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The equal collective gains value in cooperative games

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Abstract

The property of *equal collective gains* means that each player should obtain the same benefit from the cooperation of the other players in the game. We show that this property jointly with efficiency characterize a new solution, called the *equal collective gains value* (ECG-value). We introduce a new class of games, the *average productivity games*, for which the ECG-value is an imputation. For a better understanding of the new value, we also provide four alternative characterizations of it, and a negotiation model that supports it in subgame perfect equilibrium.

Keywords Shapley value · ENSC-value · Reciprocity · Equal collective gains · Balanced collective contributions

JEL Classification C71

1 Introduction

In a cooperative game, the main challenge is to find an acceptable rule to reward players with the benefits of their cooperation. Each rule can be supported either as the non cooperative equilibrium of a plausible bargaining game or as the consequence of accepting some desirable properties that the rule should satisfy. Both have been called the strategic approach and the axiomatic approach respectively.

In the present paper, we consider the case of cooperative games with transferable utility (TU-games). One of the most remarkable proposals is that of Shapley (1953). Shapley shows that properties that characterize his rule are *efficiency* (players' rewards cover the total value of the game), the *null player property* (players who do

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not contribute to the game receive nothing), *symmetry* (players who contribute the same receive the same), and *additivity* (the value of the sum of games is the sum of their values). In addition to Shapley's original characterization, other axiomatizations are due to Myerson (1980), Hart and Mas-Colell (1989, 1996), Young (1985), van den Brink (2001), Casajus (2017), and Yokote and Kongo (2017) among others.

We take the axiomatization of the Shapley value provided by Hart and Mas-Colell (1996) as starting point. There, it is shown that for TU-games the Shapley value can be determined by the properties of efficiency and *average balanced contributions.* This property says that the sum of the contributions that a player makes to the value of the remaining players must be equal to the sum that the remaining players make to her. The contribution that a player i makes to the value of another player j is understood to be the difference between what j gets in the game with and without i. Average balanced contributions is a principle of *reciprocity*: what you contribute to others is the same as what you get from others.

At this point, the following question arises immediately: Is it possible to consider separately the two principles that make up reciprocity? More precisely, (I) is there a value that satisfies *efficiency* and *equal collective contributions* (each player contributes the same to the gains of the other players)?, and (II) is there a value that satisfies *efficiency* and *equal collective gains* (each player earns the same from the contributions made by the other players)?

For question (I), (Béal et al. 2016) provide a new characterization of an already existing value, introduced by Moulin (1985), and known as the *equal allocation of non-separable contributions value* (ENSC-value). Moulin characterizes this value using a particular notion of consistency, called the *separability axiom*. Other characterizations are provided in Hwang (2006), Ju and Wettstein (2009), van den Brink and Funaki (2009), Xu et al. (2015), Sun et al. (2017) and Hou et al. (2018).

In the present paper, for question (I), we show the existence of a new solution satisfying such requirements. We call it the *equal collective gains value* (ECG-value). And this solution, as far as we know, has not yet been considered in the literature.

We introduce a new property for a game, called *average productivity (AP)*: a game (N, v) satisfies AP if the average of the marginal contributions of the members of every coalition is greater or equal to the per-capita worth of the coalition. This property does not imply monotonicity, super-additivity, or convexity of a game. We prove that in AP-games the ECG-value satisfies the minimal requirement of *individual rationality*. Moreover, we use a numerical example of an AP-game for which neither the Shapley value, nor the ENSC-value, nor the solidarity value satisfy individual rationality.

For a better understanding of this new proposal we offer four additional axiomatic charaterizations. In the first one, we use a variation of the *null play*er axiom. We say that a player is an *AP-null player* when the average of the marginal contributions of the players is equal to the per-capita worth of the coalition, for all coalitions containing her. The AP-null player axiom says that every AP-null player must receive zero. It turns out that a solution satisfies efficiency, additivity, symmetry and AP-null player if, and only if, it is the ECG-value. Second, we offer an "ordinal" version of our initial characterization with the properties of efficiency and equal collective gains, in the same vein as Casajus (2017) for the Shapley value. It only requires that the gains from the contributions made by the other players have the *same sign* (and therefore, not necessarily equal). We call it *weak equal collective gains.* We also use the *weak differential marginality*, introduced in Casajus and Yokote (2017). It indicates that whenever two agents' productivities change by the same amount, then their payoffs change in the *same direction.* We find that the ECG-value is the unique solution that satisfies efficiency, weak equal collective gains, and weak differential marginality. Moreover, since the ECG-value satisfies the stronger *differential marginality*, which is equivalent to the van den Brink's (2001) *fairness*, we also prove that the ECG-value is the unique solution that satisfies efficiency, the AP-null player axiom and fairness. Finally, we can relax the principle of equal collective gains, applying it only to symmetric players, following Yokote and Kongo (2017) for the Shapley value. We add a new property called *AP-marginality.* This property is similar in its definition to that of Young (1985) but with the average of the marginal contributions instead of the individual marginal contributions. We obtain that the ECG-value is the unique solution that satisfies efficiency, equal collective gains for symmetric players, and AP-marginality.

Complementary to the axiomatic approach, we also offer a negotiation model that supports the ECG-value. This is an alternating random proposer bargaining, very similar to that of Hart and Mas-Colell (1996) that implements the Shapley value in TU-games. The only difference resides in what happens in the event of a breakdown: In the Hart and Mas-Colell's model, with a certain probability $(1-\rho) < 1$, the proposer whose proposal has been rejected leaves the game, and negotiations restart without her; in our model, with probability $(1-\rho) < 1$, every player, proposer or not, other than the one who rejects the offer, can leave the game equally likely, and the game restarts without her. This is a non-cooperative game whose subgame perfect equilibrium offers converge to the ECG-value when $\rho \to 1$, in the class of average productivity games. In our bargaining model, this is a sufficient condition to ensure that no player has an incentive in the equilibrium to leave the negotiations without agreement.

It would be worth noting that the condition of average productivity plays a fundamental role in the axiomatic approach by means of the AP-null player property, and in its strategic support through the non-cooperative negotiation model. In the first case, indicating when a player should be considered as irrelevant, obtaining a zero payoff, and in the second case, ensuring that the equilibrium of the negotiation game corresponds to the ECG-value.

The rest of the paper is organized as follows:

In Sect. 2, we define the ECG-value and show its main characterization with efficiency and equal collective gains property. In Sect. 3 we consider some properties that the ECG-value satisfies and compare its behaviour with regard the Shapley, the ENSC and the solidarity values with the help of a numerical example. In Sect. 4 we offer four additional axiomatic characterizations of this new solution. Finally, in Sect. 5 we present a negotiation model which implements the ECG-value.

2 Equal collective gains

Let $\mathbb{N}$ denote the set of natural numbers. Let $N = \{1, 2, \ldots, n\} \subset \mathbb{N}$ be a finite set of players. A *cooperative game* with transferable utility (TU-game) is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and each nonempty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S, and it is interpreted as the total utility that the coalition S, if it forms, can obtain for its members. Let $\mathcal{G}^N$ denote the set of all TU-games with player set N and let $\mathcal{G}$ denote the set of all games. For the sake of simplicity, we write $S \cup i$, $N \backslash i$ and $v(i)$ instead of $S \cup \{i\}$, $N \backslash \{i\}$, and $v(\{i\})$ respectively. For each $S \subseteq N$, we denote the *restriction* of (N, v) to S as the game $(S, v_{|S})$, where $v_{|S}(T) = v(T)$ for all $T \subseteq S$. We also simplify the notation denoting $(S, v_{|S})$ by (S, v).

A *solution* is a function ψ that assigns to every TU-game $(N, v) \in \mathcal{G}$ a $|N|$-dimensional real vector $\psi(N, v)$ which ith component $\psi^i(N, v)$ represents an assessment made by i of her gains from participating in the game.

A basic property for a solution is that it must divide among the agents all the gains from their cooperation. This is called *efficiency*.

Definition 1 *(Efficiency (E))* $\sum_{i \in N} \psi^i(N, v) = v(N)$.

The *imputation set* $I(N, v)$ is the set of efficient and individually rational payoffs, that is

$$I(N, v) := \left\{ x \in \mathbb{R}^N : \sum_{k \in N} x^k = v(N), \text{ and } x^k \geq v(k), \text{ for all } k \in N \right\}$$

We start by considering the two-player case, i.e. $N = \{i, j\}$. A standard way of sharing the profits from cooperation between i and j is the principle of *equal division of the surplus*. This means that player i gets:

$$\psi^i(\{i, j\}, v) = v(i) + \frac{1}{2}\left[v(\{i, j\}) - v(i) - v(j)\right].$$

By *efficiency*, this implies that

$$\psi^i(\{i, j\}, v) = \psi^i(i, v) + \frac{1}{2}\left[v(\{i, j\}) - \psi^i(i, v) - \psi^j(j, v)\right],$$

and then it holds the equality

$$\psi^i(\{i, j\}, v) - \psi^i(i, v) = \psi^j(\{i, j\}, v) - \psi^j(j, v). \tag{1}$$

Here are some ways to extend this principle to the general n-person case.

For each coalition S containing players i, j, the term $\Delta^i \psi^j(S, v) = \psi^j(S, v) - \psi^j(S \backslash i, v)$ is the g*ain* of player j in coalition S due to the cooperation of player i.

Condition (1) is introduced in Myerson (1980) with the name of *balanced contributions,* in order to characterize the Shapley value.[1]

[1] This property is also used in Hart and Mas-Colell (1989) to characterize the Potential of a game (see also Calvo and Santos 1999).

Definition 2 (*Balanced contributions*) $\Delta^i \psi^j(N, v) = \Delta^j \psi^i(N, v)$ for all $i, j \in N$.

