) \leqslant \frac{1}{2} \right\}. \tag{3.45}$$

Note that, as a consequence of (1.6)*, we get*

$$h_m(X) \leqslant 1.$$

Moreover, if $h_m(X) > 0$, then $h_m(X)$ is the best constant in the isoperimetric inequality (1.20)*.*

Recall that in Section 3.4 we defined the m-Cheeger constant $h_1^m(\Omega)$ for sets $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$ (see (3.24)*; recall that ν need not be finite for that definition). In this section, the m-Cheeger constant $h_m(X)$ is, instead, a global constant of the random walk space. Observe that, since ν is a probability measure,*

$$h_m(X) \leqslant h_1^m(\Omega)$$

for any $\Omega \in \mathcal{B}$ such that $0 < \nu(\Omega) \leqslant 1/2$; and, if $h_m(X) = \frac{P_m(\Omega)}{\nu(\Omega)}$ for some $\Omega \in \mathcal{B}$ such that $0 < \nu(\Omega) \leqslant 1/2$, then $h_m(X) = h_1^m(\Omega)$ and, moreover, Ω is m-calibrable.

Having in mind (1.8)*, we have that this definition is consistent with the definition on graphs (see* [**63**]*, also* [**26**]*).*

Example 3.61. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space given in Example 1.38. Then, for $E \subset V(G)$, since

$$P_{m^G}(E) = \sum_{x \in E} \sum_{y \notin E} w_{x,y} \quad \text{and} \quad \nu_G(E) := \sum_{x \in E} d_x,$$

we have

$$\frac{P_{m^G}(E)}{\nu_G(E)} = \frac{1}{\sum_{x \in E} d_x} \sum_{x \in E} \sum_{y \notin E} w_{x,y}.$$

Therefore,

$$h_{m^G}(V(G)) = \inf \left\{ \frac{1}{\sum_{x \in E} d_x} \sum_{x \in E} \sum_{y \notin E} w_{x,y} \ : \ E \subset V(G),\ 0 < \nu_G(E) \leqslant \frac{1}{2}\nu_G(V) \right\}.$$

This minimization problem is closely related with the *balance graph cut problem* that appears in Machine Learning Theory (see [**87, 88**]).

We will now give a variational characterization of the Cheeger constant (similar to the one given for $h_1^m(\Omega)$ in Theorem 3.39) which generalizes the one obtained in [**148**] *for the particular case of finite graphs. We first recall the following definition.*

Definition 3.62. Let $(X,\mathcal{B},\nu)$ be a probability space and let $u : X \to \mathbb{R}$ be a measurable function. A number $\mu \in \mathbb{R}$ is a *median* of u (with respect to ν) if

$$\nu(\{x \in X \ : \ u(x) < \mu\}) \leqslant \frac{1}{2} \quad \text{and} \quad \nu(\{x \in X \ : \ u(x) > \mu\}) \leqslant \frac{1}{2}.$$

We denote by $\operatorname{med}_\nu(u)$ the set of all medians of u.

Remark 3.63. It is easy to see that

$$\mu \in \operatorname{med}_\nu(u) \Leftrightarrow -\nu(\{u = \mu\}) \leqslant \nu(\{u > \mu\}) - \nu(\{u < \mu\}) \leqslant \nu(\{u = \mu\}),$$

from this it follows that

$$0 \in \operatorname{med}_\nu(u) \Leftrightarrow \exists \xi \in \operatorname{sign}(u) \text{ such that } \int_X \xi(x) d\nu(x) = 0, \tag{3.46}$$

where $\operatorname{sign}(u)$ is the multivalued sign function (see (3.5)). Moreover, it follows that

$$\arg\min \left\{ \int_X |u-c| d\nu \ : \ c \in \mathbb{R} \right\} = \operatorname{med}_\nu(u). \tag{3.47}$$

Indeed, let $m \in \operatorname{med}_\nu(u)$. We may suppose that $m = 0$ (otherwise, consider $u' = u - m$). Let's see that

$$\int_X |u-c| d\nu \geqslant \int_X |u| d\nu$$

for every $c \geqslant 0$ (it then follows for any $c < 0$ by taking $u' := -u$ and $c' := -c$). First note that

$$(|u-c| - |u|)\chi_{\{u \leqslant 0\}} = c\chi_{\{u \leqslant 0\}}$$

and

$$(|u-c| - |u|)\chi_{\{u > 0\}} \geqslant -c\chi_{\{u > 0\}}$$

thus, adding and integrating these equations over X with respect to ν yields

$$\int_X |u-c| d\nu - \int_X |u| d\nu \geqslant c(\nu(\{u \leqslant 0\}) - \nu(\{u > 0\})) = c(1 - 2\nu(\{u > 0\})) \geqslant 0.$$

Definition 3.64. Let $[X,\mathcal{B},m,\nu]$ be a random walk space. We denote

$$\Pi(X) := \left\{ u \in L^1(X,\nu) \ : \ \|u\|_{L^1(X,\nu)} = 1 \text{ and } 0 \in \operatorname{med}_\nu(u) \right\}$$

and

$$\lambda_1^m(X) := \inf \left\{ TV_m(u) \ : \ u \in \Pi(X) \right\}. \tag{3.48}$$

Theorem 3.65. *If $[X,\mathcal{B},m,\nu]$ is a reversible random walk space and ν is a probability measure, then*

$$h_m(X) = \lambda_1^m(X).$$

Proof. If $D \in \mathcal{B}$ satisfies $0 < \nu(D) \leqslant \frac{1}{2}$, then $0 \in \operatorname{med}_\nu(\chi_D)$. Therefore,

$$\lambda_1^m(X) \leqslant TV_m\left(\frac{1}{\nu(D)}\chi_D\right) = \frac{1}{\nu(D)} P_m(D),$$

thus

$$\lambda_1^m(X) \leqslant h_m(X).$$

Now, for the opposite inequality, let $u \in L^1(X,\nu)$ such that $\|u\|_{L^1(X,\nu)} = 1$ and $0 \in \text{med}_\nu(u)$. Since $0 \in \text{med}_\nu(u)$, by the coarea formula (Theorem 3.5), and having in mind that the set $\{t \in \mathbb{R} \ : \ \nu(\{u = t\}) > 0\}$ is countable, we obtain that

$$\begin{aligned} TV_m(u) &= \int_{-\infty}^{+\infty} P_m(E_t(u))\,dt = \int_0^{+\infty} P_m(E_t(u))\,dt + \int_{-\infty}^0 P_m(X\setminus E_t(u))\,dt \\ &\geqslant h_m(X)\int_0^{+\infty} \nu(E_t(u))\,dt + h_m(X)\int_{-\infty}^0 \nu(X\setminus E_t(u))\,dt \\ &= h_m(X)\left(\int_X u^+(x)d\nu(x) + \int_X u^-(x)d\nu(x)\right) = h_m(X)\|u\|_{L^1(X,\nu)} = h_m(X). \end{aligned}$$

Therefore, taking the infimum over u, we get $\lambda_1^m(X) \geqslant h_m(X)$. □

Following [**63**] *and using Theorem 3.65, the next result shows that the Cheeger inequality* (3.44) *also holds in our context.*

Theorem 3.66. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and assume that ν is a probability measure. The following Cheeger inequality holds:*

$$\frac{(h_m(X))^2}{2} \leqslant \text{gap}(-\Delta_m) \leqslant 2h_m(X).$$

Proof. Let $(f_n) \subset L^2(X,\nu)$ such that $\nu(f_n) = 0$ and

$$\lim_{n\to\infty} \frac{\mathcal{H}_m(f_n)}{\|f_n\|_2^2} = \text{gap}(-\Delta_m).$$

If we take $\mu_n \in \text{med}_\nu(f_n)$, we have

$$\begin{aligned} 2\mathcal{H}_m(f_n) &= \int_X\int_X (f_n(y) - \mu_n - (f_n(x) - \mu_n))^2 dm_x(y)d\nu(x) \\ &= \int_X\int_X \left[(f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+ - ((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-)\right]^2 dm_x(y)d\nu(x) \\ &= \int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x) \\ &\quad + \int_X\int_X \left((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-\right)^2 dm_x(y)d\nu(x) \\ &\quad -2\int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)\left((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-\right) dm_x(y)d\nu(x). \end{aligned}$$

Now, an easy calculation gives

$$-\int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)\left((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-\right) dm_x(y)d\nu(x) \geqslant 0.$$

On the other hand, since $\nu(f_n) = 0$, we have

$$\int_X f_n^2(x)d\nu(x) \leqslant \int_X (f_n(x)-\mu_n)^2 d\nu(x).$$

Therefore,

$$\frac{2\mathcal{H}_m(f_n)}{\|f_n\|_2^2} \geqslant \frac{\displaystyle\int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x)}{\displaystyle\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x)}$$

$$+\frac{\displaystyle\int_X\int_X \left((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-\right)^2 dm_x(y)d\nu(x)}{\displaystyle\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x)}.$$

Having in mind that

$$\frac{a+b}{c+d} \geqslant \min\left\{\frac{a}{c}, \frac{b}{d}\right\} \quad \text{for every } a,b,c,d \in \mathbb{R}^+,$$

and

$$\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x) > 0,$$

we can assume, without loss of generality, that

$$\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) > 0,$$

and that

$$\frac{2\mathcal{H}_m(f_n)}{\|f_n\|_2^2} \geqslant \frac{\displaystyle\int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x)}{\displaystyle\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x)}.$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\int_X\int_X \left|\left((f_n(y)-\mu_n)^+\right)^2 - \left((f_n(x)-\mu_n)^+\right)^2\right| dm_x(y)d\nu(x) \\
&= \int_X\int_X \left|(f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right| \left|(f_n(y)-\mu_n)^+ + (f_n(x)-\mu_n)^+\right| dm_x(y)d\nu(x) \\
&\leqslant \left(\int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x)\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_X\int_X \left((f_n(y)-\mu_n)^+ + (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, by the invariance of ν with respect to m,

$$\int_X\int_X \left((f_n(y)-\mu_n)^+ + (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x) \leqslant 4\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x).$$

Thus,

$$\frac{2\mathcal{H}_m(f_n)}{\|f_n\|_{L^2(X,\nu)}^2} \geqslant \left(\frac{\displaystyle\frac{1}{2}\int_X\int_X \left|\left((f_n(y)-\mu_n)^+\right)^2 - \left((f_n(x)-\mu_n)^+\right)^2\right| dm_x(y)d\nu(x)}{\displaystyle\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x)}\right)^2.$$

Then, since $0 \in \operatorname{med}_\nu\left(((f_n-\mu_n)^+)^2\right)$, by Theorem 3.65, we get

$$(h_m(X))^2 \leqslant \frac{2\mathcal{H}_m(f_n)}{\|f_n\|_{L^2(X,\nu)}^2},$$

and, consequently, taking limits as $n \to \infty$, we obtain

$$\frac{(h_m(X))^2}{2} \leqslant \operatorname{gap}(-\Delta_m).$$

To prove the other inequality we assume that $\operatorname{gap}(-\Delta_m) > 0$. Now, by (1.20), we have

$$\min\{\nu(D), 1-\nu(D)\} \leqslant \frac{2}{\operatorname{gap}(-\Delta_m)} P_m(D) \quad \text{for all } D \in \mathcal{B},\ 0 < \nu(D) < 1,$$

and it follows that $\operatorname{gap}(-\Delta_m) \leqslant 2h_m(X)$. □

Let $A \in \mathcal{B}$ with $\nu(A) = \frac{1}{2}$ and $u = \chi_A - \chi_{X\setminus A}$. It is easy to see that $TV_m(u) = 2P_m(A)$ and $\mathcal{H}_m(u) = 4P_m(A)$. Hence, since $\|u\|_{L^1(X,\nu)} = \|u\|_{L^2(X,\nu)} = 1$, $\nu(u) = 0$ and $0 \in \operatorname{med}_\nu(u)$, we obtain the following result as a consequence of Theorem 3.65.

Corollary 3.67. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $A \in \mathcal{B}$ with $\nu(A) = \frac{1}{2}$ and $u = \chi_A - \chi_{X\setminus A}$. Then,*

1. $h_m(X) = \dfrac{P_m(A)}{\nu(A)} \Leftrightarrow u = \chi_A - \chi_{X\setminus A}$ *is a minimizer of* (3.48).
2. *u is a minimizer of* (3.48) *and* $\operatorname{gap}(-\Delta_m) = 2h_m(X) \Leftrightarrow u$ *is a minimizer of* (1.13).

Bringing together all the above results we have:

Theorem 3.68. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. The following statements are equivalent:*

(a) $[X, \mathcal{B}, m, \nu]$ *satisfies a Poincaré inequality,*

(b) $\operatorname{gap}(-\Delta_m) > 0$,

(c) $[X, \mathcal{B}, m, \nu]$ *satisfies an isoperimetric inequality,*

(d) $h_m(X) > 0$.

Example 3.69. It is well known (see for instance [**63**]), that for a finite weighted discrete graph G, $h_m(G) > 0$ if, and only if, G is connected. This result is not true for infinite graphs. In fact, the graph given in Example 1.77 is connected and its Cheeger constant is zero (since $\operatorname{gap}(-\Delta_m) = 0$).

3.6.1. The m-Cheeger constant and the m-eigenvalues of Δ_1^m. *In this section we will study the relations between the non-null m-eigenvalues of the 1-Laplacian and the m-Cheeger constant $h_m(X)$. We will suppose that ν is a probability measure. This does not entail a loss of generality since, if ν is finite, $\lambda_D^m = \frac{P_m(D)}{\nu(D)}$ remains unchanged if we normalise to a probability measure, and the same is true for the m-eigenvalues of the 1-Laplacian.*

Proposition 3.70. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let (λ, u) be an m-eigenpair of Δ_1^m. Then,*
(i) $\lambda = 0 \Leftrightarrow u$ *is ν-a.e. a constant, that is, $u = 1$ or $u = -1$ ν-a.e.*
(ii) $\lambda \neq 0 \Leftrightarrow$ *there exists $\xi \in \operatorname{sign}(u)$ such that* $\displaystyle\int_X \xi(x) d\nu(x) = 0$.

Observe that $(0, 1)$ and $(0, -1)$ are m-eigenpairs of the 1-Laplacian in reversible random walk spaces.

Proof. (i) By (3.40), if $\lambda = 0$, we have that $TV_m(u) = 0$ and then, by Lemma 3.6, we get that u is ν-a.e. a constant, thus, since $\|u\|_{L^1(X,\nu)} = 1$, either $u = 1$ or $u = -1$ ν-a.e.. Similarly, if u is ν-a.e. a constant then $TV_m(u) = 0$ and, by (3.40), $\lambda = 0$.
(ii) ($\Leftarrow$) If $\lambda = 0$, by (i), we have that $u = 1$ or $u = -1$ ν-a.e., and this is in contradiction with the existence of $\xi \in \operatorname{sign}(u)$ such that $\int_X \xi(x) d\nu(x) = 0$.

($\Rightarrow$) There exists $\xi \in \text{sign}(u)$ and $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric satisfying (3.39) and $\|\mathbf{g}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$. Hence, since $\mathbf{g}$ is antisymmetric, by the reversibility of ν with respect to m, we have

$$\lambda \int_X \xi(x) d\nu(x) = -\int_X \int_X \mathbf{g}(x,y)\, dm_x(y) d\nu(x) = 0.$$

Therefore, since $\lambda \neq 0$,

$$\int_X \xi(x) d\nu(x) = 0. \qquad \square$$

Proposition 3.70 and equation (3.46) yield the following result which was obtained, for finite graphs, by Hein and Bühler in [**99**].

COROLLARY 3.71. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. If (λ, u) is an m-eigenpair of Δ_1^m then*

$$\lambda \neq 0 \ \Leftrightarrow\ 0 \in \text{med}_\nu(u).$$

Observe that, by this corollary, if $\lambda \neq 0$ is an m-eigenvalue of Δ_1^m, then there exists an m-eigenvector u associated with λ such that its 0*-superlevel set $E_0(u)$ has positive ν-measure. In fact, for any m-eigenvector u associated with λ, either u or $-u$ will satisfy this condition.*

PROPOSITION 3.72. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $t \geqslant 0$. If (λ, u) is an m-eigenpair of Δ_1^m with $\lambda > 0$ and $\nu(E_t(u)) > 0$, then $\left(\lambda, \frac{1}{\nu(E_t(u))}\chi_{E_t(u)}\right)$ is an m-eigenpair of Δ_1^m, $\lambda = \lambda^m_{E_t(u)}$ and $E_t(u)$ is m-calibrable. Moreover, $\nu(E_t(u)) \leqslant \frac{1}{2}$.*

PROOF. First observe that, by Corollary 3.71, we have that $\nu(E_0(u)) \leqslant \frac{1}{2}$, thus $\nu(E_t(u)) \leqslant \frac{1}{2}$ for every $t \geqslant 0$. Moreover, since (λ, u) is an m-eigenpair, there exists $\xi \in \text{sign}(u)$ such that

$$-\lambda\xi \in \Delta_1^m u;$$

hence, there exists $\mathbf{g}(x,y) \in \text{sign}(u(y) - u(x))$ antisymmetric such that

$$-\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda\xi(x) \quad \text{for } \nu\text{-a.e. } x \in X.$$

Let $t \geqslant 0$ such that $\nu(E_t(u)) > 0$. Then,

$$\xi(x) = \begin{cases} 1 & \text{if } x \in E_t(u) \text{ (since } u(x) > t \geqslant 0 \text{ and } \xi \in \text{sign}(u)), \\ \in [-1,1] & \text{if } x \in X \setminus E_t(u), \end{cases}$$

and, therefore, $\xi \in \text{sign}(\chi_{E_t(u)})$. On the other hand,

$$\mathbf{g}(x,y) = \begin{cases} \in [-1,1] & \text{if } x, y \in E_t(u), \\ -1 & \text{if } x \in E_t(u),\ y \in X \setminus E_t(u) \text{ (since } u(x) > t \geqslant u(y)), \\ 1 & \text{if } x \in X \setminus E_t(u),\ y \in E_t(u) \text{ (since } u(y) > t \geqslant u(x)), \\ \in [-1,1] & \text{if } x, y \in X \setminus E_t(u), \end{cases}$$

and, consequently, $\mathbf{g}(x,y) \in \text{sign}(\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x))$. Therefore, $\left(\lambda, \frac{1}{\nu(E_t(u))}\chi_{E_t(u)}\right)$ is an m-eigenpair of Δ_1^m. Moreover, by Theorem 3.56, we have that $E_t(u)$ is m-calibrable. $\square$

REMARK 3.73. As a consequence of Proposition 3.50, when we search for m-eigenvalues of the 1-Laplacian we can restrict ourselves to m-eigenpairs of the form $\left(\lambda, \frac{1}{\nu(E)}\chi_E\right)$ where E is m-calibrable and not decomposable as $E = E_1 \sqcup_m E_2$ (recall Definition 1.33). Indeed, suppose that $\left(\lambda, \frac{1}{\nu(E)}\chi_E\right)$ is an m-eigenpair and $E = E_1 \sqcup_m E_2$ for some $E_1, E_2 \in \mathcal{B}_E$. Then, by (3.40), there exist $\xi \in \operatorname{sign}(\chi_E)$ and $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric such that

$$\begin{cases} -\displaystyle\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda\xi(x) \quad \text{for } \nu\text{-a.e. } x \in X, \\ \mathbf{g}(x,y) \in \operatorname{sign}(\chi_E(y) - \chi_E(x)) \quad \text{for } \nu \otimes m_x\text{-a.e. } (x,y) \in X \times X. \end{cases}$$

Then, we may take the same ξ and $\mathbf{g}(x,y)$ to see that $\left(\lambda, \frac{1}{\nu(E_1)}\chi_{E_1}\right)$ is also an m-eigenpair of Δ_1^m. Indeed, since $\lambda_E^m = \lambda_{E_1}^m$, we only need to verify that $\mathbf{g}(x,y) \in \operatorname{sign}(\chi_{E_1}(y) - \chi_{E_1}(x))$ $\nu \otimes m_x$-a.e.. For $x \in E_1$ we have:

- if $y \in E_1$, then $\chi_E(y) - \chi_E(x) = 0 = \chi_{E_1}(y) - \chi_{E_1}(x)$,
- if $y \in X\backslash E$, then $\chi_E(y) - \chi_E(x) = -1 = \chi_{E_1}(y) - \chi_{E_1}(x)$,

and, since $L_m(E_1, E_2) = 0$, we have that $\nu \otimes m_x(E_1 \times E_2) = 0$ so the condition is satisfied. Similarly for $x \in E_2$ (again $\nu \otimes m_x(E_2 \times E_1) = 0$). If $x \in X\backslash E$ then,

- if $y \in E_1$, $\chi_E(y) - \chi_E(x) = 1 = \chi_{E_1}(y) - \chi_{E_1}(x)$,
- if $y \in E_2$, $\chi_E(y) - \chi_E(x) = 1 \in \operatorname{sign}(0) = \operatorname{sign}(\chi_{E_1}(y) - \chi_{E_1}(x))$
- if $y \in X\backslash E$, $\chi_E(y) - \chi_E(x) = 0 = \chi_{E_1}(y) - \chi_{E_1}(x)$.

By Corollary 3.71, if (λ, u) is an m-eigenpair of Δ_1^m and $\lambda \neq 0$ then $u \in \Pi(X)$ (recall Definition 3.64). Now, $TV_m(u) = \lambda$ (see Remark 3.52(4)), thus, as a corollary of Theorem 3.65, we have the following result. Recall that, for finite graphs, it is well known that the first non–zero eigenvalue coincides with the Cheeger constant (see [**57**]*).*

THEOREM 3.74. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. If $\lambda \neq 0$ is an m-eigenvalue of Δ_1^m then*

$$h_m(X) \leqslant \lambda.$$

This result also follows by Proposition 3.72 since $\nu(E_0(u)) \leqslant \frac{1}{2}$.

In the next result we will see that if the infimum in (3.45) is attained then $h_m(X)$ is an m-eigenvalue of Δ_1^m.

THEOREM 3.75. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$ such that $0 < \nu(\Omega) \leqslant \frac{1}{2}$.*

(i) If Ω and $X\backslash\Omega$ are m-calibrable, then $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of Δ_1^m.

(ii) If $h_m(X) = \lambda_\Omega^m$, then Ω and $X\backslash\Omega$ are m-calibrable.

Therefore,

(iii) if $h_m(X) = \lambda_\Omega^m$, then $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of Δ_1^m.

PROOF. First of all, observe that, since $\nu(\Omega) \leqslant \frac{1}{2}$,

$$\lambda_{X\backslash\Omega}^m \leqslant \lambda_\Omega^m. \tag{3.49}$$

(i): By Theorem 3.43, since Ω is m-calibrable, there exists an antisymmetric function $\mathbf{g}_1$ in $\Omega \times \Omega$ such that

$$-1 \leqslant \mathbf{g}_1(x,y) \leqslant 1 \qquad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in \Omega \times \Omega,$$

and

$$\lambda_\Omega^m = -\int_\Omega \mathbf{g}_1(x,y)\, dm_x(y) + 1 - m_x(\Omega) \quad \text{for } \nu\text{-a.e. } x \in \Omega; \tag{3.50}$$

and, since $X\backslash\Omega$ is m-calibrable, there exists an antisymmetric function $\mathbf{g}_2$ in $(X\backslash\Omega) \times (X\backslash\Omega)$ such that

$$-1 \leqslant \mathbf{g}_2(x,y) \leqslant 1 \qquad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in (X\backslash\Omega) \times (X\backslash\Omega),$$

and

$$\lambda^m_{X\setminus\Omega} = -\int_{X\setminus\Omega} \mathbf{g}_2(x,y)\,dm_x(y) + 1 - m_x(X\setminus\Omega) \quad \text{for } \nu\text{-a.e. } x \in X\setminus\Omega. \tag{3.51}$$

Consequently, by taking

$$\mathbf{g}(x,y) = \begin{cases} \mathbf{g}_1(x,y) & \text{if } x,y \in \Omega, \\ -1 & \text{if } x \in \Omega, y \in X\setminus\Omega, \\ 1 & \text{if } x \in X\setminus\Omega, y \in \Omega, \\ -\mathbf{g}_2(x,y) & \text{if } x,y \in X\setminus\Omega, \end{cases}$$

we have that $\mathbf{g}(x,y) \in \operatorname{sign}(\chi_\Omega(y) - \chi_\Omega(x))$. Moreover, from (3.50),

$$\lambda^m_\Omega = -\int_X \mathbf{g}(x,y)\,dm_x(y) \quad \text{for } \nu\text{-a.e. } x \in \Omega,$$

and, since $\lambda^m_{X\setminus\Omega} \leqslant \lambda^m_\Omega$, from (3.51),

$$-\lambda^m_\Omega \leqslant -\lambda^m_{X\setminus\Omega} = -\int_X \mathbf{g}(x,y)\,dm_x(y) \leqslant \lambda^m_\Omega \quad \text{for } \nu\text{-a.e. } x \in X\setminus\Omega.$$

Hence, by Remark 3.52 (2), we conclude that $\left(\lambda^m_\Omega, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of Δ^m_1.

(ii): Since $h_m(X) = \frac{P_m(\Omega)}{\nu(\Omega)}$ and $0 < \nu(\Omega) \leqslant \frac{1}{2}$, we have $h_m(X) = h^m_1(\Omega) = \frac{P_m(\Omega)}{\nu(\Omega)}$ and, consequently, Ω is m-calibrable. Let us suppose that $X\setminus\Omega$ is not m-calibrable. Then, there exists $E \in \mathcal{B}_{X\setminus\Omega}$ such that $\nu(E) < \nu(X\setminus\Omega)$ and

$$\lambda^m_E < \lambda^m_{X\setminus\Omega}\ .$$

Now, this implies that $\nu(E) > \frac{1}{2}$ since, otherwise (recall (3.49)), we get

$$h_m(X) \leqslant \lambda^m_E < \lambda^m_{X\setminus\Omega} \leqslant \lambda^m_\Omega = h_m(X)$$

which is a contradiction.

Moreover, since $\nu(E) < \nu(X\setminus\Omega)$, $\lambda^m_E < \lambda^m_{X\setminus\Omega}$ also implies that

$$P_m(E) < P_m(X\setminus\Omega) = P_m(\Omega).$$

However, since $\nu(E) > \frac{1}{2}$, we have that $\nu(X\setminus E) \leqslant \frac{1}{2}$ and, consequently, taking into account that $\nu(\Omega) \leqslant \nu(X\setminus E)$, we get

$$\lambda^m_{X\setminus E} = \frac{P_m(E)}{\nu(X\setminus E)} < \frac{P_m(\Omega)}{\nu(\Omega)} = h_m(X),$$

which is also a contradiction.

Finally, *(iii)* is a direct consequence of *(i)* and *(ii)*. □

As a consequence of Proposition 3.72 and Theorem 3.75, we have the following result.

Corollary 3.76. *Let $[X,\mathcal{B},m,\nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. If $h_m(X)$ is a positive m-eigenvalue of Δ^m_1, then, for any eigenvector u associated with $h_m(X)$ and any $t \geqslant 0$ such that $\nu(E_t(u)) > 0$,*

$$\left(h_m(X), \tfrac{1}{\nu(E_t(u))}\chi_{E_t(u)}\right) \text{ is an } m\text{-eigenpair of } \Delta^m_1,$$

$\nu(E_t(u)) \leqslant \frac{1}{2}$, and

$$h_m(X) = \lambda^m_{E_t(u)}.$$

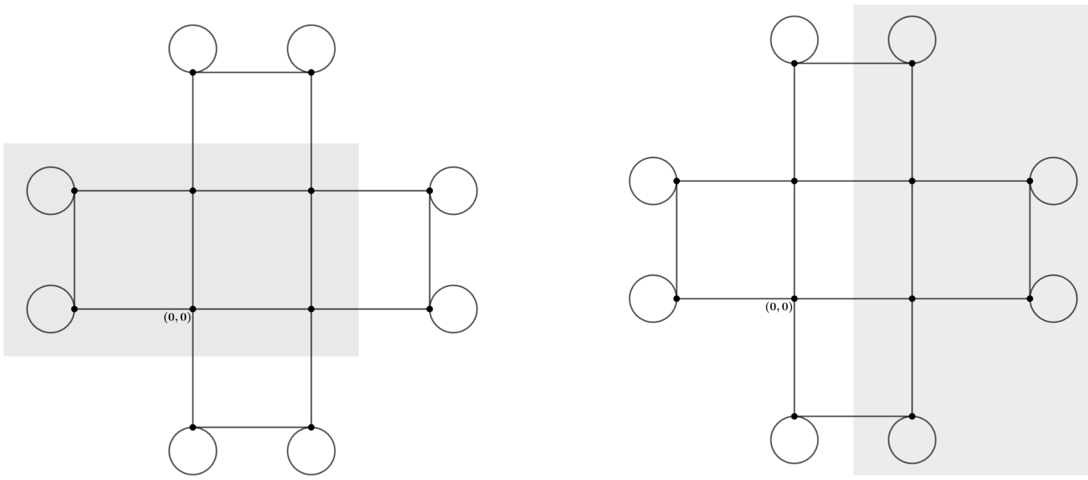
Moreover, both $E_t(u)$ and $X\setminus E_t(u)$ are m-calibrable.

Remark 3.77. For $\Omega \in \mathcal{B}$ with $\nu(\Omega) = \frac{1}{2}$ (thus $\lambda^m_\Omega = 2P_m(\Omega)$) we have that:

(1) Ω and $X\backslash\Omega$ are m-calibrable if, and only if, $\left(2P_m(\Omega), t\chi_\Omega - (2-t)\chi_{X\backslash\Omega}\right)$ is an m-eigenpair of Δ_1^m for any $t \in [0,2]$.

(2) If $h_m(X) = 2P_m(\Omega)$ then $\left(2P_m(\Omega), t\chi_\Omega - (2-t)\chi_{X\backslash\Omega}\right)$ is an m-eigenpair of Δ_1^m for all $t \in [0,2]$.

EXAMPLE 3.78. In Figure 2, following the notation in Example 3.57(2), we consider the metric random walk space $[(\Omega_2)_m, d_{\mathbb{Z}^2}, m_2, \nu_{\mathbb{Z}^2} \llcorner (\Omega_2)_m]$. In Figure 2(A), we show this space partitioned into the two m_2-calibrable sets $E = \{(-1,0),(0,0),(1,0),(-1,1),(0,1),(1,1)\}$ and $X\backslash E$, of equal measure. Hence, by the previous remark, both $(\lambda_E^{m_2}, \frac{1}{\nu(E)}\chi_E)$ and $(\lambda_E^{m_2}, \frac{1}{\nu(E)}\chi_{X\backslash E})$ are m_2-eigenpairs of $\Delta_1^{m_2}$. However, the Cheeger constant $h_{m_2}(X)$ is smaller than the eigenvalue $\lambda_E^{m_2}$ since, for $D = \{(1,-1),(1,0),(2,0),(2,1),(1,1),(1,2)\}$, we have $\lambda_D^{m_2} = \frac{1}{6}$ (see Figure 2(B)).



(A) Let E be the set formed by the vertices inside the shaded region. Then, $\lambda_E^{m_2} = \frac{1}{4}$.

(B) Let D be the set formed by the vertices inside the shaded region. Then, $\lambda_D^{m_2} = \frac{1}{6}$.

FIGURE 2. The line segments represented in the figures correspond to the edges (recall that $w_{xy} = 1$ if $d_{\mathbb{Z}^2}(x,y) = 1$). The loops that "appear" when considering m_2 are represented by circles.

REMARK 3.79. By Theorems 3.74 and 3.75, and Corollary 3.76, for finite connected weighted discrete graphs, we have that

$$h_m(X) \text{ is the first non-zero eigenvalue of } \Delta_1^{m^G} \tag{3.52}$$

(as already proved in [**57**], [**58**], and [**99**]). Then, to solve the optimal Cheeger cut problem, it is enough to find an eigenvector associated with $h_m(X)$, since then $\{E_0(u), X\backslash E_0(u)\}$ or $\{E_0(-u), X\backslash E_0(-u)\}$ is a Cheeger cut.

In the next examples we will see that (3.52) *is not true in general. We obtain infinite connected weighted discrete graphs (with finite invariant and reversible measure) for which there is no first positive m-eigenvalue.*

EXAMPLE 3.80. (1) Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated to the finite weighted discrete graph $G = (V(G), E(G))$ with vertex set $V(G) = \{x_0, x_1, \ldots, x_n, \ldots\}$ and weights defined as follows:

$$w_{x_{2n}x_{2n+1}} = \frac{1}{2^n}, \quad w_{x_{2n+1}x_{2n+2}} = \frac{1}{3^n} \quad \text{for } n = 0,1,2,\ldots \text{ and } w_{x,y} = 0 \text{ otherwise.}$$

We have $d_{x_0} = 1$, $d_{x_1} = 2$ and, for $n \geqslant 1$,

$$d_{x_{2n}} = w_{x_{2n-1}x_{2n}} + w_{x_{2n}x_{2n+1}} = \frac{1}{3^{n-1}} + \frac{1}{2^n}$$

and
$$d_{x_{2n+1}} = w_{x_{2n}x_{2n+1}} + w_{x_{2n+1}x_{2n+2}} = \frac{1}{2^n} + \frac{1}{3^n}.$$
Furthermore,
$$\nu_G(V(G)) = \sum_{i=0}^{\infty} d_{x_i} = 3 + \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} + \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{3^n} = 7.$$
Observe that the measure ν_G is not normalized to a probability measure, but this does not affect the result because the constants λ_Ω^m and the m-eigenvalues of the 1-Laplacian are independent of this normalization.

Consider $E_n := \{x_{2n}, x_{2n+1}\}$ for $n \geqslant 1$. By Remark 3.35*(2)*, we have that E_n is m^G-calibrable. On the other hand,
$$m_{x_{2n-1}}(E_n) = \frac{1}{1 + (\frac{3}{2})^{n-1}}\ , \quad m_{x_{2n+2}}(E_n) = \frac{1}{1 + \frac{3}{4}(\frac{3}{2})^{n-1}} = \lambda_{E_n}^{m^G}$$
and $m_x(E_n) = 0$ otherwise in $V(G)\backslash E_n$. Hence,
$$m_x(E_n) \leqslant \lambda_{E_n}^{m^G} \quad \text{for all } x \in V(G)\backslash E_n.$$
Then, by Theorem 3.56, we have that $(\lambda_{E_n}^{m^G}, \frac{1}{\nu(E_n)}\chi_{E_n})$ is an m^G-eigenpair of $\Delta_1^{m^G}$. Now,
$$\lim_{n\to\infty} \lambda_{E_n}^{m^G} = \lim_{n\to\infty} \frac{2^{n+1}}{2^{n+1} + 3^n} = 0.$$
Consequently, both by Theorem 3.74 and by the definition of $h_{m^G}(V(G))$, we get
$$h_{m^G}(V(G)) = 0.$$
(2) Let $0 < s < r < \frac{1}{2}$. Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space defined in Example 1.38 with vertex set $V(G) = \{x_0, x_1, \ldots, x_n, \ldots\}$ and weights defined as follows:
$$w_{x_0,x_1} = \frac{r}{1-r} + \frac{s}{1-s}, \quad w_{x_n x_{n+1}} = r^n + s^n \quad \text{for } n = 1, 2, 3, \ldots \text{ and } w_{x,y} = 0 \text{ otherwise.}$$
Then,
$$h_{m^G}(V(G)) = \frac{1-r}{1+r} \ \text{ is not an } m^G\text{-eigenvalue of } \Delta_1^{m^G}.$$
Indeed, to start with, observe that $\nu_G(V(G)) = \frac{4r}{1-r} + \frac{4s}{1-s}$,
$$\nu_G(\{x_0\}) \leqslant \frac{\nu_G(V(G))}{2},\ \nu_G(\{x_0, x_1\}) > \frac{\nu_G(V(G))}{2},$$
$$\nu_G(\{x_1\}) \leqslant \frac{\nu_G(V(G))}{2},\ \nu_G(\{x_1, x_2\}) > \frac{\nu_G(V(G))}{2},$$
and, for $E_n := \{x_n, x_{n+1}, x_{n+2}, \ldots\}$, $n \geqslant 2$,
$$\nu_G(E_n) \leqslant \frac{\nu_G(V(G))}{2}.$$
Now, for $n \geqslant 2$,
$$\lambda_{E_n}^m = \frac{r^{n-1} + s^{n-1}}{r^{n-1} + s^{n-1} + 2\left(\frac{r^n}{1-r} + \frac{s^n}{1-s}\right)} = \frac{r^{n-1} + s^{n-1}}{\frac{1+r}{1-r}r^{n-1} + \frac{1+s}{1-s}s^{n-1}}$$
decreases as n increases (therefore, the sets E_n are not m-calibrable), and
$$\lim_n \lambda_{E_n}^m = \frac{1-r}{1+r}.$$
Let us see that, for any $E \subset V(G)$ with $0 < \nu_G(E) \leqslant \frac{\nu(V(G))}{2}$, we have $\lambda_E^m > \frac{1-r}{1+r}$. Indeed, to start with, observe that if $E = \{x_0\}$ or $E = \{x_1\}$ then $\lambda_{\{x_0\}}^m = \lambda_{\{x_1\}}^m = 1 > \frac{1-r}{1+r}$. Moreover, we have that $\{x_0, x_1\} \not\subset E$ and $\{x_1, x_2\} \not\subset E$ since $\nu_G(\{x_0, x_1\}) \nleqslant \frac{\nu_G(V(G))}{2}$ and $\nu_G(\{x_1, x_2\}) \nleqslant \frac{\nu_G(V(G))}{2}$. Therefore, it remains to be seen what happens for sets E satisfying

(i) $x_0 \in E$, $x_1 \notin E$ and $x_n \in E$ for some $n \geqslant 2$,
(ii) $x_1 \in E$, $x_0 \notin E$ and $x_n \in E$ for some $n \geqslant 3$,
(iii) $x_0 \notin E$, $x_1 \notin E$ and $x_n \in E$ for some $n \geqslant 2$.
For the case (i), let $n_1 \in \mathbb{N}$ be the first index $n \geqslant 2$ such that $x_n \in E$; for the case (ii), let $n_2 \in \mathbb{N}$ be the first index $n \geqslant 3$ such that $x_n \in E$; and for the case (iii), let $n_3 \in \mathbb{N}$ be the first index $n \geqslant 2$ such that $x_n \in E$. Now, for the case (i) we have that

$$\lambda_E^m \geqslant \lambda_{\{x_0\}\cup E_{n_1}} \geqslant \lambda_{E_{n_1}}.$$

Indeed, the first equality follows from the fact that $P_m(E) \geqslant P_m(\{x_0\} \cup E_{n_1})$ and $\nu(E) \leqslant \nu(\{x_0\} \cup E_{n_1})$ and the second one follows since

$$\lambda_{\{x_0\}\cup E_{n_1}} = \frac{\frac{r}{1-r} + \frac{s}{1-s} + P_m(E_{n_1})}{\frac{r}{1-r} + \frac{s}{1-s} + \nu(E_{n_1})} > \frac{P_m(E_{n_1})}{\nu(E_{n_1})} = \lambda_{E_{n_1}}.$$

Hence, $\lambda_E^m > \frac{1-r}{1+r}$. With a similar argument we get, in the case (ii),

$$\lambda_E^m \geqslant \lambda_{\{x_1\}\cup E_{n_2}} \geqslant \lambda_{E_{n_2}} > \frac{1-r}{1+r};$$

and, in the case (iii),

$$\lambda_E^m \geqslant \lambda_{E_{n_3}} > \frac{1-r}{1+r}.$$

Consequently, $h_{m^G}(V(G)) = \frac{1-r}{1+r}$ and, by Corollary 3.76, it is not an m-eigenvalue of $\Delta_1^{m^G}$.

CHAPTER 4

(BV, L^p)-*decomposition,* $p = 1, 2$*, of functions in random walk spaces*

Let us recall the classic problem in image restoration. Given a noisy/corrupted image $f : \Omega \to \mathbb{R}$ on, for example, a rectangle Ω in $\mathbb{R}^2$, the aim is to remove the noise or corruption in order to obtain the desired "clean" image $u : \Omega \to \mathbb{R}$, which is related to the original one by

$$f = u + n,$$

*when n is the additive noise. Unfortunately, the problem of recovering u from f is ill-posed (see [**12**]). To handle this problem, Rudin, Osher and Fatemi (see [**143**]) proposed to solve the following constrained minimization problem over $BV(\Omega)$:*

$$\text{(4.1)} \qquad \textit{Minimize} \int_\Omega |Du| \quad \textit{subject to} \quad \int_\Omega u = \int_\Omega f \quad \textit{and} \quad \int_\Omega |u - f|^2 = \sigma^2.$$

The first constraint corresponds to the assumption that the noise has zero mean, and the second that its standard deviation is σ. Problem (4.1) is naturally linked to the following unconstrained problem (called the ROF-model):

$$\text{(4.2)} \qquad \min \left\{ \int_\Omega |Du| + \frac{\lambda}{2} \|u - f\|_2^2 \, : \, u \in BV(\Omega) \right\},$$

*for some Lagrange multiplier $\lambda > 0$. Chambolle and Lions ([**54**]) proved an existence and uniqueness result for (4.1), as well as the link between (4.1) and (4.2). The constant λ in (4.2) plays the role of a "scale parameter". By tweaking λ, we can select the level of detail desired in the reconstructed image.*

Following the ROF-model we obtain the following (BV, L^2)-decomposition of f:

$$\text{(4.3)} \qquad f = u_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] = \underset{(u,v) \in BV(\Omega) \times L^2(\Omega)}{\arg\min} \left\{ \int_\Omega |Du| + \frac{\lambda}{2} \|v\|_2^2 \, : \, f = u + v \right\}.$$

*An alternative variational problem arises when the L^2-fidelity term $\|f - u\|_2^2$ is replaced by the L^1-fidelity term $\|f - u\|_1$. This was proposed by Alliney (see [**3**] and [**4**]) in one dimensional spaces and was extensively studied by Chan, Esedoglu and Nikolova (see [**55**] and [**56**]):*

$$\text{(4.4)} \qquad f = u_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] \in \underset{(u,v) \in BV(\Omega) \times L^1(\Omega)}{\arg\min} \left\{ \int_\Omega |Du| + \lambda \|v\|_1 \, : \, f = u + v \right\}.$$

*The resulting (BV, L^1)-decomposition differs from the (BV, L^2)-decomposition in several important aspects which have attracted considerable attention in recent years (see [**19**], [**71**], [**78**], [**92**], [**162**] and the references therein). Let us point out that the (BV, L^1)-decomposition is contrast invariant (see [**55**]), as opposed to the (BV, L^2)-decomposition.*

*The use of neighborhood filters by Buades, Coll and Morel in [**47**], that was originally proposed by P. Yaroslavsky [**161**], has led to an extensive literature of nonlocal models in image processing (see, for instance, [**48**], [**49**], [**92**], [**109**], [**114**] and the references therein). This nonlocal ROF-model, in a simplified version, has the form*

$$\text{(4.5)} \qquad \min \left\{ \int_{\Omega \times \Omega} J(x - y)|u(x) - u(y)| dx dy + \frac{\lambda}{2} \|u - f\|_2^2 \; : \; u \in L^2(\Omega) \right\}.$$

On the other hand, an image can be seen as a weighted graph where the pixels are taken as the vertices, and the weights are related to the similarity between pixels. Depending on the problem there are different ways to define the weights, see for instance [**79**], [**101**], [**102**] *and* [**114**]. *The ROF-model in a weighted graph $G = (V(G), E(G))$ reads as follows:*

(4.6)

$$\min \left\{ \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in V(G)} |u(y) - u(x)| w_{xy} + \frac{\lambda}{2} \sum_{x \in V(G)} |u(x) - f(x)|^2 \sum_{y \sim x} w_{xy} \; : \; u \in L^2(G, \nu_G) \right\}.$$

Problems (4.5) *and* (4.6) *are particular cases of the following ROF-model in a random walk space $[X, \mathcal{B}, m, \nu]$:*

$$\min \left\{ \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) + \frac{\lambda}{2} \int_X |u(x) - f(x)|^2 d\nu(x) \; : \; u \in L^2(X, \nu) \right\},$$

which is one of the motivations for this chapter and we call the m-ROF-model.

Another problem in which we are interested is the (BV, L^1)-decomposition in a random walk space $[X, \mathcal{B}, m, \nu]$, that reads as

$$\min \left\{ \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) + \lambda \int_X |u(x) - f(x)| d\nu(x) \;\; : \;\; u \in L^1(X, \nu) \right\},$$

which has as a particular case the (BV, L^1)-decomposition in graphs.

The scale λ in the (BV, L^2)-decomposition (4.3) *is viewed as a parameter that dictates the separation of the scale decomposition $f = u_\lambda + v_\lambda$. Following Meyer* [**127**]: *"The first component u_λ is well structured and has a simple geometric description since it models the objects that are present in the image. The second component v_λ contains both the textured parts and the noise".*

In [**143**], *to solve problem* (4.1), *the gradient descent method was used, which required to solve numerically the parabolic problem*

$$\begin{cases} u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right) - \lambda(u - f) & \text{in } (0, \infty) \times \Omega, \\ \frac{Du}{|Du|} \eta = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = v_0(x) & \text{in } x \in \Omega. \end{cases} \tag{4.7}$$

Then, the denoised version of f is approached by the solution of (4.7) *as t increases. The concept of solution for which this problem is well-possed was given in* [**11**]. *We will see here that a non-local version of* (4.7) *can be used to approach the solutions of the ROF-model in the workspace of metric random walk spaces (see Theorem 4.11).*

Our aim is to study the (BV, L^p)-decomposition, $p = 1, 2$, of functions in random walk spaces, developing a general theory that can be applied, in particular, to weighted discrete graphs and nonlocal models.

4.1. The Rudin-Osher-Fatemi Model

Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space, $f \in L^2(X, \nu)$ and $\lambda > 0$. Our aim in this section is to study the m-ROF-model:

$$\min \left\{ TV_m(u) + \frac{\lambda}{2} \int_X |u(x) - f(x)|^2 d\nu(x) : \; u \in L^2(X, \nu) \right\}. \tag{4.8}$$

We will start by proving existence and uniqueness of the minimizer of problem (4.8) *as well as a characterization of this minimizer.*

Let us write

$$\mathcal{G}_m(u, f, \lambda) := TV_m(u) + \frac{\lambda}{2} \|u - f\|^2_{L^2(X,\nu)}, \quad u \in L^2(X, \nu).$$

Theorem 4.1. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. For any $f \in L^2(X, \nu)$ and $\lambda > 0$, there exists a unique minimizer u_λ of problem (4.8). Moreover, u_λ is the unique solution of the problem*

$$\lambda(u - f) \in \Delta_1^m(u). \tag{4.9}$$

Consequently, $u_\lambda \in L^2(X, \nu)$ is the solution of problem (4.8) if, and only if, there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric such that

$$\lambda(u_\lambda - f) = \mathrm{div}_m \mathbf{g} \tag{4.10}$$

and

$$\mathbf{g}(x, y) \in \mathrm{sign}(u_\lambda(y) - u_\lambda(x)) \quad \textit{for } (\nu \otimes m_x)\textit{-a.e. } (x, y) \in X \times X.$$

Proof. Let $f \in L^2(X, \nu)$ and let $\{u_n\}_{n\in\mathbb{N}} \subset L^2(X, \nu)$ be a minimizing sequence of problem (4.8), i.e.,

$$\alpha := \inf \left\{ \mathcal{G}_m(u, f, \lambda) \,:\, u \in L^2(X, \nu) \right\} = \lim_{n\to\infty} \mathcal{G}_m(u_n, f, \lambda).$$

Since

$$\|u_n\|^2_{L^2(X,\nu)} \leqslant 2 \left(\|u_n - f\|^2_{L^2(X,\nu)} + \|f\|^2_{L^2(X,\nu)} \right) \leqslant 2 \left(\frac{2}{\lambda} \mathcal{G}_m(u_n, f, \lambda) + \|f\|^2_{L^2(X,\nu)} \right),$$

we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(X, \nu)$ and we can assume that, up to a subsequence,

$$u_n \rightharpoonup u_\lambda \quad \text{weakly in } L^2(X, \nu).$$

Therefore, by the lower semi-continuity of the L^2-norm with respect to the weak convergence in $L^2(X, \nu)$ and Corollary 3.10, we have that

$$\mathcal{G}_m(u_\lambda, f, \lambda) \leqslant \liminf_{n\to\infty} \mathcal{G}_m(u_n, f, \lambda) = \alpha,$$

and, consequently, u_λ is a minimizer of problem (4.8). The uniqueness of the minimizer follows from the strict convexity of $\|\cdot\|^2_{L^2(X,\nu)}$ and the convexity of TV_m.

Since u_λ is a minimizer of problem (4.8), we have that $0 \in \partial \mathcal{G}_m(u_\lambda, f, \lambda)$. Now, if $\Phi(u) := \frac{\lambda}{2} \|u - f\|^2_{L^2(X,\nu)}$, then, by [**43**, Corollary 2.11] we have that

$$\partial \mathcal{G}_m(u, f, \lambda) = \partial \mathcal{F}_m(u) + \partial \Phi(u), \tag{4.11}$$

thus

$$0 \in \partial \mathcal{G}_m(u_\lambda, f, \lambda) = \partial \mathcal{F}_m(u_\lambda) + \lambda(u_\lambda - f),$$

which yields (4.9). Then, the characterization of u_λ follows from Theorem 3.13. □

Observe that, on account of (3.15), we have that u_λ is the solution of problem (4.8) if, and only if,

$$\begin{cases} f - u_\lambda \in G_m(X, \nu), \\ \|f - u_\lambda\|_{m,*} \leqslant \frac{1}{\lambda} \quad \textit{and} \\ \lambda \displaystyle\int_X (f - u_\lambda) u_\lambda d\nu = TV_m(u_\lambda). \end{cases} \tag{4.12}$$

As a consequence of this we have the following result.

Proposition 4.2. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^2(X, \nu)$ and $\lambda > 0$. Then, $u_\lambda = 0$ is the solution of problem (4.8) if, and only if,*

$$f \in G_m(X, \nu) \quad \textit{and} \quad \|f\|_{m,*} \leqslant \frac{1}{\lambda}.$$

For $f \in G_m(X, \nu)$, if $\|f\|_{m,*} > \frac{1}{\lambda}$, then u_λ is characterized by the following two conditions

$$\|f - u_\lambda\|_{m,*} = \frac{1}{\lambda}$$

and

$$\lambda \int_X (f - u_\lambda) u_\lambda d\nu = TV_m(u_\lambda).$$

Proof. The first part follows from (4.12). Let $f \in G_m(X, \nu)$ with $\|f\|_{m,*} > \frac{1}{\lambda}$. From (4.12) again,

$$\|\lambda(f - u_\lambda)\|_{m,*} \leqslant 1 \quad \text{and} \quad \lambda \int_X (f - u_\lambda) u_\lambda d\nu = TV_m(u_\lambda).$$

Now, since $\|f\|_{m,*} > \frac{1}{\lambda}$, we know that $u_\lambda \not\equiv 0$, thus, by (3.14), we have

$$\|\lambda(f - u_\lambda)\|_{m,*} \geqslant \frac{\lambda}{TV_m(u_\lambda)} \int_X (f - u_\lambda) u_\lambda d\nu = 1.$$

Therefore,

$$\|f - u_\lambda\|_{m,*} = \frac{1}{\lambda},$$

and we conclude the proof. □

The m-ROF-model leads to the following (BV, L^2)-decomposition:

$$f = u_\lambda + v_\lambda \ , \quad [u_\lambda, v_\lambda] = \underset{(u,v) \in L^2(X,\nu) \times L^2(X,\nu)}{\arg\min} \left\{ TV_m(u) + \frac{\lambda}{2} \|v\|^2_{L^2(X,\nu)} \ : \ f = u + v \right\}.$$

Then, bringing together the previous results we obtain:

Corollary 4.3. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^2(X, \nu)$ and $\lambda > 0$. For the (BV, L^2)-decomposition $[u_\lambda, v_\lambda]$ of f, we have:*

(i) $v_\lambda \in G_m(X, \nu)$, $\|v_\lambda\|_{m,*} \leqslant \frac{1}{\lambda}$ and $\lambda \int_X v_\lambda u_\lambda d\nu = TV_m(u_\lambda)$.

(ii) $u_\lambda = 0$ if, and only if, $v_\lambda = f$.

(iii) For $f \in G_m(X, \nu)$, if $\|f\|_{m,*} > \frac{1}{\lambda}$, then

$$\|v_\lambda\|_{m,*} = \frac{1}{\lambda} \quad \text{and} \quad \lambda \int_X v_\lambda u_\lambda d\nu = TV_m(u_\lambda).$$

Remark 4.4. (i) If λ is too small then the regularization term $TV_m(u)$ is excessively penalized and the image is over-smoothed, resulting in a loss of information in the reconstructed image. On the other hand, if λ is too large then the reconstructed image is under-regularized and noise is left in the reconstruction.

(ii) In [**149**] and [**150**], Tadmor, Nezzar and Vese propose a multiscale decomposition in order to overcome the difficulties that have been brought up in the previous point. In this regard, the space of functions $G_m(X, \nu)$ is of particular interest, since, as we have seen in Corollary 4.1, after a first decomposition $[u_\lambda, v_\lambda]$ the function v_λ is a function of $G_m(X, \nu)$ which in turn can be decomposed. The multiscale decomposition takes advantage of this fact by taking an increasing sequence of scales λ_i tending to infinity and inductively applying the (BV, L^2)-decomposition with scale parameter λ_{i+1} to v_{λ_i} so that after k-steps we have

$$f = \sum_{i=1}^{k} u_{\lambda_i} + v_{\lambda_k}, \quad \|v_{\lambda_k}\|_{m*} \leqslant \frac{1}{\lambda_k}.$$

Integrating both sides of (4.10) over X with respect to ν and using Green's formula (3.4) (with $u = 1$ and $\mathbf{z} = \mathbf{g}$) we get:

Proposition 4.5. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^2(X, \nu)$. If $u_\lambda \in L^2(X, \nu)$ is the unique minimizer of problem (4.8) then*

$$\int_X u_\lambda(x) d\nu(x) = \int_X f(x) d\nu(x).$$

Furthermore, we have the following Maximum Principle.

PROPOSITION 4.6. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f_1, f_2 \in L^2(X, \nu)$. If $[u_{i,\lambda}, v_{i,\lambda}]$ is the (BV, L^2)-decomposition of f_i, $i = 1, 2$, then*

$$\|(u_{1,\lambda} - u_{2,\lambda})^+\|_{L^2(X,\nu)} \leqslant \|(f_1 - f_2)^+\|_{L^2(X,\nu)}. \tag{4.13}$$

In particular, for $c, C \in \mathbb{R}$, if $c \leqslant f \leqslant C$ ν-a.e., and $[u_\lambda, v_\lambda]$ is the (BV, L^2)-decomposition of f, then

$$c \leqslant u_\lambda \leqslant C \quad \nu\text{-a.e.}$$

PROOF. Since $\lambda(u_{i,\lambda} - f_i) \in \Delta_1^m(u_{i,\lambda})$, $i = 1, 2$, (4.13) is a direct consequence of the complete accretivity of $-\Delta_1^m$ (see section A.7 of Appendix A).

The second part follows from (4.13) and the fact that, for a constant $k \in \mathbb{R}$, $[k, 0]$ is the (BV, L^2)-decomposition of $f = k$. □

REMARK 4.7. For the local ROF-model in $\mathbb{R}^N$ it is well known that, if $f = \chi_{B_r(0)}$, then the solution is given by

$$u_\lambda = \begin{cases} 0, & \text{for } 0 \leqslant \lambda \leqslant \frac{1}{2r}, \\ \left(1 - \frac{1}{2r\lambda}\right)\chi_{B_r(0)}, & \text{for } \lambda > \frac{1}{2r}. \end{cases}$$

For the m-ROF-model studied here, if $[X, \mathcal{B}, m, \nu]$ is m-connected, $\lambda > 0$ and $f = \chi_\Omega$, there does not exist a solution of the form $c\chi_\Omega$ whatever $\Omega \in \mathcal{B}$ such that $0 < \nu(\Omega) < 1$. Indeed, if such a solution exists then, by Proposition 4.5, we would have $c = 1$. Hence, by Theorem 4.1, we have that

$$0 \in \Delta_1^m \chi_\Omega,$$

which is not possible since $[X, \mathcal{B}, m, \nu]$ is m-connected.

However, we can have a solution of the form $c\chi_\Omega$ if $f = \chi_\Omega + h\chi_{X\setminus\Omega}$ for some m-calibrable set $\Omega \in \mathcal{B}$ and a function $h \in L^2(X, \nu)$ satisfying

$$\int_{X\setminus\Omega} h d\nu = -\frac{1}{\lambda} P_m(\Omega).$$

Indeed, let $\Omega \in \mathcal{B}$, $h \in L^2(X, \nu)$ and suppose that $f = \chi_\Omega + h\chi_{X\setminus\Omega}$. Then, we need the following equation to be satisfied:

$$\lambda(c-1)\chi_\Omega - \lambda h(x)\chi_{X\setminus\Omega} \in \Delta_1^m \chi_\Omega,$$

which, by Remarks 3.41 and 3.45, holds if $c = 1 - \frac{\lambda_\Omega^m}{\lambda}$, Ω is m-calibrable and

$$\int_{X\setminus\Omega} h d\nu = -\frac{1}{\lambda} P_m(\Omega).$$

For example, we can take $f(x) = \chi_\Omega(x) - \frac{1}{\lambda} m_x(\Omega)\chi_{X\setminus\Omega}(x)$, so that $u_\lambda = \left(1 - \frac{\lambda_\Omega^m}{\lambda}\right)\chi_\Omega$.

In the next result we construct a minimizer of (4.8) for $f = bu$, where u is a solution of $-u \in \Delta_1^m u$. Observe that, in this case, $\displaystyle\int_X f d\nu = 0$.

PROPOSITION 4.8. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $\lambda, b > 0$. If $u \in L^2(X, \nu)$ is a solution of*

$$-u \in \Delta_1^m u, \tag{4.14}$$

then $u_\lambda = \left(b - \frac{1}{\lambda}\right)^+ u$ is a minimizer of (4.8) with $f = bu$. Conversely, if $\left(b - \frac{1}{\lambda}\right) u$ is a minimizer of (4.8) with $f = bu$, then u is a solution of (4.14).

Proof. Set $a = \left(b - \frac{1}{\lambda}\right)^+$ and let $u \in L^2(X,\nu)$ be a solution of (4.14). Suppose first that $b > \frac{1}{\lambda}$, so that $a = b - \frac{1}{\lambda}$. Then,

$$\lambda(au - bu) = -u \in \Delta_1^m(u) = \Delta_1^m(au).$$

Hence, by Theorem 4.1, we have that au is a minimizer of (4.8) with $f = bu$. Now, assume that $b \leqslant \frac{1}{\lambda}$, so that $a = 0$. Since u is a solution of (4.14), there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric such that

$$-\mathrm{div}_m \mathbf{g} = u$$

and

$$\mathbf{g}(x,y) \in \mathrm{sign}(u(y) - u(x)) \quad \text{for } \nu \otimes m_x\text{-a.e. } (x,y) \in X \times X.$$

If $\mathbf{z} := \lambda b \mathbf{g}$, we have that $\|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$,

$$-\frac{1}{\lambda}\mathrm{div}_m \mathbf{z} = -b\,\mathrm{div}_m \mathbf{g} = bu,$$

and

$$\mathbf{z}(x,y) \in \mathrm{sign}(0) \quad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in X \times X.$$

Therefore,

$$-\lambda bu \in \partial\mathcal{F}_m(0),$$

and, by Theorem 4.1, we have that 0 is a minimizer of (4.8) with $f = bu$.

Suppose now that au is a minimizer of (4.8) with $f = bu$ and $b - a = \frac{1}{\lambda}$. Then, by Theorem 4.1, we have

$$-\lambda(au - bu) \in \partial\mathcal{F}_m(au).$$

Hence, u is a solution of (4.14). □

Remark 4.9. There is a formal connection between the m-ROF-model (4.8) and the total variation flow (3.18) that can be drawn as follows. Given the initial datum u_0, we consider an implicit time discretization of the total variation flow using the following iterative procedure:

$$\frac{u_n - u_{n-1}}{\Delta t} \in \Delta_1^m u_n \quad n \in \mathbb{N}. \tag{4.15}$$

Identifying the time step Δt in (4.15) with the regularization parameter in (4.8), that is, taking $\lambda = \frac{1}{\Delta t}$, we observe that each iteration in (4.15) can be equivalently approached by solving (4.8) (see (4.9)), where we take $u = u_n$ and $f = u_{n-1}$. In the next section we discuss how to solve the m-ROF-model via the gradient descent method.

4.1.1. The Gradient Descent Method. *As in [143], we can see that problem (4.8) is well-posed by using the gradient descent method. For this, one needs to solve the Cauchy problem*

$$\begin{cases} v_t \in \Delta_1^m v(t) - \lambda(v(t) - f) & \text{in } (0,T) \times X \\ v(0,x) = v_0(x) & \text{in } x \in X, \end{cases} \tag{4.16}$$

with v_0 satisfying

$$\int_\Omega v_0 = \int_\Omega f.$$

Now, problem (4.16) can be rewritten as the following abstract Cauchy problem in $L^2(X,\nu)$ (recall (4.11)):

$$v'(t) + \partial\mathcal{G}_m(v(t), f, \lambda) \ni 0, \quad v(0) = v_0. \tag{4.17}$$

Then, since $\mathcal{G}_m(\cdot, f, \lambda)$ is convex and lower semi-continuous, by the theory of maximal monotone operators ([43]), we have that, for any initial data $v_0 \in L^2(X,\nu)$, problem (4.17) has a unique strong solution. Therefore, if we define a solution of problem (4.16) as a function $v \in C(0,T;L^2(X,\nu)) \cap W^{1,1}_{loc}(0,T;L^2(X,\nu))$ such that $v(0,x) = v_0(x)$ for ν-a.e. $x \in X$ and satisfying

$$\lambda(v(t) - f) + v_t(t) \in \Delta_1^m(v(t)) \quad \text{for a.e. } t \in (0,T),$$

we have the following existence and uniqueness result.

THEOREM 4.10. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. For every $v_0 \in L^2(X, \nu)$ there exists a unique strong solution of the Cauchy problem (4.16) in $(0, T)$ for any $T > 0$. Moreover, we have the following contraction and maximum principles in any $L^q(X, \nu)$–space, $1 \leqslant q \leqslant \infty$:*

$$\|(v(t) - w(t))^+\|_{L^q(X,\nu)} \leqslant \|(v_0 - w_0)^+\|_{L^q(X,\nu)} \quad \forall\, 0 < t < T, \tag{4.18}$$

for any pair of solutions v, w of problem (4.16) with initial data v_0, $w_0 \in L^2(X, \nu)$ and noisy images f, $\hat{f} \in L^2(X, \nu)$, with $f \leqslant \hat{f}$, respectively.

Note that the contraction principle (4.18) in any L^q-space follows from the fact that the operator $\partial\mathcal{G}_m(\cdot, f, \lambda)$ is completely accretive (see section A.7 of Appendix A).