The *Shapley value* (Shapley 1953) of the game (N, v) is the payoff vector $\varphi(N, v) \in \mathbb{R}^N$ defined by

$$\varphi^i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!\,(s-1)!}{n!} \Delta^i v(S), \; i \in N, \tag{2}$$

where $s = |S|$ and $n = |N|$, and

$$\Delta^i v(S) := v(S) - v(S \backslash i),$$

represents the *marginal contribution* of player $i \in S$ to coalition $S \subseteq N$.

Alternatively, $\varphi^i(N, v)$ can be obtained recursively[2] by

$$\varphi^i(S, v) = \frac{1}{s} \Delta^i v(S) + \frac{1}{s} \sum_{j \in S \backslash i} \varphi^i(S \backslash j, v), \quad \text{for all } i \in S \subseteq N, \tag{3}$$

starting with $\varphi^i(i, v) = v(i)$, for all $i \in N$.

Theorem 3 (Myerson 1980) *There exists a unique solution on $\mathcal{G}$ satisfying* efficiency *and* balanced contributions, *and this is the Shapley value φ.*

Note that balanced contributions is a *bilateral* property: it is required for each pair of players. However, it is immediate that *balanced contributions* implies the following property:

$$\sum_{k \in N \backslash i} \Delta^k \psi^i(N, v) = \sum_{k \in N \backslash i} \Delta^i \psi^k(N, v).$$

That is, the sum of the *gains*[3] of player i due to the cooperation of the remaining players $k \in N \backslash i$ ($\sum_{k \in N \backslash i} \Delta^k \psi^i(N, v)$) is equal to the sum of the *gains* of the remaining players $k \in N \backslash i$ ($\sum_{k \in N \backslash i} \Delta^i \psi^k(N, v)$) due to the cooperation of player i. This *reciprocity principle* is called as *average balanced contributions* in Hart and Mas-Colell (1996).

Definition 4 (*Average balanced contributions*) $\sum_{k \in N \backslash i} \Delta^k \psi^i(N, v) = \sum_{k \in N \backslash i} \Delta^i \psi^k(N, v)$, for all $i \in N$.

This is a way of expressing the principle that each player should *receive from the others what she contributes to them*.

2 See Hart and Mas-Colell 1996.

3 Because $1/(n-1)$ can appear in both sides of equation, the terms "average" or "total" can be used indistinctly.

Theorem 5 (Hart and Mas-Colell 1996) *There exists a unique solution on $\mathcal{G}$ satisfying* efficiency *and* average balanced contributions*, and this is the Shapley value φ.*

Remark 6 In the class of TU-games, the properties of average balanced contributions and balanced contributions are equivalent. However, this fact is not longer true in the general class of non-transferable utility games. Indeed, average balanced contributions is weaker than balanced contributions (see note 25 in Hart and Mas-Colell 1996).

Another way of extending condition (1) to the n-player case is that *every player should contribute the same to the gains of the others*. Béal et al. (2016) define this property as follows:

Definition 7 (*Balanced collective contributions)* $\sum_{k\in N\backslash i} \Delta^i \psi^k (N, v) = \sum_{k\in N\backslash j} \Delta^j \psi^k (N, v)$, for all $i, j \in N$.

They show that there is a well-known solution in the literature that satisfies this property. This solution is the *equal allocation of nonseparable contribution value (ENSC-value) ϕ*, defined by

$$\phi^i (N, v) = \Delta^i v(N) + \frac{1}{n}\left[v(N) - \sum_{k\in N} \Delta^k v(N)\right], \quad i \in N. \tag{4}$$

Thus, ϕ rewards players according to their contribution $\Delta^i v(N)$ to the grand coalition N, and the remaining non-separable contribution $(v(N) - \sum_{k\in N} \Delta^k v(N))$ is shared equally among them in order to satisfy efficiency.

Theorem 8 (Béal et al. 2016) *There exists a unique solution on $\mathcal{G}$ satisfying* efficiency *and* balanced collective contributions*, and this is the ENSC-value ϕ.*

Alternatively, condition (1) can also be read as meaning that *each player should gain the same by the contribution of the others*. We can extend this principle for the n-player case as follows:

Definition 9 (*Equal collective gains (ECG))* $\sum_{k\in N\backslash i} \Delta^k \psi^i (N, v) = \sum_{k\in N\backslash j} \Delta^k \psi^j (N, v)$, for all $i, j \in N$.

We now show that there is a unique solution characterized by the properties of *efficiency* and *equal collective gains*.

Theorem 10 *There exists a unique solution on $\mathcal{G}$ satisfying* efficiency *and* equal collective gains*, and this is the equal collective gains value (ECG-value) χ, defined recursively by*

$$\chi^i (S, v) = \frac{1}{s}\Delta^* v(S) + \frac{1}{s-1} \sum_{k\in S\backslash i} \chi^i (S\backslash k, v), \quad \textit{for all } i \in S \subseteq N, \tag{5}$$

where[4]

$$\Delta^* v(S) = v(S) - \frac{1}{s-1} \sum_{k \in S} v(S \backslash k), \quad \textit{for all } S \subseteq N.$$

Proof Let ψ be a solution satisfying the above properties. Then, by *efficiency*, it holds that $\psi^i(i, v) = v(i) = \chi^i(i, v)$, for all $i \in N$. Now, let N be the grand coalition. By *equal collective gains* we have

$$\sum_{k \in N \backslash i} \Delta^k \psi^i(N, v) = \sum_{k \in N \backslash j} \Delta^k \psi^j(N, v) \Leftrightarrow$$
$$(n-1)\,\psi^i(N, v) - \sum_{k \in N \backslash i} \psi^i(N \backslash k, v) = (n-1)\,\psi^j(N, v) - \sum_{k \in N \backslash j} \psi^j(N \backslash k, v).$$

Adding over all $j \in N$, it holds

$$n\,(n-1)\,\psi^i(N, v) - n \sum_{k \in N \backslash i} \psi^i(N \backslash k, v) = (n-1)\,v(N) - \sum_{j \in N} \sum_{k \in N \backslash j} \psi^j(N \backslash k, v),$$

and, by *efficiency*

$$\begin{aligned}\psi^i(N, v) &= \frac{1}{n}\left[v(N) - \frac{1}{(n-1)} \sum_{k \in N} v(N \backslash k)\right] \\ &\quad + \frac{1}{n-1} \sum_{k \in N \backslash i} \psi^i(N \backslash k, v) = \chi^i(N, v).\end{aligned}$$

This formula (5) shows that $\psi^i(N, v)$ is uniquely defined and, by construction satisfies E and ECG. Hence, $\psi = \chi$. ◻

The above recursive formula (5) of $\chi(N, v)$ has its corresponding direct formulation, similar to (2) for the Shapley value, replacing $\Delta^i v(S)$ by $\frac{n}{s}\Delta^* v(S)$:

Proposition 11 *Formula* (5) *is equivalent to*

$$\chi^i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!\,(s-1)!}{(n-1)!\,s} \Delta^* v(S), \quad i \in N. \tag{6}$$

Proof We use induction on the number of players. The one person case is trivial. Assume that (6) holds for $n-1$ players. Then,

[4] It is understood that $\Delta^* v(i) = v(i)$.

$$\begin{aligned}
\chi^{i}(N,v) &= \frac{1}{n}\Delta^{*}v(N)+\frac{1}{n-1}\sum_{k\in N\backslash i}\chi^{i}(N\backslash k,v)\\
&= \frac{1}{n}\Delta^{*}v(N)+\frac{1}{n-1}\sum_{k\in N\backslash i}\sum_{\substack{S\subseteq N\backslash k\\ i\in S}}\frac{(n-1-s)!\,(s-1)!}{(n-2)!s}\Delta^{*}v(S)\\
&= \frac{1}{n}\Delta^{*}v(N)\\
&\quad+\frac{1}{n-1}\sum_{k\in N\backslash i}\left[\frac{1}{n-1}\Delta^{*}v(N\backslash k)+\sum_{\substack{S\subsetneq N\backslash k\\ i\in S}}\frac{(n-1-s)!\,(s-1)!}{(n-2)!s}\Delta^{*}v(S)\right]\\
&= \frac{1}{n}\Delta_{v}^{*}(N)+\sum_{k\in N\backslash i}\frac{1}{n-1}\frac{1}{n-1}\Delta_{v}^{*}(N\backslash k)\\
&\quad+\sum_{k\in N\backslash i}\sum_{\substack{S\subsetneq N\backslash k\\ i\in S}}\frac{1}{n-1}\frac{(n-1-s)!\,(s-1)!}{(n-2)!s}\Delta_{v}^{*}(S)\\
&= \frac{1}{n}\Delta^{*}v(N)+\sum_{k\in N\backslash i}\frac{1}{n-1}\frac{1}{n-1}\Delta^{*}v(N\backslash k)\\
&\quad+\sum_{\substack{S:|S|\leq n-2\\ i\in S}}\frac{n-s}{n-1}\frac{(n-1-s)!\,(s-1)!}{(n-2)!s}\Delta^{*}v(S)\\
&= \frac{1}{n}\Delta^{*}v(N)+\sum_{k\in N\backslash i}\frac{1}{n-1}\frac{1}{n-1}\Delta^{*}v(N\backslash k)\\
&\quad+\sum_{\substack{S:|S|\leq n-2\\ i\in S}}\frac{(n-s)!\,(s-1)!}{(n-1)!s}\Delta^{*}v(S)=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(n-s)!\,(s-1)!}{(n-1)!s}\Delta^{*}v(S).
\end{aligned}$$

□

Remark 12 A property similar to *equal collective gains* is introduced in Calvo and Gutiérrez-López (2013). It is called *equal expected total gains:*

$$\psi^{i}(N,v)+\sum_{k\in N\backslash i}\Delta^{k}\psi^{i}(N,v)=\psi^{j}(N,v)+\sum_{k\in N\backslash j}\Delta^{k}\psi^{j}(N,v),\quad (i,j\in N).$$

It is shown that this property, jointly with efficiency, characterize the *solidarity value*, introduced in Sprumont (1990) and Nowak and Radzik (1994), and defined by

$$\zeta^{i}(N,v)=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(n-s)!\,(s-1)!}{n!}\Delta^{av}v(S),\quad (i\in N),$$

where

$$\Delta^{av} v(S) = \frac{1}{s} \sum_{i \in S} \Delta^i v(S), \quad (S \subseteq N).$$

Or recursively by

$$\zeta^i(S, v) = \frac{1}{s} \Delta^{av} v(S) + \frac{1}{s} \sum_{k \in S \setminus i} \zeta^i(S \setminus k, v), \quad \text{for all } i \in S \subseteq N.$$

There is a simple interpretation of formula (5). Let ψ be a solution and (N, v) be a game. We define the *disagreement payoff* of i in coalition S, $i \in S$, denoted by $d^i_\psi(S, v)$, as $d^i_\psi(i, v) := 0$ and

$$d^i_\psi(S, v) := \frac{1}{s-1} \sum_{k \in S \setminus i} \psi^i\,(S \setminus k, v)\,, \quad \text{if } |S| \geq 2.$$

This amount is what player i expects to get with solution ψ in case there is no agreement in S and some player k other than i leaves the game.