THEOREM 4.11. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. For $f \in L^2(X, \nu)$, let $(T_\lambda(t))_{t\geqslant 0}$ be the semigroup solution of the Cauchy problem (4.16). Then, for every $v_0 \in L^2(X, \nu)$, we have*

$$\|T_\lambda(t)v_0 - u_\lambda\|_{L^2(X,\nu)} \leqslant \|v_0 - u_\lambda\|_{L^2(X,\nu)}\; e^{-\lambda t} \quad \text{for all } \; t \geqslant 0, \tag{4.19}$$

where u_λ is the unique minimizer of problem (4.8) with this same f.

PROOF. If $v(t) := T_\lambda(t)v_0$, we have

$$v_t + \lambda(v(t) - f) \in \Delta_1^m(v(t)),$$

and, by Theorem 4.1,

$$\lambda(u_\lambda - f) \in \Delta_1^m(u_\lambda).$$

Now, since $-\Delta_1^m$ is a monotone operator in $L^2(X, \nu)$, we get

$$\int_X (v(t) - u_\lambda)(-v_t - \lambda(v(t) - f) - (-\lambda(u_\lambda - f))d\nu \geqslant 0.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_X (v(t) - u_\lambda)^2 d\nu + \lambda \int_X (v(t) - u_\lambda)^2 d\nu \leqslant 0.$$

Then, integrating this ordinary differential inequality, we obtain (4.19). □

PROPOSITION 4.12. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^2(X, \nu)$ and $(T_\lambda(t))_{t\geqslant 0}$ be the semigroup solution of the Cauchy problem (4.16) and let $v_0 \in L^2(X, \nu)$ satisfying $\int_X v_0 d\nu = \int_X f d\nu$. Then,*

$$\int_X T_\lambda(t)v_0 d\nu = \int_X f d\nu \quad \textit{for all } t \geqslant 0.$$

PROOF. If $v(t) := T_\lambda(t)v_0$, we have

$$v_t + \lambda(v(t) - f) \in \Delta_1^m(v(t)).$$

Integrating over $(0, t) \times X$ with respect to $\mathcal{L}^1 \otimes \nu$ and having in mind that $\int_X v_0 d\nu = \int_X f\, d\nu$, we get

$$\lambda \int_0^t \int_X v(s) d\nu ds - \lambda t \int_X f d\nu + \int_X v(t) d\nu - \int_X f d\nu = 0.$$

Then, the function

$$z(t) := \int_0^t \int_X v(s) d\nu ds,$$

verifies

$$\begin{cases} z'(t) + \lambda z(t) = (\lambda t + 1) \displaystyle\int_X f d\nu \\ z(0) = 0, \end{cases}$$

whose unique solution is $z(t) = t \displaystyle\int_X f d\nu$. Hence,

$$\int_X v(t) d\nu = \int_X f d\nu. \qquad \square$$

4.2. The m-ROF-Model with L^1-fidelity term

In this section we will study the m-ROF-model with L^1-fidelity term, that is, given $f \in L^1(X,\nu)$ and $\lambda > 0$, we will study the minimization, over $L^1(X,\nu)$, of the energy given by the sum of the total variation and the L^1-fidelity term:

$$\min \left\{ TV_m(u) + \lambda \int_X |u - f| d\nu \;:\; u \in L^1(X,\nu) \right\}.$$

Definition 4.13. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^1(X,\nu)$ and $\lambda > 0$. We denote

$$\mathcal{E}_m(u, f, \lambda) := TV_m(u) + \lambda \int_X |u - f| d\nu, \quad u \in L^1(X,\nu),$$

and

$$\mathcal{E}_m(f, \lambda) := \inf_{u \in L^1(X,\nu)} \mathcal{E}_m(u, f, \lambda).$$

Moreover, we denote the set of minimizers of $\mathcal{E}_m(\cdot, f, \lambda)$ by:

$$M(f,\lambda) := \left\{ u \in L^1(X,\nu) \;:\; \mathcal{E}_m(u, f, \lambda) = \mathcal{E}_m(f, \lambda) \right\}.$$

Note that the set $M(f,\lambda)$ can have several elements. Due to the convexity and the lower semi-continuity of the energy functional $\mathcal{E}_m(\cdot, f, \lambda)$ we have that the set $M(f,\lambda)$ is closed and convex in $L^1(X,\nu)$.

In the local case, that is, for problem (4.4), the fact that there exists a minimizer for every datum in L^1 is a consequence of the direct method of the calculus of variations. However, in our context, we do not have sufficient compactness properties in order to apply this method. Therefore, the proof of the fact that $M(f,\lambda) \neq \emptyset$ for every $f \in L^1(X,\nu)$ will be shown after the study of the geometric problem associated with the (BV, L^1)-decomposition (which is addressed in Section 4.2.1).

We have the following Maximum Principle.

Proposition 4.14. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^1(X,\nu)$, $\lambda > 0$ and $c, C \in \mathbb{R}$, and assume that $c \leqslant f \leqslant C$ ν-a.e. Then,*

$$c \leqslant u \leqslant C \quad \nu\text{-a.e.} \quad \forall u \in M(f,\lambda).$$

Proof. Let $u \in M(f,\lambda)$. Obviously, we have that $TV_m(u \wedge C) \leqslant TV_m(u)$ and $|(u \wedge C) - f| \leqslant |u - f|$ ν-a.e. Hence,

$$TV_m(u \wedge C) + \lambda \|(u \wedge C) - f\|_{L^1(X,\nu)} \leqslant TV_m(u) + \lambda \|u - f\|_{L^1(X,\nu)}.$$

However, since $u \in M(f,\lambda)$, this inequality is, in fact, an equality, thus $\|(u \wedge C) - f\|_{L^1(X,\nu)} = \|u - f\|_{L^1(X,\nu)}$, and we conclude from this that

$$u \wedge C = u \quad \nu\text{-a.e.}$$

Similarly, it follows that $u \vee c = u$ ν-a.e. $\square$

REMARK 4.15. Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Following [**55**, Claims 4 & 5], if $\lambda_2 > \lambda_1 > 0$, then

$$\|u_{\lambda_1} - f\|_{L^1(X,\nu)} \geqslant \|u_{\lambda_2} - f\|_{L^1(X,\nu)} \quad \text{for } u_{\lambda_i} \in M(f, \lambda_i),\ i = 1, 2,$$

and the set

$$\Lambda(f) := \left\{ \lambda : \inf_{u \in M(f,\lambda)} \|u - f\|_{L^1(X,\nu)} < \sup_{u \in M(f,\lambda)} \|u - f\|_{L^1(X,\nu)} \right\}$$

is at most countable.

PROPOSITION 4.16. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. If $f \in L^1(X, \nu)$, then $\mathcal{E}_m(f, \lambda)$ is Lipschitz continuous with respect to λ.*

PROOF. Since $\mathcal{E}_m(f, \lambda)$ is defined as the pointwise infimum of a collection of increasing and linear functions in λ, we have that $\mathcal{E}_m(f, \lambda)$ is increasing and concave in λ. This, together with the fact that

$$\mathcal{E}_m(f, \lambda) \leqslant \mathcal{E}_m(0, f, \lambda) = \lambda \|f\|_{L^1(X,\nu)},$$

gives the desired property. □

LEMMA 4.17. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$ and $\lambda > 0$. Then,*

$$u_\lambda \in M(\chi_\Omega, \lambda) \Leftrightarrow \chi_X - u_\lambda \in M(\chi_{X \setminus \Omega}, \lambda).$$

PROOF. This follows easily since $\mathcal{E}_m(u, \chi_\Omega, \lambda) = \mathcal{E}_m(\chi_X - u, \chi_{X\setminus\Omega}, \lambda)$ for every $u \in L^1(X, \nu)$. □

In the next result we characterise the minimizers of $\mathcal{E}_m(\cdot, f, \lambda)$.

THEOREM 4.18. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^2(X, \nu)$, $\lambda > 0$ and $u_\lambda \in L^2(X, \nu)$. Then, $u_\lambda \in M(f, \lambda)$ if, and only if, there exists $\xi \in sign(u_\lambda - f)$ such that*

$$\lambda \xi \in \Delta_1^m(u_\lambda).$$

PROOF. We have that $\mathcal{E}_m(u, f, \lambda) = TV_m(u) + \lambda \mathcal{G}_f(u)$, with

$$\mathcal{G}_f(u) := \int_X |u - f| d\nu, \quad u \in L^1(X, \nu).$$

Let us first see that $u_\lambda \in M(f, \lambda)$ if, and only if, $0 \in \partial \mathcal{E}_m(u_\lambda, f, \lambda)$ (with $\mathcal{E}_m(\cdot, f, \lambda)$ considered as a functional in $L^2(X, \nu)$).

Suppose that $u_\lambda \in M(f, \lambda)$. Then,

$$\mathcal{E}_m(u_\lambda, f, \lambda) \leqslant \mathcal{E}_m(u, f, \lambda) \quad \forall u \in L^1(X, \nu)$$

thus, in particular, for every $u \in L^2(X, \nu)$. Therefore, $0 \in \partial \mathcal{E}_m(u_\lambda, f, \lambda)$.

Suppose now that $0 \in \partial \mathcal{E}_m(u_\lambda, f, \lambda)$. Let $u \in L^1(X, \nu)$ and $(u_k)_{k \geqslant 1} \subset L^2(X, \nu)$ such that $u_k \xrightarrow{k} u$ in $L^1(X, \nu)$. Then, since $0 \in \partial \mathcal{E}_m(u_\lambda, f, \lambda)$, we have that

$$\mathcal{E}_m(u_\lambda, f, \lambda) \leqslant \mathcal{E}_m(u_k, f, \lambda) \quad \forall k \geqslant 1,$$

and, by Proposition 3.4, we may take limits in k to obtain

$$\mathcal{E}_m(u_\lambda, f, \lambda) \leqslant \mathcal{E}_m(u, f, \lambda).$$

Hence, since $u \in L^1(X, \nu)$ was arbitrary, we get that $u_\lambda \in M(f, \lambda)$.

Now, by [**43**, Corollary 2.11], we have that

$$\partial \mathcal{E}_m(u, f, \lambda) = \partial \mathcal{F}_m(u) + \lambda \partial \mathcal{G}_f(u),$$

and then

$$u_\lambda \in M(f, \lambda) \Leftrightarrow 0 \in \partial \mathcal{F}_m(u_\lambda) + \lambda \partial \mathcal{G}_f(u_\lambda).$$

Moreover, it is not difficult to see that

$$v \in \partial \mathcal{G}_f(u_\lambda) \Leftrightarrow v \in \operatorname{sign}(u_\lambda - f),$$

thus,

$$u_\lambda \in M(f,\lambda) \Leftrightarrow \exists \xi \in \operatorname{sign}(u_\lambda - f) \text{ such that } \lambda\xi \in \Delta_1^m(u_\lambda). \qquad \square$$

REMARK 4.19. As a consequence of Theorem 4.18 and Theorem 3.13, we have that, given $u_\lambda \in L^2(X,\nu)$, $u_\lambda \in M(f,\lambda)$ if, and only if, there exists $\xi \in \operatorname{sign}(u_\lambda - f)$ and $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric satisfying

$$\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda\xi(x) \quad \text{ for } \nu\text{-a.e } x \in X,$$

and

$$\mathbf{g}(x,y) \in \operatorname{sign}(u_\lambda(y) - u_\lambda(x)) \quad \text{ for } \nu \otimes m_x\text{-a.e } (x,y) \in X \times X.$$

*Let's see that the (BV, L^1)-decomposition is contrast invariant (see [**71**] for the continuous case).*

COROLLARY 4.20. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^2(X,\nu)$, $\lambda > 0$ and $T : \mathbb{R} \to \mathbb{R}$ a nondecreasing function. If $u_\lambda \in M(f,\lambda)$, then $T(u_\lambda) \in M(T(f),\lambda)$.*

PROOF. Given $u_\lambda \in M(f,\lambda)$ we have that, by Theorem 4.18, there exists $\xi \in \operatorname{sign}(u_\lambda - f)$ such that

$$\lambda\xi \in \Delta_1^m(u_\lambda).$$

Then, since T is nondecreasing, we have that $\xi \in \operatorname{sign}(T(u_\lambda) - T(f))$ and

$$\lambda\xi \in \Delta_1^m(T(u_\lambda)).$$

Therefore, applying again Theorem 4.18, we get that $T(u_\lambda) \in M(T(f),\lambda)$. $\square$

*Like in the local case, by the "layer cake" formula (see [**55**, Proposition 5.1]), we obtain that*

$$\int_X |u-f| d\nu = \int_{-\infty}^{+\infty} \nu(\{x : u(x) > t\} \triangle \{x : f(x) > t\}) dt$$

where

$$A \triangle B := (A \backslash B) \cup (B \backslash A).$$

Therefore, by the coarea formula (Proposition 3.5), the energy functional $\mathcal{E}_m(\cdot, f, \lambda)$ can be rewritten in a geometric form in terms of the energies of the superlevel sets of u as follows.

THEOREM 4.21. *Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space and assume that ν is a probability measure. Let u, $f \in L^1(X,\nu)$ and $\lambda > 0$, then*

$$\mathcal{E}_m(u,f,\lambda) = \int_{-\infty}^{+\infty} \Big(P_m(E_t(u)) + \lambda\nu(E_t(u) \triangle E_t(f)) \Big) dt.$$

Consequently, given $\Omega \in \mathcal{B}$ and taking $f = \chi_\Omega$, by the Maximum Principle (Proposition 4.14), we get

$$\mathcal{E}_m(u,\chi_\Omega,\lambda) = \int_0^1 \Big(P_m(E_t(u)) + \lambda\nu(E_t(u) \triangle \Omega) \Big) dt. \tag{4.20}$$

4.2.1. The Geometric Problem. *Given $F \in \mathcal{B}$ and $\lambda > 0$, we consider the geometric functional*

$$\mathcal{E}_m^G(A, F, \lambda) := P_m(A) + \lambda\nu(A \triangle F), \quad A \in \mathcal{B}.$$

In view of Theorem 4.21, given $f \in L^1(X, \nu)$, one may consider the family of geometric problems

$$P(f, t, \lambda) : \quad \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, E_t(f), \lambda), \quad t \in \mathbb{R}. \tag{4.21}$$

Moreover, we can prove that a minimizer of $\mathcal{E}_m^G(\cdot, F, \lambda)$ always exists:

Proposition 4.22. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $F \in \mathcal{B}$ be a non-ν-null set and $\lambda > 0$. Then, there exists a minimizer u_λ of $\mathcal{E}_m(\cdot, \chi_F, \lambda)$. Moreover, for a.e. $t \in]0, 1[$, $E_t(u_\lambda)$ is a minimizer of $\mathcal{E}_m^G(\cdot, F, \lambda)$, and*

$$\min_{A \in \mathcal{B}} \mathcal{E}_m^G(A, F, \lambda) = \min_{u \in L^1(X,\nu)} \mathcal{E}_m(u, \chi_F, \lambda).$$

Proof. Since $\chi_F \in L^\infty(X, \nu)$, by the direct method of the calculus of variations, we have that there exists u_λ (which, by Proposition 4.14, belongs to $L^\infty(X, \nu)$) such that

$$\mathcal{E}_m(u_\lambda, \chi_F, \lambda) = \min_{u \in L^1(X,\nu)} \mathcal{E}_m(u, \chi_F, \lambda).$$

Now, by Theorem 4.21,

$$\int_0^1 \mathcal{E}_m^G(E_t(u_\lambda), F, \lambda)dt = \mathcal{E}_m(u_\lambda, \chi_F, \lambda) \leqslant \inf_{A \in \mathcal{B}} \mathcal{E}_m(\chi_A, \chi_F, \lambda) = \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, F, \lambda),$$

hence, for a.e. $t \in]0, 1[$,

$$\mathcal{E}_m^G(E_t(u_\lambda), F, \lambda) = \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, F, \lambda),$$

which concludes the proof. □

*This proposition is, in fact, a consequence of the following stronger result, which was proved in [**162**] for the local case. However, since its proof only uses properties of measures, the submodularity of the perimeter (which we have proven for the m-perimeter; see Proposition 1.59) and the local version of Theorem 4.21, the same proof yields the following result.*

Lemma 4.23. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Given $f \in L^1(X, \nu)$ and $\lambda > 0$, there exists a function $u \in L^1(X, \nu)$ such that*

$$\mathcal{E}_m^G(E_t(u), E_t(f), \lambda) = \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, E_t(f), \lambda) \quad \forall t \in \mathbb{R}.$$

Theorem 4.24. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. For $f \in L^1(X, \nu)$ and $\lambda > 0$ there exists (at least) one minimizer of the variational problem*

$$\min_{u \in L^1(X,\nu)} \mathcal{E}_m(u, f, \lambda). \tag{4.22}$$

Proof. Let u be the function obtained in Lemma 4.23. Then, by Theorem 4.21, given $v \in L^1(X, \nu)$, we have

$$\mathcal{E}_m(u, f, \lambda) = \int_{-\infty}^{+\infty} \mathcal{E}_m^G(E_t(u), E_t(f), \lambda)dt \leqslant \int_{-\infty}^{+\infty} \mathcal{E}_m^G(E_t(v), E_t(f), \lambda)dt = \mathcal{E}_m(v, f, \lambda). \quad \square$$

Observe that in Theorem 4.18 we obtained the Euler-Lagrange equation of the variational problem (4.22).

Duval, Aujol and Gousseau in [**78**, Theorem 4.2] *(with the help of* [**162**, Theorem 3.1]*) show that, in the local case, there is, in fact, an equivalence with the geometric problem. This result can be extended, on account of the submodularity of the m-perimeter (Proposition 1.59), to our nonlocal context:*

THEOREM 4.25. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^1(X, \nu)$ and $\lambda > 0$. The following assertions are equivalent:*
(i) u is a solution of Problem (4.22).
(ii) $E_t(u)$ is a solution of (4.21) for a.e. $t \in \mathbb{R}$.

In [**78**, Proposition 5.5] *it is also shown that at points where the boundary of a minimizer of the geometric problem for datum $F \subset \mathbb{R}^2$ and fidelity parameter λ does not coincide with the boundary of F, the mean curvature is $\pm\lambda$. Let us see that there is a nonlocal counterpart of this fact but where the nonlocal character of the problem gives rise to a nontrivial extension (recall Definition 1.60).*

PROPOSITION 4.26. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $\lambda > 0$, $F \in \mathcal{B}$ with $0 < \nu(F) < 1$, and $E \in \mathcal{B}$ a minimizer of $\mathcal{E}_m^G(\cdot, F, \lambda)$. Let $A \in \mathcal{B}$ with $\nu(A) > 0$.*
(1) Then,
(i) if $\nu(A \setminus E) > 0$,
$$\frac{1}{\nu(A\setminus E)} \int_{A\setminus E} \mathcal{H}^m_{\partial E}(x) d\nu(x) \geqslant -\lambda + \frac{1}{\nu(A\setminus E)} \int_{A\setminus E} m_x(A\setminus E) d\nu(x) \geqslant -\lambda.$$
(ii) if $\nu(A \cap E) > 0$,
$$\frac{1}{\nu(A\cap E)} \int_{A\cap E} \mathcal{H}^m_{\partial E}(x) d\nu(x) \leqslant \lambda - \frac{1}{\nu(A\cap E)} \int_{A\cap E} m_x(A\cap E) d\nu(x) \leqslant \lambda.$$
(2) Moreover,
(i) if $\nu(A \setminus E) > 0$ and $\nu(A\setminus F) = 0$, then
$$\frac{1}{\nu(A\setminus E)} \int_{A\setminus E} \mathcal{H}^m_{\partial E}(x) d\nu(x) \geqslant \lambda + \frac{1}{\nu(A\setminus E)} \int_{A\setminus E} m_x(A\setminus E) d\nu(x) \geqslant \lambda.$$
(ii) if $\nu(A \cap E) > 0$ and $\nu(A\cap F) = 0$, then
$$\frac{1}{\nu(A\cap E)} \int_{A\cap E} \mathcal{H}^m_{\partial E}(x) d\nu(x) \leqslant -\lambda - \frac{1}{\nu(A\cap E)} \int_{A\cap E} m_x(A\cap E) d\nu(x) \leqslant -\lambda.$$

PROOF. *(i)*: Suppose that $\nu(A\setminus E) > 0$. Since E is a minimizer of $\mathcal{E}_m^G(\cdot, F, \lambda)$, we have that
$$P_m(E) + \lambda\nu(E \triangle F) \leqslant P_m(E\cup A) + \lambda\nu((E\cup A)\triangle F),$$
i.e.,
$$\int_E \int_{X\setminus E} dm_x(y)d\nu(x) - \int_{E\cup A}\int_{X\setminus(E\cup A)} dm_x(y)d\nu(x) \leqslant \lambda\left(\nu((E\cup A)\triangle F) - \nu(E\triangle F)\right).$$
Now,
$$\nu((E\cup A)\triangle F) - \nu(E\triangle F)) = \nu(A\setminus E) - 2\nu((A\cap F)\setminus E) \leqslant \nu(A\setminus E)$$
but, if $\nu(A\setminus F) = 0$, then $\nu((A\cap F)\setminus E) = \nu(A\setminus E)$ so
$$\nu((E\cup A)\triangle F) - \nu(E\triangle F)) = -\nu(A\setminus E).$$
Moreover,
$$\begin{aligned}
&\int_E \int_{X\setminus E} dm_x(y)d\nu(x) - \int_{E\cup A}\int_{X\setminus(E\cup A)} dm_x(y)d\nu(x)\\
&= \int_E\int_{A\setminus E} dm_x(y)d\nu(x) - \int_{A\setminus E}\int_{X\setminus E} dm_x(y)d\nu(x) + \int_{A\setminus E}\int_{A\setminus E} dm_x(y)d\nu(x)\\
&= \int_{A\setminus E}\int_{E} dm_x(y)d\nu(x) - \int_{A\setminus E}\int_{X\setminus E} dm_x(y)d\nu(x) + \int_{A\setminus E}\int_{A\setminus E} dm_x(y)d\nu(x)\\
&= -\int_{A\setminus E} \mathcal{H}^m_{\partial E}(x)d\nu(x) + \int_{A\setminus E} m_x(A\setminus E)d\nu(x).
\end{aligned}$$

Consequently:

(1)

$$\frac{1}{\nu(A\backslash E)}\int_{A\backslash E}\mathcal{H}^m_{\partial E}(x)d\nu(x) \geqslant -\lambda + \frac{1}{\nu(A\backslash E)}\int_{A\backslash E} m_x(A\backslash E)d\nu(x).$$

(2) If $\nu(A\backslash F) = 0$, then

$$\frac{1}{\nu(A\backslash E)}\int_{A\backslash E}\mathcal{H}^m_{\partial E}(x)d\nu(x) \geqslant \lambda + \frac{1}{\nu(A\backslash E)}\int_{A\backslash E} m_x(A\backslash E)d\nu(x).$$

(ii): These statements follow from *(i)* by (1.12) and by taking into account that, since $P_m(E) = P_m(X\backslash E)$ and $E \bigtriangleup F = (X\backslash E) \bigtriangleup (X\backslash F)$, E is a minimizer of $\mathcal{E}^G_m(\cdot, F, \lambda)$ if, and only if, $X\backslash E$ is a minimizer of $\mathcal{E}^G_m(\cdot, X\backslash F, \lambda)$, and, further, that $A \cap F = A\backslash(X\backslash F)$ and $A \cap (X\backslash F) = A\backslash F$.

Indeed, let us see, for example, how *(1)(i)* implies *(1)(ii)*. First of all, as already mentioned, since E is a minimizer of $\mathcal{E}^G_m(\cdot, F, \lambda)$, we have that $X\backslash E$ is a minimizer of $\mathcal{E}^G_m(\cdot, X\backslash F, \lambda)$. Now, suppose that $\nu(A \cap E) > 0$. Then, since $A\backslash(X\backslash E) = A \cap E$, we may apply *(1)(i)* to the minimizer $X\backslash E$ of $\mathcal{E}^G_m(\cdot, X\backslash F, \lambda)$ to obtain the following:

$$\frac{1}{\nu(A\backslash(X\backslash E))}\int_{A\backslash(X\backslash E)}\mathcal{H}^m_{\partial(X\backslash E)}(x)d\nu(x) \geqslant -\lambda + \frac{1}{\nu(A\backslash(X\backslash E))}\int_{A\backslash(X\backslash E)} m_x(A\backslash(X\backslash E))d\nu(x),$$

which, by (1.12), can be rewritten as

$$\frac{1}{\nu(A\cap E)}\int_{A\cap E}\mathcal{H}^m_{\partial E}(x)d\nu(x) \leqslant \lambda - \frac{1}{\nu(A\cap E)}\int_{A\cap E} m_x(A\cap E)d\nu(x) \leqslant \lambda. \qquad \square$$

Corollary 4.27. *Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with a connected weighted discrete graph G, and let E, F and λ be as in the hypothesis of Proposition 4.26. Then,*

(1) we have

$$\mathcal{H}^{m^G}_{\partial E}(x) \leqslant -\lambda - \frac{w_{x,x}}{d_x} \quad \forall x \in E\backslash F, \tag{4.23}$$

and

$$\lambda + \frac{w_{x,x}}{d_x} \leqslant \mathcal{H}^{m^G}_{\partial E}(x) \quad \forall x \in F\backslash E. \tag{4.24}$$

(2) we have

$$\mathcal{H}^{m^G}_{\partial E}(x) \leqslant \lambda - \frac{w_{x,x}}{d_x} \quad \forall x \in E\cap F,$$

and

$$-\lambda + \frac{w_{x,x}}{d_x} \leqslant \mathcal{H}^{m^G}_{\partial E}(x) \quad \forall x \in X\backslash (E\cup F).$$

Proof. *(1)*: If $E\backslash F \neq \emptyset$ let $x \in E\backslash F$ and take $A = \{x\}$, so that $A \cap E = A$. Note that $\nu_G(A) > 0$ since G is connected. Then, since $A \cap F = \emptyset$, by Proposition 4.26 *(2)(ii)*, we get

$$\frac{1}{\nu_G(\{x\})}\int_{\{x\}}\mathcal{H}^{m^G}_{\partial E}(y)d\nu(y) \leqslant -\lambda - \frac{1}{\nu_G(\{x\})}\int_{\{x\}} m_y(\{x\})d\nu(y).$$

That is, $\mathcal{H}^{m^G}_{\partial E}(x) \leqslant -\lambda - m_x(\{x\})$ for every $x \in E\backslash F$, which gives (4.23). Now, (4.24) can be obtained with a similar argument by using Proposition 4.26 *(2)(i)*, or as follows: since E is a minimizer for $\mathcal{E}_{m^G}(\cdot, F, \lambda)$, it follows that $X\backslash E$ is a minimizer for $\mathcal{E}_{m^G}(\cdot, X\backslash F, \lambda)$. Consequently, from (4.23),

$$\mathcal{H}^{m^G}_{\partial(X\backslash E)}(x) \leqslant -\lambda - \frac{w_{x,x}}{d_x} \quad \forall x \in (X\backslash E)\backslash(X\backslash F),$$

that is, since $\mathcal{H}^{m^G}_{\partial(X\setminus E)}(x) = -\mathcal{H}^{m^G}_{\partial E}(x)$,

$$\mathcal{H}^{m^G}_{\partial E}(x) \geqslant \lambda + \frac{w_{x,x}}{d_x} \quad \forall x \in F\setminus E.$$

The proof of *(2)* is similar. □

With this results at hand, we obtain a priori estimates on the λ for which a set E can be a minimizer of $\mathcal{E}^G_m(\cdot, F, \lambda)$. Indeed, we must try with λ such that

$$\max\left\{ \sup_{x\in F\cap E}\left(\mathcal{H}^{m^G}_{\partial E}(x) + \frac{w_{x,x}}{d_x}\right), \sup_{x\notin F\cup E}\left(-\mathcal{H}^{m^G}_{\partial E}(x) + \frac{w_{x,x}}{d_x}\right)\right\}$$
$$\leqslant \lambda \leqslant \min\left\{ \inf_{x\in E\setminus F}\left(-\mathcal{H}^{m^G}_{\partial E}(x) - \frac{w_{x,x}}{d_x}\right), \inf_{x\in F\setminus E}\left(\mathcal{H}^{m^G}_{\partial E}(x) - \frac{w_{x,x}}{d_x}\right)\right\}.$$

Definition 4.28. Let (X, d, ν) be a metric measure space. For a measurable set $E \subset X$ we will write $x \in \partial_\nu E$ if

$$\nu(B_\epsilon(x) \cap E) > 0 \text{ and } \nu(B_\epsilon(x)\setminus E) > 0 \text{ for every } \epsilon > 0.$$

Corollary 4.29. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $m = m^{J,\Omega}$ be the random walk given in Example 1.42. Suppose further that* $\operatorname{supp}(J) = B_r(0)$. *Let E, F and λ be as in the hypothesis of Proposition 4.26 and suppose that $\partial_{\mathcal{L}^N} E$ is not empty. Let $x \in \partial_{\mathcal{L}^N} E$.*
(i) If there is a neighborhood $V \subset \Omega$ of x such that $\mathcal{L}^N(V \cap F) = 0$ then

$$-\lambda - 2\int_{\mathbb{R}^N\setminus\Omega} J(x-y)dy \leqslant \mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) \leqslant -\lambda \quad \textit{if } x \in E$$

and

$$-\lambda \leqslant \mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) \leqslant -\lambda + 2\int_{\mathbb{R}^N\setminus\Omega} J(x-y)dy \quad \textit{if } x \in \Omega\setminus E.$$

(ii) If there is a neighborhood $V \subset \Omega$ of x such that $\mathcal{L}^N(V\setminus F) = 0$ then

$$\lambda - 2\int_{\mathbb{R}^N\setminus\Omega} J(x-y)dy \leqslant \mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) \leqslant \lambda \quad \textit{if } x \in E$$

and

$$\lambda \leqslant \mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) \leqslant \lambda + 2\int_{\mathbb{R}^N\setminus\Omega} J(x-y)dy \quad \textit{if } x \in \Omega\setminus E.$$

In particular, for $x \in \partial_{\mathcal{L}^N} E$ such that $d(x, \mathbb{R}^N\setminus\Omega) \geqslant r$, either $x \in \partial_{\mathcal{L}^N} F$ or, if $x \notin \partial_{\mathcal{L}^N} F$, then $\mathcal{H}^{m^{J,\Omega}}_{\partial E}(x) = \pm\lambda$.

Proof. *(i)*: Let $x \in \partial_{\mathcal{L}^N} E$ such that we can find a neighborhood $V \subset \Omega$ of x with $\mathcal{L}^N(V \cap F) = 0$. Then, for $\epsilon > 0$ small enough, we have that $B_\epsilon(x) \subset \Omega$ and $\nu(B_\epsilon(x) \cap F) = 0$. Hence, by Proposition 4.26 *(1)(i)* and *(2)(ii)* with $A = B_\epsilon(x)$, and recalling Remark 1.61,

$$\frac{1}{\mathcal{L}^N(B_\epsilon(x)\setminus E)}\int_{B_\epsilon(x)\setminus E}\left(\int_{\Omega\setminus E} J(z-y)dy - \int_E J(z-y)dy + \int_{\mathbb{R}^N\setminus\Omega} J(z-y)dy\right)dz \geqslant -\lambda,$$

and

$$\frac{1}{\mathcal{L}^N(B_\epsilon(x)\cap E)}\int_{B_\epsilon(x)\cap E}\left(\int_{\Omega\setminus E} J(z-y)dy - \int_E J(z-y)dy - \int_{\mathbb{R}^N\setminus\Omega} J(z-y)dy\right)dz \leqslant -\lambda.$$

Now, since, $z \mapsto \int_{\Omega \setminus E} J(z-y)dy - \int_E J(z-y)dy$ and $z \mapsto \int_{\mathbb{R}^N \setminus \Omega} J(z-y)dy$ are continuous, we get, by letting ϵ tend to 0 in the above inequalities, that

$$\begin{aligned} -\lambda - \int_{\mathbb{R}^N \setminus \Omega} J(x-y)dy &\leqslant \int_{\Omega \setminus E} J(x-y)dy - \int_E J(x-y)dy \\ &\leqslant -\lambda + \int_{\mathbb{R}^N \setminus \Omega} J(x-y)dy. \end{aligned}$$

Therefore, we have that

$$-\lambda - 2\int_{\mathbb{R}^N \setminus \Omega} J(x-y)dy \leqslant \mathcal{H}_{\partial E}^{m^{J,\Omega}}(x) \leqslant -\lambda \quad \text{if } x \in E$$

and

$$-\lambda \leqslant \mathcal{H}_{\partial E}^{m^{J,\Omega}}(x) \leqslant -\lambda + 2\int_{\mathbb{R}^N \setminus \Omega} J(x-y)dy \quad \text{if } x \in \Omega \setminus E.$$

Consequently, if $d(x, \mathbb{R}^N \setminus \Omega) \geqslant r$, then $\int_{\mathbb{R}^N \setminus \Omega} J(x-y)dy = 0$ and

$$\mathcal{H}_{\partial E}^{m^{J,\Omega}}(x) = \int_{\Omega \setminus E} J(x-y)dy - \int_E J(x-y)dy = -\lambda.$$

A similar proof using Proposition 4.26 *(1)(ii)* and *(2)(i)* gives *(ii)*. □

Since the metric random walk space of the previous corollary satisfies the strong-Feller property (recall Definition 1.35), some of the results given there (though not all) will follow from the following result.

Proposition 4.30. *Let $[X, d, m, \nu]$ be a reversible metric random walk space. Let E, F and λ be as in the hypothesis of Proposition 4.26. Let $x \in \partial_\nu E$ and suppose that $[X, d, m, \nu]$ has the strong-Feller property at x. The following holds:*

(1) If there is a neighbourhood V of x such that $\nu(V \cap F) = 0$, then $\mathcal{H}_{\partial E}^m(x) = -\lambda$.

(2) If there is a neighbourhood V of x such that $\nu(V \setminus F) = 0$, then $\mathcal{H}_{\partial E}^m(x) = \lambda$.

(3) $|\mathcal{H}_{\partial E}^m(x)| \leqslant \lambda$.

In particular, if $[X, d, m, \nu]$ has the strong-Feller property, then

(1) $\mathcal{H}_{\partial E}^m(x) = -\lambda$ for every $x \in \partial_\nu E \cap int(X \setminus F)$.

(2) $\mathcal{H}_{\partial E}^m(x) = \lambda$ for every $x \in \partial_\nu E \cap int(F)$.

(3) $|\mathcal{H}_{\partial E}^m(x)| \leqslant \lambda$ for every $x \in \partial_\nu E$.

Proof. The proof follows by Proposition 4.26 and Lemma 1.64. Indeed, let us prove *(1)*. Take $A_n = B(x, \frac{1}{n}) \setminus E$, then, since $x \in \partial_\nu E$, we have that $\nu(A_n) > 0$. Therefore, by *(1)(i)* of Proposition 4.26, we have

$$\frac{1}{\nu(A_n)} \int_{A_n} \mathcal{H}_{\partial E}^m(y) d\nu(y) \geqslant -\lambda \ \text{ for } n \text{ large enough,}$$

and taking limits when $n \to \infty$, by Lemma 1.64, we get that $\mathcal{H}_{\partial E}^m(x) \geqslant -\lambda$.

To prove the opposite inequality we proceed analogously by taking $A_n = B(x, \frac{1}{n}) \cap E$ and using *(2)(ii)* of Proposition 4.26 (note that, since V is a neighborhood of x, $\nu(A_n \cap F) \leqslant \nu(V \cap F) = 0$ for n large enough). □

4.2.2. Thresholding Parameters. *In the local case it is well known (see [55]) that for $f = \chi_{B_r(0)}$ the solution u_λ of problem (4.4) is given by:*
(i) $u_\lambda = 0$ *if* $0 < \lambda \leqslant \frac{2}{r}$,
(ii) $u_\lambda = c\chi_{B_r(0)}$ *with* $0 \leqslant c \leqslant 1$ *if* $\lambda = \frac{2}{r}$,
(iii) $u_\lambda = \chi_{B_r(0)}$ *if* $\lambda \geqslant \frac{2}{r}$.

In [78, Proposition 5.2] *it is shown that this thresholding property holds true for a large class of calibrable sets in* $\mathbb{R}^2$. *Our goal now is to show that there is also a thresholding property in the nonlocal case.*

For a constant c, we will abuse notation and denote the constant function $c\chi_X$ by c whenever this is not misleading.

LEMMA 4.31. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $f \in L^1(X, \nu)$ and $\lambda_0 > 0$.*
(i) If $f \in M(f, \lambda_0)$ then

$$\{f\} = M(f, \lambda) \quad \forall \lambda > \lambda_0.$$

(ii) If $f \in L^2(X, \nu)$ and a constant $c \geqslant 0$ satisfies $c \in M(f, \lambda_0)$, then $c \in \text{med}_\nu(f)$,

$$\text{med}_\nu(f) \subset M(f, \lambda_0),$$

and

$$\text{med}_\nu(f) = M(f, \lambda) \quad \forall 0 < \lambda < \lambda_0.$$

(iii) Let $u \in L^1(X, \nu)$ and $\lambda_0 < \lambda_1$. If $u \in M(f, \lambda_0) \cap M(f, \lambda_1)$, then $u \in M(f, \lambda)$ for every $\lambda_0 \leqslant \lambda \leqslant \lambda_1$.

PROOF.
(i): Take $\lambda > \lambda_0$, then, for any $u \in L^1(X, \nu)$ such that $\nu(\{u \neq f\}) > 0$, we have

$$\mathcal{E}_m(f, f, \lambda) = TV_m(f) = \mathcal{E}_m(f, f, \lambda_0) \leqslant \mathcal{E}_m(u, f, \lambda_0) < \mathcal{E}_m(u, f, \lambda).$$

(ii): Since $c \in M(f, \lambda_0)$ we have that, by Theorem 4.18, there exists $\xi \in \text{sign}(c - f)$ and $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric satisfying

$$\int_X \mathbf{g}(x, y)\, dm_x(y) = \lambda_0 \xi(x) \quad \text{for } \nu\text{-a.e } x \in X \text{ and}$$

$$\mathbf{g}(x, y) \in \text{sign}(0) \quad \text{for } (\nu \otimes m_x)\text{-a.e. } (x, y) \in X \times X.$$

Then,

$$\int_X \xi d\nu(x) = \frac{1}{\lambda_0} \int_X \int_X \mathbf{g}(x, y)\, dm_x(y) d\nu(x) = 0,$$

so that $0 \in \text{med}_\nu(c - f)$, which is equivalent to $c \in \text{med}_\nu(f)$. Now, for $\lambda < \lambda_0$, taking $g_\lambda(x, y) = \frac{\lambda}{\lambda_0} g(x, y)$ we obtain that

$$c \in M(f, \lambda).$$

Furthermore, by (3.47), for any other $m \in \text{med}_\nu(f)$ and any $\lambda > 0$,

$$\mathcal{E}(c, f, \lambda_0) = \lambda \int_X |c - f| d\nu = \lambda \int_X |m - f| d\nu = \mathcal{E}(m, f, \lambda),$$

so that

$$\text{med}_\nu(f) \subset M(f, \lambda_0), \quad \forall 0 < \lambda \leqslant \lambda_0.$$

Now, let $m \in \text{med}_\nu(f)$, for any constant function $k \notin \text{med}_\nu(f)$, by (3.47) we have that

$$\int_X |k - f| d\nu > \int_X |m - f| d\nu$$

so $k \notin M(f, \lambda)$ for every $\lambda > 0$.

Suppose then that there exists some nonconstant function u, such that $u \in M(f, \lambda)$ for $0 < \lambda < \lambda_0$. Since ν is ergodic with respect to m we have that $TV_m(u) > 0$, thus

$$\mathcal{E}(u, f, \lambda) \leqslant \mathcal{E}(m, f, \lambda)$$

implies that

$$\int_X |u - f| d\nu < \int_X |m - f|$$

and, therefore,

$$\begin{aligned} \mathcal{E}(u, f, \lambda_0) &= \mathcal{E}(u, f, \lambda) + (\lambda_0 - \lambda) \int_X |u - f| d\nu \\ &< \mathcal{E}(m, f, \lambda) + (\lambda_0 - \lambda) \int_X |m - f| d\nu = \mathcal{E}(m, f, \lambda_0) \end{aligned}$$

which is a contradiction. Consequently,

$$\text{med}_\nu(f) = M(f, \lambda) \quad \forall 0 < \lambda < \lambda_0.$$

(iii) Follows easily. □

Proposition 4.32. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let (λ_0, u_0) be an m-eigenpair of the 1-Laplacian Δ_1^m on X with $\lambda_0 > 0$. Then, $0 \in med_\nu(u_0)$ and*

$$\begin{cases} med_\nu(u_0) = M(u_0, \lambda) & \text{if } 0 < \lambda < \lambda_0, \\ \{cu_0 : 0 \leqslant c \leqslant 1\} \cup med_\nu(u_0) \subset M(u_0, \lambda_0), & \\ \{u_0\} = M(u_0, \lambda) & \text{if } \lambda > \lambda_0. \end{cases}$$

Proof. Since (λ_0, u_0) is an m-eigenpair of the 1-Laplacian Δ_1^m with $\lambda_0 > 0$, we have that $0 \in \text{med}_\nu(u_0)$ (see Corollary 3.71). Furthermore, by the definition of m-eigenpair, we have that

$$\exists \xi_0 \in \text{sign}(u_0) \text{ such that } -\lambda_0 \xi_0 \in \Delta_1^m(u_0).$$

Hence, for $0 < c \leqslant 1$, $\xi := -\xi_0 \in \text{sign}(cu_0 - u_0)$ and $\lambda_0 \xi \in \Delta_1^m(u_0) = \Delta_1^m(cu_0)$, which implies that $cu_0 \in M(u_0, \lambda_0)$. Moreover, since $TV_m(u_0) = \lambda_0$ (see Remark 3.52) and $\|u_0\|_{L^1(X,\nu)} = 1$, we have that

$$\mathcal{E}(u_0, u_0, \lambda_0) = \lambda_0 = \mathcal{E}(0, u_0, \lambda_0).$$

Consequently, by Lemma 4.31, we get the rest of the thesis. □

Corollary 4.33. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$ such that $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)} \chi_\Omega\right)$ is an m-eigenpair (in particular, Ω is m-calibrable), then,*
(i) if $\nu(\Omega) < \frac{1}{2}\nu(X)$,

$$\begin{cases} \{0\} = M(\chi_\Omega, \lambda) & \text{if } 0 < \lambda < \lambda_\Omega^m, \\ \{c\chi_\Omega : 0 \leqslant c \leqslant 1\} \subset M(\chi_\Omega, \lambda_\Omega^m), & \\ \{\chi_\Omega\} = M(\chi_\Omega, \lambda) & \text{if } \lambda > \lambda_\Omega^m; \end{cases}$$

(ii) if $\nu(\Omega) = \frac{1}{2}\nu(X)$,

$$\begin{cases} \{c : 0 \leqslant c \leqslant 1\} = M(\chi_\Omega, \lambda) & \text{if } 0 < \lambda < \lambda_\Omega^m, \\ \{c\chi_\Omega + d\chi_{X \setminus \Omega} : 0 \leqslant d \leqslant c \leqslant 1\} \subset M(\chi_\Omega, \lambda_\Omega^m), & \\ \{\chi_\Omega\} = M(\chi_\Omega, \lambda) & \text{if } \lambda > \lambda_\Omega^m. \end{cases}$$

Proof. Note that, since $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of Δ_1^m, $\nu(\Omega) \leqslant \frac{1}{2}\nu(X)$. Now, if $\nu(\Omega) < \frac{1}{2}\nu(X)$, then $\text{med}_\nu(\chi_\Omega) = \{0\}$; and, if $\nu(\Omega) = \frac{1}{2}\nu(X)$, then $\text{med}_\nu(\chi_\Omega) = \{c : 0 \leqslant c \leqslant 1\}$. Consequently, the result follows by Proposition 4.32 and Corollary 4.20 with $T(r) = \nu(\Omega)r$.

In the case that $\nu(\Omega) = \frac{1}{2}\nu(X)$, we have

$$\{c\chi_\Omega \,:\, 0 \leqslant c \leqslant 1\} \cup \{c \,:\, 0 \leqslant c \leqslant 1\} \subset M(\chi_\Omega, \lambda_\Omega^m),$$

hence, since $M(\chi_\Omega, \lambda_\Omega^m)$ is convex, we get that

$$\{c\chi_\Omega + d\chi_{X\setminus\Omega} : 0 \leqslant d \leqslant c \leqslant 1\} \subset M(\chi_\Omega, \lambda_\Omega^m).$$

□

Proposition 4.34. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < 1$. If $\chi_\Omega \in M(\chi_\Omega, \lambda_\Omega^m)$ then $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair.*

Proof. Let us first see that Ω is m-calibrable. Indeed, for $E \in \mathcal{B}_\Omega$ with $0 < \nu(E) < \nu(\Omega)$, we have that

$$\begin{aligned} P_m(\Omega) &= \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \lambda_\Omega^m) \leqslant \mathcal{E}_m(\chi_E, \chi_\Omega, \lambda_\Omega^m) \\ &= P_m(E) + \lambda_\Omega^m(\nu(\Omega) - \nu(E)) = P_m(E) + P_m(\Omega) - \lambda_\Omega^m \nu(E), \end{aligned}$$

thus the m-calibrability of Ω follows.

Since Ω is m-calibrable, by Theorem 3.43 there exists an antisymmetric function $\mathbf{g}_0$ in $\Omega \times \Omega$ such that

$$-1 \leqslant \mathbf{g}_0(x,y) \leqslant 1 \quad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in \Omega \times \Omega,$$

and

$$\lambda_\Omega^m = -\int_\Omega \mathbf{g}_0(x,y)\, dm_x(y) + 1 - m_x(\Omega) \quad \text{for } \nu\text{-a.e. } x \in \Omega. \tag{4.25}$$

Now, if $\chi_\Omega \in M(\chi_\Omega, \lambda_\Omega^m)$, there exists $\xi_1 \in \text{sign}(0)$ such that

$$\lambda_\Omega^m \xi_1 \in \Delta_1^m(\chi_\Omega).$$

Therefore, there exists $g_1 \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric such that

$$-\int_X \mathbf{g}_1(x,y)\, dm_x(y) = -\lambda_\Omega^m \xi_1(x) \quad \text{for } \nu\text{-a.e } x \in X, \tag{4.26}$$

and

$$g_1(x,y) \in \text{sign}(\chi_\Omega(y) - \chi_\Omega(x)) \quad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in X \times X.$$

Let

$$g(x,y) := \begin{cases} g_0(x,y) & \text{if } (x,y) \in \Omega \times \Omega, \\ g_1(x,y) & \text{elsewhere} \end{cases}$$

and

$$\xi(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ -\xi_1(x) & \text{elsewhere.} \end{cases}$$

Then, (4.25) and (4.26) read as follows

$$\lambda_\Omega^m \xi(x) = -\int_X \mathbf{g}(x,y)\, dm_x(y), \quad \text{for } \nu\text{-a.e. } x \in \Omega$$

and

$$\lambda_\Omega^m \xi(x) = -\int_X \mathbf{g}(x,y)\, dm_x(y), \quad \text{for } \nu\text{-a.e } x \in X\setminus\Omega,$$

thus $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ is an m-eigenpair of Δ_1^m. □

COROLLARY 4.35. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < 1$. The following statements are equivalent:*
(i) $\chi_\Omega \in M(\chi_\Omega, \lambda_\Omega^m)$,
(ii) $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ *is an m-eigenpair,*
(iii) the following thresholding property holds:

$$\begin{cases} 0 \in M(\chi_\Omega, \lambda) & \forall\, 0 < \lambda \leqslant \lambda_\Omega^m, \\ \\ \chi_\Omega \in M(\chi_\Omega, \lambda) & \forall\, \lambda \geqslant \lambda_\Omega^m, \end{cases}$$

PROOF. The implication $(i) \Rightarrow (ii)$ follows by Proposition 4.34, while $(ii) \Rightarrow (iii)$ is a consequence of Corollary 4.33. The implication $(iii) \Rightarrow (i)$ is trivial. □

We say that a function $f \in BV_m(X, \nu) \cap L^{p'}(X, \nu)$ is maximal if the supremum in (3.6) is a maximum, that is, if there exists $\mathbf{z}_0 = \mathbf{z}_0(f) \in X_m^p(X, \nu)$ with $\|\mathbf{z}_0\|_{L^\infty(X \times X, \nu \otimes m_x)} \leqslant 1$ such that

$$TV_m(f) = \int_X f(x)(\mathrm{div}_m \mathbf{z}_0)(x) d\nu(x). \tag{4.27}$$

PROPOSITION 4.36. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. If $f \in BV_m(X, \nu)$ is a maximal function with $\mathbf{z}_0 = \mathbf{z}_0(f)$ satisfying equation (4.27), then, for $\lambda_* = \|\mathrm{div}_m z_0\|_{L^\infty(X,\nu)}$,*

$$f \in M(f, \lambda_*),$$

and, consequently, $M(f, \lambda) = \{f\}$ for all $\lambda > \lambda_*$.

PROOF. Given $u \in L^1(X, \nu)$, by Proposition 3.9, we have that

$$\mathcal{E}_m(u, f, \lambda_*) = TV_m(u) + \lambda_* \int_X |u - f| d\nu \geqslant \int_X u(x)(\mathrm{div}_m \mathbf{z}_0)(x) d\nu(x) + \lambda_* \int_X |u - f| d\nu$$
$$= \int_X f(x)(\mathrm{div}_m \mathbf{z}_0)(x) d\nu(x) + \lambda_* \int_X |u - f| d\nu + \int_X (u(x) - f(x))(\mathrm{div}_m \mathbf{z}_0)(x) d\nu(x)$$
$$\geqslant \mathcal{E}_m(f, f, \lambda_*) + \left(\lambda_* - \|\mathrm{div}_m \mathbf{z}_0\|_{L^\infty(X,\nu)}\right) \int_X |u - f| d\nu = \mathcal{E}_m(f, f, \lambda_*).$$

Therefore, $f \in M(f, \lambda_*)$. The rest of the thesis follows by Lemma 4.31. □

REMARK 4.37. Note that, since $\|\mathbf{z}_0\|_{L^\infty(X \times X, \nu \otimes m_x)} \leqslant 1$, we have that $\lambda_* \leqslant 2$.

PROPOSITION 4.38. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. For any $\Omega \in \mathcal{B}$, χ_Ω is a maximal function with $\mathbf{z}_0 = \mathbf{z}_0(\chi_\Omega)$ given by*

$$z_0(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \Omega \times \Omega, \\ -1 & \text{if } (x, y) \in (X \backslash \Omega) \times \Omega, \\ 1 & \text{if } (x, y) \in \Omega \times (X \backslash \Omega), \\ 0 & \text{if } (x, y) \in (X \backslash \Omega) \times (X \backslash \Omega). \end{cases}$$

Hence, if $\lambda_* = \|\mathrm{div}_m z_0\|_{L^\infty(X,\nu)}$, then

$$\chi_\Omega \in M(\chi_\Omega, \lambda_*),$$

and, moreover,

$$\lambda_*(\Omega) := \lambda_* = \|\chi_\Omega - m_{(.)}(\Omega)\|_{L^\infty(X,\nu)}$$

satisfies $0 < \lambda_* \leqslant 1$.

Proof. It is straightforward to see that χ_Ω is a maximal function with the given $\mathbf{z}_0$. It then follows by Proposition 4.36 that $\chi_\Omega \in M(\chi_\Omega, \lambda_*)$. Let us see the last equality. For $x \in \Omega$,

$$\begin{aligned}(\mathrm{div}_m \mathbf{z}_0)(x) &= \frac{1}{2}\left(\int_X z_0(x,y)dm_x(y) - \int_X z_0(y,x)dm_x(y)\right) = \\ &= \frac{1}{2}\int_{X\setminus\Omega} 1dm_x(y) - \frac{1}{2}\int_{X\setminus\Omega} -1dm_x(y) = m_x(X\setminus\Omega).\end{aligned}$$

Therefore,

$$\int_X \chi_\Omega(x)(\mathrm{div}_m \mathbf{z}_0)(x)d\nu(x) = \int_\Omega\int_{X\setminus\Omega} dm_x(y)d\nu(x) = P_m(\Omega).$$

Observe also that, for $x \in X\setminus\Omega$,

$$(\mathrm{div}_m \mathbf{z}_0)(x) = -m_x(\Omega).$$

Therefore,

$$\mathrm{div}_m \mathbf{z}_0(x) = \chi_\Omega(x) - m_x(\Omega),$$

and, consequently,

$$\lambda_* = \|\mathrm{div}_m \mathbf{z}_0\|_{L^\infty(X,\nu)} = \|\chi_\Omega - m_{(.)}(\Omega)\|_{L^\infty(X,\nu)}. \qquad \square$$

Remark 4.39. (i) We have that

$$\lambda_\Omega^m \leqslant \lambda_*(\Omega).$$

Otherwise, if $\lambda_*(\Omega) < \lambda_\Omega^m$, since $\chi_\Omega \in M(\chi_\Omega, \lambda_*)$, by Lemma 4.31 (i) we would have that $\chi_\Omega \in M(\chi_\Omega, \lambda_\Omega^m)$. Hence, by Proposition 4.34, $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair and then, by Proposition 4.33, $\chi_\Omega \notin M(\chi_\Omega, \lambda_*)$ which is a contradiction.

Note that, by Proposition 4.34,

$$\text{if } \lambda_\Omega^m = \lambda_*(\Omega) \text{ then } \left(\lambda_\Omega^m, \tfrac{1}{\nu(\Omega)}\chi_\Omega\right) \text{ is an } m\text{-eigenpair.}$$

Recall that, in Theorem 3.56, assuming that Ω is m-calibrable, we got that $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair under the weaker assumption that $\lambda_\Omega^m \geqslant m_x(\Omega)$ for all $x \in X\setminus\Omega$.
(ii) Furthermore,

$$\lambda_*(X\setminus\Omega) = \lambda_*(\Omega),$$

and, consequently, from the previous point,

$$\max\{\lambda_\Omega^m, \lambda_{X\setminus\Omega}^m\} \leqslant \lambda_*(\Omega).$$

Proposition 4.40. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$. There exists $\lambda(\Omega)$ satisfying*

$$max\{\lambda_\Omega^m, \lambda_{X\setminus\Omega}^m\} \leqslant \lambda(\Omega) \leqslant \lambda_*(\Omega)$$

and

$$\begin{cases} \chi_\Omega \notin M(\chi_\Omega, \lambda) & \text{if } 0 < \lambda < \lambda(\Omega), \\ \chi_\Omega \in M(\chi_\Omega, \lambda(\Omega)), & \\ \{\chi_\Omega\} = M(\chi_\Omega, \lambda) & \text{if } \lambda > \lambda(\Omega). \end{cases}$$

Furthermore,

$$\lambda(\Omega) = \lambda_\Omega^m \quad \text{if, and only if,} \quad \left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right) \text{ is an } m\text{-eigenpair,} \tag{4.28}$$

and

$$\lambda(\Omega) = \lambda_{X\setminus\Omega}^m \quad \text{if, and only if,} \quad \left(\lambda_{X\setminus\Omega}^m, \frac{1}{\nu(X\setminus\Omega)}\chi_{X\setminus\Omega}\right) \text{ is an } m\text{-eigenpair.} \tag{4.29}$$

PROOF. By Proposition 4.38, $\lambda_*(\Omega) \in \{\lambda : \chi_\Omega \in M(\chi_\Omega, \lambda)\} \neq \emptyset$. Set

$$\lambda(\Omega) := \inf\{\lambda : \chi_\Omega \in M(\chi_\Omega, \lambda)\}.$$

Then,

$$\lambda(\Omega) \leqslant \lambda_*(\Omega),$$

and, by Proposition 4.16,

$$\lambda(\Omega) = \min\{\lambda : \chi_\Omega \in M(\chi_\Omega, \lambda)\}.$$

Hence,

$$\chi_\Omega \in M(\chi_\Omega, \lambda(\Omega)),$$

and, by Lemma 4.31, $\{\chi_\Omega\} = M(\chi_\Omega, \lambda)$ for every $\lambda > \lambda(\Omega)$.

For $\lambda < \lambda_\Omega^m$, we have

$$\mathcal{E}_m(0, \chi_\Omega, \lambda) = \lambda\nu(\Omega) < P_m(\Omega) = \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \lambda)$$

so $\chi_\Omega \notin M(\chi_\Omega, \lambda)$. Moreover, for $\lambda < \lambda_{X\setminus\Omega}^m$, we have

$$\mathcal{E}_m(\chi_X, \chi_\Omega, \lambda) = \lambda\nu(X\setminus\Omega) < P_m(X\setminus\Omega) = P_m(\Omega) = \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \lambda)$$

so $\chi_\Omega \notin M(\chi_\Omega, \lambda)$. Consequently, we have that

$$\max\{\lambda_\Omega^m, \lambda_{X\setminus\Omega}^m\} \leqslant \lambda(\Omega) \leqslant \lambda_*(\Omega)\,.$$

Finally, (4.28) follows from Corollary 4.35, and (4.29) follows from Corollary 4.35 and Lemma 4.17. □

We have the following formula for the thresholding parameter $\lambda(\Omega)$.

PROPOSITION 4.41. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$, then*

$$\lambda(\Omega) = \sup\left\{\frac{P_m(\Omega) - P_m(E)}{\nu(\Omega \triangle E)} : E \in \mathcal{B},\, \nu(\Omega \triangle E) > 0\right\}.$$

PROOF. Set

$$\alpha := \sup\left\{\frac{P_m(\Omega) - P_m(E)}{\nu(\Omega \triangle E)} : E \in \mathcal{B},\, \nu(\Omega \triangle E) > 0\right\}$$

and let $E \in \mathcal{B}_\Omega$ with $\nu(\Omega \triangle E) > 0$. Then, since $\chi_\Omega \in M(\chi_\Omega, \lambda(\Omega))$, we have that

$$\mathcal{E}_m(\chi_E, \chi_\Omega, \lambda(\Omega)) = P_m(E) + \lambda(\Omega)\nu(\Omega \triangle E) \geqslant \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \lambda(\Omega)) = P_m(\Omega).$$

We obtain from this that

$$\lambda(\Omega) \geqslant \frac{P_m(\Omega) - P_m(E)}{\nu(\Omega \triangle E)},$$

and, hence,

$$\lambda(\Omega) \geqslant \alpha.$$

On the other hand, by (4.20) and the definition of α, for every $u \in L^1(X, \nu)$ we have

$$\mathcal{E}_m(u, \chi_\Omega, \alpha) = \int_0^1 \Big(P_m(E_t(u)) + \alpha\nu(E_t(u) \triangle \Omega)\Big)dt \geqslant P_m(\Omega) = \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \alpha),$$

thus $\chi_\Omega \in M(\chi_\Omega, \alpha)$, and, consequently,

$$\lambda(\Omega) \leqslant \alpha.$$

□

*It is known (see [**78**]) that a thresholding property for a set in $\mathbb{R}^2$ implies calibrability of the set. From the previous results we obtain the non-local counterpart of this result.*

PROPOSITION 4.42. *Let $[X, \mathcal{B}, m, \nu]$ be an m-connected reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < 1$, if there exists a thresholding parameter $\lambda^* > 0$ such that*

(1) $0 \in M(\chi_\Omega, \lambda) \quad \forall\, 0 < \lambda < \lambda^*$, *and*

(2) $\chi_\Omega \in M(\chi_\Omega, \lambda) \quad \forall \lambda > \lambda^*$,
then

$$\lambda(\Omega) = \lambda^* = \lambda_\Omega^m,$$

and $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ *is an m-eigenpair of Δ_1^m. In particular, Ω is m-calibrable.*

Proof. By (1), we have that

$$\mathcal{E}_m(0, \chi_\Omega, \lambda) \leqslant \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \lambda) \quad \forall\, 0 < \lambda < \lambda^*,$$

that is,

$$\lambda\nu(\Omega) \leqslant P_m(\Omega) \quad \forall\, 0 < \lambda < \lambda^*,$$

thus

$$\lambda \leqslant \lambda_\Omega^m \quad \forall\, 0 < \lambda < \lambda^*.$$

Hence, $\lambda^* \leqslant \lambda_\Omega^m$. On the other hand, by (2) and the definition of $\lambda(\Omega)$, $\lambda(\Omega) \leqslant \lambda^*$. Then, since $\lambda_\Omega^m \leqslant \lambda(\Omega)$, we get

$$\lambda_\Omega^m \leqslant \lambda(\Omega) \leqslant \lambda^* \leqslant \lambda_\Omega^m.$$

Thus, by Proposition 4.40, we have that $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of Δ_1^m. □

We now provide some results regarding a thresholding parameter under which the set of minimizers is formed by constant functions.

Proposition 4.43. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Let $\Omega \in \mathcal{B}$.*
(i) If there exists $\lambda > 0$ such that $0 \in M(\chi_\Omega, \lambda)$, then there exists $\lambda^0(\Omega)$ satisfying

$$0 < \lambda^0(\Omega) \leqslant h_1^m(\Omega)$$

and

$$\begin{cases} med_\nu(\chi_\Omega) = M(\chi_\Omega, \lambda) & \text{if } \ 0 < \lambda < \lambda^0(\Omega), \\ 0 \in M(\chi_\Omega, \lambda^0(\Omega)), & \\ 0 \notin M(\chi_\Omega, \lambda) & \text{if } \ \lambda > \lambda^0(\Omega). \end{cases}$$

(ii) If there exists $\lambda > 0$ such that $1 \in M(\chi_\Omega, \lambda)$, then there exists $\lambda^1(\Omega)$ satisfying

$$0 < \lambda^1(\Omega) \leqslant h_1^m(X \setminus \Omega)$$

and

$$\begin{cases} med_\nu(\chi_\Omega) = M(\chi_\Omega, \lambda) & \text{if } \ 0 < \lambda < \lambda^1(\Omega), \\ 1 \in M(\chi_\Omega, \lambda^1(\Omega)), & \\ 1 \notin M(\chi_\Omega, \lambda) & \text{if } \ \lambda > \lambda^1(\Omega). \end{cases}$$

Proof. (i): Let $\widetilde{\Omega} \subset \Omega$ be a measurable set, then

$$\mathcal{E}_m(\chi_{\widetilde{\Omega}}, \chi_\Omega, \lambda) - \mathcal{E}_m(0, \chi_\Omega, \lambda) = P_m(\widetilde{\Omega}) - \lambda\nu(\widetilde{\Omega}),$$

so that

$$\mathcal{E}_m(\chi_{\widetilde{\Omega}}, \chi_\Omega, \lambda) < \mathcal{E}_m(0, \chi_\Omega, \lambda) \Leftrightarrow \lambda > \lambda_{\widetilde{\Omega}}^m,$$

thus

$$0 \in M(\chi_\Omega, \lambda) \text{ implies } \lambda \leqslant h_1^m(\Omega).$$

Therefore, if we set

$$\lambda^0(\Omega) := \sup\{\lambda \,:\, 0 \in M(\chi_\Omega, \lambda)\},$$

we have that $\lambda^0(\Omega) \leqslant h_1^m(\Omega)$. Moreover, by Proposition 4.16, we have that

$$\lambda^0(\Omega) = \max\{\lambda > 0 \,:\, 0 \in M(\chi_\Omega, \lambda)\}$$

and this is the parameter that we were looking for.

(ii) follows from (i) and Lemma 4.17. □

We can set $\lambda^0(\Omega) = 0$ *if there is no* $\lambda > 0$ *such that* $0 \in M(\chi_\Omega, \lambda)$, *and* $\lambda^1(\Omega) = 0$ *if there is no* $\lambda > 0$ *such that* $1 \in M(\chi_\Omega, \lambda)$.

We have the following formula for the thresholding parameter $\lambda^0(\Omega)$.

Proposition 4.44. *Let* $[X, \mathcal{B}, m, \nu]$ *be a reversible random walk space and assume that* ν *is a probability measure. Let* $\Omega \in \mathcal{B}$, *then*

$$\lambda^0(\Omega) = \inf\left\{ \frac{P_m(E)}{\nu(\Omega) - \nu(\Omega \triangle E)} \ : E \in \mathcal{B},\, \nu(\Omega \triangle E) < \nu(\Omega) \right\}.$$

Moreover, if $0 < \nu(\Omega) < 1$ *and* $\lambda^0(\Omega) \geqslant \lambda_\Omega^m$ *then* Ω *is* m*-calibrable.*

Proof. Set

$$\alpha := \inf\left\{ \frac{P_m(E)}{\nu(\Omega) - \nu(\Omega \triangle E)} \ : E \in \mathcal{B},\, \nu(\Omega \triangle E) < \nu(\Omega) \right\}.$$

Since

$$\mathcal{E}_m(\chi_E, \chi_\Omega, \lambda^0(\Omega)) = P_m(E) + \lambda^0(\Omega)\nu(\Omega \triangle E) \quad \text{and} \quad \mathcal{E}_m(0, \chi_\Omega, \lambda^0(\Omega)) = \lambda^0(\Omega)\nu(\Omega),$$

we have that $\lambda^0(\Omega) \leqslant \alpha$. Let us see the opposite inequality. For this it is enough to prove that $0 \in M(\chi_\Omega, \alpha)$, that is

$$\mathcal{E}_m(u, \chi_\Omega, \alpha) \geqslant \mathcal{E}_m(0, \chi_\Omega, \alpha) \quad \forall u \in L^1(X, \nu).$$

By (4.20), this inequality is equivalent to

$$\int_0^1 \Big(P_m(E_t(u)) + \alpha\Big(\nu(E_t(u) \triangle \Omega) - \nu(\Omega)\Big)\Big) dt \geqslant 0 \quad \forall u \in L^1(X, \nu).$$

Now,

$$\begin{aligned}
&\int_0^1 \Big(P_m(E_t(u)) + \alpha\Big(\nu(E_t(u) \triangle \Omega) - \nu(\Omega)\Big)\Big) dt \\
&= \int_{\{t: \nu(E_t(u)\triangle\Omega) - \nu(\Omega) \geqslant 0\}} \Big(P_m(E_t(u)) + \alpha\Big(\nu(E_t(u) \triangle \Omega) - \nu(\Omega)\Big)\Big) dt \\
&\quad + \int_{\{t: \nu(E_t(u)\triangle\Omega) - \nu(\Omega) < 0\}} \Big(P_m(E_t(u)) + \alpha\Big(\nu(E_t(u) \triangle \Omega) - \nu(\Omega)\Big)\Big) dt,
\end{aligned}$$

but the first integral in the right hand side is trivially non-negative and the second one is also non-negative by the definition of α.

Let us see that, if $0 < \nu(\Omega) < 1$ and $\lambda^0(\Omega) \geqslant \lambda_\Omega^m$, then Ω is m-calibrable. Suppose that $0 \in M(\chi_\Omega, \lambda_\Omega^m)$. Then, by Theorem 4.18, there exists $\xi \in \operatorname{sign}(-\chi_\Omega)$ such that

$$\lambda_\Omega^m \xi \in \Delta_1^m 0.$$

Consequently, $\xi' := -\xi \in \operatorname{sign}(\chi_\Omega)$ satisfies $-\lambda_\Omega^m \xi' \in \Delta_1^m 0$ thus, by Remark 3.41(2), $-\lambda_\Omega^m \xi' \in \Delta_1^m \chi_\Omega$. Now, by Theorem 3.42, this is equivalent to Ω being m-calibrable. □

Remark 4.45. Note that, if $\chi_E \in M(\chi_\Omega, \lambda)$, then

$$\lambda^-(E) \leqslant \lambda \leqslant \lambda^+(E), \tag{4.30}$$

where

$$\lambda^-(E) := \sup\left\{ \frac{P_m(U) - P_m(E)}{\nu(\Omega \triangle E) - \nu(\Omega \triangle U)} : U \in \mathcal{B},\, \nu(\Omega \triangle U) > \nu(\Omega \triangle E) \right\},$$

$$\lambda^+(E) := \inf\left\{ \frac{P_m(U) - P_m(E)}{\nu(\Omega \triangle E) - \nu(\Omega \triangle U)} : U \in \mathcal{B},\, \nu(\Omega \triangle U) < \nu(\Omega \triangle E) \right\}.$$

Indeed, if $\chi_E \in M(\chi_\Omega, \lambda)$, then, for any $U \in \mathcal{B}$,

$$P_m(E) + \lambda\nu(\Omega \triangle E) \leqslant P_m(U) + \lambda\nu(\Omega \triangle U)$$

thus, if $\nu(\Omega \triangle U) > \nu(\Omega \triangle E)$, we have that

$$\lambda \geqslant \frac{P_m(U) - P_m(E)}{\nu(\Omega \triangle E) - \nu(\Omega \triangle U)},$$

and, if $\nu(\Omega \triangle U) < \nu(\Omega \triangle E)$, we have that

$$\lambda \leqslant \frac{P_m(U) - P_m(E)}{\nu(\Omega \triangle E) - \nu(\Omega \triangle U)}.$$

Furthermore, observe that, if $\chi_E \in M(\chi_\Omega, \lambda)$, then

$$P_m(E) = \inf \{ P_m(U) \, : \, U \in \mathcal{B}, \, \nu(\Omega \triangle U) = \nu(\Omega \triangle E) \} \, . \tag{4.31}$$

Conversely, (4.30) and (4.31) imply that $\chi_E \in M(\chi_\Omega, \lambda)$.

The following example proves that the minimizer when the observed image is the characteristic function of a set Ω need not be the characteristic function of a set contained in Ω. Note that in the continuous setting, when Ω is a bounded convex domain, it is known that for almost all $\lambda > 0$ there is a unique minimizer which, moreover, is the characteristic function of a set contained in Ω (see [**55**, Corollary 5.3]*). We also observe how, with the ROF-model with L^1-fidelity term, the scale space is mostly constant and makes sudden transitions at certain values of the scale parameter. In particular, we see how a set may suddenly vanish.*

Example 4.46. Consider the locally finite weighted discrete graph G with vertex set $X = \{1,2,3,4,5,6\}$ and weights $w_{1,2} = 5$, $w_{2,3} = 6$, $w_{3,4} = 2$, $w_{4,5} = 1$, $w_{5,6} = 3$ and $w_{i,j} = 0$ otherwise. Let $[X, d_G, m^G, \nu_G]$ be the associated metric random walk space and let $\Omega = \{1,2\}$.

We have that

$$\begin{cases} \{0\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } 0 < \lambda < \frac{1}{5} = \lambda^0(\Omega), \\ \{c\chi_{\{1,2,3,4\}} \, : \, c \in [0,1]\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } \lambda = \frac{1}{5}, \\ \{\chi_{\{1,2,3,4\}}\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } \frac{1}{5} < \lambda < \frac{1}{3}, \\ \{\chi_{\{1,2,3\}} + c\chi_{\{4\}} \, : \, c \in [0,1]\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } \lambda = \frac{1}{3}, \\ \{\chi_{\{1,2,3\}}\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } \frac{1}{3} < \lambda < \frac{1}{2}, \\ \{\chi_{\{1,2\}} + c\chi_{\{3\}} \, : \, c \in [0,1]\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } \lambda = \frac{1}{2}, \\ \{\chi_{\{1,2\}}\} = M(\chi_{\{1,2\}}, \lambda) & \text{for } \lambda > \frac{1}{2} = \lambda(\Omega). \end{cases}$$

Indeed, to start with, note that

$$\mathcal{E}_m(\chi_\Omega, \chi_\Omega, \lambda) = 6 =: h_1(\lambda),$$

$$\mathcal{E}_m(\chi_{\{1,2,3\}}, \chi_\Omega, \lambda) = 2 + 8\lambda =: h_2(\lambda),$$

$$\mathcal{E}_m(\chi_{\{1,2,3,4\}}, \chi_\Omega, \lambda) = 1 + 11\lambda =: h_3(\lambda),$$

and

$$\mathcal{E}_m(0, \chi_\Omega, \lambda) = 16\lambda =: h_4(\lambda).$$

We have that,

- if $0 \leqslant \lambda < \frac{1}{5}$, then $h_4(\lambda) < h_i(\lambda)$ for $i = 1,2,3$,
- if $\frac{1}{5} < \lambda < \frac{1}{3}$, then $h_3(\lambda) < h_i(\lambda)$ for $i = 1,2,4$,
- if $\frac{1}{3} < \lambda < \frac{1}{2}$, then $h_2(\lambda) < h_i(\lambda)$ for $i = 1,3,4$,

- and, if $\frac{1}{2} < \lambda \leqslant 1$, then $h_1(\lambda) < h_i(\lambda)$ for $i = 2, 3, 4$.

Moreover, for any other set $F \subseteq \{1,2,3,4,5,6\}$ different from $\{1,2\}$, $\{1,2,3\}$ and $\{1,2,3,4\}$, and for any $\lambda > 0$, we have that $\mathcal{E}_m(\chi_F, \chi_\Omega, \lambda)$ is larger than $\min\{h_1(\lambda), h_2(\lambda), h_3(\lambda), h_4(\lambda)\}$.

Following Remark 4.19, to see that $\chi_\Omega \in M(\chi_\Omega, \frac{1}{2})$, take

$$g(1,2) = -\frac{1}{10},\ g(2,3) = -1,\ g(3,4) = -1,\ g(4,5) = -\frac{1}{2},\ g(5,6) = 0$$

and

$$\xi(1) = -\frac{1}{5},\ \xi(2) = -1,\ \xi(3) = 1,\ \xi(4) = 1,\ \xi(5) = \frac{1}{4},\ \xi(6) = 0\ .$$

For $\lambda < \frac{1}{2}$, since $h_4(\lambda) > h_3(\lambda)$, we have that $\chi_\Omega \notin M(\chi_\Omega, \lambda)$. Moreover, by Lemma 4.31 (i) we get that

$$\{\chi_\Omega\} = M(\chi_\Omega, \lambda) \quad \text{for } \lambda > \frac{1}{2}.$$

Since $\mathcal{E}_m(\chi_{\{1,2,3\}}, \chi_\Omega, \frac{1}{2}) = \mathcal{E}_m(\chi_\Omega, \chi_\Omega, \frac{1}{2})$ we have that $\chi_{\{1,2,3\}} \in M(\chi_\Omega, \frac{1}{2})$ and using the convexity of $M(f, \lambda)$ we get that

$$\{\chi_{\{1,2\}} + c\chi_{\{3\}} \ :\ c \in [0,1]\} \subset M\left(\chi_{\{1,2\}}, 1/2\right).$$

Now, $\{1,2\}$ and $\{1,2,3\}$ are the unique minimizers of $\mathcal{E}_m^G(\cdot, \Omega, 1/2)$, thus, by Theorem 4.25, we have that

$$\{\chi_{\{1,2\}} + c\chi_{\{3\}} \ :\ c \in [0,1]\} = M\left(\chi_{\{1,2\}}, 1/2\right).$$

To see that $\chi_{\{1,2,3\}} \in M(\chi_\Omega, \frac{1}{3})$, take

$$g(1,2) = -\frac{1}{5},\ g(2,3) = -\frac{7}{9},\ g(3,4) = -1,\ g(4,5) = -1,\ g(5,6) = 0$$

and

$$\xi(1) = -\frac{3}{5},\ \xi(2) = -1,\ \xi(3) = 1,\ \xi(4) = 1,\ \xi(5) = \frac{3}{4},\ \xi(6) = 0\ .$$

Consequently, by Lemma 4.31 (iii) we have that $\chi_{\{1,2,3\}} \in M(\chi_\Omega, \lambda)$ for $\frac{1}{3} \leqslant \lambda \leqslant \frac{1}{2}$. Moreover, $\{1,2,3\}$ is the unique minimizer of $\mathcal{E}_m^G(\cdot, \Omega, \lambda)$ for such parameters λ thus, by Theorem 4.25, $\chi_{\{1,2,3\}}$ is the unique element in $M(\chi_\Omega, \lambda)$ for $\frac{1}{3} < \lambda < \frac{1}{2}$.

Since $\mathcal{E}_m(\chi_{\{1,2,3,4\}}, \chi_\Omega, \frac{1}{3}) = \mathcal{E}_m(\chi_{\{1,2,3\}}, \chi_\Omega, \frac{1}{3})$ we have that $\chi_{\{1,2,3,4\}} \in M(\chi_\Omega, \frac{1}{3})$ and, as above, by Theorem 4.25,

$$\{\chi_{\{1,2,3\}} + c\chi_{\{4\}} \ :\ c \in [0,1]\} = M\left(\chi_{\{1,2\}}, 1/3\right).$$

Now, to see that $\chi_{\{1,2,3,4\}} \in M(\chi_\Omega, \frac{1}{5})$, take

$$g(1,2) = -\frac{1}{5},\ g(2,3) = -\frac{8}{15},\ g(3,4) = -\frac{4}{5},\ g(4,5) = -1,\ g(5,6) = -\frac{1}{15}$$

and

$$\xi(1) = -1,\ \xi(2) = -1,\ \xi(3) = 1,\ \xi(4) = 1,\ \xi(5) = 1,\ \xi(6) = \frac{1}{3}\ .$$

Then again, by Lemma 4.31 (iii), we have that $\chi_{\{1,2,3,4\}} \in M(\chi_\Omega, \lambda)$ for $\frac{1}{5} \leqslant \lambda \leqslant \frac{1}{3}$ and as before, by Theorem 4.25, $\left\{\chi_{\{1,2,3,4\}}\right\} = M(\chi_\Omega, \lambda)$ for $\frac{1}{5} < \lambda < \frac{1}{3}$.

Finally, the fact that $\mathcal{E}_m(\chi_{\{1,2,3,4\}}, \chi_\Omega, \frac{1}{5}) = \mathcal{E}_m(0, \chi_\Omega, \frac{1}{5})$ gives, by Theorem 4.25, that

$$\{c\chi_{\{1,2,3,4\}} \ :\ c \in [0,1]\} = M\left(\chi_{\{1,2\}}, 1/5\right)$$

and, by Lemma 4.31 (ii), $\{0\} = M(\chi_\Omega, \lambda)$ for $0 \leqslant \lambda < \frac{1}{5}$.

Note that $\lambda_\Omega^m = \frac{3}{8} < \frac{1}{2} = \lambda(\Omega)$ thus, by Proposition 4.40, $(\frac{3}{8}, \frac{1}{16}\chi_\Omega)$ is not an m-eigenpair. However, Ω is m-calibrable since it consists of two points. Note also that

$$\frac{P_m(\Omega) - P_m(\{1,2,3\})}{\nu(\Omega \triangle \{1,2,3\})} = \frac{6-2}{8} = \frac{1}{2} = \lambda(\Omega),$$

$$\frac{P_m(\{1,2,3,4\})}{\nu(\Omega)-\nu(\Omega\bigtriangleup\{1,2,3,4\})}=\frac{1}{16-11}=\frac{1}{5}=\lambda^0(\Omega),$$

and, regarding Corollary 4.45,

$$\frac{P_m(\{1,2,3\})-P_m(\{1,2,3,4\})}{\nu(\Omega\bigtriangleup\{1,2,3,4\})-\nu(\Omega\bigtriangleup\{1,2,3\})}=\frac{2-1}{11-8}=\frac{1}{3}\,.$$

Finally, observe that by Corollary 4.27, since

$$\mathcal{H}^m_{\partial\{1,2,3,4\}}(4)=1-2m_4(\{1,2,3,4\}=1-2\frac{2}{3}=-\frac{1}{3},$$

in order for $\{1,2,3,4\}$ to be a minimizer of $\mathcal{E}^G_m(\cdot,\{1,2\},\lambda)$, we must have

$$\lambda\leqslant\frac{1}{3},$$

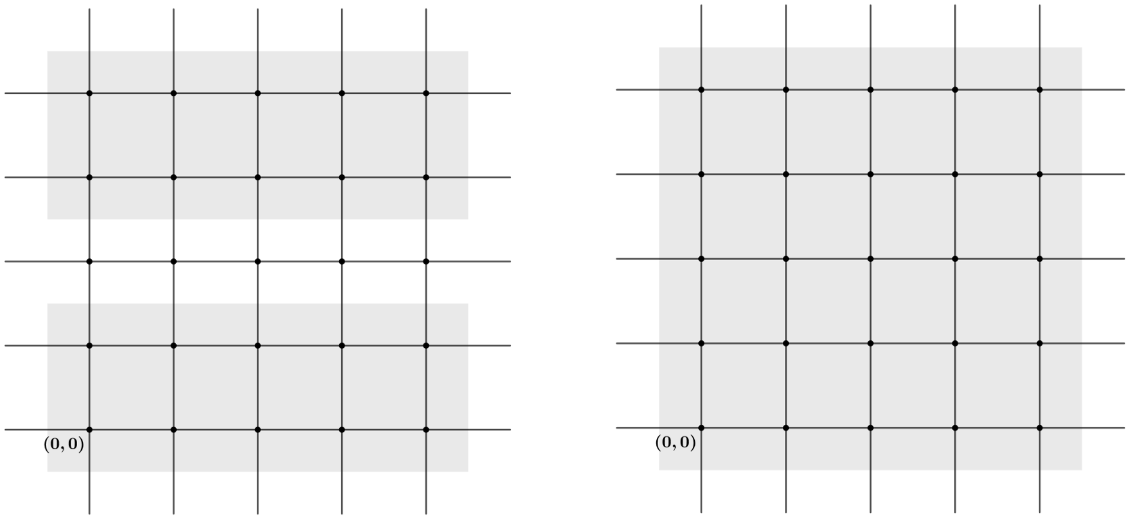
and $\frac{1}{3}$ is precisely the upper thresholding parameter for this set.

Observe that, if we add a loop at vertex 4, $w_{4,4}=\alpha>0$, the set $\{1,2,3,4\}$ can be a minimizer of $\mathcal{E}^G_m(\cdot,\{1,2\},\lambda)$ only if (by (4.23))

$$\lambda\leqslant-\mathcal{H}^m_{\partial\{1,2,3,4\}}(4)-\frac{w_{4,4}}{d_4}=\frac{1}{3+\alpha}<\frac{1}{3}\,.$$

In the following example, for which we avoid to give as much detail as in the previous one, we can see how, as the value of λ is decreased, minimizers become coarser as smaller objects merge together to form larger ones.

Example 4.47. Let $[\mathbb{Z}^2,d_{\mathbb{Z}^2},m^{\mathbb{Z}^2},\nu_{\mathbb{Z}^2}]$ be the metric random walk space given in Example 3.57*(2)* and consider the set Ω given in Figure 1a. Then, for $\frac{1}{3}<\lambda<\frac{2}{5}$, the minimizer for the ROF-model with the L^1-fidelity term and datum χ_Ω is the characteristic function of the set E represented in Figure 1b.



(A) Ω is the set formed by the points inside the shaded region.

(B) The minimizer, E, for $\frac{1}{3}<\lambda<\frac{2}{5}$ is the set formed by the points inside the shaded region.

Figure 1. The point $(0,0)$ is labelled in the graphs, and the adjacent points are represented by dots.

This set E merges together the two components of Ω. Note that

$$\mathcal{E}_m(\chi_\Omega,\chi_\Omega,\lambda)=28,$$

$$\mathcal{E}_m(\chi_E,\chi_\Omega,\lambda)=20+20\lambda,$$

and

$$\mathcal{E}_m(0, \chi_\Omega, \lambda) = 80\lambda.$$

By restricting the ambient space to a big enough bounded subset of $\mathbb{Z}^2$ and recalling Example 1.42 we obtain a finite invariant measure and the same calculations work.

4.2.3. The Gradient Descent Method. *In order to apply this method one needs to solve the Cauchy problem*

$$\begin{cases} v_t \in \Delta_1^m v(t) - \lambda \mathrm{sign}(v(t) - f) & \text{in } (0, T) \times X \\ \\ v(0, x) = v_0(x) & \text{in } X, \end{cases} \tag{4.32}$$

that can be rewritten as the following abstract Cauchy problem in $L^2(X, \nu)$

$$v'(t) + \partial \mathcal{E}_m(u, f, \lambda)(v(t)) \ni 0, \quad v(0) = v_0. \tag{4.33}$$

*Let f be in $L^1(X, \nu)$. Since $\mathcal{E}_m(\cdot, f, \lambda)$ is convex and lower semi-continuous, by the theory of maximal monotone operators ([**43**]), we have that, for any initial data $v_0 \in L^2(X, \nu)$, problem (4.33) has a unique strong solution. Therefore, if we define a solution of problem (4.32) as a function $v \in C(0, T; L^2(X, \nu)) \cap W^{1,1}_{loc}(0, T; L^2(X, \nu))$ such that $v(0, x) = v_0(x)$ for ν-a.e. $x \in X$ and such that there exists $\xi(t) \in \mathrm{sign}(v(t) - f)$ satisfying*

$$\lambda \xi(t) + v_t(t) \in \Delta_1^m(v(t)) \quad \text{for a.e. } t \in (0, T),$$

we have the following existence and uniqueness result.

THEOREM 4.48. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. For every $v_0 \in L^2(X, \nu)$ there exists a unique strong solution of the Cauchy problem (4.32) in $(0, T)$ for any $T > 0$. Moreover, we have the following contraction principle in any $L^q(X, \nu)$–space, $1 \leqslant q \leqslant \infty$:*

$$\|v(t) - w(t)\|_{L^q(X,\nu)} \leqslant \|v_0 - w_0\|_{L^q(X,\nu)} \quad \forall\, 0 < t < T, \tag{4.34}$$

for any pair of solutions v, w of problem (4.32) with initial datum v_0 and w_0, respectively.

Note that the contraction principle (4.34) in any L^q-space follows from the fact that the operator $\partial \mathcal{E}_m(\cdot, f, \lambda)$ is completely accretive. Indeed, given $(u_1, v_1), (u_2, v_2) \in \partial \mathcal{E}_m(\cdot, f, \lambda)$ and

$$p \in P_0 := \{q \in C^\infty(\mathbb{R}) \ : \ 0 \leqslant q' \leqslant 1, \quad supp(q') \text{ compact and } \ 0 \notin supp\},$$

we need to prove that

$$\int_X (v_2 - v_1) p(u_2 - u_1) d\nu \geqslant 0.$$

Now, there exist $\xi_i \in \mathrm{sign}(u_i - f)$ such that $v_i - \lambda \xi_i = w_i \in \partial \mathcal{F}_m(u_i)$, $i = 1, 2$. Then, since $\partial \mathcal{F}_m$ is a completely accretive operator and

$$\lambda \int_X (\xi_2 - \xi_1) p(u_2 - u_1) d\nu = \lambda \int_X (\xi_2 - \xi_1) p((u_2 - f) - (u_1 - f)) d\nu \geqslant 0,$$

we have that

$$\int_X (v_2 - v_1) p(u_2 - u_1) d\nu = \int_X (w_2 - w_1) p(u_2 - u_1) + \lambda \int_X (\xi_2 - \xi_1) p(u_2 - u_1) d\nu \geqslant 0.$$

Let $(T_\lambda(t))_{t \geqslant 0}$ be the semigroup in $L^2(X, \nu)$ associated with the operator $\partial \mathcal{E}_m(\cdot, f, \lambda)$, that is, $T_\lambda(t) v_0$ is the solution of problem (4.32). On account of the contraction principle we have that for any $u^* \in M(f, \lambda)$, if $\mathcal{L}_{u^*}(u) := \|u - u^*\|_{L^2(X,\nu)}$, then

$$\mathcal{L}_{u^*} \text{ is a Lyapunov functional for the semigroup } (T_\lambda(t))_{t \geqslant 0}. \tag{4.35}$$

Indeed, for $t > s$, we have

$$\begin{aligned} \mathcal{L}_{u^*}(T_\lambda(t) v_0) &= \|T_\lambda(t) v_0 - u^*\|_{L^2(X,\nu)} = \|T_\lambda(t - s)\, (T_\lambda(s) v_0) - T_\lambda(t - s) u^*\|_{L^2(X,\nu)} \\ &\leqslant \|T_\lambda(s) v_0 - u^*\|_{L^2(X,\nu)} = \mathcal{L}_{u^*}(T_\lambda(s) v_0). \end{aligned}$$

THEOREM 4.49. *Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space and assume that ν is a probability measure. Assume that $f \in L^1(X, \nu)$. Let $v_0 \in L^2(X, \nu)$ and $v(t) := T_\lambda(t)v_0$. If the ω-limit set*

$$\omega(v_0) := \{w \in L^2(X, \nu) \,:\, \exists t_n \to +\infty \ s.t. \lim_{n\to\infty} v(t_n) = w\}$$

is non-empty, then there exists $u^* \in M(f, \lambda)$ such that

$$\lim_{t\to\infty} v(t) = u^* \quad in\ L^2(X, \nu).$$

PROOF. Let $u^* \in \omega(v_0)$, then there exists $t_n \to +\infty$ such that

$$\lim_{n\to\infty} v(t_n) = u^*.$$

By [**43**, Theorem 3.2], we get

$$\frac{d^+ v}{dt}(t) + \partial\mathcal{E}_m(\cdot, f, \lambda)(v(t)) \ni 0 \quad \text{for all } t \in (0, +\infty) \tag{4.36}$$

and, by Theorem 4.24, we have that $M(f, \lambda) \neq \emptyset$ thus $0 \in R(\partial\mathcal{E}_m(\cdot, f, \lambda))$. Therefore, by [**43**, Theorem 3.10],

$$\lim_{n\to\infty} \frac{d^+ v}{dt}(t_n) = 0. \tag{4.37}$$

Since $\partial\mathcal{E}_m(\cdot, f, \lambda)$ is closed, from (4.36) and (4.37) we get

$$0 \in \partial\mathcal{E}_m(\cdot, f, \lambda)(u^*),$$

i.e., $u^* \in M(f, \lambda)$. Now, by (4.35), $\mathcal{L}_{u^*}$ is a Lyapunov functional for the semigroup $(T_\lambda(t))_{t\geq 0}$. It follows from this that

$$\lim_{t\to\infty} v(t) = u^* \quad \text{in } L^2(X, \nu).$$

□

Proving that the ω-limit set $\omega(v_0)$ is non-empty is not an easy task here. For example, one could try to proceed with the usual method of proving that the resolvent is compact, but this requires the use of regularity results which are difficult to obtain in our context due to the non-locality of the problem. Nonetheless, in finite graphs it is trivially true that the ω-limit set is non-empty. Consequently, we have the following result.

COROLLARY 4.50. *Let $[V(G), d_G, m^G, \nu_G]$ be the metric random walk space associated with a locally finite weighted discrete graph $G = (V(G), E(G))$. Suppose that ν_G is a probability measure. Then, for every $v_0 \in L^2(V(G), \nu_G)$ and for $v(t) := T_\lambda(t)v_0$, there exists $u^* \in M(f, \lambda)$ such that*

$$\lim_{t\to\infty} v(t) = u^* \quad in\ \ L^2(V(G), \nu_G).$$

CHAPTER 5

Nonlinear diffusion problems with nonlinear boundary conditions

In this chapter we study the existence and uniqueness of mild (see section A.3 of Appendix A) and strong solutions of nonlocal nonlinear diffusion problems of p-Laplacian type with nonlinear boundary conditions. The problems are posed in a subset W of a reversible random walk space $[X, \mathcal{B}, m, \nu]$. The nonlocal diffusion can hold either in W, in its nonlocal boundary $\partial_m W$, or in both at the same time. We will assume that W_m is m-connected (recall Definitions 1.51 and 1.32) and that $\nu(W_m) < \infty$. The formulations of the diffusion problems that we study are the following:

$$(5.1) \qquad \begin{cases} v_t(t,x) - \mathrm{div}_m \mathbf{a}_p u(t,x) = f(t,x), & x \in W,\ 0 < t < T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0 < t < T, \\ -\mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p} u(t,x) \in \beta(u(t,x)), & x \in \partial_m W,\ 0 < t < T, \\ v(0,x) = v_0(x), & x \in W, \end{cases}$$

and, for nonlinear dynamical boundary conditions,

$$(5.2) \qquad \begin{cases} v_t(t,x) - \mathrm{div}_m \mathbf{a}_p u(t,x) = f(t,x), & x \in W,\ 0 < t < T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0 < t < T, \\ w_t(t,x) + \mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p} u(t,x) = g(t,x), & x \in \partial_m W,\ 0 < t < T, \\ w(t,x) \in \beta(u(t,x)), & x \in \partial_m W,\ 0 < t < T, \\ v(0,x) = v_0(x), & x \in W, \\ w(0,x) = w_0(x), & x \in \partial_m W, \end{cases}$$

where γ and β are maximal monotone (multivalued) graphs in $\mathbb{R} \times \mathbb{R}$ with $0 \in \gamma(0) \cap \beta(0)$, $\mathrm{div}_m \mathbf{a}_p$ is a nonlocal Leray-Lions type operator whose model is the nonlocal p-Laplacian type diffusion operator, and $\mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p}$ is a nonlocal Neumann boundary operator (see Subsection 5.1.1 for details). In fact, we will solve these problems with greater generality, as we will not only consider them for a set W and its nonlocal boundary $\partial_m W$, but rather for any two disjoint sets Ω_1, $\Omega_2 \in \mathcal{B}$ such that their union is m-connected and of finite measure.

*These problems can be seen as the nonlocal counterpart of local diffusion problems governed by the p-Laplacian diffusion operator (or a Leray-Lions operator) where two further nonlinearities are induced by γ and β (see, for example, [**13**] and [**33**] for the local problems). In [**18**], and the references therein, one can find an interpretation of the nonlocal diffusion process involved in these kind of problems.*

*On the nonlinearities (brought about by) γ and β we do not impose any further assumptions aside from the natural one (see Ph. Bénilan, M. G. Crandall and P. Sacks' work [**33**]):*

$$0 \in \gamma(0) \cap \beta(0),$$

and (in order for diffusion to take place)

$$\nu(W)\Gamma^- + \nu(\partial_m W)\mathfrak{B}^- < \nu(W)\Gamma^+ + \nu(\partial_m W)\mathfrak{B}^+,$$

where

$$\mathit{\Gamma}^- := \inf\{\mathrm{Ran}(\gamma)\},\ \mathit{\Gamma}^+ := \sup\{\mathrm{Ran}(\gamma)\},\ \mathfrak{B}^- := \inf\{\mathrm{Ran}(\beta)\}\ \text{ and }\ \mathfrak{B}^+ := \sup\{\mathrm{Ran}(\beta)\}.$$

Therefore, we work with a rather general class of nonlocal nonlinear diffusion problems with nonlinear boundary conditions that, in particular, include the homogeneous Dirichlet boundary condition or the Neumann boundary condition.

With our general approach we are able to directly cover: obstacle problems, with unilateral or bilateral obstacles (either in W, in $\partial_m W$, or in both at the same time); the nonlocal counterpart of Stefan like problems, that involve monotone graphs like the graph inverse of

$$\theta_S(r) := \begin{cases} r & \text{if } r < 0, \\ [0,\lambda] & \text{if } r = 0, \\ \lambda + r & \text{if } r > 0, \end{cases}$$

for $\lambda > 0$ (these would take the place of γ in our general setting, and the same is true for the following examples); diffusion problems in porous media, where monotone graphs like $p_s(r) = |r|^{s-1}r$, $s > 0$, are involved; and Hele-Shaw type problems, which involve graphs like

$$H(r) := \begin{cases} 0 & \text{if } r < 0, \\ [0,1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases}$$

Moreover, if $\gamma(r) = 0$ in the first problem, then the dynamics only take place in the nonlocal boundary and we obtain the evolution problem for a nonlocal Dirichlet-to-Neumann operator as a particular case.

Motivation for the study of these nonlocal diffusion problems of p-Laplacian type involving nonlocal Neumann boundary operators is provided by the nonlocal Neumann boundary operators studied (for the linear case) in [**72**] *and* [**96**]. *Nevertheless, due to the generality of the hypotheses considered in our study, the results that we obtain lead to new existence and uniqueness results for a great range of problems. This is true even when the problems are considered on weighted discrete graphs or $\mathbb{R}^N$ with a random walk induced by a nonsingular kernel, spaces for which only some particular cases of these problems have been studied (some references are given afterwards). In these ambient spaces, and for the nonlocal p-Laplacian operator, Problem* (5.1) *has the following formulations (recall Example 1.38 and Definition 1.51, for the necessary definitions and notations):*

$$\begin{cases} v_t(t,x) = \dfrac{1}{d_x} \displaystyle\sum_{y \in V(G)} w_{x,y} |u(y) - u(x)|^{p-2}(u(y) - u(x)), & x \in W,\ 0 < t < T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0 < t < T, \\ \dfrac{1}{d_x} \displaystyle\sum_{y \in W_{m^G}} w_{x,y} |u(y) - u(x)|^{p-2}(u(y) - u(x)) \in \beta(u(t,x)), & x \in \partial_{m^G} W,\ 0 < t < T, \\ u(x,0) = u_0(x), & x \in W, \end{cases}$$

for weighted discrete graphs, and

$$\begin{cases} v_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(y-x)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x \in W,\ 0<t<T, \\ v(t,x) \in \gamma(u(t,x)), & x \in W,\ 0<t<T, \\ \displaystyle\int_{W_{m^J}} J(y-x)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy \in \beta(u(t,x)), & x \in \partial_{m^J} W,\ 0<t<T, \\ v(x,0) = v_0(x), & x \in W. \end{cases}$$

for the case of $\mathbb{R}^N$ with the random walk induced by the nonsingular kernel J. We have detailed these problems with well-known formulations in order to show the extent to which Problems (5.1) and (5.2) cover specific nonlocal problems of great interest.

Nonlinear semigroup theory will be the basis for the study of the existence and uniqueness of solutions of the above problems. This study will be developed in Section 5.3, where we prove, as a particular case of Theorem 5.22, the existence of mild solutions to Problem (5.2) for general data in L^1, and of strong solutions assuming extra integrability conditions on the data. Moreover, a contraction and comparison principle is obtained. The same is done for Problem (5.1) in Theorem 5.28. See [**23**], [**24**], [**28**], [**32**], [**43**], [**67**], [**68**] *and* [**69**] *for details on nonlinear semigroup theory; a summary of it can be found in Appendix A.*

To apply the nonlinear semigroup theory our first aim is to prove the existence and uniqueness of solutions of the problem

$$\begin{cases} \gamma(u(x)) - \mathrm{div}_m \mathbf{a}_p u(x) \ni \varphi(x), & x \in W, \\ \mathcal{N}_1^{\mathbf{a}_p} u(x) + \beta(u(x)) \ni \phi(x), & x \in \partial_m W, \end{cases} \tag{5.3}$$

for general maximal monotone graphs γ and β. This is the nonlocal counterpart of (local) quasilinear elliptic problems with nonlinear boundary conditions (see [**14**] *and* [**33**] *for the general study of the local case) and is an interesting problem in itself due to the generality with which we address it. To this aim, we will make use of the generalised Poincaré type inequalities introduced in Section 1.6, but even with this at hand we can not obtain compactness arguments like the ones used in the local theory or in fractional diffusion problems. Consequently, we have to make the most of boundedness and monotonicity arguments in order to prove our results, being the main ideas an implementation of those used in* [**14**] *and* [**33**] *(see also* [**16**] *for a very particular case). The same holds for the diffusion problems. The study of Problem (5.3) will be developed in Section 5.1, where we prove, for a more general problem, the existence of solutions (Theorem 5.15) and a contraction and comparison principle (Theorem 5.14). At the end of that section we deal with another nonlocal Neumann boundary operator.*

For linear or quasilinear elliptic problems with boundary conditions, obstacles complicate the existence of solutions. The appearance of this difficulty is better understood when one takes into account the continuity of the solution between the inside of the domain and the boundary via the trace. In fact, for a bounded smooth domain Ω in $\mathbb{R}^N$, γ with bounded domain $[0,1]$ and $\beta(r) = 0$ for all r, it is not possible to find a weak solution of

$$\begin{cases} -\Delta u + \gamma(u) \ni \varphi & \text{in } \Omega, \\ \nabla u \cdot \eta = \phi & \text{in } \partial\Omega, \end{cases}$$

for data satisfying $\varphi \leqslant 0$, $\phi \leqslant 0$ and $\phi \not\equiv 0$ (see [**14**]*). However, in our non-local setting this sort of continuity is not present and the study of these nonlocal diffusion problems with obstacles hence differs from the study of the local ones (see* [**15**] *for a detailed study of these local problems). In particular, we do not need to impose any assumptions on the nonlinearities γ and β aside from the natural ones.*

There is a very long list of references for the local elliptic and parabolic counterparts of the problems that we study; see, for example, [13], [14], [28], [29], [30], [33], [157], *and the references therein. See also* [103] *for a Hele-Shaw problem with dynamical boundary conditions and the references therein. For some particular nonlocal problems we refer to* [16], [17], [18], [34], [50], [60], [97], [106] *and* [125]. *For fractional diffusion problems we refer, for example, to* [119], *where Dirichlet and Neumann boundary conditions are considered; to* [39], [40], [65], [90] *and* [137], *where fractional porous medium equations are studied, see also J. L. Vázquez's survey* [158] *and the references therein; and to* [153] *and* [154] *for fractional diffusion problems for the Stefan problem.*

Assumption 1. *All along this chapter $[X, \mathcal{B}, m, \nu]$ is a reversible random walk space.*

5.0.1. Yosida approximation and a Bénilan-Crandall relation.

Given a maximal monotone graph ϑ in $\mathbb{R} \times \mathbb{R}$ (see [43]) and $\lambda > 0$, we denote by ϑ_λ the *Yosida approximation of* ϑ, which is given by $\vartheta_\lambda := \lambda(I - (I + \frac{1}{\lambda}\vartheta)^{-1})$.

The function ϑ_λ is maximal monotone and 2λ-Lipschitz. Moreover, $\lim_{\lambda \to +\infty} \vartheta_\lambda(s) = \vartheta^0(s)$ where

$$\vartheta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \vartheta(s) & \text{if } s \in D(\vartheta), \\ +\infty & \text{if } [s, +\infty) \cap D(\vartheta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(\vartheta) = \emptyset. \end{cases}$$

Furthermore, if $s \in D(\vartheta)$, $|\vartheta_\lambda(s)| \leqslant |\vartheta^0(s)|$ for every $\lambda > 0$, and $|\vartheta_\lambda(s)|$ is nondecreasing in λ.

Given a maximal monotone graph ϑ in $\mathbb{R} \times \mathbb{R}$ with $0 \in \vartheta(0)$, we define, for $s \in D(\vartheta)$,

$$\vartheta_+(s) := \begin{cases} \vartheta(s) & \text{if } s > 0, \\ \vartheta(0) \cap [0, +\infty] & \text{if } s = 0, \\ \{0\} & \text{if } s < 0, \end{cases}$$

and

$$\vartheta_-(s) := \begin{cases} \{0\} & \text{if } s > 0, \\ \vartheta(0) \cap [-\infty, 0] & \text{if } s = 0, \\ \vartheta(s) & \text{if } s < 0. \end{cases}$$

Note that the Yosida approximation $(\vartheta_+)_\lambda$ of ϑ_+ is nondecreasing in $\lambda > 0$ and $(\vartheta_-)_\lambda$ is nonincreasing in $\lambda > 0$. Observe also that $(\vartheta_+)_\lambda(s) = 0$ for $s \leqslant 0$ and $(\vartheta_-)_\lambda(s) = 0$ for $s \geqslant 0$, for every $\lambda > 0$, and $\vartheta_+ + \vartheta_- = \vartheta$.

The following lemma is easy to prove.

Lemma 5.1. *Let ϑ be a maximal monotone graph such that $0 \in \vartheta(0)$, $\lambda > 0$ and $r_\vartheta :=$* $\sup D(\vartheta) < +\infty$. *It holds that*

$$\vartheta_\lambda(r) = \lambda(r - r_\vartheta)$$

for every $r > r_\vartheta + \frac{1}{\lambda}\vartheta^0(r_\vartheta)$.

Remark 5.2. Given a maximal monotone graph ϑ with $0 \in D(\vartheta)$, $j_\vartheta(r) := \int_0^r \vartheta^0(s)ds$, $r \in \mathbb{R}$, defines a convex and lower semicontinuous function such that ϑ is equal to the subdifferential of j_ϑ:

$$\vartheta = \partial j_\vartheta.$$

Moreover, if $j_\vartheta{}^*$ is the Legendre transform of j_ϑ, then

$$\vartheta^{-1} = \partial j_\theta{}^*.$$

We now recall a Bénilan-Crandall relation between functions $u, v \in L^1(\Omega, \nu)$. Denote by J_0 and P_0 the following sets of functions:

$$J_0 := \{j : \mathbb{R} \to [0, +\infty] \ : \ j \text{ is convex, lower semi-continuous and } j(0) = 0\},$$

(5.4) $$P_0 := \left\{\rho \in C^\infty(\mathbb{R}) \ : \ 0 \leqslant \rho' \leqslant 1, \ supp(\rho') \text{ is compact and } 0 \notin supp(\rho)\right\}.$$

Assume that $\nu(\Omega) < +\infty$ *and let* $u, v \in L^1(\Omega, \nu)$. *The following relation between* u *and* v *is defined in* [**31**]:

$$u \ll v \quad \text{if} \int_\Omega j(u)\, d\nu \leqslant \int_\Omega j(v)\, d\nu \quad \text{for every } j \in J_0.$$

Moreover, the following equivalences are proved in [**31**, Proposition 2.2] *(we only give the particular cases that we will use):*

(5.5) $$\int_\Omega v\rho(u) d\nu \geqslant 0 \quad \forall \rho \in P_0 \iff u \ll u + \lambda v \quad \forall \lambda > 0,$$

(5.6) $$\int_\Omega v\rho(u) d\nu \geqslant 0 \quad \forall \rho \in P_0 \iff \int_{\{u<-h\}} v d\nu \leqslant 0 \leqslant \int_{\{u>h\}} v d\nu \quad \forall h > 0.$$

5.0.2. Two useful lemmas. *The proofs of the following lemmas are similar to the proof of* [**13**, Lemma 4.2].

LEMMA 5.3. *Let* $p \geqslant 1$. *Let* $[X, d, m, \nu]$ *be a reversible random walk space. Let* $A, B \in \mathcal{B}$ *be disjoint sets and assume that* $A \cup B$ *is non-*ν*-null and* m*-connected. Suppose that* $[X, \mathcal{B}, m, \nu]$ *satisfies a generalised* (p,p)*-Poincaré type inequality on* (A, B). *Let* α *and* τ *be maximal monotone graphs in* $\mathbb{R}^2$ *such that* $0 \in \alpha(0)$ *and* $0 \in \tau(0)$. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset L^p(A \cup B, \nu)$, $\{z_n\}_{n\in\mathbb{N}} \subset L^1(A, \nu)$ *and* $\{\omega_n\}_{n\in\mathbb{N}} \subset L^1(B, \nu)$ *be such that, for every* $n \in \mathbb{N}$, $z_n \in \alpha(u_n)$ ν*-a.e. in* A *and* $\omega_n \in \tau(u_n)$ ν*-a.e. in* B. *Finally, let* $Q_1 := (A \cup B) \times (A \cup B)$.

(i) Suppose that

$$\mathcal{R}^+_{\alpha,\tau} := \nu(A) \sup\{\mathrm{Ran}(\alpha)\} + \nu(B) \sup\{\mathrm{Ran}(\tau)\} = +\infty$$

and that there exists $M > 0$ *such that*

$$\int_A z_n^+ d\nu + \int_B \omega_n^+ d\nu < M \quad \forall n \in \mathbb{N}.$$

Then, there exists a constant $K = K(A, B, M, \alpha, \tau)$ *such that*

$$\|u_n^+\|_{L^p(A\cup B,\nu)} \leqslant K \left(\left(\int_{Q_1} |u_n^+(y) - u_n^+(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + 1 \right) \quad \forall n \in \mathbb{N}.$$

(ii) Suppose that

$$\mathcal{R}^-_{\alpha,\tau} := \nu(A) \inf\{\mathrm{Ran}(\alpha)\} + \nu(B) \inf\{\mathrm{Ran}(\tau)\} = -\infty$$

and that there exists $M > 0$ *such that*

$$\int_A z_n^- d\nu + \int_B \omega_n^- d\nu < M \quad \forall n \in \mathbb{N}.$$

Then, there exists a constant $\widetilde{K} = \widetilde{K}(A, B, M, \alpha, \tau)$, *such that*

$$\|u_n^-\|_{L^p(A\cup B,\nu)} \leqslant \widetilde{K} \left(\left(\int_{Q_1} |u_n^-(y) - u_n^-(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + 1 \right) \quad \forall n \in \mathbb{N}.$$

LEMMA 5.4. *Let* $p \geqslant 1$. *Let* $[X, d, m, \nu]$ *be a reversible random walk space. Let* $A, B \in \mathcal{B}$ *be disjoint sets and assume that* $A \cup B$ *is non-*ν*-null and* m*-connected. Suppose that* $[X, \mathcal{B}, m, \nu]$ *satisfies a generalised* (p,p)*-Poincaré type inequality on* (A, B). *Let* α *and* τ *be maximal monotone graphs in* $\mathbb{R}^2$ *such that* $0 \in \alpha(0)$ *and* $0 \in \tau(0)$. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset L^p(A \cup B, \nu)$, $\{z_n\}_{n\in\mathbb{N}} \subset L^1(A, \nu)$ *and* $\{\omega_n\}_{n\in\mathbb{N}} \subset L^1(B, \nu)$ *such that, for every* $n \in \mathbb{N}$, $z_n \in \alpha(u_n)$ ν*-a.e. in* A *and* $\omega_n \in \tau(u_n)$ ν*-a.e. in* B. *Finally, let* $Q_1 := (A \cup B) \times (A \cup B)$.

(i) Suppose that $\mathcal{R}^+_{\alpha,\tau} < +\infty$ and that there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_A z_n d\nu + \int_B \omega_n d\nu < M < \mathcal{R}^+_{\alpha,\tau} \quad \forall n \in \mathbb{N},$$

and

$$\max\left\{ \int_{\{x\in A\,:\, z_n<-h\}} |z_n| d\nu, \int_{x\in B\,:\,\omega_n(x)<-h\}} |\omega_n| d\nu \right\} < \frac{\mathcal{R}^+_{\alpha,\tau} - M}{8} \quad \forall n \in \mathbb{N}.$$

Then, there exists a constant $K = K(A, B, M, h, \alpha, \tau)$ such that

$$\|u_n^+\|_{L^p(A\cup B,\nu)} \leqslant K\left(\left(\int_{Q_1} |u_n^+(y) - u_n^+(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + 1 \right) \quad \forall n \in \mathbb{N}$$

(ii) Suppose that $\mathcal{R}^-_{\alpha,\tau} > -\infty$ and that there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_A z_n d\nu + \int_B \omega_n d\nu > M > \mathcal{R}^-_{\alpha,\tau} \quad \forall n \in \mathbb{N},$$

and

$$\max\left\{ \int_{\{x\in A\,:\, z_n>h\}} z_n d\nu, \int_{x\in B\,:\,\omega_n(x)>h\}} \omega_n d\nu \right\} < \frac{M - \mathcal{R}^-_{\alpha,\tau}}{8} \quad \forall n \in \mathbb{N}.$$

Then, there exists a constant $\widetilde{K} = \widetilde{K}(A, B, M, h, \alpha, \tau)$ such that

$$\|u_n^-\|_{L^p(A\cup B,\nu)} \leqslant \widetilde{K}\left(\left(\int_{Q_1} |u_n^-(y) - u_n^-(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + 1 \right) \quad \forall n \in \mathbb{N}.$$

5.1. Nonlocal stationary problems

In this section we will give our main results concerning the existence and uniqueness of solutions of the nonlocal stationary Problem (5.3). We will start by recalling the class of nonlocal Leray-Lions type operators and the Neumann boundary operators that we will be working with.

5.1.1. Nonlocal diffusion operators of Leray-Lions type and nonlocal Neumann boundary operators. *For $1 < p < +\infty$, we consider a function $\mathbf{a}_p : X \times X \times \mathbb{R} \to \mathbb{R}$ such that*

$$(x, y) \mapsto \mathbf{a}_p(x, y, r) \quad \text{is } \nu \otimes m_x\text{-measurable } \forall r \in \mathbb{R};$$

(5.7) $$\mathbf{a}_p(x, y, .) \text{ is continuous for } \nu \otimes m_x\text{-a.e } (x, y) \in X \times X;$$

(5.8) $$\mathbf{a}_p(x, y, r) = -\mathbf{a}_p(y, x, -r) \quad \text{for } \nu \otimes m_x\text{-a.e } (x, y) \in X \times X \text{ and } \forall r \in \mathbb{R};$$

(5.9) $$(\mathbf{a}_p(x, y, r) - \mathbf{a}_p(x, y, s))(r - s) > 0 \quad \text{for } \nu \otimes m_x\text{-a.e. } (x, y) \in X \times X \text{ and } \forall r \neq s;$$

there exist constants $c_p, C_p > 0$ such that

(5.10) $$|\mathbf{a}_p(x, y, r)| \leqslant C_p\left(1 + |r|^{p-1}\right) \quad \text{for } \nu \otimes m_x\text{-a.e. } (x, y) \in X \times X \text{ and } \forall r \in \mathbb{R},$$

and

(5.11) $$\mathbf{a}_p(x, y, r)r \geqslant c_p|r|^p \quad \text{for } \nu \otimes m_x\text{-a.e. } (x, y) \in X \times X \text{ and } \forall r \in \mathbb{R}.$$

This last condition implies that

$$\mathbf{a}_p(x, y, 0) = 0 \text{ and } sign_0(\mathbf{a}_p(x, y, r)) = sign_0(r) \text{ for } \nu \otimes m_x\text{-a.e. } (x, y) \in X \times X \text{ and } \forall r \in \mathbb{R}.$$

EXAMPLE 5.5. An example of a function $\mathbf{a}_p$ satisfying the above assumptions is

$$\mathbf{a}_p(x, y, r) := \frac{\varphi(x) + \varphi(y)}{2} |r|^{p-2} r,$$

being $\varphi : X \to \mathbb{R}$ a ν-measurable function satisfying $0 < c \leqslant \varphi \leqslant C$ where c and C are constants. In particular, if $\varphi(x) = 2$, $x \in X$, we have (recall Definition 1.44) that

$$\mathrm{div}_m(\mathbf{a}_p(x, y, \nabla u(x, y)))(x) = \mathrm{div}_m(\mathbf{a}_p(x, y, u(y) - u(x))(x)$$

$$= \int_X |u(y) - u(x)|^{p-2}(u(y) - u(x))dm_x(y) = \int_X |\nabla u(x, y)|^{p-2} \nabla u(x, y) dm_x(y)$$

is the (non-local) p-Laplacian operator on the random walk space $[X, \mathcal{B}, m, \nu]$.

Observe that $\mathrm{div}_m(\mathbf{a}_p(x, y, u(y) - u(x))(x)$ *defines a kind of Leray–Lions operator for the random walk* m.

We now introduce the nonlocal Neumann boundary operators that we will be working with.

DEFINITION 5.6. Let $W \in \mathcal{B}$ with $\nu(W) > 0$. The Gunzburger–Lehoucq type Neumann boundary operator on $\partial_m W$ is given by

$$\mathcal{N}_1^{\mathbf{a}_p} u(x) := -\int_{W_m} \mathbf{a}_p(x, y, u(y) - u(x)) dm_x(y), \quad x \in \partial_m W,$$

where, taking into account the supports of the measures m_x, we have that, in fact, the integral is effectively being calculated over the nonlocal tubular boundary $\partial_m W \cup \partial_m(X \backslash W)$ of W.

On the other hand, the Dipierro–Ros-Oton–Valdinoci type Neumann boundary operator on $\partial_m W$ is given by

$$\mathcal{N}_2^{\mathbf{a}_p} u(x) := -\int_{W} \mathbf{a}_p(x, y, u(y) - u(x)) dm_x(y) \quad x \in \partial_m W,$$

for which the integral is effectively being calculated over the nonlocal boundary $\partial_m(X \backslash W)$ of $X \backslash W$.

For each of these Neumann boundary operators we can look for solutions of the following problem

$$\begin{cases} \gamma(u(x)) - \mathrm{div}_m \mathbf{a}_p u(x) \ni \varphi(x), & x \in W, \\ \mathcal{N}_{\mathbf{j}}^{\mathbf{a}_p} u(x) + \beta(u(x)) \ni \phi(x), & x \in \partial_m W, \end{cases}$$

$\mathbf{j} \in \{1, 2\}$, *where we are using the simplified notation*

$$\mathrm{div}_m \mathbf{a}_p u(x) := \mathrm{div}_m(\mathbf{a}_p(x, y, u(y) - u(x))(x).$$

On account of (5.8), *we have that*

$$\mathrm{div}_m \mathbf{a}_p u(x) = \frac{1}{2} \int_X (\mathbf{a}_p(x, y, u(y) - u(x)) - \mathbf{a}_p(y, x, u(x) - u(y))) dm_x(y)$$

$$= \int_X \mathbf{a}_p(x, y, u(y) - u(x)) dm_x(y).$$

Moreover, by the reversibility of ν *with respect to* m *and recalling the definitions of* $\partial_m W$ *and* W_m *(Definition 1.51), we have that* $m_x(X \backslash W_m) = 0$ *for* ν*-a.e.* $x \in W$. *Indeed,*

$$\int_W m_x(X \backslash W_m) d\nu(x) = \int_{X \backslash W_m} m_x(W) d\nu(x) = 0.$$

Consequently, we have that, in fact,

$$\mathrm{div}_m \mathbf{a}_p u(x) = \int_{W_m} \mathbf{a}_p(x, y, u(y) - u(x)) dm_x(y) \quad \textit{for every } x \in W. \tag{5.12}$$

Lemma 5.7. *Let $\Omega \in \mathcal{B}$ with $\nu(\Omega) < \infty$ and let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega,\nu)$ such that $u_k \stackrel{k}{\longrightarrow} u \in L^p(\Omega,\nu)$ in $L^p(\Omega,\nu)$ and pointwise ν-a.e. in Ω. Suppose also that there exists $h \in L^p(\Omega,\nu)$ such that $|u_k| \leqslant h$ ν-a.e. in Ω. Then,*

$$\mathbf{a}_p(x,y,u_k(y)-u_k(x)) \stackrel{k}{\longrightarrow} \mathbf{a}_p(x,y,u(y)-u(x)) \quad \text{in } L^{p'}(\Omega\times\Omega,\nu\otimes m_x)$$

and, in particular, for ν-a.e. $x\in\Omega$,

$$\int_\Omega \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) \stackrel{k}{\longrightarrow} \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) \quad \text{in } L^{p'}(\Omega,\nu).$$

Proof. Let $A\subset\Omega$ be a ν-null set such that $|u_k(x)| \leqslant h(x) < +\infty$ for every $x\in\Omega\backslash A$ and every $k\in\mathbb{N}$, and such that $u_k(x) \stackrel{k}{\longrightarrow} u(x)$ for every $x\in\Omega\backslash A$. By (5.7), there exists a $\nu\otimes m_x$-null set $N_1\subset\Omega\times\Omega$ such that $\mathbf{a}_p(x,y,\cdot)$ is continuous for every $(x,y)\in(\Omega\times\Omega)\backslash N_1$. Therefore,

$$\mathbf{a}_p(x,y,u_k(y)-u_k(x)) \stackrel{k}{\longrightarrow} \mathbf{a}_p(x,y,u(y)-u(x))$$

for every $(x,y)\in(\Omega\times\Omega)\backslash(N_1\cup(A\times\Omega)\cup(\Omega\times A))$, where, by the reversibility of ν with respect to m, $N_1\cup(A\times\Omega)\cup(\Omega\times A)$ is also $\nu\otimes m_x$-null. Moreover, by (5.10), there exists a $\nu\otimes m_x$-null set $N_2\subset\Omega\times\Omega$ such that

$$\begin{aligned}|\mathbf{a}_p(x,y,u_k(x)-u_k(y))| &\leqslant C_p(1+|u_k(x)-u_k(y)|^{p-1}) \leqslant \widetilde{C}(1+|u_k(x)|^{p-1}+|u_k(y)|^{p-1})\\ &\leqslant \widetilde{C}(1+|h(x)|^{p-1}+|h(y)|^{p-1})\end{aligned}$$

for every $(x,y)\in(\Omega\times\Omega)\backslash(N_2\cup(A\times\Omega)\cup(\Omega\times A))$ and some constant $\widetilde{C}$, where, again, $N_2\cup(A\times\Omega)\cup(\Omega\times A)$ is $\nu\otimes m_x$-null. Then, taking $(x,y)\in(\Omega\times\Omega)\backslash(N_1\cup N_2\cup(A\times\Omega)\cup(\Omega\times A))$, we have that

$$\mathbf{a}_p(x,y,u_k(y)-u_k(x)) \stackrel{k}{\longrightarrow} \mathbf{a}_p(x,y,u(y)-u(x))$$

and

$$|\mathbf{a}_p(x,y,u_k(x)-u_k(y))| \leqslant \widetilde{C}(1+|h(x)|^{p-1}+|h(y)|^{p-1}).$$

Now, by the invariance of ν with respect to m, since $h\in L^p(\Omega,m_x)$ and $\nu(\Omega)<+\infty$, we have that $1+|h(x)|^{p-1}+|h(y)|^{p-1}\in L^{p'}(\Omega\times\Omega,\nu\otimes m_x)$ so we may apply the dominated convergence theorem to conclude. □

Remark 5.8. Taking a subsequence if necessary, the ν-a.e. pointwise convergence and the boundedness by the function h in the hypotheses are a consequence of the convergence in $L^p(\Omega,\nu)$.

5.1.2. Existence and uniqueness of solutions of doubly nonlinear stationary problems under nonlinear boundary conditions. *As mentioned in the introduction, the aim here is to study the existence and uniqueness of solutions of the problem*

$$(5.13)\qquad \begin{cases}\gamma(u(x)) - \mathrm{div}_m \mathbf{a}_p u(x) \ni \varphi(x), & x\in W,\\ \mathcal{N}_1^{\mathbf{a}_p}u(x)+\beta(u(x))\ni\phi(x), & x\in\partial_m W,\end{cases}$$

where $W\in\mathcal{B}$ is m-connected and $\nu(W_m)<+\infty$. See [14] and [33] for the reference local models. In Section 5.2 we will address this problem but with the nonlocal Neumann boundary operator $\mathcal{N}_2^{\mathbf{a}_p}$ instead.

Problem (5.13) is a particular case (recall (5.12)) of the following general, and interesting by itself, problem. Let $\Omega_1,\Omega_2\in\mathcal{B}$ be disjoint non-ν-null sets and let

$$\boxed{\Omega := \Omega_1\cup\Omega_2.}$$

Given $\varphi \in L^1(\Omega,\nu)$ we consider the problem

$$(5.14) \qquad \left(GP_\varphi^{\mathbf{a}_p,\gamma,\beta}\right) \quad \begin{cases} \gamma(u(x)) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) \ni \varphi(x), & x \in \Omega_1, \\ \beta(u(x)) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) \ni \varphi(x), & x \in \Omega_2. \end{cases}$$

For simplicity, we will generally use the notation (GP_φ) in place of $\left(GP_\varphi^{\mathbf{a}_p,\gamma,\beta}\right)$. However, we will use the more detailed notation further on. Moreover, we make the following assumptions.

ASSUMPTION 2. *We assume that $\Omega = \Omega_1 \cup \Omega_2$ is m-connected and $\nu(\Omega) < +\infty$.*

REMARK 5.9. Observe that, given an m-connected set $\Omega \in \mathcal{B}$, $m_x(\Omega) > 0$ for ν-a.e. $x \in \Omega$. Indeed, if

$$N := \{x \in \Omega \,:\, m_x(\Omega) = 0\}$$

then

$$L_m(N,\Omega) = 0$$

thus $\nu(N) = 0$.

ASSUMPTION 3. *Let*

$$\mathcal{N}_\perp^\Omega := \{x \in \Omega \,:\, (m_x \llcorner \Omega) \perp (\nu \llcorner \Omega)\}\,.$$

We assume that (recall that similar assumptions where considered in Section 1.6.1)

$$\nu\left(\mathcal{N}_\perp^\Omega\right) = 0.$$

REMARK 5.10. Note that, for $x \in \Omega$ such that $m_x(\Omega) > 0$, if $m_x \ll \nu$ (i.e., m_x is absolutely continuous with respect to ν) then $(m_x \llcorner \Omega) \not\perp (\nu \llcorner \Omega)$. Therefore, by Remark 5.9, if $m_x \ll \nu$ for ν-a.e. $x \in \Omega$ then $\nu\left(\mathcal{N}_\perp^\Omega\right) = 0$. Hence, the above condition is weaker than assuming that $m_x \ll \nu$ for ν-a.e. $x \in \Omega$.

ASSUMPTION 4. *We will assume, together with $0 \in \gamma(0) \cap \beta(0)$, that*

$$\mathcal{R}_{\gamma,\beta}^- < \mathcal{R}_{\gamma,\beta}^+,$$

where

$$\mathcal{R}_{\gamma,\beta}^- := \nu(\Omega_1)\inf\{\mathrm{Ran}(\gamma)\} + \nu(\Omega_2)\inf\{\mathrm{Ran}(\beta)\},$$
$$\mathcal{R}_{\gamma,\beta}^+ := \nu(\Omega_1)\sup\{\mathrm{Ran}(\gamma)\} + \nu(\Omega_2)\sup\{\mathrm{Ran}(\beta)\}.$$

Set

$$\boxed{Q_1 := \Omega \times \Omega.}$$

ASSUMPTION 5. *We assume that $[X,\mathcal{B},m,\nu]$ satisfies a generalised (p,p)-Poincaré type inequality on $(\Omega,\emptyset)$, i.e., given $0 < l \leqslant \nu(\Omega)$, there exists a constant $\Lambda > 0$ such that, for any $u \in L^p(\Omega,\nu)$ and any $Z \in \mathcal{B}$ with $\nu(Z) \geqslant l$,*

$$\|u\|_{L^p(\Omega,\nu)} \leqslant \Lambda\left(\left(\int_{Q_1}|u(y)-u(x)|^p dm_x(y)d\nu(x)\right)^{\frac{1}{p}} + \left|\int_Z u\,d\nu\right|\right).$$

From now in this chapter we work under Assumptions 1 to 5.