The amount, $v(S) - \sum_{k \in S} d^k_\psi(S, v)$, can be interpreted as the benefit/loss of the cooperation that remains to be shared between the coalition members, after guaranteeing themselves their disagreement payoffs. Assuming that such benefit/loss is equally distributed, it follows that

$$\psi^i\,(S, v) = \frac{1}{s} \left[v(S) - \sum_{k \in S} d^k_\psi(S, v) \right] + d^i_\psi(S, v), \quad \text{for all } i \in S \subseteq N. \tag{7}$$

It turns out that ψ is the ECG-value, because, by efficiency, we have that

$$\begin{aligned} v(S) - \sum_{k \in S} d^k_\psi(S, v) &= v(S) - \frac{1}{s-1} \sum_{k \in S} \sum_{i \in S \setminus k} \psi^k(S \setminus i, v) \\ &= v(S) - \sum_{i \in S} \frac{v(S \setminus i)}{s-1} = \Delta^* v(S). \end{aligned}$$

Moreover, Eq. (7) implies that for all $|S| \geq 2$ and all $i, j \in S$, it also holds that

$$\psi^i(S, v) - d^i_\psi(S, v) = \psi^j(S, v) - d^j_\psi(S, v).$$

Remark 13 We can introduce asymmetries between players given by a vector of positive weights $w \in \mathbb{R}^N_{++}$. For each $i \in N$, w^i is the *weight* of player i. A *weighted solution* ψ_w is a function that assigns to every game (N, v) and every weight vector w, a vector $\psi_w(N, v) \in \mathbb{R}^N$. Weights are exogeneously given and independent of the game (N, v). They can be interpreted in different ways, depending of the context.

For example, weighted solutions can support the use of asymmetries in applications where the players themselves represent groups of individuals. This is the case when the player set N is a "contraction" of the original situation in which the player set M is the union of coalitions of partners, or teams, where the cardinality of each team N_i is $|N_i| = w^i$. All the players in a team are symmetric and the team must be completed in order to be effective. Another example is the distribution of amounts of a public good among N cities. Here, it is assumed that all the citizens of each city receive the same amount of public good, so all of them are symmetric with respect to the distribution of that good, and there are no subgroups of citizens that have access to the good while others in the same city cannot consume it, so all of them are partners. The behavior of values under this kind of "replica" games has been studied in Kalai (1977), Thomson (1986), Thomson and Lensberg (1989) and Calvo et al. (2000). Under this "population" interpretation, condition (1) can be reformulated as

$$\frac{1}{w^i}\left[\psi^i\left(\{i,j\},v\right)-\psi^i\left(i,v\right)\right]=\frac{1}{w^j}\left[\psi^j\left(\{i,j\},v\right)-\psi^j\left(j,v\right)\right]. \tag{8}$$

This can be interpreted as that the *per capita gain of player i (by the contribution of player j) is equal to the per capita gain of player j (by the contribution of player i)* For the n-person case the *weighted collective gains principle* takes the following form:

Weighted collective gains: $\left(1/w^i\right)\sum_{k\in N\backslash i}\Delta^k\psi^i\left(N,v\right) = \left(1/w^j\right)\sum_{k\in N\backslash j}\Delta^k\psi^j\left(N,v\right)$, for all $i, j\in N$

We can characterize the weighted collective gains value in a similar way as the symmetric one, and then we omit the proof.

Theorem 14 *Let $w\in\mathbb{R}^N_{++}$ be a vector of positive weights. Then, there exists a unique solution on $\mathcal{G}$ satisfying* efficiency *and* weighted collective gains*, and this is the weighted collective gains value χ_w defined recursively by*

$$\chi_w^i(S,v)=\frac{w^i}{\sum_{k\in S}w^k}\Delta^*v(S)+\sum_{k\in S\backslash i}\frac{1}{s-1}\chi_w^i(S\backslash k,v),\ \ \textit{for all } i\in S\subseteq N. \tag{9}$$

3 Comparison and interpretation of the value

In this section we show some basic properties that satisfies the ECG-value, and compare its behavior with regard to the solutions mentioned above.

Building the payoff configuration $(\chi\ (S,v))_{S\subseteq N}$ following Eq. (7), it could happen that $\Delta^*v(S)<0$ for some coalition S. Then, at this step, players are negotiating losses and prefer not to reach an agreement within S, since $\chi^i\ (S,v)<d_\chi^i(S,v)$. Therefore, in order to interpret the ECG-value as the plausible outcome of a gradual negotiation process, we need to impose some restriction on (N,v) to guarantee that $\Delta^*v(S)$ is

non-negative for every $S \subseteq N$. To this end, note that

$$\Delta^* v(S) = v(S) - \sum_{k \in S} \frac{v(S \backslash k)}{s-1} = \frac{1}{s-1}\left[s v(S) - \sum_{k \in S} v(S \backslash k) - v(S) \right]$$
$$= \frac{1}{s-1}\left[\sum_{k \in S} \Delta^k v(S) - v(S) \right].$$

According to this fact, we say that a game (N, v) satisfies *average productivity (AP) if*

$$\frac{1}{s} \sum_{k \in S} \Delta^k v(S) \geq \frac{v(S)}{s}. \text{ for all } S \subseteq N. \tag{10}$$

That is, the average of the marginal contributions is greater or equal than the per-capita worth.[5] We denote by $\mathcal{G}^{AP}$ the family of TU-games satisfying average productivity.

We now show the relationship between *AP* and some monotonicity properties in TU-games.

A game (N, v) is said *totally additive* if $v(S) = \sum_{k \in S} v(k)$ for all $S \subseteq N$. A solution ψ satisfies *individual rationality* if $\psi^i(N, v) \geq v(i)$ for all $i \in N$. If (N, v) is totally additive then it holds that $\Delta^* v(S) = 0$ for all $S \subseteq N$, with $|S| \geq 2$. This implies that $\chi^i(N, v) = v(i)$, for all $i \in N$, and therefore satisfies individual rationality in totally additive games.[6] On the contrary, the solidarity value does not satisfy this property in this class of games.

A game (N, v) is said *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T$. A game (N, v) is said *per-capita monotone* if

$$\frac{v(S)}{s} \leq \frac{v(T)}{t} \quad \text{whenever } S \subseteq T. \tag{11}$$

A game (N, v) is said *convex* (Shapley 1971) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$. Equivalently, (N, v) is convex if $\Delta^i v(S \cup i) \leq \Delta^i v(T \cup i)$ for all $i \in N$ and all $S, T \subseteq N \backslash i$ with $S \subseteq T$. A convex game describes the situation of increasing returns of cooperation: the larger the coalition to which player i belongs, the larger her marginal contribution to it.

Proposition 15 *If* (N, v) *is a per-capita monotone game, then* (N, v) *satisfies AP.*

Proof Let (N, v) satisfying (11), then

$$\frac{v(S)}{s} \geq \frac{v(S \backslash k)}{s-1} \text{ for all } k \in S \;\Rightarrow\; \frac{v(S)}{s} \geq \frac{1}{s} \sum_{k \in S} \frac{v(S \backslash k)}{s-1} \;\Leftrightarrow$$

[5] The term $1/s$ on both sides of the inequality (10) has been left for interpretation reasons.

[6] The same happens for the Shapley and the ENSC values.

$$(s-1)\,v(S) \geq \sum_{k \in S} v(S \backslash k) \;\Leftrightarrow\; s v(S) - \sum_{k \in S} v(S \backslash k) \geq v(S) \;\Leftrightarrow$$
$$\sum_{k \in S} \Delta_v^k(S) \geq v(S).$$

□

Proposition 16 *If (N, v) is a convex game, then (N, v) satisfies AP.*

Proof Let a coalition $S \subseteq N$, and π be an order defined in S. Denote by $P_\pi^k = \{j \in S : \pi(j) < \pi(k)\}$ the set of predecessors of player k in the order given by π. We have that

$$\sum_{k \in S} \left[v\left(P_\pi^k \cup i\right) - v\left(P_\pi^k\right) \right] = v(S).$$

By construction, it is always true that $P_\pi^k \subseteq S \backslash k$. Therefore, if (N, v) is a convex game, it holds that

$$v\left(P_\pi^k \cup i\right) - v\left(P_\pi^k\right) \leq v(S) - v(S \backslash k),$$

and then

$$v(S) = \sum_{k \in S} \left[v\left(P_\pi^k \cup i\right) - v\left(P_\pi^k\right) \right] \leq \sum_{k \in S} \left[v(S) - v(S \backslash k) \right] .$$

□

However, the property *AP* does not imply neither monotonicity, per-capita monotonicity, nor convexity.

Example 1: Let $N = \{1, 2, 3\}$ be the player set, $\alpha \in \mathbb{R}$ and v defined as

$$v(1) = 1 + 2\alpha;\;\; v(2) = v(3) = -\alpha\;;$$
$$v\left(\{1, 2\}\right) = v(\{1, 3\}) = 1 + \alpha\;;$$
$$v\left(\{2, 3\}\right) = 2 - 2\alpha;\;\; v\left(\{1, 2, 3\}\right) = 2.$$

This game satisfies *AP* but does not satisfy any of the other properties.

Moreover, it holds that $\sum_{k \in S} \Delta^k v(S) = v(S)$, for all coalitions $S \neq \{2, 3\}$. This fact simplifies the computation of $\chi\left(\{1, 2, 3\}, v\right)$ in this example. In particular, we find that

$$\chi^1(\{1, 2, 3\}, v) = 1 + 2\alpha,\;\; \chi^2(\{1, 2, 3\}, v) = \chi^3(\{1, 2, 3\}, v) = \frac{1}{2} - \alpha.$$

It is illustrative to compare these values with those obtained with φ, ϕ, and ζ:

$$\text{Shapley:}\;\; \varphi^1(\{1, 2, 3\}, v) = \frac{2}{3} + 2\alpha,\;\; \varphi^2(\{1, 2, 3\}, v) = \varphi^3(\{1, 2, 3\}, v) = \frac{2}{3} - \alpha,$$

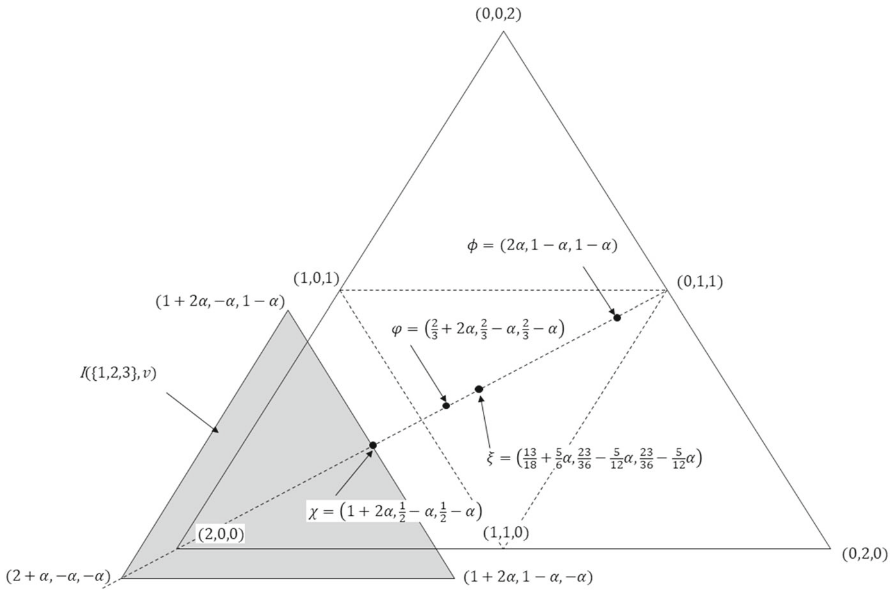

Fig. 1 The imputation set

$$\text{ENSC:}\ \ \phi^1(\{1,2,3\}, v) = 2\alpha,\ \phi^2(\{1,2,3\}, v) = \phi^3(\{1,2,3\}, v) = 1-\alpha,$$
$$\text{Solidarity:}\ \ \zeta^1(\{1,2,3\}, v) = \frac{13}{18} + \frac{5\alpha}{6},\ \zeta^2(\{1,2,3\}, v) = \zeta^3(\{1,2,3\}, v) = \frac{23}{36} - \frac{5\alpha}{12},$$

Remarkably, none of these last three values satisfy individual rationality for every $\alpha > -5/21$, unlike the ECG-value. We show this situation graphically in the following Fig. 1 for values of $\alpha \geq 0$.

The next proposition shows that if a game satisfies AP then the equal collective gains value satisfies individual rationality.

Proposition 17 *If (N, v) satisfies AP, then it holds that $\chi\,(N, v) \in I(N, v)$.*

Proof The proof is done by induction. Then one player case is trivial, because $\chi^i(i, v) = v(i)$. Assume that it holds for coalitions of size $s-1$. Therefore, by condition AP and (7), we have that

$$\chi^i(S, v) \geq d^i_\chi(S, v) = \frac{1}{s-1}\sum_{k\in S\backslash i}\chi^i(S\backslash k, v) \geq v(i),$$

as $\chi^i(S\backslash k, v) \geq v(i)$ for all $k \in S\backslash i$, by the induction hypothesis. □

What about the stability of the ECG-value? We do not have positive results on this. The *core* $C(N, v)$ of a game (N, v) is the set of efficient and coalitionally rational payoffs, that is

$$C(N, v) := \left\{ x \in \mathbb{R}^N : \sum_{k \in N} x^k = v(N), \text{ and } \sum_{k \in S} x^k \geq v(S), \text{ for all } S \subseteq N \right\}$$

Convex games have a nonempty core (Shapley 1971). Now, we show that if a game is convex, the equal collective gains value could be outside the core.

Example 2: Let $(\{1, 2, 3\}, v)$ be the game defined by $v(i) = 0$, $i = 1, 2, 3$; $v(\{1, 2\}) = 1$; $v(\{1, 3\}) = v(\{2, 3\}) = 0$; and $v(\{1, 2, 3\}) = 1$. This is a convex game and its core is given by the convex hull of $\{(1, 0, 0), (0, 1, 0)\}$. Therefore, any point x in the core must verify that $x^1 + x^2 = 1$ and $x^k \geq 0$, $k = 1, 2, 3$. The ECG-value is

$$\chi^1(\{1, 2, 3\}, v) = \frac{5}{12}, \quad \chi^2(\{1, 2, 3\}, v) = \frac{5}{12}, \quad \chi^3(\{1, 2, 3\}, v) = \frac{2}{12},$$

and then

$$\chi^1(\{1, 2, 3\}, v) + \chi^2(\{1, 2, 3\}, v) = \frac{10}{12} < 1$$

which means that $\chi(\{1, 2, 3\}, v)$ does not belong to the core.

All these values considered here belong to the large family of efficient, linear and symmetric values (ELS-values). There are several alternative characterizations of this family in the literarure (Ruiz et al. 1998; Driessen and Radzik 2003; Hernandez-Lamoneda et al. 2008; Chameni and Andjiga 2008; Chameni Nembua 2012; Casajus 2012). We recall here the Chameni's characterization:

Proposition 18 (Chameni Nembua 2012) *A value ψ^α is an ELS-value if, and only if, there exists a sequence of parameters $\alpha = \left(\left(\alpha_s^n\right)_{s=1}^n\right)_{n=1,2,\ldots}$, with $\alpha_s^n \in \mathbb{R}$ for all n and s, and $\alpha_1^n = 1$, such that*

$$\psi^{\alpha,i}(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} \Delta^{\alpha_s^n | i} v(S), \quad (i \in N), \tag{12}$$

where

$$\Delta^{\alpha_s^n | i} v(S) = \alpha_s^n \Delta^i v(S) + \frac{1 - \alpha_s^n}{s - 1} \sum_{k \in S \setminus i} \Delta^k v(S) \tag{13}$$

$$= v(S) - \alpha_s^n v(S \setminus i) - \frac{1 - \alpha_s^n}{s - 1} \sum_{k \in S \setminus i} v(S \setminus k). \tag{14}$$

When $\alpha_s^n \in [0, 1]$, the coefficients $\left(\alpha_s^n\right)_{s=1}^n$ have an intuitive interpretation: if coalition S forms, each player $i \in S$ receives a fraction α_s^n of her contribution $\Delta^i v(S)$, with the rest $\left(1 - \alpha_s^n\right) \Delta^i v(S)$ being equally shared among the remaining players in the coalition. Thus, player i receives a share α_s^n of her own contribution, plus a share $\left(1-\alpha_s^n\right)/(s-1)$ of the contribution of each of the other players in the coalition. Next proposition shows the α's parameters associated to these values.

Proposition 19

(a) *The ENSC-value* ϕ *is obtained when* $\alpha_n^n = n-1$ *and* $a_s^n = 0$ *for all* $1 < s < n$.
(b) *The Shapley value* φ *is obtained when* $\alpha_s^n = 1$ *for all* $n, s > 1$.
(c) *The ECG-value* χ *is obtained when* $\alpha_s^n = \frac{n}{s(s-1)}$ *for all* $n, s > 1$.
(d) *The solidarity value* ζ *is obtained when* $\alpha_s^n = 1/s$ *for all* $n, s > 1$.

These parameters α_s^n show the different weights that the values give to the contributions of the players to the coalitions they belong to, depending on the size of the coalition. For the Shapley value, only the own contribution $\Delta^i v(S)$ is considered, $\alpha_s^n = 1$, in all sizes s. The ECG-value gives larger weight to the own contribution $\Delta^i v(S)$ at lower size:

$$\alpha_n^n = \frac{1}{n-1} < \cdots < \frac{n}{s\,(s-1)} < \cdots < \alpha_2^n = \frac{n}{2}.$$

Something similar happens to the solidarity value:

$$\alpha_n^n = \frac{1}{n} < \cdots < \frac{1}{s} < \cdots < \alpha_2^n = \frac{1}{2}.$$

The opposite case is the ENSC-value, which gives weight only to the own contribution $\Delta^i v(N)$ in the grand coalition N: $\alpha_n^n = n-1$, and $a_s^n = 0$ otherwise.