REMARK 5.11. Observe that, in fact, Assumption 5 implies that Ω is m-connected (which is part of Assumption 2). Indeed, suppose that Assumption 5 holds but Ω is not m-connected. Then, we may find non-ν-null sets A, $B \in \mathcal{B}_\Omega$ such that $A \cup B = \Omega$ and $L_m(A,B) = 0$ (recall Definition 1.32). Let us first suppose that $\nu(A \cap B) = 0$. Then, if we define $u : \Omega \to \mathbb{R}$ by

$$u(x) := \begin{cases} \dfrac{1}{\nu(A)}, & \text{if } x \in A, \\ -\dfrac{1}{\nu(B)}, & \text{if } x \in B\backslash A, \end{cases}$$

we have that $u \in L^p(\Omega,\nu)$, $\int_\Omega u d\nu = 0$ and

$$\int_\Omega \int_\Omega |u(y)-u(x)|^p dm_x(y) d\nu(x) = 0.$$

Therefore, by Assumption 5, we get that $\|u\|_{L^p(\Omega,\nu)} = 0$ which is a contradiction.

If $\nu(A \cap B) > 0$ then $\nu(A\backslash B) > 0$, $\nu(B\backslash A) > 0$ or $\nu(A) = \nu(B) = \nu(\Omega)$. In the first two cases we have that $L_m(A\backslash B, B) \leqslant L_m(A,B) = 0$ and $L_m(A, B\backslash A) \leqslant L_m(A,B) = 0$, respectively, so we work as before. If $\nu(A) = \nu(B) = \nu(\Omega)$ then $L_m(\Omega,\Omega) = 0$ and we may take, for example, $A' = \Omega_1$ and $B' = \Omega_2$ in our previous argument.

Definition 5.12. A solution of (GP_φ) is a pair $[u,v]$ with $u \in L^p(\Omega,\nu)$ and $v \in L^{p'}(\Omega,\nu)$ such that

1. $v(x) \in \gamma(u(x))$ for ν-a.e. $x \in \Omega_1$,
2. $v(x) \in \beta(u(x))$ for ν-a.e. $x \in \Omega_2$,
3. $[(x,y) \mapsto a_p(x,y,u(y)-u(x))] \in L^{p'}(\Omega\times\Omega, \nu \otimes m_x)$,
4. and

$$v(x) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) = \varphi(x), \quad x \in \Omega.$$

A subsolution (supersolution) to (GP_φ) is a pair $[u,v]$ with $u \in L^p(\Omega,\nu)$ and $v \in L^1(\Omega,\nu)$ satisfying 1., 2., 3. and

$$v(x) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) \leqslant \varphi(x), \quad x \in \Omega,$$

$$\left(v(x) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) \geqslant \varphi(x), \quad x \in \Omega \right).$$

Remark 5.13 (Integration by parts formula). The following integration by parts formula follows by the reversibility of ν with respect to m (recall Lemma 1.48). Let $q \geqslant 1$. Let $u : X \to \mathbb{R}$ be a measurable function such that

$$[(x,y) \mapsto \mathbf{a}_p(x,y,u(y)-u(x))] \in L^q(Q_1, \nu \otimes m_x)$$

and let $w \in L^{q'}(\Omega,\nu)$, then

$$\begin{aligned} &-\int_\Omega \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) w(x) d\nu(x) \\ &= -\int_{\Omega_1} \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) w(x) d\nu(x) \\ &\quad -\int_{\Omega_2} \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) w(x) d\nu(x) \\ &= \frac{1}{2} \int_{Q_1} \mathbf{a}_p(x,y,u(y)-u(x))(w(y)-w(x)) d(\nu\otimes m_x)(x,y). \end{aligned}$$

Let us see, formally, the way in which we will use the above integration by parts formula in what follows. Suppose that we are in the following situation:

$$\begin{cases} -\displaystyle\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) = f(x), & x \in \Omega_1, \\ -\displaystyle\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x)) dm_x(y) = g(x), & x \in \Omega_2. \end{cases}$$

Then, multiplying both equations by a test function w, integrating them with respect to ν over Ω_1 and Ω_2, respectively, adding them and using the integration by parts formula we get

$$\begin{aligned}&\frac{1}{2}\int_{Q_1} \mathbf{a}_p(x,y,u(y)-u(x))(w(y)-w(x))d(\nu\otimes m_x)(x,y)\\ &=\int_{\Omega_1} f(x)w(x)d\nu(x)+\int_{\Omega_2} g(x)w(x)d\nu(x).\end{aligned}$$

Moreover, as a consequence of these computations and (5.9), taking $u=u_i$, $f=f_i$ and $g=g_i$, $i=1,2$, in the above system and given a nondecreasing function $T:\mathbb{R}\to\mathbb{R}$ we obtain

$$\begin{aligned}&\int_{\Omega_1}(f_1(x)-f_2(x))T(u_1(x)-u_2(x))d\nu(x)+\int_{\Omega_2}(g_1(x)-g_2(x))T(u_1(x)-u_2(x))d\nu(x)\\ &=\frac{1}{2}\int_{Q_1}(\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))\\ &\qquad\times(T(u_1(y)-u_2(y))-T(u_1(x)-u_2(x)))d(\nu\otimes m_x)(x,y)\geqslant 0.\end{aligned}$$

The next result gives a maximum principle for solutions of Problem (GP_φ) *given in* (5.14) *and, consequently, also for solutions of Problem* (5.13).

THEOREM 5.14 (Contraction and comparison principle). *Let* $\varphi_1,\ \varphi_2\in L^1(\Omega,\nu)$. *Let* $[u_1,v_1]$ *be a subsolution of* (GP_{φ_1}) *and* $[u_2,v_2]$ *be a supersolution of* (GP_{φ_2}). *Then,*

$$\int_\Omega (v_1-v_2)^+d\nu\leqslant\int_\Omega(\varphi_1-\varphi_2)^+d\nu. \tag{5.15}$$

Moreover, if $\varphi_1\leqslant\varphi_2$ *with* $\varphi_1\neq\varphi_2$*, then* $v_1\leqslant v_2$, $v_1\neq v_2$*, and* $u_1\leqslant u_2$ ν*-a.e. in* Ω.

Furthermore, if $\varphi_1=\varphi_2$ *and* $[u_i,v_i]$ *is a solution of* (GP_{φ_i}), $i=1,2$, *then* $v_1=v_2$ ν*-a.e. in* Ω *and* u_1-u_2 *is* ν*-a.e. equal to a constant.*

PROOF. By hypothesis we have that

$$v_1(x)-v_2(x)-\int_\Omega(\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))dm_x(y)\leqslant\varphi_1(x)-\varphi_2(x)$$

for $x\in\Omega$. Multiplying this inequality by $\frac{1}{k}T_k^+(u_1-u_2+k\operatorname{sign}_0^+(v_1-v_2))$ and integrating over Ω we get

$$\begin{aligned}&\int_\Omega(v_1(x)-v_2(x))\,\frac{1}{k}T_k^+(u_1(x)-u_2(x)+k\operatorname{sign}_0^+(v_1(x)-v_2(x)))d\nu(x)\\ &\quad-\int_\Omega\int_\Omega(\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))dm_x(y)\\ &\qquad\times\frac{1}{k}T_k^+(u_1(x)-u_2(x)+k\operatorname{sign}_0^+(v_1(x)-v_2(x)))d\nu(x)\\ &\leqslant\int_\Omega(\varphi_1(x)-\varphi_2(x))\frac{1}{k}T_k^+(u_1(x)-u_2(x)+k\operatorname{sign}_0^+(v_1(x)-v_2(x)))d\nu(x)\\ &\leqslant\int_\Omega(\varphi_1(x)-\varphi_2(x))^+d\nu(x).\end{aligned} \tag{5.16}$$

Moreover, by the integration by parts formula, we have that

$$
\begin{aligned}
&-\int_\Omega\int_\Omega (\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))dm_x(y)\\
&\qquad\qquad\times\frac{1}{k}T_k^+\left(u_1(x)-u_2(x)+k\,\mathrm{sign}_0^+(v_1(x)-v_2(x))\right)d\nu(x)\\
&=\frac{1}{2}\int_\Omega\int_\Omega(\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))\\
&\qquad\times\bigg(\frac{1}{k}T_k^+(u_1(y)-u_2(y)+k\,\mathrm{sign}_0^+(v_1(y)-v_2(y)))\\
&\qquad\qquad-\frac{1}{k}T_k^+(u_1(x)-u_2(x)+k\,\mathrm{sign}_0^+(v_1(x)-v_2(x)))\bigg)dm_x(y)d\nu(x).
\end{aligned}
$$

Now, since the integrand on the right hand side is bounded from below by an integrable function, we can apply Fatou's lemma to get (recall the last observation in Remark 5.13)

$$
\begin{aligned}
\liminf_{k\to 0^+} -\int_\Omega\int_\Omega&(\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))dm_x(y)\\
&\times\frac{1}{k}T_k^+(u_1(x)-u_2(x)+k\,\mathrm{sign}_0^+(v_1(x)-v_2(x)))d\nu(x)\geqslant 0.
\end{aligned}
$$

Hence, taking limits in (5.16), we get

$$
\begin{aligned}
&\int_\Omega (v_1(x)-v_2(x))^+\,d\nu(x)\\
&=\lim_{k\to 0^+}\int_\Omega (v_1(x)-v_2(x))\,\frac{1}{k}T_k^+(u_1(x)-u_2(x)+k\,\mathrm{sign}_0^+(v_1(x)-v_2(x)))d\nu(x)\\
&\leqslant \int_\Omega(\varphi_1(x)-\varphi_2(x))^+d\nu(x),
\end{aligned}
$$

and (5.15) is proved.

Take now $\varphi_1\leqslant\varphi_2$ with $\varphi_1\neq\varphi_2$, then, by (5.15), we have that $v_1\leqslant v_2$ ν-a.e. in Ω. Now, since $[u_1,v_1]$ is a subsolution of (GP_{φ_1}) we have that

$$
v_1(x)-\int_\Omega \mathbf{a}_p(x,y,u_1(y)-u_1(x))dm_x(y)\leqslant\varphi_1(x)
$$

thus

$$
\int_\Omega v_1(x)d\nu(x)-\underbrace{\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_1(y)-u_1(x))dm_x(y)d\nu(x)}_{=0}\leqslant\int_\Omega\varphi_1(x)d\nu(x).
$$

Therefore, with the same calculation for $[u_2,v_2]$, we have that

$$
\int_\Omega v_1(x)d\nu(x)\leqslant\int_\Omega\varphi_1(x)d\nu(x)<\int_\Omega\varphi_2(x)d\nu(x)\leqslant\int_\Omega v_2(x)d\nu(x)
$$

thus $v_1\neq v_2$. Now, since $(\varphi_1-\varphi_2)^+=0$ and $(v_1-v_2)^+=0$, from (5.16) we get that

$$
\begin{aligned}
0\geqslant&\int_\Omega (v_1(x)-v_2(x))\,\frac{1}{k}T_k^+(u_1(x)-u_2(x)))d\nu(x)\\
&-\int_\Omega\int_\Omega(\mathbf{a}_p(x,y,u_1(y)-u_1(x))-\mathbf{a}_p(x,y,u_2(y)-u_2(x)))\\
&\qquad\qquad\times\frac{1}{k}T_k^+(u_1(x)-u_2(x))dm_x(y)d\nu(x).
\end{aligned}
$$

However, $u_1(x)\leqslant u_2(x)$ for ν-a.e. $x\in\Omega$ such that $v_1(x)<v_2(x)$, so

$$
(v_1(x)-v_2(x))\,\frac{1}{k}T_k^+(u_1(x)-u_2(x)))=0
$$

for ν-a.e. $x \in \Omega$, and we have that

$$-\int_\Omega \int_\Omega (\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x)))\frac{1}{k}T_k^+(u_1(x)-u_2(x))dm_x(y)d\nu(x)$$

is non-positive. Now, recalling Remark 5.13 and (5.9), we obtain

$$\int_\Omega \int_\Omega (\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x))) \\ \times((u_1(y)-u_2(y))^+ - (u_1(x)-u_2(x))^+)dm_x(y)d\nu(x) = 0$$

thus

(5.17)
$$(\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x)))((u_1(y)-u_2(y))^+ - (u_1(x)-u_2(x))^+) = 0$$

for $(x,y) \in (\Omega\times\Omega)\backslash N$ where $N \subset \Omega\times\Omega$ is a $\nu\otimes m_x$-null set. Let $C \subset \Omega$ be a ν-null set such that the section $N_x := \{y \in \Omega : (x,y) \in N\}$ of N is m_x-null for every $x \in \Omega\backslash C$ and let's see that $u_1(x) \leqslant u_2(x)$ for every $x \in \Omega\backslash(C \cup \mathcal{N}_\perp^\Omega)$ (recall Assumption 3 for the definition of the ν-null set $\mathcal{N}_\perp^\Omega$). Suppose that there exists $x_0 \in \Omega\backslash(C\cup\mathcal{N}_\perp^\Omega)$ such that $u_1(x_0)-u_2(x_0) > 0$. Then, from (5.17) (and (5.9)) we get that $u_1(y)-u_2(y) = u_1(x_0)-u_2(x_0) > 0$ for every $y \in \Omega\backslash N_{x_0}$. Let

$$S := \{y \in \Omega : u_1(y)-u_2(y) = u_1(x_0)-u_2(x_0)\} \supset \Omega\backslash N_{x_0},$$

then, since $x_0 \notin \mathcal{N}_\perp^\Omega$ and $m_{x_0}(N_{x_0}) = 0$, we must have $\nu(S) \geqslant \nu(\Omega\backslash N_{x_0}) > 0$. Now, following the same argument as before, if $x \in S$ then $\Omega\backslash N_x \subset S$ thus $m_x(\Omega\backslash S) \leqslant m_x(N_x) = 0$ and, therefore,

$$L_m(S,\Omega\backslash S) = 0.$$

However, since Ω is m-connected and $\nu(S) > 0$ we must have $\nu(\Omega\backslash S) = 0$ thus $u_1(y)-u_2(y) = u_1(x_0)-u_2(x_0) > 0$ for ν-a.e. $y \in \Omega$. This contradicts that $v_1 \leqslant v_2$, $v_1 \neq v_2$, ν-a.e. in Ω.

Finally, suppose that $[u_1,v_1]$ and $[u_2,v_2]$ are solutions of (GP_φ) for some $\varphi \in L^1(\Omega,\nu)$. Then,

$$v_1(x)-v_2(x) - \int_\Omega (\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x)))dm_x(y) = 0$$

thus, since $v_1 = v_2$ ν-a.e. in Ω,

$$-\int_\Omega (\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x)))dm_x(y) = 0.$$

Multiplying this equation by u_1-u_2, integrating over Ω and using the integration by parts formula as in Remark 5.13 we get

$$\int_\Omega \int_\Omega (\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x))) \\ \times (u_1(y)-u_1(x)-(u_2(y)-u_2(x)))\,dm_x(y)d\nu(x) = 0$$

thus, by (5.9),

(5.18)
$$(\mathbf{a}_p(x,y,u_1(y)-u_1(x)) - \mathbf{a}_p(x,y,u_2(y)-u_2(x)))(u_1(y)-u_1(x)-(u_2(y)-u_2(x))) = 0$$

for $(x,y) \in (\Omega\times\Omega)\backslash N'$ where $N' \subset \Omega\times\Omega$ is a $\nu\otimes m_x$-null set. Let $C' \subset \Omega$ be a ν-null set such that the section $N'_x := \{y \in \Omega : (x,y) \in N'\}$ of N' is ν-null for every $x \in \Omega\backslash C'$ and let's see that there exists $L \in \mathbb{R}$ such that $u_1(x)-u_2(x) = L$ for ν-a.e. $x \in \Omega$. Let $x_0 \in \Omega\backslash C'$, $L := u_1(x_0)-u_2(x_0)$ and

$$S' := \{y \in \Omega : u_1(y)-u_2(y) = L\} \supset \Omega\backslash N'_{x_0}.$$

By (5.18) we have that $\Omega\backslash C'_{x_0} \subset S'$. Proceeding as we did before to prove that $\nu(\Omega\backslash S) = 0$ we obtain that $\nu(\Omega\backslash S') = 0$. □

In order to prove the existence of solutions of Problem (5.14) *(Theorem* 5.15*) we will first prove the existence of solutions of an approximate problem. Then we will obtain some monotonicity and boundedness properties of the solutions of these approximate problems that will allow us to pass to the limit. This method lets us get around the loss of compactness results in our setting with respect to the local setting. Indeed, we follow ideas used in* [**14**]*, but, as we have said, making the most of the monotone arguments since the Poincaré type inequalities here only produce boundedness in L^p spaces (versus the boundedness in $W^{1,p}$ spaces obtained in the local setting). This will be done in the following subsections.*

5.1.3. Existence of solutions of the approximate problem. Take $\varphi \in L^\infty(\Omega,\nu)$. Let $n, k \in \mathbb{N}$, $K > 0$ and

$$A := A_{n,k} : L^p(\Omega,\nu) \to L^{p'}(\Omega,\nu) \equiv L^{p'}(\Omega_1,\nu) \times L^{p'}(\Omega_2,\nu)$$

be defined by

$$A(u) = (A_1(u), A_2(u)),$$

where

$$\begin{aligned} A_1(u)(x) := \quad & T_K((\gamma_+)_k(u(x))) + T_K((\gamma_-)_n(u(x))) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) \\ & + \frac{1}{n}|u(x)|^{p-2}u^+(x) - \frac{1}{k}|u(x)|^{p-2}u^-(x), \end{aligned}$$

for $x \in \Omega_1$, and

$$\begin{aligned} A_2(u)(x) := \quad & T_K((\beta_+)_k(u(x))) + T_K((\beta_-)_n(u(x))) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) \\ & + \frac{1}{n}|u(x)|^{p-2}u^+(x) - \frac{1}{k}|u(x)|^{p-2}u^-(x), \end{aligned}$$

for $x \in \Omega_2$. Here, T_K is the truncation operator defined as

$$T_K(r) := \begin{cases} -K & \text{if } r < -K, \\ r & \text{if } |r| \leqslant K, \\ K & \text{if } r > K, \end{cases}$$

and $(\gamma_+)_k$, $(\gamma_-)_n$, $(\beta_+)_k$ and $(\beta_-)_n$ are Yosida approximations as defined in Subsection 5.0.1.

It is easy to see that A is continuous and, moreover, it is monotone and coercive in $L^p(\Omega,\nu)$. Indeed, the monotonicity follows by the integration by parts formula (Remark 5.13) and the coercivity follows by the following computation (where the term involving $\mathbf{a}_p$ has been removed because it is nonnegative, as shown in Remark 5.13):

$$\int_\Omega A(u)u d\nu \geqslant \frac{1}{n}||u^+||^p_{L^p(\Omega,\nu)} + \frac{1}{k}||u^-||^p_{L^p(\Omega,\nu)}.$$

Therefore, since $\varphi \in L^\infty(\Omega,\nu) \subset L^{p'}(\Omega,\nu)$, by [**42**, Corollary 30], there exist $u_{n,k} \in L^p(\Omega,\nu)$, n, $k \in \mathbb{N}$, such that

$$(A_1(u_{n,k}), A_2(u_{n,k})) = \varphi.$$

That is,

$$\begin{aligned} & T_K((\gamma_+)_k(u_{n,k}(x))) + T_K((\gamma_-)_n(u_{n,k}(x))) - \int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y) \\ & + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi(x) \quad \text{for } x \in \Omega_1, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} & T_K((\beta_+)_k(u_{n,k}(x))) + T_K((\beta_-)_n(u_{n,k}(x))) - \int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y) \\ & + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi(x) \quad \text{for } x \in \Omega_2. \end{aligned} \tag{5.20}$$

Let n, $k \in \mathbb{N}$. We start by proving that $u_{n,k} \in L^\infty(\Omega,\nu)$. Set

$$M := \left((k+n)\|\varphi\|_{L^\infty(\Omega,\nu)}\right)^{\frac{1}{p-1}}.$$

Then, multiplying (5.19) and (5.20) by $(u_{n,k}-M)^+$, integrating over Ω_1 and Ω_2, respectively, adding both equations and removing the terms which are zero, we get

(5.21)

$$\begin{aligned}
&\int_{\Omega_1} T_K((\gamma_+)_k(u_{n,k}(x)))(u_{n,k}(x)-M)^+d\nu(x) + \int_{\Omega_2} T_K((\beta_+)_k(u_{n,k}(x)))(u_{n,k}(x)-M)^+d\nu(x)\\
&\quad -\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))(u_{n,k}(x)-M)^+dm_x(y)d\nu(x)\\
&\quad +\frac{1}{n}\int_\Omega |u_{n,k}(x)|^{p-2}u_{n,k}^+(x)(u_{n,k}(x)-M)^+d\nu(x)\\
&=\int_\Omega \varphi(x)(u_{n,k}(x)-M)^+d\nu(x).
\end{aligned}$$

Now, by the integration by parts formula (recall Remark 5.13), we have that

$$\begin{aligned}
&-\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))(u_{n,k}(x)-M)^+dm_x(y)d\nu(x)\\
&=\frac{1}{2}\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))\left((u_{n,k}(y)-M)^+-(u_{n,k}(x)-M)^+\right)dm_x(y)d\nu(x)\geqslant 0.
\end{aligned}$$

Hence, removing nonnegative terms in (5.21), we get

$$\int_\Omega |u_{n,k}(x)|^{p-2}u_{n,k}^+(x)(u_{n,k}(x)-M)^+d\nu(x) \leqslant n\int_\Omega \varphi(x)(u_{n,k}(x)-M)^+d\nu(x),$$

thus

$$\int_\Omega T_K(|u_{n,k}(x)|^{p-2}u_{n,k}^+(x))(u_{n,k}(x)-M)^+d\nu(x) \leqslant n\int_\Omega \varphi(x)(u_{n,k}(x)-M)^+d\nu(x).$$

Now, subtracting $\int_\Omega M^{p-1}(u_{n,k}(x)-M)^+d\nu(x)$ from both sides of the above inequality yields

$$\begin{aligned}
&\int_\Omega \left(T_K(|u_{n,k}(x)|^{p-2}u_{n,k}^+(x)) - M^{p-1}\right)(u_{n,k}(x)-M)^+d\nu(x)\\
&\leqslant n\int_\Omega\left(\varphi(x)-\frac{1}{n}M^{p-1}\right)(u_{n,k}(x)-M)^+d\nu(x)\leqslant 0
\end{aligned}$$

and, consequently, taking $K>M$, we get

$$u_{n,k}\leqslant M \quad \nu\text{-a.e. in } \Omega.$$

Similarly, taking $w=(u_{n,k}+M)^-$, we get

$$\begin{aligned}
&\int_\Omega \left(T_K(|u_{n,k}(x)|^{p-2}u_{n,k}^-(x)) + M^{p-1}\right)(u_{n,k}(x)+M)^-d\nu(x)\\
&\geqslant k\int_\Omega\left(\varphi(x)+\frac{1}{k}M^{p-1}\right)(u_{n,k}+M)^-d\nu(x)\geqslant 0
\end{aligned}$$

which yields, taking also $K>M$,

$$u_{n,k}\geqslant -M \quad \nu\text{-a.e. in } \Omega.$$

Therefore,

$$\|u_{n,k}\|_{L^\infty(\Omega,\nu)}\leqslant M$$

as desired.

Now, taking

$$K>\max\left\{M,(\gamma_+)_k(M),-(\gamma_-)_k(-M),(\beta_+)_n(M),-(\beta_-)_n(-M)\right\},$$

equations (5.19) and (5.20) yield

$$(5.22)\quad \begin{aligned}&(\gamma_+)_k(u_{n,k}(x)) + (\gamma_-)_n(u_{n,k}(x)) - \int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)\\ &\qquad + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi(x), \quad x \in \Omega_1,\end{aligned}$$

and

$$(5.23)\quad \begin{aligned}&(\beta_+)_k(u_{n,k}(x)) + (\beta_-)_n(u_{n,k}(x)) - \int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)\\ &\qquad + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi(x), \quad x \in \Omega_2.\end{aligned}$$

Take now $\varphi \in L^{p'}(\Omega,\nu)$ and, for $n,k \in \mathbb{N}$, set

$$(5.24)\qquad \varphi_{n,k} := \sup\{\inf\{n,\varphi\},-k\}.$$

Then, since $\varphi_{n,k} \in L^\infty(\Omega,\nu)$, by the previous computations leading to (5.22) and (5.23), we have that there exists a solution $u_{n,k} \in L^\infty(\Omega,\nu)$ of the following *Approximate Problem* (5.25)–(5.26):

$$(5.25)\quad \begin{aligned}&(\gamma_+)_k(u_{n,k}(x)) + (\gamma_-)_n(u_{n,k}(x)) - \int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)\\ &\qquad + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi_{n,k}(x), \quad x \in \Omega_1,\end{aligned}$$

$$(5.26)\quad \begin{aligned}&(\beta_+)_k(u_{n,k}(x)) + (\beta_-)_n(u_{n,k}(x)) - \int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)\\ &\qquad + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi_{n,k}(x), \quad x \in \Omega_2.\end{aligned}$$

Moreover, we obtain the following estimates which will be used later on. Multiplying (5.25) and (5.26) by $\frac{1}{s}T_s(u^+_{n,k})$, integrating with respect to ν over Ω_1 and Ω_2, respectively, adding both equations, applying the integration by parts formula (Remark 5.13), and letting $s \downarrow 0$, we get, after removing some nonnegative terms, that

$$(5.27)\quad \frac{1}{n}\int_\Omega |u_{n,k}|^{p-2}u^+_{n,k}d\nu + \int_{\Omega_1}(\gamma_+)_k(u_{n,k})d\nu + \int_{\Omega_2}(\beta_+)_k(u_{n,k})d\nu \leqslant \int_\Omega \varphi^+_{n,k}d\nu \leqslant \int_\Omega \varphi^+ d\nu.$$

Similarly, multiplying by $\frac{1}{s}T_s(u^-_{n,k})$ we get

$$(5.28)\quad \frac{1}{k}\int_\Omega |u_{n,k}|^{p-2}u^-_{n,k}d\nu - \int_{\Omega_1}(\gamma_-)_n(u_{n,k})d\nu - \int_{\Omega_2}(\beta_-)_n(u_{n,k})d\nu \leqslant \int_\Omega \varphi^-_{n,k}d\nu \leqslant \int_\Omega \varphi^- d\nu.$$

5.1.4. Monotonicity of the solutions of the approximate problems. *Using that $\varphi_{n,k}$ is nondecreasing in n and nonincreasing in k, and thanks to the way in which we have approximated the maximal monotone graphs γ and β, we will obtain monotonicity properties for the solutions of the approximate problems.*

Fix $k \in \mathbb{N}$. Let $n_1 < n_2$. Multiply equations (5.25) and (5.26) with $n = n_1$ by $(u_{n_1,k} - u_{n_2,k})^+$, integrate with respect to ν over Ω_1 and Ω_2, respectively, and add both equations. Then, doing the same with $n = n_2$ and subtracting the resulting equation from the one that we have obtained for $n = n_1$ we get

$$\begin{aligned}
&\int_{\Omega_1} ((\gamma_+)_k(u_{n_1,k}(x)) - (\gamma_+)_k(u_{n_2,k}(x)))\,(u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \\
&\quad + \int_{\Omega_1} ((\gamma_-)_{n_1}(u_{n_1,k}(x)) - (\gamma_-)_{n_2}(u_{n_2,k}(x)))\,(u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \\
&\quad + \int_{\Omega_2} ((\beta_+)_k(u_{n_1,k}(x)) - (\beta_+)_k(u_{n_2,k}(x)))\,(u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \\
&\quad + \int_{\Omega_2} ((\beta_-)_{n_1}(u_{n_1,k}(x)) - (\beta_-)_{n_2}(u_{n_2,k}(x)))\,(u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \\
&\quad - \int_{\Omega}\int_{\Omega} (\mathbf{a}_p(x,y,u_{n_1,k}(y) - u_{n_1,k}(x)) - \mathbf{a}_p(x,y,u_{n_2,k}(y) - u_{n_2,k}(x))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (u_{n_1,k}(x) - u_{n_2,k}(x))^+ dm_x(y) d\nu(x) \\
&\quad + \int_{\Omega} \left(\frac{1}{n_1} |u_{n_1,k}(x)|^{p-2} u^+_{n_1,k}(x) - \frac{1}{n_2} |u_{n_2,k}(x)|^{p-2} u^+_{n_2,k}(x) \right) (u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \\
&\quad - \frac{1}{k} \int_{\Omega} \left(|u_{n_1,k}(x)|^{p-2} u^-_{n_1,k}(x) - |u_{n_2,k}(x)|^{p-2} u^-_{n_2,k}(x) \right) (u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \\
&= \int_{\Omega} (\varphi_{n_1,k}(x) - \varphi_{n_2,k}(x))\,(u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) \leqslant 0.
\end{aligned}$$

Since $(\gamma_+)_k$ and $(\beta_+)_k$ are maximal monotone the first and third summands on the left hand side are nonnegative, and the same is true for the second and fourth summands since $(\gamma_-)_{n_1} \geqslant (\gamma_-)_{n_2}$, $(\beta_-)_{n_1} \geqslant (\beta_-)_{n_2}$ and these are all maximal monotone. The fifth summand is also nonnegative as illustrated in Remark 5.13. Then, since the last two summands are obviously nonnegative, we get that, in fact,

$$\int_{\Omega} \left(\frac{1}{n_1} |u_{n_1,k}(x)|^{p-2} u^+_{n_1,k}(x) - \frac{1}{n_2} |u_{n_2,k}(x)|^{p-2} u^+_{n_2,k}(x) \right) (u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) = 0$$

and

$$\frac{1}{k} \int_{\Omega} \left(|u_{n_1,k}(x)|^{p-2} u^-_{n_1,k}(x) - |u_{n_2,k}(x)|^{p-2} u^-_{n_2,k}(x) \right) (u_{n_1,k}(x) - u_{n_2,k}(x))^+ d\nu(x) = 0$$

which together imply that

$$u_{n_1,k}(x) \leqslant u_{n_2,k}(x) \quad \text{for } \nu\text{-a.e. } x \in \Omega.$$

Similarly, we obtain that, for a fixed n, $u_{n,k}$ is ν-a.e. in Ω nonincreasing in k.

5.1.5. An L^p-estimate for the solutions of the approximate problems. Multiplying (5.25) and (5.26) by

$$u_{n,k} - \frac{1}{\nu(\Omega_1)} \int_{\Omega_1} u_{n,k} d\nu,$$

integrating with respect to ν over Ω_1 and Ω_2, respectively, adding both equations and using the integration by parts formula (Remark 5.13) we get

(5.29)

$$\begin{aligned}
&\int_{\Omega_1} ((\gamma_+)_k(u_{n,k}(x)) + (\gamma_-)_n(u_{n,k}(x))) \left(u_{n,k}(x) - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu(x) \\
&\quad + \int_{\Omega_2} ((\beta_+)_k(u_{n,k}(x)) + (\beta_-)_n(u_{n,k}(x))) \left(u_{n,k}(x) - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu(x) \\
&\quad + \frac{1}{2}\int_{\Omega}\int_{\Omega} \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))(u_{n,k}(y)-u_{n,k}(x))dm_x(y)d\nu(x) \\
&\quad + \int_{\Omega}\left(\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right)\left(u_{n,k}(x) - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu(x) \\
&= \int_{\Omega} \varphi_{n,k}(x)\left(u_{n,k}(x) - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu(x).
\end{aligned}$$

For the first summand on the left hand side of (5.29) we have

$$\begin{aligned}
&\int_{\Omega_1} ((\gamma_+)_k(u_{n,k}) + (\gamma_-)_n(u_{n,k})) \left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu \\
&= \int_{\Omega_1}\left((\gamma_+)_k(u_{n,k}) - (\gamma_+)_k\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}\right)\right)\left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu \\
&\quad + \int_{\Omega_1}\left((\gamma_-)_n(u_{n,k}) - (\gamma_-)_n\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}\right)\right)\left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu \geqslant 0,
\end{aligned}$$

and for the second

$$\begin{aligned}
&\int_{\Omega_2} ((\beta_+)_k(u_{n,k}) + (\beta_-)_n(u_{n,k})) \left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right) d\nu \\
&= \int_{\Omega_2}\left((\beta_+)_k(u_{n,k}) - (\beta_+)_k\left(\frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}\right)\right)\left(u_{n,k} - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right) d\nu \\
&\quad + \int_{\Omega_2}\left((\beta_-)_n(u_{n,k}) - (\beta_-)_n\left(\frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}\right)\right)\left(u_{n,k} - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right) d\nu \\
&\quad - \int_{\Omega_2} ((\beta_+)_k(u_{n,k}) + (\beta_-)_n(u_{n,k}))\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right) d\nu \\
&\geqslant -\int_{\Omega_2} ((\beta_+)_k(u_{n,k}) + (\beta_-)_n(u_{n,k}))\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right) d\nu.
\end{aligned}$$

Since $F_{n,k}(s) := \frac{1}{n}|s|^{p-2}s^+ - \frac{1}{k}|s|^{p-2}s^-$ is nondecreasing, for the fourth summand on the left hand side of (5.29) we have that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{n}|u_{n,k}(x)|^{p-2}u_{n,k}^+(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u_{n,k}^-(x)\right)\left(u_{n,k}(x) - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right)d\nu(x) \\
&= \int_{\Omega_1}\left(F_{n,k}(u_{n,k}(x)) - F_{n,k}\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right)\right)\left(u_{n,k}(x) - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right)d\nu(x) \\
&\quad + \int_{\Omega_2}\left(F_{n,k}(u_{n,k}(x)) - F_{n,k}\left(\frac{1}{\nu(\Omega_2)}\int_{\Omega_1} u_{n,k}d\nu\right)\right)\left(u_{n,k}(x) - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right)d\nu(x) \\
&\quad - \int_{\Omega_2} F_{n,k}(u_{n,k}(x))\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right)d\nu(x) \\
&\geqslant -\int_{\Omega_2} F_{n,k}(u_{n,k}(x))\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right)d\nu(x).
\end{aligned}$$

Finally, recalling (5.11) for the third summand in (5.29), we get

$$\begin{aligned}
&\frac{c_p}{2}\int_\Omega\int_\Omega |u_{n,k}(y) - u_{n,k}(x)|^p dm_x(y)d\nu(x) \\
&\leqslant \int_\Omega \varphi_{n,k}\left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right)d\nu \\
&\quad + \int_{\Omega_2}((\beta_+)_k(u_{n,k}) + (\beta_-)_n(u_{n,k}))\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{r,n,k}d\nu\right)d\nu \\
&\quad + \int_{\Omega_2}\left(\frac{1}{n}|u_{n,k}(x)|^{p-2}u_{n,k}^+(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u_{n,k}^-(x)\right) \\
&\qquad\qquad \times\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu - \frac{1}{\nu(\Omega_2)}\int_{\Omega_2} u_{n,k}d\nu\right)d\nu.
\end{aligned}$$

Now, by Hölder's inequality and the generalised Poincaré type inequality with $l = \nu(\Omega_1)$ (let Λ_1 denote the constant appearing in the generalised Poincaré type inequality in Assumption 5), we have that

$$\begin{aligned}
&\int_\Omega \varphi_{n,k}\left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right)d\nu \leqslant \|\varphi\|_{L^{p'}(\Omega,\nu)}\left\|u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k}d\nu\right\|_{L^p(\Omega,\nu)} \\
&\leqslant \Lambda_1\|\varphi\|_{L^{p'}(\Omega,\nu)}\left(\int_\Omega\int_\Omega |u_{n,k}(y) - u_{n,k}(x)|^p dm_x(y)d\nu(x)\right)^{\frac{1}{p}},
\end{aligned}$$

and, by (5.27), (5.28) and the generalised Poincaré type inequality with $l = \nu(\Omega_1)$ and with $l = \nu(\Omega_2)$ (let Λ_2 denote the constant appearing in the Poincaré type inequality for the latter case), we obtain that

$$\int_{\Omega_2}\left(((\beta_+)_k(u_{n,k})+(\beta_-)_n(u_{n,k}))+\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)-\frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right)$$
$$\times\left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1}u_{n,k}d\nu-\frac{1}{\nu(\Omega_2)}\int_{\Omega_2}u_{n,k}d\nu\right)d\nu$$
$$\leqslant\|\varphi\|_{L^1(\Omega,\nu)}\left|\frac{1}{\nu(\Omega_1)}\int_{\Omega_1}u_{n,k}d\nu-\frac{1}{\nu(\Omega_2)}\int_{\Omega_2}u_{n,k}d\nu\right|$$
$$\leqslant\|\varphi\|_{L^1(\Omega,\nu)}\frac{1}{\nu(\Omega)^{\frac{1}{p}}}\left(\left\|u_{n,k}-\frac{1}{\nu(\Omega_1)}\int_{\Omega_1}u_{n,k}d\nu\right\|_{L^p(\Omega,\nu)}+\left\|u_{n,k}-\frac{1}{\nu(\Omega_2)}\int_{\Omega_2}u_{n,k}d\nu\right\|_{L^p(\Omega,\nu)}\right)$$
$$\leqslant\|\varphi\|_{L^1(\Omega,\nu)}\frac{\Lambda_1+\Lambda_2}{\nu(\Omega)^{\frac{1}{p}}}\left(\int_\Omega\int_\Omega|u_{n,k}(y)-u_{n,k}(x)|^pdm_x(y)d\nu(x)\right)^{\frac{1}{p}}.$$

Therefore, bringing (5.29) and the subsequent equations together, we get
(5.30)
$$\frac{c_p}{2}\left(\int_\Omega\int_\Omega|u_{n,k}(y)-u_{n,k}(x)|^pdm_x(y)d\nu(x)\right)^{\frac{1}{p'}}\leqslant\Lambda_1\|\varphi\|_{L^{p'}(\Omega,\nu)}+\frac{\Lambda_1+\Lambda_2}{\nu(\Omega)^{\frac{1}{p}}}\|\varphi\|_{L^1(\Omega,\nu)}.$$

5.1.6. Existence of solutions of (GP_φ)**.** *Observe that a solution* $[u,v]$ *of* (GP_φ) *satisfies*

$$\int_{\Omega_1}vd\nu+\int_{\Omega_2}vd\nu=\int_\Omega\varphi,$$

therefore, since $v\in\gamma(u)$ *in* Ω_1 *and* $v\in\beta(u)$ *in* Ω_2, *we need* φ *to satisfy*

$$\mathcal{R}^-_{\gamma,\beta}\leqslant\int_\Omega\varphi d\nu\leqslant\mathcal{R}^+_{\gamma,\beta}.$$

We will prove the existence of solutions when the inequalities in the previous equation are strict, this suffices for what we need in the next section. Recall that we are working under the Assumptions 1 to 5.

Theorem 5.15. *Given* $\varphi\in L^{p'}(\Omega,\nu)$ *such that*

$$\mathcal{R}^-_{\gamma,\beta}<\int_\Omega\varphi d\nu<\mathcal{R}^+_{\gamma,\beta},$$

Problem (GP_φ) *stated in* (5.14) *has a solution.*

Observe then that any solution $[u,v]$ *of* (GP_φ) *under such assumptions will also satisfy*

$$\mathcal{R}^-_{\gamma,\beta}<\int_\Omega vd\nu<\mathcal{R}^+_{\gamma,\beta},$$

this will be used later on.

We divide the proof into three cases.

Proof of Theorem 5.15 when $\mathcal{R}^\pm_{\gamma,\beta}=\infty$. Suppose that

$$\mathcal{R}^-_{\gamma,\beta}=-\infty\ \text{ and }\mathcal{R}^+_{\gamma,\beta}=+\infty.$$

Let $\varphi\in L^{p'}(\Omega,\nu)$, $\varphi_{n,k}$ defined as in (5.24) and let $u_{n,k}\in L^\infty(\Omega,\nu)$, $n,k\in\mathbb{N}$, be solutions of the *Approximate Problem* (5.25)–(5.26).

Step A (Boundedness). Let us first see that $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ is bounded.

Step 1. We start by proving that $\{\|u^+_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ is bounded. We will see this case by case. Since $\mathcal{R}^+_{\gamma,\beta}=+\infty$, we have that $\sup\{\text{Ran}(\gamma)\}=+\infty$ or $\sup\{\text{Ran}(\beta)\}=+\infty$.

Case 1.1. Suppose that $\sup\{\mathrm{Ran}(\gamma)\} = +\infty$. Then, by (5.27) we have that

$$\int_{\Omega_1} (\gamma_+)_k(u_{n,k})d\nu < M := \int_\Omega \varphi d\nu \quad \text{for every } n,k \in \mathbb{N}.$$

Let $z^+_{n,k} := (\gamma_+)_k(u_{n,k})$ and $\widetilde{\Omega}_{n,k} := \left\{x \in \Omega_1 : z^+_{n,k}(x) < \frac{2M}{\nu(\Omega_1)}\right\}$. Then

$$\begin{aligned} 0 &\leqslant \int_{\widetilde{\Omega}_{n,k}} z^+_{n,k} d\nu = \int_{\Omega_1} z^+_{n,k} d\nu - \int_{\Omega_1 \setminus \widetilde{\Omega}_{n,k}} z^+_{n,k} d\nu \\ &\leqslant M - (\nu(\Omega_1) - \nu(\widetilde{\Omega}_{n,k}))\frac{2M}{\nu(\Omega_1)} = \nu(\widetilde{\Omega}_{n,k})\frac{2M}{\nu(\Omega_1)} - M, \end{aligned}$$

thus

$$\nu(\widetilde{\Omega}_{n,k}) \geqslant \frac{\nu(\Omega_1)}{2}.$$

Case 1.1.1. Assume that $\sup D(\gamma) = +\infty$. Let $r_0 \in \mathbb{R}$ be such that $\gamma^0(r_0) > 2M/\nu(\Omega_1)$ and let $k_0 \in \mathbb{N}$ such that

$$\frac{2M}{\nu(\Omega_1)} < (\gamma_+)_k(r_0) \leqslant \gamma^0(r_0) \quad \text{for } k \geqslant k_0. \tag{5.31}$$

Then, since in $\widetilde{\Omega}_{n,k}$ we have that $(\gamma_+)_k(u_{n,k}) = z^+_{n,k} < \frac{2M}{\nu(\Omega_1)}$, from (5.31) we get that

$$u^+_{n,k} \leqslant r_0 \quad \text{in } \widetilde{\Omega}_{n,k}, \ \text{ for every } k \geqslant k_0 \text{ and every } n \in \mathbb{N}.$$

Therefore, this bound, the generalised Poincaré type inequality with $l = \frac{\nu(\Omega_1)}{2}$ and (5.30) yield that $\{\|u^+_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ is bounded.
Case 1.1.2. If $r_\gamma := \sup D(\gamma) < +\infty$, by Lemma 5.1 we have that

$$(\gamma_+)_k(r) = k(r - r_\gamma), \quad \text{for } r \geqslant r_\gamma + \tfrac{1}{k}\gamma^0(r_\gamma).$$

Then, in $\widetilde{\Omega}_{n,k}$ we have that

$$(\gamma_+)_k(u^+_{n,k}) < \frac{2M}{\nu(\Omega_1)} \leqslant \frac{2M}{\nu(\Omega_1)} + \gamma^0(r_\gamma) = (\gamma_+)_k\left(r_\gamma + \frac{1}{k}\left(\frac{2M}{\nu(\Omega_1)} + \gamma^0(r_\gamma)\right)\right),$$

thus, for all n and k,

$$u^+_{n,k} \leqslant r_\gamma + \frac{1}{k}\left(\frac{2M}{\nu(\Omega_1)} + \gamma^0(r_\gamma)\right) \quad \text{in } \widetilde{\Omega}_{n,k}.$$

Therefore, again, this bound together with the generalised Poincaré type inequality with $l = \frac{\nu(\Omega_1)}{2}$ and (5.27) yield the thesis.
Case 1.2. If $\sup\{\mathrm{Ran}(\beta)\} = +\infty$ we proceed similarly.
Step 2. Using that $\mathcal{R}^-_{\gamma,\beta} = -\infty$ we obtain that $\{\|u^-_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ is bounded with an analogous argument.

Consequently, we get that $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ is bounded as desired.

Step B (Taking limits in n). The monotonicity properties obtained in Subsection 5.1.4 together with the boundedness of $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ allow us to apply the monotone convergence theorem to obtain $u_k \in L^p(\Omega,\nu)$, $k \in \mathbb{N}$, and $u \in L^p(\Omega,\nu)$ such that, taking a subsequence if necessary, $u_{n,k} \xrightarrow{n} u_k$ in $L^p(\Omega,\nu)$ and pointwise ν-a.e. in Ω for $k \in \mathbb{N}$, and $u_k \xrightarrow{k} u$ in $L^p(\Omega,\nu)$ and pointwise ν-a.e. in Ω.

We now want to take limits, in n and then in k, in (5.25) and (5.26). Since $u_{n,k} \xrightarrow{n} u_k$ in $L^p(\Omega,\nu)$ and pointwise ν-a.e. in Ω, we have that

$$\int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x))dm_x(y) \xrightarrow{n} \int_\Omega \mathbf{a}_p(x,y,u_k(y) - u_k(x))dm_x(y), \tag{5.32}$$

$$\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) \xrightarrow{n} 0$$

and
$$\frac{1}{k}|u_{n,k}(x)|^{p-2}u_{n,k}^-(x) \overset{n}{\longrightarrow} \frac{1}{k}|u_k(x)|^{p-2}u_k^-(x)$$
in $L^{p'}(\Omega,\nu)$ and, up to a subsequence, for ν-a.e. $x \in \Omega$. Indeed, for the second and third convergences note that, by the mean value theorem, for a, $b \in \mathbb{R}$,
$$|a^{p-1} - b^{p-1}|^{p'} \leqslant (p-1)^{p'} \max\{|a|^p, |b|^p\}^{\frac{p-2}{p-1}} |a-b|^{p'} \leqslant (p-1)^{p'} (|a|^p + |b|^p)^{\frac{1}{(p-1)'}} |a-b|^{p'}$$
thus, by Hölder's inequality,
$$\|u_{n,k}^{p-1} - u_k^{p-1}\|_{L^{p'}(\Omega,\nu)} \leqslant (p-1)^{p'} \left(\|u_{n,k}\|_{L^p(\Omega,\nu)} + \|u_k\|_{L^p(\Omega,\nu)}\right)^{\frac{p-2}{p}} \|u_{n,k} - u_k\|_{L^p(\Omega,\nu)}$$
hence $u_{n,k}^{p-1} \overset{n}{\to} u_k^{p-1}$ in $L^{p'}(\Omega,\nu)$. Moreover, since $\{u_{n,k}\}$ is nonincreasing in n, we have that $|u_{n,k}| \leqslant \max\{|u_{1,k}|, |u_k|\}$ ν-a.e. in Ω, for every n, $k \in \mathbb{N}$, so Lemma 5.7 yields the convergence (5.32) in $L^{p'}(\Omega,\nu)$.

Now, isolating $(\gamma_+)_k(u_{n,k}) + (\gamma_-)_n(u_{n,k})$ and $(\beta_+)_k(u_{n,k}) + (\beta_-)_n(u_{n,k})$ in equations (5.25) and (5.26), respectively, and taking the positive parts, we get that
$$(\gamma_+)_k(u_{n,k}(x)) = \left(\int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x))dm_x(y) + \frac{1}{n}|u_{n,k}(x)|^{p-2}u_{n,k}^+(x) \right.$$
$$\left. - \frac{1}{k}|u_{n,k}(x)|^{p-2}u_{n,k}^-(x) + \varphi_{n,k}(x)\right)^+$$
for $x \in \Omega_1$, and
$$(\beta_+)_k(u_{n,k}(x))$$
$$= \left(\int_\Omega \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x))dm_x(y) + \frac{1}{n}|u_{n,k}(x)|^{p-2}u_{n,k}^+(x) \right.$$
$$\left. - \frac{1}{k}|u_{n,k}(x)|^{p-2}u_{n,k}^-(x) + \varphi_{n,k}(x)\right)^+$$
for $x \in \Omega_2$. Therefore, since the right hand sides of these equations converge in $L^{p'}(\Omega_1,\nu)$ and $L^{p'}(\Omega_2,\nu)$ (and also ν-a.e. in Ω_1 and Ω_2), respectively, we have that there exist $z_k^+ \in L^{p'}(\Omega_1,\nu)$ and $\omega_k^+ \in L^{p'}(\Omega_2,\nu)$ such that $(\gamma_+)_k(u_{n,k}) \overset{n}{\to} z_k^+$ in $L^{p'}(\Omega_1,\nu)$ and pointwise ν-a.e. in Ω_1, and $(\beta_+)_k(u_{n,k}) \overset{n}{\to} \omega_k^+$ in $L^{p'}(\Omega_2,\nu)$ and pointwise ν-a.e. in Ω_2. Moreover, since $(\gamma_+)_k$ and $(\beta_+)_k$ are maximal monotone graphs, $z_k^+ = (\gamma_+)_k(u_k)$ ν-a.e. in Ω_1, and $\omega_k^+ = (\beta_+)_k(u_k)$ ν-a.e. in Ω_2.

Similarly, taking the negative parts, we get that
$$\exists \lim_{n\to+\infty} (\gamma_-)_n(u_{n,k}(x)) = z_k^-(x) \quad \text{in } L^{p'}(\Omega_1,\nu) \text{ and for } \nu\text{-a.e. } x \in \Omega_1,$$
and
$$\exists \lim_{n\to+\infty} (\beta_-)_n(u_{n,k}(x)) = \omega_k^-(x) \quad \text{in } L^{p'}(\Omega_2,\nu) \text{ and for } \nu\text{-a.e. } x \in \Omega_2.$$
Moreover, by [**33**, Lemma G], $z_k^- \in \gamma_-(u_k)$ and $\omega_k^- \in \beta_-(u_k)$. Therefore, we have obtained that
$$z_k^+(x) + z_k^-(x) - \int_\Omega \mathbf{a}_p(x,y,u_k(y) - u_k(x))dm_x(y) - \frac{1}{k}|u_k(x)|^{p-2}u_k^-(x) = \varphi_k(x), \tag{5.33}$$
for ν-a.e. $x \in \Omega_1$, and
$$\omega_k^+(x) + \omega_k^-(x) - \int_\Omega \mathbf{a}_p(x,y,u_k(y) - u_k(x))dm_x(y) - \frac{1}{k}|u_k(x)|^{p-2}u_k^-(x) = \varphi_k(x) \tag{5.34}$$
for ν-a.e. $x \in \Omega_2$.

Step C (Taking limits in k). Now again, isolating $z_k^+ + z_k^-$ and $\omega_k^+ + \omega_k^-$ in equations (5.33) and (5.34), respectively, and taking the positive and negative parts as above, we get that there exist $z^+ \in L^{p'}(\Omega_1,\nu)$, $z^- \in L^{p'}(\Omega_1,\nu)$, $\omega^+ \in L^{p'}(\Omega_2,\nu)$ and $\omega^- \in L^{p'}(\Omega_2,\nu)$ such that $z_k^+ \overset{k}{\to} z^+$ and $z_k^- \overset{k}{\to} z^-$ in $L^{p'}(\Omega_1,\nu)$ and pointwise ν-a.e. in Ω_1, and $\omega_k^+ \overset{k}{\to} \omega^+$ and $\omega_k^- \overset{k}{\to} \omega^-$

in $L^{p'}(\Omega_2,\nu)$ and pointwise ν-a.e. in Ω_2. In addition, by the maximal monotonicity of γ_- and β_-, $z^- \in \gamma_-(u)$ and $\omega^- \in \beta_-(u)$ ν-a.e. in Ω_1 and Ω_2, respectively. Moreover, by [**33**, Lemma G], $z^+ \in \gamma_+(u)$ and $\omega^+ \in \beta_+(u)$ ν-a.e. in Ω_1 and Ω_2, respectively.

Consequently,

$$z(x) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \varphi(x) \text{ for } \nu\text{-a.e. } x \in \Omega_1,$$

and

$$\omega(x) - \int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \varphi(x) \text{ for } \nu\text{-a.e. } x \in \Omega_2,$$

where $z = z^+ + z^- \in \gamma(u)$ ν-a.e. in Ω_1 and $\omega = \omega^+ + \omega^- \in \beta(u)$ ν-a.e. in Ω_2. The proof of existence in this case is done. □

Proof of Theorem 5.15 when $\mathcal{R}^{\pm}_{\gamma,\beta} < \infty$. Suppose that

$$-\infty < \mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta} < +\infty.$$

Let $\varphi \in L^{p'}(\Omega,\nu)$, and assume that it satisfies

$$\mathcal{R}^-_{\gamma,\beta} < \int_\Omega \varphi d\nu < \mathcal{R}^+_{\gamma,\beta}.$$

Then, for $\varphi_{n,k}$ defined as in (5.24), there exist $M_1, M_2 \in \mathbb{R}$ and $n_0, k_0 \in \mathbb{N}$ such that

$$\mathcal{R}^-_{\gamma,\beta} < M_2 < \int_\Omega \varphi_{n,k} d\nu < M_1 < \mathcal{R}^+_{\gamma,\beta} \tag{5.35}$$

for every $n \geqslant n_0$ and $k \geqslant k_0$. For $n,k \in \mathbb{N}$ let $u_{n,k} \in L^\infty(\Omega,\nu)$ be the solution to the *Approximate Problem* (5.25)–(5.26), and let

$$M_3 := \sup_{n,k\in\mathbb{N}} \left\| u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k} d\nu \right\|_{L^p(\Omega,\nu)} < +\infty. \tag{5.36}$$

Observe that M_3 is finite by the generalised Poincaré type inequality together with (5.30). Let $k_1 \in \mathbb{N}$ such that $k_1 \geqslant k_0$ and $M_1 + \frac{1}{k} M_3 \nu(\Omega)^{\frac{1}{p(p-1)}} < \mathcal{R}^+_{\gamma,\beta}$ for every $k \geqslant k_1$.

Step D (Boundedness in n and passing to the limit in n) Let us see that, for each $k \in \mathbb{N}$, $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_n$ is bounded. Fix $k \geqslant k_1$ and suppose that $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_n$ is not bounded. Then, by (5.36), we have that

$$\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k} d\nu \overset{n\to+\infty}{\longrightarrow} +\infty.$$

Thus, since $u_{n,k}$ is nondecreasing in n, there exists $n_1 \geqslant n_0$ such that

$$\begin{aligned} u^-_{n,k} &\leqslant \left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k} d\nu\right)^- + \left(\frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k} d\nu\right)^- \\ &= \left(u_{n,k} - \frac{1}{\nu(\Omega_1)}\int_{\Omega_1} u_{n,k} d\nu\right)^-, \end{aligned}$$

for every $n \geqslant n_1$, thus

$$\|u^-_{n,k}\|_{L^p(\Omega,\nu)} \leqslant M_3 \text{ for every } n \geqslant n_1.$$

Consequently, $\|u^-_{n,k}\|_{L^{p-1}(\Omega,\nu)} \leqslant M_3 \nu(\Omega)^{\frac{1}{p(p-1)}}$ for $n \geqslant n_1$. Then, with this bound and (5.35) at hand, integrating (5.25) and (5.26) with respect to ν over Ω_1 and Ω_2, respectively, adding both equations and removing some nonnegative terms we get

$$\int_{\Omega_1} \underbrace{(\gamma_+)_k(u_{n,k}(x)) + (\gamma_-)_n(u_{n,k}(x))}_{z_{n,k}(x)} d\nu(x) + \int_{\Omega_2} \underbrace{(\beta_+)_k(u_{n,k}(x)) + (\beta_-)_n(u_{n,k}(x))}_{\omega_{n,k}(x)} d\nu(x)$$
$$\leqslant \underbrace{M_1 + \frac{1}{k} M_3 \nu(\Omega)^{\frac{1}{p(p-1)}}}_{M_4} < \mathcal{R}^+_{\gamma,\beta}.$$

Therefore, for each $n \in \mathbb{N}$, either

$$\int_{\Omega_1} z_{n,k} d\nu < \sup\{\mathrm{Ran}(\gamma)\}\nu(\Omega_1) - \frac{\delta}{2} \tag{5.37}$$

or

$$\int_{\Omega_2} \omega_{n,k} d\nu < \sup\{\mathrm{Ran}(\beta)\}\nu(\Omega_2) - \frac{\delta}{2}, \tag{5.38}$$

where $\delta := \mathcal{R}^+_{\gamma,\beta} - M_4 > 0$.

For $n \in \mathbb{N}$ such that (5.37) holds let

$$K_{n,k} := \left\{ x \in \Omega_1 \ : \ z_{n,k}(x) < \sup\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)} \right\}.$$

Then,

$$\int_{K_{n,k}} z_{n,k} d\nu = \int_{\Omega_1} z_{n,k} d\nu - \int_{\Omega_1 \setminus K_{n,k}} z_{n,k} d\nu < -\frac{\delta}{4} + \nu(K_{n,k}) \left(\sup\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)} \right),$$

and

$$\int_{K_{n,k}} z_{n,k} d\nu \geqslant \inf\{\mathrm{Ran}(\gamma)\}\nu(K_{n,k}).$$

Therefore,

$$\nu(K_{n,k}) \left(\sup\{\mathrm{Ran}(\gamma)\} - \inf\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)} \right) \geqslant \frac{\delta}{4},$$

thus $\nu(K_{n,k}) > 0$, $\sup\{\mathrm{Ran}(\gamma)\} - \inf\{\mathrm{Ran}(\gamma)\} - \dfrac{\delta}{4\nu(\Omega_1)} > 0$ and

$$\nu(K_{n,k}) \geqslant \frac{\delta/4}{\sup\{\mathrm{Ran}(\gamma)\} - \inf\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)}}.$$

Note that, if $\sup\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)} \leqslant 0$ then $z_{n,k} \leqslant 0$ in $K_{n,k}$, thus $u^+_{n,k} = 0$ in $K_{n,k}$ and, consequently, $\|u^+_{n,k}\|_{L^p(K_{n,k},\nu)} = 0$. Therefore, by the generalised Poincaré type inequality and (5.30) we get that $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_n$ is bounded, which is a contradiction. We may therefore suppose that $\sup\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)} > 0$. Then, for $k_2 \geqslant k_1$ large enough so that $\sup\{\mathrm{Ran}((\gamma_+)_k)\} > \sup\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)}$ for $k \geqslant k_2$,

$$\|u^+_{n,k}\|_{L^p(K_{n,k},\nu)} \leqslant \nu(K_{n,k})^{\frac{1}{p}} (\gamma_+)^{-1}_k \left(\sup\{\mathrm{Ran}(\gamma)\} - \frac{\delta}{4\nu(\Omega_1)} \right)$$

and by the generalised Poincaré's inequality and (5.30) we get that $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_n$ is bounded, which is a contradiction. Similarly for $n \in \mathbb{N}$ such that (5.38) holds.

We have obtained that $\{\|u_{n,k}\|_{L^p(\Omega,\nu)}\}_n$ is bounded for each $k \in \mathbb{N}$. Therefore, since $\{u_{n,k}\}_n$ is nondecreasing in n, we may apply the monotone convergence theorem to obtain $u_k \in L^p(\Omega,\nu)$, $k \in \mathbb{N}$, such that, taking a subsequence if necessary, $u_{n,k} \stackrel{n}{\to} u_k$ in $L^p(\Omega,\nu)$ and pointwise ν-a.e. in Ω for $k \in \mathbb{N}$. Proceeding now like in *Step B* of the previous proof we get: $z^+_k \in L^{p'}(\Omega_1,\nu)$ and $\omega^+_k \in L^{p'}(\Omega_2,\nu)$ such that $z^+_k \in \gamma_+(u_k)$ and $\omega^+_k \in \beta_+(u_k)$ ν-a.e. in Ω_1 and

Ω_2, respectively; and $z_k^- \in L^{p'}(\Omega_1,\nu)$ and $\omega_k^- \in L^{p'}(\Omega_2,\nu)$ with $z_k^- \in \gamma_-(u_k)$ and $\omega_k^- \in \beta_-(u_k)$, ν-a.e. Ω_1 and Ω_2, respectively, and such that

$$z_k^+(x) + z_k^-(x) - \int_\Omega \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) - \frac{1}{k}|u_k(x)|^{p-2}u_k^-(x) = \varphi_k(x), \tag{5.39}$$

for ν-a.e. $x \in \Omega_1$, and

$$\omega_k^+(x) + \omega_k^-(x) - \int_\Omega \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) - \frac{1}{k}|u_k(x)|^{p-2}u_k^-(x) = \varphi_k(x) \tag{5.40}$$

for ν-a.e. $x \in \Omega_2$.

Step E (Boundedness in k and pass to the limit in k) We will now see that $\{\|u_k\|_{L^p(\Omega,\nu)}\}_k$ is bounded. Since $u_k^+ \leqslant u_1^+$, it is enough to see that $\{\|u_k^-\|_{L^p(\Omega,\nu)}\}_k$ is bounded.

Now, (5.39) and (5.40) yield

$$\int_{\Omega_1} \underbrace{z_k^+(x) + z_k^-(x)}_{z_k(x)} d\nu(x) + \int_{\Omega_2} \underbrace{\omega_k^+(x) + \omega_k^-(x)}_{\omega_k(x)} d\nu(x) \geqslant M_2 > \mathcal{R}^-_{\gamma,\beta}.$$

Therefore, for each $k \in \mathbb{N}$, either

$$\int_{\Omega_1} z_k d\nu > \inf\{\mathrm{Ran}(\gamma)\}\nu(\Omega_1) + \frac{\delta'}{2} \tag{5.41}$$

or

$$\int_{\Omega_2} \omega_k d\nu > \inf\{\mathrm{Ran}(\beta)\}\nu(\Omega_2) + \frac{\delta'}{2}, \tag{5.42}$$

where $\delta' := M_2 - \mathcal{R}^-_{\gamma,\beta} > 0$.

For $k \in \mathbb{N}$ such that (5.41) holds let $K_k := \{x \in \Omega_1 \ : \ z_k(x) > \inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)}\}$. Then

$$\begin{aligned}
\int_{K_k} z_k d\nu &= \int_{\Omega_1} z_k d\nu - \int_{\Omega_1 \setminus K_k} z_k d\nu \\
&> \left(\inf\{\mathrm{Ran}(\gamma)\}\nu(\Omega_1) + \frac{\delta'}{2}\right) - (\nu(\Omega_1) - \nu(K_k))\left(\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)}\right) \\
&= \frac{\delta'}{4} + \nu(K_k)\left(\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)}\right),
\end{aligned}$$

and

$$\int_{K_k} z_k d\nu \leqslant \sup\{\mathrm{Ran}(\gamma)\}\nu(K_k).$$

Therefore,

$$\nu(K_k)\left(\sup\{\mathrm{Ran}(\gamma)\} - \inf\{\mathrm{Ran}(\gamma)\} - \frac{\delta'}{4\nu(\Omega_1)}\right) \geqslant \frac{\delta'}{4},$$

thus $\nu(K_k) > 0$, $\sup\{\mathrm{Ran}(\gamma)\} - \inf\{\mathrm{Ran}(\gamma)\} - \dfrac{\delta'}{4\nu(\Omega_1)} > 0$ and

$$\nu(K_k) \geqslant \frac{\delta'/4}{\sup\{\mathrm{Ran}(\gamma)\} - \inf\{\mathrm{Ran}(\gamma)\} - \frac{\delta'}{4\nu(\Omega_1)}}.$$

Now, if $\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)} \geqslant 0$ then $z_k \geqslant 0$ in K_k, thus $u_{n,k}^- = 0$ in K_k and $\|u_k^-\|_{L^p(K_k,\nu)} = 0$; so by the generalised Poincaré type inequality and (5.30) we get that $\{\|u_k\|_{L^p(\Omega,\nu)}\}_n$ is bounded. If $\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)} < 0$, then

$$\|u_k^-\|_{L^p(K_{n,k},\nu)} \leqslant -\nu(K_k)^{\frac{1}{p}}\gamma_-^{-1}\left(\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)}\right)$$

and by the generalised Poincaré inequality and (5.30) we get that $\{\|u_k\|_{L^p(\Omega,\nu)}\}_k$ is bounded. Similarly for $k \in \mathbb{N}$ such that (5.42) holds.