4 Axiomatic characterizations

We here propose four additional axiomatizations of the equal collective gains value. The first one is done with *additivity* and a modification of the *null player* axiom, similar to that in Shapley (1953) for his value. The second one with a *relaxation* of the *equal collective gains* property, similar to that in Casajus (2017) for the Shapley value. This relaxation involves intra-personal utility comparisons but avoids inter-personal utility comparisons. The third one with the (van den Brink 2001)'s *fairness*, and the fourth one, applying the *equal collective gains property* only *to symmetric players*, following Yokote and Kongo (2017) for the Shapley value.

We need some additional definitions. Two players $i, j \in N$ are *symmetric* in (N, v) if $v\,(S \cup i) = v\,(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *null player* in (N, v) if $\Delta^i v\,(S) = 0$ for all $S \subseteq N$, $i \in S$. Player $i \in N$ is a *dummy player* in (N, v) if $\Delta^i v\,(S) = v(i)$ for all $S \subseteq N$, $i \in S$. For any two games (N, v) and $\left(N, v'\right)$ and $a, b \in \mathbb{R}$, the game $\left(N, av + bv'\right)$ is defined by $(av + bv')(S) = av(S) + bv'(S)$ for all $S \subseteq N$.

Consider the following properties of a solution ψ in $\mathcal{G}^N$:

Additivity (A): For all (N, v) and $(N, w) \in \mathcal{G}^N$, $\psi\,(N, v + w) = \psi\,(N, v) + \psi\,(N, w)$.

Symmetry (S): For all $(N, v) \in \mathcal{G}^N$ and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v), then $\psi^i(N, v) = \psi^j(N, v)$.

Null player axiom (N): For all $(N, v) \in \mathcal{G}^N$ and all $i \in N$, if i is a null player in (N, v), then $\psi^i(N, v) = 0$.

Dummy player axiom (D): For all $(N, v) \in \mathcal{G}^N$ and all $i \in N$, if i is a dummy player in (N, v), then $\psi^i(N, v) = v(i)$.

The following theorem is due to Shapley.

Theorem 20 (Shapley 1953) *A solution* ψ *on* $\mathcal{G}^N$ *satisfies* efficiency, additivity, symmetry *and* the null player axiom *if and only if* ψ *is the Shapley value.*

Many variations of the null player axiom have been successfully used to characterize other solutions (see Nowak and Radzik 1994; Ju et al. 2007; Kamijo and Kongo 2012; Chameni Nembua 2012; Casajus and Huettner 2014; van den Brink and Funaki 2015; Béal et al. 2015; Radzik and Driessen 2016). We recall the null player variation introduced in Nowak and Radzik (1994) for the characterization of the solidarity value: Player $i \in N$ is an *A-null player* in (N, v) if $\Delta^{av} v(S) = 0$ for all coalitions $S \subseteq N$ containing i, where $\Delta^{av} v(S) := 1/s \sum_{k \in S} \Delta^k v(S)$.

Theorem 21 (Nowak and Radzik 1994) *A solution* ψ *on* $\mathcal{G}^N$ *satisfies* efficiency, additivity, symmetry *and* the A-null player axiom *if and only if* ψ *is the solidarity value.*

We introduce here a close version of the A-null player axiom. We say that player $i \in N$ is an *AP-dummy player* in (N, v) if

$$\Delta^{av} v(S) = \frac{v(S)}{s}$$

for all coalitions $S \subseteq N$ containing i, $|S| \geq 2$. Notice that $\Delta^{av} v(S) = v(S)/s$ is equivalent to $\Delta^* v(S) = 0$. We say that a player is an *AP-null player* in (N, v) if i is an AP-dummy player and $v(i) = 0$. It is clear that the equal collective gains value satisfies efficiency, additivity, symmetry and the following axioms:

Definition 22 (*AP-Dummy player axiom (AP-D)*) For all $(N, v) \in \mathcal{G}^N$ and all $i \in N$, if i is an *AP*-dummy player in (N, v), then $\psi^i(N, v) = v(i)$.

Definition 23 (*AP-Null player axiom (AP-N)*) For all $(N, v) \in \mathcal{G}^N$ and all $i \in N$, if i is an *AP*-null player in (N, v), then $\psi^i(N, v) = 0$.

We define a new basis for $\mathcal{G}^N$, denoted by $\{(N, v_T)\}_{\emptyset \neq T \subseteq N}$. For all $\emptyset \neq T \subseteq N$, (N, v_T) is defined by

$$v_T(S) = \begin{cases} \binom{|S|-1}{|T|-1}^{-1}, & \text{if } S \supseteq T, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

It is easy to see that $\Delta^*_{v_T}(S) = 0$ for all coalitions $S \neq T$. Therefore, it is immediate that all players in $N \setminus T$ are AP-null players in the game (N, v_T), so they receive a zero payoff, and all players in T are symmetric so they receive the same payoff. Thus, it holds that

$$\chi^i(N, v_T) = \begin{cases} \frac{1}{|T|} v_T(N), & \text{if } i \in T, \\ 0, & \text{otherwise.} \end{cases}$$

We now present our first characterization of χ.

Theorem 24 *A solution ψ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* the AP-null player axiom *if and only if ψ is the equal collective gains value.*

Proof We omit it because it parallels that for the Shapley value by Shapley (1953), or for the solidarity value by Nowak and Radzik (1994). □

Casajus (2017) introduces a relaxation of the balanced contributions property called the *weak balanced contributions property*.

Definition 25 (*Weak balanced contributions property (WBC)*) sign $\left(\Delta^i \psi^j(N, v)\right) =$ sign $\left(\Delta^j \psi^i(N, v)\right)$ for all $i, j \in N$.

Recall that the sign function, sign: $\mathbb{R} \to \{-1, 0, 1\}$ is given by sign$(x) = 1$ for $x > 0$, sign$(0) = 0$, and sign$(x) = -1$ for $x < 0$.

As Casajus pointed "Since the balanced contributions property equates the differences of two players' payoffs, it implicitly involves the interpersonal comparison of utilities. Inter-personal utility comparison, however, is often criticized from the viewpoint of utility theory." Therefore, relaxing this principle, requiring that these payoffs' variations only change in the *same direction*, avoid this kind of *quantitative* interpersonal utility comparisons, resting only in a *qualitative* comparison. It turns out that there exists a large class of solutions that satisfy efficiency and the weak balanced contributions property, among them the subclass of weighted Shapley values. Adding *weak differential marginality* to efficiency and weak balanced contributions, Casajus (2017, Theorem 2) recovers the Shapley value.

The principle of *differential marginality* was introduced in Casajus (2011) to characterize the Shapley value together with efficiency and the null player property. It indicates that whenever two agents' productivities change by the same amount, then their payoffs change in the same amount:

Definition 26 *Differential marginality*, (DM): For all (N, v) and $(N, w) \in \mathcal{G}^N$, and $i, j \in N$, if $v(S \cup i) - w(S \cup i) = v(S \cup j) - w(S \cup j)$, for all $S \subseteq N \setminus \{i, j\}$, then

$$\psi^i(N, v) - \psi^i(N, w) = \psi^j(N, v) - \psi^j(N, w).$$

Weak differential marginality is a relaxation of differential marginality. It indicates that whenever two agents' productivities change by the same amount, then their payoffs change in the *same direction*:

Definition 27 (*Weak differential marginality (WDM)*) For all (N, v) and $(N, w) \in \mathcal{G}^N$, and $i, j \in N$, if $v(S \cup i) - w(S \cup i) = v(S \cup j) - w(S \cup j)$, for all $S \subseteq N \setminus \{i, j\}$, then

$$\operatorname{sign}\left(\psi^i(N, v) - \psi^i(N, w)\right) = \operatorname{sign}\left(\psi^j(N, v) - \psi^j(N, w)\right).$$

Casajus and Yokote (2017) prove that for games with more than two players, the Casajus' characterization (Casajus 2011) can be improved by using weak differential marginality instead of differential marginality.

Theorem 28 (Casajus and Yokote 2017) *Let $|N| \neq 2$. The Shapley value is the unique solution that satisfies* efficiency*, the* null player property *and* weak differential marginality.

If the null player property is strengthened into the dummy player property, this theorem also holds for $|N| = 2$.

Casajus (2017) proves the following theorem that rests on the fact that efficiency and the weak balanced contributions property imply the dummy player property (Lemma 1, Casajus 2017).

Theorem 29 (Casajus 2017) *The Shapley value is the unique solution on $\mathcal{G}$ that satisfies the properties of* efficiency*,* weak balanced contributions*, and* weak differential marginality.

Following Casajus (2017), we suggest a relaxation of the equal collective gains property, called the *weak equal collective gains* property in order to characterize the ECG-value χ with efficiency and weak differential marginality.

Definition 30 (*Weak equal collective gains property (WECG)*)

$$\operatorname{sign}\left(\sum_{k \in N \setminus i} \Delta^k \psi^i(N, v)\right) = \operatorname{sign}\left(\sum_{k \in N \setminus j} \Delta^k \psi^j(N, v)\right), \text{ for all } i, j \in N.$$

As χ satisfies equal collective gains property, its obvious that χ also satisfies *WECG*. We now prove that χ also satisfies differential marginality.