Now, proceeding as in *Step C* of the previous proof, we finish this proof. □

Finally, we give the proof of the remaining case. Some of the arguments here differ from those of the above cases.

Proof of Theorem 5.15 in the mixed case. Let us see the existence for

$$-\infty < \mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta} = +\infty, \tag{5.43}$$

or

$$-\infty = \mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta} < +\infty. \tag{5.44}$$

Suppose that (5.43) holds and let $\varphi \in L^{p'}(\Omega,\nu)$ satisfying

$$\mathcal{R}^-_{\gamma,\beta} < \int_\Omega \varphi d\nu.$$

If (5.44) holds and we have $\varphi \in L^{p'}(\Omega,\nu)$ satisfying $\displaystyle\int_\Omega \varphi d\nu < \mathcal{R}^+_{\gamma,\beta}$, the argument is analogous.

Let $\varphi_{n,k}$ be defined as in (5.24) and let $u_{n,k} \in L^\infty(\Omega,\nu)$, $n,k \in \mathbb{N}$, be the solution to the *Approximate Problem* (5.25)–(5.26). Then, by Lemma 5.3 together with (5.27), we have that $\{\|u^+_{n,k}\|_{L^p(\Omega,\nu)}\}_{n,k}$ is bounded. However, for a fixed $k \in \mathbb{N}$, since $u_{n,k}$ is nondecreasing in n we have that $\{\|u^-_{n,k}\|_{L^p(\Omega,\nu)}\}_n$ is bounded. Therefore, proceeding as in *Step B* of the first case, we obtain $u_k \in L^p(\Omega,\nu)$, z^+_k, $z^-_k \in L^{p'}(\Omega_1,\nu)$ and ω^+_k, $\omega^-_k \in L^{p'}(\Omega_2,\nu)$, $k \in \mathbb{N}$, such that

$$z^+_k(x) + z^-_k(x) - \int_\Omega \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) - \frac{1}{k}|u_k(x)|^{p-2}u^-_k(x) = \varphi_k(x), \tag{5.45}$$

for ν-a.e. $x \in \Omega_1$, and

$$\omega^+_k(x) + \omega^-_k(x) - \int_\Omega \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) - \frac{1}{k}|u_k(x)|^{p-2}u^-_k(x) = \varphi_k(x) \tag{5.46}$$

where, for $k \in \mathbb{N}$,

$$z^+_k = (\gamma_+)_k(u_k), \ \ z^-_k \in \gamma_-(u_k) \ \ \nu\text{-a.e. in } \Omega_1,$$

and

$$\omega^+_k = (\beta_+)_k(u_k), \ \ \omega^-_k \in \beta_-(u_k) \ \ \nu\text{-a.e. in } \Omega_2.$$

We will now see that $\{\|u_k\|_{L^p(\Omega,\nu)}\}_k$ is bounded. Proceeding as in *Step E* of the previous proof and using the same notation, we get that for each $k \in \mathbb{N}$, either

$$\int_{\Omega_1} z_k d\nu > \inf\{\mathrm{Ran}(\gamma)\}\nu(\Omega_1) + \frac{\delta'}{2} \tag{5.47}$$

or

$$\int_{\Omega_2} \omega_k d\nu > \inf\{\mathrm{Ran}(\beta)\}\nu(\Omega_2) + \frac{\delta'}{2}. \tag{5.48}$$

We now proceed by dividing the proof into cases. However, we first need the following estimate. Let $\rho \in P_0$ (recall (5.4)). Multiplying equations (5.45) and (5.46) by $\rho(u^+_k)$, integrating with respect to ν over Ω_1 and Ω_2, respectively, and using the integration by parts formula (Remark 5.13) we get, after removing some nonnegative terms,

$$\int_\Omega \left(z^+_k \chi_{\Omega_1} + w^+_k \chi_{\Omega_2}\right)\rho(u^+_k)d\nu \leqslant \int_\Omega \varphi^+_k \rho(u^+_k)d\nu.$$

Therefore, from (5.6), we get that, for any $h > 0$,

$$\int_{\{u^+_k > h\}} \left(z^+_k \chi_{\Omega_1} + w^+_k \chi_{\Omega_2}\right)d\nu \leqslant \int_{\{u^+_k > h\}} \varphi^+_k d\nu.$$

Now,

$$\int_{\{u_k^+>h\}} \varphi_k^+ d\nu \leqslant \big(\nu(\{u_k^+ > h\})\big)^{1/p} \left(\int_\Omega |\varphi^+|^{p'} d\nu\right)^{1/p'}$$

and

$$\nu(\{u_k^+ > h\}) \leqslant \int_\Omega \frac{|u_k^+|^p}{h^p} d\nu \leqslant \int_\Omega \frac{|u_1^+|^p}{h^p} d\nu,$$

which implies that

$$\int_{\{u_k^+>h\}} \left(z_k^+ \chi_{\Omega_1} + w_k^+ \chi_{\Omega_2}\right) d\nu \leqslant \frac{1}{h} \|\varphi\|_{L^{p'}(\Omega,\nu)} \|u_1^+\|_{L^p(\Omega,\nu)}. \tag{5.49}$$

Case 1. For $k \in \mathbb{N}$ such that (5.47) holds, let

$$K_k := \left\{ x \in \Omega_1 \,:\, z_k(x) > \inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)} \right\}.$$

Then,

$$\int_{K_k} z_k d\nu = \int_{\Omega_1} z_k d\nu - \int_{\Omega_1 \setminus K_k} z_k d\nu > \frac{\delta'}{4} + \nu(K_k) \left(\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)} \right). \tag{5.50}$$

Case 1.1. Suppose that $\sup D(\gamma) = +\infty$. Taking $h > 0$ such that

$$\frac{1}{h} \|\varphi\|_{L^{p'}(\Omega,\nu)} \|u_1^+\|_{L^p(\Omega,\nu)} < \delta'/8,$$

we have that, by (5.49),

$$\int_{K_k} z_k d\nu = \int_{K_k \cap \{u_k > h\}} z_k d\nu + \int_{K_k \cap \{u_k \leqslant h\}} z_k d\nu \leqslant \frac{\delta'}{8} + \nu(K_k)\gamma^0(h).$$

Therefore, recalling (5.50), we get

$$\frac{\delta'}{4} + \nu(K_k) \left(\inf\{\mathrm{Ran}(\gamma)\} + \frac{\delta'}{4\nu(\Omega_1)} \right) < \frac{\delta'}{8} + \nu(K_k)\gamma^0(h)$$

thus

$$\frac{\delta'}{8} < \nu(K_k) \left(\gamma^0(h) - \inf\{\mathrm{Ran}(\gamma)\} - \frac{\delta'}{4\nu(\Omega_1)} \right).$$

Consequently, $\nu(K_k) > 0$, $\gamma^0(h) - \inf\{\mathrm{Ran}(\gamma)\} - \dfrac{\delta'}{4\nu(\Omega_1)} > 0$ and

$$\nu(K_k) \geqslant \frac{\delta'/4}{\gamma^0(h) - \inf\{\mathrm{Ran}(\gamma)\} - \frac{\delta'}{4\nu(\Omega_1)}}.$$

From here we conclude as in the previous proof.
Case 1.2. Suppose now that $\sup D(\gamma) = r_\gamma < +\infty$.
Case 1.2.1. If, moreover, $\sup D(\beta) = r_\beta < +\infty$, by Lemma 5.1,

$$(\gamma_+)_k(r) = k(r - r_\gamma)^+ \quad \text{for } r \geqslant r_\gamma + \tfrac{1}{k}\gamma^0(r_\gamma) =: r_\gamma^k, \tag{5.51}$$

and

$$(\beta_+)_k(r) = k(r - r_\beta)^+ \quad \text{for } r \geqslant r_\beta + \tfrac{1}{k}\beta^0(r_\beta) =: r_\beta^k. \tag{5.52}$$

Let's suppose that $r_\gamma \leqslant r_\beta$ (if $r_\beta \leqslant r_\gamma$ we proceed analogously) and let $\Psi_k(r) := k(r - r_\beta^k)^+$. Let $\rho \in \mathcal{P}_0$. Multiplying equations (5.45) and (5.46) by $\rho(\Psi_k(u_k))$, integrating with respect to ν over Ω_1 and Ω_2, respectively, adding them and applying the integration by parts formula as illustrated in Remark 5.13 we get (after removing the nonnegative term involving $\mathbf{a}_p$)

$$\int_\Omega \left(z_k^+ \chi_{\Omega_1} + \omega_k^+ \chi_{\Omega_2}\right) \rho(\Psi_k(u_k)) d\nu \leqslant \int_\Omega \varphi_k^+ \rho(\Psi_k(u_k)) d\nu$$

thus, using equations (5.51) and (5.52) and noting that $\rho(\Psi(u_k)) > 0$ only if $u_k > r_\beta^k$, we have

$$\begin{aligned}\int_\Omega k\left(u_k - r_\beta\right)^+ \rho(\Psi_k(u_k))d\nu &\leqslant \int_\Omega \left(k(u_k-r_\gamma)^+\chi_{\Omega_1} + k(u_k-r_\beta)^+\chi_{\Omega_2}\right)\rho(\Psi_k(u_k))d\nu \\ &\leqslant \int_\Omega \varphi_k^+\rho(\Psi_k(u_k))d\nu \leqslant \int_\Omega \varphi^+\rho(\Psi_k(u_k))d\nu \ .\end{aligned}$$

Therefore, by (5.5), we get that

$$k(u_k - r_\beta)^+ \ll k(u_k - r_\beta)^+ + \lambda(\varphi^+ - k(u_k - r_\beta^k)^+)$$

for every $\lambda > 0$. In particular, for $\lambda = 1$,

$$k(u_k - r_\beta)^+ \ll k(u_k - r_\beta)^+ + \varphi^+ - k(u_k - r_\beta^k)^+. \tag{5.53}$$

Now, $k(u_k(x) - r_\beta)^+ + \varphi^+(x) - k(u_k(x) - r_\beta^k)^+$ is equal to

$$\begin{cases} \varphi^+(x) & \text{for } x \in \Omega \text{ such that } u_k(x) \leqslant r_\beta, \\ \varphi^+(x) + k(u_k(x) - r_\beta) & \text{for } x \in \Omega \text{ such that } r_\beta \leqslant u_k(x) \leqslant r_\beta^k, \\ \varphi^+(x) + k(r_\beta^k - r_\beta) & \text{for } x \in \Omega \text{ such that } u_k(x) \geqslant r_\beta^k, \end{cases}$$

thus $0 \leqslant k(u_k(x) - r_\beta)^+ + \varphi^+(x) - k(u_k(x) - r_\beta^k)^+ \leqslant \varphi^+(x) + \beta^0(r_\beta)$ for every $x \in \Omega$. Consequently, by (5.53), $\|k(u_k - r_\beta)^+\|_{L^{p'}(\Omega,\nu)} \leqslant \|\varphi^+ + \beta^0(r_\beta)\|_{L^{p'}(\Omega,\nu)}$ thus, up to a subsequence, $k(u_k - r_\beta) \stackrel{k}{\rightharpoonup} \omega \in L^{p'}(\Omega,\nu)$ weakly in $L^{p'}(\Omega,\nu)$.

Let's see that, up to a subsequence, $z_k^+ \stackrel{k}{\rightharpoonup} z \in L^{p'}(\Omega_1,\nu)$ weakly in $L^{p'}(\Omega_1,\nu)$. As above, given $\rho \in \mathcal{P}_0$, multiplying equations (5.45) and (5.46) by $\rho(z_k^+ + \omega_k^+)$ we get

$$\int_\Omega \left(z_k^+\chi_{\Omega_1} + \omega_k^+\chi_{\Omega_2}\right)\rho(z_k^+ + \omega_k^+)d\nu \leqslant \int_\Omega \varphi^+\rho(z_k^+ + \omega_k^+)d\nu.$$

Therefore, reasoning as before, we get

$$z_k^+ + \omega_k^+ \ll z_k^+ + \omega_k^+ + (\varphi^+ - (z_k^+\chi_{\Omega_1} + \omega_k^+\chi_{\Omega_2})) = \varphi^+ + z_k^+\chi_{\Omega_2} + \omega_k^+\chi_{\Omega_1}$$

thus

$$\|z_k^+\|_{L^{p'}(\Omega_1,\nu)} + \|z_k^+\|_{L^{p'}(\Omega_2,\nu)} \leqslant \|z_k^+\|_{L^{p'}(\Omega_2,\nu)} + \|\omega_k^+\|_{L^{p'}(\Omega_1,\nu)} + \|\varphi^+\|_{L^{p'}(\Omega_1,\nu)}$$

which yields

$$\|z_k^+\|_{L^{p'}(\Omega_1,\nu)} \leqslant \|\omega_k^+\|_{L^{p'}(\Omega_1,\nu)} + \|\varphi^+\|_{L^{p'}(\Omega_1,\nu)}.$$

We conclude because, by the previous computations,

$$\|\omega_k^+\|_{L^{p'}(\Omega_1\cap\{u_k\geqslant r_\beta^k\},\nu)} = \|k(u_k - r_\beta)^+\|_{L^{p'}(\Omega_1\cap\{u_k\geqslant r_\beta^k\},\nu)}$$

is uniformly bounded and $\|\omega_k^+\|_{L^{p'}(\Omega_1\cap\{u_k<r_\beta^k\},\nu)} \leqslant \|\beta^0(r_\beta)\|_{L^{p'}(\Omega_1,\nu)} < +\infty$.

Finally, by the Dunford-Pettis Theorem (see, for example, [**7**, Theorem 1.38]), $\{z_k\}_k$ is an equi-integrable family and, therefore (see [**7**, Proposition 1.27]),

$$\lim_{h\to+\infty}\sup_{k\in\mathbb{N}}\int_{\{x\in\Omega_1\,:\,z_k(x)>h\}} z_k d\nu = 0.$$

Consequently, we may find $h > 0$ such that

$$\sup_{k\in\mathbb{N}}\int_{\{x\in\Omega_1\,:\,z_k(x)>h\}} z_k d\nu < \frac{\delta'}{8}.$$

Then again,

$$\int_{K_k} z_k d\nu = \int_{K_k\cap\{z_k>h\}} z_k d\nu + \int_{K_k\cap\{z_k\leqslant h\}} z_k d\nu \leqslant \frac{\delta'}{8} + \nu(K_k)h, \tag{5.54}$$

and we finish as in the previous proof.

Case 1.2.2. Suppose that $\sup D(\beta) = +\infty$. Set $r_0 := r_\gamma^1 = r_\gamma + \gamma^0(r_\gamma)$, which obviously satisfies $r_0 \geqslant r_\gamma^k = r_\gamma + \frac{1}{k}\gamma^0(r_\gamma)$ for every $k \in \mathbb{N}$. Then, since $(\gamma_+)_k(r_0) \uparrow +\infty$ there exists $k_0 \in \mathbb{N}$ such that $(\gamma_+)_k(r_0) \geqslant \beta^0(r_0) \geqslant (\beta_+)_k(r_0)$ for every $k \geqslant k_0$. Therefore, recalling that the Yosida approximation $(\beta_+)_k$ is k-Lipschitz, we have that $(\beta_+)_k(r) \leqslant k(r - r_0)^+ + (\gamma_+)_k(r_0) \leqslant k(r - r_\gamma)^+ = (\gamma_+)_k(r)$ for every $r \geqslant r_0$ and $k \geqslant k_0$. Therefore, we proceed as in the previous case but with $\widehat{\Psi}_k(r) := ((\beta_+)_k(r) - \beta^0(r_0))^+$ instead of Ψ_k to obtain (noting that $\rho(\widehat{\Psi}_k(u_k)) > 0$ only if $u_k > r_0$)

$$\begin{aligned}\int_\Omega (\beta_+)_k(u_k)\rho(\widehat{\Psi}_k(u_k))d\nu &\leqslant \int_\Omega \left(k(u_k - r_\gamma)^+\chi_{\Omega_1} + (\beta_+)_k(u_k)\chi_{\Omega_2}\right)\rho(\widehat{\Psi}_k(u_k))d\nu \\ &\leqslant \int_\Omega \varphi_k^+\rho(\widehat{\Psi}_k(u_k))d\nu \leqslant \int_\Omega \varphi^+\rho(\widehat{\Psi}_k(u_k))d\nu\end{aligned}$$

for every $k \geqslant k_0$. Again, as before,

$$(\beta_+)_k(u_k) \ll (\beta_+)_k(u_k) + \varphi^+ - ((\beta_+)_k(u_k) - \beta^0(r_0))^+, \ k \geqslant k_0,$$

but $(\beta_+)_k(u_k) + \varphi^+ - ((\beta_+)_k(u_k) - \beta^0(r_0))^+$ is equal to

$$\begin{cases} \varphi^+(x) + (\beta_+)_k(u_k) & \text{for } x \in \Omega \text{ such that } (\beta_+)_k(u_k(x)) \leqslant \beta^0(r_0) \\ \varphi^+(x) + \beta^0(r_0) & \text{for } x \in \Omega \text{ such that } (\beta_+)_k(u_k(x)) > \beta^0(r_0) \end{cases}$$

thus $0 \leqslant (\beta_+)_k(u_k) + \varphi^+ - ((\beta_+)_k(u_k) - \beta^0(r_0))^+ \leqslant \varphi^+ + \beta^0(r_0)$ in Ω. Consequently, $\|(\beta_+)_k(u_k)\|_{L^{p'}(\Omega,\nu)} \leqslant \|\varphi^+ + \beta^0(r_0)\|_{L^{p'}(\Omega,\nu)}$ and we can get, as in the previous case, that (5.54) holds for some $h > 0$.

Case 2. For $k \in \mathbb{N}$ such that (5.48) holds, let

$$\tilde{K}_k := \{x \in \Omega_2 \ : \ w_k(x) > \inf\{\text{Ran}(\beta)\} + \frac{\delta'}{4\nu(\Omega_2)}\}$$

and proceed similarly. □

Remark 5.16.

(i) Taking limits in (5.30) we obtain that, if $[u,v]$ is a solution of $(GP_\varphi^{\mathbf{a}_p,\gamma,\beta})$, then

$$\frac{c_p}{2}\left(\int_\Omega\int_\Omega |u(y) - u(x)|^p dm_x(y)d\nu(x)\right)^{\frac{1}{p'}} \leqslant \Lambda_1\|\varphi\|_{L^{p'}(\Omega,\nu)} + \frac{\Lambda_1 + \Lambda_2}{\nu(\Omega)^{\frac{1}{p}}}\|\varphi\|_{L^1(\Omega,\nu)}$$

where c_p is the constant in (5.11), and Λ_1 and Λ_2 come from the generalised Poincaré type inequality and depend only on p, Ω_1 and Ω_2.

(ii) Observe that, on account of (5.10) and the above estimate, we have

$$\left(\int_\Omega \left|\int_\Omega \mathbf{a}_p(x,y,u(y) - u(x))dm_x(y)\right|^{p'} d\nu(x)\right)^{\frac{1}{p'}} \leqslant C_p\nu(\Omega) + \frac{2C_p}{c_p}(2\Lambda_1 + \Lambda_2)\|\varphi\|_{L^{p'}(\Omega,\nu)}.$$

Therefore, since $[u,v]$ is a solution of $(GP_\varphi^{\mathbf{a}_p,\gamma,\beta})$,

$$\|v\|_{L^{p'}(\Omega,\nu)} \leqslant C_p\nu(\Omega) + \left(\frac{2C_p}{c_p}(2\Lambda_1 + \Lambda_2) + 1\right)\|\varphi\|_{L^{p'}(\Omega,\nu)}.$$

(iii) When $\varphi = 0$ in Ω_2, we can easily get that $v \ll \varphi$ in Ω_1.

5.2. Other boundary conditions

We can now ask for existence and uniqueness of solutions of the following problem (which was introduced in Section 5.1.1)

$$(5.55) \qquad \begin{cases} \gamma(u(x)) - \text{div}_m\mathbf{a}_p u(x) \ni \varphi(x), & x \in W, \\ \mathcal{N}_2^{\mathbf{a}_p}u(x) + \beta(u(x)) \ni \phi(x), & x \in \partial_m W, \end{cases}$$

or, of the more general problem,

$$\begin{cases} \gamma(u(x)) - \displaystyle\int_{W\cup\Omega_2} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) \ni \varphi(x), & x\in\Omega_1 = W, \\ \mathcal{N}_2^{\mathbf{a}_p}u(x) + \beta(u(x)) \ni \phi(x), & x \in \Omega_2 \subseteq \partial_m W. \end{cases}$$

Recall that $\mathcal{N}_2^{\mathbf{a}_p}$ *is defined as follows*

$$\mathcal{N}_2^{\mathbf{a}_p}u(x) := -\int_W \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y), \quad x\in\partial_m W,$$

which involves integration with respect to ν *only over* W*, or more specifically over* $\partial_m(X\backslash W)$*.*

For Problem (5.55) *we know that, in general, we do not have an appropriate Poincaré type inequality to work with (see Remark 1.86). Therefore, other techniques must be used to obtain the existence of solutions. In the particular case of* $\gamma(r) = \beta(r) = r$ *this is done in Section 5.5 by exploiting further monotonicity techniques.*

However, if a generalised Poincaré type inequality (Definition 1.81) is satisfied on $(A = \Omega_1, B = \Omega_2)$ *(this holds, for example, for finite graphs even if* $\Omega_2 = \partial_m W$*), we could solve the above problem by using the same techniques that we have used to solve Problem* (5.13)*. Indeed, we work analogously but with the integration by parts formula given for* Q_2 *in Remark 5.17 below.*

In any case, one could try to solve the stationary problem for both types of boundary conditions for data in $L^{q'}(\Omega,\nu)$*, where* $\max\{p-1,1\} < q < p$*, by using a generalised* (q,p)*-Poincaré type inequality. In Theorem 1.88 we give sufficient conditions for this generalised Poincaré type inequality to hold.*

Remark 5.17. Let $\Omega := \Omega_1 \cup \Omega_2$ and

$$\boxed{Q_2 := (\Omega\times\Omega)\backslash(\Omega_2\times\Omega_2).}$$

The following integration by parts formula holds: Let $q \geqslant 1$. Let u be a ν-measurable function such that

$$[(x,y)\mapsto \mathbf{a}_p(x,y,u(y)-u(x))] \in L^q(Q_2, \nu\otimes m_x)$$

and let $w \in L^{q'}(\Omega,\nu)$, then

$$\begin{aligned} &-\int_{\Omega_1}\int_{\Omega} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)w(x)d\nu(x) \\ &\quad -\int_{\Omega_2}\int_{\Omega_1} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)w(x)d\nu(x) \\ &= \frac{1}{2}\int_{Q_2} \mathbf{a}_p(x,y,u(y)-u(x))(w(y)-w(x))d(\nu\otimes m_x)(x,y). \end{aligned}$$

Remark 5.18. It is possible to consider this type of problems but with the random walk and the nonlocal Leray-Lions operator having a different behaviour on each subset Ω_i, $i = 1,2$. For example, one could consider a problem, posed in $\Omega_1\cup\Omega_2 \subset \mathbb{R}^N$, such as the following

$$\begin{cases} \gamma(u(x)) - \displaystyle\int_{\Omega_1} \mathbf{a}_p^1(x,y,u(y)-u(x))J_1(x-y)dy \\ \qquad\qquad - \displaystyle\int_{\Omega_2} \mathbf{a}_p^3(x,y,u(y)-u(x))J_3(x-y)dx \ni \varphi(x), \quad x\in\Omega_1, \\ \beta(u(x)) - \displaystyle\int_{\Omega_1} \mathbf{a}_p^3(x,y,u(y)-u(x))J_3(x-y)dy \\ \qquad\qquad - \displaystyle\int_{\Omega_2} \mathbf{a}_p^2(x,y,u(y)-u(x))J_2(x-y)dx \ni \varphi(x), \quad x\in\Omega_2, \end{cases}$$

where J_i are kernels like the ones in Example 1.37, and $\mathbf{a}_p^i$ are functions like the one in Subsection 5.1.1, $i = 1, 2, 3$. This could be done by obtaining a Poincaré type inequality involving $\frac{1}{\alpha_0} J_0$, where J_0 is the minimum of the previous three kernels and $\alpha_0 = \int_{\mathbb{R}^N} J_0(z)dz$. This idea has been used in [**52**] to study an homogenization problem.

5.3. Doubly nonlinear diffusion problems

We will study two kinds of nonlocal p-Laplacian type diffusions problems. In one of them we cover nonlocal nonlinear diffusion problems with nonlinear dynamical boundary conditions and on the other we tackle nonlinear boundary conditions. We work under the Assumptions 1 to 5 used in Subsection 5.1.2.

5.3.1. Nonlinear dynamical boundary conditions. *Our aim in this section is to study the following diffusion problem*

$$(5.56)\quad \begin{cases} v_t(t,x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t,y)-u(t,x))dm_x(y) = f(t,x), & x \in \Omega_1,\ 0<t<T, \\ v(t,x) \in \gamma(u(t,x)), & x \in \Omega_1,\ 0<t<T, \\ w_t(t,x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t,y)-u(t,x))dm_x(y) = g(t,x), & x \in \Omega_2,\ 0<t<T, \\ w(t,x) \in \beta(u(t,x)), & x \in \Omega_2,\ 0<t<T, \\ v(0,x) = v_0(x), & x \in \Omega_1, \\ w(0,x) = w_0(x), & x \in \Omega_2, \end{cases}$$

of which Problem (5.2) is a particular case and which covers the case of dynamic evolution on the boundary $\partial_m W$ when $\beta \neq \mathbb{R} \times \{0\}$. This includes, in particular, for $\gamma = \mathbb{R} \times \{0\}$, the problem where the dynamic evolution occurs only on the boundary:

$$\begin{cases} -\mathrm{div}_m \mathbf{a}_p u(t,x) = f(t,x), & x \in W,\ 0<t<T, \\ w_t(t,x) + \mathcal{N}_{\mathbf{1}}^{\mathbf{a}_p} u(t,x) = g(t,x), & x \in \partial_m W,\ 0<t<T, \\ w(t,x) \in \beta(u(t,x)), & x \in \partial_m W,\ 0<t<T, \\ w(0,x) = w_0(x), & x \in \partial_m W. \end{cases}$$

See [**13**] *for the reference local model.*

Note that we may abbreviate Problem (5.56) by using v instead of (v,w) and f instead of (f,g) as

$$(5.57)\quad \begin{cases} v_t(t,x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t,y)-u(t,x))dm_x(y) = f(t,x), & x \in \Omega,\ 0<t<T, \\ v(t,x) \in \gamma(u(t,x)), & x \in \Omega_1,\ 0<t<T, \\ v(t,x) \in \beta(u(t,x)), & x \in \Omega_2,\ 0<t<T, \\ v(0,x) = v_0(x), & x \in \Omega. \end{cases}$$

To solve this problem we will use nonlinear semigroup theory. To this end we introduce a multivalued operator associated with Problem (5.57) that allows us to rewrite it as an abstract Cauchy problem. Observe that this operator will be defined on

$$L^1(\Omega,\nu) \times L^1(\Omega,\nu) \equiv \left(L^1(\Omega_1,\nu) \times L^1(\Omega_2,\nu)\right) \times \left(L^1(\Omega_1,\nu) \times L^1(\Omega_2,\nu)\right).$$

Definition 5.19. We say that $(v,\hat{v}) \in \mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta}$ if $v,\hat{v} \in L^1(\Omega,\nu)$, and there exists $u \in L^p(\Omega,\nu)$ with

$$u \in \mathrm{Dom}(\gamma) \text{ and } v \in \gamma(u) \quad \nu\text{-a.e. in } \Omega_1,$$

and

$$u \in \mathrm{Dom}(\beta) \text{ and } v \in \beta(u) \quad \nu\text{-a.e. in } \Omega_2,$$

such that

$$(x,y) \mapsto a_p(x,y,u(y)-u(x)) \in L^{p'}(\Omega\times\Omega, \nu\otimes m_x)$$

and

$$-\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \hat{v} \quad \text{in} \quad \Omega;$$

that is, $[u,v]$ is a solution of $(GP_{v+\hat{v}})$ (see (5.14) and Definition 5.12).

On account of the results given in Subsection 5.1.2 (Theorems 5.14 and 5.15) we have the following result.

Theorem 5.20. *The operator $\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta}$ is T-accretive in $L^1(\Omega,\nu)$ (see Definition A.47) and satisfies the range condition*

$$\left\{\varphi \in L^{p'}(\Omega,\nu) : \mathcal{R}^-_{\gamma,\beta} < \int_\Omega \varphi d\nu < \mathcal{R}^+_{\gamma,\beta}\right\} \subset R(I+\lambda\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta}) \quad \forall\lambda>0.$$

With respect to the domain of such operator we can prove the following result. Recall that $\Gamma^- = \inf\{\mathrm{Ran}(\gamma)\}$, $\Gamma^+ = \sup\{\mathrm{Ran}(\gamma)\}$, $\mathfrak{B}^- = \inf\{\mathrm{Ran}(\beta)\}$ and $\mathfrak{B}^+ = \sup\{\mathrm{Ran}(\beta)\}$.

Theorem 5.21. *It holds that*

$$\overline{D(\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta})}^{L^{p'}(\Omega,\nu)} = \left\{v \in L^{p'}(\Omega,\nu) : \Gamma^- \leqslant v \leqslant \Gamma^+ \text{ in } \Omega_1,\ \mathfrak{B}^- \leqslant v \leqslant \mathfrak{B}^+ \text{ in } \Omega_2\right\}.$$

Therefore, we also have that

$$\overline{D(\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta})}^{L^1(\Omega,\nu)} = \left\{v \in L^1(\Omega,\nu) : \Gamma^- \leqslant v \leqslant \Gamma^+ \text{ in } \Omega_1,\ \mathfrak{B}^- \leqslant v \leqslant \mathfrak{B}^+ \text{ in } \Omega_2\right\}.$$

Proof. It is obvious that

$$\overline{D(\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta})}^{L^{p'}(\Omega,\nu)} \subset \left\{v \in L^{p'}(\Omega,\nu) : \Gamma^- \leqslant v \leqslant \Gamma^+ \text{ in } \Omega_1,\ \mathfrak{B}^- \leqslant v \leqslant \mathfrak{B}^+ \text{ in } \Omega_2\right\}.$$

For the other inclusion it is enough to see that

$$\left\{v \in L^\infty(\Omega,\nu) : \Gamma^- \leqslant v \leqslant \Gamma^+ \text{ in } \Omega_1,\ \mathfrak{B}^- \leqslant v \leqslant \mathfrak{B}^+ \text{ in } \Omega_2\right\} \subset \overline{D(\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta})}^{L^{p'}(\Omega,\nu)}.$$

Suppose first that γ and β satisfy

$$\Gamma^- < 0, \quad \Gamma^+ > 0,$$
$$\mathfrak{B}^- = 0, \quad \mathfrak{B}^+ > 0.$$

It is enough to see that for any $v \in L^\infty(\Omega,\nu)$ such that there exist $m_1 < 0$, $\widetilde{m_i} \in \mathbb{R}$, $\widetilde{M_i} \in \mathbb{R}$, $M_i > 0$, $i = 1,2$, satisfying

$$\Gamma^- < m_1 < \widetilde{m_1} \leqslant v \leqslant \widetilde{M_1} < M_1 < \Gamma^+ \text{ in } \Omega_1,$$
$$0 < \widetilde{m_2} \leqslant v \leqslant \widetilde{M_2} < M_2 < \mathfrak{B}^+ \text{ in } \Omega_2,$$

it holds that $v \in \overline{D(\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta})}^{L^{p'}(\Omega,\nu)}$.

By the results in Subsection 5.1.6 we know that, for $n \in \mathbb{N}$, there exists $u_n \in L^p(\Omega,\nu)$ and $v_n \in L^{p'}(\Omega,\nu)$ such that $[u_n,v_n]$ is a solution of $\left(GP_v^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$, i.e., $v_n \in \gamma(u_n)$ ν-a.e. in Ω_1, $v_n \in \beta(u_n)$ ν-a.e. in Ω_2 and

$$v_n(x) - \frac{1}{n}\int_\Omega \mathbf{a}_p(x,y,u_n(y)-u_n(x))dm_x(y) = v(x) \quad \text{for } \nu\text{-a.e. } x \in \Omega.$$

In other words, $(v_n, n(v - v_n)) \in \mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta}$ or, equivalently,

$$v_n := \left(I + \frac{1}{n}\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta}\right)^{-1}(v) \in D(\mathcal{B}_{\mathbf{a}_p}^{m,\gamma,\beta}).$$

Let us see that $v_n \xrightarrow{n} v$ in $L^{p'}(\Omega,\nu)$.

Let $a_{m_1} \leqslant 0$ and $a_{M_1} \geqslant 0$ such that

$$m_1 \in \gamma(a_{m_1}) \text{ and } M_1 \in \gamma(a_{M_1}),$$

and let $b_{M_2} \geqslant 0$ such that

$$M_2 \in \beta(b_{M_2}).$$

Set

$$\widehat{v}(x) := \begin{cases} M_1, & x \in \Omega_1, \\ M_2, & x \in \Omega_2, \end{cases}$$

$$\widehat{u}(x) := \begin{cases} a_{M_1}, & x \in \Omega_1, \\ b_{M_2}, & x \in \Omega_2, \end{cases}$$

and

$$\widehat{\varphi}_n(x) := \begin{cases} M_1 - \dfrac{1}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,\widehat{u}(y) - \widehat{u}(x))dm_x(y), & x \in \Omega_1, \\ M_2 - \dfrac{1}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,\widehat{u}(y) - \widehat{u}(x))dm_x(y), & x \in \Omega_2. \end{cases}$$

Then, $[\widehat{u},\widehat{v}]$ is a solution of $\left(GP_{\widehat{\varphi}_n}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$.

Similarly, for

$$\widetilde{v}(x) := \begin{cases} m_1, & x \in \Omega_1, \\ 0, & x \in \Omega_2, \end{cases}$$

$$\widetilde{u}(x) := \begin{cases} a_{m_1}, & x \in \Omega_1, \\ 0, & x \in \Omega_2, \end{cases}$$

and

$$\widetilde{\varphi}_n(x) := \begin{cases} m_1 - \dfrac{1}{n}\displaystyle\int_{\Omega_2} \mathbf{a}_p(x,y,-a_{m_1})dm_x(y), & x \in \Omega_1, \\ \dfrac{1}{n}\displaystyle\int_{\Omega_1} \mathbf{a}_p(x,y,-a_{m_1})dm_x(y), & x \in \Omega_2, \end{cases}$$

we have that $[\widetilde{u},\widetilde{v}]$ is a solution of $\left(GP_{\widetilde{\varphi}_n}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$.

Now, recalling (5.10), we have that there exists $n_0 \in \mathbb{N}$ such that

$$v \leqslant \widetilde{M_1}\chi_{\Omega_1} + \widetilde{M_2}\chi_{\Omega_2} < \widehat{\varphi}_n \text{ in } \Omega$$

and

$$v \geqslant \widetilde{m_1}\chi_{\Omega_1} + \widetilde{m_2}\chi_{\Omega_2} > \widetilde{\varphi}_n \text{ in } \Omega$$

for $n \geqslant n_0$. Consequently, by the maximum principle (Theorem 5.14) we obtain that

$$\widetilde{u} \leqslant u_n \leqslant \widehat{u},$$

thus

$$\left\{\|u_n\|_{L^\infty(\Omega,\nu)}\right\}_n \text{ is bounded.}$$

Finally, since

$$v_n(x) - v(x) = \frac{1}{n}\int_\Omega \mathbf{a}_p(x,y,u_n(y) - u_n(x))dm_x(y) \qquad \nu\text{-a.e. in } \Omega,$$

we conclude that, on account of (5.10),

$$v_n \xrightarrow{n} v \text{ in } L^{p'}(\Omega,\nu).$$

The other cases follow similarly, we will see two of them. Note that, since $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$, it is not possible to have $\gamma = \mathbb{R}\times\{0\}$ and $\beta = \mathbb{R}\times\{0\}$ simultaneously. For example, suppose that we have

$$\Gamma^- = 0, \quad \Gamma^+ > 0,$$

$$\mathfrak{B}^- = 0, \quad \mathfrak{B}^+ > 0.$$

We will use the same notation. Let $v \in L^\infty(\Omega,\nu)$ such that there exist $\widetilde{m_i} \in \mathbb{R}$, $\widetilde{M_i} \in \mathbb{R}$, $M_i > 0$, $i = 1,2$, satisfying

$$0 < \widetilde{m_1} \leqslant v \leqslant \widetilde{M_1} < M_1 < \Gamma^+ \text{ in } \Omega_1,$$

$$0 < \widetilde{m_2} \leqslant v \leqslant \widetilde{M_2} < M_2 < \mathfrak{B}^+ \text{ in } \Omega_2.$$

As before, the results in Subsection 5.1.6 ensure that there exist $u_n \in L^p(\Omega,\nu)$ and $v_n \in L^{p'}(\Omega,\nu)$, $n\in\mathbb{N}$, such that $[u_n,v_n]$ is a solution of $\left(GP_v^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$. Let $a_{M_1} \geqslant 0$ and $b_{M_2} \geqslant 0$ such that

$$M_1 \in \gamma(a_{M_1}) \text{ and } M_2 \in \beta(b_{M_2}).$$

Now again, let

$$\widehat{v}(x) := \begin{cases} M_1, & x \in \Omega_1, \\ M_2, & x \in \Omega_2, \end{cases}$$

$$\widehat{u}(x) := \begin{cases} a_{M_1}, & x \in \Omega_1, \\ b_{M_2}, & x \in \Omega_2, \end{cases}$$

and

$$\widehat{\varphi}_n(x) := \begin{cases} M_1 - \dfrac{1}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,\widehat{u}(y)-\widehat{u}(x))dm_x(y), & x \in \Omega_1, \\ M_2 - \dfrac{1}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,\widehat{u}(y)-\widehat{u}(x))dm_x(y), & x \in \Omega_2. \end{cases}$$

Then, as before, $[\widehat{u},\widehat{v}]$ is a solution of $\left(GP_{\widehat{\varphi}_n}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$.

Now, taking $\widetilde{v}$, $\widetilde{u}$ and $\widetilde{\varphi}$ all equal to the null function in Ω and recalling that $\mathbf{a}_p(x,y,0) = 0$ for every $x,y \in X$, we obviously have that $[\widetilde{u},\widetilde{v}]$ is a solution of $\left(GP_0^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$. Consequently, again by the second part of the maximum principle, we obtain, as desired, that $0 \leqslant u_n \leqslant \widehat{v}$ for n large enough.

Finally, as a further example of a case which does not follow exactly with the same argument, suppose that $\gamma := \mathbb{R}\times\{0\}$ and, for example,

$$\mathfrak{B}^- = 0, \; \mathfrak{B}^+ > 0.$$

In this case we have to take $0 \not\equiv v \in L^\infty(\Omega,\nu)$ such that $v = 0$ in Ω_1 and such that there exists $M_2 > 0$ satisfying

$$0 \leqslant v < M_2 \text{ in } \Omega_2.$$

As in the previous cases, there exist $u_n \in L^p(\Omega,\nu)$ and $v_n \in L^{p'}(\Omega,\nu)$, $n \in \mathbb{N}$, such that $[u_n,v_n]$ is a solution of $\left(GP_v^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$. Let $b_{M_2} \geqslant 0$ such that $M_2 \in \beta(b_{M_2})$,

$$\widehat{v}(x) := \begin{cases} 0, & x \in \Omega_1, \\ M_2, & x \in \Omega_2, \end{cases}$$

$$\widehat{u}(x) := b_{M_2}, \; x \in \Omega,$$

and

$$\varphi_n(x) := \begin{cases} 0, & x \in \Omega_1, \\ M_2, & x \in \Omega_2. \end{cases}$$

Then, $[\hat{u}, \hat{v}]$ is a solution of $\left(GP_{\varphi_n}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$. Finally, take $\widetilde{v}$ and $\widetilde{u}$ again equal to the null function in Ω so that $[\widetilde{u}, \widetilde{v}]$ is a solution of $\left(GP_0^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}\right)$. Consequently, for n large enough, we get that $0 \leqslant u_n \leqslant \hat{v}$. □

In the next result we state the existence and uniqueness of solutions of Problem (5.57).

THEOREM 5.22. *Let $T > 0$. For any $v_0 \in L^1(\Omega, \nu)$ and $f \in L^1(0, T; L^1(\Omega, \nu))$ such that*

$$\Gamma^- \leqslant v_0 \leqslant \Gamma^+ \quad \text{in } \Omega_1,$$
$$\mathfrak{B}^- \leqslant v_0 \leqslant \mathfrak{B}^+ \quad \text{in } \Omega_2,$$

and

$$\mathcal{R}^-_{\gamma,\beta} < \int_\Omega v_0 d\nu + \int_0^t \int_\Omega f d\nu dt < \mathcal{R}^+_{\gamma,\beta} \quad \forall 0 \leqslant t \leqslant T, \tag{5.58}$$

there exists a unique mild solution $v \in C([0, T] : L^1(\Omega, \nu))$ of Problem (5.57).

Let v and $\widetilde{v}$ be the mild solutions of Problem (5.57) with respective data v_0, $\widetilde{v}_0 \in L^1(\Omega, \nu)$ and f, $\widetilde{f} \in L^1(0, T; L^1(\Omega, \nu))$, we have

$$\begin{aligned} \int_\Omega (v(t,x) - \widetilde{v}(t,x))^+ d\nu(x) \quad &\leqslant \int_\Omega (v_0(x) - \widetilde{v}_0(x))^+ d\nu(x) \\ &+ \int_0^t \int_\Omega \left(f(s,x) - \widetilde{f}(s,x)\right)^+ d\nu(x) ds, \quad \forall 0 \leqslant t \leqslant T. \end{aligned}$$

If, in addition to the previous assumptions on the data, we impose that (see Remark 5.2)

$$v_0 \in L^{p'}(\Omega, \nu), \; f \in L^{p'}(0, T; L^{p'}(\Omega, \nu)) \;\; \text{and} \; \int_{\Omega_1} j^*_\gamma(v_0) d\nu + \int_{\Omega_2} j^*_\beta(v_0) d\nu < +\infty, \tag{5.59}$$

then the mild solution v belongs to $W^{1,1}(0, T; L^{p'}(\Omega, \nu))$ and satisfies

$$\begin{cases} \partial_t v(t) + \mathcal{B}^{m,\gamma,\beta}_{\mathbf{a}_p} v(t) \ni f(t) & \text{for a.e. } t \in (0, T), \\ v(0) = v_0, \end{cases}$$

that is, v is a strong solution.

PROOF. We start by proving the existence of mild solutions. Let $n \in \mathbb{N}$ and consider the partition

$$t_0^n = 0 < t_1^n < \cdots < t_{n-1}^n < t_n^n = T$$

where $t_i^n := iT/n$, $i = 1, \ldots, n$. Now, let $f_i^n \in L^{p'}(\Omega, \nu)$, $i = 1, \ldots n$, and $v_0^n \in L^{p'}(\Omega, \nu)$ such that

$$\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \|f(t) - f_i^n\|_{L^1(\Omega,\nu)} dt \leqslant \frac{T}{n} \tag{5.60}$$

and

$$\|v_0 - v_0^n\|_{L^1(\Omega,\nu)} \leqslant \frac{T}{n}. \tag{5.61}$$

Then, setting

$$f_n(t) := f_i^n, \quad \text{for } t \in]t_{i-1}^n, t_i^n], \; i = 1, \ldots, n,$$

we have that

$$\int_0^T \|f(t) - f_n\|_{L^1(\Omega,\nu)} dt \leqslant \frac{T}{n}.$$

From the results in Subsection 5.1.6 we will see that, for n large enough, we may recursively find a solution $[u_i^n, v_i^n]$ of $\left(GP^{\frac{T}{n}\mathbf{a}_p,\gamma,\beta}_{\frac{T}{n}f_i^n+v_{i-1}^n}\right)$, $i = 1, \ldots, n$, in other words,

$$v_i^n(x) - \frac{T}{n}\int_\Omega \mathbf{a}_p(x, y, u_i^n(y) - u_i^n(x))dm_x(y) = \frac{T}{n}f_i^n(x) + v_{i-1}^n(x),\ x \in \Omega,$$

or, equivalently,

$$\frac{v_i^n(x) - v_{i-1}^n(x)}{T/n} - \int_\Omega \mathbf{a}_p(x, y, u_i^n(y) - u_i^n(x))dm_x(y) = f_i^n(x),\ x \in \Omega, \tag{5.62}$$

with $v_i^n(x) \in \gamma(u_i^n(x))$ for ν-a.e. $x \in \Omega_1$ and $v_i^n(x) \in \beta(u_i^n(x))$ for ν-a.e. $x \in \Omega_2$, $i = 1, \ldots, n$. That is, we may find the unique solution v_i^n of the time discretization scheme associated with (5.57):

$$v_i^n + \frac{T}{n}\mathcal{B}^{m,\gamma,\beta}_{\mathbf{a}_p}(v_i^n) \ni \frac{T}{n}f_i^n + v_{i-1}^n \quad \text{for } i = 1, \ldots, n.$$

However, to apply the results in Subsection 5.1.6, we must ensure that

$$\mathcal{R}^-_{\gamma,\beta} < \int_\Omega \left(\frac{T}{n}f_i^n + v_{i-1}^n\right) d\nu < \mathcal{R}^+_{\gamma,\beta} \tag{5.63}$$

holds for each step. For the first step we need that

$$\mathcal{R}^-_{\gamma,\beta} < \int_\Omega v_0^n d\nu + \frac{T}{n}\int_\Omega f_1^n d\nu < \mathcal{R}^+_{\gamma,\beta}$$

holds so that condition (5.63) is satisfied. Integrating (5.62) with respect to ν over Ω we get

$$\int_\Omega v_1^n d\nu = \int_\Omega v_0^n d\nu + \frac{T}{n}\int_\Omega f_1^n d\nu$$

thus

$$\frac{T}{n}\int_\Omega f_2^n d\nu + \int_\Omega v_1^n d\nu = \frac{T}{n}\sum_{j=1}^{2}\int_\Omega f_j^n d\nu + \int_\Omega v_0^n d\nu,$$

so that, for the second step, we need

$$\mathcal{R}^-_{\gamma,\beta} < \frac{T}{n}\sum_{j=1}^{2}\int_\Omega f_j^n d\nu + \int_\Omega v_0^n d\nu < \mathcal{R}^+_{\gamma,\beta}.$$

Therefore, we recursively obtain that, for each n and each step $i = 1, \ldots, n$, the following must be satisfied:

$$\mathcal{R}^-_{\gamma,\beta} < \frac{T}{n}\sum_{j=1}^{i}\int_\Omega f_j^n d\nu + \int_\Omega v_0^n d\nu < \mathcal{R}^+_{\gamma,\beta}.$$

However, taking n large enough, we have that this holds thanks to (5.58), (5.60) and (5.61).

Therefore,

$$v_n(t) := \begin{cases} v_0^n, & \text{if } t \in [t_0^n, t_1^n], \\ v_i^n, & \text{if } t \in]t_{i-1}^n, t_i^n],\ i = 2, \ldots, n, \end{cases}$$

is a T/n-approximate solution of Problem (5.57) as defined in nonlinear semigroup theory. Consequently, by nonlinear semigroup theory (see [**28**], [**24**, Theorem 4.1] or Theorem A.23) and on account of Theorem 5.20 and Theorem 5.21 we have that Problem (5.57) has a unique mild solution $v(t) \in C([0, T]; L^1(\Omega, \nu))$ with

$$v_n(t) \xrightarrow{n} v(t) \quad \text{in } L^1(\Omega, \nu) \text{ uniformly for } t \in [0, T]. \tag{5.64}$$

Uniqueness and the maximum principle for mild solutions is guaranteed by the T-accretivity of the operator.

Let's now see that $v(t)$ is a strong solution of Problem (5.57) when (5.59) holds. Note that, since $v_0 \in L^{p'}(\Omega,\nu)$, we may take $v_0^n = v_0$ for every $n \in \mathbb{N}$ in the previous computations and $f_i^n \in L^{p'}(\Omega,\nu)$, $i = 1,\ldots n$, additionally satisfying

$$\sum_{i=1}^{n} \int_{t_{i-1}^n}^{t_i^n} \|f(t) - f_i^n\|_{L^{p'}(\Omega,\nu)}^{p'} dt \leqslant \frac{T}{n},$$

thus, by Remark 5.16, we get that, in fact, $v \in L^{p'}(0,T,L^{p'}(\Omega,\nu))$. Indeed,

$$\int_0^T \|v\|_{L^{p'}(\Omega,\nu)} dt \leqslant K \left(1 + \int_0^T \|v_0\|_{L^{p'}(\Omega,\nu)} dt \right)$$

for some constant K.

Multiplying equation (5.62) by u_i^n and integrating over Ω with respect to ν we obtain

$$\begin{aligned} &\int_\Omega \frac{v_i^n(x) - v_{i-1}^n(x)}{T/n} u_i^n(x) d\nu(x) - \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_i^n(y) - u_i^n(x)) dm_x(y) u_i^n(x) d\nu(x) \\ &\qquad = \int_\Omega f_i^n(x) u_i^n(x) d\nu(x). \end{aligned} \tag{5.65}$$

Now, since $v_i^n(x) \in \gamma(u_i^n(x))$ for ν-a.e. $x \in \Omega_1$ and $v_i^n(x) \in \beta(u_i^n(x))$ for ν-a.e. $x \in \Omega_2$, we have that (see Remark 5.2)

$$\begin{cases} u_i^n(x) \in \gamma^{-1}(v_i^n(x)) = \partial j_\gamma^*(v_i^n(x)) & \text{for } \nu\text{-a.e. } x \in \Omega_1, \\ u_i^n(x) \in \beta^{-1}(v_i^n(x)) = \partial j_\beta^*(v_i^n(x)) & \text{for } \nu\text{-a.e. } x \in \Omega_2. \end{cases}$$

Consequently,

$$\begin{cases} j_\gamma^*(v_{i-1}^n(x)) - j_\gamma^*(v_i^n(x)) \geqslant (v_{i-1}^n(x) - v_i^n(x)) u_i^n(x) & \text{for } \nu\text{-a.e. } x \in \Omega_1, \\ j_\beta^*(v_{i-1}^n(x)) - j_\beta^*(v_i^n(x)) \geqslant (v_{i-1}^n(x) - v_i^n(x)) u_i^n(x) & \text{for } \nu\text{-a.e. } x \in \Omega_2. \end{cases}$$

Therefore, from (5.65) it follows that

$$\begin{aligned} &\frac{1}{T/n} \int_{\Omega_1} (j_\gamma^*(v_i^n(x)) - j_\gamma^*(v_{i-1}^n(x))) d\nu(x) + \frac{1}{T/n} \int_{\Omega_2} (j_\beta^*(v_i^n(x)) - j_\beta^*(v_{i-1}^n(x))) d\nu(x) \\ &\quad - \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_i^n(y) - u_i^n(x)) u_i^n(x) dm_x(y) d\nu(x) \\ &\leqslant \int_\Omega f_i^n(x) u_i^n(x) d\nu(x), \end{aligned}$$

$i = 1,\ldots,n$. Then, integrating this equation over $]t_{i-1}^n, t_i^n]$ and adding for $1 \leqslant i \leqslant n$ we get

$$\begin{aligned} &\int_{\Omega_1} (j_\gamma^*(v_n^n(x)) - j_\gamma^*(v_0(x))) d\nu(x) + \int_{\Omega_2} (j_\beta^*(v_n^n(x)) - j_\beta^*(v_0(x))) d\nu(x) \\ &\quad - \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_i^n(y) - u_i^n(x)) dm_x(y) u_i^n(x) d\nu(x) dt \\ &\leqslant \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \int_\Omega f_i^n(x) u_i^n(x) d\nu(x) dt, \end{aligned}$$

which, recalling the definitions of f_n, u_n and v_n, and integrating by parts, can be rewritten as

$$\begin{aligned}&\int_{\Omega_1}(j_\gamma^*(v_n^n(x))-j_\gamma^*(v_0(x)))d\nu(x)+\int_{\Omega_2}(j_\beta^*(v_n^n(x))-j_\beta^*(v_0(x)))d\nu(x)\\&+\frac{1}{2}\int_0^T\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x))(u_n(t)(y)-u_nt)(x))dm_x(y)d\nu(x)dt\\&\leqslant\int_0^T\int_\Omega f_n(t)(x)u_n(t)(x)d\nu(x)dt.\end{aligned}\tag{5.66}$$

This, together with (5.11) and the fact that j_γ^* and j_β^* are nonnegative, yields

$$\begin{aligned}&\frac{c_p}{2}\int_0^T\int_\Omega\int_\Omega|u_n(t)(y)-u_n(t)(x)|^p dm_x(y)d\nu(x)dt\\&\leqslant\frac{1}{2}\int_0^T\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x))(u_n(t)(y)-u_n(t)(x))dm_x(y)d\nu(x)dt\\&\leqslant\int_{\Omega_1}(j_\gamma^*(v_0(x)))d\nu(x)+\int_{\Omega_2}(j_\beta^*(v_0(x)))d\nu(x)+\int_0^T\int_\Omega f_n(t)(x)u_n(t)(x)d\nu(x)dt\\&\leqslant\int_{\Omega_1}(j_\gamma^*(v_0(x)))d\nu(x)+\int_{\Omega_2}(j_\beta^*(v_0(x)))d\nu(x)+\int_0^T\|f_n(t)\|_{L^{p'}(\Omega,\nu)}\|u_n(t)\|_{L^p(\Omega,\nu)}dt.\end{aligned}$$

Therefore, for any $\delta>0$, by (5.59) and Young's inequality, there exists $C(\delta)>0$ such that

$$\int_0^T\int_\Omega\int_\Omega|u_n(t)(y)-u_n(t)(x)|^p dm_x(y)d\nu(x)dt\leqslant C(\delta)+\delta\int_0^T\|u_n(t)\|_{L^p(\Omega,\nu)}^p dt.\tag{5.67}$$

Now, by (5.64), if $\mathcal{R}_{\gamma,\beta}^+=+\infty$, there exists $M>0$ and $n_0\in\mathbb{N}$ such that

$$\sup_{t\in[0,T]}\int_\Omega v_n^+(t)(x)d\nu(x)<M,\ \forall n\geqslant n_0,$$

and, if $\mathcal{R}_{\gamma,\beta}^+<+\infty$, there exist $M\in\mathbb{R}$, $h>0$ and $n_0\in\mathbb{N}$ such that

$$\sup_{t\in[0,T]}\int_\Omega v_n(t)(x)d\nu(x)<M<\mathcal{R}_{\gamma,\beta}^+,$$

and

$$\sup_{t\in[0,T]}\int_{\{x\in\Omega\,:\,v_n(t)(x)<-h\}}|v_n(t)(x)|d\nu(x)<\frac{\mathcal{R}_{\gamma,\beta}^+-M}{8},\ \forall n\geqslant n_0.$$

Consequently, Lemma 5.3 and Lemma 5.4 yield

$$\|u_n^+(t)\|_{L^p(\Omega,\nu)}\leqslant C_2\left(\left(\int_\Omega\int_\Omega|u_n^+(t)(y)-u_n^+(t)(x)|^p dm_x(y)d\nu(x)\right)^{\frac{1}{p}}+1\right)$$

for some constant $C_2>0$. Similarly, we may find $C_3>0$ such that

$$\|u_n^-(t)\|_{L^p(\Omega,\nu)}\leqslant C_3\left(\left(\int_\Omega\int_\Omega|u_n^-(t)(y)-u_n^-(t)(x)|^p dm_x(y)d\nu(x)\right)^{\frac{1}{p}}+1\right).$$

Consequently, by (5.67), choosing δ small enough, we deduce that $\{u_n\}_n$ is bounded in $L^p(0,T;L^p(\Omega,\nu))$. Therefore, there exists a subsequence, which we continue to denote by $\{u_n\}_n$, and $u\in L^p(0,T;L^p(\Omega,\nu))$ such that

$$u_n\overset{n}{\rightharpoonup}u\ \text{ weakly in } L^p(0,T;L^p(\Omega,\nu)).$$

Then, since γ and β are maximal monotone graphs, we conclude that $v(t)(x)\in\gamma(u(t)(x))$ for $\mathcal{L}^1\times\nu$-a.e. $(t,x)\in(0,T)\times\Omega_1$ and $v(t)(x)\in\beta(u(t)(x))$ for $\mathcal{L}^1\times\nu$-a.e. $(t,x)\in(0,T)\times\Omega_2$.

Note that, since, by (5.67),

$$\left\{\int_0^T \int_\Omega \int_\Omega |u_n(t)(y) - u_n(t)(x)|^p dm_x(y) d\nu(x) dt\right\}_n \quad \text{is bounded,}$$

then, by (5.10), we have that $\{[(t,x,y) \mapsto \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x))]\}_n$ is bounded in $L^{p'}(0,T;L^{p'}(\Omega\times\Omega,\nu\otimes m_x))$ so we may take a further subsequence, which we still denote in the same way, such that

$$[(t,x,y) \mapsto \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x))] \xrightarrow{n} \Phi, \quad \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega\times\Omega,\nu\otimes m_x)).$$

Note that, for any $\xi \in L^p(\Omega,\nu)$, by the integrations by parts formula we know that

$$\begin{aligned} &- \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x)) \xi(x) dm_x(y) d\nu(x) \\ &= \frac{1}{2} \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x))(\xi(y) - \xi(x)) dm_x(y) d\nu(x) \end{aligned}$$

for $t \in [0,T]$, thus taking limits as $n \to \infty$ we have

$$-\int_\Omega \int_\Omega \Phi(t,x,y)\xi(x) dm_x(y) d\nu(x) = \frac{1}{2}\int_\Omega \int_\Omega \Phi(t,x,y)(\xi(y) - \xi(x)) dm_x(y) d\nu(x). \tag{5.68}$$

Now, from (5.62) we have that

$$\frac{v_n(t)(x) - v_n(t - T/n)(x)}{T/n} - \int_\Omega \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x)) dm_x(y) = f_n(t)(x) \tag{5.69}$$

for $t \in [0,T]$ and $x \in \Omega$. Let $\Psi \in W_0^{1,1}(0,T;L^p(\Omega,\nu))$, then

$$\begin{aligned} &\int_0^T \frac{v_n(t)(x) - v_n(t - T/n)(x)}{T/n} \Psi(t)(x) dt \\ &= -\int_0^{T-T/n} v_n(t)(x) \frac{\Psi(t+T/n)(x) - \Psi(t)(x)}{T/n} dt + \int_{T-T/n}^T \frac{v_n \Psi(t)(x)}{T/n} dt - \int_0^{T/n} \frac{v_0 \Psi(t)(x)}{T/n} dt \end{aligned}$$

for $x \in \Omega$. Therefore, multiplying (5.69) by Ψ, integrating over $(0,T)\times\Omega$ with respect to $\mathcal{L}^1 \times \nu$ and taking limits we get

$$\begin{aligned} &-\int_0^T \int_\Omega v(t)(x) \frac{d}{dt}\Psi(t)(x) d\nu(x) dt - \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y) dm_x(y) \Psi(t)(x) d\nu(x) dt \\ &= \int_0^T \int_\Omega f(t)(x) \Psi(t)(x) d\nu(x) dt. \end{aligned} \tag{5.70}$$

Therefore, taking $\Psi(t)(x) = \psi(t)\xi(x)$, where $\psi \in W_0^{1,1}(0,T)$ and $\xi \in L^p(\Omega,\nu)$, we obtain that

$$\int_0^T v(t)(x)\psi'(t) dt = -\int_0^T \int_\Omega \Phi(t,x,y)\psi(t) dm_x(y) dt - \int_0^T f(t)(x)\psi(t) dt, \quad \text{for } \nu\text{-a.e. } x \in \Omega.$$

It follows that $v \in W^{1,1}(0,T;L^{p'}(\Omega,\nu))$ and

$$v'(t)(x) - \int_\Omega \Phi(t,x,y) dm_x(y) = f(t) \quad \text{for a.e. } t \in (0,T) \text{ and } \nu\text{-a.e. } x \in \Omega.$$

Hence, to conclude it remains to prove that

$$\int_\Omega \Phi(t,x,y) dm_x(y) = \int_\Omega \mathbf{a}_p(x,y,u(t)(y) - u(t)(x)) dm_x(y)$$

for $\mathcal{L}^1 \times \nu$-a.e. $(t,x) \in [0,T] \times \Omega$. To this aim we make use of the following claim that will be proved later on,

$$\begin{aligned}(5.71)\quad &\limsup_n \int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x))(u_n(t)(y)-u_n(t)(x)))dm_x(y)d\nu(x)dt \\ &\leqslant \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y)(u(t)(y)-u(t)(x)))dm_x(y)d\nu(x)dt.\end{aligned}$$

Now, let $\rho \in L^p(0,T;L^p(\Omega,\nu))$. By (5.9) we have

$$\begin{aligned}&\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,\rho(t)(y)-\rho(t)(x)) \\ &\qquad\qquad \times(u_n(t)(y)-\rho(t)(y)-(u_n(t)(x)-\rho(t)(x)))dm_x(y)d\nu(x)dt \\ &\leqslant \int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x)) \\ &\qquad\qquad \times(u_n(t)(y)-\rho(t)(y)-(u_n(t)(x)-\rho(t)(x)))dm_x(y)d\nu(x)dt\end{aligned}$$

thus, taking limits as $n \to \infty$ and using (5.71), we obtain

$$\begin{aligned}&\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,\rho(t)(y)-\rho(t)(x)) \\ &\qquad\qquad \times(u(t)(y)-\rho(t)(y)-(u(t)(x)-\rho(t)(x)))dm_x(y)d\nu(x)dt \\ &\leqslant \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y)(u(t)(y)-\rho(t)(y)-(u(t)(x)-\rho(t)(x)))dm_x(y)d\nu(x)dt\end{aligned}$$

which, integrating by parts and recalling (5.68) becomes

$$\begin{aligned}&\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,\rho(t)(y)-\rho(t)(x))dm_x(y)(u(t)(x)-\rho(t)(x))d\nu(x)dt \\ &\geqslant \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y)dm_x(y)(u(t)(x)-\rho(t)(x))d\nu(x)dt.\end{aligned}$$

To conclude, take $\rho = u \pm \lambda\xi$ for $\lambda > 0$ and $\xi \in L^p(0,T;L^p(\Omega,\nu))$ to get

$$\begin{aligned}&\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,(u\pm\lambda\xi)(t)(y)-(u\pm\lambda\xi)(t)(x))dm_x(y)\xi(t)(x)d\nu(x)dt \\ &\geqslant \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y)dm_x(y)\xi(t)(x)d\nu(x)dt\end{aligned}$$

which, letting $\lambda \to 0$ yields

$$\begin{aligned}&\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u(t)(y)-u(t)(x))dm_x(y)\xi(t)(x)d\nu(x)dt \\ &= \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y)dm_x(y)\xi(t)(x)d\nu(x)dt\end{aligned}$$

for any $\xi \in L^p(0,T;L^p(\Omega,\nu))$. Therefore,

$$\int_\Omega \mathbf{a}_p(x,y,u(t)(y)-u(t)(x))dm_x(y) = \int_\Omega \Phi(t,x,y)dm_x(y)$$

for $\mathcal{L}^1 \times \nu$-a.e. $(t,x) \in [0,T] \times \Omega$.