Proposition 31 *The ECG-value χ satisfies differential marginality.*

Proof Let (N, v) and $(N, w) \in \mathcal{G}^N$, and $i, j \in N$, such that $v(S \cup i) - w(S \cup i) = v(S \cup j) - w(S \cup j)$, for all $S \subseteq N \setminus \{i, j\}$. Then, it holds that

$$\Delta_v^*(S \cup i) - \Delta_w^*(S \cup i) = \Delta_v^*(S \cup j) - \Delta_w^*(S \cup j), \text{ for all } S \subseteq N \setminus \{i, j\}.$$

Then,

$$\chi^i(N, v) - \chi^i(N, w) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!\,(s-1)!}{(n-1)!s} \left(\Delta_v^*(S) - \Delta_w^*(S)\right)$$

$$\begin{aligned}
&= \sum_{\substack{S\subseteq N\\ i\in S, j\notin S}} \frac{(n-s)!\,(s-1)!}{(n-1)!s}\left(\Delta_v^*(S)-\Delta_w^*(S)\right)\\
&\quad + \sum_{\substack{S\subseteq N\\ i\in S, j\in S}} \frac{(n-s)!\,(s-1)!}{(n-1)!s}\left(\Delta_v^*(S)-\Delta_w^*(S)\right)\\
&= \sum_{\substack{S\subseteq N\\ j\in S, i\notin S}} \frac{(n-s)!\,(s-1)!}{(n-1)!s}\left(\Delta_v^*(S)-\Delta_w^*(S)\right)\\
&\quad + \sum_{\substack{S\subseteq N\\ i\in S, j\in S}} \frac{(n-s)!\,(s-1)!}{(n-1)!s}\left(\Delta_v^*(S)-\Delta_w^*(S)\right)\\
&= \chi^j(N,v)-\chi^j(N,w).
\end{aligned}$$

□

This fact implies that χ also satisfies weak differential marginality.

We now prove our second characterization.

Theorem 32 *The ECG-value χ is the unique solution on $\mathcal{G}$ that satisfies* efficiency*, the* weak equal collective gains property *and* weak differential marginality.

Proof *Existence*. We already know that χ satisfies efficiency, weak differential marginality and weak equal collective gains.

Uniqueness. Let ψ be a solution satisfying the above axioms. Let (N, v) be a game. If $|N| = 1$, by E, $\psi^i(N, v) = v(i) = \chi^i(N, v)$. Suppose $|N| \geq 2$. We show that $\psi = \chi$ by induction.

First, we define the game (N, w) as

$$\begin{cases} w(N\backslash i) = \frac{n-1}{n}\Delta_v^*(N) + v(N\backslash i), \text{ for all } i \in N,\\ w(S) = v(S), \text{ for all } S \neq N\backslash i,\ i \in N. \end{cases}$$

For all $i, j \in N$, it holds that

$$v(S\cup i) - w(S\cup i) = v(S\cup j) - w(S\cup j) = \begin{cases} -\frac{n-1}{n}\Delta_v^*(N), & \text{for } S = N\setminus\{i,j\},\\ 0, & \text{otherwise}. \end{cases}$$

Therefore, by *WDM*, $\text{sign}\left(\psi^i(N,v) - \psi^i(N,w)\right) = \text{sign}\left(\psi^j(N,v) - \psi^j(N,w)\right)$, for all $i, j \in N$, and then, by E and $v(N) = w(N)$, necessarily $\psi^i(N, v) = \psi^i(N, w)$ for all $i \in N$.

Assume by induction that $\psi^i(N\backslash k, w) = \chi^i(N\backslash k, w)$ for all $k \in N$. By construction of χ, it holds that $\chi^i(N\backslash k, w) = \frac{1}{n}\Delta_v^*(N) + \chi^i(N\backslash k, v)$. Applying *WECG* and the induction hypothesis, we obtain that

$$\text{sign}\left(\psi^i(N,w) - \frac{1}{n-1}\sum_{k\in N\backslash i}\chi^i(N\backslash k, w)\right)$$

$$= \text{sign}\left(\psi^j(N,w) - \frac{1}{n-1}\sum_{k\in N\setminus j}\chi^j(N\setminus k, w)\right), \text{ for all } j \in N.$$

Therefore,

$$\begin{aligned}
&\text{sign}\left(\psi^i(N,w) - \frac{1}{n-1}\sum_{k\in N}\chi^i(N\setminus k, w)\right)\\
&= \text{sign}\left(\sum_{j\in N}\left(\psi^j(N,w) - \frac{1}{n-1}\sum_{k\in N\setminus j}\chi^j(N\setminus k, w)\right)\right)\\
&= \text{sign}\left(w(N) - \frac{1}{n-1}\sum_{k\in N} w(N\setminus k)\right) = \text{sign}\left(\Delta_v^*(N) - \Delta_v^*(N)\right) = 0.
\end{aligned}$$

This implies, for all $i \in N$,

$$\begin{aligned}
\psi^i(N,v) = \psi^i(N,w) &= \frac{1}{n-1}\sum_{k\in N\setminus i}\chi^i(N\setminus k, w)\\
&= \frac{1}{n}\Delta_v^*(N) + \frac{1}{n-1}\sum_{k\in N\setminus i}\chi^i(N\setminus k, v) = \chi^i(N,v).
\end{aligned}$$

□

Remark 33 The axiom system in Theorem 32 is independent. Indeed:

(1) The solution ψ given by $\psi^i(N,v) = 0$ for all $i \in N$ and $(N,v) \in \mathcal{G}$ satisfies the axioms of Theorem 32 except efficiency.
(2) The Shapley value satisfies the axioms of Theorem 32 except the *weak equal collective gains* property.
(3) For $(N,v) \in \mathcal{G}^N$, let $N_0(v) = \{i \in N : i \text{ is a } AP - \text{null player in } (N,v)\}$. The solution ψ given by

$$\psi^i(N,v) = \begin{cases} \frac{v(N)}{|N\diagdown N_0(v)|}, & i \in N\diagdown N_0(v),\\ 0, & i \in N_0(v).\end{cases}$$

satisfies the axioms of Theorem 32 except weak differential marginality.

It can also be proved the following proposition.

Proposition 34 *If a solution satisfies efficiency and the weak equal collective gains property, then it also satisfies the AP-dummy player axiom.*

Proof Let ψ be a solution satisfying E and *WECG*, and let $i \in N$ be a AP-*dummy player* in(N,v). If $|N| = 1$, by E, $\psi^i(N,v) = v(i)$. Suppose $|N| \geq 2$ and assume by induction that $\psi_v^i(N \setminus k) = v(i)$ for all $k \in N$. By *WECG*,

$$\text{sign}\left(\psi^i(N, v) - \frac{1}{n-1}\sum_{k\in N\setminus i}\psi^i(N\setminus k, v)\right)$$
$$= \text{sign}\left(\psi^j(N, v) - \frac{1}{n-1}\sum_{k\in N\setminus j}\psi^j(N\setminus k, v)\right), \text{ for all } j \in N.$$

Therefore,

$$\text{sign}\left(\psi^i(N, v) - \frac{1}{n-1}\sum_{k\in N\setminus i}\psi^i(N\setminus k, v)\right)$$
$$= \text{sign}\left(\sum_{j\in N}\left(\psi^j(N, v) - \frac{1}{n-1}\sum_{k\in N\setminus j}\psi^j(N\setminus k, v)\right)\right)$$
$$= \text{sign}\left(v(N) - \frac{1}{n-1}\sum_{k\in N} v(N\setminus k)\right) = \text{sign}\left(\Delta_v^*(N)\right) = 0.$$

That is,

$$\psi^i(N, v) - \frac{1}{n-1}\sum_{k\in N\setminus i}\psi^i(N \setminus k, v) = 0,$$

and by the induction hypothesis, $\psi^i(N, v) = v(i)$. □

Proposition 31 shows that the ECG-value satisfies differential marginality. Casajus (2011) proves that *DM* is equivalent to the van den Brink (2001) *fairness*:

Definition 35 (*Fairness (BF)*) If $i, j \in N$ are symmetric in $(N, w) \in \mathcal{G}^N$ then, $\psi^i(N, v + w) - \psi^i(N, v) = \psi^j(N, v + w) - \psi^j(N, v)$, for all $(N, v) \in \mathcal{G}^N$.

van den Brink (2001) proves that fairness, efficiency and the null player axiom characterize the Shapley value. We now prove a similar characterization of the ECG-value.

Theorem 36 *The ECG-value* χ *is the unique solution on* $\mathcal{G}^N$ *that satisfies* efficiency, *the* AP-null player axiom *and* fairness.

Proof *Existence*. We already know that χ satisfies efficiency and the *AP*-null player axiom. Moreover, Proposition 31 shows that the ECG-value satisfies differential marginality and Casajus (2011) proves that *DM* is equivalent to fairness.

Uniqueness. Let ψ be a solution satisfying the above axioms.

First, we show that the AP-null player axiom and fairness imply symmetry. Indeed, for the *null game* $(N, v_0) \in \mathcal{G}^N$ given by $v_0(S) = 0$ for all $S \subseteq N$, the AP-null player axiom implies that $\psi^i(N, v_0) = 0$ for all $i \in N$. Let (N, v) be a game. If $i, j \in N$ are symmetric in (N, v), then fairness implies that

$$\begin{aligned}\psi^i(N, v) &= \psi^i(N, v + v_0) = \psi^i(N, v + v_0) - \psi^i(N, v_0) \\ &= \psi^j(N, v + v_0) - \psi^j(N, v_0) = \psi^j(N, v).\end{aligned}$$

Thus, ψ satisfies symmetry.

On the other hand, Casajus (2011), Proposition 6 proves that for $|N| \neq 2$, efficiency, *null game* ($\psi^i(N, v_0) = 0$ for all $i \in N$, and differential marginality imply additivity. Since the AP-null player axiom implies null game, and fairness is equivalent to *DM*, then ψ satisfies *A* for $|N| \neq 2$, which, in view of Theorem 24, proves the claim.

Remains the uniqueness for $|N| = 2$. Let $N = \{1, 2\}$ and $(N, v) \in \mathcal{G}^N$. We now use the fact that the games $\{(N, v_T)\}_{\emptyset \neq T \subseteq N}$ form a basis for $\mathcal{G}^N$. Thus, $v = \alpha_N v_N + \alpha_1 v_{\{1\}} + \alpha_2 v_{\{2\}}$, where the constants α_T are uniquely determined by the game (N, v). Define $(N, w) \in \mathcal{G}^N$ as $w = -\alpha_N v_N - \alpha_2 \left(v_{\{1\}} + v_{\{2\}}\right)$. We have that $v + w = (\alpha_1 - \alpha_2)\, v_{\{1\}}$ and players 1 and 2 are symmetric in (N, w). By *BF* and *E*,

$$\begin{aligned}&\psi^1(N, v) - \psi^2(N, v) = \psi^1(N, v + w) - \psi^2(N, v + w) \text{ and} \\ &\psi^1(N, v) + \psi^2(N, v) = v(N).\end{aligned}$$

Since player 2 is an AP-null player in $(N, v + w)$, $\psi(N, v + w)$ is uniquely determined by E and AP-N. Hence, $\psi(N, v)$ is unique too. □

We now introduce a different relaxation of the *equal collective gains* property, applying it only to symmetric players, following Yokote and Kongo (2017).

Definition 37 (*Equal collective gains property for symmetric players (ECGS)*) For all $(N, v) \in \mathcal{G}^N$ and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v), then

$$\sum_{k \in N \setminus i} \Delta^k \psi^i(N, v) = \sum_{k \in N \setminus j} \Delta^k \psi^j(N, v).$$

Adding a new axiom, called *AP-marginality,* to efficiency and the equal collective gains property for symmetric players, we characterize the ECG-value. This new property is similar to that of Young (1985) but with the average of the marginal contributions instead of the individual marginal contributions.

Definition 38 (*AP-marginality*) For all (N, v) and $\left(N, v'\right) \in \mathcal{G}^N$, if for some player $i \in N$, we have $\Delta^*(v, S \cup i) = \Delta^*(v', S \cup i)$, for all $S \subseteq N \setminus i$, then $\psi^i(N, v) = \psi^i\left(N, v'\right)$.

We now prove our fourth characterization.

Theorem 39 *The ECG-value χ is the unique solution on $\mathcal{G}$ that satisfies* efficiency*, the* equal collective gains property for symmetric players *and* AP-marginality.

Proof *Existence*. It only remains to prove that χ satisfies *AP-marginality*, but this is straightforward taking into account formula 6.

Uniqueness. Let ψ be a solution satisfying the above axioms. Let (N, v) be a game.

If $|N| = 1$, by *E*, $\psi^i\,(N, v) = v(i) = \chi^i(N, v)$. Suppose $|N| \geq 2$. We show that $\psi = \chi$ by induction. Assume that $\psi\,(S, v) = \chi(S, v)$ for all (S, v) with $|S| < n$.

Let (N, v_0) be the game defined as $v_0(S) = 0$ for all $S \subseteq N$. First, we prove that $\psi^i(N, v_0) = 0$ for all $i \in N$. Indeed, by *ECGS* we have that

$$\psi^i\,(N, v_0) - \frac{1}{n-1}\sum_{k\in N\backslash i}\psi^i\,(N\backslash k, v_0) = \psi^j\,(N, v_0)$$
$$-\frac{1}{n-1}\sum_{k\in N\backslash i}\psi^j\,(N\backslash k, v_0)\,, \text{ for all } i, j \in N.$$

By the induction hypothesis, $\psi^i\,(N \setminus k, v_0) = \chi^i\,(N\backslash k, v_0) = 0$, for all $i, k \in N$, therefore, $\psi^i\,(N, v_0) = \psi^j\,(N, v_0)\,$, for all $i, j \in N$, and by *E*, we obtain $\psi^i\,(N, v_0) = 0$ for all $i \in N$.

If $i \in N$ is an $AP-$null player in (N, v), then $\Delta^*(v, S \cup i) = 0 = \Delta^*(v_0, S \cup i)$, for all $S \subseteq N \diagdown i$, thus by *AP-marginality*, $\psi^i\,(N, v) = \psi^i\,(N, v_0) = 0 = \chi^i(N, v)$. Hence, it only remains to show that $\psi^i(N, v)$ is uniquely determined when $i \in N$ is not an *AP*-null player.

Now consider the game $(N, \alpha v_T)$ with $\alpha \neq 0$ and $\emptyset \neq T \subseteq N$. If $|T| = 1$, by *efficiency*, $\psi^i(N, \alpha v_T) = \alpha v_T(N)$ for $\{i\} = T$. Suppose that $|T| \geq 2$, then by *ECGS* we have that

$$\psi^i\,(N, \alpha v_T) - \frac{1}{n-1}\sum_{k\in N\backslash i}\psi^i\,(N\backslash k, \alpha v_T) = \psi^j\,(N, \alpha v_T)$$
$$-\frac{1}{n-1}\sum_{k\in N\backslash i}\psi^j\,(N\backslash k, \alpha v_T)\,, \text{ for all } i, j \in T, \tag{16}$$

and

$$\chi^i\,(N, \alpha v_T) - \frac{1}{n-1}\sum_{k\in N\backslash i}\chi^i\,(N\backslash k, \alpha v_T) = \chi^j\,(N, \alpha v_T)$$
$$-\frac{1}{n-1}\sum_{k\in N\backslash i}\chi^j\,(N\backslash k, \alpha v_T)\,, \text{ for all } i, j \in T, \tag{17}$$

By the induction hypothesis, $\psi^i\,(N\backslash k, \alpha v_T) = \chi^i\,(N\backslash k, \alpha v_T)$, for all $i, k \in N$, therefore,

$$\psi^i\,(N, \alpha v_T) - \chi^i\,(N, \alpha v_T) = \psi^j\,(N, \alpha v_T) - \chi^j\,(N, \alpha v_T)\,, \text{ for all } i, j \in T, \tag{18}$$

and, by *E*, $\psi^i\,(N, \alpha v_T) = \chi^i\,(N, \alpha v_T)$, for all $i \in T$.

We now use the fact that the games $\{(N, v_T)\}_{\emptyset \neq T \subseteq N}$ form a basis for $\mathcal{G}^N$. Thus,

$$(N, v) = \sum_{\emptyset \neq T \subseteq N} (N, \alpha_T v_T),$$

where the constants α_T are uniquely determined by the game (N, v). Let $I(N, v) = \{T \subseteq N : \alpha_T \neq 0\}$. We now proceed by induction over $|I(N, v)|$. We already know that $\psi(N, v)$ is uniquely determined when $|I(N, v)| \leq 1$. Suppose that it is true for every game (N, v) with $|I(N, v)| \leq k$. Let (N, v) be a game with $|I(N, v)| = k + 1$. Then, we have $k + 1$ nonempty coalitions $T_1, \ldots, T_{k+1}$ such that

$$(N, v) = \sum_{j=1}^{k+1} \left(N, \alpha_{T_j} v_{T_j}\right).$$

Let $T = T_1 \cap \cdots \cap T_{k+1}$ and suppose that $i \notin T$. Define a new game (N, v') as

$$(N, v') = \sum_{j: i \in T_j} \left(N, \alpha_{T_j} v_{T_j}\right).$$

Then, $\left|I(N, v')\right| \leq k$ and $\Delta^*(v, S \cup i) = \Delta^*(v', S \cup i)$, for all $S \subseteq N \diagdown i$, thus by *AP-marginality*, $\psi^i(N, v) = \psi^i(N, v')$, but $\psi^i(N, v')$ is uniquely determined by induction hypothesis. Suppose now that $i \in T$. If $|T| = 1$, by *E*, $\psi^i(N, v)$ is uniquely determined. If $|T| \geq 2$, by *ECGS* and proceeding in the same way as before in (16), (17) and (18), we obtain

$$\psi^i(N, v) - \chi^i(N, v) = \psi^j(N, v) - \chi^j(N, v), \text{ for all } i, j \in T,$$

and by *efficiency*, since $\psi^k(N, v)$ is uniquely determined for all $k \in N \diagdown T$, we conclude that $\psi^i(N, v)$ is also uniquely determined for all $i \in T$. □

5 Strategic support

In this section we show a negotiation model that brings a complementary support for the equal collective gains value. This in the tradition of the well-known "Nash program" as he points out

> "…The two approaches to the problem, via the negotiation model or via the axioms, are complementary. Each helps to justify and clarify the other."
> (Nash 1953, p. 128).

The definition (6) suggests the following negotiation model to implement the ECG-value. This is a non cooperative game of alternating offers by a random proposer, similar to that introduced in Hart and Mas-Colell (1996).

Let $(N, v) \in G^N$ be a TU-game and $0 \leq \rho < 1$ be a fixed parameter:

In each *round* there is a set $S \subseteq N$ of *active* players, and a *proposer* $i \in S$. In the first round, the active set is $S = N$. The proposer is chosen at random from S, with all players in S being equally likely to be selected. The proposer $i \in S$ makes an offer $\left(a^j_{S,i}\right)_{j\in S}$, where $a^j_{S,i}$ is the proposal made by i to j in coalition S. The offer must be feasible, i.e. $\sum_{j\in N} a^j_{S,i} \leq v(S)$. If all members of S accept the offer -they are asked in some prespecified order- then the game ends with these payoffs. If the offer is rejected by even one member j of $S\backslash i$, then, with probability ρ, we move to the next round where the set of active players again is S, and, with probability $1-\rho$, a *breakdown* occurs: a player k in S other than the responder j, being equally likely to be selected, leaves the game obtaining a payoff of $v(k)$, and the set of active players becomes $S\backslash k$.

The only difference with the Hart and Mas-Colell model is that in there, when breakdown occurs, the proposer i leaves the game obtaining a payoff of *zero*. In this way the SP (subgame perfect) equilibrium offers converge to the Shapley value when $\rho \to 1$ in the class of monotonic TU-games.

In the next theorem we offer the characterization of the equilibrium proposals.