Let's prove claim (5.71). By (5.66) and Fatou's lemma, we have

(5.72)
$$\limsup_n \frac{1}{2}\int_0^T\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x))(u_n(t)(y)-u_n(t)(x))dm_x(y)d\nu(x)dt$$
$$\leqslant -\int_{\Omega_1}(j_\gamma^*(v(T)(x))-j_\gamma^*(v(0)(x)))d\nu(x)-\int_{\Omega_2}(j_\beta^*(v(T)(x))-j_\beta^*(v(0)(x)))d\nu(x)$$
$$+\int_0^T\int_\Omega f(t)(x)u(t)(x)d\nu(x)dt.$$

Moreover, by (5.70), we have that

(5.73)
$$\int_0^T v(t)(x)\frac{d}{dt}\Psi(t)(x)dt=\int_0^T F(t)(x)\Psi(t)(x)dt, \text{ for } \nu\text{-a.e. } x\in\Omega,$$

where F is given by

(5.74)
$$F(t)(x)=-\int_\Omega \phi(t,x,y)dm_x(y)-f(t)(x),\ x\in\Omega.$$

Let $\psi\in W_0^{1,1}(0,T)$, $\psi\geqslant 0$, $\tau>0$ and
$$\eta_\tau(t)(x)=\frac{1}{\tau}\int_t^{t+\tau}u(s)(x)\psi(s)ds,\quad t\in[0,T],\, x\in\Omega.$$

Then, for τ small enough we have that $\eta_\tau\in W_0^{1,1}(0,T;L^p(\Omega,\nu))$ so we may use it as a test function in (5.73) to obtain
$$\begin{aligned}\int_0^T F(t)(x)\eta_\tau(t)(x)dt &= \int_0^T v(t)(x)\frac{d}{dt}\eta_\tau(t)(x)\\ &= \int_0^T v(t)(x)\frac{u(t+\tau)(x)\psi(t+\tau)-u(t)(x)\psi(t)}{\tau}dt\\ &= \int_0^T \frac{v(t-\tau)(x)-v(t)(x)}{\tau}u(t)(x)\psi(t)dt.\end{aligned}$$

Now, since
$$\gamma^{-1}(r)=\partial j_{\gamma^{-1}}(r)=\partial\left(\int_0^r(\gamma^{-1})^0(s)ds\right)$$
and $u(t)\in\gamma^{-1}(v(t))$ in Ω_1 and $u(t)\in\beta^{-1}(v(t))$ in Ω_2, we have
$$(v(t-\tau)(x)-v(t)(x))u(t)(x)\leqslant\int_{v(t)(x)}^{v(t-\tau)(x)}(\gamma^{-1})^0(s)ds,\quad \text{for } \nu\text{-a.e. } x\in\Omega_1,$$
and
$$(v(t-\tau)(x)-v(t)(x))u(t)(x)\leqslant\int_{v(t)(x)}^{v(t-\tau)(x)}(\beta^{-1})^0(s)ds,\quad \text{for } \nu\text{-a.e. } x\in\Omega_2,$$
thus
$$\int_0^T\int_\Omega F(t)(x)\eta_\tau(t)(x)d\nu(x)dt$$
$$\leqslant\frac{1}{\tau}\int_0^T\int_{\Omega_1}\int_{v(t)(x)}^{v(t-\tau)(x)}(\gamma^{-1})^0(s)dsd\nu(x)\psi(t)dt+\frac{1}{\tau}\int_0^T\int_{\Omega_2}\int_{v(t)(x)}^{v(t-\tau)(x)}(\beta^{-1})^0(s)dsd\nu(x)\psi(t)dt$$
$$=\int_0^T\int_{\Omega_1}\int_0^{v(t)(x)}(\gamma^{-1})^0(s)dsd\nu(x)\frac{\psi(t+\tau)-\psi(t)}{\tau}dt$$
$$+\int_0^T\int_{\Omega_2}\int_0^{v(t)(x)}(\beta^{-1})^0(s)dsd\nu(x)\frac{\psi(t+\tau)-\psi(t)}{\tau}dt$$

which, letting $\tau \to 0^+$ yields

$$\begin{aligned}
&\int_0^T \int_\Omega F(t)(x)u(t)(x)\psi(t)d\nu(x)dt \\
&\leqslant \int_0^T \int_{\Omega_1} \int_0^{v(t)(x)} (\gamma^{-1})^0(s)dsd\nu(x)\psi'(t)dt + \int_0^T \int_{\Omega_2} \int_0^{v(t)(x)} (\beta^{-1})^0(s)dsd\nu(x)\psi'(t)dt \\
&= \int_0^T \int_{\Omega_1} j_{\gamma^{-1}}(v(t)(x))d\nu(x)\psi'(t)dt + \int_0^T \int_{\Omega_2} j_{\beta^{-1}}(v(t)(x))d\nu(x)\psi'(t)dt \\
&= \int_0^T \int_{\Omega_1} j_\gamma^*(v(t)(x))d\nu(x)\psi'(t)dt + \int_0^T \int_{\Omega_2} j_\beta^*(v(t)(x))d\nu(x)\psi'(t)dt.
\end{aligned}$$

Taking

$$\tilde{\eta}_\tau(t)(x) = \frac{1}{\tau}\int_t^{t+\tau} u(s-\tau)(x)\psi(s)ds$$

yields the opposite inequality so that, in fact,

$$\begin{aligned}
&\int_0^T \int_\Omega F(t)(x)u(t)(x)d\nu(x)\psi(t)dt \\
&= \int_0^T \int_{\Omega_1} j_\gamma^*(v(t)(x))d\nu(x)\psi'(t)dt + \int_0^T \int_{\Omega_2} j_\beta^*(v(t)(x))d\nu(x)\psi'(t)dt.
\end{aligned}$$

Then,

$$-\frac{d}{dt}\left(\int_{\Omega_1} j_\gamma^*(v(t)(x))d\nu(x) + \int_{\Omega_2} j_\beta^*(v(t)(x))d\nu(x)\right) = \int_\Omega F(t)(x)u(t)(x)d\nu(x) \tag{5.75}$$

in $\mathcal{D}'(]0,T[)$, thus, in particular,

$$\int_{\Omega_1} j_\gamma^*(v(t)(x))d\nu(x) + \int_{\Omega_2} j_\beta^*(v(t)(x))d\nu(x) \in W^{1,1}(0,T).$$

Therefore, integrating from 0 to T in (5.75) and recalling (5.74) we get

$$\begin{aligned}
&\int_0^T \int_\Omega \int_\Omega \Phi(t,x,y)u(t)(x)dm_x(y)d\nu(x)dt \\
&= -\int_{\Omega_1} (j_\gamma^*(v(T)(x)) - j_\gamma^*(v(0)(x)))d\nu(x) - \int_{\Omega_2} (j_\beta^*(v(T)(x)) - j_\beta^*(v(0)(x)))d\nu(x) \\
&\quad + \int_0^T \int_\Omega f(t)(x)u(t)(x)d\nu(x)dt
\end{aligned}$$

which, together with (5.72), yields the claim (5.71). □

Observe that we have imposed the compatibility condition (5.58) *because, for a strong solution, we have that*

$$\int_\Omega v_0 d\nu + \int_0^t \int_\Omega f(t)dtd\nu = \int_\Omega v(t)d\nu, \ \textit{for } t \in [0,T].$$

EXAMPLE 5.23. Let $W \subset X$ be a ν-measurable set such that W_m is m-connected. Given $f \in L^1(\partial_m W, \nu)$, we say that a function $u \in L^1(W_m, \nu)$ is an $\mathbf{a}_p$-lifting of f to $W_m = W \cup \partial_m W$ if

$$\begin{cases} -\mathrm{div}_m \mathbf{a}_p u(x) = 0, & x \in W, \\ u(x) = f(x), & x \in \partial_m W. \end{cases}$$

We define the Dirichlet-to-Neumann operator $\mathfrak{D}_{\mathbf{a}_p} \subset L^1(\partial_m W, \nu) \times L^1(\partial_m W, \nu)$ as follows: $(f,\psi) \in \mathfrak{D}_{\mathbf{a}_p}$ if

$$\mathcal{N}_1^{\mathbf{a}_p} u(x) = \psi(x), \quad x \in \partial_m W,$$

where u is an $\mathbf{a}_p$-lifting of f to W_m.

Then, rewriting the operator $\mathfrak{D}_{\mathbf{a}_p}$ as $\mathcal{B}^{m,\gamma,\beta}_{\mathbf{a}_p}$ for $\gamma(r) = 0$ and $\beta(r) = r$, $r \in \mathbb{R}$, ($\Omega_1 = W$ and $\Omega_2 = \partial_m W$), by the results in this subsection we have that $\mathfrak{D}_{\mathbf{a}_p}$ is T-accretive in $L^1(\partial_m W, \nu)$ (it is easy to see that, in fact, in this situation, it is completely accretive), it satisfies the range condition

$$L^{p'}(\partial_m W, \nu) \subset R(I + \mathfrak{D}_{\mathbf{a}_p}),$$

and it has dense domain. The non-homogeneous Cauchy evolution problem for this nonlocal Dirichlet-to-Neumann operator is a particular case of Problem (5.57):

$$\begin{cases} -\mathrm{div}_m \mathbf{a}_p(u)(x) = 0, & x \in W,\ 0 < t < T, \\ u_t(t,x) + \mathcal{N}_1^{\mathbf{a}_p} u(t,x) = g(t,x), & x \in \partial_m W,\ 0 < t < T, \\ w(0,x) = w_0(x), & x \in \partial_m W. \end{cases}$$

See, for example, [**9**], [**10**], [**98**], and the references therein, for local evolution problems with the p-Dirichlet-to-Neumann operator, see [**34**] for the nonlocal problem with convolution kernels.

5.3.2. Nonlinear boundary conditions. *In this subsection our aim is to study the following diffusion problem*

$$\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right) \quad \begin{cases} v_t(t,x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t,y)-u(t,x))dm_x(y) = f(t,x), & x \in \Omega_1,\ 0 < t < T, \\ v(t,x) \in \gamma(u(t,x)), & x \in \Omega_1,\ 0 < t < T, \\ \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t,y)-u(t,x))dm_x(y) \in \beta(u(t,x)), & x \in \Omega_2,\ 0 < t < T, \\ v(0,x) = v_0(x), & x \in \Omega_1, \end{cases}$$

that in particular covers Problem (5.1)*. See* [**33**] *for the reference local model.*

We will assume that

$$\Gamma^- = \inf\{\mathrm{Ran}(\gamma)\} < \Gamma^+ = \sup\{\mathrm{Ran}(\gamma)\}$$

since, otherwise, we do not have an evolution problem. Hence, $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$*. Moreover we will also assume that*

$$\mathfrak{B}^- = \inf\{\mathrm{Ran}(\beta)\} < \mathfrak{B}^+ = \sup\{\mathrm{Ran}(\beta)\},$$

since the case $\mathfrak{B}^- = \mathfrak{B}^+$ *(*$\beta = \mathbb{R} \times \{0\}$*) is treated with more generality in Subsection 5.3.1.*

We will again make use of nonlinear semigroup theory. To this end we introduce the corresponding operator associated with $\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right)$*, which is now defined in* $L^1(\Omega_1,\nu) \times L^1(\Omega_1,\nu)$*.*

Definition 5.24. We say that $(v,\hat{v}) \in B^{m,\gamma,\beta}_{\mathbf{a}_p}$ if $v, \hat{v} \in L^1(\Omega_1,\nu)$ and there exist $u \in L^p(\Omega,\nu)$ and $w \in L^1(\Omega_2,\nu)$ with

$$u \in \mathrm{Dom}(\gamma) \text{ and } v \in \gamma(u) \quad \nu\text{-a.e. in } \Omega_1,$$

and

$$u \in \mathrm{Dom}(\beta) \text{ and } w \in \beta(u) \quad \nu\text{-a.e. in } \Omega_2,$$

such that

$$(x,y) \mapsto a_p(x,y,u(y)-u(x)) \in L^{p'}(Q_1, \nu \otimes m_x)$$

and

$$\begin{cases} -\displaystyle\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \hat{v} & \text{in } \Omega_1, \\ w - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = 0 & \text{in } \Omega_2; \end{cases}$$

that is, $[u,(v,w)]$ is a solution of $(GP_{(v+\hat{v},\mathbf{0})})$, where $\mathbf{0}$ is the null function in Ω_2 (see (5.14) and Definition 5.12).

Set

$$\mathcal{R}^-_{\gamma,\lambda\beta} := \nu(\Omega_1)\varGamma^- + \lambda\nu(\Omega_2)\mathfrak{B}^-,$$
$$\mathcal{R}^+_{\gamma,\lambda\beta} := \nu(\Omega_1)\varGamma^+ + \lambda\nu(\Omega_2)\mathfrak{B}^+.$$

On account of the results given in Subsection 5.1.2 (Theorems 5.14 and 5.15) we have:

Theorem 5.25. *The operator $B^{m,\gamma,\beta}_{\mathbf{a}_p}$ is T-accretive in $L^1(\Omega_1,\nu)$ and satisfies the range condition*

$$\left\{\varphi\in L^{p'}(\Omega_1,\nu) : \mathcal{R}^-_{\gamma,\lambda\beta} < \int_{\Omega_1}\varphi d\nu < \mathcal{R}^+_{\gamma,\lambda\beta}\right\} \subset R(I+\lambda B^{m,\gamma,\beta}_{\mathbf{a}_p}) \quad \forall\lambda>0.$$

Remark 5.26. Observe that, if $\mathcal{R}^-_{\gamma,\beta} = -\infty$ and $\mathcal{R}^+_{\gamma,\beta} = +\infty$, then the closure of $B^{m,\gamma,\beta}_{\mathbf{a}_p}$ is m-T-accretive in $L^1(\Omega_1,\nu)$ (see Definitions A.19 and A.47).

With respect to the domain of this operator we prove the following result.

Theorem 5.27.

$$\overline{D(B^{m,\gamma,\beta}_{\mathbf{a}_p})}^{L^{p'}(\Omega_1,\nu)} = \{v\in L^{p'}(\Omega_1,\nu) : \varGamma^- \leqslant v \leqslant \varGamma^+\}.$$

Therefore, we also have

$$\overline{D(B^{m,\gamma,\beta}_{\mathbf{a}_p})}^{L^{1}(\Omega_1,\nu)} = \{v\in L^{1}(\Omega_1,\nu) : \varGamma^- \leqslant v \leqslant \varGamma^+\}.$$

Proof. It is obvious that

$$\overline{D(B^{m,\gamma,\beta}_{\mathbf{a}_p})}^{L^{p'}(\Omega_1,\nu)} \subset \left\{v\in L^{p'}(\Omega_1,\nu) : \varGamma^- \leqslant v \leqslant \varGamma^+ \text{ in } \Omega_1\right\}.$$

For the other inclusion it is enough to see that

$$\left\{v\in L^{\infty}(\Omega_1,\nu) : \varGamma^- \leqslant v \leqslant \varGamma^+ \text{ in } \Omega_1\right\} \subset \overline{D(B^{m,\gamma,\beta}_{\mathbf{a}_p})}^{L^{p'}(\Omega_1,\nu)}.$$

We work on a case-by-case basis.
(A) Suppose that $\varGamma^- < 0 < \varGamma^+$. It is enough to see that for any $v\in L^\infty(\Omega_1,\nu)$ such that there exist $m\in\mathbb{R}$, $\tilde{m}<0$, $\widetilde{M}>0$, $M\in\mathbb{R}$ satisfying

$$\varGamma^- < m < \tilde{m} < v < \widetilde{M} < M < \varGamma^+ \quad \text{in } \Omega_1$$

it holds that $v\in \overline{D(B^{m,\gamma,\beta}_{\mathbf{a}_p})}^{L^{p'}(\Omega_1,\nu)}$.

By the results in Subsection 5.1.6 we know that, for $n\in\mathbb{N}$, there exist $u_n\in L^p(\Omega,\nu)$, $v_n\in L^{p'}(\Omega_1,\nu)$ and $w_n\in L^{p'}(\Omega_2,\nu)$, such that $[u_n,(v_n,\frac{1}{n}w_n)]$ is a solution of $\left(GP^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta}_{(v,\mathbf{0})}\right)$, i.e., $v_n\in\gamma(u_n)$ ν-a.e. in Ω_1, $w_n\in\beta(u_n)$ ν-a.e. in Ω_2 and

$$\begin{cases} v_n(x) - \dfrac{1}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_n(y)-u_n(x))dm_x(y) = v(x), & \text{for } x\in\Omega_1,\\[2ex] w_n(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u_n(y)-u_n(x))dm_x(y) = 0, & \text{for } x\in\Omega_2.\end{cases}$$

In other words, $(v_n, n(v-v_n))\in B^{m,\gamma,\beta}_{\mathbf{a}_p}$ or, equivalently,

$$v_n := \left(I+\frac{1}{n}B^{m,\gamma,\beta}_{\mathbf{a}_p}\right)^{-1}(v)\in D(B^{m,\gamma,\beta}_{\mathbf{a}_p}).$$

Let us see that $v_n \xrightarrow{n} v$ in $L^{p'}(\Omega_1,\nu)$.

(A1) Suppose first that $\sup D(\beta) = +\infty$. Take $a_M > 0$ such that $M \in \gamma(a_M)$ and let $N \in \beta(a_M)$. Let

$$\widehat{v}(x) := \begin{cases} M, & x \in \Omega_1, \\ N, & x \in \Omega_2, \end{cases}$$

$$\widehat{u}(x) := a_M, \ x \in \Omega,$$

and

$$\varphi(x) := \begin{cases} M, & x \in \Omega_1, \\ 0, & x \in \Omega_2. \end{cases}$$

Then, $[\widehat{u}, \widehat{v}]$ is a supersolution of $\left(GP_{\varphi}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta} \right)$ and $(v, \mathbf{0}) \leqslant \varphi$ thus, by the maximum principle (Theorem 5.14),

$$u_n \leqslant \widehat{u} = a_M \text{ in } \Omega, \ \forall n \in \mathbb{N}.$$

(A2) Suppose now that $\sup D(\beta) = r_\beta < +\infty$. Again, by the results in Subsection 5.1.6 we know that, for $n \in \mathbb{N}$, there exist $\widetilde{u}_n \in L^p(\Omega, \nu)$, $\widetilde{v}_n \in L^{p'}(\Omega_1, \nu)$ and $\widetilde{w}_n \in L^{p'}(\Omega_2, \nu)$, such that $[\widetilde{u}_n, (\widetilde{v}_n, \frac{1}{n}\widetilde{w}_n)]$ is a solution of $\left(GP_{(M,\mathbf{0})}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta} \right)$. Therefore, by the maximum principle (Theorem 5.14),

$$v_n \leqslant \widetilde{v}_n \quad \text{in } \Omega_1.$$

Now, since $\widetilde{v}_n \ll M$ in Ω_1 (recall Remark 5.16*(iii)*), we have that $\widetilde{v}_n \leqslant M$ and, consequently, also $v_n \leqslant M$. Hence, since $M \leqslant \widetilde{M} < \Gamma^+$, we get that

$$u_n \leqslant \inf \left(\gamma^{-1}(\widetilde{M})\right) \text{ in } \Omega_1,$$

but we also have

$$u_n \leqslant r_\beta \text{ in } \Omega_2, \ \forall n \in \mathbb{N}.$$

(B) For $\Gamma^- < 0 = \Gamma^+$: let $\Gamma^- < m < \widetilde{m} < 0$, and $v \in L^\infty(\Omega_1, \nu)$ be such that

$$\widetilde{m} \leqslant v < 0.$$

As in the previous case, by the results in Subsection 5.1.6, we know that, for $n \in \mathbb{N}$, there exist $u_n \in L^p(\Omega, \nu)$, $v_n \in L^{p'}(\Omega_1, \nu)$ and $w_n \in L^{p'}(\Omega_2, \nu)$, such that $[u_n, (v_n, \frac{1}{n}w_n)]$ is a solution of $\left(GP_{(v,\mathbf{0})}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta} \right)$. Then, since for the null function $\mathbf{0}$ in Ω, we have that $[\mathbf{0}, \mathbf{0}]$ is a solution of $\left(GP_{\mathbf{0}}^{\frac{1}{n}\mathbf{a}_p,\gamma,\beta} \right)$ and $v < 0$, the maximum principle yields

$$u_n \leqslant 0 \text{ in } \Omega, \ \forall n \in \mathbb{N}.$$

Therefore, in all the cases, $\{u_n\}_n$ is $L^\infty(\Omega, \nu)$-bounded from above. With a similar reasoning we obtain that, in any of these cases, $\{u_n\}_n$ is also $L^\infty(\Omega, \nu)$-bounded from below. Then, since

$$v_n(x) - v(x) = \frac{1}{n} \int_\Omega \mathbf{a}_p(x, y, u_n(y) - u_n(x)) dm_x(y) \quad \text{in } \ \Omega_1,$$

we obtain that

$$v_n \xrightarrow{n} v \ \text{ in } L^{p'}(\Omega_1, \nu)$$

as desired. □

In the following theorem we prove the existence and uniqueness of solutions of Problem $\left(DP_{f,v_0}^{\mathbf{a}_p,\gamma,\beta} \right)$. *Recall that* $\Gamma^- < \Gamma^+$ *and* $\mathfrak{B}^- < \mathfrak{B}^+$.

Theorem 5.28. *Let $T > 0$. Let $v_0 \in L^1(\Omega_1, \nu)$ and $f \in L^1(0,T;L^1(\Omega_1,\nu))$. Assume*

$$\Gamma^- \leqslant v_0 \leqslant \Gamma^+ \text{ in } \Omega_1,$$

and

$$\text{either } \mathcal{R}^+_{\gamma,\beta} = +\infty \quad \text{or} \quad \int_{\Omega_1} f(x,t)d\nu(x) \leqslant \nu(\Omega_2)\mathfrak{B}^+ \quad \forall 0 < t < T,$$

and

$$\text{either } \mathcal{R}^-_{\gamma,\beta} = -\infty \quad \text{or} \quad \int_{\Omega_1} f(x,t)d\nu(x) \geqslant \nu(\Omega_2)\mathfrak{B}^- \quad \forall 0 < t < T.$$

Then, there exists a unique mild solution $v \in C([0,T] : L^1(\Omega_1,\nu))$ of $\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right)$.

Let v and $\widetilde{v}$ be the mild solutions of the problem with respective data v_0, $\widetilde{v}_0 \in L^1(\Omega_1,\nu)$ and f, $\widetilde{f} \in L^1(0,T;L^1(\Omega_1,\nu))$, we have

$$\begin{aligned}\int_{\Omega_1} (v(t,x) - \widetilde{v}(t,x))^+ \, d\nu(x) \quad &\leqslant \int_{\Omega_1} (v_0(x) - \widetilde{v}_0(x))^+ \, d\nu(x) \\ &\quad + \int_0^t \int_{\Omega_1} \left(f(s,x) - \widetilde{f}(s,x)\right)^+ d\nu(x)ds, \quad \forall 0 \leqslant t \leqslant T.\end{aligned}$$

Under the additional assumptions

$$v_0 \in L^{p'}(\Omega_1,\nu) \text{ and } f \in L^{p'}(0,T;L^{p'}(\Omega_1,\nu)) \text{ with}$$

(5.76)
$$\begin{gathered}\int_{\Omega_1} j^*_\gamma(v_0)d\nu < +\infty \text{ and} \\ \int_{\Omega_1} v_0^+ d\nu + \int_0^T \int_{\Omega_1} f(s)^+ d\nu dt < \nu(\Omega_1)\Gamma^+, \\ \int_{\Omega_1} v_0^- d\nu + \int_0^T \int_{\Omega_1} f(s)^- d\nu dt < -\nu(\Omega_1)\Gamma^-,\end{gathered}$$

the mild solution v belongs to $W^{1,1}(0,T;L^{p'}(\Omega_1,\nu))$ and satisfies the equation

$$\begin{cases} \partial_t v(t) + B^{m,\gamma,\beta}_{\mathbf{a}_p} v(t) \ni f(t) & \text{for a.e. } t \in (0,T), \\ v(0) = v_0, \end{cases}$$

that is, v is a strong solution.

The proof of this result differs, strongly at some points, from the proof of Theorem 5.22.

Proof. We start by proving the existence of mild solutions. Let $n \in \mathbb{N}$. Consider the partition $t_0^n = 0 < t_1^n < \cdots < t_{n-1}^n < t_n^n = T$ where $t_i^n := iT/n$, $i = 0, \ldots, n$.

Now, since $\mathfrak{B}^- < \mathfrak{B}^+$, thanks to the assumptions in the theorem we can take $v_0^n \in L^{p'}(\Omega_1,\nu)$ and $f_i^n \in L^{p'}(\Omega_1,\nu)$, $i = 1, \ldots n$, such that

$$\|v_0 - v_0^n\|_{L^1(\Omega_1,\nu)} \leqslant \frac{T}{n}$$

(5.77)
$$\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \|f(t) - f_i^n\|_{L^1(\Omega_1,\nu)} dt \leqslant \frac{T}{n}$$

and

$$\nu(\Omega_2)\mathfrak{B}^- < \int_{\Omega_1} f_i^n d\nu < \nu(\Omega_2)\mathfrak{B}^+.$$

Then, setting

$$f_n(t) := f_i^n, \quad \text{for } t \in]t_{i-1}^n, t_i^n], \; i = 1, \ldots, n,$$

we have that
$$\int_0^T \|f(t)-f_n(t)\|_{L^1(\Omega_1,\nu)}dt \leqslant \frac{T}{n}.$$

Using the results in Subsection 5.1.6, we will see that, for n large enough, we may recursively find a solution $[u_i^n,(v_i^n,\frac{T}{n}w_i^n)]$ of $\left(GP^{\frac{T}{n}\mathbf{a}_p,\gamma,\frac{T}{n}\beta}_{(\frac{T}{n}f_i^n+v_{i-1}^n,\mathbf{0})}\right)$, $i=1,\ldots,n$, so that

$$\text{(5.78)}\qquad \begin{cases} v_i^n(x)-\dfrac{T}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_i^n(y)-u_i^n(x))dm_x(y)=\dfrac{T}{n}f_i^n(x)+v_{i-1}^n(x), & x\in\Omega_1\\ w_i^n(x)-\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_i^n(y)-u_i^n(x))dm_x(y)=0, & x\in\Omega_2, \end{cases}$$

or, equivalently,

$$\text{(5.79)}\qquad \begin{cases} \dfrac{v_i^n(x)-v_{i-1}^n(x)}{T/n}-\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_i^n(y)-u_i^n(x))dm_x(y)=f_i^n(x), & x\in\Omega_1\\ w_i^n(x)-\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_i^n(y)-u_i^n(x))dm_x(y)=0, & x\in\Omega_2, \end{cases}$$

with $v_i^n(x)\in\gamma(u_i^n(x))$ for ν-a.e. $x\in\Omega_1$ and $w_i^n(x)\in\beta(u_i^n(x))$ for ν-a.e. $x\in\Omega_2$, $i=1,\ldots,n$. That is, we may find the unique solution v_i^n of the time discretization scheme associated with $\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right)$.

To apply these results we must ensure that
$$\mathcal{R}^-_{\gamma,\frac{T}{n}\beta}<\int_{\Omega_1}\left(\frac{T}{n}f_i^n+v_{i-1}^n\right)d\nu<\mathcal{R}^+_{\gamma,\frac{T}{n}\beta}$$
holds for each step, but this holds true thanks to the choice of the f_i^n, $i=1,\ldots,n$.

Therefore, we have that
$$v_n(t):=\begin{cases} v_0^n, & \text{if } t\in[t_0^n,t_1^n],\\ v_i^n, & \text{if } t\in]t_{i-1}^n,t_i^n],\ i=2,\ldots,n, \end{cases}$$
is a $\frac{T}{n}$-approximate solution of Problem $\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right)$. Consequently, by nonlinear semigroup theory (see [**28**], [**24**, Theorem 4.1] or Theorem A.23) and on account of Theorem 5.25 and Theorem 5.27 we have that $\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right)$ has a unique mild solution $v(t)\in C([0,T];L^1(\Omega_1,\nu))$ with

$$\text{(5.80)}\qquad v_n(t)\xrightarrow{n} v(t)\ \text{ in } L^1(\Omega_1,\nu) \text{ uniformly for } t\in[0,T].$$

Uniqueness and the maximum principle for mild solutions is guaranteed by the T-accretivity of the operator.

We now prove, step by step, that these mild solutions are strong solutions of Problem $\left(DP^{\mathbf{a}_p,\gamma,\beta}_{f,v_0}\right)$ under the set of assumptions given in (5.76)

Step 1. Suppose first that $\mathcal{R}^-_{\gamma,\beta}=-\infty$ and $\mathcal{R}^+_{\gamma,\beta}=+\infty$.

In the construction of the mild solution, we now take $v_0^n=v_0$ (since $v_0\in L^{p'}(\Omega_1,\nu)$) and the functions $f_i^n\in L^{p'}(\Omega_1,\nu)$, $i=1,\ldots n$, additionally satisfying
$$\sum_{i=1}^n\int_{t_{i-1}^n}^{t_i^n}\|f(t)-f_i^n\|^{p'}_{L^{p'}(\Omega_1,\nu)}dt\leqslant\frac{T}{n}$$
and
$$\nu(\Omega_2)\mathfrak{B}^-<\int_{\Omega_1}f_i^n d\nu<\nu(\Omega_2)\mathfrak{B}^+.$$

Multiplying both equations in (5.79) by u_i^n, integrating with respect to ν the first one over Ω_1 and the second one over Ω_2, and adding them, we obtain

$$\begin{aligned}&\int_{\Omega_1} \frac{v_i^n(x) - v_{i-1}^n(x)}{T/n} u_i^n(x) d\nu(x) + \int_{\Omega_2} w_i^n(x) u_i^n(x) d\nu(x)\\ &\quad - \int_\Omega \int_\Omega \mathbf{a}_p(x, y, u_i^n(y) - u_i^n(x)) u_i^n(x) dm_x(y) d\nu(x)\\ &= \int_{\Omega_1} f_i^n(x) u_i^n(x) d\nu(x).\end{aligned}$$

Then, since $w_i^n(x) \in \beta(u_i^n(x))$ for ν-a.e. $x \in \Omega_2$ the second term on the left hand side is nonnegative and integrating by parts the third term we get

$$\begin{aligned}&\int_{\Omega_1} \frac{v_i^n(x) - v_{i-1}^n(x)}{T/n} u_i^n(x) d\nu(x)\\ &\quad + \frac{1}{2} \int_\Omega \int_\Omega \mathbf{a}_p(x, y, u_i^n(y) - u_i^n(x))(u_i^n(y) - u_i^n(x)) dm_x(y) d\nu(x)\\ &\leqslant \int_{\Omega_1} f_i^n(x) u_i^n(x) d\nu(x).\end{aligned} \tag{5.81}$$

Now, since $v_i^n(x) \in \gamma(u_i^n(x))$ for ν-a.e. $x \in \Omega_1$, we have that

$$u_i^n(x) \in \gamma^{-1}(v_i^n(x)) = \partial j_\gamma^*(v_i^n(x)) \quad \text{for } \nu\text{-a.e. } x \in \Omega_1.$$

Consequently,

$$j_\gamma^*(v_{i-1}^n(x)) - j_\gamma^*(v_i^n(x)) \geqslant (v_{i-1}^n(x) - v_i^n(x)) u_i^n(x) \quad \text{for } \nu\text{-a.e. } x \in \Omega_1.$$

Therefore, from (5.81) it follows that

$$\begin{aligned}&\frac{n}{T} \int_{\Omega_1} (j_\gamma^*(v_i^n(x)) - j_\gamma^*(v_{i-1}^n(x))) d\nu(x)\\ &\quad + \frac{1}{2} \int_\Omega \int_\Omega \mathbf{a}_p(x, y, u_i^n(y) - u_i^n(x))(u_i^n(y) - u_i^n(x)) dm_x(y) d\nu(x)\\ &\leqslant \int_{\Omega_1} f_i^n(x) u_i^n(x) d\nu(x),\end{aligned}$$

$i = 1, \ldots, n$. Then, integrating this equation over $]t_{i-1}, t_i]$ and adding for $1 \leqslant i \leqslant n$ we get

$$\begin{aligned}&\int_{\Omega_1} (j_\gamma^*(v_n^n(x)) - j_\gamma^*(v_0(x))) d\nu(x)\\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \int_\Omega \mathbf{a}_p(x, y, u_i^n(y) - u_i^n(x))(u_i^n(y) - u_i^n(x)) dm_x(y) d\nu(x) dt\\ &\leqslant \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{\Omega_1} f_i^n(x) u_i^n(x) d\nu(x) dt,\end{aligned}$$

which, recalling the definitions of f_n, u_n and v_n, can be rewritten as

$$\begin{aligned}&\int_{\Omega_1} (j_\gamma^*(v_n^n(x)) - j_\gamma^*(v_0(x))) d\nu(x)\\ &\quad + \frac{1}{2} \int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x, y, u_n(t)(y) - u_n(t)(x))(u_n(t)(y) - u_n(t)(x)) dm_x(y) d\nu(x) dt\\ &\leqslant \int_0^T \int_{\Omega_1} f_n(t)(x) u_n(t)(x) d\nu(x) dt.\end{aligned} \tag{5.82}$$

This, together with (5.11) and the fact that j_γ^* is nonnegative, yields

$$\begin{aligned}
&\frac{c_p}{2}\int_0^T\int_\Omega\int_\Omega |u_n(t)(y)-u_n(t)(x)|^p dm_x(y)d\nu(x)dt\\
&\leqslant \frac{1}{2}\int_0^T\int_\Omega\int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x))(u_n(t)(y)-u_n(t)(x))dm_x(y)d\nu(x)dt\\
&\leqslant \int_{\Omega_1} j_\gamma^*(v_0(x))d\nu(x)+\int_0^T\int_{\Omega_1} f_n(t)(x)u_n(t)(x)d\nu(x)dt\\
&\leqslant \int_{\Omega_1} j_\gamma^*(v_0(x))d\nu(x)+\int_0^T \|f_n(t)\|_{L^{p'}(\Omega_1,\nu)}\|u_n(t)\|_{L^p(\Omega_1,\nu)}dt.
\end{aligned}$$

Therefore, for any $\delta>0$, by (5.76) and Young's inequality, there exists $C(\delta)>0$ such that, in particular,

$$\int_0^T\int_\Omega\int_\Omega |u_n(t)(y)-u_n(t)(x)|^p dm_x(y)d\nu(x)dt \leqslant C(\delta)+\delta\int_0^T \|u_n(t)\|^p_{L^p(\Omega_1,\nu)}dt. \tag{5.83}$$

Observe also that, for any $n\in\mathbb{N}$ and $i\in\{1,\ldots,n\}$, and for $t\in]t_{i-1}^n,t_i^n]$ if $i\geqslant 2$, or $t\in[t_0^n,t_1^n]$ if $i=1$,

$$\int_{\Omega_1} v_n^+(t)d\nu+\int_0^{t_i^n}\int_{\Omega_2} w_n^+(s)d\nu ds \leqslant \int_{\Omega_1} v_0^+ d\nu+\int_0^{t_i^n}\int_{\Omega_1} f_n^+(s)d\nu ds. \tag{5.84}$$

Indeed, multiplying the first equation in (5.78) by $\frac{1}{r}T_r^+(u_i^n)$ and integrating with respect to ν over Ω_1, then multiplying the second by $\frac{T}{n}\frac{1}{r}T_r^+(u_i^n)$ and integrating with respect to ν over Ω_2, adding both equations, removing the nonnegative term involving $\mathbf{a}_p$ (recall Remark 5.13) and letting $r\downarrow 0$, we get that

$$\int_{\Omega_1}(v_i^n)^+d\nu+\frac{T}{n}\int_{\Omega_2}(w_i^n)^+d\nu\leqslant \int_{\Omega_1}(v_{i-1}^n)^+d\nu+\frac{T}{n}\int_{\Omega_1}(f_i^n)^+d\nu,$$

i.e.,

$$\int_{\Omega_1}(v_i^n)^+d\nu\leqslant \int_{\Omega_1}(v_{i-1}^n)^+d\nu+\frac{T}{n}\int_{\Omega_1}(f_i^n)^+d\nu-\frac{T}{n}\int_{\Omega_2}(w_i^n)^+d\nu.$$

Therefore,

$$\int_{\Omega_1}(v_i^n)^+d\nu\leqslant \int_{\Omega_1}(v_0^n)^+d\nu+\sum_{j=1}^i\frac{T}{n}\int_{\Omega_1}(f_j^n)^+d\nu-\sum_{j=1}^i\frac{T}{n}\int_{\Omega_2}(w_j^n)^+d\nu$$

which is equivalent to (5.84).

Now, by (5.80), if $\Gamma^+=+\infty$, there exists $M>0$ such that

$$\sup_{t\in[0,T]}\int_{\Omega_1} v_n^+(t)(x)d\nu(x)<M \quad \text{for every } n\in\mathbb{N}.$$

Consequently, Lemma 5.3 applied for $A=\Omega_1$, $B=\emptyset$ and $\alpha=\gamma$, yields

$$\|u_n^+(t)\|_{L^p(\Omega_1,\nu)}\leqslant K_2\left(\left(\int_{\Omega_1}\int_{\Omega_1}|u_n^+(t)(y)-u_n^+(t)(x)|^p dm_x(y)d\nu(x)\right)^{\frac{1}{p}}+1\right)$$

for every $n\in\mathbb{N}$ and $0\leqslant t\leqslant T$, and for some constant $K_2>0$.

Suppose now that $\Gamma^+<+\infty$. Then, by (5.84) we have that, for any $n\in\mathbb{N}$ and $i\in\{1,\ldots,n\}$, and for $t\in]t_{i-1}^n,t_i^n]$ if $i\geqslant 2$, or $t\in[t_0^n,t_1^n]$ if $i=1$,

$$\int_{\Omega_1} v_n^+(t)d\nu\leqslant\int_{\Omega_1} v_0^+d\nu+\int_0^{t_i^n}\int_{\Omega_1} f_n^+(s)d\nu ds$$

thus, by the assumptions in (5.76) and by (5.77), we have that there exists $M \in \mathbb{R}$ such that

$$\sup_{t\in[0,T]} \int_{\Omega_1} v_n(t)d\nu \leqslant M < \nu(\Omega_1)\Gamma^+$$

for n sufficiently large and, by (5.80), such that

$$\sup_{t\in[0,T]} \int_{\{x\in\Omega_1 : v_n(t)<-h\}} |v_n(t)|d\nu < \frac{\nu(\Omega_1)\Gamma^+ - M}{8}$$

for n sufficiently large. Therefore, we may apply Lemma 5.4 for $A = \Omega_1$, $B = \emptyset$ and $\alpha = \gamma$ to conclude that there exists a constant $K_2' > 0$ such that

$$\|u_n^+(t)\|_{L^p(\Omega_1,\nu)} \leqslant K_2' \left(\left(\int_{\Omega_1} \int_{\Omega_1} |u_n^+(t)(y) - u_n^+(t)(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + 1 \right), \quad \forall 0 \leqslant t \leqslant T,$$

for n sufficiently large.

Similarly, we may find $K_3 > 0$ such that

$$\|u_n^-(t)\|_{L^p(\Omega_1,\nu)} \leqslant K_3 \left(\left(\int_{\Omega_1} \int_{\Omega_1} |u_n^-(t)(y) - u_n^-(t)(x)|^p dm_x(y) d\nu(x) \right)^{\frac{1}{p}} + 1 \right), \quad \forall 0 \leqslant t \leqslant T,$$

for n sufficiently large.

Consequently, by the generalised Poincaré type inequality together with (5.83) for δ small enough, we get

$$\int_0^T \|u_n(t)\|_{L^p(\Omega,\nu)} \, dt \leqslant K_4, \quad \forall n \in \mathbb{N},$$

for some constant $K_4 > 0$, that is, $\{u_n\}_n$ is bounded in $L^p(0,T;L^p(\Omega,\nu))$. Therefore, there exists a subsequence, which we continue to denote by $\{u_n\}_n$, and $u \in L^p(0,T;L^p(\Omega,\nu))$ such that

$$u_n \overset{n}{\rightharpoonup} u \quad \text{weakly in } L^p(0,T;L^p(\Omega,\nu)).$$

Note that, since $\left\{ \int_0^T \int_\Omega \int_\Omega |u_n(t)(y) - u_n(t)(x)|^p dm_x(y) d\nu(x) dt \right\}_n$ is bounded, then, by (5.10), we have that $\{[(t,x,y) \mapsto \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x))]\}_n$ is bounded in $L^{p'}(0,T;L^{p'}(\Omega\times\Omega,\nu\otimes m_x))$ so we may take a further subsequence, which we continue to denote in the same way, such that

$$[(t,x,y) \mapsto \mathbf{a}_p(x,y,u_n(t)(y) - u_n(t)(x))] \overset{n}{\rightharpoonup} \Phi, \quad \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega\times\Omega,\nu\otimes m_x)).$$

Now, let $\Psi \in W_0^{1,1}(0,T;L^p(\Omega,\nu))$, then

$$\int_0^T \frac{v_n(t)(x) - v_n(t-T/n)(x)}{T/n} \Psi(t)(x) dt$$

$$= -\int_0^{T-T/n} v_n(t)(x) \frac{\Psi(t+T/n)(x) - \Psi(t)(x)}{T/n} dt + \int_{T-T/n}^T \frac{v_n \Psi(t)(x)}{T/n} dt - \int_0^{T/n} \frac{z_0 \Psi(t)(x)}{T/n}$$

for $x \in \Omega_1$. Therefore, multiplying both equations in (5.79) by Ψ, integrating the first one over Ω_1 and the second one over Ω_2 with respect to ν, adding them, and taking limits as $n \to +\infty$ we get that

$$\begin{aligned} &-\int_0^T \int_{\Omega_1} v(t)(x) \frac{d}{dt} \Psi(t)(x) d\nu(x) dt + \int_0^T \int_{\Omega_2} w(t)(x) \Psi(t)(x) d\nu(x) dt \\ &- \int_0^T \int_\Omega \int_\Omega \Phi(t,x,y) dm_x(y) \Psi(t)(x) d\nu(x) dt \\ &= \int_0^T \int_{\Omega_1} f(t)(x) \Psi(t)(x) d\nu(x) dt. \end{aligned}$$

Therefore, taking $\Psi(t)(x) = \psi(t)\xi(x)$, where $\psi \in W_0^{1,1}(0,T)$ and $\xi \in L^p(\Omega,\nu)$, we obtain that

$$\int_0^T v(t)(x)\psi'(t)dt = -\int_0^T \int_\Omega \Phi(t,x,y)\psi(t)dm_x(y)dt - \int_0^T f(t)(x)\psi(t)dt$$

for ν-a.e. $x \in \Omega_1$.

It follows that $v \in W^{1,1}(0,T;L^1(\Omega_1,\nu))$. Then, by Remark 5.26, we conclude that the mild solution v is, in fact, a strong solution (see [**28**] or Corollary A.29). Hence we have that (5.85)

$$v'(t)(x) - \int_\Omega \mathbf{a}_p(x,y,u(t)(y)-u(t)(x))dm_x(y) = f(t)(x) \quad \text{for a.e. } t \in [0,T] \text{ and } \nu\text{-a.e. } x \in \Omega_1.$$

Let's see, for further use, that $\displaystyle\int_{\Omega_1} j_\gamma^*(v(t))d\nu \in W^{1,1}(0,T)$. By (5.82) and Fatou's lemma, we have

$$\limsup_n \frac{1}{2}\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_n(t)(y)-u_n(t)(x))(u_n(t)(y)-u_n(t)(x))dm_x(y)d\nu(x)dt$$
$$\leqslant -\int_{\Omega_1} (j_\gamma^*(v(T)(x)) - j_\gamma^*(v(0)(x)))d\nu(x) + \int_0^T \int_{\Omega_1} f(t)(x)u(t)(x)d\nu(x)dt.$$

Moreover, by (5.85), we have that

$$\int_0^T v(t)(x)\frac{d}{dt}\Psi(t)(x)dt = \int_0^T F(t)(x)\Psi(t)(x)dt, \tag{5.86}$$

where F is given by

$$F(t)(x) = -\int_\Omega \mathbf{a}_p(x,y,u(t)(y)-u(t)(x))dm_x(y) - f(t)(x),\ x \in \Omega_1.$$

Let $\psi \in W_0^{1,1}(0,T)$, $\psi \geqslant 0$, $\tau > 0$ and

$$\eta_\tau(t)(x) = \frac{1}{\tau}\int_t^{t+\tau} u(s)(x)\psi(s)ds, \quad t \in [0,T],\ x \in \Omega_1.$$

Then, for τ small enough we have that $\eta_\tau \in W_0^{1,1}(0,T;L^p(\Omega_1,\nu))$ so we may use it as a test function in (5.86) to obtain

$$\begin{aligned}
\int_0^T \int_{\Omega_1} F(t)(x)\eta_\tau(t)(x)d\nu(x)dt &= \int_0^T \int_{\Omega_1} v(t)(x)\frac{d}{dt}\eta_\tau(t)(x)d\nu(x)dt \\
&= \int_0^T \int_{\Omega_1} v(t)(x)\frac{u(t+\tau)(x)\psi(t+\tau) - u(t)(x)\psi(t)}{\tau}d\nu(x)dt \\
&= \int_0^T \int_{\Omega_1} \frac{v(t-\tau)(x) - v(t)(x)}{\tau}u(t)(x)\psi(t)d\nu(x)dt.
\end{aligned}$$

Now, since

$$\gamma^{-1}(r) = \partial j_{\gamma^{-1}}(r) = \partial\left(\int_0^r (\gamma^{-1})^0(s)ds\right)$$

and $u(t) \in \gamma^{-1}(v(t))$ ν-a.e. in Ω_1, we have

$$(v(t-\tau)(x) - v(t)(x))u(t)(x) \leqslant \int_{v(t)(x)}^{v(t-\tau)(x)} (\gamma^{-1})^0(s)ds, \quad \text{for } \nu\text{-a.e. } x \in \Omega_1,$$

and thus, for ν-a.e. $x\in\Omega_1$ we have

$$\begin{aligned}\int_0^T\int_{\Omega_1} F(t)(x)\eta_\tau(t)(x)d\nu(x)dt &\leqslant \frac{1}{\tau}\int_0^T\int_{\Omega_1}\int_{v(t)(x)}^{v(t-\tau)(x)}(\gamma^{-1})^0(s)ds\psi(t)d\nu(x)dt\\ &=\int_0^T\int_{\Omega_1}\int_0^{v(t)(x)}(\gamma^{-1})^0(s)ds\frac{\psi(t+\tau)-\psi(t)}{\tau}d\nu(x)dt,\end{aligned}$$

which, letting $\tau\to 0^+$ yields

$$\begin{aligned}\int_0^T\int_{\Omega_1} F(t)u(t)(x)\psi(t)d\nu(x)dt &\leqslant \int_0^T\int_{\Omega_1}\int_0^{v(t)(x)}(\gamma^{-1})^0(s)ds\Psi'(t)d\nu(x)dt\\ &=\int_0^T\int_{\Omega_1} j_{\gamma^{-1}}(v(t)(x))\psi'(t)d\nu(x)dt\\ &=\int_0^T\int_{\Omega_1} j_\gamma^*(v(t)(x))\psi'(t)d\nu(x)dt.\end{aligned}$$

Taking

$$\widetilde{\eta}_\tau(t)(x)=\frac{1}{\tau}\int_t^{t+\tau}u(s-\tau)\Psi(s)ds,\quad t\in[0,T],\, x\in\Omega_1,$$

yields the opposite inequalities so that, in fact,

$$\int_0^T\int_{\Omega_1} F(t)(x)u(t)(x)d\nu(x)\psi(t)dt=\int_0^T\int_{\Omega_1} j_\gamma^*(v(t)(x))d\nu(x)\psi'(t)dt,$$

i.e.,

$$-\frac{d}{dt}\int_{\Omega_1} j_\gamma^*(v(t)(x))d\nu(x)=\int_{\Omega_1}F(t)(x)u(t)(x)d\nu(x)\quad\text{in } \mathcal{D}'(]0,T[),$$

thus, in particular,

$$\int_{\Omega_1} j_\gamma^*(v)d\nu\in W^{1,1}(0,T). \tag{5.87}$$

Step 2. Suppose now that, either $\mathcal{R}^-_{\gamma,\beta}=-\infty$ and $\mathcal{R}^+_{\gamma,\beta}<+\infty$, or $\mathcal{R}^-_{\gamma,\beta}>-\infty$ and $\mathcal{R}^+_{\gamma,\beta}=+\infty$. Recall that we are assuming the hypotheses in (5.76) and that $v_0^n=v_0$ for every $n\in\mathbb{N}$. Suppose first that $\mathcal{R}^-_{\gamma,\beta}=-\infty$ and $\mathcal{R}^+_{\gamma,\beta}<+\infty$. Then, for $k\in\mathbb{N}$, let $\beta^k:\mathbb{R}\to\mathbb{R}$ be the following maximal monotone graph

$$\beta^k(r):=\begin{cases}\beta(r) & \text{if } r<k,\\ [\beta^0(k),\mathfrak{B}^+] & \text{if } r=k,\\ \mathfrak{B}^++r-k & \text{if } r>k.\end{cases}$$

We have that $\beta^k\to\beta$ in the sense of maximal monotone graphs. Indeed, given $\lambda>0$ and $s\in\mathbb{R}$ there exists $r\in\mathbb{R}$ such that $s\in r+\lambda\beta(r)$ thus, for $k>r$, we have that $s\in r+\lambda\beta(r)=r+\lambda\beta^k(r)$, i.e., $r=(I+\lambda\beta)^{-1}(s)=(I+\lambda\beta^k)^{-1}(s)$.

By Step 1 we know that, since $\mathcal{R}^-_{\gamma,\beta^k}=-\infty$ and $\mathcal{R}^+_{\gamma,\beta^k}=+\infty$, there exists a strong solution $v_k\in W^{1,1}(0,T;L^1(\Omega_1,\nu))$ of Problem $\left(DP^{\mathbf{a}_p,\gamma,\beta^k}_{f-\frac{1}{k},v_0}\right)$, i.e., there exist $u_k\in L^p(0,T;L^p(\Omega,\nu))$ and $w_k\in L^{p'}(0,T;L^{p'}(\Omega_2,\nu))$ such that

(5.88)
$$\begin{cases}(v_k)_t(t)(x)-\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_k(t)(y)-u_k(t)(x))dm_x(y)=f(t)(x)-\frac{1}{k}, & x\in\Omega_1,\ 0<t<T,\\ w_k(t)(x)-\displaystyle\int_\Omega \mathbf{a}_p(x,y,u_k(t)(y)-u_k(t)(x))dm_x(y)=0, & x\in\Omega_2,\ 0<t<T,\end{cases}$$

with $v_k \in \gamma(u_k)$ ν-a.e. in Ω_1 and $w_k \in \beta^k(u_k)$ ν-a.e. in Ω_2. Let's see that

$$u_k \leqslant u_{k+1}, \quad \nu\text{-a.e. in } \Omega,\ k \in \mathbb{N}, \tag{5.89}$$

and

$$v_k \leqslant v_{k+1}, \quad \nu\text{-a.e. in } \Omega_1,\ k \in \mathbb{N}. \tag{5.90}$$

Going back to the construction of the mild solution, in this case of $\left(DP^{\mathbf{a}_p,\gamma,\beta^k}_{f-\frac{1}{k},v_0}\right)$, for each step $n \in \mathbb{N}$ and for each $i \in \{1,\ldots,n\}$, we have that there exists $u^n_{k,i} \in L^p(\Omega,\nu)$, $v^n_{k,i} \in L^{p'}(\Omega_1,\nu)$ and $w^n_{k,i} \in L^{p'}(\Omega_2,\nu)$ such that

$$\begin{cases} v^n_{k,i}(x) - \dfrac{T}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,u^n_{k,i}(y)-u^n_{k,i}(x))dm_x(y) = \dfrac{T}{n}\left(f^n_i(x) - \dfrac{1}{k}\right) + v^n_{k,i-1}(x), & x \in \Omega_1 \\ w^n_{k,i}(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u^n_{k,i}(y)-u^n_{k,i}(x))dm_x(y) = 0, & x \in \Omega_2, \end{cases}$$

with $v^n_{k,i} \in \gamma(u^n_{k,i})$ ν-a.e. in Ω_1 and $w^n_{k,i} \in \beta^k(u^n_{k,i})$ ν-a.e. in Ω_2. Let

$$z^n_{k,i} := \begin{cases} w^n_{k+1,i} & \text{if } u^n_{k+1,i} < k, \\ \mathfrak{B}^+ & \text{if } u^n_{k+1,i} = k, \\ \beta^k(u^n_{k+1,i}) & \text{if } u^n_{k+1,i} > k, \end{cases}$$

for $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$ (observe that $\beta^k(r)$ is single-valued for $r > k$ and coincides with $\beta^{k+1}(r) = \beta(r)$ for $r < k$). It is clear that $z^n_{k,i} \in \beta^k(u^n_{k+1,i})$ and, since $\beta^k \geqslant \beta^{k+1}$, we have that $z^n_{k,i} \geqslant w^n_{k+1,i}$. Then, for $n \in \mathbb{N}$ and $x \in \Omega_1$,

$$\begin{aligned} & v^n_{k+1,1}(x) - \frac{T}{n}\int_\Omega \mathbf{a}_p(x,y,u^n_{k+1,1}(y)-u^n_{k+1,1}(x))dm_x(y) = \frac{T}{n}\left(f^n_1(x) - \frac{1}{k+1}\right) + v_0(x) \\ & > \frac{T}{n}\left(f^n_1(x) - \frac{1}{k}\right) + v_0(x) = v^n_{k,1}(x) - \frac{T}{n}\int_\Omega \mathbf{a}_p(x,y,u^n_{k,1}(y)-u^n_{k,1}(x))dm_x(y) \end{aligned}$$

and, for $n \in \mathbb{N}$ and $x \in \Omega_2$,

$$\begin{aligned} & z^n_{k,1}(x) - \int_\Omega \mathbf{a}_p(x,y,u^n_{k+1,1}(y)-u^n_{k+1,i}(x))dm_x(y) \\ & \geqslant w^n_{k+1,1}(x) - \int_\Omega \mathbf{a}_p(x,y,u^n_{k+1,1}(y)-u^n_{k+1,1}(x))dm_x(y) \\ & = 0 = w^n_{k,1}(x) - \int_\Omega \mathbf{a}_p(x,y,u^n_{k,1}(y)-u^n_{k,1}(x))dm_x(y). \end{aligned}$$

Hence, by the maximum principle (Theorem 5.14),

$$v^n_{k,1} \leqslant v^n_{k+1,1} \quad \text{and} \quad u^n_{k,1} \leqslant u^n_{k+1,1} \quad \nu\text{-a.e.}$$

Proceeding in the same way we get that

$$v^n_{k,i} \leqslant v^n_{k+1,i} \quad \text{and} \quad u^n_{k,i} \leqslant u^n_{k+1,i} \quad \nu\text{-a.e.}$$

for each $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$. From here we get (5.89) and (5.90).

Since $\gamma^{-1}(r) = \partial j^*_\gamma(r)$ and $u_k(t) \in \gamma^{-1}(v_k(t))$ ν-a.e. in Ω_1, we have

$$\int_{\Omega_1} (v_k(t-\tau)(x) - v_k(t)(x))u_k(t)(x)d\nu(x) \leqslant \int_{\Omega_1} j^*_\gamma(v_k(t-\tau)(x)) - j^*_\gamma(v_k(t)(x))d\nu(x).$$

Integrating this equation over $[0,T]$, dividing by τ, letting $\tau \to 0^+$ and recalling that, by (5.87), $\int_{\Omega_1} j^*(v_k)d\nu \in W^{1,1}(0,T)$, we get

$$\begin{aligned} -\int_0^T \int_{\Omega_1} (v_k)_t(t)(x)u_k(t)(x)d\nu(x)dt &\leqslant \int_{\Omega_1} j^*(v(0)(x)) - j^*(v_k(T)(x))d\nu(x) \\ &\leqslant \int_{\Omega_1} j^*(v(0)(x))d\nu(x). \end{aligned}$$

Therefore, multiplying (5.88) by u_k and integrating with respect to ν we get

$$\begin{aligned} &\frac{1}{2}\int_0^T \int_\Omega \int_\Omega \mathbf{a}_p(x,y,u_k(t,y)-u_k(t)(x))(u_k(t)(y)-u_k(t)(x))dm_x(y)d\nu(x)dt \\ &\leqslant \int_0^T \int_{\Omega_1} \left(f(t)(x) - \frac{1}{k}\right) u_k(t)(x)d\nu(x)dt + \int_{\Omega_1} j^*(v(0)(x))d\nu(x). \end{aligned}$$

Now, working as in the previous step, since $\Gamma^+ < \infty$, we get that $\left\{\|u_k\|^p_{L^p(0,T;L^p(\Omega,\nu))}\right\}_k$ is bounded. Then, by the monotone convergence theorem we get that there exists $u \in L^p(0,T;L^p(\Omega,\nu))$ such that $u_k \xrightarrow{k} u$ in $L^p(0,T;L^p(\Omega,\nu))$. From this we get, by [**33**, Lemma G], that $v(t)(x) \in \gamma(u(t)(x))$ for a.e. $t \in [0,T]$ and ν-a.e. $x \in \Omega_1$.

Therefore, (5.88) and Lemma 5.7 (note that, by the monotonicity of $\{u_k\}$, we have that $|u_k| \leqslant \max\{|u_1|,|u|\} \in L^p(\Omega,\nu)$) yield that $(v_k)_t$ converges strongly in $L^{p'}(0,T;L^{p'}(\Omega_1,\nu))$ and w_k converges strongly in $L^{p'}(0,T;L^{p'}(\Omega_2,\nu))$. In particular, $v \in W^{1,1}(0,T;L^1(\Omega_1,\nu))$, $w(t)(x) \in \beta(u(t)(x))$ for a.e. $t \in [0,T]$ and ν-a.e. $x \in \Omega_2$, and

$$\begin{cases} v_t(t)(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t)(y)-u(t)(x))dm_x(y) = f(t)(x), & x \in \Omega_1,\ 0<t<T, \\ w(t)(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u(t)(y)-u(t)(x))dm_x(y) = 0, & x \in \Omega_2,\ 0<t<T. \end{cases}$$

The case $\mathcal{R}^-_{\gamma,\beta} > -\infty$ and $\mathcal{R}^+_{\gamma,\beta} = +\infty$ follows similarly by taking

$$\widetilde{\beta}^k := \begin{cases} \mathfrak{B}^- + r + k & \text{if } r < -k, \\ [\mathfrak{B}^-, \beta^0(-k)] & \text{if } r = -k, \\ \beta(r) & \text{if } r > -k. \end{cases}$$

instead of β^k, $k \in \mathbb{N}$.

Step 3. Finally, assume that both $\mathcal{R}^-_{\gamma,\beta}$ and $\mathcal{R}^+_{\gamma,\beta}$ are finite. We define, for $k \in \mathbb{N}$,

$$\widetilde{\beta}^k := \begin{cases} \mathfrak{B}^- + r + k & \text{if } r < -k, \\ [\mathfrak{B}^-, \beta^0(-k)] & \text{if } r = -k, \\ \beta(r) & \text{if } r > -k. \end{cases}$$

By the previous step we have that, for k large enough such that $f + \frac{1}{k}$ satisfies

$$\int_{\Omega_1} v_0^+ d\nu + \int_0^T \int_{\Omega_1} \left(f(s)^+ + \frac{1}{k}\right) d\nu ds < \nu(\Omega_1)\Gamma^+,$$

there exists a strong solution $v_k \in W^{1,1}(0,T;L^1(\Omega_1,\nu))$ of Problem $\left(DP^{\mathbf{a}_p,\gamma,\widetilde{\beta}^k}_{f+\frac{1}{k},v_0}\right)$, i.e., there exist $u_k \in L^p(0,T;L^p(\Omega,\nu))$ and $w_k \in L^{p'}(0,T;L^{p'}(\Omega_2,\nu))$ such that

$$\begin{cases} (v_k)_t(t)(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u_k(t)(y)-u_k(t)(x))dm_x(y) = f(t)(x) + \frac{1}{k}, & x \in \Omega_1,\ 0<t<T, \\ w_k(t)(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u_k(t)(y)-u_k(t)(x))dm_x(y) = 0, & x \in \Omega_2,\ 0<t<T, \end{cases}$$

with $v_k \in \gamma(u_k)$ ν-a.e. in Ω_1 and $w_k \in \widetilde{\beta}^k(u_k)$ ν-a.e. in Ω_2.