Theorem 40 *Let $(N, v) \in \mathcal{G}^{AP}$. Then, for each specification of the parameter $\rho \in \mathbb{R}$ with $0 \leq \rho < 1$, there is an SP equilibrium. The proposals corresponding to an SP equilibrium are always accepted (i.e., at any information set where a player responds, it accepts the proposal made by the proposer), and they are characterized by:*

(P.1) $a^i_{S,i}(\rho) = v(S) - \sum_{j\in S\backslash i} a^j_{S,i}(\rho)$ *for each* $i \in S \subseteq N$*; and*

(P.2) $a^j_{S,i}(\rho) = \rho a^j_S(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash j} a^j_{S\backslash k}(\rho)$, *for each* $i, j \in S$ *with* $i \neq j$, *and each* $S \subseteq N$;
where $a_S(\rho) = \frac{1}{s}\sum_{j\in S} a_{S,j}(\rho)$. *Moreover, these proposals are unique and* $a^k_S(\rho) \geq v(k)$ *for each* $k \in S$.

Condition (P.2) says that i proposes to each $j \in S\backslash i$ the expected payoff that j would get in the continuation of the game in case of rejection; and (P.1) says that i gets for itself the remainder up to complete $v(S)$. Both conditions, (P.1) and (P.2), imply efficiency of the proposals, i.e. $\sum_{j\in S} a^j_{S,i}(\rho) = v(S)$, and hence the averages of the proposals are also efficient, i.e. $\sum_{j\in S} a^j_S(\rho) = v(S)$.

Proof The proof is done by induction. The one-player case is immediate. Assume that it is true for less than s players. Let $a_{S,i}(\rho)$, for $i \in S \subseteq N$, be the proposals of a given SP equilibrium, and denote by $c_S \in \mathbb{R}^S$ the expected payoff vector for the members of S in the subgame where S is the set of active players. By (P.1) and (P.2) it holds that $\sum_{j\in S} c^j_S = v(S)$. The induction hypothesis implies that $c_{S\backslash k} = a_{S\backslash k}(\rho)$ for all $k \in S$.

Let $d_{S,i} \in \mathbb{R}^S$ be defined by

$$d^j_{S,i} := \rho c^j_S + (1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash j} a^j_{S\backslash k}(\rho), \quad (j \in S\backslash i),$$

and

$$d^i_{S,i} := v(S) - \sum_{j\in S\setminus i} d^j_{S,i}.$$

The amount $d^j_{S,i}$ is the expected payoff of j following a rejection of i's proposal. Hence, rejecting this proposal, player j gets at most $d^j_{S,i}$, then he has no incentive to reject it. Therefore, $d_{S,i}$ is the best proposal for i among the proposals that will be accepted if i is the proposer. In addition, any proposal of i which is rejected by some j yields to i at most[7]

$$\begin{aligned}
&\rho c^i_S(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\setminus j} a^i_{S\setminus k}(\rho)\\
&\quad = \rho c^i_S(\rho) + (1-\rho)\frac{1}{s-1}\left[v(i) + \sum_{k\in S\setminus j\setminus i} a^i_{S\setminus k}(\rho)\right]
\end{aligned}$$

But

$$\begin{aligned}
&d^i_{S,i} - \left(\rho c^i_S(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\setminus j} a^i_{S\setminus k}(\rho)\right)\\
&\quad = v(S) - \sum_{j\in S\setminus i}\left(\rho c^j_S + (1-\rho)\frac{1}{s-1}\sum_{k\in S\setminus j} a^j_{S\setminus k}(\rho)\right)\\
&\quad = (1-\rho)\, v(S) - (1-\rho)\frac{1}{s-1}\sum_{j\in S}\sum_{k\in S\setminus j} a^j_{S\setminus k}(\rho)\\
&\qquad + \left(\rho c^i_S(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\setminus i} a^i_{S\setminus k}(\rho)\right)\\
&\qquad - \left(\rho c^i_S(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\setminus j} a^i_{S\setminus k}(\rho)\right)\\
&\quad = (1-\rho)\left(v(S) - \frac{1}{s-1}\sum_{j\in S} v(S\setminus j)\right)\\
&\qquad + (1-\rho)\frac{1}{s-1}\left[\sum_{k\in S\setminus i} a^i_{S\setminus k}(\rho) - \sum_{k\in S\setminus j} a^i_{S\setminus k}(\rho)\right].
\end{aligned}$$

[7] Recall that $a^i_{S\setminus i}(\rho) = v(i)$ by the rules of the negotiating game.

By the AP condition we have that

$$v(S) - \frac{1}{s-1} \sum_{j \in S} v(S \backslash j) \geq 0,$$

and, by induction,

$$\sum_{k \in S \backslash i} a^i_{S \backslash k}(\rho) - \sum_{k \in S \backslash j} a^i_{S \backslash k}(\rho) = a^i_{S \backslash j}(\rho) - v(i) \geq 0.$$

Hence, $d^i_{S,i} \geq \left(\rho c^i_S(\rho) + (1-\rho)\frac{1}{s-1} \sum_{k \in S \backslash j} a^i_{S \backslash k}(\rho) \right)$, and then player i has no incentive to make proposals that will be rejected. Therefore, player i will propose $a_{S,i}(\rho) = d_{S,i}$ and the proposal will be accepted for every responder j. Thus, it follows that $d_{S,i}$ satisfies (P.1) and (P.2) for all $i \in S$. Therefore it holds that $c_S = a_S(\rho)$. To see that $a^i_S(\rho) \geq v(i)$, note that the following strategy will guarantee to i a payoff of at least $v(i)$: accept only if offered at least $v(i)$, and, when proposing, propose $v(i)$ for himself. □

Remark 41 In the Hart and Mas-Colell model, monotonicity of v is a sufficient condition to guarantee that neither, proposer or respondent, has incentive to follow a strategy to leave the game. The average productivity condition plays the same role in our negotiation model, ensuring that no one has incentive to leave the game.

Theorem 42 *Let* $(N, v) \in \mathcal{G}^{AP}$. *Then,*

(1) *for every* $\rho \in \mathbb{R}$, $(0 \leq \rho < 1)$ *there is a unique SP equilibrium. Moreover, for all* $i \in S \subseteq N$, *the SP equilibrium average payoff vector* $a^i_S(\rho)$ *equals the equal collective gains value* $\chi^i(S, v)$*; and*
(2) *when* $\rho \to 1$ *it holds that* $\left| a^i_{S,i}(\rho) - a^i_{S,j}(\rho) \right| \to 0$, *for all* $j \in S \backslash i$.

(1) says that the equilibrium proposals coincide with the ECG-value in the *average* and (2) that these equilibrium proposals coincide *exactly* when $\rho \to 1$.

Proof (1) Existence of the SP equilibrium follows from Theorem (40). Now, let $i \in S \subseteq N$. By (P.1) and (P.2) we have that

$$\begin{aligned} s a^i_S(\rho) &= \left(v(S) - \sum_{j \in S \backslash i} a^j_{S,i}(\rho) \right) + \sum_{j \in S \backslash i} a^i_{S,j}(\rho) \\ &= \left(v(S) - \sum_{j \in S \backslash i} \left[\rho a^j_S(\rho) + (1-\rho)\frac{1}{s-1} \sum_{k \in S \backslash j} a^j_{S \backslash k}(\rho) \right] \right) \\ &\quad + (s-1) \left[\rho a^i_S(\rho) + (1-\rho)\frac{1}{s-1} \sum_{k \in S \backslash i} a^i_{S \backslash k}(\rho) \right] \end{aligned}$$

$$
\begin{aligned}
&= v(S) - \sum_{j\in S} \rho a_S^j(\rho) + s\rho a_S^i(\rho) - (1-\rho)\sum_{j\in S\backslash i}\frac{1}{s-1}\sum_{k\in S\backslash j} a_{S\backslash k}^j(\rho)\\
&\quad - (1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho) + s(1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho)\\
&= (1-\rho)v(S) + s\rho a_S^i(\rho) - (1-\rho)\frac{1}{s-1}\sum_{j\in S}\sum_{k\in S\backslash j} a_{S\backslash k}^j(\rho)\\
&\quad + s(1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho).
\end{aligned}
$$

Therefore,

$$
sa_S^i(\rho) = v(S) - \frac{1}{s-1}\sum_{k\in S} v(S\backslash k) + s\frac{1}{s-1}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho),
$$

and then

$$
a_S^i(\rho) = \frac{1}{s}\left[v(S) - \frac{1}{s-1}\sum_{k\in S} v(S\backslash k)\right] + \frac{1}{s-1}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho). \tag{19}
$$

For the one-player case, $i \in N$, it is immediate that $a_i^i(\rho) = v(i) = \chi^i(i, v)$ and it is unique. Assume by induction that the equality holds for all $S\backslash k$, $k \in S$. Therefore, Eq. (19) implies that $a_S^i(\rho) = \chi^i(S, v)$. Moreover, as $\chi^i(S\backslash k, v)$ are also unique, it follows that $a_S^i(\rho)$ is also unique.

(2) For all $j \in S\backslash i$ we have

$$
\begin{aligned}
\left|a_{S,i}^i(\rho) - a_{S,j}^i(\rho)\right| &= \left| v(S) - \sum_{j\in S\backslash i}\left[\rho a_S^j(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash j} a_{S\backslash k}^j(\rho)\right]\right.\\
&\quad \left. - \left[\rho a_S^i(\rho) + (1-\rho)\frac{1}{s-1}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho)\right]\right|\\
&= (1-\rho)\left| v(S) - \frac{1}{s-1}\sum_{k\in S} v(S\backslash k)\right|.
\end{aligned}
$$

Then it holds that

$$
\lim_{\rho\to 1}\left[(1-\rho)\left|v(S) - \frac{1}{s-1}\sum_{k\in S} v(S\backslash k)\right|\right] = 0.
$$

$\square$

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