Going back to the construction of the mild solution, in this case of $\left(DP^{\mathbf{a}_p,\gamma,\widetilde{\beta}^k}_{f+\frac{1}{k},v_0}\right)$, for each step $n \in \mathbb{N}$ and for each $i \in \{1,\ldots,n\}$, we have that there exists $u^n_{k,i} \in L^p(\Omega,\nu)$, $v^n_{k,i} \in L^{p'}(\Omega_1,\nu)$ and $w^n_{k,i} \in L^{p'}(\Omega_2,\nu)$ such that

$$\begin{cases} v^n_{k,i}(x) - \dfrac{T}{n}\displaystyle\int_\Omega \mathbf{a}_p(x,y,u^n_{k,i}(y)-u^n_{k,i}(x))dm_x(y) = \dfrac{T}{n}\left(f^n_i(x)+\dfrac{1}{k}\right) + v^n_{k,i-1}(x), & x \in \Omega_1 \\ w^n_{k,i}(x) - \displaystyle\int_\Omega \mathbf{a}_p(x,y,u^n_{k,i}(y)-u^n_{k,i}(x))dm_x(y) = 0, & x \in \Omega_2, \end{cases}$$

where $v^n_{k,i} \in \gamma(u^n_{k,i})$ ν-a.e. in Ω_1 and $w^n_{k,i} \in \widetilde{\beta}^k(u^n_{k,i})$ ν-a.e. in Ω_2. Let

$$z^n_{k,i} := \begin{cases} w^n_{k+1,i} & \text{if } u^n_{k+1,i} > -k, \\ \mathfrak{B}^- & \text{if } u^n_{k+1,i} = -k, \\ \widetilde{\beta}^k(u^n_{k+1,i}) & \text{if } u^n_{k+1,i} < -k, \end{cases}$$

for $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$ (observe that $\widetilde{\beta}^k(r)$ is single-valued for $r < -k$ and coincides with $\widetilde{\beta}^{k+1}(r) = \beta(r)$ for $r > -k$). It is clear that $z^n_{k,i} \in \widetilde{\beta}^k(u^n_{k+1,i})$ and, since $\widetilde{\beta}^k \leqslant \widetilde{\beta}^{k+1}$, we have that $z^n_{k,i} \leqslant w^n_{k+1,i}$, $i \in \{1,\ldots,n\}$. Then, for $n \in \mathbb{N}$ and $x \in \Omega_1$,

$$\begin{aligned} &v^n_{k+1,1}(x) - \frac{T}{n}\int_\Omega \mathbf{a}_p(x,y,u^n_{k+1,1}(y)-u^n_{k+1,1}(x))dm_x(y) = \frac{T}{n}\left(f^n_1(x)+\frac{1}{k+1}\right) + v^n_0(x) \\ &< \frac{T}{n}\left(f^n_1(x)+\frac{1}{k}\right) + v^n_0(x) = v^n_{k,1}(x) - \frac{T}{n}\int_\Omega \mathbf{a}_p(x,y,u^n_{k,1}(y)-u^n_{k,1}(x))dm_x(y) \end{aligned}$$

and, for $n \in \mathbb{N}$ and $x \in \Omega_2$,

$$\begin{aligned} &z^n_{k,1}(x) - \int_\Omega \mathbf{a}_p(x,y,u^n_{k+1,1}(y)-u^n_{k+1,i}(x))dm_x(y) \\ &\leqslant w^n_{k+1,1}(x) - \int_\Omega \mathbf{a}_p(x,y,u^n_{k+1,1}(y)-u^n_{k+1,1}(x))dm_x(y) \\ &= 0 = w^n_{k,1}(x) - \int_\Omega \mathbf{a}_p(x,y,u^n_{k,1}(y)-u^n_{k,1}(x))dm_x(y). \end{aligned}$$

Hence, by the maximum principle (Theorem 5.14),

$$v^n_{k,1} \geqslant v^n_{k+1,1} \quad \text{and} \quad u^n_{k,1} \geqslant u^n_{k+1,1} \quad \nu\text{-a.e.}.$$

Proceeding in the same way we get that, for $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$,

$$v^n_{k,i} \geqslant v^n_{k+1,i} \quad \text{and} \quad u^n_{k,i} \geqslant u^n_{k+1,i} \quad \nu\text{-a.e.}.$$

Therefore,

$$u_k \geqslant u_{k+1}, \quad \nu\text{-a.e. in } \Omega,\ k \in \mathbb{N},$$

and

$$v_k \geqslant v_{k+1}, \quad \nu\text{-a.e. in } \Omega_1,\ k \in \mathbb{N}.$$

We can now conclude, as in the previous step, that

$$\int_0^T \|u_k^-(t)\|_{L^p(\Omega_1,\nu)}dt \leqslant K_5 \left(\int_0^T \left(\int_{\Omega_1}\int_{\Omega_1} |u_k^-(t)(y) - u_k^-(t)(x)|^p dm_x(y)d\nu(x) \right)^{\frac{1}{p}} dt + 1 \right)$$

for some constant $K_5 > 0$. Moreover, by the monotonicity of $\{u_k\}$, we get that

$$\left\{ \int_0^T \|u_k^+(t)\|_{L^p(\Omega_1,\nu)}dt \right\}_k$$

is bounded. From this point we can finish the proof as in the previous step. □

5.4. A particular case

Let $W \in \mathcal{B}$ and consider the following problem:

$$(5.91) \qquad \begin{cases} u_t(t,x) = \mathrm{div}_m \mathbf{a}_p u(t,x), & x \in W,\ 0 < t < T, \\ \mathcal{N}_1^{\mathbf{a}_p} u(t,x) = \varphi(x), & x \in \partial_m W,\ 0 < t < T, \\ u(0,x) = u_0(x), & x \in W. \end{cases}$$

This is, of course, a particular case of the problem studied in the previous section. Indeed, it corresponds to the choice $\Omega_1 = W$, $\Omega_2 = \partial_m W$, $\gamma(r) = r$ and $\beta(r) = 0$. With this particular choice we gain the complete accretivity of the associated operator (see Theorem 5.32) which allows us to prove a stronger version of Theorem 5.28 (see Theorem 5.34).

Since we will continue to work with this particular choice of Ω_1, Ω_2, γ and β in the next section, let us rewrite some of the definitions and results. We continue to work under Assumptions 1, 2, 3 and 5.

Let

$$Q_1 := W_m \times W_m \quad \textit{and} \quad Q_2 := Q_1 \backslash (\partial_m W \times \partial_m W).$$

In this particular case the integration by parts formula takes the following form (recall Lemma 1.48; the case $\mathbf{j} = 2$ will be used in the next section).

Proposition 5.29. *Let $\mathbf{j} \in \{1,2\}$. Let $u : W_m \to \mathbb{R}$ be a measurable function such that*

$$(x,y) \mapsto \mathbf{a}_p(x,y,u(y)-u(x)) \in L^q(Q_{\mathbf{j}}, \nu \otimes m_x)$$

and let $w \in L^{q'}(W_m)$, then

$$-\int_W \mathrm{div}_m \mathbf{a}_p u(x) w(x) d\nu(x) + \int_{\partial_m W} \mathcal{N}_{\mathbf{j}}^{\mathbf{a}_p} u(x) w(x) d\nu(x)$$
$$= \frac{1}{2} \int_{Q_{\mathbf{j}}} \mathbf{a}_p(x,y,u(y)-u(x))(w(y)-w(x)) d(\nu \otimes m_x)(x,y).$$

Moreover, if $u \in L^p(W_m,\nu)$, then

$$\int_W \mathrm{div}_m \mathbf{a}_p u(x) d\nu(x) = \int_{\partial_m W} \mathcal{N}_{\mathbf{j}}^{\mathbf{a}_p} u(x) d\nu(x).$$

Remark 5.30. Let us see, formally (as in Remark 5.13), the way in which the previous proposition will be used. Suppose that we are in the following situation:

$$\begin{cases} -\mathrm{div}_m \mathbf{a}_p u(x) = f(x), & x \in W, \\ \mathcal{N}_{\mathbf{j}}^{\mathbf{a}_p} u(x) = g(x), & x \in \partial_m W, \end{cases}$$

for $\mathbf{j} = 1$ or 2. Then, as in Remark 5.13, multiplying by a test function w, we get

$$\frac{1}{2} \int_{Q_{\mathbf{j}}} \mathbf{a}_p(x,y,u(y)-u(x))(w(y)-w(x)) d(\nu \otimes m_x)(x,y)$$
$$= \int_W f(x) w(x) d\nu(x) + \int_{\partial_m W} g(x) w(x) d\nu(x).$$

Moreover, if

$$\begin{cases} -\mathrm{div}_m \mathbf{a}_p u_i(x) = f_i(x), & x \in W, \\ \mathcal{N}_{\mathbf{j}}^{\mathbf{a}_p} u_i(x) = g_i(x), & x \in \partial_m W, \end{cases}$$

$i = 1, 2$, then, for a nondecreasing function $T : \mathbb{R} \to \mathbb{R}$, we obtain

$$\int_W (f_1(x) - f_2(x)) T(u_1(x) - u_2(x)) d\nu(x) + \int_{\partial_m W} (g_1(x) - g_2(x)) T(u_1(x) - u_2(x)) d\nu(x)$$
$$= \frac{1}{2} \int_{Q_{\mathbf{j}}} (\mathbf{a}_p(x, y, u_1(y) - u_1(x)) - \mathbf{a}_p(x, y, u_2(y) - u_2(x))) \times$$
$$\times (T(u_1(y) - u_2(y)) - T(u_1(x) - u_2(x))) d(\nu \otimes m_x)(x, y) \geqslant 0.$$

To study problem (5.91) we define the following operator in $L^1(W, \nu) \times L^1(W, \nu)$ associated with the problem. Observe that this operator does not correspond exactly to the one defined in 5.19. In particular, note that it depends on φ and the space of definition is $L^1(W, \nu)$ and not $L^1(W_m, \nu)$.

Definition 5.31. Let $\varphi \in L^1(\partial_m W, \nu)$. We say that $(u, v) \in B^m_{\mathbf{a}_p, \varphi}$ if $u, v \in L^1(W, \nu)$ and there exists $\overline{u} \in L^p(W_m, \nu)$ (that we will denote equally as u) such that $\overline{u}_{|W} = u$,

$$(x, y) \mapsto a_p(x, y, u(y) - u(x)) \in L^{p'}(Q_1, \nu \otimes m_x)$$

and

$$\begin{cases} -\mathrm{div}_m \mathbf{a}_p u = v & \text{in } W, \\ \mathcal{N}_1^{\mathbf{a}_p} u = \varphi & \text{in } \partial_m W; \end{cases}$$

that is,

$$v(x) = - \int_{W_m} \mathbf{a}_p(x, y, u(y) - u(x)) dm_x(y), \quad x \in W,$$

and

$$\varphi(x) = - \int_{W_m} \mathbf{a}_p(x, y, u(y) - u(x)) dm_x(y), \quad x \in \partial_m W.$$

Theorem 5.32. *Let $\varphi \in L^{p'}(\partial_m W, \nu)$. The operator $B^m_{\mathbf{a}_p, \varphi}$ is completely accretive (see section A.7 of Appendix A) and satisfies the range condition*

$$L^{p'}(W, \nu) \subset R(I + B^m_{\mathbf{a}_p, \varphi}). \tag{5.92}$$

Consequently, $B^m_{\mathbf{a}_p, \varphi}$ is m-*completely accretive in $L^{p'}(W, \nu)$.*

Proof. To prove the complete accretivity of the operator $B^m_{\mathbf{a}_p, \varphi}$ we need to show that, if $(u_i, v_i) \in B^m_{p, \varphi}$, $i = 1, 2$, and $q \in P_0$, then

$$\int_\Omega (v_1(x) - v_2(x)) q(u_1(x) - u_2(x)) d\nu(x) \geqslant 0.$$

In fact, by the integration by parts formula given in Proposition 5.29 and having in mind that, for both $i = 1$ and 2,

$$\varphi(x) = - \int_{W_m} \mathbf{a}_p(x, y, u_i(y) - u_i(x)) dm_x(y), \quad x \in \partial_m W,$$

we get (see also Remark 5.30)

$$\int_W (v_1(x) - v_2(x)) q(u_1(x) - u_2(x)) dx$$
$$= \frac{1}{2} \int_{Q_1} (\mathbf{a}_p(x, y, u_1(y) - u_1(x)) - \mathbf{a}_p(x, y, u_2(y) - u_2(x)))$$
$$\times (q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))) d(\nu \otimes m_x)(x, y) \geqslant 0\,.$$

The range condition (5.92) follows from Theorem 5.15. □

With a similar proof to that of Theorem 5.27 we obtain:

THEOREM 5.33. *Let $\varphi \in L^{p'}(\partial_m W, \nu)$. Then,*

$$\overline{D(B^m_{\mathbf{a}_p,\varphi})}^{L^{p'}(W,\nu)} = L^{p'}(W,\nu).$$

The following theorem is a consequence of the previous results thanks to Theorems A.25, A.45 and A.48.

THEOREM 5.34. *Let $\varphi \in L^{p'}(\partial_m W, \nu)$ and $T > 0$. For any $u_0 \in \overline{D(B^m_{\mathbf{a}_p,\varphi})}^{L^{p'}(W,\nu)} = L^{p'}(W,\nu)$ there exists a unique mild solution $u(t,x)$ of Problem (5.91). Moreover, for any $q \geqslant p'$ and $u_{0,i} \in L^q(W,\nu)$, $i = 1, 2$, we have the following contraction principle for the corresponding mild solutions u_i:*

$$\|(u_1(t,.) - u_2(t,.))^+\|_{L^q(W,\nu)} \leqslant \|(u_{0,1} - u_{0,2})^+\|_{L^q(W,\nu)} \quad \textit{for any} \quad 0 \leqslant t < T.$$

If $u_0 \in D(B^m_{\mathbf{a}_p,\varphi})$ then the mild solution is a strong solution.

REMARK 5.35. It is natural to ask whether $u \in L^\infty(W_m, \nu)$ whenever u is the solution of the problem

$$\begin{cases} u(x) - \mathrm{div}_m \mathbf{a}_p u(x) = v(x), & x \in W, \\ \mathcal{N}_1^{\mathbf{a}_p} u(x) = \varphi(x), & x \in \partial_m W, \end{cases}$$

with $v \in L^\infty(W,\nu)$ and $\varphi \in L^\infty(\partial_m W, \nu)$. In the next example we will see that this is not true in general.

EXAMPLE 5.36. Let $V(G) := \{x_0, x_1, \ldots, x_n, \ldots\}$, $w_{x_0,x_n} = w_{x_n,x_0} = \frac{1}{7^n}$ for $n \in \mathbb{N}$, $w_{x_n,x_n} = \frac{1}{3^n} - \frac{1}{7^n}$ for $n \in \mathbb{N}$ and $w_{x,y} = 0$ otherwise. Consider the metric random walk space $[V(G), d_G, m^G, \nu_G]$ associated with this infinite weighted discrete graph. Note that this graph is not locally finite. Then,

$$d_{x_0} = \frac{1}{6}, \quad d_{x_n} = \frac{1}{3^n}, \quad m_{x_0} = \sum_{n \geqslant 1} \frac{6}{7^n} \delta_{x_n},$$

$$m_{x_n} = \left(\frac{3}{7}\right)^n \delta_{x_0} + \left(1 - \left(\frac{3}{7}\right)^n\right) \delta_{x_n}, \quad n \geqslant 1,$$

and

$$\nu = \frac{1}{6}\delta_{x_0} + \sum_{n \geqslant 1} \frac{1}{3^n} \delta_{x_n}, \quad \nu(V) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{2}{3}.$$

Let $1 < p < +\infty$ and $W := \{x_0\}$, and denote $\nu_G := \nu$ and $m := m^G$, so that $\partial_m W = \{x_1, \ldots, x_n, \ldots\}$.

Let $a_p(x,y,r) = |r|^{p-2} r$, define $u : W_m \to \mathbb{R}$ by

$$u(x) := \begin{cases} 0 & \text{if } x = x_0 \\ 2^{\frac{n}{p-1}} & \text{if } x = x_n, \ n \geqslant 1, \end{cases}$$

$v : W \to \mathbb{R}$ by $v(x_0) = -\frac{12}{5}$ and $\varphi : \partial_m W \to \mathbb{R}$ by $\varphi(x_n) = (\frac{6}{7})^n$, $n \geqslant 1$. Then $u, v \in L^\infty(W,\nu)$, $\varphi \in L^\infty(\partial_m W, \nu)$ and $(x,y) \mapsto a_p(x,y,u(y) - u(x)) \in L^{p'}(Q_1, \nu \otimes m_x)$.

Now,

$$\begin{aligned} & u(x_0) - \int_{W_m} \mathbf{a}_p(x_0, y, u(y) - u(x_0)) dm_{x_0}(y) \\ & = u(x_0) - \sum_{n \geqslant 1} |u(x_n) - u(x_0)|^{p-2} (u(x_n) - u(x_0)) m_{x_0}(\{x_n\}) \\ & = u(x_0) - \sum_{n \geqslant 1} u(x_n)^{p-1} \frac{6}{7^n} = - \sum_{n \geqslant 1} 2^n \frac{6}{7^n} = -\frac{12}{5} = v(x_0) \end{aligned}$$

and

$$-\int_{W_m} \mathbf{a}_p(x_n, y, u(y) - u(x_n)) dm_{x_n}(y) = -|u(x_0) - u(x_n)|^{p-2}(u(x_0) - u(x_n)) m_{x_n}(\{x_0\}) = = u(x_n)^{p-1} \left(\frac{3}{7}\right)^n = \left(\frac{6}{7}\right)^n = \varphi(x_n).$$

Therefore, u is a solution of the Neumann problem

$$\begin{cases} u - \mathrm{div}_m \mathbf{a}_p u = v & \text{in } W, \\ \mathcal{N}_m^{\mathbf{a}_p} u = \varphi & \text{in } \partial_m W. \end{cases}$$

Note that $v \in L^\infty(W,\nu)$ and $\varphi \in L^\infty(\partial_m W,\nu)$ but $u \notin L^\infty(W_m,\nu)$. Note also that $u \in L^p(W_m,\nu)$ for sufficiently large p ($p > \frac{1}{\log(\frac{3}{2})}$).

Consequently, since $v \in L^\infty(W,\nu)$ and $\varphi \in L^\infty(\partial_m W,\nu)$ but, for $p \leqslant \frac{1}{\log(\frac{3}{2})}$,

$$u = (I + B^m_{\mathbf{a}_p,\varphi})^{-1} v$$

does not belong to $L^p(W_m,\nu)$, we have that this metric random walk space does not satisfy a (p,p)-Poincaré type inequality on $(W,\varnothing)$.

5.5. Neumann boundary conditions of Dipierro–Ros-Oton–Valdinoci type

Let $W \in \mathcal{B}$. In this section we will study the evolution problem

$$(5.93)\qquad \begin{cases} u_t(t,x) = \mathrm{div}_m \mathbf{a}_p u(t,x), & x \in W,\ 0 < t < T, \\ \mathcal{N}_2^{\mathbf{a}_p} u(t,x) = \varphi(x), & x \in \partial_m W,\ 0 < t < T, \\ u(0,x) = u_0(x), & x \in W, \end{cases}$$

and the following associated Neumann problem

$$\begin{cases} u(x) - \mathrm{div}_m \mathbf{a}_p u(x) = \varphi(x), & x \in W, \\ \mathcal{N}_2^{\mathbf{a}_p} u(x) = \varphi(x), & x \in \partial_m W, \end{cases}$$

but we will do so without requiring that a Poincaré type inequality holds (i.e., we drop Assumption 5). Note that, as in the previous section, this would correspond to taking $\Omega_1 = W$ and $\Omega_2 = \partial_m W$, thus

$$Q_2 = (W_m \times W_m) \backslash (\partial_m W \times \partial_m W).$$

Since we will not assume that a Poincaré type inequality holds, a price has to be paid by restricting the functions φ which may appear in the Neumann boundary condition to the following space of functions.

Definition 5.37. We define

$$L^{m,\infty}(\partial_m W,\nu) := \left\{ \varphi : \partial_m W \to \mathbb{R} \ : \ \varphi \text{ is measurable and } \frac{\varphi}{m_{(.)}(W)} \in L^\infty(\partial_m W,\nu) \right\}.$$

Remark 5.38. Note that $L^{m,\infty}(\partial_m W,\nu) \subset L^\infty(\partial_m W,\nu)$.

Suppose that $[V, d_G, m^G, \nu_G]$ is the metric random walk space associated with a locally finite weighted discrete graph as described in Example 1.38 and let $W \subset V$. Then, if $\partial_m W \subset V$ is a finite set, we have that $L^{m,\infty}(\partial_m W,\nu) = L^\infty(\partial_m W,\nu)$.

Consider now the metric random walk space $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ given in Example 1.37. Let $W \subset \mathbb{R}^N$ be a bounded domain and denote

$$W_r := \{x \in \mathbb{R}^N : \mathrm{dist}(x,W) < r\}.$$

Suppose that $\mathrm{supp}(J) \supseteq B(0,R)$. Then,

$$\{\varphi \in L^\infty(\partial_m W,\nu) \ : \ \mathrm{supp}(\varphi) \subset W_r,\ r < R\} \subset L^{m,\infty}(\partial_m W,\nu).$$

Indeed, let $\varphi \in L^\infty(\partial_m W,\nu)$ such that $\operatorname{supp}(\varphi) \neq \emptyset$ and $\operatorname{supp}(\varphi) \subset W_r$ for some $r < R$. It is enough to see that there exists $\delta > 0$ such that

$$m_x^J(W) = \int_W J(x-y)dy \geqslant \delta > 0$$

for every $x \in \operatorname{supp}(\varphi)$. Suppose otherwise that there exists a sequence $(x_n) \subset \operatorname{supp}(\varphi)$ such that $\lim_n \int_W J(x_n - y)dy = 0$, then, since $\partial_m W$ is bounded, there exists a subsequence of (x_n) converging to $x_0 \in \operatorname{supp}(\varphi)$. Therefore, by the continuity of J and applying Fatou's Lemma we get that $\int_W J(x_0 - y)dy = 0$. However, this is not possible because $\operatorname{dist}(x_0, W) \leqslant r < R$ and, therefore, since W is open, we have that $\mathcal{L}^N(B(x_0,R)\cap W) > 0$ with $B(x_0,R)\cap W \subset \operatorname{supp}(J(x_0 - .))$ so

$$\int_W J(x_0-y)dy \geqslant \int_{B(x_0,R)\cap W} J(x_0-y)dy > 0.$$

In particular, characteristic functions of sets $A \subset W_r$ with $r < R$, belong to $L^{m,\infty}(\partial_m W,\nu)$.

To study problem (5.93), we define the following associated operator in $L^1(W,\nu)\times L^1(W,\nu)$.

Definition 5.39. Let $1 < p < \infty$ and $\varphi \in L^{m,\infty}(\partial_m W,\nu)$. We say that $(u,v) \in A^m_{\mathbf{a}_p,\varphi}$ if $u, v \in L^1(W,\nu)$, and there exists a measurable function $\overline{u}$ in W_m with $\overline{u}_{|W} = u$ (that we denote equally as u) satisfying

$$m_{(.)}(W)|u|^{p-1} \in L^1(\partial_m W,\nu),$$

$$(x,y) \mapsto a_p(x,y,u(y)-u(x)) \in L^1(Q_2, \nu\otimes m_x),$$

and

$$\begin{cases} -\operatorname{div}_m \mathbf{a}_p u = v & \text{in } W, \\ \mathcal{N}_2^{\mathbf{a}_p} u = \varphi & \text{in } \partial_m W, \end{cases}$$

that is,

$$v(x) = -\int_{W_m} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y), \quad x \in W,$$

and

$$\varphi(x) = -\int_W \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y), \quad x \in \partial_m W.$$

Remark 5.40. Let $1 < p < \infty$ and $\varphi \in L^{m,\infty}(\partial_m W,\nu)$.

1. Let $u \in L^p(W,\nu)$ and, for $x \in \partial_m W$, let $\Psi_x : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Psi_x(r) := -\int_W a_p(x,y,u(y)-r)dm_x(y).$$

Then, since Ψ is ν-a.e. increasing (recall (5.9)), the equation

$$-\int_W \mathbf{a}_p(x,y,u(y)-r)dm_x(y) = \varphi(x), \tag{5.94}$$

has a unique solution $r =: \overline{u}(x)$ for ν-a.e. $x \in \partial_m W$ (and $\overline{u}$ is easily seen to be measurable).

2. As a consequence, the extension of u to the boundary $\partial_m W$ in Definition 5.39 is unique.

3. Let us see that, if $u \in L^\infty(W,\nu)$, then

$$\|\overline{u}\|_{L^\infty(\partial_m W,\nu)} \leqslant \|u\|_{L^\infty(W,\nu)} + \frac{1}{c^{\frac{1}{p-1}}} \left\| \frac{\varphi}{m_{(.)}(W)} \right\|^{\frac{1}{p-1}}_{L^\infty(\partial_m W,\nu)}.$$

Indeed, let us denote $u(x) := \overline{u}(x)$, $x \in \partial_m W$, and suppose that $\|u\|_{L^\infty(\partial_m W,\nu)} > \|u\|_{L^\infty(W,\nu)}$, otherwise the result is trivial. Let $0 < \varepsilon < \|u\|_{L^\infty(\partial_m W,\nu)} - \|u\|_{L^\infty(W,\nu)}$,

$$A^+ := \{x \in \partial_m W : u(x) > \|u\|_{L^\infty(\partial_m W,\nu)} - \varepsilon\}$$

and

$$A^- := \{x \in \partial_m W : u(x) < -\|u\|_{L^\infty(\partial_m W,\nu)} + \varepsilon\}.$$

Suppose first that $\nu(A^-) > 0$ and let

$$B := \{y \in W : u(y) < -\|u\|_{L^\infty(W,\nu)}\}.$$

Then, since $\nu(B) = 0$, we have that $\nu \otimes m_x(A^- \times B) = \nu \otimes m_x(B \times A^-) = 0$. Now,

$$u(y) - u(x) > \|u\|_{L^\infty(\partial_m W,\nu)} - \|u\|_{L^\infty(W,\nu)} - \varepsilon > 0$$

for every $(x, y) \in A^- \times (W \backslash B)$. Therefore, since, by (5.11), $\mathbf{a}_p(x, y, r) \geqslant cr^{p-1}$ $\nu \otimes m_x$-a.e. and for every $r \geqslant 0$, we have that

$$\mathbf{a}_p(x, y, u(y) - u(x)) \geqslant c(\|u\|_{L^\infty(\partial_m W,\nu)} - \|u\|_{L^\infty(W,\nu)} - \varepsilon)^{p-1}$$

for $\nu \otimes m_x$-a.e. $(x, y) \in A^- \times (W \backslash B)$. Now, integrating (5.94) over A^- with respect to ν, we get:

$$\begin{aligned} &-\int_{A^-}\int_W \mathbf{a}_p(x, y, u(y) - u(x))dm_x(y)d\nu(x) \\ &\geqslant \int_{A^-} \varphi(x)d\nu(x) = \int_{A^-} \frac{\varphi(x)}{m_x(W)} m_x(W)d\nu(x) \\ &\geqslant -\left\| \frac{\varphi}{m_{(.)}(W)} \right\|_{L^\infty(\partial_m W,\nu)} \int_{A^-} m_x(W)d\nu(x). \end{aligned}$$

but, by the previous computations,

$$\begin{aligned} &-\int_{A^-}\int_W \mathbf{a}_p(x, y, u(y) - u(x))dm_x(y)d\nu(x) \\ &\leqslant -c(\|u\|_{L^\infty(\partial_m W,\nu)} - \|u\|_{L^\infty(W,\nu)} - \varepsilon)^{p-1} \int_{A^-} m_x(W)d\nu(x), \end{aligned}$$

thus

$$c(\|u\|_{L^\infty(\partial_m W,\nu)} - \|u\|_{L^\infty(W,\nu)} - \varepsilon)^{p-1} \leqslant \left\| \frac{\varphi}{m_{(.)}(W)} \right\|_{L^\infty(\partial_m W,\nu)}$$

and the result follows since ε was arbitrarily small. If $\nu(A^-) = 0$ then $\nu(A^+) > 0$ and we would proceed analogously.

THEOREM 5.41. *Let $\varphi \in L^{m,\infty}(\partial_m W, \nu)$. The operator $A^m_{\mathbf{a}_p,\varphi}$ is completely accretive and satisfies the range condition*

$$L^{p'}(W, \nu) \subset R(I + A^m_{\mathbf{a}_p,\varphi}). \tag{5.95}$$

PROOF. The proof of the complete accretivity of $A^m_{\mathbf{a}_p,\varphi}$ follows similarly to that of $B^m_{\mathbf{a}_p,\varphi}$ (Theorem 5.32).

Let us see that $A^m_{\mathbf{a}_p,\varphi}$ satisfies the range condition (5.95), that is, let us prove that, for $\phi \in L^{p'}(W, \nu)$, there exists $u \in D(A^m_{\mathbf{a}_p,\varphi})$ such that

$$u + A^m_{\mathbf{a}_p,\varphi} u \ni \phi.$$

We divide the proof into two steps.

Step 1. Assume that $\phi \in L^\infty(W, \nu)$. Working as in Subsection 5.1.3 but with

$$A_1(u)(x) = T_K(u)(x) - \int_{W_m} \mathbf{a}_p(x, y, u(y) - u(x))dm_x(y) + \frac{1}{n}|u(x)|^{p-2}u^+(x) - \frac{1}{k}|u(x)|^{p-2}u^-(x),$$

for $x \in W$, and

$$A_2(u)(x) := -\int_W \mathbf{a}_p(x, y, u(y) - u(x))dm_x(y) + \frac{1}{n}|u(x)|^{p-2}u^+(x) - \frac{1}{k}|u(x)|^{p-2}u^-(x),$$

for $x \in \partial_m W$, we have that, for $k, n \in \mathbb{N}$ and $K > 0$, there exist $u_{n,k} \in L^p(W_m, \nu)$ such that

$$T_K(u_{n,k})(x) - \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) \\ + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \phi(x),$$

$x \in W$, and

$$- \int_W \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi(x),$$

$x \in \partial_m W$. Now, let $M > 0$. Multiplying the first equation by $(u_{n,k} - M)^+$ and integrating over W with respect to ν, by Proposition 5.29, we get that, after removing some positive terms,

$$\int_W T_K(u_{n,k})(u_{n,k} - M)^+ d\nu + \frac{1}{n}\int_{\partial_m W} |u_{n,k}|^{p-2}u^+_{n,k}(u_{n,k} - M)^+ d\nu \\ \leqslant \int_W \phi(u_{n,k} - M)^+ d\nu + \int_{\partial_m W} \varphi(u_{n,k} - M)^+ d\nu.$$

Therefore, taking

$$M = M_{\phi,\varphi,n,k} := \max\left\{\|\phi\|_{L^\infty(W,\nu)}, \left(n\|\varphi\|_{L^\infty(\partial_m W,\nu)}\right)^{\frac{1}{p-1}}, \left(k\|\varphi\|_{L^\infty(\partial_m W,\nu)}\right)^{\frac{1}{p-1}}\right\}$$

we get that

$$\int_W (T_K(u_{n,k}) - M)(u_{n,k} - M)^+ d\nu + \frac{1}{n}\int_{\partial_m W} (|u_{n,k}|^{p-2}u^+_{n,k} - M^{p-1})(u_{n,k} - M)^+ d\nu \\ \leqslant \int_W (\phi - M)(u_{n,k} - M)^+ d\nu + \int_{\partial_m W} (\varphi - \frac{1}{n}M^{p-1})(u_{n,k} - M)^+ d\nu \leqslant 0$$

and, consequently, taking $K > M$, we get that

$$u_{n,k} \leqslant M \quad \nu - \text{a.e. in } W_m;$$

and, similarly, we get that

$$u_{n,k} \geqslant -M \quad \nu - \text{a.e. in } W_m.$$

Hence,

$$\|u_{n,k}\|_{L^\infty(W_m,\nu)} \leqslant M.$$

Therefore, we have

$$u_{n,k}(x) - \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) \\ + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \phi(x), \tag{5.96}$$

for $x \in W$; and

$$- \int_W \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) \\ + \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x) = \varphi(x), \tag{5.97}$$

for $x \in \partial_m W$.

Let us now see that $\|u_{n,k}\|_{L^\infty(W_m,\nu)}$ is uniformly bounded in n and k. First, working as in the proof of Remark 5.40.*3*, we prove that

$$\|u_{n,k}\|_{L^\infty(\partial_m W,\nu)} \leqslant \|u_{n,k}\|_{L^\infty(W,\nu)} + \frac{1}{C^{\frac{1}{p-1}}}\left\|\frac{\varphi}{m_{(.)}(W)}\right\|^{\frac{1}{p-1}}_{L^\infty(\partial_m W,\nu)}$$

for every n, $k \geqslant 1$. Indeed, define A^- as in that remark and integrate (5.97) over A^- with respect to ν (note that the term involving $\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x) - \frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x)$ does not affect the reasoning). The same can be done with A^+. Therefore, it is enough to see that $\|u_{n,k}\|_{L^\infty(W,\nu)}$ is uniformly bounded in n and k. Let

$$K := \frac{1}{c^{\frac{1}{p-1}}} \left\| \frac{\varphi}{m_{(.)}(W)} \right\|_{L^\infty(\partial_m W,\nu)}^{\frac{1}{p-1}},$$

so that $\|u_{n,k}\|_{L^\infty(\partial_m W,\nu)} \leqslant \|u_{n,k}\|_{L^\infty(W,\nu)} + K$. Now, if all of the $u_{n,k}$ are ν-null the result is trivial. Therefore, fix some $u_{n,k} \not\equiv 0$ and $0 < \varepsilon < \|u_{n,k}\|_{L^\infty(W,\nu)}$. Let

$$A_\star^+ := \{x \in W : u_{n,k}(x) > \|u_{n,k}\|_{L^\infty(W,\nu)} - \varepsilon\}$$

and

$$A_\star^- := \{x \in W : u_{n,k}(x) < -\|u_{n,k}\|_{L^\infty(W,\nu)} + \varepsilon\}.$$

Suppose first that $\nu(A_\star^+) > 0$. Integrating over $A_\star^+$ in (5.96) we get:

$$\int_{A_\star^+} u_{n,k}(x) d\nu(x) - \int_{A_\star^+} \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) d\nu(x)$$
$$+\frac{1}{n} \int_{A_\star^+} |u_{n,k}(x)|^{p-2} u_{n,k}^+(x) d\nu(x) = \int_{A_\star^+} \phi(x) d\nu(x) \leqslant \nu(A_\star^+) \|\phi\|_{L^\infty(W,\nu)}.$$

Consequently, dividing by $\nu(A_\star^+)$, we have

$$\|u_{n,k}\|_{L^\infty(W,\nu)} - \varepsilon - \frac{1}{\nu(A_\star^+)} \int_{A_\star^+} \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) d\nu(x) \leqslant \|\phi\|_{L^\infty(W,\nu)}.$$

Now, $\mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) \leqslant \mathbf{a}_p(x, y, K+\varepsilon) \leqslant C\left(1 + (K+\varepsilon)^{p-1}\right)$ for $\nu \otimes m$-a.e. $(x, y) \in A_\star^+ \times W_m$, thus, since $m_x(W_m) = 1$ for ν-a.e. $x \in W$,

$$\frac{1}{\nu(A_\star^+)} \int_{A_\star^+} \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) d\nu(x) \leqslant C\left(1 + (K+\varepsilon)^{p-1}\right),$$

and, since $\varepsilon > 0$ is arbitrarily small, we conclude that

$$\|u_{n,k}\|_{L^\infty(W,\nu)} \leqslant \|\phi\|_{L^\infty(W,\nu)} + C\left(1 + \frac{1}{c} \left\| \frac{\varphi}{m_{(.)}(W)} \right\|_{L^\infty(\partial_m W,\nu)}\right)$$

where the right hand side does not depend on n or k. If $\nu(A_\star^+) = 0$ then $\nu(A_\star^-) > 0$ and we proceed similarly, that is, integrate over $A_\star^-$ in (5.96) to obtain that

$$\int_{A_\star^-} u_{n,k}(x) d\nu(x) - \int_{A_\star^-} \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) d\nu(x)$$
$$- \frac{1}{k} \int_{A_\star^-} |u_{n,k}(x)|^{p-2} u_{n,k}^-(x) d\nu(x)$$
$$= \int_{A_\star^-} \phi(x) d\nu(x) \geqslant -\nu(A_\star^-) \|\phi\|_{L^\infty(W,\nu)}.$$

Then, dividing by $\nu(A_\star^-)$, we have

$$-\|u_{n,k}\|_{L^\infty(W,\nu)} + \varepsilon - \frac{1}{\nu(A_\star^-)} \int_{A_\star^-} \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(y) - u_{n,k}(x)) dm_x(y) d\nu(x) \geqslant -\|\phi\|_{L^\infty(W,\nu)}$$

which, using (5.8), is equivalent to

$$\|u_{n,k}\|_{L^\infty(W,\nu)} \leqslant \|\phi\|_{L^\infty(W,\nu)} + \varepsilon + \frac{1}{\nu(A_\star^-)} \int_{A_\star^-} \int_{W_m} \mathbf{a}_p(x, y, u_{n,k}(x) - u_{n,k}(y)) dm_x(y) d\nu(x).$$

Now, $\mathbf{a}_p(x, y, u_{n,k}(x) - u_{n,k}(y)) \leqslant \mathbf{a}_p(x, y, K+\varepsilon) \leqslant C\left(1 + (K+\varepsilon)^{p-1}\right)$ for $\nu \otimes m$-a.e. $(x, y) \in A_\star^- \times W_m$ and we conclude as before.

Now, let us see that $\{u_{n,k} \llcorner W\}$ is ν-a.e. nondecreasing in n, and ν-a.e. nonincreasing in k. Let $n' < n$. Multiplying (5.96) for $u_{n',k}$ and $u_{n,k}$ by $(u_{n',k} - u_{n,k})^+$, integrating over W with respect to ν, and subtracting we obtain

$$\begin{aligned}
&\int_W \big(u_{n',k}(x) - u_{n,k}(x)\big) \big(u_{n',k}(x) - u_{n,k}(x)\big)^+ d\nu(x) \\
&\quad + \int_W \left(\frac{1}{n'}|u_{n',k}(x)|^{p-2}u^+_{n',k}(x) - \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)\right) \big(u_{n',k}(x) - u_{n,k}(x)\big)^+ d\nu(x) - \\
&\quad - \int_W \frac{1}{k}\left(|u_{n',k}(x)|^{p-2}u^-_{n',k}(x) - |u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right) \big(u_{n',k}(x) - u_{n,k}(x)\big)^+ d\nu(x) \\
&\quad - \int_W \int_{W_m} \mathbf{a}_p(x,y,u_{n',k}(y) - u_{n',k}(x)) \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ dm_x(y)d\nu(x) \\
&\quad + \int_W \int_{W_m} \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x)) \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ dm_x(y)d\nu(x) \\
&= \int_W \phi(x) \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ d\nu(x) - \int_W \phi(x) \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ d\nu(x) = 0.
\end{aligned}$$

Now, by Proposition 5.29 with $w(x) = (u_{n',k}(x) - u_{n,k}(x))^+$ and recalling (5.9) (see also Remark 5.30), and then using (5.97) we obtain

$$\begin{aligned}
&- \int_W \int_{W_m} \mathbf{a}_p(x,y,u_{n',k}(y) - u_{n',k}(x)) \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ dm_x(y)d\nu(x) \\
&\quad + \int_W \int_{W_m} \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x)) \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ dm_x(y)d\nu(x) \\
&= \frac{1}{2}\int_{Q_2} (\mathbf{a}_p(x,y,u_{n',k}(y) - u_{n',k}(x)) - \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x))) \\
&\qquad\qquad \times \big((u_{n',k}(y)) - u_{n,k}(y))^+ - (u_{n',k}(x) - u_{n,k}(x))^+\big) dm_x(y)d\nu(x) \\
&\quad - \int_{\partial_m W} \int_{W_m} (\mathbf{a}_p(x,y,u_{n',k}(y) - u_{n',k}(x)) - \mathbf{a}_p(x,y,u_{n,k}(y) - u_{n,k}(x))) \\
&\qquad\qquad \times \big(u_{n',k}(x)) - u_{n,k}(x)\big)^+ dm_x(y)d\nu(x) \\
&\geqslant \int_{\partial_m W} \left(\frac{1}{n'}|u_{n',k}(x)|^{p-2}u^+_{n',k}(x) - \frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)\right) \big(u_{n',k}(x) - u_{n,k}(x)\big)^+ d\nu(x) \\
&\quad - \frac{1}{k}\int_{\partial_m W} \left(|u_{n',k}(x)|^{p-2}u^-_{n',k}(x) - |u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right) \big(u_{n',k}(x) - u_{n,k}(x)\big)^+ d\nu(x)\,.
\end{aligned}$$

Consequently,

$$\int_W \left(u_{n',k}(x)-u_{n,k}(x)\right)\left(u_{n',k}(x)-u_{n,k}(x)\right)^+ d\nu(x)$$
$$+\int_W \left(\frac{1}{n'}|u_{n',k}(x)|^{p-2}u^+_{n',k}(x)-\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)\right)\left(u_{n',k}(x)-u_{n,k}(x)\right)^+ d\nu(x)$$
$$-\int_W \frac{1}{k}\left(|u_{n',k}(x)|^{p-2}u^-_{n',k}(x)-|u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right)\left(u_{n',k}(x)-u_{n,k}(x)\right)^+ d\nu(x)$$
$$+\int_{\partial_m W} \left(\frac{1}{n'}|u_{n',k}(x)|^{p-2}u^+_{n',k}(x)-\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)\right)\left(u_{n',k}(x)-u_{n,k}(x)\right)^+ d\nu(x)$$
$$-\frac{1}{k}\int_{\partial_m W}\left(|u_{n',k}(x)|^{p-2}u^-_{n',k}(x)-|u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right)\left(u_{n',k}(x)-u_{n,k}(x)\right)^+ d\nu(x)\leqslant 0\,.$$

Therefore, since the last four summands on the left hand side are non-negative we get that

$$\int_W \left(u_{n',k}(x)-u_{n,k}(x)\right)\left(u_{n',k}(x)-u_{n,k}(x)\right)^+ d\nu(x)\leqslant 0$$

so $u_{n,k}\llcorner W$ is ν-a.e. nondecreasing in n. Similarly, we get that $u_{n,k}\llcorner W$ is ν-a.e. nonincreasing in k.

Let us see that these monotonicities also hold in $\partial_m W$. Recall that

$$-\int_W \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)=\varphi(x)-\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)+\frac{1}{k}|u_{n,k}(x)|^{p-2}u^-_{n,k}(x),$$

for $x\in\partial_m W$. Now, let $N\subset X$ be a ν-null set such that, for every $x\in X\backslash N$,

$$(\mathbf{a}_p(x,y,r)-\mathbf{a}_p(x,y,s))(r-s)>0\quad\text{for } m_x\text{-a.e. } y\in X \text{ and for all } r\neq s.$$

Then, for a fixed $k\in\mathbb{N}$, let $n'<n$, $x\in\partial_m W\backslash N$ and suppose that $u_{n',k}(x)>u_{n,k}(x)$. Since $(u_{n,k})\llcorner W$ is ν-a.e. nondecreasing in n, by the absolute continuity of m_x with respect to ν, we have that $(u_{n,k})\llcorner W$ is m_x-a.e. nondecreasing in n, therefore

$$\begin{aligned}0&<-\int_W\left(\mathbf{a}_p(x,y,u_{n',k}(y)-u_{n',k}(x))-\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))\right)dm_x(y)\\&=\frac{1}{n}|u_{n,k}(x)|^{p-2}u^+_{n,k}(x)-\frac{1}{n'}|u_{n',k}(x)|^{p-2}u^+_{n',k}(x)\\&\quad+\frac{1}{k}\left(|u_{n',k}(x)|^{p-2}u^-_{n',k}(x)-|u_{n,k}(x)|^{p-2}u^-_{n,k}(x)\right)\leqslant 0\end{aligned}$$

which is a contradiction. Consequently, $u_{n,k}\llcorner\partial_m W$ is ν-a.e. nondecreasing in n. Similarly, $u_{n,k}\llcorner\partial_m W$ is ν-a.e. nonincreasing in k.

Then, for ν-a.e. $x\in W_m$, we can pass to the limit in n, and then in k, in (5.96) and (5.97), to get $u\in L^\infty(W_m,\nu)$ such that

$$u(x)-\int_{W_m}\mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)=\phi(x),\quad x\in W,$$

and

$$-\int_W \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)=\varphi(x),\quad x\in\partial_m W.$$

Therefore, for $\phi\in L^\infty(W,\nu)$ the range condition holds.

Step 2. Let us now take $\phi\in L^{p'}(W,\nu)$. Let $\phi_{n,k}:=\sup\{\inf\{\phi,n\},-k\}$, which is nondecreasing in n and nonincreasing in k. By Step 1, there exists a solution $u_{n,k}\in L^\infty(W_m,\nu)$ of

$$u_{n,k}+A^m_{\mathbf{a}_p,\varphi}(u_{n,k})\ni\phi_{n,k},$$

that is,

$$u_{n,k}(x)-\int_{W_m}\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)=\phi_{n,k}(x),\quad x\in W,$$

and

$$-\int_W \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)=\varphi(x),\quad x\in\partial_m W.$$

Let us see the monotonicity properties of $u_{n,k}$. By the complete accretivity, we have that

$$\left\| \left(u_{n',k'}-u_{n,k}\right)^{\pm} \right\|_{L^p(W,\nu)} \leqslant \left\| \left(\phi_{n',k'}-\phi_{n,k}\right)^{\pm} \right\|_{L^p(W,\nu)} \tag{5.98}$$

and

$$\left\| \left(u_{n',k'}-u_{n,k}\right)^{\pm} \right\|_{L^{p'}(W,\nu)} \leqslant \left\| \left(\phi_{n',k'}-\phi_{n,k}\right)^{\pm} \right\|_{L^{p'}(W,\nu)}.$$

This implies, for example, that if $n'<n$ then $u_{n',k}\leqslant u_{n,k}$ ν-a.e. in W thus, as before, $u_{n,k}\llcorner W$ is ν-a.e. nondecreasing in n and ν-a.e. nonincreasing in k. Moreover, it also implies the convergence of $u_{n,k}\llcorner W$ in $L^p(W,\nu)$.

On the other hand, for $n'<n$, we have

$$\int_W \left(\mathbf{a}_p(x,y,u_{n',k}(y)-u_{n',k}(x))-\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))\right)dm_x(y)=0$$

for every $x\in\partial_m W$ and, therefore, the same reasoning as before yields that $u_{n,k}(x)$ is nondecreasing in n and nonincreasing in k for ν-a.e. $x\in\partial_m W$.

We want to pass to the limit in

$$\text{(5.99)}\quad \begin{cases} u_{n,k}(x)-\displaystyle\int_{W_m}\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)=\phi_{n,k}(x), & x\in W, \qquad \text{(a)}\\ -\displaystyle\int_W \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)=\varphi(x), & x\in\partial_m W. \qquad \text{(b)}\end{cases}$$

We start by letting $n\to+\infty$. Let's see that we can take limits as $n\to+\infty$ in the terms involving $\mathbf{a}_p$ (this follows similarly to the proof of Lemma 5.7, but with the added difficulty of having to see what happens on the m-boundary of W). Fix $k\in\mathbb{N}$. By (5.98), we have that $u_{n,k}\xrightarrow{n}u_k$ in $L^p(W,\nu)$. Hence, there exists $h_k\in L^p(W,\nu)$ such that

$$|u_{n,k}|\leqslant h_k \quad \nu\text{-a.e. in } W \text{ for every } n\in\mathbb{N}.$$

By the invariance of ν with respect to m we may take a ν-null set $B\in\mathcal{B}$ such that, for every $x\in\partial_m W\backslash B$,

(a) $u_{n,k}\xrightarrow{n}u_k$ in $L^p(W,m_x)$,

(b) $h_k\in L^p(W,m_x)$,

(c) $|u_{n,k}(y)|\leqslant h_k(y)$ for m_x-a.e. $y\in W$ and for every $n\in\mathbb{N}$,

(d) $\varphi(x)<+\infty$, and

(e) $\mathbf{a}_p(x,y,r)(=-\mathbf{a}_p(x,y,-r))\geqslant c_p r^{p-1}$ for m_x-a.e. $y\in W$ and every $r>0$ (recall (5.11)).

Now, let's see that $u_{n,k}(x)$ has a finite limit for every $x\in\partial_m W\backslash B$. Suppose otherwise that there exists $x\in\partial_m W\backslash B$ such that $u_{n,k}(x)\to+\infty$. Then, given $M>0$, there exists n_0 such that, for $n\geqslant n_0$, $u_{n,k}(x)>M$. Hence, for $n\geqslant n_0$,

$$-\mathbf{a}_p(x,\cdot,u_{n,k}(\cdot)-u_{n,k}(x))\geqslant-\mathbf{a}_p(x,\cdot,h_k(\cdot)-M)\in L^{p'}(W,m_x)\quad m_x\text{-a.e. in } W,$$

so we may apply Fatou's lemma to obtain:

$$\begin{aligned}&\int_W \liminf_n -\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)\\ &\leqslant \liminf_n \int_W -\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)=\varphi(x)<+\infty\end{aligned}$$

which is a contradiction. Indeed, since $u_{n,k}(x)\to+\infty$ and $u_{n,k}(y)\leqslant h_k(y)<+\infty$ for m_x-a.e $y\in W$, we have that, for n large enough,

$$-\mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))\geqslant c_p(u_{n,k}(x)-u_{n,k}(y))^{p-1}\xrightarrow{n}+\infty \quad \text{for } \nu\text{-a.e. } y\in W.$$

Therefore, $u_{n,k}(x) \to u_k(x) < +\infty$ for ν-a.e. $x \in \partial_m W$ (thus, in particular, u_k is measurable on $\partial_m W$), and we can use the dominated convergence theorem (we omit the details since they are similar to those in the proof of Lemma 5.7) to pass to the limit in (5.99)(b) for ν-a.e. $x \in \partial_m W$, obtaining:

$$-\int_W \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) = \varphi(x).$$

It remains to take limits as $n \to +\infty$ in (5.99)(a). Now,

$$\int_{W_m} \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y) = u_{n,k}(x) - \phi_{n,k}(x), \quad x \in W, \tag{5.100}$$

thus, by the monotonicity of $\{u_{n,k}\}_n$ (and (5.9)), we have that, for ν-a.e. $x \in W$,

$$\int_{W_m} \mathbf{a}_p(x,y,u_{n,k}(y)-u_k(x))dm_x(y) \leqslant u_{n,k}(x) - \phi_{n,k}(x).$$

Therefore, since the right hand side of this equation converges to a finite limit ν-a.e. in W as $n \to +\infty$, we get that, for ν-a.e. $x \in W$, $\{\mathbf{a}_p(x,\cdot,u_{n,k}(\cdot)-u_k(x))\}_n \subset L^\infty(W_m,m_x)$ is an m_x-a.e. nondecreasing sequence of functions with uniformly bounded m_x-integrals (recall that $u_{n,k} \in L^\infty(W_m,\nu)$, hence $u_{n,k} \in L^\infty(W_m,m_x)$ ν-a.e.). Consequently, by the monotone convergence theorem, we have that, for ν-a.e. $x \in W$, $a_p(x,\cdot,u_k(\cdot)-u_k(x)) \in L^1(W_m,m_x)$ and (recall (5.7))

$$\int_{W_m} \mathbf{a}_p(x,y,u_{n,k}(y)-u_k(x))dm_x(y) \xrightarrow{n} \int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y). \tag{5.101}$$

Moreover, by (5.11), we get that $|u_k|^{p-1} \in L^1(W_m,\nu)$.

On the other hand, by (5.100) and the monotonicity of $\{u_{n,k}\}_n$, we also get that, for ν-a.e. $x \in W$,

$$\int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_{n,k}(x))dm_x(y) \geqslant u_{n,k}(x) - \phi_{n,k}(x).$$

Hence, for ν-a.e. $x \in W$, $\{\mathbf{a}_p(x,\cdot,u_k(\cdot)-u_{n,k}(x))\}_n \subset L^1(W_m,m_x)$ is a nondecreasing sequence of functions with uniformly bounded m_x-integrals and we may again apply the monotone convergence theorem to obtain

$$\int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_{n,k}(x))dm_x(y) \xrightarrow{n} \int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y). \tag{5.102}$$

Consequently, since, for ν-a.e. $x \in W$,

$$\begin{aligned}\int_{W_m} &\mathbf{a}_p(x,y,u_{n,k}(y)-u_k(x))dm_x(y)\\ &\leqslant \int_{W_m} \mathbf{a}_p(x,y,u_{n,k}(y)-u_{n,k}(x))dm_x(y)\\ &\qquad\leqslant \int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_{n,k}(x))dm_x(y),\end{aligned}$$

by (5.101) and (5.102), we can also pass to the limit in (5.99)(b) to get that, for ν-a.e. $x \in W$,

$$u_k(x) - \int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) = \phi_k(x).$$

In particular, since u_k, $\phi_k \in L^1(W,\nu)$, the previous equation yields

$$\int_{W_m} \mathbf{a}_p(x,y,u_k(y)-u_k(x))dm_x(y) \in L^1(W,\nu),$$

thus $\mathbf{a}_p(x,y,u_k(y)-u_k(x)) \in L^1(W \times W_m, \nu \otimes m_x)$. Therefore, by the reversibility of ν with respect to m (and (5.8)) we get that $\mathbf{a}_p(x,y,u_k(y)-u_k(x)) \in L^1(Q_2,\nu \otimes m_x)$.

Finally, we take limits as $k \to +\infty$. We may repeat the previous reasoning to obtain that $u_k(x) \to u(x) > -\infty$ for ν-a.e. $x \in \partial_m W$. Consequently, we have that $u_k \to u$ in $L^p(W,\nu)$ and u_k tends to a measurable ν-a.e. finite function u in $\partial_m W$. Then, we apply the monotone convergence theorem in the same way to get:

$$\begin{cases} u(x) - \displaystyle\int_{W_m} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \phi(x), & x \in W \\ -\displaystyle\int_{W} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \varphi(x), & x \in \partial_m W, \end{cases} \tag{5.103}$$

where

$$\mathbf{a}_p(x,y,u(y)-u(x)) \in L^1(Q_2, \nu \otimes m_x).$$

By (5.11), we have that

$$c|u(y)-u(x)|^{p-1} \leqslant |\mathbf{a}_p(x,y,u(y)-u(x))|,$$

thus

$$|u(y)|^{p-1} \leqslant \tilde{C}\left(|\mathbf{a}_p(x,y,u(y)-u(x))| + |u(x)|^{p-1}\right)$$

for every x, $y \in W_m$ and some constant $\tilde{C}$. Therefore, since $m_x(W_m) = 1$ for $x \in W$,

$$\begin{aligned} &\int_{W_m} m_x(W)|u(x)|^{p-1} d\nu(x) = \int_{W_m}\int_W |u(x)|^{p-1} dm_x(y)d\nu(x) \\ &= \int_W \int_{W_m} |u(y)|^{p-1} dm_x(y) d\nu(x) \\ &\leqslant \tilde{C}\left(\int_{Q_2} |\mathbf{a}_p(x,y,u(y)-u(x))| d(\nu\otimes m_x)(x,y) + \int_W |u(x)|^{p-1} d\nu(x)\right) < +\infty. \end{aligned}$$

This implies, in particular, that

$$m_{(\cdot)}(W)|u|^{p-1} \in L^1(\partial_m W, \nu). \tag{5.104}$$

□

Remark 5.42 (**Regularity for $p \geqslant 2$**). In the context of Theorem 5.41, let us see that, for $p \geqslant 2$,

$$\mathbf{a}_p(x,y,u(y)-u(x)) \in L^{p'}(Q_2, \nu \otimes m_x) \tag{5.105}$$

and

$$m_{(\cdot)}(W)|u|^p \in L^1(\partial_m W, \nu). \tag{5.106}$$

Indeed, by (5.104), since $0 \leqslant m_{(\cdot)}(W) \leqslant 1$,

$$m_{(\cdot)}^{p-1}(W)|u|^{p-1} \in L^1(\partial_m W, \nu).$$

Therefore,

$$m_{(\cdot)}(W)u \in L^{p-1}(\partial_m W, \nu) \subset L^1(\partial_m W, \nu).$$

Hence, we get that

$$u\varphi = m_{(\cdot)}(W)u \frac{\varphi}{m_{(\cdot)}(W)} \in L^1(\partial_m W, \nu).$$

Now, multiplying the first equation in (5.103) by $T_k u(x)$, integrating over W and then integrating by parts,

$$\begin{aligned} &\int_W u T_k u d\nu + \frac{1}{2}\int_{Q_2} \mathbf{a}_p(x,y,u(y)-u(x))(T_k u(y) - T_k u(x)) d(\nu\otimes m_x)(x,y) \\ &= \int_W \phi T_k u d\nu + \int_{\partial_m W} \varphi T_k u d\nu. \end{aligned}$$

Hence, letting $k \to \infty$, by Fatou's lemma,

$$(x,y) \mapsto \mathbf{a}_p(x,y,u(y)-u(x))(u(y)-u(x)) \in L^1(Q_2, \nu \otimes m_x),$$

and this is equivalent, on account of (5.10) and (5.11), to (5.105). Moreover, in this situation, we can repeat the argument used to obtain (5.104) but using p instead of $p-1$, to get (5.106). Indeed,

$$\int_{W_m} m_x(W)|u(x)|^p d\nu(x) = \int_{W_m}\int_W |u(x)|^p dm_x(y)d\nu(x)$$
$$= \int_W \int_{W_m} |u(y)|^p dm_x(y)d\nu(x)$$
$$\leqslant \tilde{C}\left(\int_{Q_2} \mathbf{a}_p(x,y,u(y)-u(x))(u(y)-u(x))d(\nu\otimes m_x)(x,y) + \int_W |u(x)|^p d\nu(x)\right) < +\infty.$$

With respect to the domain of the operator $A^m_{\mathbf{a}_p,\varphi}$, we have the following result.

THEOREM 5.43. *Let $\varphi \in L^{m,\infty}(\partial_m W,\nu)$. Then, we have*

$$L^\infty(W,\nu) \subset D(A^m_{\mathbf{a}_p,\varphi})$$

and, consequently,

$$\overline{D(A^m_{\mathbf{a}_p,\varphi})}^{L^{p'}(W,\nu)} = L^{p'}(W,\nu).$$

PROOF. Take $u \in L^\infty(W,\nu)$. By Remark 5.40.*1* & *3*, there exists an extension of u to $\partial_m W$ (which we continue to denote by u) satisfying

$$-\int_W \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \varphi(x), \quad x \in \partial_m W,$$

and, moreover,

$$u \in L^\infty(\partial_m W,\nu).$$

Therefore, for $x \in W$,

$$\phi(x) = -\int_{W_m} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)$$

defines a function in $L^1(W,\nu)$, and we have that

$$u \in D(A^m_{\mathbf{a}_p,\varphi}).$$

□

THEOREM 5.44. *Let $p \geqslant 2$ and assume that $\varphi \in L^{m,\infty}(\partial_m W,\nu)$. Then*

$$L^{p-1}(W,\nu) \subset D(A^m_{\mathbf{a}_p,\varphi}).$$

PROOF. Suppose that $p > 2$ (the case $p = 2$ follows by a similar, but simpler, argument).

Given $u \in L^{p-1}(W,\nu)$, denote again by u the unique extension of u to the boundary $\partial_m W$ satisfying

$$-\int_W \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) = \varphi(x),\ x \in \partial_m W.$$

Then, for $x \in \partial_m W$, we have

$$u(x)\int_{\{y\in W:|u(y)-u(x)|>1\}} \frac{\mathbf{a}_p(x,y,u(y)-u(x))}{u(y)-u(x)} dm_x(y)$$
$$= \varphi(x) + \int_{\{y\in W:|u(y)-u(x)|\leqslant 1\}} \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)$$
$$+ \int_{\{y\in W:|u(y)-u(x)|>1\}} \frac{\mathbf{a}_p(x,y,u(y)-u(x))}{u(y)-u(x)} u(y)dm_x(y),$$

and, consequently, by (5.10) and taking into account (5.11),

$$\begin{aligned}&|u(x)|\int_{\{y\in W:|u(y)-u(x)|>1\}}\frac{\mathbf{a}_p(x,y,u(y)-u(x))}{u(y)-u(x)}dm_x(y)\\&\leqslant|\varphi(x)|+2C+2C\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-2}|u(y)|dm_x(y).\end{aligned}\tag{5.107}$$

Now, by (5.11),

$$\begin{aligned}&c\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-2}dm_x(y)\\&\leqslant\int_{\{y\in W:|u(y)-u(x)|>1\}}\frac{\mathbf{a}_p(x,y,u(y)-u(x))}{u(y)-u(x)}dm_x(y).\end{aligned}$$

Hence, by (5.107), we get

$$\begin{aligned}&c|u(x)|\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-2}dm_x(y)\\&\leqslant|\varphi(x)|+2C+2C\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-2}|u(y)|dm_x(y).\end{aligned}\tag{5.108}$$

Let us now see that

$$\mathbf{\Theta}:=\int_{\partial_m W}\int_W|u(y)-u(x)|^{p-1}dm_x(y)d\nu(x)<+\infty.\tag{5.109}$$

By (5.108) and the reversibility of ν, we have

$$\begin{aligned}&\int_{\partial_m W}\int_W|u(y)-u(x)|^{p-1}dm_x(y)d\nu(x)\\&=\int_{\partial_m W}\int_{\{y\in W:|u(y)-u(x)|\leqslant 1\}}|u(y)-u(x)|^{p-1}dm_x(y)d\nu(x)\\&\quad+\int_{\partial_m W}\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-1}dm_x(y)d\nu(x)\\&\leqslant\nu(W_m)+\int_{\partial_m W}\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-2}|u(y)|dm_x(y)d\nu(x)\\&\quad+\int_{\partial_m W}|u(x)|\int_{\{y\in W:|u(y)-u(x)|>1\}}|u(y)-u(x)|^{p-2}dm_x(y)\nu(x)\\&\leqslant\left(1+2\frac{C}{c}\right)\nu(W_m)+\frac{1}{c}\int_{\partial_m W}|\varphi(x)|d\nu(x)\\&\quad+\left(1+2\frac{C}{c}\right)\int_{\partial_m W}\int_W|u(y)-u(x)|^{p-2}|u(y)|dm_x(y)d\nu(x).\end{aligned}$$

Now, by using Hölder's inequality, with exponents $\frac{p-1}{p-2}$ and $p-1$, and the reversibility of ν, we get

$$\begin{aligned}&\int_{\partial_m W}\int_W|u(y)-u(x)|^{p-2}|u(y)|dm_x(y)d\nu(x)\\&\leqslant\left(\int_{\partial_m W}\int_W|u(y)-u(x)|^{p-1}dm_x(y)d\nu(x)\right)^{\frac{p-2}{p-1}}\left(\int_{\partial_m W}\int_W|u(y)|^{p-1}dm_x(y)d\nu(x)\right)^{\frac{1}{p-1}}\\&\leqslant\left(\int_{\partial_m W}\int_W|u(y)-u(x)|^{p-1}dm_x(y)d\nu(x)\right)^{\frac{p-2}{p-1}}\left(\int_W|u(x)|^{p-1}d\nu(x)\right)^{\frac{1}{p-1}}.\end{aligned}$$

Therefore,

$$\mathbf{\Theta} \leqslant \left(1+2\frac{C}{c}\right)\nu(W_m) + \frac{1}{c}\int_{\partial_m W}|\varphi(x)|d\nu(x) + \left(1+2\frac{C}{c}\right)\mathbf{\Theta}^{\frac{p-2}{p-1}}\left(\int_W |u(x)|^{p-1}d\nu(x)\right)^{\frac{1}{p-1}}$$

and, consequently, $\mathbf{\Theta}$ is finite. Observe that an explicit upper bound, depending on $\|\varphi\|_{L^1(W,\nu)}$ and $\|u\|_{L^{p-1}(W,\nu)}$, can be stated.

Furthermore, we obtain the following regularity of u on the boundary:

$$\int_{\partial_m W} m_x(W)|u(x)|^{p-1}d\nu(x) < +\infty. \tag{5.110}$$

Indeed, since $|u(y)|^{p-1} \leqslant \tilde{C}\left(|u(y)-u(x)|^{p-1} + |u(x)|^{p-1}\right)$ for some constant $\tilde{C}$ and every x, $y \in W_m$, we have that

$$\begin{aligned}
&\int_{W_m} m_x(W)|u(x)|^{p-1}d\nu(x) = \int_{W_m}\int_W |u(x)|^{p-1}dm_x(y)d\nu(x)\\
&= \int_W\int_{W_m}|u(y)|^{p-1}dm_x(y)d\nu(x)\\
&\leqslant \tilde{C}\left(\int_{Q_2}|u(y)-u(x)|^{p-1}d(\nu\otimes m_x)(x,y) + \int_W |u(x)|^{p-1}d\nu(x)\right),
\end{aligned}$$

thus (5.110) holds.

Let us finally see that, for $x \in W$,

$$\phi(x) := -\int_{W_m}\mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)$$

belongs to $L^1(W,\nu)$. Indeed,

$$-\int_{W_m}\mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)$$

$$= -\int_W \mathbf{a}_p(x,y,u(y)-u(x))dm_x(y) - \int_{\partial_m W}\mathbf{a}_p(x,y,u(y)-u(x))dm_x(y).$$

Now, the first summand on the right hand side belongs to $L^1(W,\nu)$. Let us see that the second one also belongs to $L^1(W,\nu)$. Since

$$\begin{aligned}
&\int_W\left|\int_{\partial_m W}\mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)\right|d\nu(x)\\
&\leqslant \int_W\int_{\partial_m W}|\mathbf{a}_p(x,y,u(y)-u(x))|dm_x(y)d\nu(x)\\
&\leqslant \int_W\int_{\partial_m W} C(1+|u(y)-u(x)|^{p-1})dm_x(y)d\nu(x)\\
&\leqslant C\nu(W_m) + C\int_W\int_{\partial_m W}|u(y)-u(x)|^{p-2}|u(y)|dm_x(y)d\nu(x)\\
&\quad + C\int_W\int_{\partial_m W}|u(y)-u(x)|^{p-2}dm_x(y)|u(x)|d\nu(x),
\end{aligned}$$

we have that, by Tonelli-Hobson's theorem, $x \mapsto -\int_{\partial_m W}\mathbf{a}_p(x,y,u(y)-u(x))dm_x(y)$ belongs to $L^1(W,\nu)$ if the following functions belong to $L^1(W,\nu)$:

$$x \longmapsto \int_{\partial_m W}|u(y)-u(x)|^{p-2}u(y)dm_x(y)$$

and

$$x \longmapsto \int_{\partial_m W} |u(y)-u(x)|^{p-2} dm_x(y) u(x).$$

With regard to the first function, by Hölder's inequality and the reversibility of ν with respect to m, we have that

$$\begin{aligned}
&\int_W \left| \int_{\partial_m W} |u(y)-u(x)|^{p-2} u(y) dm_x(y) \right| d\nu(x) \\
&\leqslant \int_W \int_{\partial_m W} |u(y)-u(x)|^{p-2} |u(y)| dm_x(y) d\nu(x) \\
&\leqslant \left(\int_W \int_{\partial_m W} |u(y)-u(x)|^{p-1} dm_x(y) d\nu(x) \right)^{\frac{p-2}{p-1}} \left(\int_W \int_{\partial_m W} |u(y)|^{p-1} dm_x(y) d\nu(x) \right)^{\frac{1}{p-1}} \\
&= \left(\int_{\partial_m W} \int_W |u(y)-u(x)|^{p-1} dm_x(y) d\nu(x) \right)^{\frac{p-2}{p-1}} \left(\int_{\partial_m W} \int_W |u(x)|^{p-1} dm_x(y) d\nu(x) \right)^{\frac{1}{p-1}} \\
&= \left(\int_{\partial_m W} \int_W |u(y)-u(x)|^{p-1} dm_x(y) d\nu(x) \right)^{\frac{p-2}{p-1}} \left(\int_{\partial_m W} m_x(W) |u(x)|^{p-1} d\nu(x) \right)^{\frac{1}{p-1}}
\end{aligned}$$

which is finite by (5.109) and (5.110). The second one also belongs to $L^1(W,\nu)$ since, by (5.109) (using the reversibility of ν with respect to m), $x \mapsto \int_{\partial_m W} |u(y)-u(x)|^{p-2} dm_x(y) \in L^{(p-1)'}(W,\nu)$, and $u \in L^{p-1}(W,\nu)$. □

The following theorem is a consequence of the above results thanks to Theorems A.25, A.45 and A.48.

Theorem 5.45. *Let $\varphi \in L^{m,\infty}(\partial_m W,\nu)$ and $T>0$. For any $u_0 \in \overline{D(A^m_{\mathbf{a}_p,\varphi})}^{L^{p'}(W,\nu)} = L^{p'}(W,\nu)$ there exists a unique mild solution $u(t,x)$ of Problem (5.93). Moreover, for any $q \geqslant p'$ and $u_{0i} \in L^q(W,\nu)$, $i=1,2$, we have the following contraction principle for the corresponding mild solutions u_i:*

$$\|(u_1(t,.)-u_2(t,.))^+\|_{L^q(W,\nu)} \leqslant \|(u_{0,1}-u_{0,2})^+\|_{L^q(W,\nu)} \quad \text{for any } 0 \leqslant t < T.$$

If $u_0 \in D(A^m_{\mathbf{a}_p,\varphi})$, then the mild solution is a strong solution. In particular, if $u_0 \in L^\infty(W,\nu)$, Problem (5.93) has a unique strong solution. For $p \geqslant 2$ this is true for data in $L^{p-1}(W,\nu)$.

APPENDIX A

Nonlinear semigroups

This appendix is part of [**12**, Appendix A].

A.1. Introduction

In this appendix we outline some of the main points of the theory of nonlinear semigroups and evolution equations governed by accretive operators. We refer the reader to [**23**], [**28**], [**32**], [**43**], [**67**], [**68**], [**69**]. *Our main objective will be the study of evolution problems of the form:*

$$\text{(A.1)}\qquad \begin{cases} u'(t) + Au(t) = f(t) & \text{on } (0,T), \\ u(0) = u_0, \end{cases}$$

where X is a Banach space, $f : (0,T) \to X$ and $A : D(A) \to X$ is an operator.

Let us give one example of how to write a PDE problem as a problem in the form (A.1).

Example A.1. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the classical initial-boundary problem for the heat equation, that is, the problem

$$\text{(A.2)}\qquad \begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \Delta w(x,t) & \text{in } \Omega \times (0,\infty), \\ w(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ w(x,0) = f(x) & \text{in } \Omega. \end{cases}$$

Write $u(t) = w(\cdot,t)$, regarded as a function of x, and take X to be a space of functions on Ω, for example, $X = L^p(\Omega)$ for some $p \geqslant 1$ or $X = C(\overline{\Omega})$. Suppose we are in this last case. Let A be the operator with domain

$$D(A) := \left\{ v \in C(\overline{\Omega}) \ : \ \Delta v \in C(\overline{\Omega}) \ \text{ and } \ v(x) = 0 \ \ \forall\, x \in \partial\Omega \right\}$$

and defined by $Av := -\Delta v$, for $v \in D(A)$. Then, we can write the problem (A.2) in the form (A.1). Note that the boundary condition of (A.2) is absorbed into the domain of the operator A and into the requirement that $u(t) \in D(A)$ for all $t \geqslant 0$.

A.2. Abstract Cauchy problems

From now on, X will be a real Banach space with norm denoted by $\| \ \|$ and dual X^*.

We will use multivalued nonlinear operators, not only because they permit the obtainment of a coherent theory, but also because it is often necessary in applications. So let us recall some notations and basic facts concerning multivalued operators.

A mapping $A : X \to 2^X$ from X into 2^X (the collection of subsets of X) will be called an operator in X. For $x \in X$, Ax denotes the value of A at x, $D(A) := \{x \in X \, : \, Ax \neq \varnothing\}$ will be called the effective domain of A, and $R(A) := \bigcup \{Ax \, : \, x \in D(A)\}$ its range.

If A is an operator in X, it determines the subset

$$G(A) := \{(x,y) \in X \times X \, : \, y \in Ax\},$$

called the graph of A. Conversely, a subset G of $X \times X$ determines a unique operator A whose graph is G, i.e., the operator A is given by $Ax := \{y \, : \, (x,y) \in G\}$. Whenever it is convenient we will identify an operator with its graph.

Given two operators A and B in X and $\alpha \in \mathbb{R}$, we define new operators $A+B$, αA and A^{-1} as follows:

$$(A+B)x := Ax + Bx,$$

$$(\alpha A)x := \alpha(Ax),$$

$$A^{-1}x := \{y \in X \,:\, x \in Ay\}.$$

The closure of the operator A, denoted by $\overline{A}$, is defined to be the closure of the graph of A in $X \times X$, that is:

$$y \in \overline{A}x \Leftrightarrow \exists y_n \in Ax_n \,:\, x_n \to x,\ y_n \to y.$$

Before proceeding we fix some notation: By $L^1(a,b;X)$ we denote the vector space of all Bochner integrable functions $f : [a,b] \to X$ with respect to the Lebesgue measure (i.e., the strong measurable functions f such that $\int_a^b \|f(t)\|\ dt < +\infty$). If I is an interval in $\mathbb{R}$, $L^1_{loc}(I;X)$ is the space of those functions $f : I \to X$ which are Bochner integrable on compact subintervals of I. As in the case of real functions, if $f \in L^1(a,b;X)$, for almost all $t \in (a,b)$ one has

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{t-h}^{t+h} \|f(s) - f(t)\| ds = 0. \tag{A.3}$$

If (A.3) holds t is called a Lebesgue point of f.

The space $W^{1,1}(a,b;X)$ consists of those functions f which have the form

$$f(t) = f(0) + \int_0^t h(s)\ ds \tag{A.4}$$

for some $h \in L^1(a,b;X)$. It is well known that $W^{1,1}(a,b;X)$ consists of exactly those absolutely continuous functions $f : [a,b] \to X$ which are differentiable a.e. on $[a,b]$ and if (A.4) holds, then $f'(t) = h(t)$ a.e.

In a general Banach space X, the absolute continuity of a function $f : [a,b] \to X$ does not imply the existence of $f'(t)$ almost everywhere. When this happens it is said that the Banach space X has the Radon-Nikodym property. For instance, every reflexive Banach space has the Radon-Nikodym property. However, there are important Banach spaces like $L^1(\Omega)$, $L^\infty(\Omega)$ or $C(\overline{\Omega})$ without the Radon-Nikodym property.

As we mentioned before, our aim is to study evolution problems of the form:

$$\begin{cases} u'(t) + Au(t) \ni f(t), & t \in (0,T), \\ u(0) = x, \end{cases} \tag{A.5}$$

where $f : (0,T) \to X$ and A is an operator in X. A problem of the form (A.5) is called an abstract Cauchy problem, and it will be denoted by (CP)$_{x,f}$. In the homogeneous case, that is, for $f = 0$, we will write (CP)$_x$ instead of (CP)$_{x,0}$.

Definition A.2. A function u is called a *strong solution* of (CP)$_{x,f}$ if

$$\begin{cases} u \in C([0,T];X) \cap W^{1,1}_{loc}((0,T);X), \\ u' + Au(t) \ni f(t) \ \text{ for a.e. } t \in (0,T), \\ u(0) = x. \end{cases}$$

*However, the more adequate notion of solution for (CP)$_{x,f}$ in general Banach spaces is the concept of mild solution, introduced by M.G. Crandall and T.M. Liggett in [**69**] and Ph. Bénilan in [**28**], which is studied in the next section.*

A.3. Mild solutions

Let A be an operator in X and $f \in L^1(a,b;X)$. Roughly speaking a mild solution of the problem

$$u' + Au \ni f \qquad \text{on} \quad [a,b]$$

is a continuous function $u \in C([a,b];X)$ which is the uniform limit of solutions of time-discretized problems, given by the implicit Euler scheme of the form

$$\frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}} + Av(t_i) \ni f_i,$$

where f_i are approximations of f when $|t_i - t_{i-1}| \to 0$. Therefore, the underlying idea of the notion of mild solution is simple and from the point of view of numerical analysis, even classical. Formally, the definition is as follows.

Definition A.3. Let $\varepsilon > 0$. An *ε-discretization* of $u' + Au \ni f$ on $[a,b]$ consists of a partition $t_0 < t_1 < \cdots < t_N$ and a finite sequence $f_1, f_2, \ldots, f_N$ of elements of X such that,

$$a \leqslant t_0 < t_1 < \cdots < t_N \leqslant b, \quad \text{with}$$

$$t_i - t_{i-1} \leqslant \varepsilon, \; i = 1, \ldots, N, \quad t_0 - a \leqslant \varepsilon \quad \text{and} \quad b - t_N \leqslant \varepsilon.$$

and

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\| \, ds \leqslant \varepsilon.$$

We will denote this discretization by $D_A(t_0, \ldots, t_N; f_1, \ldots, f_N)$.

A *solution of the discretization* $D_A(t_0, \ldots, t_N; f_1, \ldots, f_N)$ is a piecewise constant function $v : [t_0, t_N] \to X$ whose values $v(t_0) = v_0$, $v(t) = v_i$ for $t \in]t_{i-1}, t_i]$, $i = 1, \ldots, N$, satisfy

$$\frac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i \ni f_i, \qquad i = 1, \ldots, N.$$

A *mild solution* of $u' + Au \ni f$ on $[a,b]$ is a continuous function $u \in C([a,b];X)$ such that, for each $\varepsilon > 0$ there is $D_A(t_0, \ldots, t_N; f_1, \ldots, f_N)$ an ε-discretization of $u' + Au \ni f$ on $[a,b]$ which has a solution v satisfying

$$\|u(t) - v(t)\| \leqslant \varepsilon \qquad \text{for} \quad t_0 \leqslant t \leqslant t_N.$$

It is easy to see that if u is a mild solution of $u' + Au \ni f$ on $[a,b]$ and $[c,d] \subset [a,b]$, then $u \llcorner [c,d]$ is a mild solution of $u' + Au \ni f$ on $[c,d]$. Therefore, the following definition is consistent.

Definition A.4. Let I an interval of $\mathbb{R}$, and $f \in L^1_{loc}(I;X)$. A *mild solution* of $u' + Au \ni f$ on I is a function $u \in C(I;X)$ whose restriction to each compact subinterval $[a,b]$ of I is a mild solution of $u' + Au \ni f$ on $[a,b]$.

In the next result we will see that mild solutions generalize the concept of strong solutions.

Theorem A.5. *Let $f \in L^1_{loc}(I;X)$ and u be a strong solution of $u' + Au \ni f$ on I. Then u is a mild solution of $u' + Au \ni f$ on I.*

The heart of the proof of the above theorem is the following result concerning the approximation of Bochner integrals by Riemann sums in a strong sense.

Lemma A.6. *Let Y be a Banach space, $g \in L^1(a,b;Y)$ and K be a subset of $[a,b]$ such that $[a,b] \backslash K$ has measure zero. Then, given $\delta > 0$, there is a partition $a = t_0 < t_1 < \cdots < t_N \leqslant b$ satisfying:*

$$t_i \in K \text{ and } t_i \text{ is a Lebesgue point of } g \text{ for all } i = 1, \ldots, N,$$

$$b - t_N < \delta \quad \text{and} \quad t_i - t_{i-1} < \delta, \quad i = 1, \ldots, N,$$

and

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|g(t) - g(t_i)\| \, dt < \delta.$$

The converse of Theorem A.5 is false; mild solutions need not be strong solutions. The next result collects some of the properties of mild solutions.

THEOREM A.7. *Let A be an operator in X and $f \in L^1_{loc}(I;X)$. Then:*

(i) If u is a mild solution of $u' + Au \ni f$ on I, then $u(t) \in \overline{D(A)}$ for all $t \in I$.

(ii) Let I_1, I_2 be subintervals of I with $I \subset \overline{I_1 \cup I_2}$. If $u \in C(I;X)$ is a mild solution of $u' + Au \ni f$ on I_1 and on I_2, then u is a mild solution of $u' + Au \ni f$ on I.

(iii) Let $\overline{A}$ be the closure of the operator A. Then, u is a mild solution of $u' + Au \ni f$ on I if, and only if, u is a mild solution of $u' + \overline{A}u \ni f$ on I.

(iv) Let $\{u_n\} \subset C(I;X)$, $\{f_n\} \subset L^1_{loc}(I;X)$ and u_n be a mild solution of $u_n' + Au_n \ni f_n$ on I. Assume $u \in C(I;X)$, $f \in L^1_{loc}(I;X)$ and for each compact subinterval $[a,b]$ of I,

$$\lim_{n\to\infty} \left(\int_a^b \|f_n(t) - f(t)\| \, dt + \sup_{a \leqslant t \leqslant b} \|u_n(t) - u(t)\| \right) = 0,$$

then u is a mild solution of $u' + Au \ni f$ on I.

DEFINITION A.8. Let D be a subset of X. A family of mappings $S(t) : D \to D$ $(t \geqslant 0)$ satisfying:

$$S(t+s)x = S(t)S(s)x \qquad \text{for all } \ t, s \geqslant 0, \ x \in D,$$

$$\lim_{t\to 0} S(t)x = x \qquad \text{for } \ x \in D,$$

is called a *strongly continuous semigroup* on D.

One may now associate with every operator A in X a strongly continuous semigroup $(S^A(t))_{t \geqslant 0}$ by the following definition:

$$D\left(S^A\right) := \{x \in X \ : \ \exists! \text{ mild solution } u_x \text{ of } u' + Au \ni 0 \text{ on } (0, +\infty) \text{ with } u_x(0) = x\}.$$

For $t \geqslant 0$ and $x \in D(S^A)$, we set

$$S^A(t)x := u_x(t).$$

It is an immediate consequence of the properties of mild solutions that, in fact, $(S^A(t))_{t \geqslant 0}$ is a strongly continuous semigroup on $D(S^A)$.

In the linear case, that is, if $S(t) \in \mathcal{L}(X)$, the strongly continuous semigroups are called C_0-semigroups. In this situation, each C_0-semigroup $(S(t))_{t \geqslant 0}$ has associated its infinitesimal generator B defined by

$$Bx := \lim_{t\to 0} \frac{S(t)x - x}{t} \qquad \text{for } \ x \in D(B)$$

and

$$D(B) := \left\{ x \in X \ : \ \exists \lim_{t\to 0} \frac{S(t)x - x}{t} \right\}.$$

In the linear case it is well known that:

"$-A$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t \geqslant 0}$ of bounded linear operators on X if, and only if, A is linear, closed and $D(S^A) = X$, and then $S^A(t) = S(t)$ for all $t \geqslant 0$."

This motivates the development of a nonlinear semigroup theory analogous to the classical linear one. We will see that in the nonlinear case the situation is very different to the linear one, and has more difficulties.

A.4. Accretive operators

We are going to introduce now the class of operators for which we can obtain existence and uniqueness results of mild solutions.

The existence of mild solutions requires, as we pointed out before, the existence of solutions of discretized equations of the form

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i, \qquad i = 1, \ldots, N$$

or, equivalently,

$$x_i + (t_i - t_{i-1})Ax_i \ni (t_i - t_{i-1})f_i + x_{i-1}, \qquad i = 1, \ldots, N. \tag{A.6}$$

Then, to solve (A.6) we need the inverse of the operator $(I + \lambda A)$ to be a singlevalued operator. Operators satisfying this property are the following:

Definition A.9. An operator A in X is *accretive* if

$$\|x - \hat{x}\| \leqslant \|x - \hat{x} + \lambda(y - \hat{y})\|, \quad \text{whenever} \;\; \lambda > 0 \;\; \text{and} \;\; (x, y), (\hat{x}, \hat{y}) \in A.$$

Note that A is accretive if and only if for $\lambda > 0$ and $z \in X$, $x + \lambda y = z$ has at most one solution $(x, y) \in A$ and the relations $x + \lambda y = z$, $(x, y) \in A$, $\hat{x} + \lambda\hat{y} = \hat{z}$, $(\hat{x}, \hat{y}) \in A$ imply

$$\|x - \hat{x}\| = \left\|(I + \lambda A)^{-1} z - (I + \lambda A)^{-1} \hat{z}\right\| \leqslant \|z - \hat{z}\|.$$

Therefore, we have:

"A is accretive if, and only if, $(I + \lambda A)^{-1}$ is a singlevalued nonexpansive map for $\lambda \geqslant 0$"

If A is accretive, we denote $J_\lambda^A := (I + \lambda A)^{-1}$ and we call J_λ^A the resolvent of A. Note that $D(J_\lambda^A) = R(I + \lambda A)$.

It is easy to see that if β is an operator in $\mathbb{R}$, then β is accretive if, and only if, $(y - \hat{y})(x - \hat{x}) \geqslant 0$ for all $(x, y), (\hat{x}, \hat{y}) \in \beta$. Thus, if β is univalued, then β is accretive if, and only if, β is nondecreasing. The following operators are examples of accretive operators in $\mathbb{R}$:

$$sign_0(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r = 0, \\ 1 & \text{if } r > 0, \end{cases}$$

and

$$\text{sign}(r) = \begin{cases} -1 & \text{if } r < 0, \\ [-1, 1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases}$$

In order to verify the accretivity of a given operator, it is useful to take into account alternative characterizations of this property. To do that we need to introduce the bracket and the duality map.

For each $\lambda \neq 0$ define $[\cdot,\cdot]_\lambda : X \times X \to \mathbb{R}$ by

$$[x, y]_\lambda := \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

For fixed $(x, y) \in X \times X$, $\lambda \mapsto [x, y]_\lambda$ is nondecreasing for $\lambda > 0$. Indeed, if $\lambda \geqslant \mu > 0$ then

$$\|x + \mu y\| = \left\|\left(1 - \frac{\mu}{\lambda}\right) x + \frac{\mu}{\lambda}(x + \lambda y)\right\| \leqslant \left(1 - \frac{\mu}{\lambda}\right)\|x\| + \frac{\mu}{\lambda}\|x + \lambda y\|,$$

it follows this that $[x, y]_\mu \leqslant [x, y]_\lambda$. Therefore, for every $(x, y) \in X \times X$ we can define:

$$[x, y] := \lim_{\lambda \downarrow 0} [x, y]_\lambda = \inf_{\lambda > 0} [x, y]_\lambda.$$

The number $[x, y]$ is the right-hand derivative of the norm of x in the direction y. In the next proposition we collect some of the useful properties of the bracket $[\cdot,\cdot]$.

Proposition A.10. *If $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$, then* *(i)* $[\cdot,\cdot] : X \times X \to \mathbb{R}$ *is upper-semi-continuous*

(ii) $[\alpha x, \beta y] = |\beta|[x,y]$ *if* $\alpha \cdot \beta > 0$

(iii) $[x, \alpha x + y] = \alpha\|x\| + [x,y]$

(iv) $[x,y] \geqslant 0$ *if and only if* $\|x + \lambda y\| \geqslant \|x\|$ *for* $\lambda \geqslant 0$

(v) $|[x,y]| \leqslant \|y\|$ *and* $[0,y] = \|y\|$

(vi) $[x,y] \geqslant -[x,-y]$

(vii) $[x, y+z] \leqslant [x,y] + [x,z]$

(viii) *Let* $u :]a,b[\to \mathbb{R}$ *and* $t_0 \in]a,b[$*, such that* u *is differentiable at* t_0*, then* $t \mapsto \|u(t)\|$ *is differentiable at* t_0 *if, and only if,* $[u(t_0), u'(t_0)] = -[u(t_0), -u'(t_0)]$*. In this case*

$$\frac{d}{dt}\|u(t)\|_{|t=t_0} = \left[u(t_0), u'(t_0)\right].$$

As a consequence of (iv) of the above proposition we obtain the following characterization of accretive operators.

Corollary A.11. *An operator* A *in* X *is accretive if, and only if,*

$$[x - \hat{x}, y - \hat{y}] \geqslant 0$$

whenever $(x,y), (\hat{x},\hat{y}) \in A$*.*

In some specific Banach spaces the bracket $[\cdot,\cdot]$ *can be computed explicitly. We give some examples.*

Example A.12. Suppose $(H, (\ |\ ))$ is a Hilbert space. Then, for $x, y \in H$,

$$(\|x + \lambda y\| - \|x\|)(\|x + \lambda y\| + \|x\|) = \|x + \lambda y\|^2 - \|x\|^2 = 2\lambda(x|y) + \lambda^2\|y\|^2.$$

Dividing this equality by λ yields

$$(\|x + \lambda y\| + \|x\|)[x,y]_\lambda = 2(x|y) + \lambda\|y\|^2,$$

so we find

$$\|x\|[x,y] = (x|y).$$

Then, by Corollary A.11, it follows that an operator A in H is accretive if, and only if,

$$(x - \hat{x}|y - \hat{y}) \geqslant 0 \quad \text{for all } (x,y), (\hat{x},\hat{y}) \in A. \tag{A.7}$$

An operator in a Hilbert space satisfying (A.7) is called *monotone* and therefore in Hilbert spaces monotone and accretive operators coincide.

Example A.13. Let $X = L^p(\Omega)$ where $1 < p < \infty$. By the convexity of the map $t \mapsto |t|^p$, and applying the Dominated Convergence Theorem, it is easy to see that

$$[f,g] = \|f\|_p^{1-p} \int_\Omega g|f|^{p-1}\,\mathrm{sign}_0(f).$$

In the case $p = 1$, i.e., for $X = L^1(\Omega)$, we have

$$[f,g] = \int_\Omega g\ \mathrm{sing}_0(f) + \int_{\{f=0\}} |g|.$$

The formulas for the bracket given in the above examples are very useful to prove that some operators are accretive. Another useful tool to study the accretivity of some operators is the duality map $\mathcal{J} : X \to 2^{X^*}$*, defined as*

$$\mathcal{J}(x) := \{x^* \in X^* : \|x^*\| \leqslant 1,\ \langle x, x^*\rangle = \|x\|\}.$$

By Hanh-Banach's Theorem, we have $\mathcal{J}(x) \neq \emptyset$ *for every* $x \in X$*.*

Given $x^* \in \mathcal{J}(x)$*, since* $\|x^*\| \leqslant 1$*, we have*

$$|\langle x^*, x + \lambda y\rangle| \leqslant \|x + \lambda y\|$$

and

$$\langle x^*, y\rangle = \frac{1}{\lambda}\left(\langle x^*, x+\lambda y\rangle - \|x\|\right) \leqslant [x,y]_\lambda.$$

Hence

$$\langle x^*, y\rangle \leqslant [x,y] \qquad \forall\, x^* \in \mathcal{J}(x).$$

On the other hand, if $V = LIN\{x,y\}$ *and we define* $\xi^* \in V^*$ *by*

$$\langle \xi^*, \alpha x + \beta y\rangle := \alpha\|x\| + \beta[x,y],$$

then, by Hanh-Banach Theorem, there exists $x^* \in X^*$ *such that* $x^*|_V = \xi^*$*, so*

$$\langle x^*, x\rangle = -\|x\| \quad \text{and} \quad \langle x^*, y\rangle = [x,y].$$

Moreover, it is not so difficult to see that $\|x^*\| \leqslant 1$*, thus* $x^* \in \mathcal{J}(x)$*. Consequently, we have the following result.*

PROPOSITION A.14. *For* $x, y \in X$

$$[x,y] = \max_{x^*\in\mathcal{J}(x)} \langle x^*, y\rangle.$$

As a consequence of the above proposition and Corollary A.11, we have the following characterization of accretive operators.

COROLLARY A.15. *An operator* A *in* X *is accretive if, and only if, whenever* $(x,y),(\hat{x},\hat{y}) \in A$*, there exists* $x^* \in \mathcal{J}(x-\hat{x})$ *such that*

$$\langle x^*, y-\hat{y}\rangle \geqslant 0.$$

EXAMPLE A.16. Let $X = L^p(\Omega)$ where $1 < p < \infty$, then by Hölder inequality we have

$$\mathcal{J}(f) = \text{sign}_0(f)|f|^{p-1}\|f\|_p^{1-p}.$$

In $L^1(\Omega)$, we have

$$\mathcal{J}(f) = \text{sign}(f) = \left\{g \in L^\infty(\Omega) \ : \ |g| \leqslant 1, \quad gf = |f| \ \ \text{a.e.}\right\}.$$

Given $w \in \mathbb{R}$*, we define:*

$$\mathcal{A}(w) := \left\{A \subset X \times X \ : \ A + wI \ \ \text{is accretive}\right\}.$$

PROPOSITION A.17. *Let* A *be an operator in* X*. The following statements are equivalent:*

(i) $A \in \mathcal{A}(w)$.

(ii) $(1-\lambda w)\|x-\hat{x}\| \leqslant \|x-\hat{x}+\lambda(y-\hat{y})\| \ \ \forall\,\lambda < 0,\ (x,y),(\hat{x},\hat{y}) \in A.$

(iii) $[x-\hat{x}, y-\hat{y}] + w\|x-\hat{x}\| \geqslant 0.$

(iv) *For* $\lambda > 0$*,* $\lambda w < 1$*,* $J_\lambda^A = (I+\lambda A)^{-1}$ *is Lipschitz continuous with Lipschitz constant* $\dfrac{1}{1-\lambda w}$.

(v) *For* $(x,y),(\hat{x},\hat{y}) \in A$*, there exists* $x^* \in \mathcal{J}(x-\hat{x})$ *such that*

$$\langle x^*, y-\hat{y}\rangle + w\|x-\hat{x}\| \geqslant 0.$$

We have that accretivity implies uniqueness of the strong solutions. More precisely, we have the following result.

THEOREM A.18. *Let* $f, \hat{f} \in L^1(0,T;X)$*,* $A \in \mathcal{A}(w)$ *and let* u *and* $\hat{u}$ *be strong solutions of* $u' + Au \ni f$ *and* $\hat{u}' + A\hat{u} \ni \hat{f}$*, respectively, on* $[0,T]$*. Then,*

$$\begin{aligned}\|u(t)-\hat{u}(t)\| &\leqslant e^{wt}\|u(0)-\hat{u}(0)\| + \int_0^t e^{w(t-s)}\left[u(s)-\hat{u}(s), f(s)-\hat{f}(s)\right]\,ds \\ &\leqslant e^{wt}\|u(0)-\hat{u}(0)\| + \int_0^t e^{w(t-s)}\|f(s)-\hat{f}(s)\|\,ds\end{aligned}$$

for $t \in [0, T]$.

In particular, the strong solutions of $(CP)_{x,f}$ *are unique.*

We have seen that the accretivity of the operator A *implies uniqueness of the solution* x_i *of the discretized equation*

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i, \qquad i = 1, \ldots, N,$$

which, if they exist, are given by

$$x_i = J^A_{(t_i - t_{i-1})}\left((t_i - t_{i-1})f_i + x_{i-1}\right) \qquad i = 1, \ldots, N.$$

This formula indicates that apart from accretivity one should expect a range condition (i.e., a condition on $R(I + \lambda A) = D(J^A_\lambda)$*) to hold in order to get existence of solution as well. This motivates the following definition.*

Definition A.19. An operator A is called m-*accretive* in X if, and only if, A is accretive and $R(I + \lambda A) = X$ for all $\lambda > 0$.

Applying the Banach Fixed Point Theorem it is not hard to see that if A *is accretive then* A *is* m*-accretive if there exists* $\lambda > 0$ *such that* $R(I + \lambda A) = X$*. Moreover, if* A *is accretive, we have that (see also* [**32**, Proposition 2.18]*)*

$$\overline{R(I + A)} \subset R(I + \overline{A}). \tag{A.8}$$

Indeed, given $f \in \overline{R(I + A)}$*, there exists* $\{f_n\}_{n\geqslant 1} \subset R(I + A)$ *such that* $f_n \stackrel{n}{\to} f$*. Then, if* $g_n := (I + A)^{-1} f_n$*, by the accretivity of* A*, we get that* $g_n \stackrel{n}{\to} g$*. Now,* $(g_n, f_n - g_n) \in A$ *so* $(g, f - g) \in \overline{A}$ *and, therefore,* $f \in R(I + \overline{A})$*.*

It is easy to see that each m*-accretive operator* A *in* X *is maximal accretive in the sense that every accretive extension of* A *coincides with* A*. In general, the converse is not true, but it is true in Hilbert spaces due to the following classical result of G. Minty* [**130**]*:*

Minty's Theorem. *Let* H *be a Hilbert space and* A *an accretive operator in* H*. Then,* A *is* m*-accretive if, and only if,* A *is maximal monotone.*

One of the most important examples of maximal monotone operators in Hilbert spaces comes from optimization theory, they are the subdifferentials of convex functions which we introduce next.

Let $(H, (\ |\ ))$ *be a Hilbert space and* $\varphi : H \to (-\infty, +\infty]$*. We denote*

$$D(\varphi) := \{x \in H \ :\ \varphi(x) \neq +\infty\} \qquad \text{(effective domain)}.$$

We say that φ *is proper if* $D(\varphi) \neq \emptyset$ *and that* φ *is convex if*

$$\varphi(\alpha x + (1 - \alpha)y) \leqslant \alpha\varphi(x) + (1 - \alpha)\varphi(y)$$

for all $\alpha \in [0, 1]$ *and* $x, y \in H$*.*

Some of the properties of φ *are reflected in its epigraph defined by*

$$\mathrm{Epi}(\varphi) := \{(x, r) \in H \times \mathbb{R} \ :\ r \geqslant \varphi(x)\}.$$

For instance, φ *is convex if, and only if,* $\mathrm{Epi}(\varphi)$ *is a convex subset of* $H \times \mathbb{R}$*; and* φ *is lower-semi-continuous if, and only if,* $\mathrm{Epi}(\varphi)$ *is closed.*

The subdifferential $\partial\varphi$ *of* φ *is the operator defined by*

$$w \in \partial\varphi(z) \iff \varphi(x) \geqslant \varphi(z) + (w|x - z) \qquad \forall\, x \in H.$$

Observe that $0 \in \partial\varphi(z) \iff \varphi(x) \geqslant \varphi(z) \quad \forall\, x \in H \iff$

$$\varphi(z) = \min_{x \in D(\varphi)} \varphi(x).$$

Therefore, we have that $0 \in \partial\varphi(z)$ *is the Euler-Lagrange equation of the variational problem*

$$\varphi(z) = \min_{x \in D(\varphi)} \varphi(x).$$

If $(z,w),(\hat{z},\hat{w}) \in \partial\varphi$, *then* $\varphi(z) \geqslant \varphi(\hat{z}) + (\hat{w}|z-\hat{z})$ *and* $\varphi(\hat{z}) \geqslant \varphi(z) + (\hat{w}|\hat{z}-z)$. *Adding this inequalities we get*

$$(w-\hat{w}|z-\hat{z}) \geqslant 0.$$

Thus, $\partial\varphi$ *is a monotone operator. Now, if* φ *is convex, lower-semi-continuous and proper, it can be proved that* $\partial\varphi$ *is maximal monotone and* $\overline{D(\partial\varphi)} = \overline{D(\varphi)}$ *(see* [**23**] *and* [**43**]*).*

Given K *a closed convex subset of* H, *the indicator function of* K *is defined by*

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then, it is easy to see that the subdifferential is characterized as follows:

$$v \in \partial I_K(u) \Leftrightarrow u \in K \quad \text{and} \quad (v, w-u) \leqslant 0 \quad \forall\, w \in K.$$

As we mentioned, in the linear case, the existence and uniqueness of mild solutions is equivalent to the fact that $-A$ *is the infinitesimal generator of a* C_0*-semigroup. Now, there are classical results connecting this fact with the* m*-accretivity of the operator* A, *for example:*

Theorem A.20. (**Lumer-Phillips Theorem**) $-A$ *is the infinitesimal generator of a* C_0*-semigroup* $(S(t))_{t\geqslant 0}$ *of linear contractions on* X *if, and only if,* A *is linear,* m*-accretive and* $\overline{D(A)} = X$. *Moreover, we have the Hille-Yosida exponential formula*

$$S(t)x = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x.$$

A first extension to the nonlinear case of this type of results has been given by Y. Komura in [**108**].

Komura Theorem

(i) Let A *be a maximal monotone operator in the Hilbert space* H. *Then* $\overline{D(A)}$ *is a closed convex subset of* H *and* $D(S^A) = \overline{D(A)}$.

(ii) Given some closed convex set $C \subset H$ *and an strongly continuous semigroup of contractions* $(S(t))_{t\geqslant 0}$ *on* C, *then there exists a unique maximal monotone operator* A *in* H *such that* $\overline{D(A)} = C$ *and* $S^A(t) = S(t)$ *for all* $t \geqslant 0$.

This result has been extended to some Banach spaces with good geometrical properties, but it turns out to be false in general Banach spaces. The good extension to nonlinear operators in general Banach spaces was done by Crandall-Liggett ([**69**]*) and Ph. Bénilan (*[**28**]*) at the beginning of the 1970's. In the next section we give the outline of this theory.*

A.5. Existence and uniqueness theorem

Suppose A *is an operator in* X *and* $f \in L^1(0,T;X)$. *Consider the abstract Cauchy problem*

$$(CP)_{x_0,f} \begin{cases} u'(t) + Au(t) \ni f(t) & \text{on} \quad t \in (0,T), \\ u(0) = x \end{cases}$$

Definition A.21. An ε-*approximate solution* of $(\mathrm{CP})_{x_0,f}$ is a solution v of an ε-discreti– zation $D_A(0 = t_0, \ldots, t_N, f_1, \ldots f_N)$ of $u' + Au \ni f$ on $[0,T]$ with $\|v(0) - x_0\| < \varepsilon$.

It follows from this definition that u *is a mild solution of* $(CP)_{x_0,f}$ *on* $[0,T]$ *if, and only if,* $u \in C([0,T];X)$ *and for each* $\varepsilon > 0$ *there is an* ε*-approximate solution* v *of* $(CP)_{x_0,f}$ *such that* $\|u(t) - v(t)\| < \varepsilon$ *on the domain of* v.

Definition A.22. Suppose that for each $\varepsilon > 0$ there are ε-approximate solutions of $(\mathrm{CP})_{x_0,f}$ on $[0,T]$. We say that the ε-*approximate solutions converge* on $[0,T]$ as $\varepsilon \downarrow 0$ to $u \in C([0,T];X)$ if there exists a function $\psi : [0,+\infty[\to [0,+\infty[$ with $\lim_{\varepsilon\downarrow 0} \psi(\varepsilon) = 0$ such that $\|u(t) - v(t)\| \leqslant \psi(\varepsilon)$ whenever $\varepsilon > 0$, v is a ε-approximate solution of $(\mathrm{CP})_{x_0,f}$ on $[0,T]$ and t is in the domain of v.

Theorem A.23. *Suppose that $A \in \mathcal{A}(w)$, $f \in L^1(0,T;X)$ and $x_0 \in \overline{D(A)}$. If the problem* $(\mathrm{CP})_{x_0,f}$ *has an ε-approximate solution on $[0,T]$ for every $\varepsilon > 0$, then it has a unique mild solution on $[0,T]$ to which the ε-approximate solutions of* $(\mathrm{CP})_{x_0,f}$ *converge as $\varepsilon \downarrow 0$.*

This theorem was proved by Ph. Bénilan in his Thesis ([28]) as an extension of Crandall-Liggett's Theorem (which corresponds to $f = 0$). We also have the following result.

Theorem A.24. *Let A be an accretive operator in X and let u be a mild solution of $u' + Au \ni 0$ on $[0,T]$. Then:*

(i) If v is an ε-approximate solution of $u' + Au \ni 0$ on $[0,T]$ with $[0,s]$ in its domain and $(x,y) \in A$, then

$$\|u(t) - v(s)\| \leqslant 2\|u(0) - x\| + \|y\||t-s| \qquad 0 \leqslant s,t \leqslant T.$$

(ii) If $\hat{u}$ is a mild solution of $\hat{u}' + A\hat{u} \ni 0$ on $[0,T]$, then

$$\|u(t) - \hat{u}(t)\| \leqslant \|u(0) - \hat{u}(0)\| \qquad 0 \leqslant t \leqslant T.$$

Theorem A.23 tells us that, for accretive operators, to have existence and uniqueness of mild solutions it is enough to have existence of ε-approximate solutions for each $\varepsilon > 0$. Now, we have seen this is the case for m*-accretive operators, consequently, we have the following result.*

Theorem A.25. *Let A be an operator in X, $f \in L^1(0,T;X)$ and $x_0 \in \overline{D(A)}$. If $A + wI$ is* m*-accretive, then the problem*

$$u' + Au \ni f \quad \text{on} \quad [0,T], \qquad u(0) = x_0$$

has a unique mild solution u on $[0,T]$.

Recall that

$$D(S^A) := \{x \in X \,:\, \exists! \text{ mild solution } u_x \text{ of } u' + Au \ni 0 \text{ on } (0,+\infty) \text{ with } u_x(0) = x\},$$

and for $t \geqslant 0$ and $x \in D(S^A)$, $S^A(t)x := u_x(t)$. From now on, we denote $S^A(t)$ by e^{-tA}, and we call $(e^{-tA})_{t\geqslant 0}$ the semigroup generated by $-A$.

As a consequence of Theorem A.24, if A is accretive, then $(e^{-tA})_{t\geqslant 0}$ is a contraction semigroup, i.e.,

$$\|e^{-tA}x - e^{-tA}\hat{x}\| \leqslant \|x - \hat{x}\| \qquad \forall\, x, \hat{x} \in D(S^A),\ \forall\, t \geqslant 0.$$

Moreover, by the properties of mild solutions, it is easy to see that $D(S^A)$ is closed and, by Theorem A.24, we have that the map

$$(t,x) \mapsto e^{-tA}x \quad \text{is continuous in } [0,+\infty) \times D(S^A).$$

As a consequence of Theorem A.25 we have that if A is m*-accretive in X, then $D(S^A) = \overline{D(A)}$ and $(e^{-tA})_{t\geqslant 0}$ is a contraction semigroup in $\overline{D(A)}$.*

Let us see now that in the homogeneous case we can weaken the m*-accretivity of the operator and get an explicit representation of the mild solution. Suppose for the moment that A is* m*-accretive. Let $\lambda > 0$ and let v be a solution of the discretization $D_A(0,\lambda,2\lambda,\ldots,N\lambda;0,\ldots,0)$ satisfying $v(0) = x_0$. Due to the fact that the discretization has a constant step size λ, the difference equation for v is equivalent to*

$$\text{(A.9)} \qquad \begin{cases} v(t) = x_0 \quad \text{for} \quad -\lambda < t \leqslant 0 \\[2mm] \dfrac{v(t) - v(t-\lambda)}{\lambda} + Av(t) \ni 0 \quad \text{for} \quad 0 < t \leqslant N\lambda. \end{cases}$$

Moreover, $v(k\lambda) = J_\lambda^A v((k-1)\lambda)$ or, iterating

$$v(k\lambda) = (J_\lambda^A)^k v(0) = (J_\lambda^A)^k x_0.$$

Then in order to solve (A.9) we only need that $\overline{D(A)} \subset D(J_\lambda^A)$ for $\lambda > 0$ and of course the accretivity of the operator A.

DEFINITION A.26. An accretive operator A satisfies the *range condition* if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$.

THEOREM A.27. **(Crandall-Liggett Theorem)** *If A is accretive and satisfies the range condition, then $-A$ generates a semigroup of contractions $(e^{-tA})_{t\geqslant 0}$ on $\overline{D(A)}$ and:*

(i) For $x_0 \in \overline{D(A)}$ and $0 \leqslant t < \infty$,

$$\lim_{\lambda\downarrow 0, k\lambda\to t} (J_\lambda^A)^k x_0 = e^{-tA}x_0$$

holds uniformly for t on compact subintervals of $[0,\infty)$.

(ii) If $x_0 \in \overline{D(A)}$, $t > 0$ and $n \in \mathbb{N}$, then

$$\left\| (J_{t/n}^A)^n x_0 - e^{-tA}x_0 \right\| \leqslant \frac{t}{\sqrt{n}} \|y\| + 2\|x_0 - x\|$$

for every $(x,y) \in A$.

From either (i) or (ii) of the last theorem we deduce

$$e^{-tA}x = \lim_{n\to\infty} \left(I + \frac{t}{n}A \right)^{-n} x \quad \text{for} \quad x \in \overline{D(A)}. \tag{A.10}$$

This representation of the semigroup $(e^{-tA})_{t\geqslant 0}$ is called the exponential formula by analogy with the formula $\lim_{n\to\infty}(1 + \frac{t}{n}a)^{-n} = e^{-ta}$ for $a \in \mathbb{C}$.

*Observe the analogy of (A.10) with the exponential formula given by the Lumer-Phillips Theorem for the linear case. Now, there are strong differences between the linear and nonlinear cases. For instance, in the linear case, $-A$ is the infinitesimal generator of the C_0-semigroup $(e^{-tA})_{t\geqslant 0}$, and in the nonlinear case there are examples of operators A satisfying the assumptions of Crandall-Liggett's Theorem, such that the domain of the infinitesimal generator of the semigroup $(e^{-tA})_{t\geqslant 0}$ is empty (see [**69**]).*

A.6. Regularity of the mild solution

As we have already pointed out mild solutions may not satisfy any additional regularity properties, in general, they can not be interpreted as a solution of the Cauchy problem in a pointwise sense, and they are not strong solutions.

Nevertheless, the question of whether under certain additional assumptions one may obtain more regularity of mild solutions arises naturally. This will be done now. We emphasize, before this, that even in applications one does not want to be limited to strong solutions, since there are important partial differential equations which simply do not have strong solutions.

A basic fact is the following consistence between the accretivity of A and the differentiability of mild solutions of $u' + Au \ni f$.

THEOREM A.28. *Let A be an accretive operator in X, $f \in L^1(0,T;X)$ and u be a mild solution of $u' + Au \ni f$ on $[0,T]$. Suppose that u has a right derivative $\dfrac{d^+u}{dt}(\tau)$ at $\tau \in]0,T[$ and*

$$\lim_{h\downarrow 0} \frac{1}{h} \int_\tau^{\tau+h} \|f(t) - f(\tau)\| \, dt = 0,$$

that is, τ is a right Lebesgue point of f. Then, the operator $\hat{A}$ given by

$$\hat{A}x = Ax \quad \text{for} \quad x \neq u(\tau)$$

$$\hat{A}u(\tau) = Au(\tau) \cup \left\{ f(\tau) - \frac{d^+u}{dt}(\tau) \right\}$$

is accretive.

Since every m*-accretive operator is maximal accretive, as a consequence of the above theorem we have the following result.*

Corollary A.29. *Suppose that A is an* m*-accretive operator in X, $f \in L^1(0,T;X)$ and u is a mild solution of $u' + Au \ni f$ on $[0,T]$. Then,*

(i) if u is differentiable at $t \in (0,T)$ and t is a right Lebesgue point of f,

$$u'(t) + Au(t) \ni f(t).$$

(ii) if $u \in W^{1,1}(0,T;X)$, u is a strong solution of $u' + Au \ni f$ on $[0,T]$.

Then, the problem is:

Problem: When does a mild solution belongs to $W^{1,1}(0,T;X)$?

We denote by $BV(0,T;X)$ the subspace of function in $L^1(0,T;X)$ which are of bounded variation, i.e., $f \in BV(0,T;X)$ if $f \in L^1(0,T;X)$ and

$$\mathrm{Var}(f,T) := \limsup_{h \downarrow 0} \int_0^{T-h} \frac{\|f(\tau+h) - f(\tau)\|}{h}\, d\tau < +\infty.$$

The main conditions guaranteeing that a mild solution is in $W^{1,1}(0,T;X)$ are given by the following result.

Proposition A.30. *Let A be an accretive operator in X, $f \in BV(0,T;X)$ and $x \in D(A)$. If u is a mild solution of $(CP)_{x,f}$ on $[0,T]$, then u is locally Lipschitz continuous on $[0,T)$. Moreover, if X has the Radon-Nikodym property, then $u \in W^{1,1}(0,T;X)$ and, consequently, u is a strong solution of $(CP)_{x,f}$ on $[0,T]$.*

In the case that the operator is the subdifferential of a convex lower semi-continuous function in a Hilbert space, we have good regularity. More precisely, we have the following result.

Theorem A.31. *Let H be a Hilbert space and $\varphi : H \to (-\infty, +\infty]$ a proper, convex and lower semi-continuous function such that $\mathrm{Min}\,\varphi = 0$, and let $K := \{v \in H : \varphi(v) = 0\}$. Assume that $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\partial\varphi)}$, then the mild solution $u(t)$ of*

$$\begin{cases} u' + \partial\varphi(u) \ni f \quad \text{on } [0,T], \\ u(0) = u_0, \end{cases}$$

is a strong solution and we have the following estimates,

$$\|u'(t)\|_{L^2(\delta,T;H)} \leqslant \|f\|_{L^2(0,T;H)} + \frac{1}{\sqrt{2\delta}} \int_0^\delta \|f(t)\|\, dt + \frac{1}{\sqrt{2\delta}} \mathrm{dist}\,(u_0, K)$$

for $0 < \delta < T$, and

$$\left(\int_0^T \|u'(t)\|^2 t\, dt \right)^{\frac{1}{2}} \leqslant \left(\int_0^T \|f(t)\|^2 t\, dt \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \int_0^T \|f(t)\|^2\, dt + \frac{1}{\sqrt{2}}\, \mathrm{dist}\,(u_0, K)\,.$$

Moreover, for almost all $t \in [0,T]$, we have

$$\frac{d}{dt}\varphi(u(t)) = (h | u'(t)) \qquad \forall\, h \in \partial\varphi(u(t)).$$

In the homogeneous case, i.e., $f = 0$, we have

$$\|u'(t)\|_{L^\infty(\delta,T;H)} \leqslant \frac{1}{\delta}\|u_0\| \quad \text{for} \quad 0 < \delta < T.$$

A.7. Completely accretive operators

*Many nonlinear semigroups that appear in the applications are also order-preserving and contractions in every L^p. Ph. Bénilan and M. Crandall introduced in [**31**] a class of operators, named completely accretive, for which the semigroup generated by the Crandall-Ligget's exponential formula enjoys these properties. In this section we outline some of the main points given in [**31**].*

Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space and let $M(\Omega)$ denote the space of measurable functions from Ω into $\mathbb{R}$. We denote by $L(\Omega)$ the space

$$\begin{aligned} L(\Omega) &:= L^1(\Omega) + L^\infty(\Omega) \\ &= \left\{ u \in M(\Omega) \ : \ \int_\Omega (|u| - k)^+ < \infty \ \text{ for some } k > 0 \right\}; \end{aligned}$$

$L(\Omega)$ is exactly the subset of $M(\Omega)$ on which the functional

$$\|u\|_{1+\infty} := \inf\{\|f\|_1 + \|g\|_\infty \ : f, g \in M(\Omega), \ f + g = u\}$$

is finite and $L(\Omega)$ equipped with $\| \ \|_{1+\infty}$ is a Banach space.

Let

$$\begin{aligned} L_0(\Omega) &:= \{u \in L(\Omega) \ : \ \mu(\{|u| > k\}) < \infty \text{ for any } k > 0\} \\ &= \left\{ u \in M(\Omega) \ : \ \int_\Omega (|u| - k)^+ < \infty \ \text{ for any } k > 0 \right\}. \end{aligned}$$

$L_0(\Omega)$ is a closed subspace of $L(\Omega)$; in fact, it is the closure in $L(\Omega)$ of the linear span of the set of characteristic functions of sets of finite measure. Hereafter, $L_0(\Omega)$ carries the norm $\| \ \|_{1+\infty}$; it is then a Banach space. With the natural paring $\langle u, v\rangle = \int_\Omega uv$, the dual space of $L_0(\Omega)$ is isometrically isomorphic to

$$L^{1\cap\infty}(\Omega) := L^1(\Omega) \cap L^\infty(\Omega),$$

when in $L^{1\cap\infty}(\Omega)$ we consider the norm

$$\|u\|_{1\cap\infty} := \max\{\|u\|_1, \|u\|_\infty\}.$$

Given $u, v \in M(\Omega)$, we shall write

$$u \ll v \quad \textit{if, and only if,} \ \int_\Omega j(u)dx \leqslant \int_\Omega j(v)dx$$

for all $j \in J_0$, where

$$J_0 = \{j : \mathbb{R} \to [0, \infty], \ \text{convex}, \ \text{l.s.c.}, \ j(0) = 0\}$$

(l.s.c. is an abbreviation for lower semi-continuous function).

Definition A.32. A functional $N : M(\Omega) \to (-\infty, +\infty]$ is called *normal* if $N(u) \leqslant N(v)$ whenever $u \ll v$.

A map $S : D(S) \subset M(\Omega) \to M(\Omega)$ is a *complete contraction* if it is an N-contraction for every normal functional N, i.e., if

$$N(Su - Sv) \leqslant N(u - v) \quad \text{ for } u, v \in D(S).$$

A Banach space $(X, \| \ \|_X)$, with $X \subset M(\Omega)$ is a *normal Banach space* if it has the following property:

$$u \in X, \ v \in M(\Omega), \ v \ll u \ \Rightarrow \ v \in X \ \text{ and } \ \|v\|_X \leqslant \|u\|_X.$$

Simple examples of normal Banach spaces are: $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, $L(\Omega)$, $L_0(\Omega)$ and $L^{1\cap\infty}(\Omega)$.

Proposition A.33. *Let $S : D(S) \subset M(\Omega) \to M(\Omega)$ and assume that*

$$u, v \in D(S) \text{ and } k \geqslant 0 \ \Rightarrow \ u \wedge (v + k) \text{ or } v \vee (u - k) \in D(S).$$

Then, S is a complete contraction if, and only if, it is order-preserving and a contraction for $\| \ \|_1$ and $\| \ \|_\infty$.

Definition A.34. Let A be an operator in $M(\Omega)$. We shall say that A is *completely accretive* if

$$u-\hat{u} \ll u-\hat{u}+\lambda(v-\hat{v}) \quad \text{for all } \lambda>0 \text{ and all } (u,v),(\hat{u},\hat{v}) \in A.$$

In other words, A is completely accretive if

$$N(u-\hat{u}) \leqslant N\left(u-\hat{u}+\lambda(v-\hat{v})\right)$$

for all $\lambda>0$, all $(u,v),(\hat{u},\hat{v}) \in A$ and every normal functional N in $M(\Omega)$.

The definition of completely accretive operators does not refer explicitly to topologies or norms. However, if A is completely accretive in $M(\Omega)$ and $A \subset X \times X$, where X is a subspace of $M(\Omega)$ whose norm is given by a normal function, then A is accretive in X. Choices for X might be $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$.

Let

$$P_0=\left\{q \in C^{\infty}(\mathbb{R}):\ 0 \leqslant q' \leqslant 1,\ \textit{supp}(q') \textit{ is compact and } 0 \notin \operatorname{supp}(q)\right\}.$$

The following result, which is a generalization of one due to H. Brezis and W. Strauss ([45]), provides a very useful characterization of the complete accretivity.

Proposition A.35. *Let $u \in L_0(\Omega)$, $v \in L(\Omega)$. Then,*

$$u \ll u+\lambda v \quad \forall\, \lambda>0 \ \Leftrightarrow\ \int_{\Omega} q(u)v \geqslant 0 \quad \forall\, q \in P_0.$$

Observe that $L^p(\Omega) \subset L_0(\Omega)$ for any $1 \leqslant p<\infty$. If $\mu(\Omega)<\infty$ then $L_0(\Omega)=L(\Omega)=L^1(\Omega)$. Consequently, from the above proposition the following characterization follows.

Corollary A.36. *If $A \subseteq L^p(\Omega) \times L^p(\Omega)$, $1 \leqslant p<\infty$, then A is completely accretive if, and only if,*

$$\int_{\Omega} q(u-\hat{u})(v-\hat{v}) \geqslant 0 \quad \textit{for any } \ q \in P_0,\ (u,v),(\hat{u},\hat{v}) \in A.$$

Proposition A.37. *Let $u \in L_0(\Omega)$. Then,*

(i) $\{v \in M(\Omega)\ : v \ll u\}$ is a weakly sequentially compact subset of $L_0(\Omega)$.

(ii) Let $(X, \|\,\|_X)$ be a normal Banach space satisfying $X \subset L_0(\Omega)$ and having the property

$$u_n \ll u \in X,\ n=1,2,\ldots,\ \textit{and } u_n \to u \textit{ a.e. } \Rightarrow \|u_n-u\|_X \to 0. \tag{A.11}$$

If $\{u_n\}$ is sequence satisfying $u_n \ll u \in X$ for $n=1,2,\ldots,$ and $u_n \rightharpoonup u$ weakly in $L_0(\Omega)$, then $\|u_n-u\|_X \to 0$.

Remark A.38. The assumption (A.11) is satisfied for $X=L^p(\Omega)$, $1 \leqslant p<+\infty$.

Definition A.39. Let X a linear subspace of $M(\Omega)$. An operator A in X is m-*completely accretive* in X if A is completely accretive and $R(I+\lambda A)=X$ for $\lambda>0$.

Remark A.40. The above definition does not require X to be a Banach space and so does not require A to be m-accretive in any Banach space. However, if A is completely accretive then it is accretive in $L(\Omega)$ and if A is m-completely accretive in a subspace X of $L(\Omega)$, then the closure $\overline{A}$ of A in $L(\Omega)$ is completely accretive and m-accretive in the closure $\overline{X}$ of X in $L(\Omega)$. We also note that if A is completely accretive in a subspace X of $M(\Omega)$ and $R(I+\lambda A)=X$ for some $\lambda>0$, the only completely accretive operator B in X which extends A is A.

Proposition A.41. *Let X be a normal Banach space, $X \subset L_0(\Omega)$, and A be a completely accretive operator in X. Then, if there exists $\lambda>0$ for which $R(I+\lambda A)$ is dense in $L_0(\Omega)$, the operator $A^X := \overline{A} \cap (X \times X)$ is the unique m-completely accretive extension of A in X.*

Definition A.42. Let A be an operator in $L_0(\Omega)$. Then A° is the restriction of A defined by

$$v \in A^\circ u \;\Leftrightarrow\; v \in Au \;\text{ and }\; v \ll w, \qquad \forall\, w \in Au.$$

In the case that X is a normal Banach space and A is m-completely accretive in X, by Crandall-Ligget's Theorem, A generates a contraction semigroup in X given by the exponential formula

$$e^{-tA}u_0 = X - \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} u_0 \quad \text{for any } u_0 \in \overline{D(A)}^X.$$

Now, since $\overline{A}$ is m-completely accretive in $\overline{X}$ endowed with the norm of $L(\Omega)$, we may also consider the semigroup $e^{-t\overline{A}}$ on $\overline{D(A)}$. We have the following relation between these two semigroups.

Proposition A.43. *Let X be a normal Banach space and A an m-completely accretive operator in X. Then, we have*

(i) e^{-tA} *is a complete contraction for* $t \geqslant 0$.

(ii) e^{-tA} *is the restriction of* $e^{-t\overline{A}}$ *to* $\overline{D(A)}^X$ *and* $e^{-t\overline{A}}$ *is the closure of* e^{-tA} *in* $L(\Omega)$.

(iii) $e^{-t\overline{A}}(\overline{D(A)} \cap X) \subset \overline{D(A)} \cap X$.

As a consequence of (iii) of the above proposition, if we denote by $S^A(t)$ the restriction of $e^{-t\overline{A}}$ to $\overline{D(A)} \cap X$, we have $S^A(t)$ is given by the exponential formula

$$S^A(t)u = L(\Omega) - \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} u \quad \text{for } \; u \in \overline{D(A)} \cap X.$$

Theorem A.44. *Let X be a normal Banach space with $X \subset L_0(\Omega)$ and A an m-completely accretive operator in X. Then, we have*

(i)

$$D(A) = \left\{ u \in \overline{D(A)} \cap X \;\; : \exists v \in X \;\, s.t.\; \frac{S^A(t)u - u}{t} \ll v \text{ for small } t > 0 \right\}.$$

(ii) $S^A(t)D(A) \subset D(A) \quad$ *for* $\quad t > 0$.

(iii) If $u \in D(A)$, then

$$\frac{u - S^A(t)u}{t} \ll v \;\text{ for } t > 0 \;\text{ and }\; v \in Au$$

and

$$L(\Omega) - \lim_{t\to 0} \frac{S^A(t)u - u}{t} = -A^\circ u.$$

Corollary A.45. *Assume that $\mu(\Omega) < \infty$. If $A \subseteq L^1(\Omega) \times L^1(\Omega)$ is an m-completely accretive operator in $L^1(\Omega)$, then, for every $u_0 \in D(A)$, the mild solution $u(t) = e^{-tA}u_0$ of the problem*

$$\begin{cases} \dfrac{du}{dt} + Au \ni 0, \\ u(0) = u_0, \end{cases}$$

is a strong solution.

The following result is a variant of the regularizing effect of the homogeneous evolution equation obtained in [**31**] *in the* m-*completely accretive case.*

Theorem A.46. *In addition to the assumptions of Theorem A.44, assume that A is positively homogeneous of degree $0 < m \neq 1$, i.e, $A(\lambda u) = \lambda^m Au$ for $u \in D(A)$. Then, for $u \in \overline{D(A)} \cap X$ and $t > 0$, we have $S^A(t)u \in D(A)$ and*

$$|A^\circ S^A(t)u| \leqslant 2\frac{|u|}{|m-1|t}.$$

To finish we summarize some results about T-accretive operators.

Definition A.47. Let X be a Banach lattice and $S : D(S) \subset X \to X$. We say that S is a *T-contraction* if

$$\|(Su - Sv)^+\| \leqslant \|(u-v)^+\| \quad \text{for } u, v \in D(S).$$

Let A be an operator in X. We say that A is *T-accretive* if

$$\|(u-\hat{u})^+\| \leqslant \|(u - \hat{u} + \lambda(v - \hat{v}))^+\| \quad \text{for } (u,v),(\hat{u},\hat{v}) \in A \ \text{ and } \lambda > 0.$$

It is clear that a T-contraction is order-preserving; moreover, if A is a T-accretive operator then its resolvents $(I + \lambda A)^{-1}$ are single-valued and order preserving. Indeed, A is T-accretive if, and only if, its resolvents are T-contractions. Contractions are not in general T-contractions and conversely. Actually, T-contractions are contractions if the norm satisfies

$$\|u^+\| \leqslant \|v^+\| \quad \text{and} \quad \|u^-\| \leqslant \|v^-\| \quad \text{implies} \quad \|u\| \leqslant \|v\|$$

for $u, v \in X$. This is the case for the spaces $X = L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Therefore, in $L^p(\Omega)$ every T-accretive operator is an accretive operator and also every completely accretive operator is a T-accretive operator. The mild solutions of the abstract Cauchy problems associated with T-accretive operators satisfy a contraction principle. More precisely, we have the following result.

Theorem A.48. *Let X be a Banach lattice and A a* m*-accretive operator in X. Then, the following are equivalent:*

(i) A is T-accretive.

(ii) If $f, \hat{f} \in L^1(0,T;X)$, and u, $\hat{u}$ are mild solutions of $u' + Au \ni f$ and $\hat{u}' + A\hat{u} \ni \hat{f}$ on $[0,T]$, then for $0 \leqslant s \leqslant t \leqslant T$

$$\|(u(t) - \hat{u}(t))^+\| \leqslant \|(u(s) - \hat{u}(s))^+\| + \int_s^t [u(\tau) - \hat{u}(\tau), f(\tau) - \hat{f}(\tau)]_+ \, d\tau,$$

where

$$[u,v]_+ := \lim_{\lambda \downarrow 0} \frac{\|(u + \lambda v)^+\| - \|u^+\|}{\lambda}.$$

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